Mathematical Logic: Introduction

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Wechat Group

Course Information

Textbooks:

- *A Mathematical Introduction to Logic (Second Edition)*, Herbert B. Enderton
- *Logic in Computer Science: Modelling and Reasoning about Systems (Second Edition)*, Michael Huth and Mark Ryan

Teaching Staff:

- *Lecturer*: Yuting Wang
  - Office hours: Monday 16:00-18:00
  - Location: IEEE Building No.1, Room 203
- *Teaching Assistants*:
  - Ling Zhang
  - Siyu Liu
  - Jinhua Wu

Grading

- Assignments (50pt) + Quizzes (10pt) + Final Exam (40pt)

Why Study Logic?

Logic is the mathematics of Computer Science
as
Calculus is the mathematics of Physics.

Overview

- What is Logic?

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- What is Logic?
- Uses of Logic

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- Uses of Logic
- Limitations of Logic

What is Logic?

Definition of Logic Systems

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- **Formal Languages**: what expressions are legal with respect to the syntax? (Known as *propositions*)

- **Proof Systems**: deriving propositions from existing ones via *proof rules*.

A Simple Example of Informal Reasoning

Laozi by Zhang Lu, Ming Dynasty

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- Conclusion: Laozi is awake

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A Simple Example of Logical Proofs

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Complete Proofs

- $A =$ "Laozi is asleep"
- $B =$ "Laozi is awake"
- $C =$ "Laozi is a man"

$$\dfrac{\textbf{not } A \qquad \dfrac{C \qquad \dfrac{\textbf{for any } X \text{ ,}\textbf{if } X \text{ is a man, } \textbf{then } X \text{ is either asleep or awake}}{\textbf{If } C \textbf{ then } (A \textbf{ or } B)}}{A \textbf{ or } B}}{B}$$

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We had a glimpse of syntax and proof rules, but what do all these mean?

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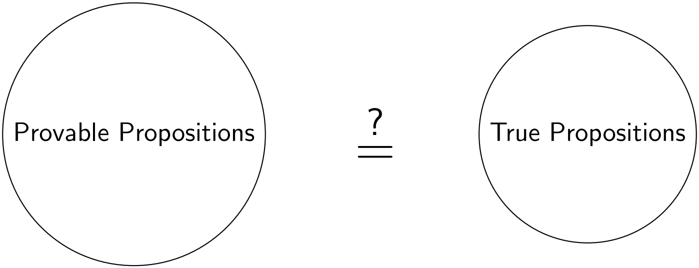
Provability vs. Truth:



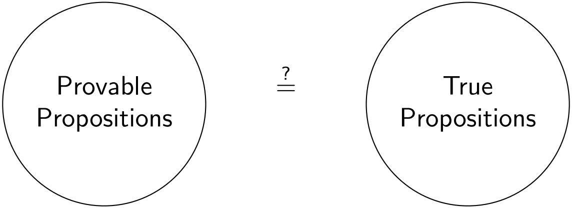
Syntax vs. Semantics

Dichotomy in Logic:

- Syntax vs. Semantics
- Form vs. Essence
- Sense vs. Denotation
- Proofs vs. Models
- ...

Logic vs. Computation

- Provability
- Truth



Is there a mechanical way to decide whether a proposition is provable?

Uses of Logic

Mathematics

Rigorous mathematical reasoning:

Terrance Tao's three stages of mathematical educations:

- *Pre-rigorous stage*. Informal, examples, fuzzy, hand-waving...
- *Rigorous stage*. Think in a precise and formal manner...
- *Post-rigorous stage*. Unconciously rigorous, intuition, "big picture"...

https://terrytao.wordpress.com/career-advice/theres-more-to-mathematics-than-rigour-and-proofs/

