One- and Two-Dimensional Ising Model

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Abstract

In this summary report of a seminar presentation, we explore the one- and two-dimensional Ising model, which play a foundational role in the study of phase transitions and critical phenomena in statistical physics. We begin with a brief historical overview of Ernst Ising and the development of his model. The one-dimensional Ising model is derived using a combinatorial approach, highlighting the absence of a phase transition. We then shift our focus to the two-dimensional case, where we apply the Peierls argument, the mean-field approximation, and the transfer matrix method, which ultimately leads to Onsager's exact solution. Despite involving only a discrete symmetry, this solution remarkably reveals a second-order phase transition at finite temperature without an external magnetic field. We qualitatively analyze the results and highlight the key physical differences between the two models.

1 Historical Background

Ernst Ising was born on 10 May 1900 in Cologne, Germany. He studied physics and mathematics at the University of Göttingen, 1920 he moved to Hamburg, where he began his doctoral studies under the supervision of Wilhelm Lenz, who proposed him to study a simple binary model to describe ferromagnetism [3].

In his 1924 PhD thesis, Ising solved the one-dimensional version of the model (in the absence of an external magnetic field) exactly. His solution showed that no phase transition occurs in one dimension [6]. Based on limited calculations, he incorrectly concluded that the absence of phase transition in one dimension would also hold in higher dimensions. As a result, his work initially received little attention [5].

Interest in the Ising model was revived in 1936, when Rudolf Peierls argued that a phase transition must occur in two or more dimensions [8]. This insight sparked a series of developments by several researchers, including Kramers and Wannier, which eventually resulted in the groundbreaking result by Lars Onsager in 1944 [3]. Onsager provided an exact solution of the two-dimensional Ising model without an external magnetic field and showed that it exhibits a second-order phase transition, including an exact expression for the critical temperature [7].

Meanwhile, Ernst Ising remained unaware of the growing significance of his model. With the rise of the Nazi regime, he was forced to flee Germany and later emigrated to the United States, where he became a professor. Despite his early contribution, he never returned to academic publishing. Ernst Ising passed away on 11 May 1998 [3].

2 What is the Ising Model

In the 1920s, physicists sought to understand the phenomenon of ferromagnetism. Ferromagnetism is the property of certain materials to become spontaneously magnetized at low temperatures, even in the absence of an external magnetic field. The central idea was to describe this behavior using a simple microscopic model: a lattice of elementary magnetic moments (*spins*), which interact only with their nearest neighbors, as shown in figure 1.

Each spin can take on one of two possible configurations: it can point either "up" or "down", often represented mathematically as $S_i = \pm 1$. A natural question arises: What is the ground state of such a system? The lowest energy configuration is achieved when all spins are aligned, either up or down, thereby minimizing the internal energy of interaction [6]. Consequently, the model exhibits two degenerate ground states.

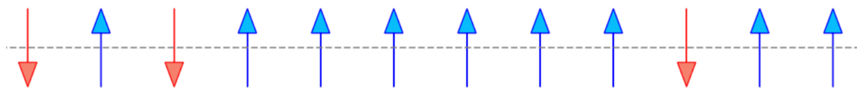

Figure 1: One dimensional Ising model: Ising chain of N spins.

However, in nature, ferromagnetic materials tend to spontaneously align in a particular direction when cooled below a certain temperature, breaking this symmetry [3]. The Ising model aims to capture this phase transition from a disordered, high temperature state where spins are randomly oriented, to an ordered, low temperature state with macroscopic magnetization.

Another central question addressed by the model is: at what temperature does this transition occur? This temperature is known as the *critical temperature* T_c. This simple yet powerful idea laid the foundation for what is now known as the Ising model, a cornerstone of modern statistical physics with applications far beyond magnetism.

3 One dimensional Ising

3.1 A combinatorial view point

Consider a one-dimensional lattice consisting of N equally spaced sites, each occupied by a spin. Figure 1. Each spin can take one of two values: *spin up* S_+ (denoted by ν_1 and assigned the value $+$) or *spin down* S_- (denoted by ν_2 and assigned the value $-$). The total number of spins is given by:

$$N = \nu_1 + \nu_2. \tag{1}$$

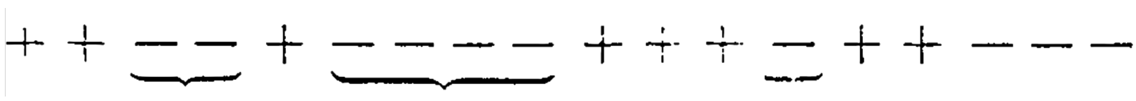

Figure 2: One dimensional lattice consisting with negative spins embedded between regions of positive spins [6].

Let the variable s denote the number of gaps embedded between regions of positive spins. That is, each of these s positions corresponds to a contiguous group of negative spins bounded by positive ones as shown in figure 2.

To keep track of the orientation at the boundary, we assume that the spin chain starts with a positive spin on the left. Depending on whether the chain ends with a positive or negative spin, we introduce an auxiliary variable $\delta \in \{0, 1\}$. This binary indicator ensures consistency in accounting for the number of alternating regions. In particular, the number of intervals available for distributing negative spin clusters is given by $s + \delta$.

