First-Order Logic: Syntax and Deductive Calculi

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First-Order Logic

Start reading (to keep up with lecture):

- Enderton, Chapter 2.0, 2.1
- Logic In Computer Science, Chapter 2.3

Example

Question

Premises:

- If it is raining or it is snowing then the sun is not shining.
- It is raining

Conclusion: The sun is not shining

Is the conclusion a semantic consequence of the premises?

Answer

Yes.
Let A, B, and C represent "it is raining", "it is snowing" and "the sun is shining". We can prove

$$\{A \vee B \rightarrow \neg C, A\} \models \neg C$$

Another Example

Question

Premises:

- All men are mortal
- Socrates is a man.

Conclusion: Socrates is mortal.

Is the conclusion a semantic consequence of the premises?

Atomic sentences contains assertions on infinite domains.

All men are mortal.

We need a more power logic to handle this kind of reasoning.

Syntax and Semantics of First-Order Logic

Recall that there are two parts to a logic:

- Syntax. It provides
  - A description of the language, and
  - Other syntactic constructs (we will see later)
- Semantics. It provides
  - A way of assigning *meaning* to *valid expressions*
  - In sentential logic, the meaning will be either TRUE or FALSE
  - In a first-order logic, the meaning may vary significantly

Let's Begin with the Syntax

Syntax of a First-Order Language $\mathbb{L}$

We start with the symbols of a first-order language $\mathbb{L}$.

There are two types of symbols:

- Logical Symbols, and
- Non-logical Symbols, also called Parameters

Logical Symbols

In a first-order language $\mathbb{L}$, we have the following logic symbols:

1. The two symbols (and), called parentheses;
2. $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$ and $\neg$. These are logical connective symbols;
3. $v_1, v_2, \ldots, v_n, \ldots$. An enumerable list of symbols called variables;
4. $=$. The identity or equality symbol. It may or may not be present in a particular first-order language.

Note that this $=$ is the equality symbol, not semantic equality.

Parameters

In a first-order language $\mathbb{L}$, we have the following parameters:

1. $\forall$. This is called the universal quantifier;
2. $\exists$. This is called the existential quantifier;
3. For each $n > 0$, there is a set (possibly empty) of objects called n-ary (or n-place) predicate symbols;
   - $=$ is a 2-ary predicate symbols;
4. For each $n > 0$, there is a set (possibly empty) of objects called n-ary (or n-place) function symbols;
5. A set (possibly empty) of objects called constant symbols.

By a *symbol* we mean either a logical symbol or a parameter.

Example

If a student takes the math logic course and a concept is taught in this course, then the student knows it.

The above sentence is translated into

$$\forall x\, \forall y\, (Student(x) \wedge Takes(x, MathLogic) \wedge Concept(y) \wedge Taught(y, MathLogic) \\ \rightarrow Knows(x, y)).$$

- Constant symbols: MathLogic
- 1-ary Predicates: Student, Concept
- 2-ary Predicates: Takes, Taught, Knows

Example: Set Theory

Set theory may be described by the following language:

- Equality: Yes;
- Predicate Symbols: a 2-place predicate symbol $\in$;
- Constant Symbols: the empty set $\emptyset$;
- Function Symbols: None.

Example

What does the following formula state?

$$\forall s\ \exists s'\ (s \in s' \land \neg(s' = \emptyset))$$

Example: Arithmetic on Natural Numbers

Arithmetic on natural numbers may be described by the following language:

- Equality: Yes;
- Predicate Symbols: a 2-ary predicate symbol $<$ and 1-ary symbol *Prime*;
- Constant Symbols: 0, 1, 2, . . . ;
- Function Symbols: 2-ary function symbols $+$ and $\times$.

Example

What does the following formula state?

$$\forall n\, (2 < n \rightarrow \exists n_1\; n_2\, (n = n_1 + n_2 \wedge Prime(n_1) \wedge Prime(n_2)))$$

Expressions

Like in sentential logic, an expression in a language $\mathbb{L}$ is a finite sequence of symbols.

