ISING MODEL

MODOU LAMIN SINGHATEH

SEMINAR

16. MAY 2025

Outline

Biography

Prof. Ernst Ising

10 May 1900, Cologne → 1920 Wilhem Lenz → 1922 Hamburg, PhD → 1925 1D Ising Model → 1933 School teacher → 1939 Luxembourg → 1947 USA → 1948 Professor, Illinois → Retired 1976 → 11 May 1998 He passed away

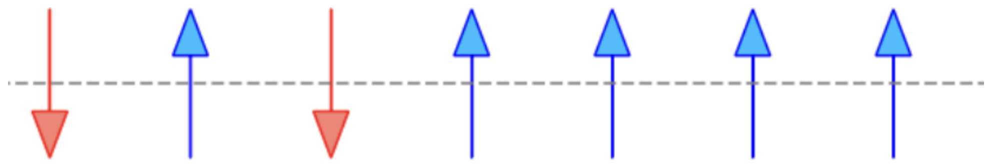
Model development

- 1925 1D Ising model
- 1928 Heisenberg
- 1936 R. Peierls 2D Ising model
- 1941 Kramer and Wannier
- 1942 Onsager 2D solution

S.G. Brush, History of the Lenz-Ising model, Rev. Mod. Phys. 39, 883 (1967)

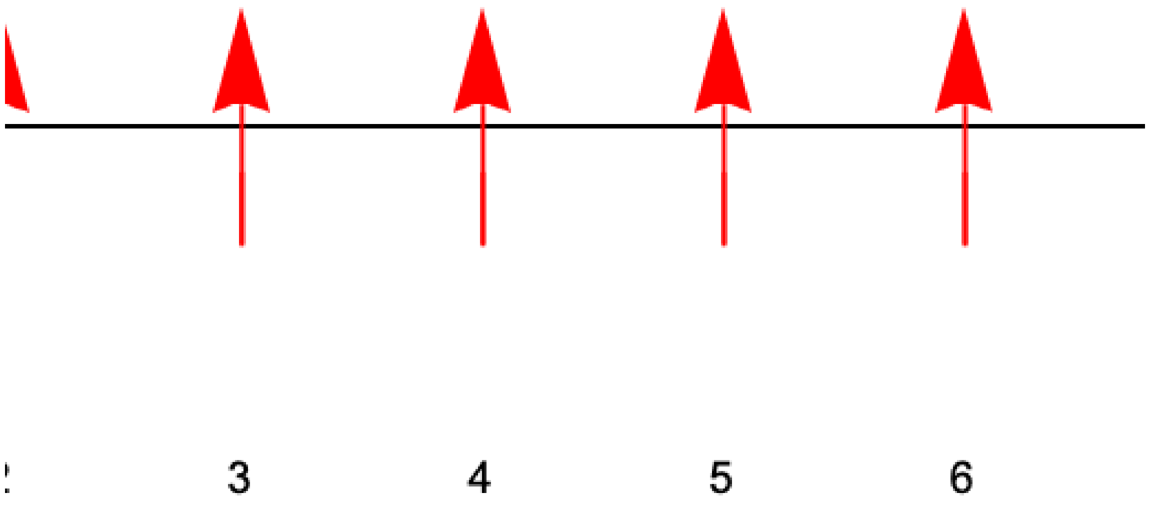
WHAT IS THE ISING MODEL

- Ferromagnetism
- Binary spins
- Interaction energy
- Phase transition
- Partition function



S.G. Brush, History of the Lenz-Ising model, Rev. Mod. Phys. 39, 883 (1967)

1D ISING MODEL



Ising's Method

Total # of elementary magnets

$$N = \nu_1 + \nu_2$$

positive negative

s gap

$$\delta \in \{0, 1\}$$

Combinatoric of choosing

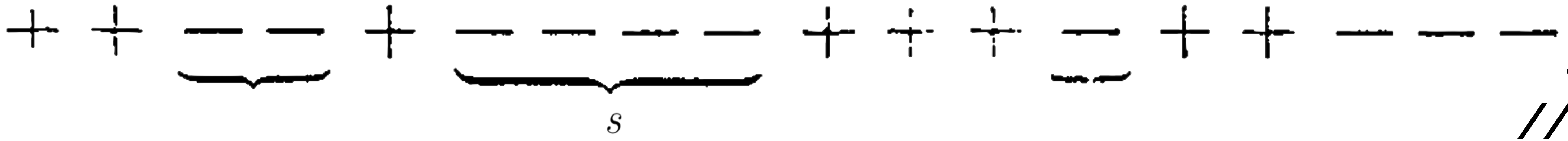
$$\binom{\nu_1 - 1}{s} \cdot \binom{\nu_2 - 1}{s + \delta - 1}$$

Total Energy

$$E = (2s + \delta) \cdot \varepsilon + (v_2 - v_1) \cdot B$$

Partition function

$$Z = \sum_{\nu_1, \nu_2, s, \delta} \left\{ \binom{\nu_1 - 1}{s}\binom{\nu_2 - 1}{s + \delta - 1} + \binom{\nu_2 - 1}{s}\binom{\nu_1 - 1}{s + \delta - 1} \cdot e^{-\beta((2s+\delta)\varepsilon + (\nu_2 - \nu_1)B)} \right\}$$