With the first spin fixed to be positive, the number of ways placing s groups of negative spins among the ν_1 positive spins is given by:

$$\binom{\nu_1 - 1}{s} \tag{2}$$

In each such configuration, the ν_2 negative spins can be distributed among the $s + \delta$ available positions in

$$\binom{\nu_2 - 1}{s + \delta - 1} \tag{3}$$

Dividing indistinguishable objects into distinguishable groups. The total number of such arrangements is given by the product:

$$\binom{\nu_1 - 1}{s} \cdot \binom{\nu_2 - 1}{s + \delta - 1} \tag{4}$$

Analogously, the first fixed spin could be negative, with all other combinatorial considerations remaining the same. In this case, the roles of ν_1 and ν_2 are interchanged, and the total number of arrangements becomes:

$$\binom{\nu_2 - 1}{s} \cdot \binom{\nu_1 - 1}{s + \delta - 1} \tag{5}$$

As mentioned earlier, the ground state corresponds to a configuration in which all spins are aligned, resulting in zero interaction energy due to the absence of spin flips. Any deviation from this ordered state accounts to an internal energy cost. Specifically, if there are s groups of flipped spins, the system pays an energy cost proportional to the number of spin-flip boundaries, given by $(2s + \delta)$, where each boundary contributes a finite energy cost $\varepsilon > 0$. Therefore, the total interaction energy due to spin mismatches can be written as:

$$E_{\text{int}} = (2s + \delta)\varepsilon \tag{6}$$

In addition, we may place the system in an external magnetic field B, which tends to align spins either parallel or anti-parallel to the field direction. This adds a Zeeman energy term that enhances the energetic difference between spin-up and spin-down states.

$$E_{\text{mag}} = -mB(\nu_1 - \nu_2) = mB(\nu_2 - \nu_1) \tag{7}$$

where the sign depends on whether the field aligns with the positive or negative spins. The total energy of the system then becomes:

$$E = (2s + \delta)\varepsilon + (\nu_2 - \nu_1)mB \tag{8}$$

Partition function

To analyze the statistical behavior of the system, we introduce the partition function. It serves as a central object in statistical mechanics, including the probabilities of all possible spin configurations. By summing over these configurations, each weighted by their corresponding Boltzmann factor β, it is defined as:

$$Z = \sum_i e^{-\beta E_i} \tag{9}$$

where $\beta = \frac{1}{k_B T}$ denotes the inverse temperature (with k_B the Boltzmann constant), and E is the total energy of the configuration.

Using the combinatorial results above, the partition function becomes:

$$Z = \sum_{\nu_1,\nu_2,s,\delta} \left[\binom{\nu_1 - 1}{s} \binom{\nu_2 - 1}{s + \delta - 1} + \binom{\nu_2 - 1}{s} \binom{\nu_1 - 1}{s + \delta - 1} \right] e^{-\beta[(2s+\delta)\varepsilon + (\nu_2 - \nu_1)mB]} \tag{10}$$

We notice that summing over all configurations can be computationally challenging, while introducing a function that encodes the dependence on the number of lattice sites N significantly simplifies the process. Therefore, we define a mathematical variable x, which has no physical meaning, and construct the generating function as:

$$F(x) = \sum_{N=0}^{\infty} Z(N)\, x^N \tag{11}$$

In many cases, especially for models with translational symmetry and regular nearest-neighbor interactions, the generating function can be expressed as a geometric series. For the one-dimensional Ising model under appropriate assumptions (e.g., no boundary effects), this takes the form [6]:

$$F(x) = \frac{2x\left[\cos(\alpha) - (1 - e^{-\beta\varepsilon})x\right]}{1 - 2\cos(\alpha)x + (1 - e^{-2\beta\varepsilon})x^2} \tag{12}$$

where $\alpha = \beta B$, with B the external magnetic field and ε the interaction energy between neighboring spins.

Using partial fraction decomposition, the coefficient of x^N yields the partition function [6]:

$$Z(N) = c_1 \left(\cos\alpha + \sqrt{\sin^2\alpha + e^{-2\beta\varepsilon}} \right)^N + c_2 \left(\cos\alpha - \sqrt{\sin^2\alpha + e^{-2\beta\varepsilon}} \right)^N \tag{13}$$

In the thermodynamic limit $N \to \infty$, the second term becomes negligible since its base is strictly less than one. Furthermore, the prefactor c_1 tends to 1. Hence, the partition function simplifies to:

$$Z(N) \approx \left(\cos\alpha + \sqrt{\sin^2\alpha + e^{-2\beta\varepsilon}} \right)^N . \tag{14}$$

From this, the free energy per spin is obtained as:

$$f(B,T) = -\frac{1}{\beta}\log Z(N)^{1/N} = -\frac{1}{\beta}\log\left(\cos\alpha + \sqrt{\sin^2\alpha + e^{-2\beta\varepsilon}}\right) \tag{15}$$

Taking the derivative of the free energy with respect to the external magnetic field B, we obtain the magnetization:

$$M = -\frac{\partial f}{\partial B} = \frac{\sin\alpha}{\sqrt{\sin^2\alpha + e^{-2\beta\varepsilon}}} \tag{16}$$

We observe that this function vanishes when $B = 0$, in other words, $\alpha = 0$. This means that the system the system exhibits no spontaneous magnetization in the absence of an external magnetic field, regardless of the temperature. Therefore, no phase transition in one dimension at any finite temperature, see figure 4. This is precisely the result that Ernst Ising obtained in his original 1924 thesis.

