Sentential Logic: Syntax and Semantics

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- Read Enderton's, Chapter 1, to keep up with lectures
- Chapters to read are described on Canvas

Syntax

The Language of Sentential Logic

In general, there are two parts to a language:

- Syntax. It provides
  - *Symbols* of the language
  - *Grammars* characterizing *well-formed formulas* (wffs)
- Semantics. It provides
  - A way to assign *meaning* to well-formed formulas
  - In sentential logic, the meaning assigned will be either TRUE or FALSE

The Logical Symbols

The symbols are divided into *logical symbols* and *non-logical symbols*.

Logical symbols include

- *Sentential Connectives*.

Symbol	Name	English
$\neg$	negation symbol	not
$\wedge$	conjunction symbol	and
$\vee$	disjunction symbol	or
$\rightarrow$	conditional symbol	if ... then ...
$\leftrightarrow$	biconditional symbol	if and only if

- *Parentheses*.

Symbol	Name
(	left parenthesis
)	right parenthesis

The Non-Logical Symbols

Non-logical symbols is the following enumerable set of elements:

$$A_1, A_2, \ldots, A_n, \ldots$$

They are also called:

- *sentence symbols*;
- *parameters*;
- *propositional symbols*.

We call A_n the n-th sentence symbol.

Summary of Symbols

Symbol	Name	Class	English
$\neg$	negation symbol	logical	not
$\wedge$	conjunction symbol	logical	and
$\vee$	disjunction symbol	logical	or
$\rightarrow$	conditional symbol	logical	if ... then ...
$\leftrightarrow$	biconditional symbol	logical	if and only if
(	left parenthesis	logical	
)	right parenthesis	logical	
A_1	first sentence symbol	non-logical	
A_2	first second symbol	non-logical	
$\ldots$	first second symbol	non-logical	

Expressions

Definition (Expressions)

An *expression* is a finite sequence of symbols.

Example

$A_3 \neg (\rightarrow$ is an expression of length 4.

Question

How many expressions are there?

Answer

The set of expressions is *enumerable*.

Notation: Expression Symbols

We often use the following symbols to represent expressions:

$$\alpha, \beta, \sigma, \ldots$$

Well-formed Formulas

Definition (Well-formed Formulas)

A well-formed formula (or simply *formula* or *wff*) is an expression built up from sentence symbols by applying some finite times of formula building operations.

Definition (Formula Building Operations)

Formula building operations include:

- $\xi_{\neg}(\alpha) = (\neg\alpha)$
- $\xi_{\wedge}(\alpha, \beta) = (\alpha \wedge \beta)$
- $\xi_{\vee}(\alpha, \beta) = (\alpha \vee \beta)$
- $\xi_{\rightarrow}(\alpha, \beta) = (\alpha \rightarrow \beta)$
- $\xi_{\leftrightarrow}(\alpha, \beta) = (\alpha \leftrightarrow \beta)$

Examples

Example

The following is a well-formed formula:

$$((\neg A_3) \vee (A_8 \leftrightarrow A_3))$$

Question

Which of the following formulas are well-formed?

- A_7
- $A_7 \to A_3$
- $(A_7 \to (A_3))$
- $(\neg A_7 \to A_3)$
- $((\neg A_7) \Longrightarrow A_3)$

Formalization of Propositions

Statements in natural languages can be formalized as wffs:

- Sentence symbols represent basic facts
- Sentential connectives represent logical relations

Example

Question

Given the following English sentence:

> **If Jones did not meet Smith last night, then either Smith left the city, or Jones is lying.**

How to formalize it as a wff?

Answer

- Use A_1 to represent "Jones met Smith last night"
- Use A_2 to represent "Smith left the city"
- Use A_3 to represent "Jones is lying"

The above sentence is formalized as

$$((\neg A_1) \to (A_2 \vee A_3))$$

Example

Question

Given the following English sentence:

> **Laozi is a man and not asleep. Furthermore, if Laozi is a man, then he is either asleep or awake.**

Let

- $A_1 =$ Laozi is a man;
- $A_2 =$ Laozi is asleep;
- $A_3 =$ Laozi is awake;

How to formalize the above sentence as a wff?

Proof by Induction

Induction on Natural Numbers

A property P about natural numbers is a subset of $\mathbb{N}$. We would like to prove that P holds for all natural numbers, i.e.,:

$$\dot{\forall} n \in \mathbb{N}, n \in P.$$

Proof by induction on n:

- Base case: show $0 \in P$ holds;
- Inductive case: for any $n \in \mathbb{N}$, assume $n \in P$ holds, show $n + 1 \in P$ holds.

Example

Example

Prove that for any $n \in \mathbb{N}$,

$$1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Proof.

Let $P = \{n \in \mathbb{N} \mid 1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}\}$. Prove $\dot\forall n \in \mathbb{N}, n \in P$ by induction on n:

- Base case: $0 \in P$ is true;
- Inductive case: if $n \in P$ is true, then $n + 1 \in P$ is true.