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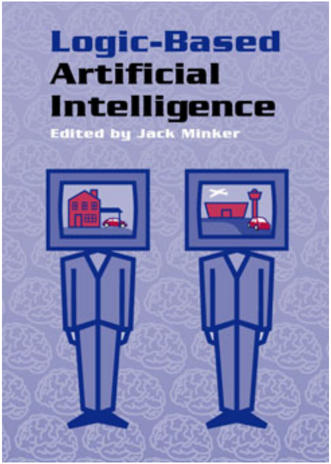
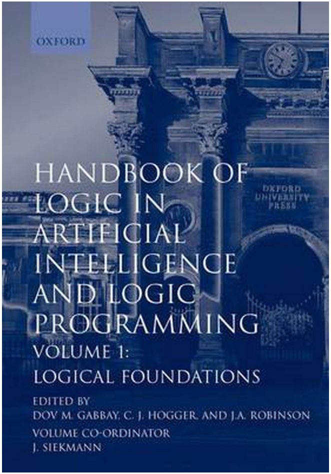
Mathematical logic is the tool for reaching the rigrous stage from the pre-rigorous one and for advancing to the final stage.

Artificial Intelligence

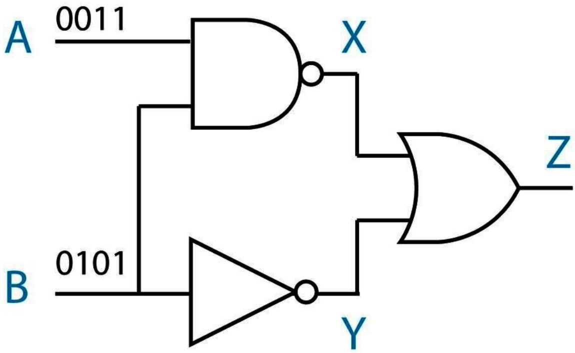
- Inference of Knowledge
- Automated Reasoning
- Logic Programming





Hardware Engineering

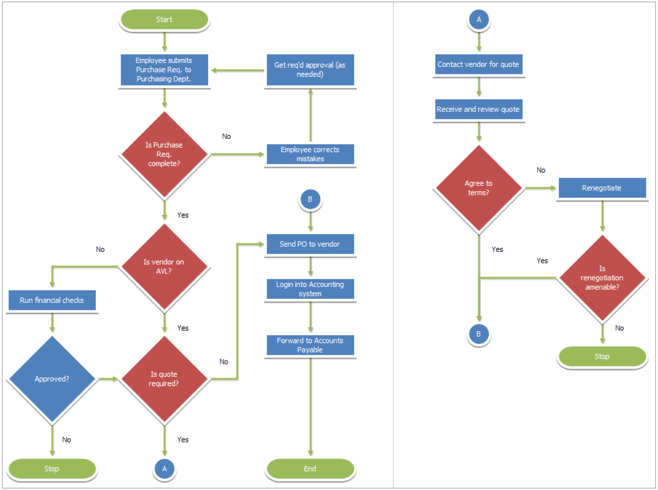
Logic Gates:



Source: `https://glossaryweb.com/wp-content/uploads/2018/03/logic-gate.jpg`

Programming Languages

Semantics and Program Verification



Source: `https://www.softwaretestinghelp.com/flowchart-software/`

Many Other Applications

- **Software Engineering**: Specification and Verification
- **Databases**: Relational Algebra and SQL
- **Algorithms and Theory of Computation**: complexity and computability
- ...

https://www.cs.cmu.edu/~rwh/papers/unreasonable/basl.pdf

Limitations of Logic

Axiomatizable Systems

An axiomatizable system consists of

- A decidable set of axioms, and
- Inference rules

An Example

Axioms Γ:

- Axiom 1: Laozi is a man
- Axiom 2: Laozi is not asleep
- Axiom 3: If X is a man, then X is either asleep or awake

Inference Rules:

- $$\frac{\textbf{for any } X,\ P\ X}{P\ t}\ X = t$$
- $$\frac{A \quad \textbf{If } A \textbf{ then } B}{B}$$
- $$\frac{\textbf{not } A \quad A \textbf{ or } B}{B}$$

Consistency and Completeness

We write $\Gamma \vdash P$ to denote that P is derivable (provable) from Γ.

