Sets, Functions and Relations

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Notations

Get familiar with the following notations

- $P \ \& \ Q$ is an abbreviation of: P and Q;
- $P \mid\mid Q$ is an abbreviation of: P or Q;
- $\sim P$ is an abbreviation of: not P;
- $P \implies Q$ is an abbreviation of: if P then Q;
- $P \iff Q$ is an abbreviation of: P if and only if Q;
- $\dot{\exists} x, P$ is an abbreviation of: there exists an x such that P;
- $\dot{\forall} x, P$ is an abbreviation of: for all x such that P;

Sets

A set is a collection of elements.

Example

$\mathbb{N}$ (natural numbers), $\mathbb{Z}$ (integers), $\mathbb{Q}$ (rationals), $\mathbb{R}$ (reals)

Given a statement P about x, $\{x \mid P(x)\}$ is a set of objects such that $P(x)$ is true.

Example

- $\{x \mid x \in \mathbb{N}\ \&\ x \text{ is divisible by } 2\}$;
- Alternatively: $\{x \in \mathbb{N} \mid x \text{ is divisible by } 2\}$;

Set Operations

- $A \cup B = \{x \mid x \in A \mid\mid x \in B\}$;
- $A \cap B = \{x \mid x \in A \;\&\; x \in B\}$;
- $A \setminus B = \{x \mid x \in A \;\&\; x \notin B\}$;
- $A \times B = \{\langle x, y\rangle \mid x \in A \;\&\; y \in B\}$;
- $A \subseteq B$ if every member of A is a member of B;
- $A \subset B$ if $A \subseteq B \;\&\; A \neq B$;

Relations

Given n sets $A_1, \ldots, A_n$, a relation over them is a subset of $A_1 \times \ldots \times A_n$.

Definition (Binary Relations)

A binary relation R is a relation over $A \times B$ given some A and B.

- The domain of R (written $dom(R)$) is $\{x \mid \exists y, \langle x, y \rangle \in R\}$;
- The range of R (written $rng(R)$) is $\{y \mid \exists x, \langle x, y \rangle \in R\}$;

Example

$<$ is the relation $\{\langle x, y \rangle \in \mathbb{N} \times \mathbb{N} \mid x \text{ is less than } y\}$

A binary relation R on A is

- reflexive iff $\langle x, x \rangle \in R$ for every $x \in A$;
- symmetric iff $\langle x, y \rangle \in R \to \langle y, x \rangle \in R$;
- transitive iff $\langle x, y \rangle \in R \;\&\; \langle y, z \rangle \in R \implies \langle x, z \rangle \in R$

An equivalence relation is a relation satisfying all three properties.

Functions

Definition (Functions)

A function $f : A \to B$ is a binary relation over $A \times B$ satisfying the following property:

- its domain is A;
- for every $x \in A$, there is a unique $y \in B$ s.t. $\langle x, y \rangle \in f$.

We write $f(x)$ for the value in B related to x by f.

One-to-One Correspondence

Definition (One-to-One Correspondence)

A function $f : A \to B$ is

- one-to-one (injective) if for every $x, y \in A$, $f(x) = f(y) \Longrightarrow x = y$;
- onto (surjective) if for every $y \in B$, there is some $x \in A$ s.t. $f(x) = y$;
- an one-to-one correspondence (bijective) between A and B if f is both one-to-one and onto.

One-to-One Correspondence (Cont'd)

Example

$\mathbb{Z}$ is one-to-one correspondent to $\mathbb{N}$.

We use the notion of one-to-one correspondence between infinite sets to talk about the size of infinite sets.

Finite Sets

Definition

- The set X is finite if there is a natural number n and a one-to-one correspondence between X and $\{0, \ldots, n\}$;
- The set X is infinite if it is not finite.

Enumerable Sets

Definition

- The set X is enumerable if there is a one-to-one correspondence between X and $\mathbb{N}$;

Example

$\mathbb{Z}$ is enumerable.

