Sentential Logic: Deductive Calculi, Soundness and Completeness

Yuting Wang

John Hopcroft Center for Computer Science
Shanghai Jiao Tong University

Oct 16, 2023

We have presented two aspects of Sentential Logic:

- Syntax
- Semantics

We have also discussed what are valid formulas (tautologies) in Sentential Logic.

Question

What is the "syntactic" counterpart of tautology in Sentential Logic?

Do we have a symbolic way to describe what are provable?

Deductive Calculi

Proofs are (purely) syntactic constructs that capture derivability of facts.

Deductive calculi provide descriptions of proofs in logic.

Many forms exist for deductive calculi:

- Hilbert-Style Calculi
- Natural Deductions (Type Theory)
- Sequent Calculi
- Proof Nets (Linear Logic)
- Deep Inference
- ...

Natural Deduction for Sentential Logic

We introduce natural deduction for sentential logic.

Natural deduction contains

- A set of wffs called logical axioms and
- A set of rules of inference for deriving new facts.

We then systematically generate a set of wffs from the logical axioms by using the rule of inference. They are called provable (or derivable) wffs.

Defining Characteristic of ND

1. Proofs are formed from sub-proofs

   A fundamental part of natural deduction, and what (according to most writers on the topic) sets it apart from other proof methods, is the notion of a "subproof" — parts of a proof in which the argumentation depends on temporary premises (hypotheses "assumed for the sake of argument")[1]

2. Meaning of logical connectives are defined by *use*.

3. Harmony between rules for "definitions" and "uses".

[1] https://plato.stanford.edu/entries/natural-deduction/

Soundness and Completeness

Our goal is to show the following are equivalent:

- The set of provable wffs of Sentential Logic.
- The set of tautologies of Sentential Logic.

This is accomplished by proving the following two theorems:

Theorem (Soundness)

Every provable wff is a tautology.

Theorem (Completeness)

Every tautology is provable.

The Logical Axioms

In our formulation of natural deduction, the only logical axiom is the Law of Excluded Middle.

Definition (Law of Excluded Middle)

For any wff α,

$$\alpha \vee \neg\alpha.$$

Theorem (Validity of LEM)

For any wff α, $\alpha \vee \neg\alpha$ is a tautology.

Inference Rules

Basic inference rules have the following form:

$$\frac{\alpha_1 \quad \dots \quad \alpha_n}{\alpha}$$

- α is called the conclusion of the rule
- $\alpha_1, \dots, \alpha_n$ are called the premises of the rule
- **Interpretation**: If there are proofs for $\alpha_1, \dots, \alpha_n$, then a proof for α is constructed by applying this rule.

Remark

- There is only *one* conclusion
- There may be one or *more than one* premises

Inference Rules for Conjunctions

A finite set of rules are defined for every sentential connective. They are divided into introduction and elimination rules:

- *Introduction rules*: sentential connectives are introduced in conclusions;
- *Elimination rules*: sentential connectives occurring in premises are eliminated in conclusions.

Introduction Rules for $\wedge$:

$$\frac{\alpha \quad \beta}{\alpha \wedge \beta} \ \wedge\text{-}I$$

Elimination Rules for $\wedge$:

$$\frac{\alpha \wedge \beta}{\alpha} \ \wedge\text{-}E_1 \qquad \frac{\alpha \wedge \beta}{\beta} \ \wedge\text{-}E_2$$

Instances of Inference Rules

An inference rule is a template that captures a collection of instances:

Example

The following are instances of the $\wedge$-I rule:

$$\frac{A_0 \quad A_1}{A_0 \wedge A_1} \qquad \frac{A_3 \quad A_8 \to A_3}{A_3 \wedge (A_8 \to A_3)} \qquad \frac{A_1 \wedge A_2 \quad A_2 \vee A_3}{(A_1 \wedge A_2) \wedge (A_2 \vee A_3)}$$

Question

The following are which rules' instances?

$$\frac{A_0 \wedge A_1}{A_0} \qquad \frac{A_3 \wedge (A_8 \to A_3)}{A_8 \to A_3} \qquad \frac{(A_1 \wedge A_2 \wedge A_3) \wedge (A_2 \wedge A_3)}{A_2 \wedge A_3}$$