Example

$\forall\neg\rightarrow v_1 v_2 v_4$ is an expression.

Terms

Definition

Given any n-ary function symbol f, the term-building operation $\mathcal{F}_f$ is defined by:

$$\mathcal{F}_f(\sigma_1, \ldots, \sigma_n) = f(\sigma_1, \ldots, \sigma_n)$$

We call σ_i the arguments to f.

Definition

A term is an expression built up from constant symbols and variables by applying some finite times of term-building operations.

Example

Example

Suppose:

- f is a 2-ary function symbol;
- g is a 3-ary function symbol;
- c_1 and c_2 are constant symbols.

Then $g(f(c_1, c_2), v_3, c_1)$ is a term.

Atomic Formulas

Definition

An expression is an atomic formula if it is of the form $P(t_1, \ldots, t_n)$ where $t_1, \ldots, t_n$ are terms, and P is a n-ary predicate symbol.

Example

- $=(v_7, v_3)$ is an atomic formula; We often write is in infix form: $v_7 = v_3$;
- If c_1 and c_3 are constant symbols and f is a 2-ary function symbol, then $=(f(c_1, v_7), c_3)$ or $f(c_1, v_7) = c_3$ is an atomic formula.

Well-Formed Formulas

Definition

The *formula-building operations* include the following:

- $\xi_{\neg}(\alpha) = (\neg\alpha)$
- $\xi_{\wedge}(\alpha, \beta) = (\alpha \wedge \beta)$
- $\xi_{\vee}(\alpha, \beta) = (\alpha \vee \beta)$
- $\xi_{\rightarrow}(\alpha, \beta) = (\alpha \rightarrow \beta)$
- $\xi_{\leftrightarrow}(\alpha, \beta) = (\alpha \leftrightarrow \beta)$
- $\mathcal{Q}_i(\gamma) = \forall v_i \; \gamma$
- $\mathcal{P}_i(\gamma) = \exists v_i \; \gamma$

Definition

A well-formed formula (wff, or simply formula) is an expression built up from atomic formulas by applying some finite times of term-building operations.

Example

Example

- $(\neg\forall\ v_3 = (v_1, v_2))$ is a wff.
- $(\forall v_3\ = \neg(v_1, v_2))$ is not a wff.

Abbreviations

For simplicity, we adopt the following abbreviations for wffs:

1. We may omit outermost parentheses;
2. $\forall$, $\exists$ applies to as little as possible;
3. $\neg$ applies to as little as possible, subject to (2)
4. $\wedge$ and $\vee$ apply to as little as possible, subject to (3)
5. $\rightarrow$ and $\leftrightarrow$ apply to as little as possible, subject to (4)
6. When one sentential connective is used repeatedly, grouping is to the right.
7. 2-ary predicate and function symbols are often write in infix form (e.g., $=, \times, +, \in$, etc.).

Example: Mortality of Men

Assume the language $\mathbb{L}$ contains the following symbols:

- *Man*: 1-ary predicate symbol for asserting whether a being is a man;
- *Mt*: 1-ary predicate symbol for asserting whether a being is a mortal;

Question

How to interpret the sentence "All men are mortal"?

Answer

- $\forall v_1$ [if v_1 is a man then v_1 is mortal];
- $\forall v_1$ [v_1 is a man] $\to$ [v_1 is mortal];
- $\forall v_1\ (Man(v_1) \to Mt(v_1))$.

Example: Set Theory

In set theory, we have the symbols $=$, $\emptyset$ and $\in$.

Question

How to interpret the sentence "There is no set of which every set is a member" in first-order logic?

Answer

- ($\neg$[There is some set of which every set is a member]);
- ($\neg\exists v_1$[Every set is a member of v_1]);
- ($\neg\exists v_1 \forall v_2$[$v_2$ is a member of v_1]);
- $(\neg\exists v_1 \forall v_2 \; v_2 \in v_1)$;

Example: Elementary Arithmetic

In elementary arithmetic, we have the symbols $<$, $+$, $\times$, 0 and 1.

