First-Order Logic: Semantics

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First-Order Logic

Start reading (to keep up with lecture):

- Enderton, Chapter 2.2

Structures

Definition of Structures

Let $\mathbb{L}$ be a first-order language.

Definition

A structure $\mathfrak{A}$ for $\mathbb{L}$ consists of:

- a non-empty set called the universe (or *domain*) of the structure and usually written as $|\mathfrak{A}|$;
- for each n-ary predicate symbol P of $\mathbb{L}$, other than $\doteq$, an n-ary relation $P^{\mathfrak{A}}$ on $|\mathfrak{A}|$;
- $\doteq^{\mathfrak{A}}$ is the identity relation on $|\mathfrak{A}|$, i.e., $\doteq^{\mathfrak{A}} = \{(a, b) \mid a, b \in |\mathfrak{A}| \text{ and } a = b\}$;
- for each n-ary function symbol f of $\mathbb{L}$, an n-ary operation on the universe, i.e., an n-ary function $f^{\mathfrak{A}} : \underbrace{|\mathfrak{A}| \times \ldots \times |\mathfrak{A}|}_{n} \to |\mathfrak{A}|$;
- for each constant symbol c of $\mathbb{L}$, $c^{\mathfrak{A}} \in |\mathfrak{A}|$.

Notation and Terminology

- $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{M}$, $\mathfrak{N}$, $\mathfrak{Q}$, $\mathfrak{R}$ and $\mathfrak{Z}$, are the usual names we will use for structures. These are the *fraktur* (Gothic) fonts.
- What $P^{\mathfrak{A}}$ (where $P \neq \doteq$) changes with the structure, but $\doteq^{\mathfrak{A}}$ is always the identity relation on $|\mathfrak{A}|$.
- We say P *denotes* (or stands for) $P^{\mathfrak{A}}$ in the structure $\mathfrak{A}$. Similar terminology is used for function symbols and constant symbols.

Example

Let $\mathbb{L}$ be the first-order language that has:

- $\dot{+}$ and $\dot{\times}$ (2-ary function symbols);
- $\dot{<}$ (a 2-ary predicate symbol);
- $\dot{0}$ and $\dot{1}$ (constant symbols), and
- $\dot{=}$.

Let $\mathfrak{N}_1$ be the structure for $\mathbb{L}$ such that:

- $|\mathfrak{N}_1| = \mathbb{N}$;
- $\dot{+}^{\mathfrak{N}_1} = +$(the addition function on $\mathbb{N}$);
- $\dot{\times}^{\mathfrak{N}_1} = \times$(the multiplication function on $\mathbb{N}$);
- $\dot{<}^{\mathfrak{N}_1} = \{(a, b) \mid a, b \in \mathbb{N} \text{ and } a < b\}$;
- $\dot{0}^{\mathfrak{N}_1} = 0$;
- $\dot{1}^{\mathfrak{N}_1} = 1$.

We can describe this structure simply as $\mathfrak{N}_1 = \{\mathbb{N}, <, +, \times, 0, 1\}$.

Example (Cont'd)

Let $\mathfrak{N}_2$ be the structure for the same language:

- $|\mathfrak{N}_2| = \mathbb{N}$;
- $\dot{+}^{\mathfrak{N}_2} = \times$;
- $\dot{\times}^{\mathfrak{N}_2} = +$;
- $\dot{<}^{\mathfrak{N}_2} = \{(a, b) \mid a, b \in \mathbb{N} \text{ and } a > b\}$;
- $\dot{0}^{\mathfrak{N}_2} = 1$;
- $\dot{1}^{\mathfrak{N}_2} = 0$.

Example (Cont'd)

Let $\mathfrak{R}$ be the structure for the same language:

- $|\mathfrak{R}| = \mathbb{R}$;
- $\dot{+}^{\mathfrak{R}} = +$(addition on the real numbers);
- $\dot{\times}^{\mathfrak{R}} = \times$(multiplication on the real numbers);
- $\dot{<}^{\mathfrak{R}} =<$ (on the real numbers);
- $\dot{0}^{\mathfrak{R}} = 0$;
- $\dot{1}^{\mathfrak{R}} = 1$.

Special Status of $\doteq$

We have been very careful in distinguishing between things in the language $\mathbb{L}$ and things outside of $\mathbb{L}$.

For example, $\doteq$ is a symbol in the language, while $=$ is not.

Question

Why does Enderton no distinguish between the two?

Example: Directed Graph

Let $\mathbb{L}$ be the first-order language that (in addition to the symbols required in every first-order language) only has a 2-ary predicate symbol $\dot{E}$.

Let $\mathfrak{B}$ be the structure for $\mathbb{L}$ such that:

- $|\mathfrak{B}| = \{a, b, c, d\}$;
- $\dot{E}^{\mathfrak{B}} = \{\langle a, b\rangle, \langle b, a\rangle, \langle b, c\rangle, \langle c, c\rangle\}$.