Inference Rules for Disjunctions

Introduction Rules for $\vee$:

$$\frac{\alpha}{\alpha \vee \beta}\ \vee\text{-}I_1 \qquad \frac{\beta}{\alpha \vee \beta}\ \vee\text{-}I_2$$

Example (Instances)

$$\frac{A_0}{A_0 \vee A_1} \qquad \frac{A_8 \rightarrow A_3}{A_3 \vee (A_8 \rightarrow A_3)} \qquad \frac{A_1 \wedge A_2 \wedge A_3}{(A_1 \wedge A_2 \wedge A_3) \vee (A_2 \vee A_3)}$$

Inference Rules for Disjunctions

Elimination Rules for $\vee$:

$$\dfrac{\alpha \vee \beta \qquad \begin{matrix}[\alpha] \\ \vdots \\ \delta\end{matrix} \qquad \begin{matrix}[\beta] \\ \vdots \\ \delta\end{matrix}}{\delta} \;\vee\text{-}E$$

Interpretation: If there is a proof for $\alpha \vee \beta$ and

- there is a proof for δ by assuming α;
- there is also a proof for δ by assuming β;

then a proof for δ without assuming α or β is constructed by applying this rule.

Note: $[\alpha]$ and $[\beta]$ are called discharged assumptions.

Remark (Intuition)

If we know α or β holds, and from either of which we can derive δ, then δ holds.

Inference Rules for Implications

Introduction and Elimination Rules for $\rightarrow$:

$$\dfrac{\begin{matrix}[\alpha] \\ \vdots \\ \beta\end{matrix}}{\alpha \rightarrow \beta}\ \rightarrow\text{-}I \qquad \dfrac{\alpha \rightarrow \beta \quad \alpha}{\beta}\ \rightarrow\text{-}E$$

Interpretation for $\rightarrow$-*I*: If there is a proof for β by assuming α, then a proof for $\alpha \rightarrow \beta$ without assuming α is constructed by applying this rule.

Remark

$\rightarrow$-*E* is also known as **Modus Ponens** (method of putting by placing).

Inference Rules for Negation

Introduction and Elimination Rules for $\neg$:

$$\dfrac{\begin{matrix}[\alpha] \\ \vdots \\ \beta \wedge \neg\beta\end{matrix}}{\neg\alpha}\ \neg\text{-}I \qquad \dfrac{\beta \quad \neg\beta}{\alpha}\ \neg\text{-}E$$

Interpretations:

- $\neg$-*I*: If there is a proof for a contradiction ($\beta \wedge \neg\beta$) by assuming α, then a proof for $\neg\alpha$ is constructed by applying this rule.
- $\neg$-*E*: If a contradiction is reached (there are proofs for both β and $\neg\beta$), then any wff α can be proved.

Remark

$\neg$-*E* is also known as **Principle of Explosion** or **ex falso quodlibet** (from contradiction, anything follows).

Inference Rules for If-and-only-if

Introduction and Elimination Rules for $\leftrightarrow$:

$$\frac{\alpha \to \beta \quad \beta \to \alpha}{\alpha \leftrightarrow \beta}\ \leftrightarrow\text{-}I \qquad \frac{\alpha \leftrightarrow \beta}{\alpha \to \beta}\ \leftrightarrow\text{-}E_1 \qquad \frac{\alpha \leftrightarrow \beta}{\beta \to \alpha}\ \leftrightarrow\text{-}E_2$$

Remark

The above rules are similar to the rules for $\wedge$ because $\alpha \leftrightarrow \beta$ is logically equivalent to $(\alpha \to \beta) \wedge (\beta \to \alpha)$.

Proof Trees

Definition (Proof Trees)

Given a wff α, a proof tree for α is constructed by applying instances of inference rules for a finite amount of times until all assumptions are discharged.

$$\dfrac{\dfrac{\begin{matrix}[\sigma_0] \\ \vdots \\ \beta_0\end{matrix} \quad \dots \quad \begin{matrix}\{\sigma_l\} \\ \vdots \\ \beta_j\end{matrix}}{\alpha_0} \quad \dots \quad \dfrac{\begin{matrix}[\delta_0] \\ \vdots \\ \gamma_0\end{matrix} \quad \dots \quad \begin{matrix}[\delta_p] \\ \vdots \\ \gamma_j\end{matrix}}{\alpha_n}}{\alpha}$$

Given a proof tree for α,

- it has α as its root;
- its leaf nodes are either logical axioms or discharged assumptions.