E. Ising. "Beitrag zur Theorie des Ferromagnetismus". In: Z. Physik 31, 253 (1925)

Ising's Method

$$\alpha = \beta B$$

$$F(x) = \sum_{N=0}^{\infty} Z(N) x^{\mathrm{N}} \qquad F(x) = \frac{2x\left[\cos\alpha - (1 - \exp(-\beta\epsilon))x\right]}{1 - 2\cos\alpha \cdot x + (1 - \exp(-2\beta\epsilon))x^2}$$

Partial fractional decomposition

$$Z(N) = c_1\left(\cos(\alpha) + \sqrt{\sin^2(\alpha) + e^{\frac{-2\varepsilon}{k_{\mathrm{B}}T}}}\right)^N + c_2\left(\cos(\alpha) - \sqrt{\sin^2(\alpha) + e^{\frac{-2\varepsilon}{k_{\mathrm{B}}T}}}\right)^N$$

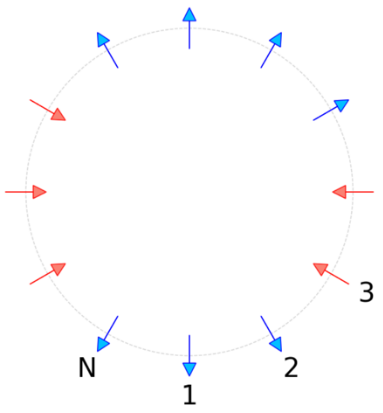
$$= \left(\cos(\alpha) + \sqrt{\sin^2(\alpha) + e^{\frac{-2\varepsilon}{k_{\mathrm{B}}T}}}\right)^N$$

Magnetisation

$$M = -\frac{\partial}{\partial B} f(B,T) = \frac{N\sin(\alpha)}{\sqrt{\sin^2(\alpha) + e^{\frac{-2\varepsilon}{k_{\mathrm{B}}T}}}}$$

E. Ising. "Beitrag zur Theorie des Ferromagnetismus". In: Z. Physik 31, 253 (1925)

TRANSFER FUNCTION



Hamiltonian

$$\mathcal{H} = -J \sum_{\langle ij \rangle} S_i\, S_j - B\mu \sum_i S_i$$

Definition

$$T_{i,i+1} := e^{K S_i S_{i+1} + \frac{1}{2} H (S_i + S_{i+1})}$$

$$K = \beta J$$

$$T = \begin{pmatrix} e^{K+H} & e^{-K} \\ e^{-K} & e^{K-H} \end{pmatrix}$$

Partition function

$$Z_N = \sum_{\{S_i\}} e^{-\beta \mathcal{H}} = \lambda_1^N + \lambda_2^N$$

$$\lambda_{1,2} = e^K \cosh H \pm \sqrt{e^{2K} \cosh^2 H - 2 \sinh 2K}$$

$$Z_N = (2 \cosh K)^N$$

Free Energy

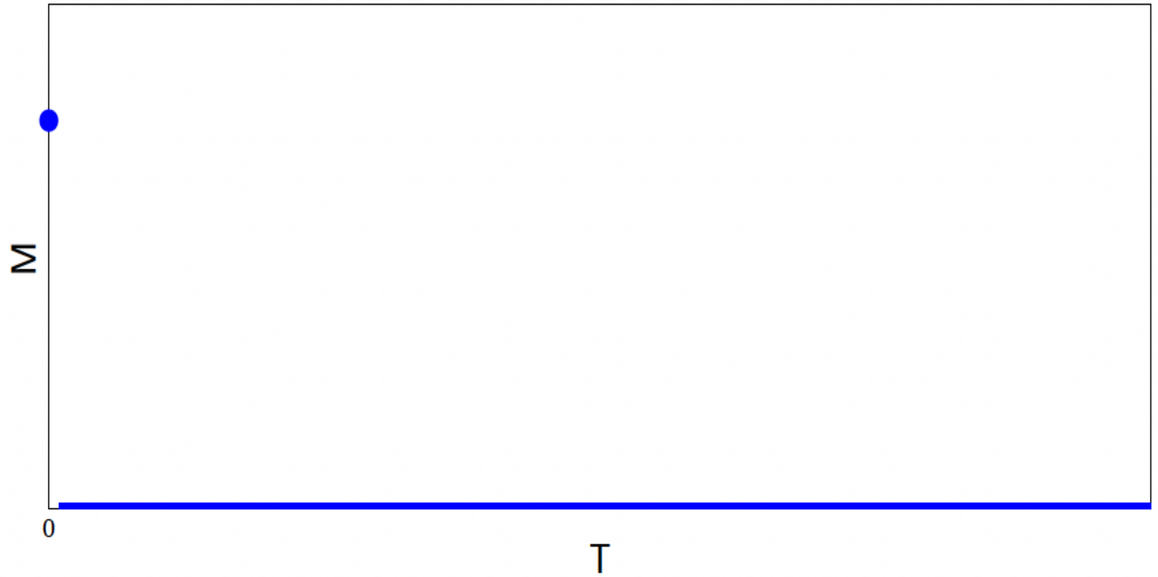
$$F = -k_B T N \ln \left(2 \cosh \frac{J}{k_B T} \right)$$