$\square$

Structural Induction

Inductive Definitions:

- Atomic building blocks;
- Constructor for building bigger definitions from smaller ones.

Induction on the structure of construction.

Example

Given any *full* binary-tree with n non-leaf nodes, it must have $n + 1$ leaf nodes.

Induction on Well-formed Formulas

Well-formed formulas is a form of inductive definitions with

- Basic building blocks (e.g., sentence symbols for wff)
- Closing operations (e.g., formula building operations for wff)

Theorem (Induction Principle)

A property S about wff is a set of wffs. If

1. every sentence symbol is in S, and
2. for every wff α and wff β, if α and β are in S then each of the following is in S:
   - $(\neg\alpha)$;
   - $(\alpha \wedge \beta)$;
   - $(\alpha \vee \beta)$;
   - $(\alpha \rightarrow \beta)$;
   - $(\alpha \leftrightarrow \beta)$.

then S is the set of all wffs, i.e., property S holds for all wffs.

Examples of Proof by Inductions

Proposition

Every wff has one of the following forms:

$$A, (\neg\alpha), (\alpha \wedge \beta), (\alpha \vee \beta), (\alpha \to \beta), (\alpha \leftrightarrow \beta) \tag{1}$$

where A is a sentence symbol and α and β are wffs.

Proof.

Let $S = \{\sigma \mid \sigma \text{ is a wff and } \sigma \text{ has the form in (1)}\}$. Proof by induction. □

Proposition

Every wff has the same number of left parentheses as right parentheses.

Parsing of Formulas

Given any expression α, if it is a well-formed formula, the following algorithm identifies it as such and constructs a tree with α at the top.

Algorithm

On input α, begin with a tree with a single node α.

1. If all the leaf nodes are sentence symbols, we are done. Otherwise, select a leaf node that is not a sentence symbol.
2. The first symbol must be (. If the second symbol is $\neg$, then we scan for a non-empty expression β with balanced left and right parentheses. Moreover, β must be followed by a). We create a child node for β and go back to (1).
3. If the second symbol is not $\neg$, we scan for (β where β is a balanced expression. β must be followed by $\wedge, \vee, \rightarrow$ or $\leftrightarrow$. We scan the remaining symbols for σ) where σ is balanced. We create two child nodes for β and σ and go back to (1).

Example

The following is a parse tree of $((\neg A_3) \vee (A_8 \leftrightarrow A_3))$

$$
\begin{tikzcd}
 & & ((\neg A_3) \vee (A_8 \leftrightarrow A_3)) & & \\
(\neg A_3) \arrow[urr] & & & (A_8 \leftrightarrow A_3) \arrow[ul] & \\
A_3 \arrow[u] & & A_8 \arrow[ur] & & A_3 \arrow[ul]
\end{tikzcd}
$$

Abbreviations

For simplicity, we adopt the following abbreviations for wffs:

1. We may omit outermost parentheses;
2. $\neg$ applies to as little as possible;
3. $\wedge$ and $\vee$ apply to as little as possible, subject to (2)
4. $\rightarrow$ and $\leftrightarrow$ apply to as little as possible, subject to (3)
5. When one sentential connective is used repeatedly, grouping is to the right.

Examples

Abbreviation	Formula
$A \to B$	$(A \to B)$
$\neg A \vee B$	$((\neg A) \vee B)$
$A \wedge B \to C \wedge D$	$((A \wedge B) \to (C \wedge D))$
$A \to B \to C \to D$	$(A \to (B \to (C \to D)))$

Semantics

From Syntax to Semantics

How to establish logical facts?

Question

For example, how do we know $(A_1 \wedge A_2) \to A_1$ is true?

Answer

By interpreting $(A_1 \wedge A_2) \to A_1$ into mathematical domains!

Truth Assignments

The math domain is a set $\{T, F\}$ of truth values:

- T, called *truth*
- F, called *falsity*

Definition (Truth Assignment)

A truth assignment for a set $\mathcal{S}$ of sentence symbols is a function

$$v : \mathcal{S} \to \{T, F\}$$

Extended Truth Assignments

Definition (Extended Truth Assignment)

Let $\bar{\mathcal{S}}$ be the set of wffs that can be built up from $\mathcal{S}$ by formula-building operations. Let v be a truth assignment for $\mathcal{S}$. An extension $\bar{v}$ of v

$$\bar{v} : \bar{\mathcal{S}} \rightarrow \{T, F\}$$

assigns truth values to every wff in $\bar{\mathcal{S}}$, as follows (where $\alpha, \beta \in \mathcal{S}$):

- $\bar{v}(A) = v(A)$ if $A \in \mathcal{S}$;
- $\bar{v}((\neg\alpha)) = \begin{cases} T & \text{if } \bar{v}((\alpha)) = F \\ F & \text{otherwise.} \end{cases}$
- ...