This denotes a directed graph (See Enderton, page 82)

Example

The wff $\exists x \forall y, \neg \dot{E}(y, x)$ denotes

> *There is a vertex x such that for any vertex y, no edge points from y to x.*

Question

How do we show $\exists x \forall y, \neg \dot{E}(y, x)$ is true in $\mathfrak{B}$?

Given a formula φ and a structure $\mathfrak{A}$, how do we define "φ is true in $\mathfrak{A}$",
Or equally speaking, "$\mathfrak{A}$ satisfies φ"?

Assignment of Values to Terms

Let $\mathfrak{A}$ be a structure for the language $\mathbb{L}$. Let V be the set of variables, and T be the set of terms of $\mathbb{L}$.

Definition (Assignment Functions)

An assignment for $\mathfrak{A}$ is a function $s : V \to |\mathfrak{A}|$.

Definition (Assignment to Terms)

An assignment $s : V \to |\mathfrak{A}|$ is extended to a function $\bar{s} : T \to |\mathfrak{A}|$ as follows:

- $\bar{s}(v) = s(v)$ if v is a variable;
- $\bar{s}(c) = c^{\mathfrak{A}}$ if c is a constant symbol;
- $\bar{s}(f(t_1, \ldots, t_n)) = f^{\mathfrak{A}}(\bar{s}(t_1), \ldots, \bar{s}(t_n))$ if f is an n-ary function symbol and $t_1, \ldots, t_n$ are terms.

Example

Look the language $\mathbb{L}$ of the earlier example. Let s be an assignment function for the structure $\mathfrak{N}_1$ such that $s(v_3) = 5$. Then

- $\overline{s}(\dot{+}(\dot{\times}(\dot{0}, v_3), \dot{1})) = 1$
- $\overline{s}(\dot{+}(\dot{\times}(\dot{1}, \dot{1}), v_3)) = 6$

Changing the Assignment Function

Let:

- s be an assignment function,
- x be a variable, and
- $a \in |\mathfrak{A}|$.

$s(x|a)$ is the new assignment, where for every variable y,

$$s(x|a)(y) = \begin{cases} s(y) & \text{if } y \neq x \\ a & \text{if } y = x \end{cases}$$

Example

If $y \neq x$ then

- $s(x|a)(y|b)(x) = a$
- $s(x|a)(y|b)(y) = b$
- $s(x|a)(x|b)(x) = b$

Satisfaction in First-Order Logic

Given a first-order language $\mathbb{L}$:

- let $\mathfrak{A}$ be a structure for $\mathbb{L}$,
- let s be an assignment for $\mathfrak{A}$, and
- let φ be a wff in $\mathbb{L}$.

We shall talk about what it means for $\mathfrak{A}$ to satisfy φ with s, written as

$$\models_{\mathfrak{A}} \varphi[s]$$

Informally, it means:

> *The translation of φ determined by $\mathfrak{A}$, where a variable x is translated as $s(x)$, is true.*

Satisfaction for Atomic Formula

Definition

Let:

- $\mathfrak{A}$ be a structure for $\mathbb{L}$,
- s be an assignment for $\mathfrak{A}$, and
- $P(t_1, \ldots, t_n)$ be an atomic wff.

Then

- $\vDash_{\mathfrak{A}} P(t_1, \ldots, t_n)[s]$ iff $(\overline{s}(t_1), \ldots, \overline{s}(t_n)) \in P^{\mathfrak{A}}$ (when $P \neq \doteq$);
- $\vDash_{\mathfrak{A}} \doteq(t_1, t_2)[s]$ iff $\overline{s}(t_1) = \overline{s}(t_2)$.

Satisfaction for Well-Formed Formula

Definition

- For an atomic formula, we have already given its definition;
- Suppose $\models_{\mathfrak{A}} \alpha[s]$ and $\models_{\mathfrak{A}} \beta[s]$ have been defined. Then
  - $\models_{\mathfrak{A}} (\alpha \wedge \beta)[s]$ iff $\models_{\mathfrak{A}} \alpha[s]$ and $\models_{\mathfrak{A}} \beta[s]$;
  - $\models_{\mathfrak{A}} (\alpha \vee \beta)[s]$ iff $\models_{\mathfrak{A}} \alpha[s]$ or $\models_{\mathfrak{A}} \beta[s]$;
  - $\models_{\mathfrak{A}} \neg\alpha[s]$ iff not $\models_{\mathfrak{A}} \alpha[s]$;
  - $\models_{\mathfrak{A}} \alpha \to \beta[s]$ iff $\models_{\mathfrak{A}} \alpha[s] \Longrightarrow \models_{\mathfrak{A}} \beta[s]$;
  - $\models_{\mathfrak{A}} (\alpha \leftrightarrow \beta)[s]$ iff $\models_{\mathfrak{A}} \alpha[s] \Longleftrightarrow \models_{\mathfrak{A}} \beta[s]$;
  - $\models_{\mathfrak{A}} \forall x\ \alpha[s]$ iff for all $a \in |\mathfrak{A}|, \models_{\mathfrak{A}} \alpha[s(x|a)]$.
  - $\models_{\mathfrak{A}} \exists x \alpha[s]$ iff there is some $a \in |\mathfrak{A}|, \models_{\mathfrak{A}} \alpha[s(x|a)]$

If $\models_{\mathfrak{A}} \varphi[s]$, we say

- $\mathfrak{A}$ satisfies φ with s, or
- s satisfies φ in the structure $\mathfrak{A}$, or
- φ is true in $\mathfrak{A}$ with s.