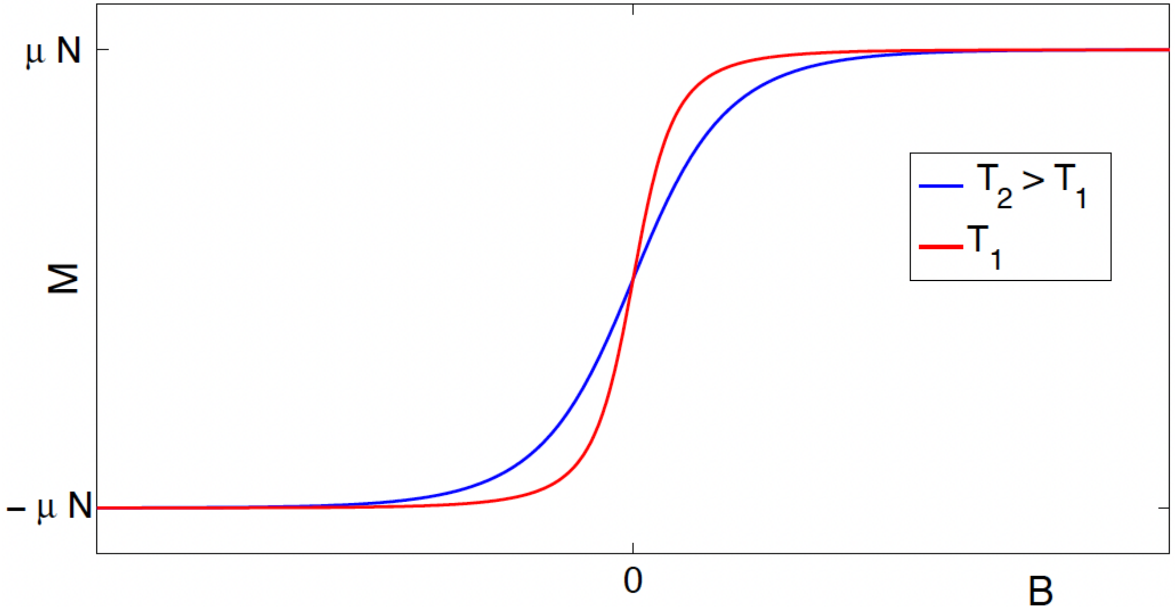
Magnetisation

$$M(T,B) = \frac{1}{Z}\sum_{\{S_i\}}\left(\mu\sum_i S_i\right)e^{-\beta\mathcal{H}} = \mu\partial_H \ln Z_N = \frac{\mu N}{\lambda_1}\partial_H\lambda_1 = \frac{\mu N \sinh H}{\sqrt{\cosh^2 H - 2e^{-2K}\sinh 2K}}$$

$$M^2 = \mu^2 N^2 \lim_{j\to\infty}\langle S_i S_{i+j}\rangle = \begin{cases}\mu^2 N^2 & T=0\\ 0 & T>0\end{cases}$$

$$M(T\neq 0, B\to\pm\infty)\to\pm\mu N$$



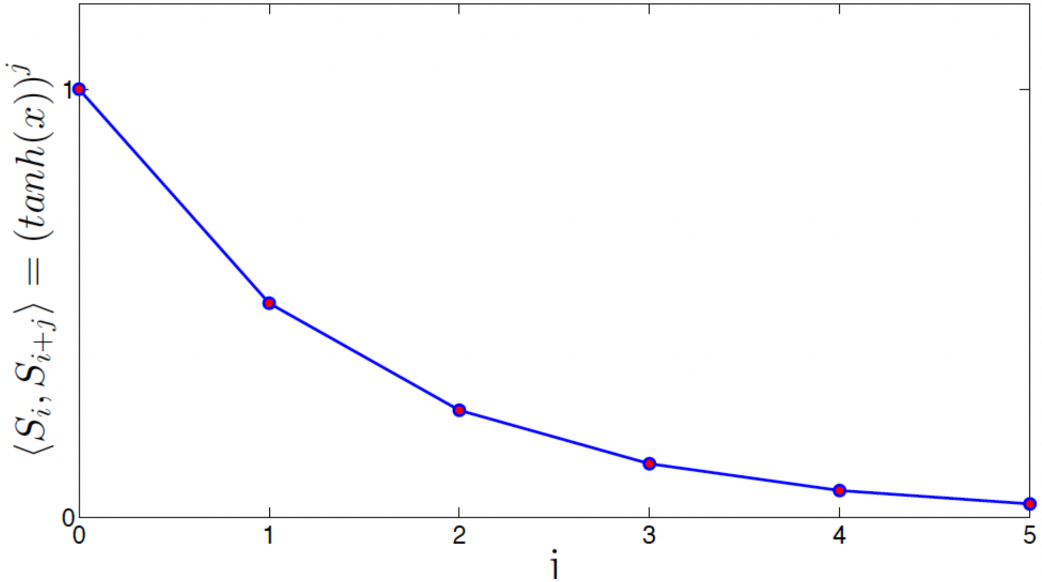


Spin Correlations

$$\langle S_i S_{i+j} \rangle = \frac{1}{Z_N} \sum_{\{S_i\}} (S_i S_{i+j}) e^{-\beta \mathcal{H}}$$