Extended Truth Assignments (Cont'd)

Definition (Extended Truth Assignment)

- ...
- $\bar{v}((\alpha \wedge \beta)) = \begin{cases} T & \text{if } \bar{v}(\alpha) = T \text{ and } \bar{v}(\beta) = T \\ F & \text{otherwise.} \end{cases}$
- $\bar{v}((\alpha \vee \beta)) = \begin{cases} T & \text{if } \bar{v}(\alpha) = T \text{ or } \bar{v}(\beta) = T \\ F & \text{otherwise.} \end{cases}$
- $\bar{v}((\alpha \to \beta)) = \begin{cases} F & \text{if } \bar{v}(\alpha) = T \text{ and } \bar{v}(\beta) = F \\ T & \text{otherwise.} \end{cases}$
- $\bar{v}((\alpha \leftrightarrow \beta)) = \begin{cases} T & \text{if } \bar{v}(\alpha) = \bar{v}(\beta) \\ F & \text{otherwise.} \end{cases}$

Examples of Truth Assignment

Question

The following is a well-formed formula α:

$$((\neg A_3) \vee (A_8 \leftrightarrow A_3))$$

Let $\mathcal{S} = \{A_3, A_8\}$ and $v : \mathcal{S} \to \{T, F\}$:

$$v(A) = \begin{cases} T & \text{if } A = A_3 \\ F & \text{if } A = A_8 \end{cases}$$

What is the value of $\bar{v}(\alpha)$?

More Example of Truth Assignment

Question

The following is a well-formed formula α:

$$((A_2 \to (A_1 \to A_6)) \leftrightarrow ((A_2 \wedge A_1) \to A_6))$$

Let

$$v(A_1) = T$$
$$v(A_2) = T$$
$$v(A_6) = F$$

What is the value of $\bar{v}(\alpha)$?

More Truth Values

Remark

- $\bar{v}((\neg\alpha)) = T \iff$ not $\bar{v}(\alpha) = T$
- $\bar{v}((\alpha \wedge \beta)) = T \iff \bar{v}(\alpha) = T \;\&\; \bar{v}(\beta) = T$
- $\bar{v}((\alpha \vee \beta)) = T \iff \bar{v}(\alpha) = T \;||\; \bar{v}(\beta) = T$
- $\bar{v}((\alpha \to \beta)) = T \iff \bar{v}(\alpha) = T \Longrightarrow \bar{v}(\beta) = T$
- $\bar{v}((\alpha \leftrightarrow \beta)) = T \iff \bar{v}(\alpha) = T \iff \bar{v}(\beta) = T$

Determinacy of Truth Assignments

Theorem

For every v_1 and v_2, and wff α, if

$$v_1(A) = v_2(A)$$

for every sentence symbol A that occurs in α, then

$$\bar{v}_1(\alpha) = \bar{v}_2(\alpha).$$

In other words, the value of a wff α under a truth assignment v is completely determined by the values of v on the (finite set of) sentence symbols that occur in α.

Truth Tables

By the previous theorem, to determine the value $\bar{v}(\alpha)$ we only need to know the value of v on the sentence symbols that occur in α.

This leads to the method of truth tables. We write out the truth tables for $\neg$, $\wedge$, $\vee$, $\rightarrow$ and $\leftrightarrow$. Then we use these truth tables to write out the truth tables for more complicated wffs.

α	β	$\neg\alpha$	$\alpha \wedge \beta$	$\alpha \vee \beta$	$\alpha \rightarrow \beta$	$\alpha \leftrightarrow \beta$
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

Table: Truth Tables

Example

Question

How to construct the truth tables for the following formulas?

- $(\neg(A \vee B))$
- $((\neg A) \wedge (\neg B))$

Satisfiability

We use upper case Greek letters, such as Σ, Γ, Δ and Π, to stand for *sets of wffs*.

Notation: $\Sigma; \alpha$ stands for $\Sigma \cup \{\alpha\}$.

Definition

- v satisfies α if $\bar{v}(\alpha) = T$;
- v satisfies Σ if $\bar{v}(\alpha) = T$ for every $\alpha \in \Sigma$. In other words, v satisfies Σ if v satisfies every member of Σ.

Definition (Satisfiability)

- α is satisfiable if there is there exists some v that satisfies α;
- Σ is satisfiable if there is there exists some v that satisfies Σ.

Examples

Are the following wffs satisfiable?

Example

- $\neg A_3 \wedge (A_1 \leftrightarrow A_3)$
- $A_1 \wedge (\neg(A_1 \rightarrow A_3)) \wedge A_3$
- $(A_1 \rightarrow A_2 \rightarrow A_3) \leftrightarrow ((A_1 \wedge A_2) \rightarrow A_3)$

Are the following sets of wffs satisfiable?

Example

- $\{\neg A_3, A_1 \leftrightarrow A_3\}$
- $\{A_1, \neg(A_1 \rightarrow A_3), A_3\}$

Translations into Sentential Logic

Question

Propositions:

1. If the store is open today, then Mary is going.
2. John is going to the store today if and only if Mary isn't.
3. If it is not raining today, then John is going to the store.
4. The store is open today if and only if it is not raining.

