Buffon's Needle Simulation

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1 Introduction

Buffon's Needle is a classic probability experiment that involves dropping a needle of a certain length L onto a floor with equally spaced parallel lines, separated by a distance D where $D > L$. The goal is to calculate the probability that the needle will cross one of the lines.

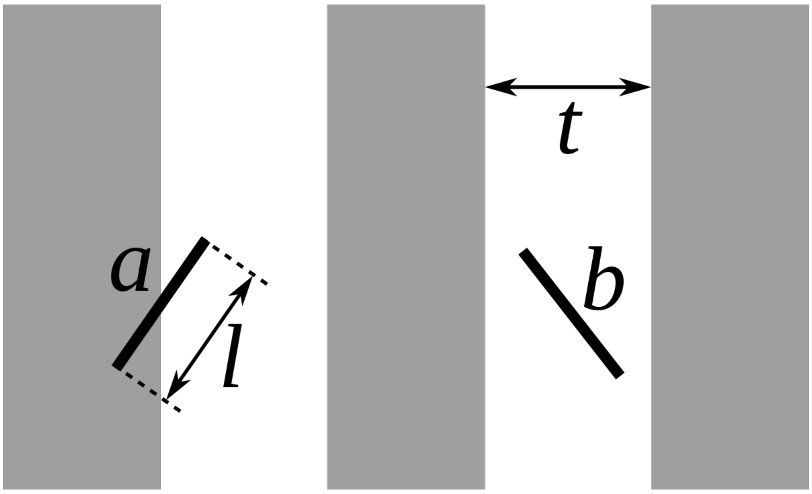

Figure 1: Buffon's needle problem

2 Analysis

Let P be the probability that the needle crosses one of the lines. We can derive this probability by considering the orientation of the needle relative to the lines on the floor.

2.1 Orientation of the Needle

The orientation of the needle upon landing can be described by two parameters:

- The distance of the midpoint of the needle to the nearest line, denoted as x.
- The angle at which the needle is dropped with respect to the lines, denoted as θ.

Let x be a random variable representing the distance from the midpoint of the needle to the nearest line, uniformly distributed between 0 and $D/2$.

$$f_X(x) = \begin{cases} \frac{2}{D} & : \ 0 \leq x \leq \frac{D}{2} \\ 0 & : \text{elsewhere.} \end{cases}$$

Let θ be a random variable representing the angle at which the needle is dropped, uniformly distributed between 0 and $\pi/2$.

$$f_\Theta(\theta) = \begin{cases} \frac{2}{\pi} & : \ 0 \leq \theta \leq \frac{\pi}{2} \\ 0 & : \text{elsewhere.} \end{cases}$$

2.2 Needle Crossing Probability

For the needle to cross a line, it must satisfy the condition that $x \leq L/2 \cdot \sin(\theta)$. If this condition is met, the needle crosses a line. Therefore, the probability of the needle crossing a line is given by:

$$P = \int_0^{\pi/2} \int_0^{L/2 \cdot \sin(\theta)} \frac{2}{\pi D} \, dx \, d\theta \tag{1}$$

Solving this integral, we get:

$$P = \frac{2L}{\pi D} \tag{2}$$

3 Monte Carlo Simulation Approach

The Monte Carlo simulation method is used to estimate the probability of a needle crossing one of the lines in the Buffon's Needle problem. This approach involves randomly dropping the needle multiple times and counting the number of times it crosses a line. The estimated probability is then calculated based on the simulation results.

3.1 Simulation Procedure

To perform the Monte Carlo simulation, we follow these steps:

1. Choose the number of simulations, N, which represents the number of times the needle will be dropped.
2. For each simulation, follow these steps:
   (a) Randomly generate the distance of the midpoint of the needle to the nearest line, x, from a uniform distribution between 0 and $D/2$.
   (b) Randomly generate the angle at which the needle is dropped, θ, from a uniform distribution between 0 and $\pi/2$.
   (c) Calculate the actual lower and upper bounds of the needle's position relative to the line based on x and θ as follows:
   $$\text{Lower Bound} = x - \frac{L}{2} \sin(\theta)$$
   $$\text{Upper Bound} = x + \frac{L}{2} \sin(\theta)$$
   (d) Check if the needle crosses a line by comparing its upper and lower bounds with the line positions. If it does, count it as a crossing event.
3. Calculate the estimated probability of a needle crossing a line as the ratio of the total number of crossing events to the total number of simulations:
   $$P \approx \frac{\text{Total Crossings}}{N}$$

4 Simulation Results

In our simulation, we looked at five different setups to see when Buffon's Needle is more or less likely to cross a line. We began with a standard configuration where the needle was half the distance between lines. Then, we tested shorter and longer needles (4 and 6 units long), a very short needle (0.05 units), and an almost-as-long-as-the-gap needle (9.99 units). These setups helped us understand how needle length affects the chance of crossing a line, from typical to exceptional cases. To observe the results on different number of Needles, we run simulation on $10^2, 10^3, 10^4, 10^5, 10^6$, needles and track how many of them crossed the line.

$$Accuracy(\%) = \left(1 - \frac{|\text{Actual Probability} - \text{Estimated Probability}|}{\text{Actual Probability}}\right) \times 100$$

4.1 Observations and Conclusions

We see that as the number of needles increases i.e. sample size , our estimates get closer to the real probabilities. This indicates that the Monte Carlo method is reliable way to achieve analytic solution from simulation.