- We represent a natural number n by

  $$\underbrace{1 + \ldots + 1}_{n \text{ times}} + 0$$

  Therefore, $2 + 1 = 3$ is interpreted as

  $$(1 + 1 + 0) + (1 + 0) = (1 + 1 + 1 + 0)$$

- We interpret the sentence "any non-zero natural number is the successor of some natural number" as follows:
  - $\forall v_1$[if v_1 is non-zero, then v_1 is the successor of some number]
  - $\forall v_1 \; (\neg v_1 = 0 \rightarrow \exists v_2 \; v_1 = 1 + v_2)$

Summary of Syntax

We introduced the symbols of a first-order language $\mathbb{L}$, and definitions of:

- Terms
- Atomic Formulas
- Well-Formed Formulas (wffs)

Natural Deduction for First-Order Logic

Rules and Axioms

Similar to proposition but have introduction and elimination rules for quantifiers

- Law of Excluded Middle
- Basic Rules + Quantifier Rules

All the concepts related to deductive calculi also work for first-order logic:

- (Partial) Proof trees
- Provability: $\Sigma \vdash \alpha$
- Derived Rules
- ...

Basic Rules

$$\frac{\alpha \quad \beta}{\alpha \wedge \beta}\ \wedge\text{-}I \qquad \frac{\alpha \wedge \beta}{\alpha}\ \wedge\text{-}E_1 \qquad \frac{\alpha \wedge \beta}{\beta}\ \wedge\text{-}E_2$$

$$\frac{\alpha}{\alpha \vee \beta}\ \vee\text{-}I_1 \qquad \frac{\beta}{\alpha \vee \beta}\ \vee\text{-}I_2 \qquad \frac{\alpha \vee \beta \quad \begin{array}{c}[\alpha]\\ \vdots\\ \delta\end{array} \quad \begin{array}{c}[\beta]\\ \vdots\\ \delta\end{array}}{\delta}\ \vee\text{-}E$$

$$\frac{\begin{array}{c}[\alpha]\\ \vdots\\ \beta\end{array}}{\alpha \to \beta}\ \to\text{-}I \qquad \frac{\alpha \to \beta \quad \alpha}{\beta}\ \to\text{-}E \qquad \frac{\begin{array}{c}[\alpha]\\ \vdots\\ \beta \wedge \neg\beta\end{array}}{\neg\alpha}\ \neg\text{-}I \qquad \frac{\beta \quad \neg\beta}{\alpha}\ \neg\text{-}E$$

$$\frac{\alpha \to \beta \quad \beta \to \alpha}{\alpha \leftrightarrow \beta}\ \leftrightarrow\text{-}I \qquad \frac{\alpha \leftrightarrow \beta}{\alpha \to \beta}\ \leftrightarrow\text{-}E_1 \qquad \frac{\alpha \leftrightarrow \beta}{\beta \to \alpha}\ \leftrightarrow\text{-}E_2$$

Substitutions: Terms

Definition

Let u be a term, x be a variable, and t be a term. u_t^x is the result of replacing every occurrence of x in u by t.

Example

Let $\mathbb{L}$ be a language with a 1-ary function symbol f and 2-ary function symbol g, and a constant c. Let $u = g(f(c), x)$ and $t = g(c, x)$. Then

$$u_t^x = g(f(c), g(c, x)).$$

Substitutions: Formulas

Definition

For a wff α, variable x, and term t, let α^x_t be the result of replacing every free occurrence of x in α by t. More precisely, we define α^x_t as follows:

- If α is atomic, say $\alpha = P(u_1, \dots, u_n)$. Then
$$\alpha^x_t = P({u_1}^x_t, \dots, {u_n}^x_t);$$
- If $\alpha = (\neg\beta)$, then
$$\alpha^x_t = (\neg\beta^x_t)$$
- If $\alpha = (\beta \to \gamma)$, then
$$\alpha^x_t = (\beta^x_t \to \gamma^x_t).$$
- ...