Is the above set of propositions satisfiable?

Question

If we add the proposition "It is not raining today" to the set, is it still satisfiable?

Example: Empty Set

Question

Does every v satisfies $\emptyset$?

Answer

Yes!

$$v \text{ satisfies } \emptyset \iff \forall \alpha,\ \underbrace{\alpha \in \emptyset}_{assumption} \implies \bar{v}(\alpha) = T.$$

The right side is true since the assumption is false.

When does the truth of one formula implies that of another?

Tautological Implications

Definition (Tautological Implication)

- A set of wffs Σ tautologically implies α when every truth assignment satisfying Σ also satisfies α;
- $\Sigma \vDash \alpha$ denotes that Σ tautologically implies α;
- $\alpha \vDash \beta$ denotes that $\{\alpha\} \vDash \beta$.

If $\Sigma \vDash \alpha$, we call α a tautological consequence of Σ.

Tautological implication is also called *semantic implication* or *semantic entailment*.

Examples

The following statements hold:

Example

- $\{A_1, A_1 \rightarrow A_3\} \models A_3$
- $\neg(A_1 \rightarrow A_3) \models \neg A_3$
- $A_1 \rightarrow A_2 \rightarrow A_3 \models (A_1 \wedge A_2) \rightarrow A_3$

Translations into Sentential Logic

Question

- Premises:
  1. If you are healthy then you are happy.
  2. You are healthy.
- Conclusion: You are happy.

Is the conclusion a tautological consequence of the premises?

Answer

Yes.

We translate into sentential logic as follows:

- Let A stand for 'you are healthy';
- Let B stand for 'you are happy';
- Then (1) is translated as: $A \to B$;
- Let $\Sigma = \{A \to B, A\}$. Then $\Sigma \vDash B$.

Another Example

Question

- Premises:
  1. If you are healthy then you are happy.
  2. You are happy.
- Conclusion: You are healthy.

Is the conclusion a tautological consequence of the premises?

Answer

No.

- Let A and B be as before;
- Let $\Sigma = \{A \to B, B\}$. Then $\Sigma \not\models A$.

Note that

- we are not asking if 'You are healthy' is true;
- we are asking if the *reasoning* is correct.

More Example

Question

Consider the two sentences:

(1) $\emptyset \vDash \alpha \implies \emptyset \vDash \beta$;

(2) $\emptyset \vDash \alpha \to \beta$.

Answer the following questions:

(a) Does (1) imply (2)?

(b) Does (2) imply (1)?

Answer

- (a) is false. An counterexample is when $\alpha = A$ and $\beta = \neg A$.
- (b) is true.

Unsatisfiable Assumption in Tautological Implication

Question

Given any α and β, does the following tautological implication holds?

$$\{\neg\alpha, \alpha\} \models \beta$$

Answer

Yes. We need to show, for all v :

$$\underbrace{v \text{ satisfies } \{\neg\alpha, \alpha\}}_{\textit{assumption}} \Longrightarrow v \text{ satisfies } \beta$$

However, the assumption does not hold for any v. Therefore, the conclusion trivially holds.

Caution: Incorrect Notations

Be careful to not confuse *syntax* with *semantics*.

- $\alpha = T$ is incorrect notation. $\bar{v}(\alpha) = T$ is correct notation.
- $v(\Sigma) = T$ is incorrect. 'v satisfies Σ' is correct.

Application: Knowledge Inference

Knowledge Base

A *knowledge base* can be thought as an "intelligent" database that can be queried and expanded.

Example

$$
\begin{aligned}
KB = \{ & \text{"If you eat vegetables, then you are healthy"}, \\
& \text{"If you eat meat, then you are happy"}, \\
& \text{"You eat vegetables"} \}
\end{aligned}
$$

Operations:

- *Ask*: ask if a piece of knowledge is true
- *Tell*: tell a (possibly new) fact that KB may learn

Ask a Knowledge Base

Example

$$
\begin{aligned}
KB = \{ & \text{“If you eat vegetables, then you are healthy”}, \\
& \text{“If you eat meat, then you are happy”}, \\
& \text{“You eat vegetables”} \}
\end{aligned}
$$

Ask the KB “Are you healthy?”

Possible answers:

- Yes.
- No.
- I do not know.

Tell a Knowledge Base

Example

$$
\begin{aligned}
KB = \{ &\text{“If you eat vegetables, then you are healthy”},\\
&\text{“If you eat meat, then you are happy”},\\
&\text{“You eat vegetables”} \}
\end{aligned}
$$

Tell the KB “You eat meat”

Possible answers:

- I already know.
- It is impossible.
- I learned something new.

KB in Sentential Logic

Translate knowledge bases and questions into wffs

Example

$$KB = \{A \rightarrow B, C \rightarrow D, A\}$$

Operations:

- *Ask*: is B true?
- *Tell*: C is true

Models

Definition

A truth assignment v is a model of α if it v satisfies α.