Example

Let $\mathfrak{N} = (\mathbb{N}, <, +, \times, 0, 1)$. This is our abbreviated way of saying:

- $\mathbb{L}$ has a binary predicate symbol $\dot{<}$, 2-ary function symbols $\dot{+}$ and $\dot{\times}$, constant symbols $\dot{0}$ and $\dot{1}$, but no other predicate symbols (except for $\doteq$), functions symbols or constant symbols;
- $\mathfrak{N}$ is the structure for $\mathbb{L}$:
  - whose universe is $\mathbb{N}$;
  - $\dot{<}^{\mathfrak{N}} =<$;
  - $\dot{+}^{\mathfrak{N}} = +$;
  - $\dot{\times}^{\mathfrak{N}} = \times$;
  - $\dot{0}^{\mathfrak{N}} = 0$ and $\dot{1}^{\mathfrak{N}} = 1$.

Similarly, let $\mathfrak{Z} = (\mathbb{Z}, <, +, \times, 0, 1)$. Note both $\mathfrak{N}$ and $\mathfrak{Z}$ are structures for the same language $\mathbb{L}$.

Example (Cont'd)

Question

Let φ be the wff

$$\forall x(\neg x \dot{<} \dot{0})$$

Which of the following judgments holds?

- For every $s : V \to \mathbb{N}$, $\vDash_{\mathfrak{N}} \varphi[s]$;
- For every $s : V \to \mathbb{Z}$, $\vDash_{\mathfrak{Z}} \varphi[s]$.

More Examples

Let $\mathfrak{R} = (\mathbb{R}, <, +, \times, 0, 1)$.

Question

Let φ be the wff

$$\forall x \forall y (x \dot{<} y \to \exists z \, (x \dot{<} z \wedge z \dot{<} y))$$

Then which of the following is true?

- For every $s : V \to \mathbb{Z}$, $\models_{\mathfrak{Z}} \varphi[s]$
- For every $s : V \to \mathbb{R}$, $\models_{\mathfrak{R}} \varphi[s]$

Example: Directed Graph

Let $\mathbb{L}$ be the first-order language that (in addition to the symbols required in every first-order language) only has a 2-ary predicate symbol $\dot{E}$.

Let $\mathfrak{B}$ be the structure for $\mathbb{L}$ such that:

- $|\mathfrak{B}| = \{a, b, c, d\}$;
- $\dot{E}^{\mathfrak{B}} = \{\langle a, b\rangle, \langle b, a\rangle, \langle b, c\rangle, \langle c, c\rangle\}$.

Question

Let $\sigma = \exists x \forall y, \neg \dot{E}(y, x)$. For every assignment $s : V \to |\mathfrak{B}|$, does $\models_{\mathfrak{B}} \sigma[s]$ hold?

More Examples

Let

- φ_1 be $\forall x(\neg x \dot{<} y)$, and
- φ_2 be $\forall x(\neg x \dot{<} \dot{0})$.

Then

(1) $\models_{\mathfrak{N}} \varphi_1[s]$ iff for all $a \in \mathbb{N}, s(y) \leq a$;

(2) $\models_{\mathfrak{N}} \varphi_2[s]$ iff for all $a \in \mathbb{N}, 0 \leq a$;

Note that

- (1) is true iff $s(y) = 0$, so whether it is true or not depend on s, whereas
- (2) is true for *all* s.

How do free occurrences of variables affect satisfaction?

Satisfaction Depends Only on Variables that Occur Free

Lemma

Let $\mathfrak{A}$ be a structure for $\mathbb{L}$, s_1 and s_2 be two assignment for $\mathfrak{A}$ and t be a term of $\mathbb{L}$.
If $s_1(x) = s_2(x)$ for every x that occurs in t, then

$$\overline{s_1}(t) = \overline{s_2}(t)$$

Theorem

Let $\mathfrak{A}$ be a structure for $\mathbb{L}$, s_1 and s_2 be two assignment for $\mathfrak{A}$ and φ be a wff of $\mathbb{L}$.
If $s_1(x) = s_2(x)$ for every x that occurs free in φ, then

$$\models_{\mathfrak{A}} \varphi[s_1] \Longleftrightarrow \models_{\mathfrak{A}} \varphi[s_2]$$

Satisfaction for Sentences

Corollary

If σ is a sentence then either:

(1) $\models_{\mathfrak{A}} \sigma[s]$ for every assignment s, or

(2) $\not\models_{\mathfrak{A}} \sigma[s]$ for every assignment s.

