Assignment 1

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Problem 1

1

Because A is countable, A is either finite or enumerable.

If A is finite, then there exists $n \in \mathbb{N}$ and a bijection f between $\{0, 1, ..., n\}$ and A. We construct the listing $a_0, a_1, ..., a_n, ...$ as follows: $\forall i \in \mathbb{N}$, $a_i = f(i \mod (n+1))$. Then for any i, $a_i \in A$ because $\mathrm{Ran}\,(f) = A$; for any $x \in A$, $a_{f^{-1}(x)} = x$, so it is a listing of A.

If A is enumerable, then there exists bijection f between $\mathbb{N}$ and A. Then the listing $a_0, a_1, ..., a_n, ...$ is defined by $a_i = f(i)$. Because $\mathrm{Ran}\,(f) = A$, for any i, $a_i \in A$; f is a surjection so for any $x \in A$ there exists $n \in \mathbb{N}$ s.t. $f(n) = x$, and therefore $a_n = x$. Consequently, $a_0, a_1, ...$ is a listing of A.

2

Suppose $a_0, a_1, ..., a_n, ...$ is a listing of A. For any $x \in A$, there exists $n \in \mathbb{N}$ s.t. $a_n = x$. Define $f : A \to \mathbb{N}$ satisfying $f(x) = \min\{n \in \mathbb{N} | a_n = x\}$. Then f is an injection because a_n is unique for any $n \in \mathbb{N}$.

If A is finite, then A is countable.

If A is infinite, then $\mathrm{Ran}\,(f)$ is an infinite subset of $\mathbb{N}$. We further define $g : \mathbb{N} \to A$ s.t. $g(n) = x$ iff $f(x)$ is the n-th smallest element in $\mathrm{Ran}\,(f)$. Then g is obviously injection. g is also surjection because for any $x \in A$, there exists n s.t. $a_n = x$, and therefore there exists $m \leq n$ s.t. $f(x) = m$. Therefore there are at most m elements smaller than $f(x)$. In other words, there exists $k \leq m$ s.t. $g(k) = x$. Now we conclude that g is a bijection, so A is enumerable, and thus countable.

Problem 2

Let $a_0, a_1, ..., a_n, ...$ be a sequence such that $a_i = f(i)$ for any $i \in \mathbb{N}$. We prove that this sequence is a listing of A.

- $\forall i \in \mathbb{N}$, $a_i = f(i) \in A$.
- $\forall a \in A$, because f is a surjection, $\exists n \in \mathbb{N}$ s.t. $f(n) = a$. Therefore, there exists $a_n = a$.

Therefore, $a_0, a_1, ..., a_n, ...$ is a listing of A. Using the conclusion of problem 1, we know that A is countable.

Problem 3

We prove by induction.

(Base step) Expressions of length $n = 1$ are exactly the alphabet, which is enumerable.

(Induction step) Expressions of length $n + 1$ can be considered as a combination of two parts: the preceding part with length n and a suffix with length 1. Mathematically, $S_{n+1} = S_n \times S_1$. As introduced in the class, the Cartesian product of two enumerable sets are also enumerable. Therefore, for any finite $n \in \mathbb{N}^+$, if S_n is enumerable, then S_{n+1} is enumerable, too.

By induction, we have shown that for any $n \in \mathbb{N}^+$, S_n is enumerable.

Problem 4

1

$$((\neg (A_2 \wedge A_3)) \rightarrow (\neg A_1)) \tag{1}$$

Explanation: The sentence is "not A_1 unless A_2 and A_3", and "not A unless B" means as long as B isn't true, A can't be true, which is translated into $\neg B \rightarrow \neg A$.

2

$$(A_1 \rightarrow (A_2 \vee (\neg A_3))) \tag{2}$$

Explanation: The sentence is "if A_1 then A_2 or not A_3" where "if ... then ..." is translated to $(... \rightarrow ...)$.

Problem 5

We prove by induction that the length of a wff without negation is $4k - 3$ if there are k sentence symbols.

(Base case) For the wff with only one sentence symbol A, the only valid wff is the sentence symbol itself, which has length $1 = 4 \times 1 - 3$.

(Induction step) Assume that $\forall i \leq k$, the length of wffs without negation is $4i - 3$ if there are i sentence symbols, we consider the case when $i = k + 1$. Suppose α is a wff with no negation and $k + 1$ sentence symbols, then $\alpha = (\beta \Box \gamma)$ where β and γ are wffs and $\Box$ is one of $\{\wedge, \vee, \rightarrow, \leftrightarrow\}$. Here the number of the wffs n_β and n_γ satisfy $n_\beta + n_\gamma = k + 1$. Because n_β and n_γ are positive integers, we know $n_\beta \leq k$ and $n_\gamma \leq k$. Therefore, the length of α is $n_\alpha = 1 + (4n_\beta - 3) + 1 + (4n_\gamma - 3) + 1 = 4(n_\beta + n_\gamma) - 3 = 4k - 3$.

Then by induction, the length of a wff without negation is $4k - 3$ if there are k sentence symbols. Therefore there are more than a quarter sentence symbols.