We shall use $\{\alpha\}$ to denote the axiom α in a leaf node.

Construction of Proof Trees

Two basic approaches:

- Start from leaves and build towards the root;
- Reduce the root until every leaf either is discharged or becomes an axiom.

Example

Construct a proof tree of $A \to (B \to A \wedge B)$ starting from leaves:

$$\frac{A \quad B}{A \wedge B}\ \wedge\text{-}I \qquad \frac{\dfrac{A \quad [B]}{A \wedge B}\ \wedge\text{-}I}{B \to A \wedge B}\ \to\text{-}I \qquad \frac{\dfrac{\dfrac{[A] \quad [B]}{A \wedge B}\ \wedge\text{-}I}{B \to A \wedge B}\ \to\text{-}I}{A \to (B \to A \wedge B)}\ \to\text{-}I$$

Construction of Proof Trees

A common strategy:

1. Bottom-up: Apply introduction rules whenever possible;
2. Top-down: Use common mathematical reasoning;

Construction of Proof Trees

Example

Construction a proof tree of $A \rightarrow (B \rightarrow A \wedge B)$ starting from the root:

$$\dfrac{\begin{array}{c}[A]\\ \vdots \\ B \rightarrow (A \wedge B)\end{array}}{A \rightarrow (B \rightarrow A \wedge B)} \rightarrow\text{-}I \qquad \dfrac{\dfrac{\begin{array}{c}[A] \quad [B]\\ \vdots \\ A \wedge B\end{array}}{B \rightarrow (A \wedge B)} \rightarrow\text{-}I}{A \rightarrow (B \rightarrow A \wedge B)} \rightarrow\text{-}I \qquad \dfrac{\dfrac{\dfrac{[A] \quad [B]}{A \wedge B} \wedge\text{-}I}{B \rightarrow A \wedge B} \rightarrow\text{-}I}{A \rightarrow (B \rightarrow A \wedge B)} \rightarrow\text{-}I$$

Uses of Assumptions

An assumption may be used zero, one or more times:

Example

In the following proof, assumption C is not used while assumption $A \wedge B$ is used for once.

$$\dfrac{\dfrac{\dfrac{[A \wedge B]}{A}\ \wedge\text{-}E_1}{C \to A}\ \to\text{-}I}{A \wedge B \to C \to A}\ \to\text{-}I$$

Uses of Assumptions

An assumption may be used zero, one or more times:

Example

In the following proof, assumption $A \wedge B$ is used twice.

$$
\dfrac{\dfrac{[A \to B \to C] \quad \dfrac{[A \wedge B]}{A}\ \wedge\text{-}E_1}{B \to C}\ \to\text{-}E \qquad \dfrac{[A \wedge B]}{B}\ \wedge\text{-}E_2}{\dfrac{\dfrac{C}{A \wedge B \to C}\ \to\text{-}I}{(A \to B \to C) \to (A \wedge B \to C)}\ \to\text{-}I}\ \to\text{-}E
$$

Example

Example

A proof tree for $((A \to B) \to C) \to (A \to B \to C)$

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{[B]}{A \to B}\ \to\text{-}I \qquad [(A \to B) \to C]}{C}\ \to\text{-}E}{B \to C}\ \to\text{-}I}{A \to B \to C}\ \to\text{-}I}{((A \to B) \to C) \to (A \to B \to C)}\ \to\text{-}I
$$

Here, assumption A is not used.

Discharged Assumptions as Conclusions

Discharged assumptions may be directly used as conclusions.

Example

$$\frac{[A]}{A \to A} \to\text{-}I$$

Question

What about the proof of $A \to A \to A$?

Answer

Only one of the assumption A is used.

Example

Example

A proof tree for $\neg A \vee B \rightarrow (A \rightarrow B)$

$$\dfrac{\dfrac{[\neg A \vee B] \qquad \dfrac{[A] \quad [\neg A]}{B}\,\neg\text{-}E \qquad [B]}{B}\,\vee\text{-}E}{\dfrac{A \rightarrow B}{\neg A \vee B \rightarrow (A \rightarrow B)}\,\rightarrow\text{-}I}$$

Here, the assumption B is used as a conclusion.