In case (1), we say σ is true in $\mathfrak{A}$, and in case (2), we say σ is false in $\mathfrak{A}$.

Thus if σ is a sentence then whether or note $\models_{\mathfrak{A}} \sigma[s]$ does not depend on s. So we can just write $\models_{\mathfrak{A}} \sigma$ or $\not\models_{\mathfrak{A}} \sigma$.

Satisfiability and Validity

Satisfiability

Definition

- The wff φ is satisfiable if there is some structure $\mathfrak{A}$ and some assignment $s : V \to |\mathfrak{A}|$ such that $\models_{\mathfrak{A}} \varphi[s]$.
- The set Γ of wffs is satisfiable if there is some structure $\mathfrak{A}$ and some assignment $s : V \to |\mathfrak{A}|$ such that $\models_{\mathfrak{A}} \varphi[s]$ for every φ in Γ.

Algorithms

Question

Is there an algorithm for determining satisfiability?

In other words, is there an algorithm that on input a wff φ will give an output of "yes" if φ is satisfiable and output "no", otherwise?

Is There a Compactness Theorem for First-Order Logic?

Question

Is the following statement true?

For every first-order language $\mathbb{L}$, and every set Γ of wffs of $\mathbb{L}$, if every finite subset of Γ is satisfiable then Γ is satisfiable.

Answer

Yes! But we have to wait for a while to see the answer.

Valid Wffs

Some wffs are satisfied in every structure under every assignment s.

Definition

φ is valid iff $\models_{\mathfrak{A}} \varphi[s]$ for every structure $\mathfrak{A}$ for $\mathbb{L}$ and every assignment function s for $\mathfrak{A}$.

Corollary

A sentence σ is valid iff it is true in every structure.

Theorem

φ is not satisfiable iff $\neg\varphi$ is valid.

Examples

Which of the following are valid?

- $x \dot{=} x$
- $\exists x\; x \dot{=} x$
- $\forall x \exists y\; x \dot{\neq} y$
- $\dot{P}(x) \vee \neg \dot{P}(x)$
- $\neg \exists x\; x \dot{\neq} x$

More Examples

Which of the following are valid?

- $\forall x(\dot{P}(x) \to \exists y \dot{P}(y))$
- $\dot{P}(x) \to \exists x \dot{P}(x)$
- $\dot{P}(x) \to \forall x \dot{P}(x)$
- $\exists x \dot{P}(x) \to \forall x \dot{P}(x)$
- $\exists x(\dot{P}(x) \to \forall x \dot{P}(x))$

Sentences for Classifying Structures

Earlier Example

Let σ be the sentence $\forall x \forall y (x \dot{<} y \to \exists z \, x \dot{<} z \land z \dot{<} y)$.
Then σ is true in $\mathfrak{R}$ but false in $\mathfrak{Z}$.

Sentences that Distinguish Between Structures

Let:

- $\mathfrak{N} = (\mathbb{N}, <)$;
- $\mathfrak{Z} = (\mathbb{Z}, <)$;
- $\mathfrak{Q} = (\mathbb{Q}, <)$;
- $\mathfrak{R} = (\mathbb{R}, <)$.

Question

For each pair of these structures, can you find a sentence in this language that is true in one and false in the other?

Elementary Equivalence

Definition

Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures for the same language $\mathbb{L}$. $\mathfrak{A}$ and $\mathfrak{B}$ are elementarily equivalent (written $\mathfrak{A} \equiv \mathfrak{B}$) if for every sentence σ of $\mathbb{L}$

$$\models_{\mathfrak{A}} \sigma \Longleftrightarrow \models_{\mathfrak{B}} \sigma.$$

Remark

We have just seen that in the language with $\dot{<}$:

- $\mathfrak{N} \not\equiv \mathfrak{Z}$;
- $\mathfrak{Z} \not\equiv \mathfrak{Q}$;
- $\mathfrak{Z} \not\equiv \mathfrak{R}$.

Comparing $\mathfrak{Q}$ and $\mathfrak{R}$

Question

Is it true that $\mathfrak{Q}$ and $\mathfrak{R}$ are elementarily equivalent in a language $\mathbb{L}$?

Answer

Perhaps the answer is not so easy!

Models of Sentences

Definition

- $\mathfrak{A}$ is a model of the sentence σ if $\models_{\mathfrak{A}} \sigma$, i.e., if σ is true in $\mathfrak{A}$;
- $\mathfrak{A}$ is a model of a set Σ of sentences if $\mathfrak{A}$ is a model of every member of Σ, i.e., every sentence in Σ is true in $\mathfrak{A}$.