Let us now approach this result from a different perspective using the transfer matrix. This approach not only confirms the absence of phase transitions in one dimension, but also provides a powerful formalism for obtaining exact solutions in two dimensions without an external magnetic field.

3.2 Transfer matrix approach

We begin by writing the Hamiltonian of the one-dimensional Ising model with nearest-neighbor interactions and an external magnetic field as:

$$H = -J\sum_{\langle i,j\rangle} S_i S_j - B\mu\sum_i S_i, \tag{17}$$

where:

- J is the coupling constant between spins,
- μ is the magnetic moment (which we set to $\mu = 1$ for simplicity),
- B is the external magnetic field,
- $\langle i,j\rangle$ denotes a sum over nearest neighbors,
- $S_i \in \{-1,+1\}$ is the spin at site i.

We can write our Hamiltonian as:

$$\beta\mathcal{H} = -K\sum_{\langle i,j\rangle} S_i S_j - H\sum_i S_i, \tag{18}$$

where we define:

$$K = \beta J, \quad H = \beta B.$$

To simplify boundary effects, we assume periodic boundary conditions [10]: $S_{N+1} = S_1$.

In this case, the lattice becomes a ring (figure 3), and we can define a transfer matrix that encodes the Boltzmann weight between neighboring spins. The matrix elements of the transfer matrix T

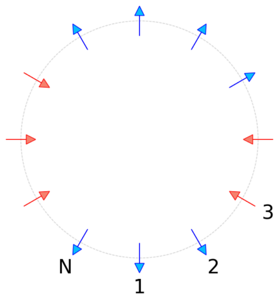

Figure 3: With periodic boundary conditions the one-dimensional Ising chain becomes a ring

are:

$$T_{S_i,S_{i+1}} = e^{KS_iS_{i+1}+\frac{H}{2}(S_i+S_{i+1})} \tag{19}$$

This allows us to write the Boltzmann factor of the full configuration as a product of transfer matrices:

$$e^{-\beta\mathcal{H}} = T_{1,2}T_{2,3}\cdots T_{N,1}, \tag{20}$$

where each $T_{i,i+1}$ is a 2×2 matrix.

$$T = \begin{pmatrix} e^{K+H} & e^{-K} \\ e^{-K} & e^{K-H} \end{pmatrix}$$

The canonical partition function becomes:

$$Z_N = \sum_{\{S_i\}} e^{-\beta H} = \sum_{S=\pm 1} \langle S_1|T^N|S_1\rangle = \mathrm{tr}(T^N). \tag{21}$$

Since T is a symmetric 2×2 matrix, it can be diagonalized. The trace becomes turn to be the eigenvalues λ_1 and λ_2. Therefore, Z_N becomes:

$$Z_N = \lambda_1^N + \lambda_2^N. \tag{22}$$

To find the eigenvalues of the transfer matrix T, we solve the characteristic equation:

$$\det(T-\lambda I) = 0,$$

where subtracting λ from the diagonal, we have:

$$T - \lambda I = \begin{pmatrix} e^{K+H} - \lambda & e^{-K} \\ e^{-K} & e^{K-H} - \lambda \end{pmatrix}.$$

The determinant is then:

$$\begin{aligned} \det(T-\lambda I) &= (e^{K+H}-\lambda)(e^{K-H}-\lambda) - e^{-2K} \\ &= e^{2K} - \lambda(e^{K+H}+e^{K-H}) + \lambda^2 - e^{-2K}. \end{aligned}$$

Thus, the characteristic polynomial becomes:

$$\lambda^2 - \lambda(e^{K+H} + e^{K-H}) + (e^{2K} - e^{-2K}) = 0.$$

The eigenvalues of the transfer matrix T are given by the quadratic formula:

$$\lambda_{1,2} = e^K \cosh(H) \pm \sqrt{e^{2K}\cosh^2(H) - 2\sinh(2K)}.$$

In the thermodynamic limit $N \to \infty$, the largest eigenvalue dominates. The partition function turn out to be:

$$Z_N \sim \lambda_1^N. \tag{23}$$

In the special case of zero external magnetic field $H = 0$,

The partition function for very large N becomes

$$Z_N \approx (2\cosh(K))^N.$$

With this, one can compute any thermodynamic quantity of interest.