Axioms in Proof Trees

Axioms may occur at leaves of proof trees.

Example

A proof tree for $(A \to B) \to \neg A \vee B$.

$$\dfrac{\dfrac{\{A \vee \neg A\} \qquad \dfrac{\dfrac{[A \to B] \quad [A]}{B}\ \to\text{-}E}{\neg A \vee B}\ \vee\text{-}I_2 \qquad \dfrac{[\neg A]}{\neg A \vee B}\ \vee\text{-}I_1}{\neg A \vee B}\ \vee\text{-}E}{(A \to B) \to \neg A \vee B}\ \to\text{-}I$$

Here, $A \vee \neg A$ is an axiom.

Examples

Question

How to construct proof trees for the following wffs?

- $(A \to B) \to (\neg B \to \neg A)$;
- $(\neg B \to \neg A) \to (A \to B)$;
- $A \to \neg\neg A$;
- $\neg\neg A \to A$;
- $\neg(A \land B) \to \neg A \lor \neg B$;
- $\neg A \lor \neg B \to \neg(A \land B)$;

Partial Proof Trees

Definition (Partial Proof Trees)

A partial proof tree for α is an incomplete proof tree for α with some leaves $\beta_1, \ldots, \beta_n$ that are neither axioms or discharged assumptions.

Example

$$\dfrac{\dfrac{[(P \to R) \vee (Q \to R)] \quad \dfrac{[P \to R] \quad P}{R}\,\to\text{-}E \quad \dfrac{[Q \to R] \quad Q}{R}\,\to\text{-}E}{R}\,\vee\text{-}E}{(P \to R) \vee (Q \to R) \to R}\,\to\text{-}I$$

Here, P and Q are assumptions not discharged.

Composition of Partial Proof Trees

Lemma

Given a partial proof tree T_1 for α with undischarged assumptions $\beta_1, \ldots, \beta_n$ and T_2 for β_1 with undischarged assumptions $\gamma_1, \ldots, \gamma_m$, T_1 and T_2 may be composed into a partial proof tree for α with undischarged assumptions $\gamma_1, \ldots, \gamma_m, \beta_2, \ldots, \beta_n$.

Example

Let $\alpha = ((A \to B) \to C) \to (A \to B \to C)$. Prove $\neg\neg\alpha$ by constructing a partial proof tree with assumption α and then prove α.

$\neg$ introduction

The following is $\neg$ introduction rule, not proof by contradiction:

$$\dfrac{\begin{array}{c}[\alpha] \\ \vdots \\ \beta \wedge \neg\beta\end{array}}{\neg\alpha}\ \neg\textit{-I}$$

Example

- For every set A, $A \prec \mathcal{P}(A)$.
- $\sqrt{2}$ is irrational.

Proof by Contradiction

Definition (Proof by Contradiction)

To prove α, assume $\neg\alpha$ is false. In the logic, that is $\neg\neg\alpha \to \alpha$.

Remark

Proof by contradiction is equivalent to Law of Excluded Middle.

Theorem

Let us assume LEM is NOT an axiom, then

- If there is a proof tree for $\alpha \vee \neg\alpha$ for any α, then so it is for $\neg\neg\alpha \to \alpha$ for any α;
- If there is a proof tree for $\neg\neg\alpha \to \alpha$ for any α, then so it is for $\alpha \vee \neg\alpha$ for any α;

Provability

Definition

- α is provable (derivable) from a set Σ of assumptions if there is a partial proof tree for α whose undischarged assumptions are in Σ. It is written as $\Sigma \vdash \alpha$.
- α is provable (derivable) if $\emptyset \vdash \alpha$. It is also written as $\vdash \alpha$.

We shall write $\alpha \vdash \beta$ for $\{\alpha\} \vdash \beta$.

Example

- $\{\alpha, \beta\} \vdash (\alpha \to \sigma) \vee (\beta \to \sigma) \to \sigma$;
- $(\neg\alpha \to \neg\beta) \vdash (\beta \to \alpha)$;
- $\vdash \alpha \leftrightarrow \neg\neg\alpha$;
- $\alpha \to \beta \vdash \alpha \vee \sigma \to \beta \vee \sigma$.