Question

Let $\mathfrak{R} = (\mathbb{R}, <, +, \times, 0, 1)$ and $\mathfrak{Q} = (\mathbb{Q}, <, +, \times, 0, 1)$. Is there a sentence that is true in $\mathfrak{R}$, but not in $\mathfrak{Q}$?

Answer

Yes. Let σ be $\exists x\; x \dot{\times} x \doteq \dot{1} \dot{+} \dot{1}$.

Example

Let $\mathbb{L}$ be a first-order language with 2-ary predicate symbols $\dot{P}$ and $\doteq$. Given a structure for $\mathbb{L}$, we have:

- $\mathfrak{A}$ is a model of $\forall x \forall y \; x \doteq y$ iff $|\mathfrak{A}|$ contains exactly one element;
- $\mathfrak{A}$ is a model of $\forall x \forall y \; \dot{P}(x, y)$ iff $\dot{P}^{\mathfrak{A}} = |\mathfrak{A}| \times |\mathfrak{A}|$;
- $\mathfrak{A}$ is a model of $\forall x \forall y \; \neg \dot{P}(x, y)$ iff $\dot{P}^{\mathfrak{A}} = \emptyset$;
- $\mathfrak{A}$ is a model of $\forall x \exists y \; \dot{P}(x, y)$ iff the domain of $\dot{P}^{\mathfrak{A}}$ is $|\mathfrak{A}|$.

We notice that a sentence may denote a class of structures (i.e., its models).

Reflexivity, Symmetry and Transitivity

Definition

Let R be a binary relation.

- R is symmetric if for every a and b,

$$(a, b) \in R \implies (b, a) \in R$$

- R is transitive if for every a, b, and c,

$$(a, b) \in R \implies (b, c) \in R \implies (a, c) \in R$$

- R is reflexive on the set A if for all $a \in A$,

$$(a, a) \in R$$

- R satisfies trichotomy on A if for all $a, b, c \in A$, exactly one of the following is true:

$$(a, b) \in R, \qquad (b, a) \in R, \qquad a = b$$

Linear Ordering

Definition

A binary relation R is a linear ordering on A if R is transitive and satisfies trichotomy on A.

Definition

Let $\mathbb{L}$ be the language with a binary relation symbol $\dot{R}$ and $\doteq$ (and no other symbols). Let $\mathfrak{A} = (A, R)$, i.e., ($A = |\mathfrak{A}|$ and $R = \dot{R}^{\mathfrak{A}}$).

- $\mathfrak{A}$ is *transitive* if R is transitive;
- $\mathfrak{A}$ is a *linearly ordered* structure if R is a linear ordering on A.

See the discussion on Page 93 of Enderton's.

Examples

Each of the following is a linearly ordered structure:

- $(\mathbb{N}, <)$;
- $(\mathbb{Z}, <)$;
- $(\mathbb{R}, <)$.

Also, each of the following is linearly ordered structure:

- $(\mathbb{N}, >)$;
- $(\mathbb{Z}, >)$;
- $(\mathbb{R}, >)$.

Question

Is $(\mathbb{N}, \leq)$ a linearly ordered structure?

What Can Sentences Say About Structures

Question

Let $\mathfrak{A} = (A, R)$.

- $\mathfrak{A}$ is transitive iff $\models_{\mathfrak{A}} \sigma$, where $\sigma =?$;
- $\mathfrak{A}$ is linearly ordered iff $\models_{\mathfrak{A}} \sigma$, where $\sigma =?$;
- $dom(R) = A$ iff $\models_{\mathfrak{A}} \sigma$, where $\sigma =?$;
- $rng(R) = A$ iff $\models_{\mathfrak{A}} \sigma$, where $\sigma =?$;
- R is a function iff $\models_{\mathfrak{A}} \sigma$, where $\sigma =?$.

See the discussion on Page 93 of Enderton's for some of the answers.

How can we characterize relations in *structures* by looking at wffs in first-order logic?

Abbreviations

Definition

Let φ be a wff such that all variables occurring free in φ are included among $v_1, \ldots, v_k$. Given $a_1, \ldots, a_k \in |\mathfrak{A}|$,

$$\models_{\mathfrak{A}} \varphi[\![a_1, \ldots, a_k]\!]$$

means $\models_{\mathfrak{A}} \varphi[s]$ for some $s : V \to |\mathfrak{A}|$ such that $s(v_i) = a_i (1 \leq i \leq k)$.

Example

Let $\mathfrak{N} = (\mathbb{N}, <, +, \times, 0, 1)$. We have

- $\models_{\mathfrak{N}} \forall v_2 (\neg v_2 \dot{<} v_1)[\![0]\!]$;
- $\not\models_{\mathfrak{N}} \forall v_2 (\neg v_2 \dot{<} v_1)[\![2]\!]$.

Relations Defined by Wffs

Definition

Let

- $\mathfrak{A}$ be a structure, and
- φ be a wff and n be such that the variables occurring free in φ are included among $v_1, \ldots, v_n$.