Therefore, the free energy of the one-dimensional Ising model in the absence of an external magnetic field is given as:

$$F = -k_B T N \ln\left[2\cosh\left(\frac{J}{k_B T}\right)\right], \tag{24}$$

From this expression, we can already suspect that the system will not exhibit phase transition, because the free energy is an analytic function of temperature for all $T > 0$. There is no singularity or divergence, which are characteristic features of second-order phase transitions.

Moreover, the magnetization M is defined as:

$$M(T,B) = \frac{1}{Z}\sum_{\{S_i\}}\left(\mu\sum_i S_i\right)e^{-\beta H}, \tag{25}$$

which, through the thermodynamic relation, becomes:

$$M = \mu\,\partial_H \ln Z_N = \frac{\mu N}{\lambda_1}\,\partial_H\lambda_1 = \frac{\mu N\sinh(H)}{\sqrt{\cosh^2(H) - 2e^{-2K}\sinh(2K)}}$$

We see again that for $M(T \neq 0, B = 0)$, the magnetization vanishes, i.e no spontaneous magnetization at any finite temperature as depicted in figure 4.

For strong external fields, the magnetization saturates, meaning that all spins align with the field direction, reaching maximal magnetization as shown in figure 5:

$$\lim_{B\to\infty} M(T \neq 0, B) = \mu N$$

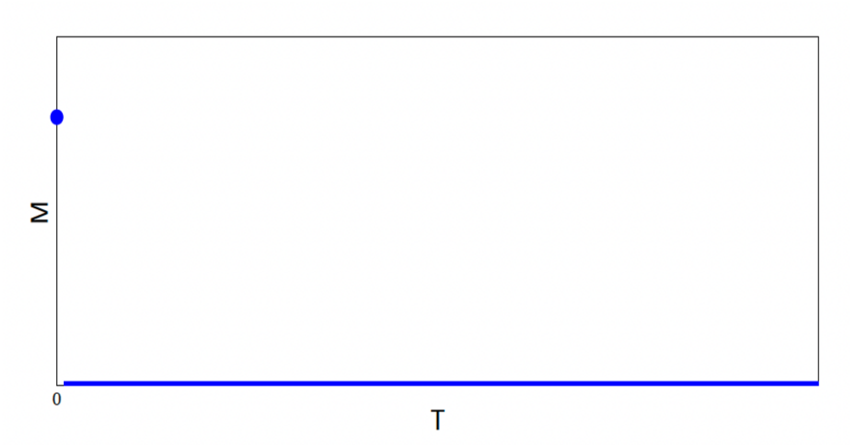

Figure 4: Magnetisation M as a function of temperature T. For $T \neq= 0$ M vanishes and shows a jumping behaviour at $T = 0$ [10].

Also as $T \to 0$ limit, with an external magnet the system becomes a step function:

$$\lim_{T\to 0} M(T,B) = \begin{cases} -\mu N, & B < 0 \\ \mu N, & B > 0 \end{cases}$$

showing a sharp transition in spin alignment with respect to the sign of the magnetic field, as shown in figure 5.

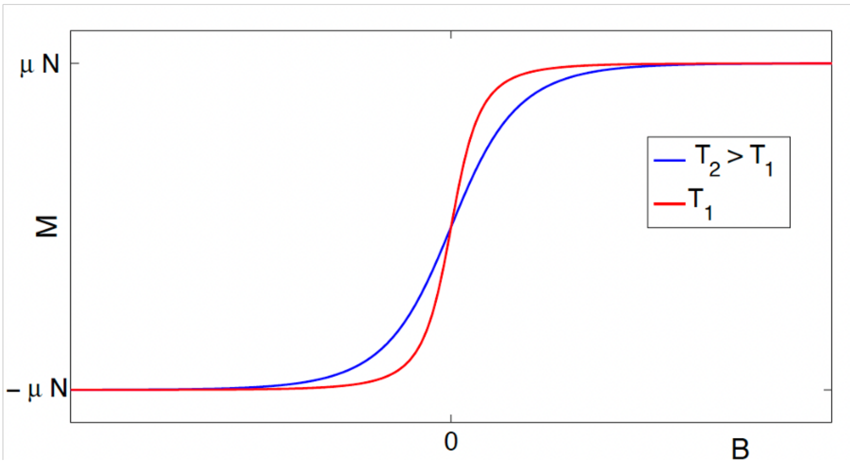

Figure 5: The magnetization M as a function of magnetic field B plotted for different temperatures [10].

This observation is fully consistent with Ernst Ising's original result from his 1924 PhD thesis, in which he used a combinatorial approach to demonstrate that the one-dimensional model does not exhibit spontaneous magnetization. Consequently, as discussed earlier, the system undergoes no phase transition at any finite temperature.

4 Two-Dimensional Ising Model

In the two-dimensional Ising model, we consider a lattice in the xy-plane consisting of rows and columns of spins. Each spin interacts with its nearest neighbors, in this case, the four adjacent spins in the up, down, left, and right directions [10].

One of the earliest intuitive explanations for the existence of a phase transition at finite temperature in the two-dimensional Ising model was the argument developed by Rudolf Peierls in 1936[3].