$$\langle S_i S_{i+j} \rangle = (\tanh K)^j = e^{-j/\xi}$$

$$\xi = -(\ln(\tanh(K)))^{-1}$$

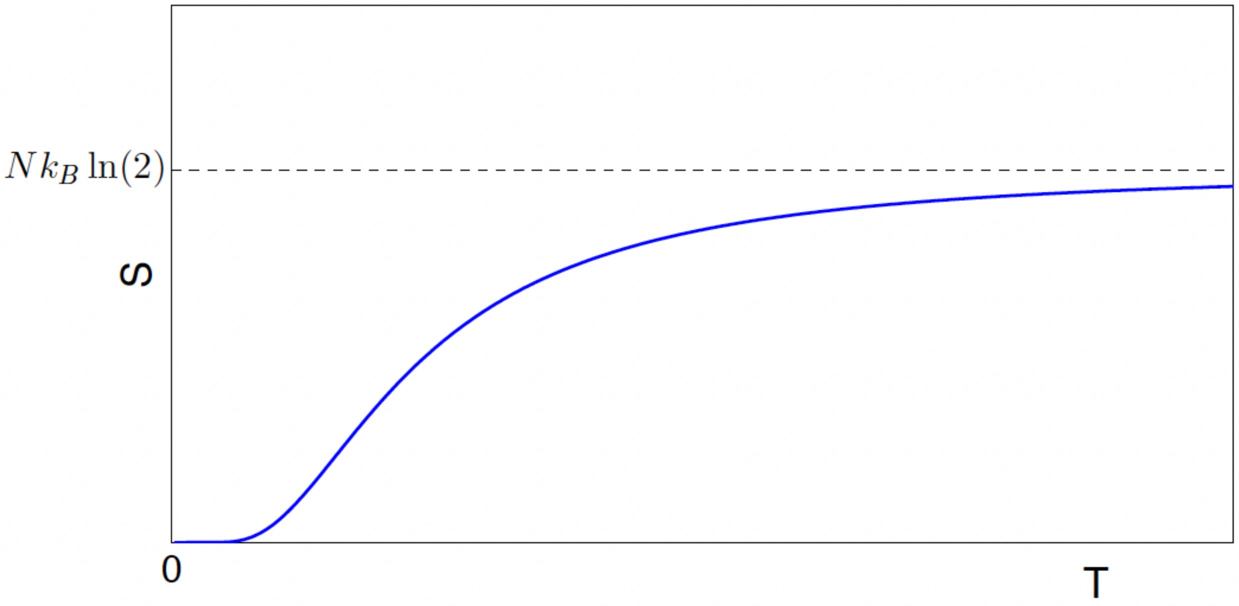


Entropy

$$S = -\frac{\partial F}{\partial T} = N k_B \left[\ln\left(2\cosh K\right) - K \tanh K\right]$$

$$S \overset{T \to \infty,\ K \to 0}{\longrightarrow} N k_B \ln 2$$

$$S \overset{T \to 0,\ K \to \infty}{\longrightarrow} N k_B (K - K) = 0$$



Recap

- Combinatoric
- No spontaneous magnetisation at finite T
- Ising’s conclusion for higher dimension
- Correlation decay exponentially

Transfer matrix

↓

Eigenvalue problem

↓

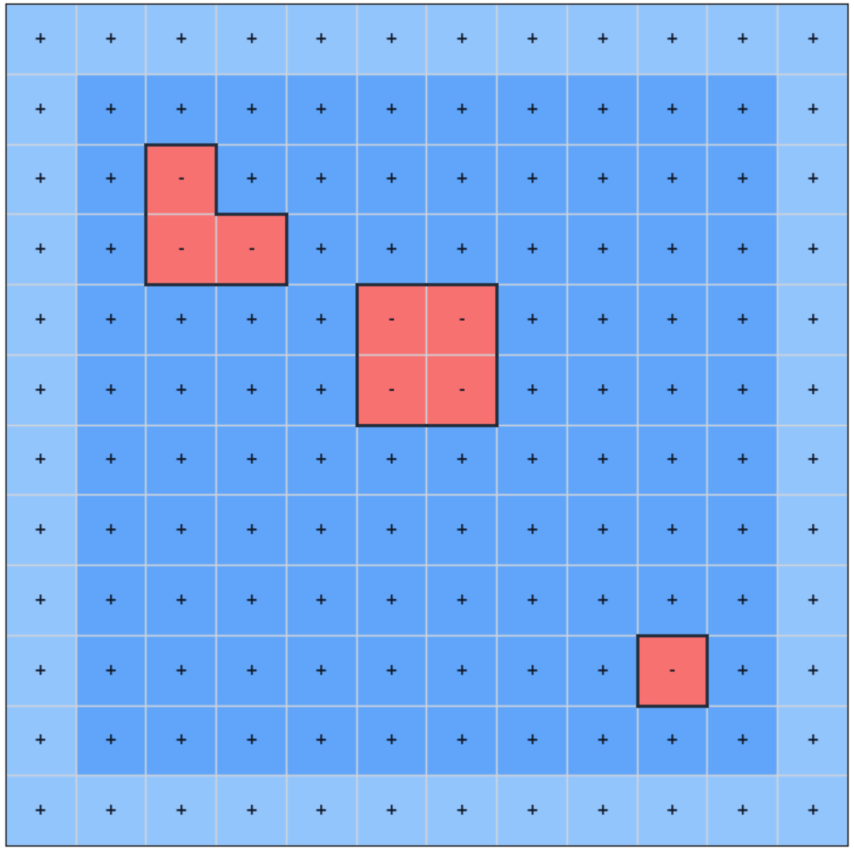
Free energy is analytic

↓

Magnetisation saturates with external B Field

2D ISING MODEL

Peierls argument



- domain walls
- Shape fluctuation
- Phase transition
- Correlation does not quickly decay

$$T > T_c = 2J/(\ln 3k_B)$$

R. Peierls, Ising’s model of ferromagnetism, Proc. Cambridge Phil. Soc., Band 32, 477 (1936)

Mean field approximation

Energy

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} s_i s_j - \mu B \sum_i s_i \; = \frac{1}{2} JNqm^2 - (Jqm + B) \sum_i s_i$$

