Mathematical Logic: Assignment 5

Dec 4, 2023

Attention: To get full credits, you *must provide explanations to your answers*! You will get at most $1/3$ of the points if you only provide the final results without any explanation.

1. (8pt) Let $\mathfrak{N} = (\mathbb{N}, +, \times, 0, 1, <)$. Let s be an assignment for $\mathfrak{N}$ such that $s(v_n) = 2n$. Are the following statement true or not? Give explanations to your answers (Hint: remember that assignments can only affect free occurrences of variables).

   - (2pt) $\vDash_{\mathfrak{N}} \exists v_0, v_0 \dot{+} v_0 \doteq v_1[s]$;
   - (2pt) $\vDash_{\mathfrak{N}} \exists v_0, v_0 \dot{\times} v_0 \doteq v_1[s]$;
   - (2pt) $\vDash_{\mathfrak{N}} \forall v_0 \exists v_1 \ v_0 \doteq v_1[s]$;
   - (2pt) $\vDash_{\mathfrak{N}} \forall v_0 \forall v_1 \ v_0 \dot{+} \dot{1} \dot{<} v_1 \to \exists v_2 \ v_0 \dot{<} v_2 \wedge v_2 \dot{<} v_1[s]$;

2. (5pt) Given a language $\mathbb{L}$ with two 1-ary predicate symbols P and Q, prove that

   $$\neg \exists x \ (P(x) \wedge Q(x)) \to \forall x \ (Q(x) \to \neg P(x))$$

   is valid (i.e., true in any structure for $\mathbb{L}$).

3. (5pt) Given a language $\mathbb{L}$ with an 2-ary predicate symbol R, define a sentence σ such that, for any structure $\mathfrak{A}$, $\vDash_{\mathfrak{A}} \sigma$ iff $R^{\mathfrak{A}}$ is a function.

4. (12pt) For each of the following relations, give a wff that defines it in the structure $\mathfrak{A} = (\mathbb{N}, +, \times)$ (Assume this language has $\dot{+}$, $\dot{\times}$ and $\doteq$ with standard interpretations in $\mathfrak{A}$).

   - (4pt) $\{0, 1\}$;
   - (4pt) $\{2\}$;
   - (4pt) $\{n \in \mathbb{N} \mid n \text{ is an even number}\}$ (Hint: you may reuse the previous defining formula φ and use $\varphi(x)$ to denote the renaming of free occurrences of v_1 in φ to x.)