Assignment 5

杨乐天

Problem 1

With assignment $s(v_n) = 2n$, we know that freely occurring v_1 is assigned as $s(v_1) = 2 = 1\dot{+}1$.

- $\vDash_{\mathfrak{N}} \exists v_0, v_0\dot{+}v_0\dot{=}v_1[s]$ holds because if we let $v_0 = 1$, then $v_0\dot{+}v_0 = 1\dot{+}1 = s(v_1)$.
- $\vDash_{\mathfrak{N}} \exists v_0, v_0\dot{\times}v_0\dot{=}v_1[s]$ doesn't hold. If we let $f(x) = x \times x = x^2$ with domain $\mathbb{N}$, we know $f(x)$ is monotonically incremental and one-to-one. Then given $f(1) = 1\dot{<}s(v_1)$, $s(v_1)\dot{<}f(2) = 4$, and since there isn't any $a \in \mathbb{N}$ such that $1\dot{<}a$ and $a\dot{<}2$, we know the proposition doesn't hold.
- $\vDash_{\mathfrak{N}} \forall v_0\exists v_1, v_0\dot{=}v_1[s]$ holds. The v_0 and v_1 here are not occurring free, so the truth value of the statement isn't affected by s. For any $a \in \mathbb{N}$, let $b = a \in \mathbb{N}$, then $(v_0\dot{=}v_1)[s(v_0|a)(v_1|b)]$ holds naturally.
- $\vDash_{\mathfrak{N}} \forall v_0\forall v_1, v_0\dot{+}\dot{1}\dot{<}v_1 \to \exists v_2, v_0\dot{<}v_2 \wedge v_2\dot{<}v_1[s]$ holds. We have to prove $v_0\dot{+}\dot{1}\dot{<}v_1 \to \exists v_2, v_0\dot{<}v_2 \wedge v_2\dot{<}v_1[s(v_0|a)(v_1|b)]$ for every $a, b \in \mathbb{N}$. If the atomic formula $v_0\dot{+}\dot{1}\dot{<}v_1$ is true, then $a + 1 < b$. In this case, $v_0\dot{<}v_2 \wedge v_2\dot{<}v_1[s(v_0|a)(v_1|b)(v_2|c)]$ is true by letting $c = a + 1$. Otherwise, if the atomic formula $v_0\dot{+}\dot{1}\dot{<}v_1$ is false, then the original formula holds naturally.

Problem 2

For any structure $\mathfrak{A}$, we have to prove $\vDash_{\mathfrak{A}} \neg\exists x(Px \wedge Qx) \to \forall x(Qx \to \neg Px)$.

If $\vDash_{\mathfrak{A}} \neg\exists x(Px \wedge Qx)$, then $\vDash_{\mathfrak{A}} \exists x(Px \wedge Qx)$ doesn't hold. Further decomposing it, there doesn't exist any $a \in |\mathfrak{A}|$ s.t. $\vDash_{\mathfrak{A}} (Pa \wedge Qa)$, i.e. for any $a \in |\mathfrak{A}|$, $\vDash_{\mathfrak{A}} Pa$ is false or $\vDash_{\mathfrak{A}} Qa$ is false. Therefore, $\vDash_{\mathfrak{A}} Qa \to \neg Pa$, and thus $\vDash_{\mathfrak{A}} \forall x(Qx \to \neg Pa)$.

Problem 3

A binary relation $R \subset |\mathfrak{A}|^2$ becomes a function iff. for any a there is unique b satisfying $(a, b) \in R$.

$$\forall a\exists b(R(a, b) \wedge \forall c(c \neq b \to \neg R(a, c)))$$

Problem 4

- $\phi_1 : v_0\dot{\times}v_0\dot{=}v_0$
- $\phi_2 : (v_0\dot{+}v_0\dot{=}v_0\dot{\times}v_0) \wedge \neg(v_0\dot{\times}v_0\dot{=}v_0)$
- $\phi_3 : \exists x\exists y(\phi_2(y) \wedge x\dot{\times}y\dot{=}v_0)$