Substitutions: Formulas (Cont'd)

Definition

- If $\alpha = \forall y\ \beta$ then

$$\alpha^x_t = \begin{cases} \alpha & \text{if } y = x \\ \forall y\ \beta^x_t & \text{if } y \neq x \end{cases}$$

- If $\alpha = \exists y\ \beta$ then

$$\alpha^x_t = \begin{cases} \alpha & \text{if } y = x \\ \exists y\ \beta^x_t & \text{if } y \neq x \end{cases}$$

Examples

Example

- $\varphi^x_x = \varphi$
- $(Q(x) \to \forall x\ P(x))^x_y = Q(y) \to \forall x\ P(x)$
- If α is $\exists y\ \forall x\ P(x, y) \to y = x$, then

$$\alpha^x_t = \exists y\ \forall x\ P(x, y) \to y = t.$$

- If $\alpha = \neg\forall y\ x = y$, then $\forall x\ \alpha \to \alpha^x_z$ is

$$\forall x(\neg\forall y\ x = y) \to (\neg\forall y\ z = y).$$

$\forall$-Quantifier Rules (First Attempt)

$$\frac{\alpha}{\forall x\ \alpha}\ \forall\text{-}I \qquad \frac{\forall x\ \alpha}{\alpha^x_t}\ \forall\text{-}E$$

- $\forall$-I: if we can prove α independent of x, then we can prove $\forall x\ \alpha$;
- $\forall$-E: given $\forall x\ \alpha$, α should hold for any t for x.

Question

How to prove Socrates is mortal?

$$\dfrac{\dfrac{\forall x\ (Man(x) \rightarrow Mt(x))}{Man(Socrates) \rightarrow Mt(Socrates)}\ \forall\text{-}E \qquad Man(Socrates)}{Mt(Socrates)}\ \rightarrow\text{-}E$$

Example

Prove $\forall x\ A(x) \rightarrow (\forall x\ B(x) \rightarrow \forall y\ (A(y) \wedge B(y)))$.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\forall x\ A(x)]}{A(y)}\ \forall\text{-}E \qquad \dfrac{[\forall x\ B(x)]}{B(y)}\ \forall\text{-}E}{A(y) \wedge B(y)}\ \wedge\text{-}I}{\forall y\ (A(y) \wedge B(y))}\ \forall\text{-}I}{\forall x\ B(x) \rightarrow \forall y\ (A(y) \wedge B(y))}\ \rightarrow\text{-}I}{\forall x\ A(x) \rightarrow (\forall x\ B(x) \rightarrow \forall y\ (A(y) \wedge B(y)))}\ \rightarrow\text{-}I
$$

Example

Prove $\forall x\ A(x) \rightarrow \forall x\ (A(x) \vee B(x))$:

$$
\dfrac{\dfrac{\dfrac{\dfrac{[\forall x\ A(x)]}{A(x)}\ \forall\text{-}I}{A(x) \vee B(x)}\ \vee\text{-}I_1}{\forall x\ (A(x) \vee B(x))}\ \forall\text{-}I}{\forall x\ A(x) \rightarrow \forall x\ (A(x) \vee B(x))}\ \rightarrow\text{-}I
$$

Problem with $\forall$-I

How to prove that if Socrates is mortal then God is mortal?

$$\frac{Mt(Socrates)}{\overline{Mt(God)}}$$

Problem: α is NOT independent of x in $\forall$-I

Free Occurrence of Variables

Definition

- The variable x occurs free in an atomic formula φ iff it occurs in φ;
- x occurs free in $\neg\alpha$ iff x occurs free in α;
- x occurs free in $\alpha \to \beta$ iff x occurs free in α or in β;
- x occurs free in $\alpha \wedge \beta$ iff x occurs free in α or in β;
- ...
- x occurs free in $\forall y\, \alpha$ iff x occurs free in α and $x \neq y$.
- x occurs free in $\exists y\, \alpha$ iff x occurs free in α and $x \neq y$.

Sentences

Definition (Sentences)

α is a sentence iff no variable occurs free in α.