Partition function

$$Z = e^{-\frac{1}{2}\beta JNqm^2} \left(e^{-\beta B_{\text{eff}}} + e^{\beta B_{\text{eff}}}\right)^N \qquad B_{\text{eff}} = B + Jqm$$

$$= e^{-\frac{1}{2}\beta JNqm^2} 2^N \cosh^N \beta B_{\text{eff}}$$

Self-Consistency

$$m = \tanh(\beta B + \beta Jqm)$$

David Tong. Lectures on Statistical Physics. 2012

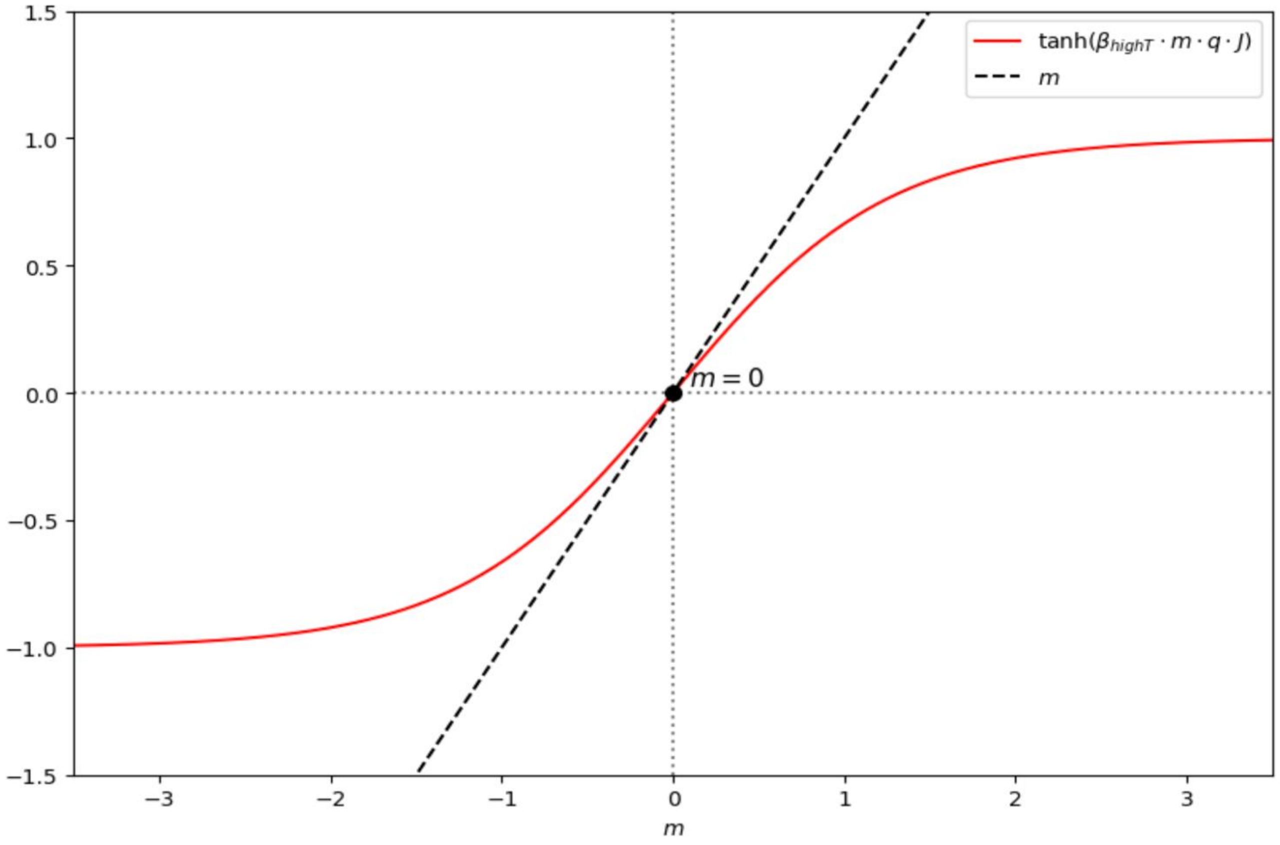
$\mathbf{B=0}$

Mean-field self consistency equation

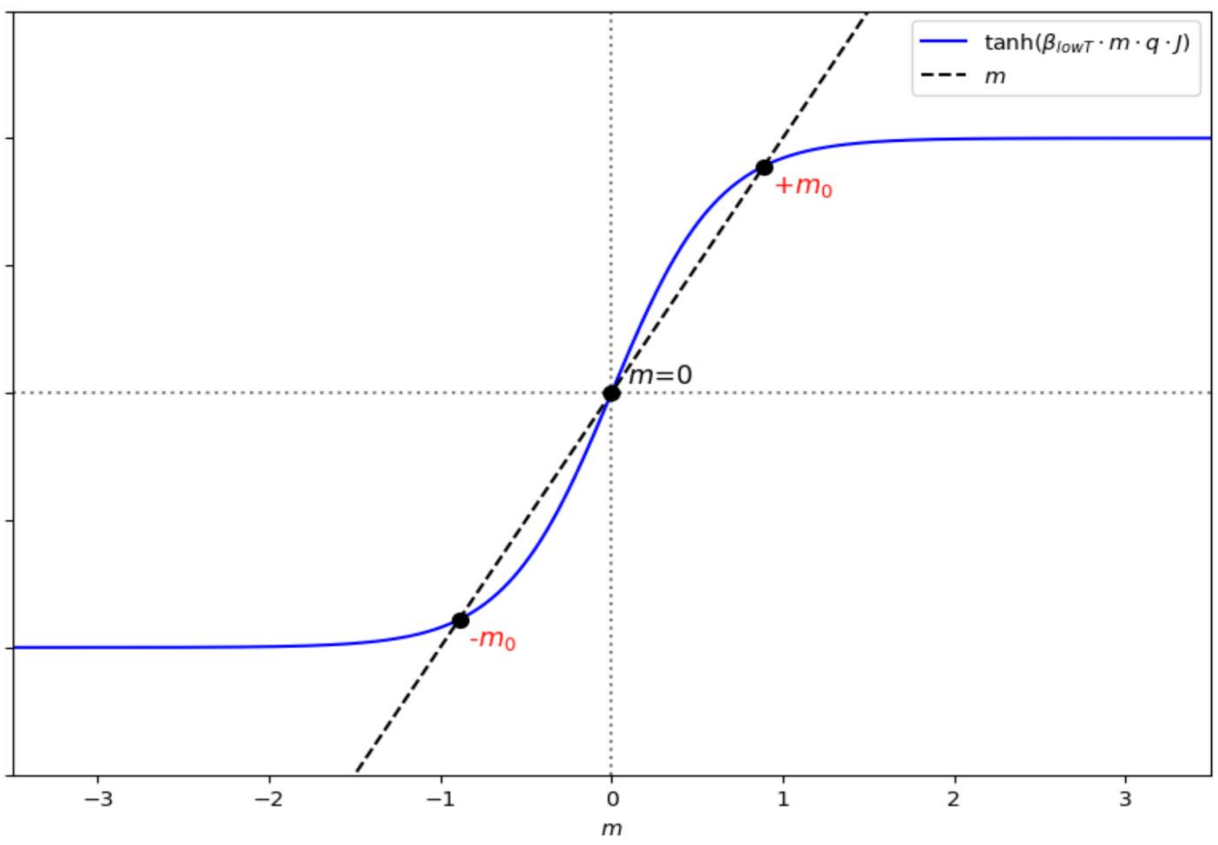
High temperature

$$m = \tanh(\beta J q m)$$

Low temperature



$\tanh(Jqm\beta)$ for $Jq\beta < 1$



$\tanh(Jqm\beta)$ for $Jq\beta > 1$

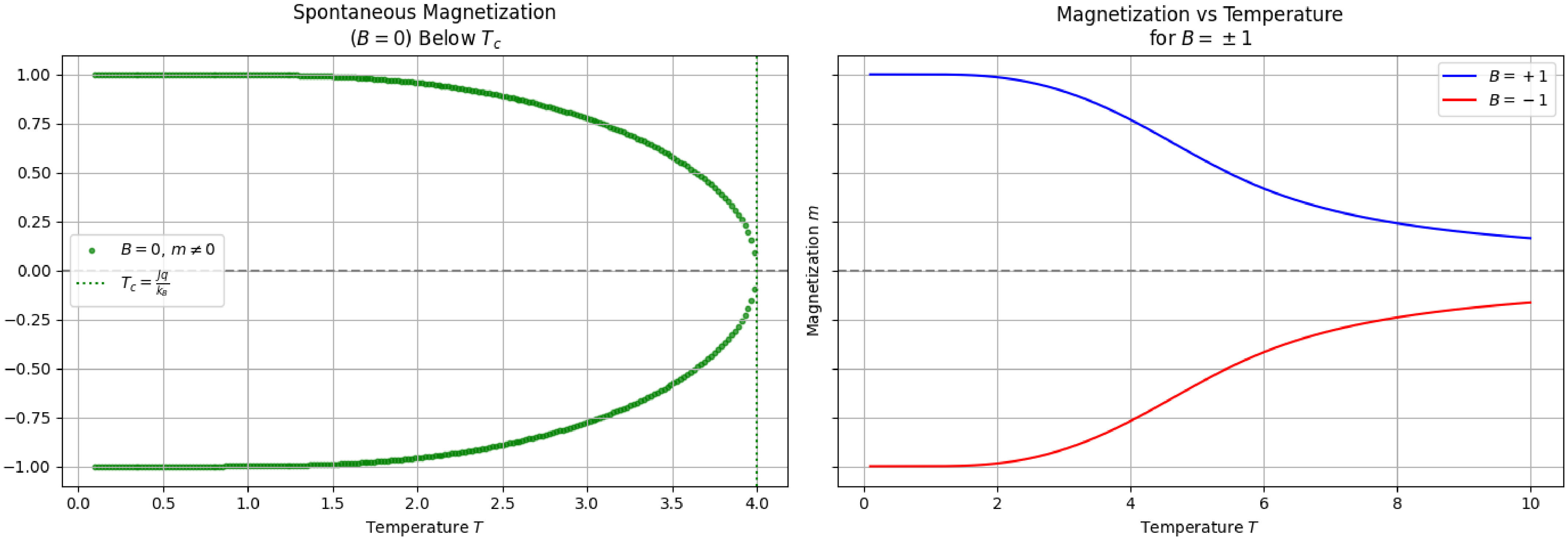
Critical temperature

$$k_B T_c = Jq$$