An axiomatizable system with axioms Γ is

- consistent if there is no P s.t. $\Gamma \vdash P$ and $\Gamma \vdash$ **not** P;
- complete if for every P, either $\Gamma \vdash P$ or $\Gamma \vdash$ **not** P.

Remarks:

- Consistency ensures no contradiction can be derived;
- Completeness means any proposition is decidable w.r.t. Γ.

Incompleteness

Gödel's Incompleteness Theorem:

Given a sufficiently expressive consistent axiomatizable system F,

- there is some proposition P such that neither P or **not** P is provable in F;
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Corollary of the incompleteness theorem:

> *The consistency of F, when expressed as a sentence, is intuitively true. Therefore, it is often called "true but not provable".*

Implications in Mathematics

David Hilbert's program:

- *To establish mathematics as both complete and decidable*
- *To find absolute proofs of consistency, i.e. proofs that establish the consistency of an axiomatizable system without assuming the consistency of another axiomatizable system.*

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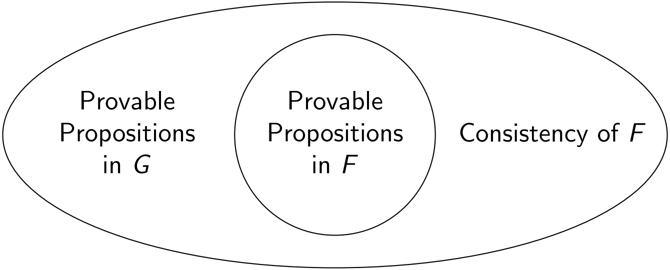
Gödel's Incompleteness Theorem gives a definitive negative answer to Hilbert's program.



Implication in Computer Science

No sufficiently powerful and consistent formal (computational) system is decidable.

In fact, incompleteness informally is equivalent to the fact that there exists *effectively enumerable* sets that are not *effectively decidable*.

Incompleteness is related to the undecidability of Turing's Halting problem.

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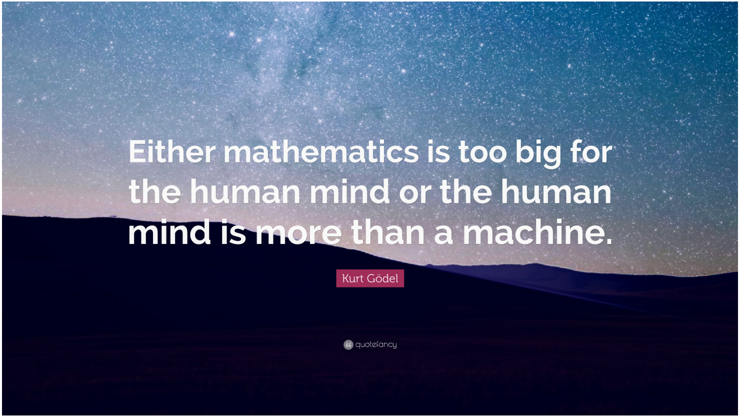
Conclusion:

Minds and Machines are Inherently Different!

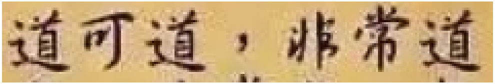

Either mathematics is too big for the human mind or the human mind is more than a machine.
Kurt Gödel
quotefancy

Objections to Lucas' Argument

The argument depends on knowing human mind is **consistent**. What if mind is **inconsistent**?

See interesting discussions at
https://www.sabinasz.net/godels-incompleteness-theorem-and-its-implications-for-artificial-intelligence

Connections to Eastern Philosophies



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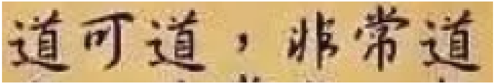

Course Contents

Part 1: Preliminaries

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Part 2: Propositional Logic

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Part 4: Applications in CS

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- **Automatic Proof Deduction**
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