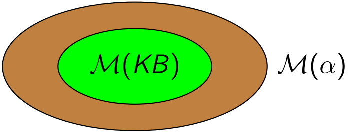
The models of α form a set $\mathcal{M}(\alpha)$.

Example

Let A stand for 'you are healthy' and B stand for 'you are happy'.

$$\begin{aligned}\mathcal{M}(A \vee B) = \{&\{A : T, B : T\},\\ &\{A : F, B : T\},\\ &\{A : T, B : F\}\}\end{aligned}$$

Entailment



Definition (Entailment)

KB entails α if $\mathcal{M}(KB) \subseteq \mathcal{M}(\alpha)$.

Remark

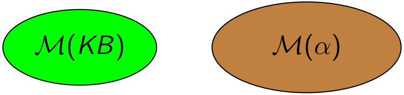
α contains no new information with respect to KB.

Example

$$KB = \{\text{“You are healthy and happy”}\}$$

$\alpha =$ “You are healthy”.

Contradiction



Definition (Contradiction)

α contradicts KB if $\mathcal{M}(KB) \cap \mathcal{M}(\alpha) = \emptyset$.

Remark

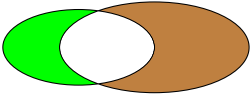
No agreement between α and KB.

Example

$$KB = \{\text{“You are healthy and happy”}\}$$

$\alpha =$ “You are unhealthy”.

Contingency



$\mathcal{M}(KB)$ $\mathcal{M}(\alpha)$

Definition (Contingency)

α is contingent on KB if $\emptyset \subset \mathcal{M}(KB) \cap \mathcal{M}(\alpha) \subset \mathcal{M}(KB)$

Remark

α tells something new.

Example

$$KB = \{\text{“You are healthy and happy”}\}$$

$\alpha =$ “You eat vegetables”.

Knowledge Base

Tell KB a fact α and get an answer:

- I already know. (Entailment)
- It is impossible. (Contradiction)
- I learned something new. (Contingency)

Ask the DB if a statement is true:

- Yes. (Entailment)
- No. (Contradiction)
- I do not know. (Contingency)

Example

Consider the previous examples...

Equivalent Logical Denotations

How to denote entailment, contradiction and contingency using sentential logic?

Use tautological implication:

- Entailment: $KB \models \alpha$;
- Contradiction: $KB \models \neg\alpha$;
- Contingency: $KB \not\models \alpha$ and $KB \not\models \neg\alpha$.

Use (un)satisfiability:

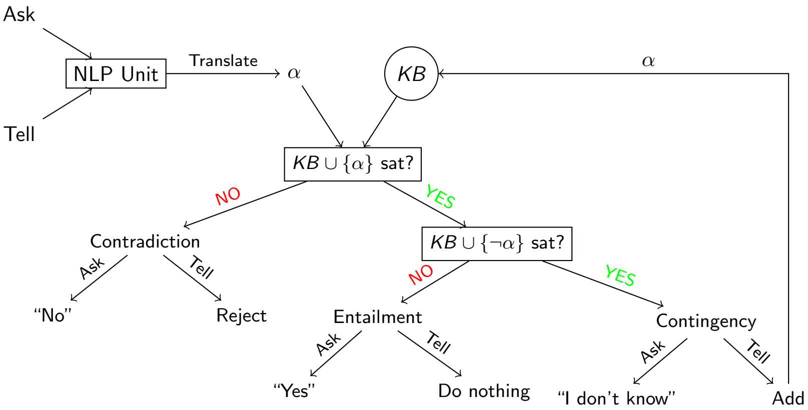
Question

Ask the questions:

- Is $KB \cup \{\alpha\}$ satisfiable?
- Is $KB \cup \{\neg\alpha\}$ satisfiable?

How do they relate to entailment, contradiction and contingency?

Building a ChatBot



Question

How to build an algorithm for checking satisfiability?

When is a formula always true
no matter what the truth assignment is?

Tautologies

Unconditional truth of wffs:

Definition

- α is a tautology if $\emptyset \vDash \alpha$.
- $\vDash \alpha$ denotes that $\emptyset \vDash \alpha$

Remark

- α is a tautology iff for every v, $\bar{v}(\alpha) = T$;
- α is a tautology iff $\neg\alpha$ is *not satisfiable*;
- α is satisfiable iff $\neg\alpha$ is *not a tautology*.

Recognizing the Tautologies

You should be able to recognize simple tautologies by using the method of *truth tables*.

Example

- $(A \wedge (A \rightarrow B)) \rightarrow B$;
- $(A \vee (B \wedge C)) \leftrightarrow ((A \vee B) \wedge (A \vee C))$.

More Examples of Tautologies

Question

Which of the following are tautologies?