Listings of Sets

Definition

Let A be a set. $a_0, \ldots, a_n, \ldots$ is a listing of A if

- each a_i is in A, and
- every member of A is equal to a_n for some $n \in \mathbb{N}$.

Theorem

The set A is enumerable iff there is some listing without repetitions of A.

Example

$\mathbb{Q}$ is enumerable.

Countable Sets

Definition

- The set X is countable if it is finite or enumerable;
- The set X is uncountable if it is not countable.

Theorem

- The set X is countable iff there is a one-to-one mapping $f : X \to \mathbb{N}$;
- The set X is countably infinite iff it is enumerable;

Listings of Countable Sets

Theorem

- The set A is countable and nonempty iff there is some listing with possible repetitions of A.

More about Countable Sets

Proposition

If A is enumerable and $B \subseteq A$ then B is countable.

There are lots of Countable Sets

Theorem

1. A and B countable implies $A \cup B$, $A \cap B$ and $A \times B$ are countable.
2. if each of $A_0, \ldots, A_n, \ldots$ is countable then the union, i.e. $\bigcup\{A_n \mid n \in \mathbb{N}\}$, is countable.
3. if A is countable and non-empty then the set of all finite sequences of members of A is countable.

How do we find an uncountable set?

The Power Set Operation

Definition

The power set of the set A is:

$$\mathcal{P}(A) = \{X \mid X \subseteq A\}$$

Theorem (Cantor's Theorem)

$\mathcal{P}(\mathbb{N})$ is uncountable.

Proof.

By a Diagonal Argument!

□

Corollary

$\mathbb{R}$ is uncountable.

Is $\mathcal{P}(\mathbb{N})$ the biggest set?

Domination of Sets

Definition

- $A \preceq B$ if there is a one-to-one function $f : A \to B$;
- $A \prec B$ if $A \preceq B$ but not $B \preceq A$;
- $A \equiv B$ if $A \preceq B$ and $B \preceq A$.

Remark

- $A \preceq B$ is our way of saying A is no bigger than B;
- $A \prec B$ is our way of saying A is smaller than B;
- $A \equiv B$ is one way of saying A and B have the same size.

Cantor-Schröder-Bernstein Theorem

Two different ways to say A and B have the same size:

1. $A \equiv B$, and
2. There is a one-to-one correspondence between A and B.

Are they equivalent?

Theorem (Cantor-Schröder-Bernstein)

$A \equiv B$ iff there is a one-to-one correspondence between A and B.

Cantor's Theorem

Theorem (Cantor)

For every set A, $A \prec \mathcal{P}(A)$.

Proof.

Show that any $f : A \to \mathcal{P}(A)$ cannot be surjective. $\square$

Corollary

1. For every A there is a B s.t. $A \prec B$;
2. $\mathcal{P}(\mathbb{N})$ is uncountable.

Sets of Functions

Definition

${}^{A}B$ is the set of all functions that map A into B.

Example

${}^{A}\{0,1\}$ is the set of all functions that map A into $\{0,1\}$. Suppose $f \in^{A} \{0,1\}$. Let $X = \{x \in A \mid f(x) = 1\}$. Then f is the characteristic function of X, i.e. $f = C_X$. Thus ${}^{A}\{0,1\}$ is the set of all characteristic functions of subsets of A.

Characteristic Functions and Power Set

Exercise

Find a one-to-one correspondence between ${}^{A}\{0, 1\}$ and $\mathcal{P}(A)$.

A special case:

Theorem

There is a one-to-one correspondence between ${}^{\mathbb{N}}\{0, 1\}$ and $\mathcal{P}(\mathbb{N})$.

In particular,

Theorem

${}^{\mathbb{N}}\{0, 1\}$ is uncountable.

More Uncountable Sets

Theorem

$^{\mathbb{N}}\mathbb{N}$ is uncountable.

The Continuum Hypothesis

Question

Is there a set A s.t. $\mathbb{N} \prec A \prec \mathcal{P}(\mathbb{N})$?

If there is such a set A it would be “bigger” than $\mathbb{N}$ but “smaller” than $\mathcal{P}(\mathbb{N})$.