Mean-field self consistency equation

$$m = \tanh(\beta B + \beta Jqm)$$

- $B \neq 0$



$$m \approx \beta Jqm - \frac{1}{3}(\beta Jqm)^3 + \ldots$$

$$m_0 \sim \pm(T_c - T)^{1/2} \qquad \beta = \frac{1}{8}$$

$$m \sim B^{1/3} \qquad \delta = \frac{1}{15}$$

David Tong. Lectures on Statistical Physics. 2012

Onsager's 1944 proof

High temperature limit

$$Z = \sum_{\{S_i\}} e^{K\sum_{\langle ij\rangle} S_i\, S_j} = \sum_{\{S_i\}} \prod_{\langle ij\rangle} e^{KS_i\, S_j}$$

$$\prod_{\langle ij\rangle} e^{KS_iS_j} = (\cosh K)^{N_b} \prod_{\langle ij\rangle} (1 + xS_iS_j\,)$$

Taylor expansion

$$Z = (\cosh K)^{N_b} \sum_{\{S_i\}} \prod_{\langle ij\rangle} (1 + xS_iS_j\,)$$

$$\begin{aligned} e^{KS_iS_j} &= \cosh K + S_iS_j \sinh K \\ &= \cosh K(1 + xS_iS_j) \end{aligned}$$

$$Z = \left(\frac{1}{1-x^2}\right)^N \sum_{\{S_i\}} \prod_{\langle ij\rangle} (1 + xS_iS_j)$$

$$x := \tanh K$$

Free Energy

Specific heat

$$c = (K)^2 \frac{\partial^2 F}{\partial^2 K}$$

Critical point

$$K_c = \frac{1}{2}\ln(1+\sqrt{2}) \approx 0.4407$$

Critical temperature

$$T_c = \frac{2J}{k_B \ln(1+\sqrt{2})} \approx 2.269 J/k_B$$

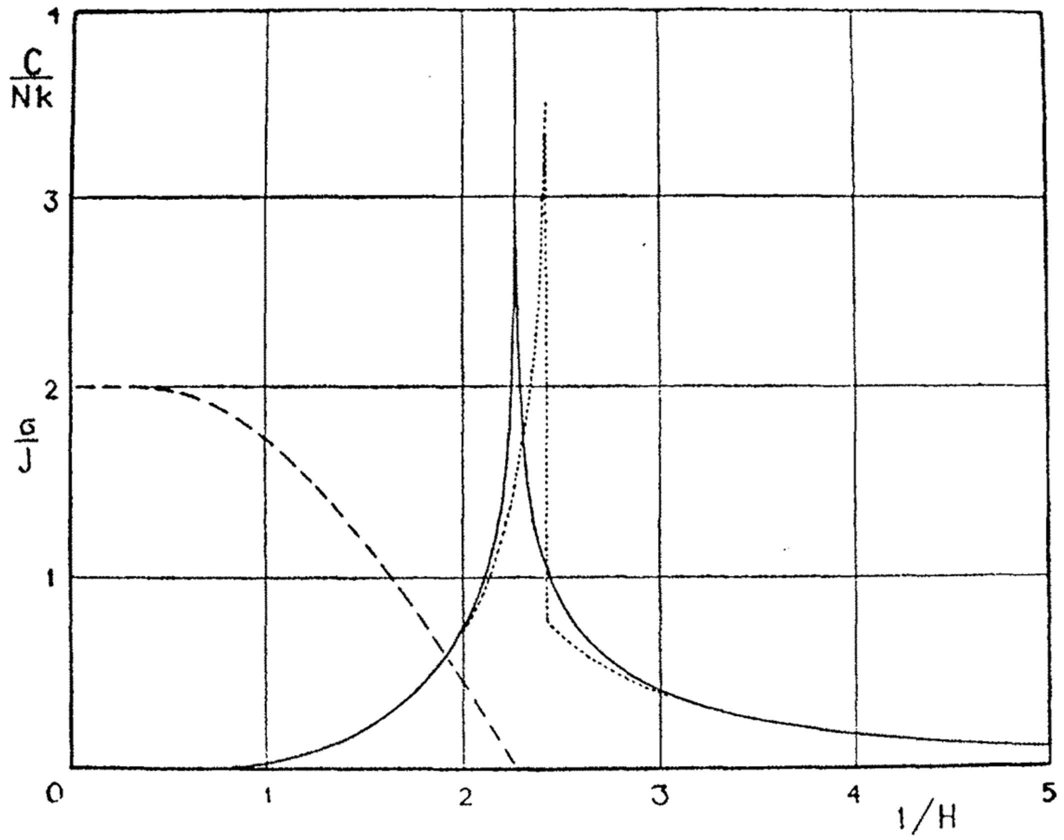


FIG. 6. Properties of quadratic crystal. — — — — — — Boundary tension σ between regions of opposite order. ———— Specific heat C. - - - - - - - Approximate computation of C by Kramers and Wannier.

