Mathematical Logic: Assignment 3

Oct 31, 2022

Attention: To get full credits, you *must provide explanations to your answers*! You will get at most $1/3$ of the points if you only provide the final results without any explanation.

1. (5pt) Prove the compactness theorem from its corollary. That is, given the following propositions:

   (a) For any $\Gamma \vDash \alpha$, there is a finite set $\Delta \subseteq \Gamma$ such that $\Delta \vDash \alpha$;

   (b) For any finitely satisfiable set Γ, Γ is satisfiable.

   Prove that (a) implies (b). (Hint: consider proof by contradiction)

2. (8pt) Write down proof trees for the following propositions (attention: you must write down the whole tree without using any shortcut or derived rules):

   - (2pt) $(\neg A \vee \neg B) \to \neg(A \wedge B)$;
   - (3pt) $(B \to (A \leftrightarrow \neg A)) \to \neg B$;
   - (3pt) $((P \to Q) \to P) \to P$.

   (Hint: some may need Law of Excluded Middle)

3. (6pt) Prove the following facts about provability:

   - (3pt) If $\Gamma \vdash \alpha$ and $\Delta; \alpha \vdash \beta$, then $\Gamma \cup \Delta \vdash \beta$;
   - (3pt) If $\Gamma; \alpha \vdash \beta \leftrightarrow \neg\beta$, then $\Gamma \vdash \neg\alpha$.

   (Hint: you need to construct new proof trees from existing trees)

4. (6pt) Prove the following properties:

   - (3pt) If $\vdash \alpha \vee \neg\alpha$ for any α, then $\vdash (\alpha \to \beta) \to (\neg\alpha \vee \beta)$ for any α and β;
   - (3pt) If $\vdash (\alpha \to \beta) \to (\neg\alpha \vee \beta)$ for any α and β, then $\vdash \alpha \vee \neg\alpha$ for any α.

   (Hint: note that $\vdash \alpha$ means α has a proof tree. Think about our proof of the equality between LEM and proof by contradiction in the class)

5. (5pt) Suppose we introduce a logical constant $\perp$ representing "false". Then $\neg\alpha$ can be replaced by $\alpha \to \perp$, meaning α should never be true (otherwise, the conclusion is false). The $\neg$ rules are replaced by introduction and elimination rules for $\perp$ as follows:

$$\cfrac{\begin{array}{c}[\alpha] \\ \vdots \\ \perp\end{array}}{\alpha \to \perp}\ (\perp - I) \qquad \frac{\perp}{\alpha}\ (\perp - E)$$

   Here, $\perp$-I is denotes that if we can prove $\perp$ by assuming α, then we can prove $\alpha \to \perp$. $\perp$-E is the principle of explosion: if we can prove false ($\perp$), then anything (any α) follows.

(a) (2pt) Translate $\neg A \vee \neg B \rightarrow \neg(A \wedge B)$ into a wff without $\neg$ by $\neg\alpha \equiv \alpha \rightarrow \perp$;

(b) (3pt) Construct a proof tree for the transformed wff with the newly introduced rules.