It's important to note that in some extreme cases, like when the needle is incredibly short or almost as long as the distance between lines, the simulations still do a remarkable job of getting close to the actual values with the condition of the number of simulation needs to be larger compared to other scenarios. Results fluctuates for the low number of simulations in extreme setups.

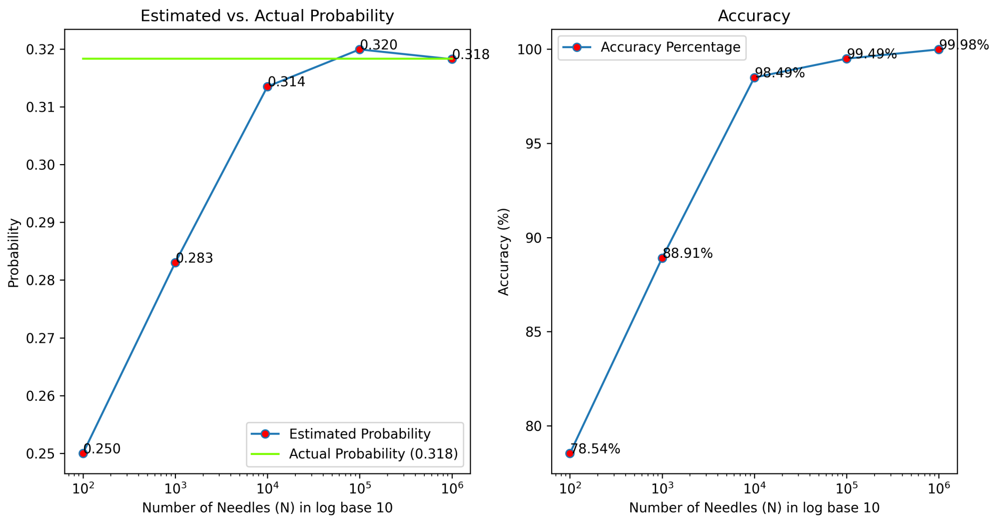