Derived Rules

Definition

A derived rule has the form

$$\frac{\beta_1 \quad \dots \quad \beta_n}{\alpha}$$

It is a partial proof tree for α with undischarged assumptions $\beta_1, \dots, \beta_n$, i.e., $\{\beta_1, \dots, \beta_n\} \vdash \alpha$.

Example

$$\frac{\neg\neg\alpha}{\alpha} \qquad \frac{\alpha}{\neg\neg\alpha} \qquad \frac{\alpha \to \beta \quad \neg\beta}{\neg\alpha} \qquad \frac{\alpha \to \beta}{\neg\alpha \lor \beta}$$

Properties of Provability

Which of the following are true?

- If α is a provable then $\Sigma \vdash \alpha$ for every Σ;
- If $\alpha \in \Sigma$ then $\Sigma \vdash \alpha$;
- If $\Sigma \vdash \alpha$ and $\Sigma \vdash \alpha \to \beta$ then $\Sigma \vdash \beta$;
- If $\Sigma \vdash \alpha$ and $\alpha \vdash \beta$ then $\Sigma \vdash \beta$;
- If $\Sigma \vdash \alpha$ then for all β, $\Sigma \vdash \beta \to \alpha$;
- If $\Sigma \vdash \alpha$ and $\Sigma \vdash \beta$ then $\Sigma \vdash \alpha \wedge \beta$;
- If $\Sigma \vdash \alpha$ or $\Sigma \vdash \beta$ then $\Sigma \vdash \alpha \vee \beta$;
- $\Sigma \vdash \alpha \to \beta$ iff $\Sigma; \alpha \vdash \beta$;
- If $\Sigma \vdash \alpha$ and $\Sigma \subseteq \Delta$ the $\Delta \vdash \alpha$.

Connection between Proofs and Truths

Soundness of Natural Deduction

Theorem (Soundness of Natural Deduction)

For any Σ and α, if $\Sigma \vdash \alpha$ then $\Sigma \vDash \alpha$.

Proof.

By induction on the height of the proof tree. $\square$

Corollary

For any α, if α is provable then α is a tautology.

Soundness is useful to prove non-existence of proofs.

Consistency

Definition

A deductive calculus is consistent if for any wff α, it is not possible that $\vdash \alpha$ and $\vdash \neg\alpha$.

Theorem

Natural deduction for sentential logic is consistent.

Truth Tables and Provability

Lemma

Given a wff α, let $A_1, \ldots, A_n$ be the sentence symbols occurring in α. Let I be a row in the truth table of α. Let

$$\hat{A}_i = \begin{cases} A_i & \text{the value of } A_i \text{ is } T \text{ in } I \\ \neg A_i & \text{the value of } A_i \text{ is } F \text{ in } I \end{cases}$$

Then,

- $\hat{A}_1, \ldots, \hat{A}_n \vdash \alpha$ if the value of α is T in I;
- $\hat{A}_1, \ldots, \hat{A}_n \vdash \neg\alpha$ if the value of α is F in I.

Proof.

By induction on α. □

Completeness of Natural Deduction

Theorem

For any α, if $\vDash \alpha$ then $\vdash \alpha$.

Proof.

Let $A_1, \dots, A_n$ be the sentence symbols occurring in α. There are 2^n lines of truth tables each of which has T as the value of α. By the previous lemma, we have

$$\{\hat{A}_1, \dots, \hat{A}_n\} \vdash \alpha$$

for every line. By repeatedly applying $\vee$-E we get $\vdash \alpha$. $\square$

Completeness of Natural Deduction

Theorem (Completeness of Natural Deduction)

For any Σ and α, if $\Sigma \vDash \alpha$ then $\Sigma \vdash \alpha$.

Proof.

By Compactness, there is a finite $\Delta \subseteq \Sigma$ s.t. $\Delta \vDash \alpha$. Assume $\Delta = \{\beta_1, \ldots, \beta_n\}$, we successively prove

1. $\vDash \beta_1 \to \ldots \to \beta_n \to \alpha$;
2. $\vdash \beta_1 \to \ldots \to \beta_n \to \alpha$;
3. $\{\beta_1, \ldots, \beta_n\} \vdash \alpha$;
4. $\Sigma \vdash \alpha$.

$\square$