Remark

We often use σ or τ to stand for sentences.

Examples

Question

Which variables occur free in the following?

- $0 < 1$
  None, so this is a sentence.
- $\forall x\ (\neg x < y)$
  y occurs free, but x *does not*.
- $\forall x(\neg x < 0)$
  No variable occurs free, so this is a sentence.
- $\forall x \forall y(x < y \to \exists z\ (x < z \land z < y))$
  None, so this is a sentence.

$\forall$-Quantifier Rules (Second Attempt)

$$\frac{\alpha^x_y}{\forall x\ \alpha}\ \forall\text{-}I \qquad \frac{\forall x\ \alpha}{\alpha^x_t}\ \forall\text{-}E$$

- $\forall$-I now carries a *side condition*: y must not occur free in any assumption not discharged or in $\forall x\ \alpha$.

Question

Now, can we prove if Socrates is mortal then God is mortal?

Problem with $\forall$-E

Assume that for any object there is something different. Then there is an object that is not itself.

$$\frac{\forall x\ \exists y\ x \neq y}{\exists y\ y \neq y}$$

Problem: the free occurrence y is captured by the existential quantifier!

Substitutability

Definition

Let α be a wff, x be a variable, and t be a term. t is substitutable for x in α if

- α is atomic, or
- $\alpha = (\neg\beta)$ and t is substitutable for x in β, or
- $\alpha = (\beta \to \gamma)$ and t is substitutable for x in both β and γ, or
- ...
- $\alpha = \forall y\ \beta$ and either
  - x does not occur free in $\forall y\ \beta$;
  - x does occur free in $\forall y\ \beta$, and t is substitutable for x in β, and y does not occur in t.
- Similarly for $\alpha = \exists y\ \beta$.

$\forall$-Quantifier Rules (Final Attempt)

- $\forall$-introduction rule:

$$\frac{\alpha^x_y}{\forall x\ \alpha}\ \forall\text{-}I$$

It applies only if y does not occur free in any assumption not discharged or in $\forall x\ \alpha$, and y is substitutable for x in α.

- $\forall$-elimination rule:

$$\frac{\forall x\ \alpha}{\alpha^x_t}\ \forall\text{-}E$$

It applies only if t is substitutable for x in α.

Examples

Prove $\forall x\, \forall y\, \alpha \rightarrow \forall y\, \forall x\, \alpha$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\forall x\, \forall y\, \alpha]}{\forall y\, \alpha^x_w}\ \forall\text{-}E}{\alpha^{x\,y}_{w\,z}}\ \forall\text{-}E}{\forall x\, \alpha^y_z}\ \forall\text{-}I}{\forall y\, \forall x\, \alpha}\ \forall\text{-}I}{\forall x\, \forall y\, \alpha \rightarrow \forall y\, \forall x\, \alpha}\ \rightarrow\text{-}I$$

Note that x and y must pick different instances of variables in $\forall$-I. What if this is violated?

Try prove if everyone trust itself, then everyone trusts everyone.

Example

Suppose we have defined E (Even) and O (Odd) such that

- $\forall n\,(\neg E(n) \to O(n))$;
- $\forall n\,(O(n) \to \neg E(n))$;

Prove $\forall n\,(E(n) \vee O(n))$.

$$
\dfrac{\dfrac{\{E(m) \vee \neg E(m)\} \qquad \dfrac{[E(m)]}{E(m) \vee O(m)}\;\vee\text{-}I_1 \qquad \dfrac{\dfrac{\dfrac{\forall n\,(\neg E(n) \to O(n))}{\neg E(m) \to O(m)}\;\forall\text{-}E \qquad [\neg E(m)]}{O(m)}\;\to\text{-}E}{E(m) \vee O(m)}\;\vee\text{-}I_2}{E(m) \vee O(m)}\;\vee\text{-}E}{\forall n\,(E(n) \vee O(n))}\;\forall\text{-}I
$$

Question

How to prove $\forall n\,\neg(E(n) \wedge O(n))$?