- $\neg A \to A$;
- $\neg(\neg A) \to A$;
- $((A \to B) \to C) \to (A \to (B \to C))$;
- $\neg(A \leftrightarrow B) \to ((A \wedge \neg B) \vee (\neg A \wedge B))$

Tautological Equivalence

Definition (Tautological Equivalence)

- Two wffs α and β are tautologically equivalent if both $\alpha \models \beta$ and $\beta \models \alpha$ hold;
- $\alpha \models\!\!\!\dashv \beta$ means α and β are tautologically equivalent.

Example

- $A \models\!\!\!\dashv \neg(\neg A)$
- $A_1 \to A_2 \models\!\!\!\dashv \neg A_1 \vee A_2$
- $\neg(A_1 \vee A_2) \models\!\!\!\dashv (\neg A_1) \wedge (\neg A_2)$
- $A_1 \to A_2 \to A_3 \models\!\!\!\dashv (A_1 \wedge A_2) \to A_3$

Tautological equivalence is also called *semantic equivalence*.

Application of Tautological Equivalence

Proposition

The following are equivalent:

- α and β are tautologically equivalent
- For every v, $\bar{v}(\alpha) = \bar{v}(\beta)$.
- α and β have the same truth table.

We can use tautological equivalence to derive truthfulness of wffs: if $\alpha \vDash\Dashv \beta$, we can freely replace one for the other in deriving the truth of some formula σ where α and β occur.

Example

$((A \to B) \to C) \to (A \to (B \to C))$

Properties of Satisfaction and Tautological Implication

Which of the following are true?

- If α is a tautology then $\Sigma \models \alpha$ for every Σ;
- If $\alpha \in \Sigma$ then $\Sigma \models \alpha$;
- If $\Sigma \models \alpha$ and $\Sigma \models \alpha \rightarrow \beta$ then $\Sigma \models \beta$;
- If $\Sigma \models \alpha$ and $\alpha \models \beta$ then $\Sigma \models \beta$;
- If $\Sigma \models \alpha$ then for all β, $\Sigma \models \beta \rightarrow \alpha$;
- If $\Sigma \models \alpha$ and $\Sigma \models \beta$ then $\Sigma \models \alpha \wedge \beta$;
- If $\Sigma \models \alpha$ or $\Sigma \models \beta$ then $\Sigma \models \alpha \vee \beta$;
- $\Sigma \not\models \alpha$ iff $\Sigma \cup \{\neg\alpha\}$ is satisfiable;
- $\Sigma \models \alpha$ iff $\Sigma \cup \{\neg\alpha\}$ is not satisfiable;
- $\Sigma \models \alpha \rightarrow \beta$ iff $\Sigma; \alpha \models \beta$;
- If Σ is not satisfiable then for every α, $\Sigma \models \alpha$.

More Properties

- If $\Sigma \vDash \alpha$ and $\Sigma \subseteq \Delta$ the $\Delta \vDash \alpha$;
- If Σ is satisfiable then every subset of Σ is satisfiable;
- If every subset of Σ is satisfiable then Σ is satisfiable;
- If every finite subset of Σ is satisfiable then Σ is satisfiable;
- If $\Sigma \vDash \alpha$ then there is a finite subset Δ of Σ such that $\Delta \vDash \alpha$.

Are all the connectives necessary?

Completeness of Connectives

Definition
A subset $\mathcal{C}$ of logical connectives is complete if any wff is tautologically equivalent to some wff using only connectives in $\mathcal{C}$.

Lemma
$\{\neg, \rightarrow\}$ is complete.

Proof.
$\alpha \vee \beta \vDash\Dashv (\neg\alpha) \rightarrow \beta$. Similar arguments for other cases. $\square$

Remark
$\{\wedge, \rightarrow\}$ is not complete.

Completeness of Connectives

Lemma

Both $\{\neg, \wedge\}$ and $\{\neg, \vee\}$ are complete.

Proof.

$\alpha \vee \beta \vDash\dashv \neg(\neg\alpha \wedge \neg\beta)$. Similar arguments for other cases. $\square$

A common and useful set of complete connectives is $\{\wedge, \vee, \neg\}$.

Let us take a look at Disjunctive Normal Form (DNF) and Conjunctive Normal Form (CNF).

Disjunctive Normal Form

Definition

The wff α is in disjunctive normal form if $\alpha = \gamma_1 \vee \ldots \vee \gamma_k$ where each γ_i is a conjunction

$$\gamma_i = \beta_{i1} \wedge \ldots \wedge \beta_{in_i}$$

where each β_{ij} is either a sentence symbol or the negation of a sentence symbol.

Question

Which of the following wffs is in disjunctive normal form?

- $(A_3 \wedge \neg A_1 \wedge A_3) \vee (A_5 \wedge A_5 \wedge A_6)$
- $A_3 \wedge \neg A_1 \wedge A_3$
- $A_3 \vee \neg A_1 \vee A_3$
- $(A_3 \vee A_1) \wedge A_1$

Conjunctive Normal Form

Definition

The wff α is in conjunctive normal form if $\alpha = \gamma_1 \wedge \ldots \wedge \gamma_k$ where each γ_i is a disjunction

$$\gamma_i = \beta_{i1} \vee \ldots \vee \beta_{in_i}$$

where each β_{ij} is either a sentence symbol or the negation of a sentence symbol.