The n-ary relation defined by φ in $\mathfrak{A}$ is

$$\{(a_1, \ldots, a_n) \mid \models_{\mathfrak{A}} \varphi[\![a_1, \ldots, a_n]\!]\}$$

Examples

Example

- Let $\mathfrak{R} = (\mathbb{R}, <, +, \times, 0, 1)$. The 1-ary relation $\{a \in \mathbb{R} \mid 0 \leq a\}$ is defined by
$$\exists v_2 (v_1 \doteq v_2 \times v_2)$$
in $\mathfrak{R}$;
- Let $\mathfrak{N} = (\mathbb{N}, <, +, \times, 0, 1)$. The 2-ary relation $\{(a, b) \mid a < b\}$ is defined by
$$\exists v_3 (v_1 \dot{+} (\dot{1} \dot{+} v_3) \doteq v_2)$$
in $\mathfrak{N}$.

Definable Relations

Definition

- The relation R is definable in the structure $\mathfrak{A}$ if there is some wff φ that defines it in $\mathfrak{A}$.
- Let f be a n-ary function f whose domain is a subset of $\underbrace{|\mathfrak{A}| \times \ldots \times |\mathfrak{A}|}_{n}$ and whose range is a subset of $|\mathfrak{A}|$. f is definable in $\mathfrak{A}$ if the (n+1)-ary relation

$$\{(a_1, \ldots, a_n, b) \mid f(a_1, \ldots, a_n) = b\}$$

is definable in $\mathfrak{A}$.

Examples

Example

Let $\mathfrak{N} = (\mathbb{N}, <, +, \times, 0, 1)$.

- $v_1 \dot{+} v_2 \dot{=} v_3$ defines $\{(a, b, c) \mid a + b = c\}$, which is the same as the function f, where $f(a, b) = a + b$.
- $v_1 \dot{+} v_3 \dot{=} v_2$ defines $\{(a, b, c) \mid a + c = b\}$, which is the same as the function f, where

$$f(a, b) = \begin{cases} b - a & \text{if } a \leq b \\ \text{Undefined} & \text{Otherwise} \end{cases}$$

What Relations are Definable in a Structure?

Proposition

Let $\mathfrak{A}$ be a structure for $\mathbb{L}$.

- $|\mathfrak{A}|$ is definable (by $v_1 \dot{=} v_1$, if $\dot{=}$ is in $\mathbb{L}$);
- $\emptyset$ is definable (by $v_1 \dot{\neq} v_1$, if $\dot{=}$ is in $\mathbb{L}$);
- $=$ is definable (by $v_1 \dot{=} v_2$, if $\dot{=}$ is in $\mathbb{L}$);
- for every n-ary predicate symbol $\dot{P}$, $\dot{P}^{\mathfrak{A}}$ is definable (by $\dot{P}(v_1, \ldots, v_n)$);
- for every n-ary function symbol f, $f^{\mathfrak{A}}$ is definable (by $f(v_1, \ldots, v_n) \dot{=} v_{n+1}$);
- fore every constant symbol c, the singleton $\{c^{\mathfrak{A}}\}$ is definable (by $v_1 \dot{=} c$).

What happens if $\dot{=}$ is not in $\mathbb{L}$?

Relations Definable in a Structure

Proposition

- If P and and Q are n-ary relations that are definable in $\mathfrak{A}$, then so are: the complement of P, $P \cup Q$, $P \cap Q$, $P \setminus Q$.
- If the $n+1$-ary relation R is definable in $\mathfrak{A}$ then so are the n-ary relations

$$\{(a_1, \ldots, a_n) \mid \text{there exists } b \in |\mathfrak{A}|, (a_1, \ldots, a_n, b) \in R\}$$
$$\{(a_1, \ldots, a_n) \mid \text{there exists } b \in |\mathfrak{A}|, (b, a_1, \ldots, a_n) \in R\}$$

In particular, if R is a binary relation that is definable in $\mathfrak{A}$ then $dom(R)$ and $rng(R)$ is definable.

Definable Subsets of $\mathfrak{N}$

Which of the following subsets of $\mathbb{N}$ are definable in $\mathfrak{N} = (\mathbb{N}, <)$?

- $\emptyset$.
- $\mathbb{N}$.
- $\{0\}$.
- $\{1\}$.
  - Let $\varphi(x)$ be the result of replacing v_1 in φ with x;
  - We obtain the defining wff as follows:

$$\begin{aligned} a = 1 &\iff a \neq 0 \text{ and } \forall b \in \mathbb{N}, (b < a \implies b = 0) \\ &\iff \models_{\mathfrak{A}} \neg\varphi[\![a]\!] \text{ and } \models_{\mathfrak{A}} \forall v_3 (v_3 \dot{<} v_1 \to \varphi(v_3))[\![a]\!] \\ &\iff \models_{\mathfrak{A}} \neg\varphi \wedge \forall v_3 (v_3 \dot{<} v_1 \to \varphi(v_3))[\![a]\!] \end{aligned}$$

So the formula $\neg\varphi \wedge \forall v_3 (v_3 \dot{<} v_1 \to \varphi(v_3))$ defines $\{1\}$.