$\exists$-Quantifier Rules

Those rules are dual of $\forall$-rules:

- $\exists$-introduction rule:

$$\frac{\alpha^x_t}{\exists x\ \alpha}\ \exists\text{-}I$$

It applies only if t is substitutable for x in α.

- $\exists$-elimination rule:

$$\frac{\exists x\ \alpha \qquad \begin{matrix}[\alpha^x_y] \\ \vdots \\ \beta\end{matrix}}{\beta}\ \exists\text{-}E$$

It applies only if y does not occur free in any assumption not discharged, in $\exists x\ \alpha$, or in β, and y is substitutable for x in α.

Example

Prove $\exists x\ (A(x) \wedge B(x)) \rightarrow \exists x\ A(x)$:

$$
\dfrac{\dfrac{[\exists x\ (A(x) \wedge B(x))] \qquad \dfrac{\dfrac{[A(y) \wedge B(y)]}{A(y)}\ \wedge\text{-}E_1}{\exists x\ A(x)}}{\exists x\ A(x)}\ \exists\text{-}E}{\exists x\ (A(x) \wedge B(x)) \rightarrow \exists x\ A(x)}\ \rightarrow\text{-}I
$$

Question

- Prove $\exists x\ (A(x) \vee B(x)) \leftrightarrow \exists x\ A(x) \vee \exists x\ B(x)$;

Example

Prove $\forall x\ (A(x) \to \neg B(x)) \to \neg\exists x\ (A(x) \wedge B(x))$

$$
\cfrac{
\cfrac{
[\exists x\ (A(x) \wedge B(x))]
\qquad
\cfrac{
\cfrac{[A(y) \wedge B(y)]}{B(y)}\ \wedge\text{-}E_2
\qquad
\cfrac{
\cfrac{[\forall x\ (A(x) \to \neg B(x))]}{A(y) \to \neg B(y)}\ \forall\text{-}E
\qquad
\cfrac{[A(y) \wedge B(y)]}{A(y)}
}{\neg B(y)}\ \to
}{B(y) \wedge \neg B(y)}\ \neg\text{-}E
}{
\cfrac{B(y) \wedge \neg B(y)}{\neg\exists x\ (A(x) \wedge B(x))}\ \neg\text{-}I
}\ \wedge\text{-}I
}{\forall x\ (A(x) \to \neg B(x)) \to \neg\exists x\ (A(x) \wedge B(x))}\ \to\text{-}I
$$

Is the above proof correct?

Example

Show that the following two different expression of "nobody trusts a politician" are equivalent ($P(x)$ denotes x is a politician and $Trusts(y, x)$ denotes y trusts x):

- $\neg \exists x \exists y \, (P(x) \wedge T(y, x))$
- $\forall x \, (P(x) \rightarrow \forall y \, (\neg T(y, x)))$

Challenge

"It is impossible that there is a barber that shaves all and only people who do not shave themselves."

Informal argument:

- Suppose a barber shaves himself. By assumption he does not shave himself;
- Suppose a barber does not shave himself. By assumption he should shave himself.

Challenge: turn the reasoning above into natural deduction.

Connection of $\forall$ ($\exists$) with $\wedge$ ($\vee$)

$\forall$ is really glorified $\wedge$. If the domain of x is finite $(t_1, \ldots, t_n)$, then $\forall x\ \alpha = \alpha^x_{t_1} \wedge \ldots \wedge \alpha^x_{t_n}$.

$$\frac{\alpha^x_{t_1} \quad \ldots \quad \alpha^x_{t_n}}{\alpha^x_{t_1} \wedge \ldots \wedge \alpha^x_{t_n}}\ \forall\text{-}I \qquad \frac{\alpha^x_{t_1} \wedge \ldots \wedge \alpha^x_{t_n}}{\alpha^x_{t_i}}\ \forall\text{-}E$$

$\exists$ is really glorified $\vee$. If the domain of x is finite $(t_1, \ldots, t_n)$, then $\exists x\ \alpha = \alpha^x_{t_1} \vee \ldots \vee \alpha^x_{t_n}$.