L. Onsager; A Two-Dimensional Model with an Order-Disorder Transition, Phys. Rev. 1944

Application

Dynamics of stock market

Generative neural networks (RBM)

Quantum Annealing for Cancer Classification

DNA

MONTE CARLO SIMULATION OF ISING MODEL

Summary

Combinatoric and transfer matrix

No phase transition in 1D

Peierls argument

Mean field approximation

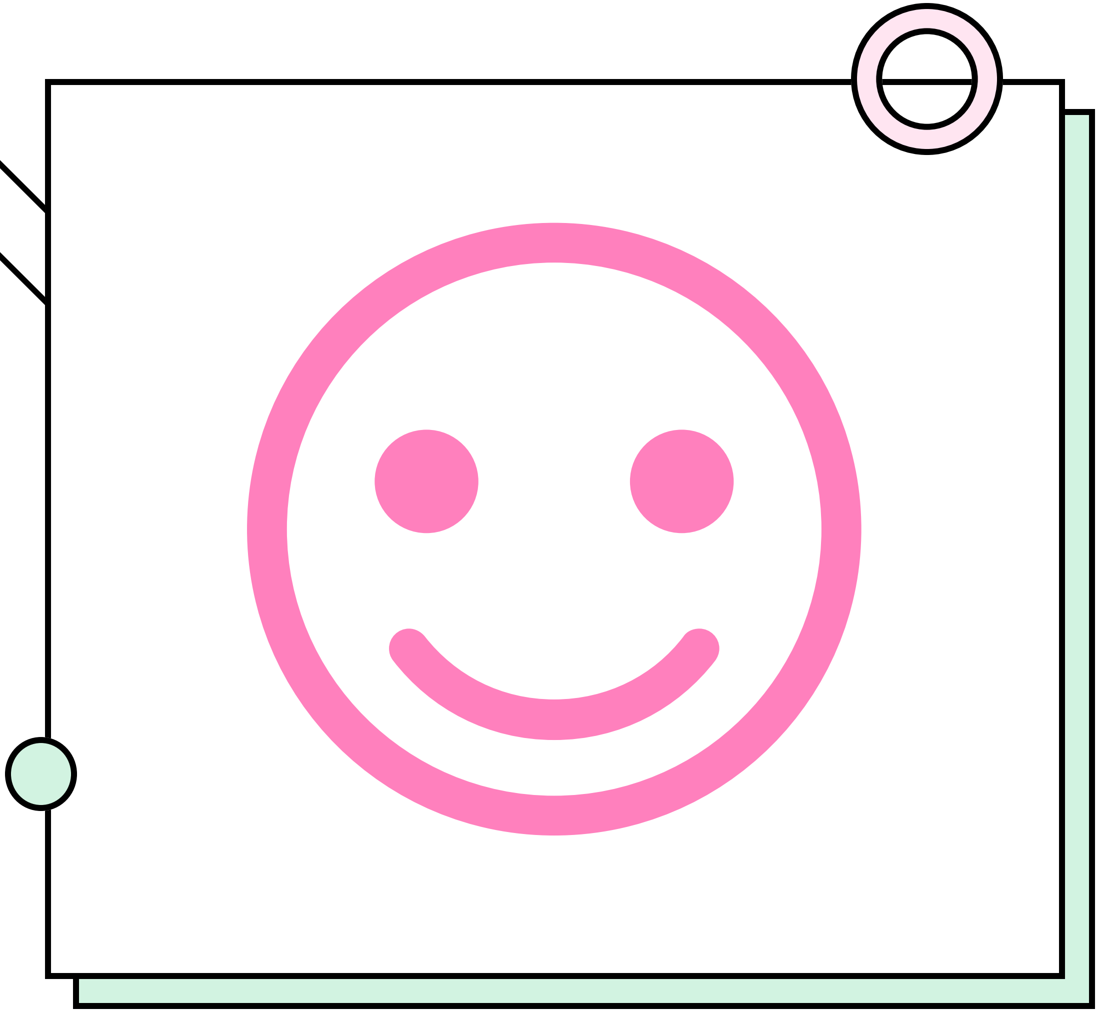
Onsagers

Application

$$T_c^{\text{Peierls}} = 1.6J/k_B$$

$$T_c^{\text{MF}} = \frac{qJ}{k_B} = \frac{4J}{k_B}$$

$$T_c^{\text{Onsager}} = \frac{2J}{k_B \ln(1+\sqrt{2})} \approx 2.269J/k_B$$



THANK YOU

Kramers-Wannier Duality

$$e^{-2\beta J} \quad \longleftrightarrow \quad \tanh \beta J$$

$$\sinh^2(2\beta_c J) = 1$$

$$\sinh 2\tilde{\beta} J = \frac{1}{\sinh 2\beta J}$$

$$2\beta_c J = \sinh^{-1}(1) = \ln(1+\sqrt{2}),$$

$$T_c = \frac{2J}{k_B \ln(1+\sqrt{2})} \approx 2.269 J/k_B$$