More Definable Subset of $\mathfrak{A}$

Which of the following subsets of $\mathbb{N}$ are definable in $\mathfrak{N} = (\mathbb{N}, <)$?

- $\{n\}$ for each $n \in \mathbb{N}$.
- Every finite subset of $\mathbb{N}$.
- Every cofinite subset of $\mathbb{N}$.
- Every subset of $\mathbb{N}$.

How Many Relations are Definable?

Lemma

(1) Given a structure $\mathfrak{A}$, the set of definable relations is *enumerable*;

(2) Not every subset of $\mathbb{N}$ is definable.

Proof.

For (1), note that the set of wffs is enumerable, and every wff may define only one relation.

For (2), note that the set of all subsets of $\mathbb{N}$ is uncountable. Therefore, some subset may not match a wff. $\square$

Definable Subsets in General

Which of the following subsets of $\mathbb{R}$ are definable in $\mathfrak{R} = (\mathbb{R}, <)$?

- $\emptyset$.
- $\mathbb{R}$.
- Anything else?

Question

More generally, given a first-order language $\mathbb{L}$ and a structure $\mathfrak{A}$ for $\mathbb{L}$, how do we figure out which relations in $\mathfrak{A}$ are definable?

Given any wff φ, how do we relate its satisfactions in different structures?

Homomorphisms

Definition

Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures for $\mathbb{L}$. A homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ is a function $h : |\mathfrak{A}| \to |\mathfrak{B}|$ such that:

- for every n-ary predicate symbol R, other than $\doteq$, and $a_1, \ldots, a_n \in |\mathfrak{A}|$,

$$(a_1, \ldots, a_n) \in R^{\mathfrak{A}} \iff (h(a_1), \ldots, h(a_n)) \in R^{\mathfrak{B}};$$

- for every n-ary function symbol f and $a_1, \ldots, a_n \in |\mathfrak{A}|$,

$$h(f^{\mathfrak{A}}(a_1, \ldots, a_n)) = f^{\mathfrak{B}}(h(a_1), \ldots, h(a_n));$$

- for every constant symbol c,

$$h(c^{\mathfrak{A}}) = c^{\mathfrak{B}}.$$

Homomorphisms

Definition

- h is a homomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$ if h is a homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ and h maps $\mathfrak{A}$ onto $\mathfrak{B}$;
- A homomorphism h from $\mathfrak{A}$ into $\mathfrak{B}$ is an isomorphism if h is one-to-one;
- The structures $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic, written $\mathfrak{A} \cong \mathfrak{B}$, if there is some isomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$;
- An automorphism of $\mathfrak{A}$ is an isomorphism of $\mathfrak{A}$ onto $\mathfrak{A}$.

Examples

Example

Let $\mathfrak{A} = (\mathbb{N}, <^{\mathbb{N}}, +^{\mathbb{N}})$ and $\mathfrak{B} = (\mathbb{E}, <^{\mathbb{E}}, +^{\mathbb{E}})$.
Here $\mathbb{E}$ is the set of even non-negative integers, $<^{\mathbb{E}}$ is the "less than" relation on $\mathbb{E}$, etc.
Then h is an isomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$, where for all $n \in \mathbb{N}$,

$$h(n) = 2n.$$

Examples

Example

Let $\mathfrak{A} = (\mathbb{N}, <^{\mathbb{N}}, +^{\mathbb{N}})$ and $\mathfrak{B} = (\mathbb{O}, <^{\mathbb{O}}, +^{\mathbb{O}})$.
Here $\mathbb{O}$ is the set of odd non-negative integers, $<^{\mathbb{O}}$ is the "less than" relation on $\mathbb{O}$, etc.

- Then an isomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$ is: There is NONE!
- In fact, $\mathfrak{B}$ is not even a structure, because $\mathbb{O}$ is not closed under addition.

Automorphisms of $\mathfrak{R} = (\mathbb{R}, <)$

Let $\mathfrak{R} = (\mathbb{R}, <)$. Which of the following functions h are automorphisms of $\mathfrak{R}$?

- The identity function.
- $h(a) = a + 3$
- $h(a) = a - 3$
- $h(a) = 2a$
- $h(a) = -a$
- $h(a) = k \times a + l$
- $h(a) = a^3$
- $h(a) = a^2$

Automorphisms of $\mathfrak{N} = (\mathbb{N}, <)$

Let $\mathfrak{N} = (\mathbb{N}, <)$.