$$\frac{\alpha^x_{t_i}}{\alpha^x_{t_1} \vee \ldots \vee \alpha^x_{t_n}}\ \exists\text{-}I \qquad \frac{\alpha^x_{t_1} \vee \ldots \vee \alpha^x_{t_n} \quad \begin{array}{c}[\alpha^x_{t_1}]\\ \vdots \\ \beta\end{array} \quad \ldots \quad \begin{array}{c}[\alpha^x_{t_n}]\\ \vdots \\ \beta\end{array}}{\beta}\ \exists\text{-}E$$

Alternative Form of Natural Deduction

Basic Rules

Rules for deriving *judgements* of the form $\Sigma \vdash \alpha$:

$$\frac{}{\alpha \vdash \alpha}\ \textit{Initial} \qquad \frac{\Delta \vdash \alpha \quad \Delta \subseteq \Sigma}{\Sigma \vdash \alpha}\ \textit{Weakening}$$

$$\frac{\Sigma \vdash \alpha \quad \Sigma \vdash \beta}{\Sigma \vdash \alpha \wedge \beta}\ \wedge\text{-}\mathit{I} \qquad \frac{\Sigma \vdash \alpha \wedge \beta}{\Sigma \vdash \alpha}\ \wedge\text{-}\mathit{E}_1 \qquad \frac{\Sigma \vdash \alpha \wedge \beta}{\Sigma \vdash \beta}\ \wedge\text{-}\mathit{E}_2$$

$$\frac{\Sigma \vdash \alpha}{\Sigma \vdash \alpha \vee \beta}\ \vee\text{-}\mathit{I}_1 \qquad \frac{\Sigma \vdash \beta}{\Sigma \vdash \alpha \vee \beta}\ \vee\text{-}\mathit{I}_2$$

$$\frac{\Sigma \vdash \alpha \vee \beta \quad \Sigma; \alpha \vdash \delta \quad \Sigma; \beta \vdash \delta}{\Sigma \vdash \delta}\ \vee\text{-}\mathit{E}$$

More Basic Rules

$$\frac{\Sigma; \alpha \vdash \beta}{\Sigma \vdash \alpha \to \beta} \to\text{-}I \qquad \frac{\Sigma \vdash \alpha \to \beta \quad \Sigma \vdash \alpha}{\Sigma \vdash \beta} \to\text{-}E$$

$$\frac{\Sigma; \alpha \vdash \beta \wedge \neg\beta}{\Sigma \vdash \neg\alpha} \neg\text{-}I \qquad \frac{\Sigma \vdash \beta \quad \Sigma \vdash \neg\beta}{\Sigma \vdash \alpha} \neg\text{-}E$$

$$\frac{\Sigma \vdash \alpha \to \beta \quad \Sigma \vdash \beta \to \alpha}{\Sigma \vdash \alpha \leftrightarrow \beta} \leftrightarrow\text{-}I$$

$$\frac{\Sigma \vdash \alpha \leftrightarrow \beta}{\Sigma \vdash \alpha \to \beta} \leftrightarrow\text{-}E_1 \qquad \frac{\Sigma \vdash \alpha \leftrightarrow \beta}{\Sigma \vdash \beta \to \alpha} \leftrightarrow\text{-}E_2$$

$\forall$-Quantifier Rules

- $\forall$-introduction rule:

$$\frac{\Sigma \vdash \alpha^x_y}{\Sigma \vdash \forall x\ \alpha}\ \forall\text{-}I$$

It applies only if y does not occur free in Σ or in $\forall x\ \alpha$.

- $\forall$-elimination rule:

$$\frac{\Sigma \vdash \forall x\ \alpha}{\Sigma \vdash \alpha^x_t}\ \forall\text{-}E$$

It applies only if t is substitutable for x in α.

$\exists$-Quantifier Rules

- $\exists$-introduction rule:

$$\frac{\Sigma \vdash \alpha^x_t}{\Sigma \vdash \exists x\ \alpha}\ \exists\text{-}I$$

It applies only if t is substitutable for x in α.