4.1 Peierls' Argument

Consider a square lattice in which all spins are initially aligned in the up direction. Flipping a finite region of spins creates a small island of down spins embedded in a sea of up spins. This introduces a boundary, or domain wall, between the flipped region and the surrounding spins (Figure 6).

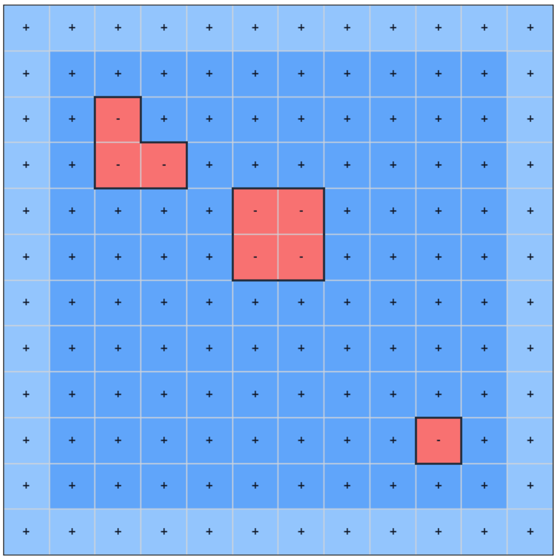

Figure 6: A two-dimensional lattice with a region of flipped spins, illustrating Peierls' argument for a phase transition in the 2D Ising model.

Let L denote the length of the closed contour (in lattice steps) that surrounds the flipped domain. Each flipped spin along this boundary contributes an energy penalty, so the total energy change is:

$$\Delta E = 2JL \tag{26}$$

where J is the nearest-neighbor coupling constant.

Imagine the number of possible shape fluctuations a domain wall can undergo, treating the domain boundary as a closed random walk on the two-dimensional lattice (as shown in figure 6). At each step (neglecting backtracking and self-intersections), the walker has approximately 3 choices. Therefore, the number of possible domain wall configurations of length L scales as 3^L, and the associated entropy is given by:

$$\Delta S = k_B \ln(3^L) = k_B L \ln 3 \tag{27}$$

From this, the change in free energy is:

$$\Delta F = \Delta E - T\Delta S = L(2J - k_B T \ln 3) \tag{28}$$

This shows that:

- If $T < T_c = \frac{2J}{k_B \ln 3}$, then $\Delta F > 0$, such domain fluctuations are energetically suppressed i.e energy wins (which favors order).

- If $T > T_c$, then $\Delta F < 0$, and entropy dominates, therefore fluctuations proliferate and the system becomes disordered.

This predicts a critical temperature T_c:

$$T_c = \frac{2J}{k_B \ln 3} \tag{29}$$

above which the system no longer maintains long-range magnetic order.

4.2 Mean field approximation

Mean field approximation (MFA) is another simplified way to study the two-dimensional Ising model by replacing interactions between neighboring spins with their average effect. This transforms the many-body problem into a single-particle problem in an effective field [12].

Let $m = \langle S \rangle$ be the average magnetization. For two neighboring spins S_i and S_j, we write:

$$S_i S_j = [(S_i - m) + m][(S_j - m) + m] \tag{30}$$

$$= m(S_i + S_j) - m^2 + (\text{fluctuation terms}) \tag{31}$$

The fluctuation terms are small and can be neglected [12]. Substituting into the Ising Hamiltonian:

$$H \approx -Jqm \sum_i S_i + \frac{1}{2} JqNm^2 - B \sum_i S_i \tag{32}$$

where N is the number of spins and q is the number of nearest neighbors. The linear terms combine into an *effective magnetic field*:

$$B_{\text{eff}} = B + Jqm$$

The partition function becomes:

$$Z = e^{-\frac{1}{2}\beta JNqm^2} \left[2\cosh(\beta B_{\text{eff}})\right]^N \tag{33}$$

Since B_{eff} depends on m, we determine the magnetization self-consistently by:

$$m = \tanh\left(\beta(B + Jqm)\right) \tag{34}$$

This equation can be solved numerically for various temperatures and fields. Its solutions can be visualized graphically by plotting both sides and finding intersections.

Phase transition at $\mathbf{B = 0}$

Setting $B = 0$, the self-consistency equation reduces to:

$$m = \tanh(\beta Jqm)$$

From figure 7, we observe that at high temperatures ($T > T_c$) without an external magnetic field, the only solution is $m = 0$. This is an unstable and corresponds to a disordered state, where spins