- The identity function is an automorphism of $\mathfrak{N}$; what about others?
- Suppose h is an automorphism of $\mathfrak{N}$, $h(0) = 0$
- Suppose h is an automorphism of $\mathfrak{N}$, $h(1) = 1$
- In general, if h is an automorphism of $\mathfrak{N}$, $h(n) = n$
- Therefore, the identity function is the only automorphism of $\mathfrak{N}$.

Substructures

A special kind of isomorphisms:

Definition

Let $\mathfrak{A} = (A, \dots)$ and $\mathfrak{B} = (B, \dots)$ be structures for $\mathbb{L}$. $\mathfrak{A}$ is a substructure of $\mathfrak{B}$ (written $\mathfrak{A} \subseteq \mathfrak{B}$) if:

- $A \subseteq B$;
- for every k-ary predicate symbol P:

  $$P^{\mathfrak{A}} = P^{\mathfrak{B}} \cap A^k$$

  (Note this is not the same as saying $P^{\mathfrak{A}} \subseteq P^{\mathfrak{B}}$);
- for every k-ary function symbol f and every k-tuple $(a_1, \dots, a_k)$ of elements of A:

  $$f^{\mathfrak{A}}(a_1, \dots, a_k) = f^{\mathfrak{B}}(a_1, \dots, a_k);$$

- for every constant symbol c, $c^{\mathfrak{A}} = c^{\mathfrak{B}}$.

Examples

Example

Let

- $\mathfrak{N} = (\mathbb{N}, <^{\mathbb{N}}, +^{\mathbb{N}}, \times^{\mathbb{N}})$
- $\mathfrak{E} = (\mathbb{E}, <^{\mathbb{E}}, +^{\mathbb{E}}, \times^{\mathbb{E}})$.

Then $\mathfrak{E}$ is a substructure of $\mathfrak{N}$.

Question

Let

- $\mathfrak{A} = (\{0, 1, 2, 3\}, P^{\mathfrak{A}})$, where $P^{\mathfrak{A}} = \{0, 1, 2\}$;
- $\mathfrak{B} = (\{0, 1\}, P^{\mathfrak{B}})$, where $P^{\mathfrak{B}} = \{0\}$.

Is $\mathfrak{B}$ is a substructure of $\mathfrak{A}$?

Answer

No. Because $P^{\mathfrak{A}} \cap \{0, 1\} = \{0, 1\} \neq \{0\} = P^{\mathfrak{B}}$.

Notation: Function Composition

Definition

If f and g are functions, then $f \circ g$ is the composition of f and g. That is,

$$f \circ g(a) = f(g(a)).$$

Example

Suppose $s : V \to |\mathfrak{A}|$ is an assignment function for $\mathfrak{A}$, and h is a homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$. Then $h \circ s$ is an assignment function for $\mathfrak{B}$.

The Value of Terms Under a Homomorphism

Lemma

Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures for the language $\mathbb{L}$.
Let h be a homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$, and $s : V \to |\mathfrak{A}|$ be an assignment for $\mathfrak{A}$. Then for every term t of $\mathbb{L}$,

$$h(\bar{s}(t)) = \overline{h \circ s}(t).$$

Proof.

By induction on t. $\square$

The Homomorphism Theorem

Theorem (The Homomorphism Theorem)

Let h be a homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ and s be an assignment function for $\mathfrak{A}$. The statement

$$\models_{\mathfrak{A}} \varphi[s] \iff \models_{\mathfrak{B}} \varphi[h \circ s]$$

(a) is true for every quantifier-free wff φ not containing $\doteq$;
(b) is true for every quantifier-free wff φ if h is one-to-one;
(c) is true for every wff φ not containing $\doteq$ if h is onto;
(d) is true for every wff φ if h is an isomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$ (i.e., $\mathfrak{A} \cong \mathfrak{B}$.

Proof.

By induction on φ. $\square$

Corollaries of the Homomorphism Theorem

Corollary

If $\mathfrak{A} \cong \mathfrak{B}$, then $\mathfrak{A} \equiv \mathfrak{B}$.

Question

Do you think the converse is true?

Answer

No. Take $\mathfrak{R} = (\mathbb{R}, <)$ and $\mathfrak{Q} = (\mathbb{Q}, <)$ as an counter example.

Corollaries of the Homomorphism Theorem

Corollary (Automorphism Theorem)

Let h be an automorphism of $\mathfrak{A}$. Let R be an n-rary relation on $|\mathfrak{A}|$ that is definable in $\mathfrak{A}$. For every n-tuple $(a_1, \ldots, a_n)$ of elements of $\mathfrak{A}$:

$$(a_1, \ldots, a_n) \in R \Longleftrightarrow (h(a_1), \ldots, h(a_n)) \in R.$$

We often use this lemma to show certain relations are not definable:

Example

Let $\mathfrak{R} = (\mathbb{R}, <)$. Its subset $\mathbb{N}$ is not definable in $\mathfrak{R}$ because $h(a) = a^3$ is an automorphism of $\mathfrak{R}$.