- $\exists$-elimination rule:

$$\frac{\Sigma \vdash \exists x\ \alpha \qquad \Sigma; \alpha^x_y \vdash \beta}{\Sigma \vdash \beta}\ \exists\text{-}E$$

It applies only if y does not occur free in Σ, in $\exists x\ \alpha$, or in β.

Examples

Prove

$$\vdash \forall x\,(A(x) \to \neg B(x)) \to \neg\exists x\,(A(x) \wedge B(x)).$$

Exercises: Reprove the other examples using rules for deriving judgments.

Type Theory

Principle of Type Theories

Curry-Howard Correspondence or Propositions-as-types:

Remark

The following two statements are equivalent:

- e is a proof tree for the formula α;
- e is a program of type α.

In other words:

- Proofs are programs;
- Propositions (formulas) are types.

A Simple Theory of Types

Deriving judgments of the form $\Sigma \vdash e : \alpha$ where $\Sigma = \{x_1 : \alpha_1, \ldots, x_n : \alpha_n\}$, meaning that a program e whose free variables are in $\{x_1, \ldots, x_n\}$ has type α.

$$e = x \mid \lambda x.e' \mid (e_1\ e_2)$$

$$\frac{}{x : \alpha \vdash x : \alpha}\ \mathit{Initial} \qquad \frac{\Delta \vdash \alpha \quad \Delta \subseteq \Sigma}{\Sigma \vdash \alpha}\ \mathit{Weakening}$$

$$\frac{\Sigma, x : \alpha \vdash e : \beta}{\Sigma \vdash \lambda x.e : \alpha \to \beta}\ \to\text{-}I (x \notin \Sigma)$$

$$\frac{\Sigma \vdash e_1 : \alpha \to \beta \quad \Sigma \vdash e_2 : \alpha}{\Sigma \vdash (e_1\ e_2) : \beta}\ \to\text{-}E$$

Enrichment with Basic Types

We add a basic type called *nat* to denote natural numbers and the typing rules for their operations

$$e = x \mid \lambda x.e' \mid (e_1\ e_2) \mid i \mid e_1 + e_2 \mid e_1 * e_2$$

$$\frac{i \in \{0, 1, 2, \ldots\}}{\vdash i : nat}\ Const$$

$$\frac{\Sigma \vdash e_1 : nat \quad \Sigma \vdash e_2 : nat}{\Sigma \vdash e_1 + e_2 : nat}\ Add$$

$$\frac{\Sigma \vdash e_1 : nat \quad \Sigma \vdash e_2 : nat}{\Sigma \vdash e_1 * e_2 : nat}\ Mult$$

Examples

Derive the following typing judgments

- $$x : nat, y : nat \vdash (\lambda x.\lambda y.x * x + y * y) : nat \to nat \to nat$$
- $$x : nat \vdash ((\lambda x.\lambda y.x * x + y * y)\ x) : nat \to nat$$
- $$x : nat \vdash (((\lambda x.\lambda y.x * x + y * y)\ x)\ 2) : nat$$

Advanced Type Theories

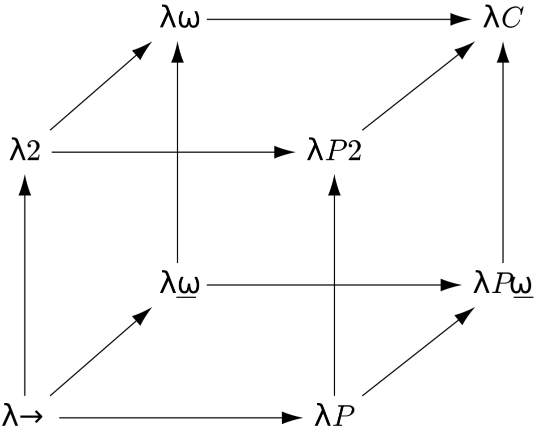

Figure: Lambda Cube

https://en.wikipedia.org/wiki/Lambda_cube