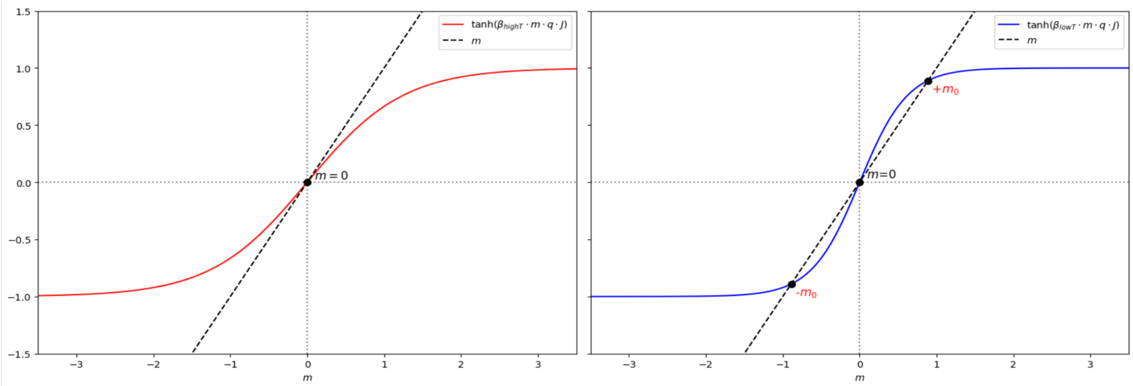

Figure 7: Self-consistency equation at high (left) and low (right) temperatures, with $B = 0$.

are randomly oriented. However, as the temperature decreases below the critical temperature T_c, two symmetric non-zero solutions emerge, indicating the onset of spontaneous magnetization, shown on right side of the graph. The critical temperature is given by:

$$T_c^{\mathrm{MF}} = \frac{Jq}{k_B} \tag{35}$$

Mathematically speaking, we can also expand the right-hand side of the self-consistency equation for small m for $B = 0$:

$$\tanh(x) \approx x - \frac{1}{3}x^3 + \cdots$$

we find the slope near $m = 0$ is βJq. The solution behavior depends on this slope (Figure 7):

- For $\beta Jq < 1$ (i.e., $T > T_c$), only the trivial solution $m = 0$ exists. This corresponds to the high-temperature, disordered phase where entropy dominates over energy (Figure 7 left).
- For $\beta Jq > 1$ (i.e., $T < T_c$), three solutions exist: $m = 0$ (unstable, as in the high-temperature case), and $m = \pm m_0$ (stable), indicating spontaneous symmetry breaking (Figure 8 left). In this regime, the effect of spin interactions overcomes thermal agitation. Furthermore, as $\beta \to \infty$ (i.e., $T \to 0$), we find $m_0 \to 1$, meaning that all spins align in the same direction.

The critical temperature where this transition occurs is [12]:

$$k_B T_c = Jq \tag{36}$$

Phase transition at B $\neq$ 0

We now consider the behavior of the magnetization in the presence of an external magnetic field B:

$$m = \tanh\left(\beta(B + Jqm)\right)$$

Two important differences arise compared to the zero-field case:

- At finite B, the system no longer exhibits a true phase transition. As the temperature increases, the magnetization decreases smoothly but never vanishes (Figure 8 right):

$$m \to \frac{B}{k_B T} \quad \text{as} \quad T \to \infty$$

- At low temperatures, the system tends to minimize its energy by aligning all spins with the external field. Similar to the $B = 0$ case, the self-consistency equation may still admit three solutions (by continuity), but only one is stable—the solution with the same sign as the magnetic field [12].

In essence, when there is no external magnetic field ($B = 0$), the Ising model exhibits a second-order phase transition at the critical temperature T_c. Below this point, the system can spontaneously magnetize in either direction ($m = \pm m_0$).

In contrast, for $B \neq 0$, no true phase transition occurs as temperature varies, the system consistently favors the direction aligned with the sign of B. However, if the temperature is held below T_c and the external field is slowly varied from negative to positive, the magnetization undergoes a sudden reversal.

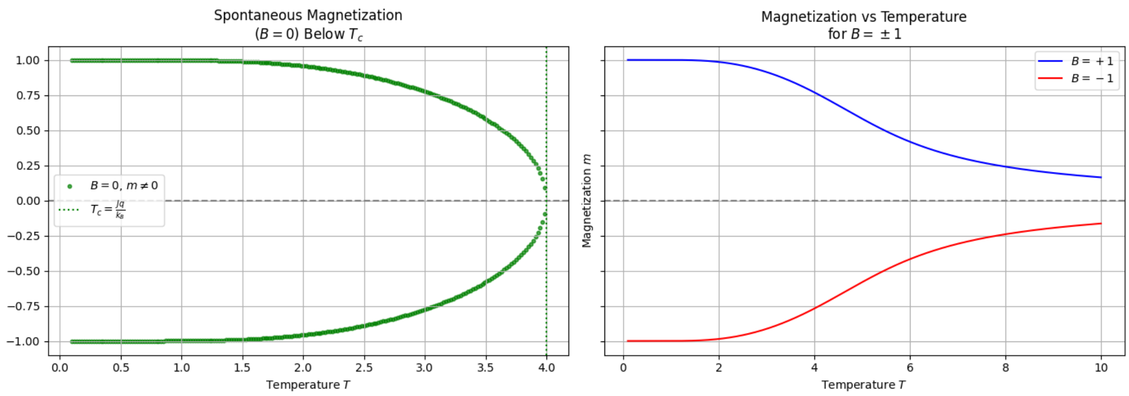

Figure 8: A phase transition at $B = 0$ (left) and no phase transition at $B \neq 0$ (right).

