Mathematical Logic: Assignment 4

Nov 14, 2023

Attention: To get full credits, you *must provide explanations to your answers*! You will get at most $1/3$ of the points if you only provide the final results without any explanation.

1. (6pt) Suppose we introduce a logical constant $\perp$ representing "false". Then $\neg$ is no longer needed as $\neg\alpha$ can be replaced by $\alpha \to \perp$. The $\neg$ rules are replaced by introduction and elimination rules for $\perp$ as follows:

$$\dfrac{\begin{array}{c}[\alpha] \\ \vdots \\ \perp\end{array}}{\alpha \to \perp}\ (\perp - I) \qquad \dfrac{\perp}{\alpha}\ (\perp - E)$$

   Assume that for any truth assignment $\bar{v}(\perp) = F$. Prove the soundness of natural deduction for the inductive cases when the bottom rule is $\perp - I$ and $\perp - E$. Remember the soundness theorem is stated as follows:

   If $\Sigma \vdash \alpha$ then $\Sigma \vDash \alpha$.

   The proof proceeds by induction on the height of the partial proof tree for $\Sigma \vdash \alpha$. You need to show the following:

   - (3pt) Assume the tree looks like

$$\dfrac{\begin{array}{c}\gamma_1 \quad \dots \quad \gamma_n \quad [\alpha] \\ \vdots \\ \perp\end{array}}{\alpha \to \perp}\ \perp - I$$

     where $\{\gamma_1, \dots, \gamma_n\} \subseteq \Sigma$. Assume soundness holds for proof trees with a smaller height, prove it holds for the whole tree.
   - (3pt) Prove a similar argument when the last rule is $\perp - E$.

2. (4pt) In sentential logic, prove that
   - (2pt) $(A \to B) \vee (A \wedge \neg B)$ is provable;
   - (2pt) $(A \to B) \vee (A \wedge B)$ is not provable. (Hint: you may use the soundness and completeness theorems)

3. (6pt) List the variables occurring free in the following wffs (where Q and R are 1-ary predicate symbols; P is a 2-ary predicate symbol; f is a 2-ary function symbol)
   - (2pt) $\forall y\ (P\ x\ y \to \forall x\ P\ x\ y)$;
   - (2pt) $\forall x\ (Q\ y \to \exists\ y\ P\ x\ z)$;
   - (2pt) $(\neg\exists\ y\ R\ (f\ y\ z)) \wedge (\forall x\ \forall y\ R\ (f\ y\ z))$

4. (4pt) Formalize the reasoning of "It is impossible that there is a barber that shaves all and only people who do not shave themselves." in natural deduction.

5. (6pt) Prove the following in natural deduction:

   - (3pt) $\neg\exists x\ (A(x) \wedge B(x)) \to \forall x\ (A(x) \to \neg B(x))$
   - (3pt) $\forall x\ (P(x) \to \forall y\ (\neg T(y, x))) \to \neg\exists x \exists y\ (P(x) \wedge T(y, x))$

6. (4pt) Formalize the following statements and prove in natural deduction the last statement by assuming the first three statements (i.e., build a partial proof tree with them as assumptions not discharged). $Y(x)$, $H(x)$ and $A(x)$ denotes x is young, healthy and active, respectively. $B(x)$ denotes x likes basketball:

   - Every young and healthy person likes baseball.
   - Every active person is healthy.
   - Someone is young and active.
   - Therefore, someone likes baseball.