Figure 2: Simulation Results for D = 10, L = 5

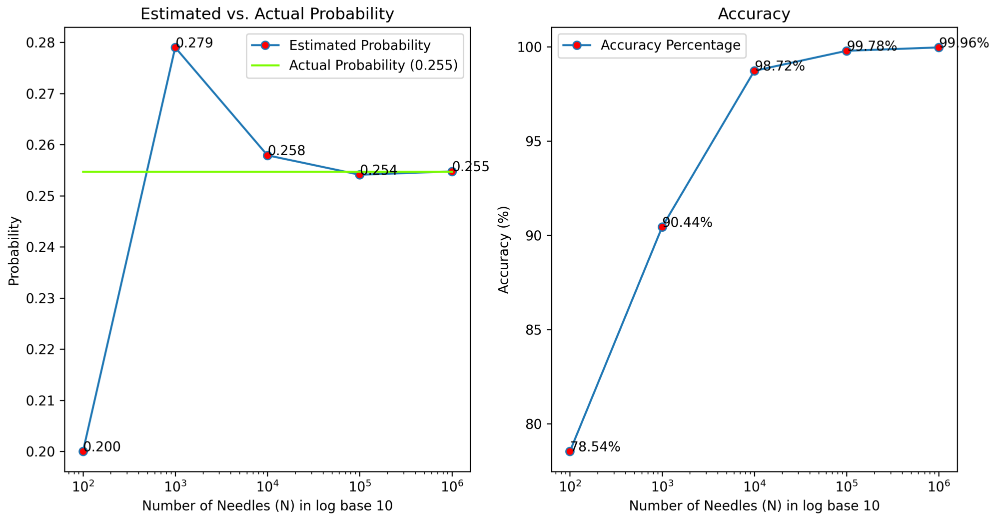

Figure 3: Simulation Results for D = 10, L = 4

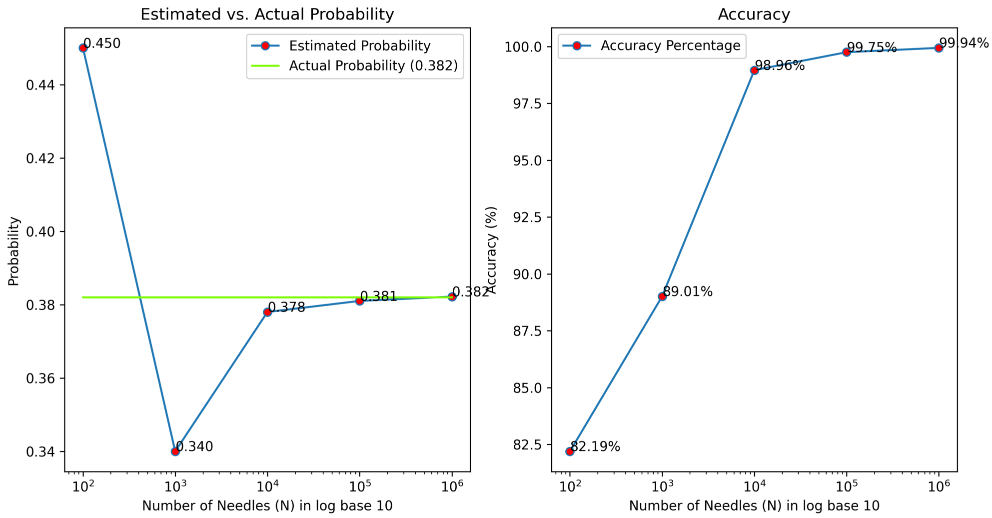

Figure 4: Simulation Results for D = 10, L = 6

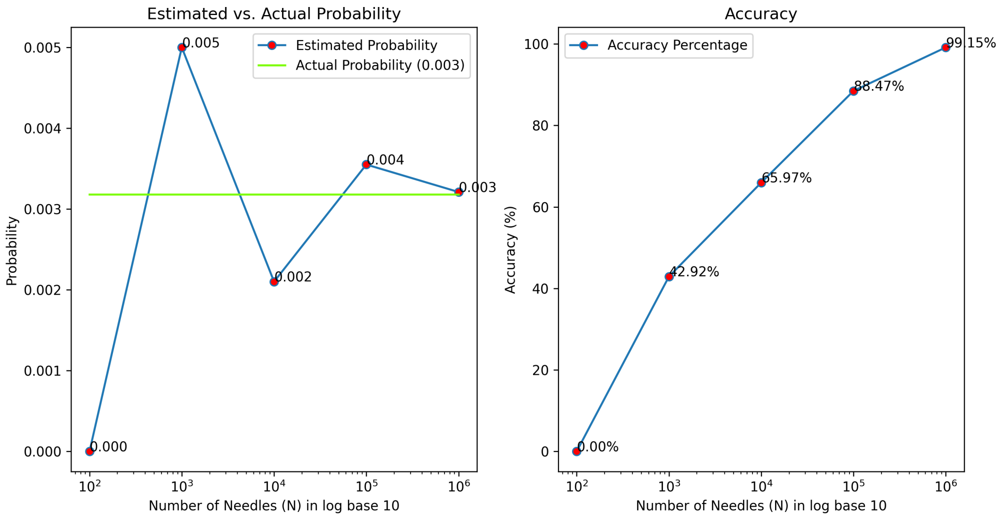

Figure 5: Simulation Results for D = 10, L = 0.05

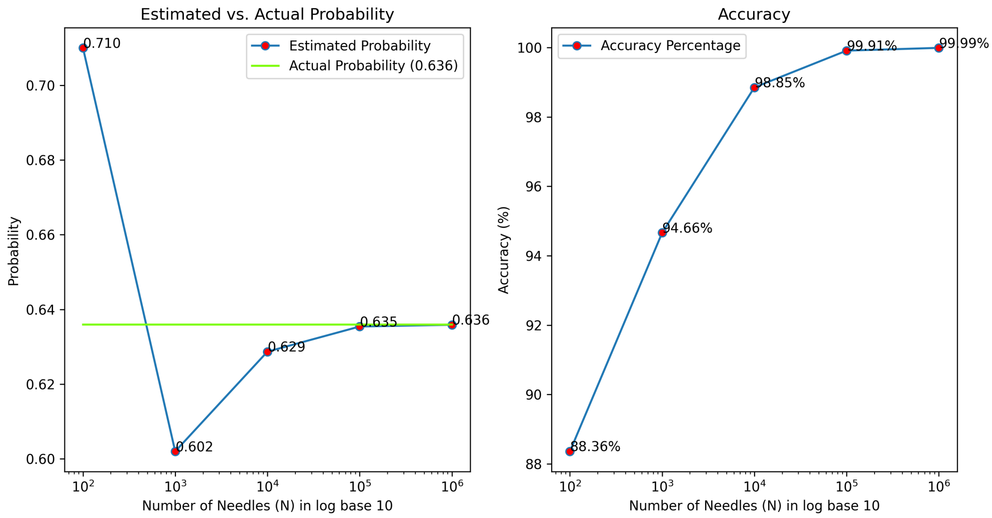

Figure 6: Simulation Results for D = 10, L = 9.99