Furthermore, one can also calculate the critical exponents and will find that they deviates from the exact solution, which is valid because we are doing a mean field approximation.

4.3 Onsager's 1944 solution

The breakthrough in solving the two-dimensional Ising model came from Lars Onsager in 1944. His key insight was to map the spin system onto a fermionic problem, which allowed an exact treatment of the model for $B = 0$ [9] [10].

Onsager also built upon the transfer matrix method, which we previously encountered for the $1D$ Ising model [7]. In the $2D$ case, the Hamiltonian takes the form:

$$\beta H = -K \sum_{r,c} S_{r,c} S_{r+1,c} - K \sum_{r,c} S_{r,c} S_{r,c+1} \tag{37}$$

where $K = \beta J$, and the sums run over all rows r and columns c of the lattice [10]. This expression accounts for nearest-neighbor interactions both horizontally and vertically.

The partition function is written as:

$$Z = \sum_{\{S_i\}} e^{K \sum_{\langle i,j \rangle} S_i S_j} = \sum_{\{S_i\}} \prod_{\langle i,j \rangle} e^{K S_i S_j} \tag{38}$$

Each exponential term can be simplified using the identity:

$$e^{KS_iS_j} = \cosh K\left(1 + xS_iS_j\right), \quad \text{with } x = \tanh K \tag{39}$$

This transforms the problem into a product of terms involving spin pair correlations. The simplification of the exponential factors makes the model amenable to further algebraic techniques, eventually allowing Onsager to solve it exactly in terms of integrals over momentum space [10]. The free energy involves integrals that cannot be simplified and is given by:

$$-\beta f = \ln 2 - \ln(1-x^2) + \frac{1}{2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{dq\,dp}{(2\pi)^2}\ln\left[(1+x^2)^2 - 2x(1-x^2)(\cos p + \cos q)\right] \tag{40}$$

We observe that there is no closed-form solution for the integral. The integrand develops a singularity when the argument of the logarithm vanishes, indicating a non-analyticity in the free energy [10]. This occurs when $x_c = \sqrt{2} - 1$. Since $x = \tanh K = \tanh\left(\frac{J}{k_B T}\right)$, we can invert this relation to find the critical coupling K_c:

$$K_c = \frac{1}{2}\ln(1+\sqrt{2}) \approx 0.4407 \tag{41}$$

Furthermore, we can also look at the specific heat c, exactly as Onsager did, which is defined as:

$$c = K^2\frac{\partial^2 f}{\partial K^2} \tag{42}$$

Onsager numerically evaluated the specific heat and found that it diverges at the critical point, which is also a clear signature of a second-order phase transition (see figure 9). The singularity in the specific heat also aligns precisely with the critical coupling K_c, from which one can calculate the critical temperature T_c:

$$T_c = \frac{2J}{k_B \ln(1+\sqrt{2})} \approx \frac{2.269J}{k_B} \tag{43}$$

This marks the point where the system undergoes a continuous (second-order) phase transition from an ordered to a disordered state.

5 Applications of the Ising Model

Although originally developed to describe ferromagnetism, the Ising model has proven remarkably versatile across many disciplines. Its underlying binary structure makes it suitable for modeling a wide range of systems that exhibit cooperative or competing interactions.

In this context of modeling stock markets and financial systems, it provides a conceptual framework for understanding collective behavior and interactions among market participants, such as herding, panic selling, and speculative bubbles.

Mapping Financial concepts to the Ising framework

In this analogy, the key elements of the Ising model are reinterpreted as follows [11]:

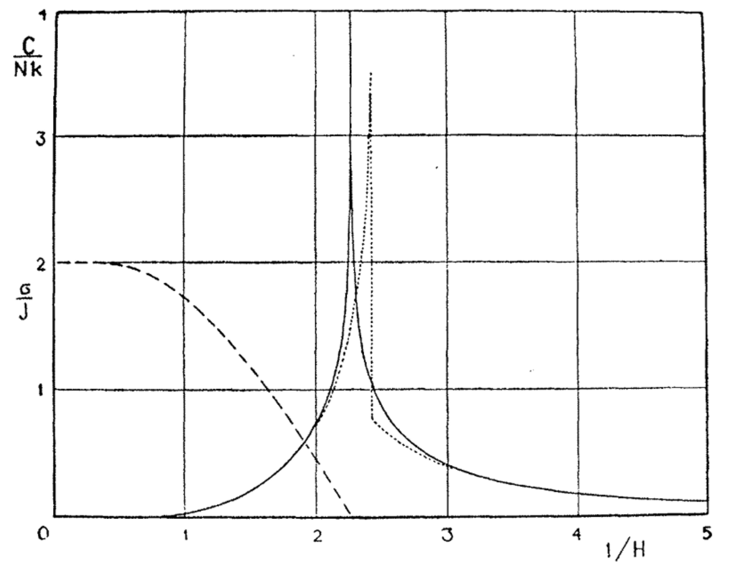

FIG. 6. Properties of quadratic crystal. — — — — — — Boundary tension σ between regions of opposite order. ———— Specific heat C. - - - - - - - - - - - - Approximate computation of C by Kramers and Wannier.