Question

Which of the following wffs is in conjunctive normal form?

- $(A_3 \vee \neg A_1 \vee A_3) \wedge (A_5 \vee A_5 \vee A_6)$
- $A_3 \wedge \neg A_1 \wedge A_3$
- $A_3 \vee \neg A_1 \vee A_3$
- $(A_3 \vee A_2) \vee (A_1 \wedge A_2)$

Completeness of Disjunctive Normal Forms

Theorem

Every wff is tautologically equivalent to a wff in disjunctive normal form.

Proof.

Construct the disjunctive normal form *truth tables*.

(1) Given a wff α containing sentence symbols $A_1, \ldots, A_n$, create its truth table;

(2) For every row i with a value T, create a wff $\gamma_i = \beta_1 \wedge \ldots \wedge \beta_n$ where

$$\beta_j = \begin{cases} A_j & \text{if } A_j \text{ is assigned T at row } i \\ \neg A_j & \text{if } A_j \text{ is assigned F at row } i \end{cases}$$

(3) We have $\alpha \models\!\!\!\!\dashv \gamma_1 \vee \ldots \vee \gamma_k$

□

Example

Compute the disjunctive normal form of $(A_1 \to A_2) \wedge A_3$:

A_1	A_2	A_3	$(A_1 \to A_2) \wedge A_3$
T	T	T	T
T	T	F	F
T	F	T	F
T	F	F	F
F	T	T	T
F	T	F	F
F	F	T	T
F	F	F	F

The disjunctive normal form is

$$(A_1 \wedge A_2 \wedge A_3) \vee (\neg A_1 \wedge A_2 \wedge A_3) \vee (\neg A_1 \wedge \neg A_2 \wedge A_3)$$

Completeness of Conjunctive Normal Forms

Lemma

Every wff in disjunctive normal form is tautologically equivalent to a wff in conjunctive normal form.

Proof.

- Given $\alpha = \gamma_1 \vee \ldots \vee \gamma_n$ in disjunctive normal form. If every γ_i is a sentence symbol or the negation of a sentence symbol, we are done.
- Otherwise, there is some $\gamma_i = \beta_{i_1} \wedge \beta_{i_2}$. Then,

$$\alpha \mathrel{\models\!\!\!\!\!\dashv} (\beta_{i_1} \wedge \beta_{i_2}) \vee \alpha' \mathrel{\models\!\!\!\!\!\dashv} (\beta_{i_1} \vee \alpha') \wedge (\beta_{i_2} \vee \alpha')$$

  where α' is the disjunction of $\{\gamma_k \mid k \neq i\}$.
- Recursively repeat the above steps on $\beta_{i_1} \vee \alpha'$ and $\beta_{i_2} \vee \alpha'$.

□

Completeness of Conjunctive Normal Forms

Theorem

Every wff is tautologically equivalent to a wff in conjunctive normal form.

Proof.

(1) Given α, construct its disjunctive normal form;

(2) Construct the equivalent conjunctive normal form.

□

Example

We compute the conjunctive normal form of $(\neg A_1 \wedge A_2) \vee (A_2 \wedge A_3)$:

$$\begin{array}{ll} & (\neg A_1 \wedge A_2) \vee (A_2 \wedge A_3) \\ \models\!\!\!\!\!\!\!\;\;\dashv & (\neg A_1 \vee (A_2 \wedge A_3)) \wedge (A_2 \vee (A_2 \wedge A_3)) \\ \models\!\!\!\!\!\!\!\;\;\dashv & ((A_2 \wedge A_3) \vee \neg A_1) \wedge ((A_2 \wedge A_3) \vee A_2) \\ \models\!\!\!\!\!\!\!\;\;\dashv & ((A_2 \vee \neg A_1) \wedge (A_3 \vee \neg A_1)) \wedge ((A_2 \wedge A_3) \vee A_2) \\ \models\!\!\!\!\!\!\!\;\;\dashv & ((A_2 \vee \neg A_1) \wedge (A_3 \vee \neg A_1)) \wedge ((A_2 \vee A_2) \wedge (A_3 \vee A_2)) \\ \models\!\!\!\!\!\!\!\;\;\dashv & (A_2 \vee \neg A_1) \wedge (A_3 \vee \neg A_1) \wedge (A_2 \vee A_2) \wedge (A_3 \vee A_2) \end{array}$$

SAT Solving

Recall that a critical component of our ChatBot is checking satisfiability.

Given a set of propositions Σ, transform it into an CNF formula. Then run a SAT solving algorithm.

- Brute force search
- DPLL
- CDCL
- Many others (See Chapter 1.6 in "Logic in Computer Science")

Remark
SAT solving is a realization of *Model Checking* in sentential logic

Compactness

Finite Satisfiability

Recall two earlier questions that were left unanswered:

Question

- If every finite subset of Σ is satisfiable, must Σ be satisfiable?
- If $\Sigma \vDash \alpha$, must there be a finite subset Δ of Σ such that $\Delta \vDash \alpha$?