Figure 9: Specific heat diverges at the critical point, indicating a phase transition [7].

- Spins $S_i = \pm 1$ represent the decisions of individual traders or investors:

$$S_i = +1 \quad \text{(Buy)}, \quad S_i = -1 \quad \text{(Sell)}$$

- Interaction term J models peer influence or herding: traders are influenced by the behavior of others. A positive coupling $J > 0$ encourages conformity (alignment of decisions).
- External field B corresponds to external market forces or public information, such as news, economic indicators, or policy changes, which bias the collective decision in one direction.
- The energy of the system reflects market tension: configurations with many agents in disagreement (e.g., half buying, half selling) have higher energy.

This mechanism helps explain contagion effects during financial stress. The model also captures how sudden transitions can occur, resembling market crashes or speculative bubbles. When many traders abruptly shift behavior in response to a small change, the system undergoes a kind of phase transition. Furthermore, it reflects the clustering of volatility, where periods of high market fluctuations are followed by similarly turbulent phases [1]. This happens because the system can dwell in metastable configurations where local disagreement leads to instability. Finally, the model illustrates how markets can behave near criticality: when the system is near a critical point, small disturbances, like minor news or trades can lead to large-scale reactions. This mirrors the heightened sensitivity of financial systems during times of uncertainty [2].

Machine Learning:

Another remarkable modern applications of the Ising model is in the architecture of Restricted Boltzmann Machines (RBMs), a type of generative neural network used in machine learning [13]. An RBM consists of two layers of binary units: a visible layer representing observed data and a hidden layer capturing latent features [4]. The units, like spins in the Ising model, take binary values (usually ± 1 or $0/1$), and the model defines an energy function similar to the Ising Hamiltonian:

$$E(v,h) = -\sum_{i,j} v_i w_{ij} h_j - \sum_i b_i v_i - \sum_j c_j h_j$$

where v_i and h_j are the visible and hidden unit states, w_{ij} are interaction weights (analogous to coupling constants in the Ising model), and b_i, c_j are bias terms.

The joint probability distribution over the visible and hidden units follows a Boltzmann distribution:

$$P(v,h) = \frac{1}{Z} e^{-E(v,h)}$$

where Z is the partition function, analogous to the one in statistical physics. The system is trained by adjusting the weights and biases to minimize the difference between the model's distribution and the data distribution, a process akin to seeking thermal equilibrium [13].

5.1 Monte Carlo Simulation

To illustrate the dynamic behavior of the Ising model under thermal fluctuations, we can perform a Monte Carlo simulation using the Metropolis algorithm as shown in figure 10. Implementations of simulations can be found on platforms GitHub.[1]

Figure 10: Monte Carlo simulation of $2D$ Ising model

[1] https://github.com/ModouLS/Ising-Model.git

1. Initialize the spin configuration of the lattice (random or uniform).
2. Choose a lattice site i at random.
3. Compute the energy change ΔE if the spin at site i is flipped.
4. Apply the Metropolis acceptance criterion:

$$P_{\text{accept}} = \begin{cases} 1 & \text{if } \Delta E \leq 0 \\ e^{-\Delta E/(k_B T)} & \text{if } \Delta E > 0 \end{cases}$$

5. If the move is accepted, flip the spin; otherwise, leave it unchanged.
6. Repeat steps 2–5 for many iterations to reach thermal equilibrium.
7. After equilibration, compute physical observables such as magnetization, energy, specific heat, and susceptibility by averaging over configurations.

6 Conclusion

In this report, we explored the Ising model in both one and two dimensions. For the one-dimensional Ising model, where spins are arranged linearly and interact only with their nearest neighbors, a combinatorial approach revealed that the free energy remains analytic at all finite temperatures. This result confirms the absence of spontaneous magnetization for any $T > 0$, and consequently, no phase transition occurs in one dimension.

In contrast, the two-dimensional Ising model, with spins on a square lattice interacting with four nearest neighbors, exhibits qualitatively different behavior. Using Peierls' argument, we showed that below a certain critical temperature, domain walls become energetically unfavorable, enabling long-range magnetic order and spontaneous magnetization.

Within the mean-field approximation, the self-consistency equation shows that for temperatures above the critical point $T_c = Jq/k_B$, there is no stable solution, corresponding to a disordered phase. Below T_c, two symmetric, non-zero stable solutions emerge, indicating spontaneous symmetry breaking and a second-order phase transition. When an external magnetic field is introduced, the symmetry is explicitly broken, and the system continuously favors magnetization in the direction of the field.

In 1944, Onsager famously provided an exact solution for the two-dimensional Ising model in the absence of an external magnetic field. He demonstrated a singularity in the free energy and confirmed the existence of a phase transition through the divergence of the specific heat at the critical temperature T_c. Despite this remarkable result, no analytical solution has been found for the Ising model in higher dimensions. In such cases, our understanding relies primarily on numerical methods. This is due to the increased complexity of the interaction terms, which makes the mathematical treatment of higher-dimensional systems significantly more challenging and computationally expensive.

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