To answer these question, we start with some definitions:

Definition (Finite Satisfiability)

Σ is finitely satisfiable if every finite subset of Σ is satisfiable.

Examples

Question

Suppose Δ is finitely satisfiable, which of the following are possible?

- $\{\alpha, \neg\alpha\} \subseteq \Delta$
- $\{\gamma, \neg(\beta \to \gamma)\} \subseteq \Delta$
- $\{\neg\beta, \neg(\beta \to \gamma)\} \subseteq \Delta$
- $\{\beta, \beta \to \gamma, \neg\gamma\} \subseteq \Delta$

Remark

Suppose:

- Δ is finitely satisfiable, and
- for every α, $\alpha \in \Delta$ or $\neg\alpha \in \Delta$

Then $\alpha \in \Delta$ iff $\neg\alpha \notin \Delta$.

Compactness Theorem

Theorem

If Σ is finitely satisfiable then Σ is satisfiable.

Proof.

We break down the proof into the following steps:

- From Σ, construct its superset Δ such that
  - (a) Δ is finitely satisfiable, and
  - (b) For every wff α, $\alpha \in \Delta$ or $\neg\alpha \in \Delta$.
- Show Δ is satisfiable.

$\square$

The above proof is supported by the following lemmas.

Compactness Theorem

Lemma (1)

If Δ is finitely satisfiable then for every wff α, either

- $\Delta \cup \{\alpha\}$ is finitely satisfiable, or
- $\Delta \cup \{\neg\alpha\}$ is finitely satisfiable.

Proof.

Assume neither conclusion is true, show there is a contradiction. □

Compactness Theorem

Lemma (2)

If Σ is finitely satisfiable then there is a $\Delta \supseteq \Sigma$ that has the following properties:

(a) Δ is finitely satisfiable, and

(b) For every wff α, $\alpha \in \Delta$ or $\neg\alpha \in \Delta$.

Proof.

Enumerate all wffs $\alpha_1, \alpha_2, \ldots$ (why this is possible?). Construct Δ_i by recursion as follows:

- $\Delta_0 = \Sigma$;
- $$\Delta_{i+1} = \begin{cases} \Delta_i \cup \{\alpha_{i+1}\} & \text{if this is finitely satisfiable} \\ \Delta_i \cup \{\neg\alpha_{i+1}\} & \text{otherwise} \end{cases}$$

Let $\Delta = \bigcup \Delta_i$. Show it has the properties (a) and (b). □

Compactness Theorem

Lemma (3)

Let Δ be a set of wffs such that

(a) Δ is finitely satisfiable, and

(b) For every wff α, $\alpha \in \Delta$ or $\neg\alpha \in \Delta$.

Then Δ is satisfiable.

Proof.

Define the assignment v as follows:

$$v(A) = \begin{cases} T & A \in \Delta \\ F & A \notin \Delta \end{cases}$$

Show for any α, $\bar{v}(\alpha) = T \iff \alpha \in \Delta$. We then have v satisfies Δ.

$\square$

Corollary of the Compactness Theorem

Corollary

If $\Sigma \models \tau$ then there is a finite subset Δ of Σ such that $\Delta \models \tau$.

Enumerability Results for Tautological Implication

Theorem

If Σ is an enumerable set of wffs, then the set of tautological consequences of Σ is enumerable.

Observation: let α be a tautological consequence of Σ (i.e., $\Sigma \models \alpha$). By compactness, there exists a finite subset Δ of Σ s.t. $\Delta \models \alpha$.

Enumerability Results for Tautological Implication

Theorem

If Σ is an enumerable set of wffs, then the set of tautological consequences of Σ is enumerable.

Proof.

- Let $\beta_1, \ldots, \beta_n, \ldots$ be an enumeration of Σ;
- Let $\Delta_n = \{\beta_1, \ldots, \beta_n\}$;
- Let $\alpha_1, \ldots, \alpha_m, \ldots$ be an enumeration of all wffs.

We construct the following table:

	α_1	α_2	α_3	$\ldots$	α_m	$\ldots$
1	$\Delta_1 \models \alpha_1$	$\Delta_1 \models \alpha_2$	$\Delta_1 \models \alpha_3$	$\ldots$	$\Delta_1 \models \alpha_m$	$\ldots$
2	$\Delta_2 \models \alpha_1$	$\Delta_2 \models \alpha_2$	$\Delta_2 \models \alpha_3$	$\ldots$	$\Delta_2 \models \alpha_m$	$\ldots$
$\ldots$						
n	$\Delta_n \models \alpha_1$	$\Delta_n \models \alpha_2$	$\Delta_n \models \alpha_3$	$\ldots$	$\Delta_n \models \alpha_m$	$\ldots$
$\ldots$						

$\square$