

Assignment 3

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Problem 1

Suppose (a) holds but (b) doesn't. Then there exists a finitely satisfiable Γ that is not satisfiable.

Notice that $\Sigma \models \alpha$ is equivalent to $\Sigma; \neg\alpha$ is unsatisfiable. Then the corollary is exactly saying that if $\Sigma; \beta$ is unsatisfiable, then there exists finite $\Delta \subseteq \Sigma$ such that $\Delta; \beta$ is unsatisfiable, where β is any wff.

Since Γ is unsatisfiable, then we let $\Gamma = \Gamma' \cup \{\beta\}$ such that $\Gamma' \cap \{\beta\} = \emptyset$. Then we know from the corollary that there exists finite $\Delta' \subseteq \Gamma'$ such that $\Delta'; \beta$ is unsatisfiable. However, $\Delta'; \beta$ is a finite subset of Γ , which contradicts with the assumption that Γ is finitely satisfiable.

Problem 2

$$\begin{array}{c}
 \textcircled{2} \frac{[\neg A \wedge B]}{A} \textcircled{1} \frac{[\neg A \wedge B]}{\neg B} \\
 \textcircled{2} \frac{B}{\neg(A \wedge B)} \textcircled{1} \frac{\neg B}{\neg(A \wedge B)} \\
 \textcircled{1} \frac{B \wedge \neg B}{\neg(A \wedge B)} \rightarrow \neg I \\
 \textcircled{1} \frac{(\neg A \vee \neg B) \rightarrow \neg(A \wedge B)}{(\neg A \vee \neg B) \rightarrow \neg(A \wedge B)} \rightarrow \neg I \\
 \textcircled{1} \frac{}{(\neg A \vee \neg B) \rightarrow \neg(A \wedge B)} \textcircled{2} \frac{}{(\neg A \wedge B)}
 \end{array}$$

$$\begin{array}{c}
 \textcircled{2} \frac{[\neg B] \quad [\neg B \rightarrow (A \leftrightarrow \neg A)]}{A \leftrightarrow \neg A} \rightarrow E \\
 \textcircled{1} \frac{A \leftrightarrow \neg A}{A \rightarrow \neg A} \leftrightarrow E \\
 \textcircled{1} \frac{A \rightarrow \neg A}{\neg A \rightarrow A} \rightarrow E \\
 \textcircled{1} \frac{\neg A \rightarrow A}{A \wedge \neg A} \wedge I \\
 \textcircled{1} \frac{A \wedge \neg A}{\neg(A \wedge \neg A)} \neg E \\
 \textcircled{1} \frac{\neg(A \wedge \neg A)}{(B \rightarrow (A \leftrightarrow \neg A)) \rightarrow \neg B} \rightarrow \neg I \\
 \textcircled{1} \frac{(B \rightarrow (A \leftrightarrow \neg A)) \rightarrow \neg B}{B \rightarrow A \leftrightarrow \neg A} \rightarrow \neg I \\
 \textcircled{1} \frac{}{B} \textcircled{2} \frac{[\neg B] \quad [\neg B \rightarrow (A \leftrightarrow \neg A)]}{A \leftrightarrow \neg A} \rightarrow E \\
 \textcircled{2} \frac{A \leftrightarrow \neg A}{A \rightarrow \neg A} \rightarrow E \\
 \textcircled{2} \frac{A \rightarrow \neg A}{\neg A \rightarrow A} \neg E
 \end{array}$$

$$\begin{array}{c}
 \textcircled{2} \frac{[\neg P] \quad [\neg P]}{P} \neg E \\
 \textcircled{1} \frac{[\neg(P \rightarrow Q) \rightarrow P]}{P \rightarrow Q} \rightarrow \neg I \\
 \textcircled{1} \frac{P \rightarrow Q \quad P \rightarrow Q}{P} V-E \\
 \textcircled{1} \frac{P}{P} V-E \\
 \textcircled{1} \frac{P}{((P \rightarrow Q) \rightarrow P) \rightarrow P} \rightarrow I \\
 \textcircled{1} \frac{}{((P \rightarrow Q) \rightarrow P) \rightarrow P} \textcircled{2} \frac{}{(P \rightarrow Q) \rightarrow P}
 \end{array}$$

Problem 3

3.a

We already have partial proof tree $\frac{\sigma_1 \sigma_2 \dots \sigma_n}{\alpha}$ where $\{\sigma_1, \sigma_2, \dots, \sigma_n\} \subseteq \Sigma$ and $\frac{\alpha \delta_1 \dots \delta_m}{\beta}$ where $\{\delta_1, \dots, \delta_m\} \subseteq \Delta$.

Then we can construct new tree $\frac{\frac{\sigma_1 \sigma_2 \dots \sigma_n}{\alpha} \delta_1 \dots \delta_m}{\beta}$ as a partial proof tree of $\Sigma \cup \Delta \vdash \beta$.

3.b

We have

$$\frac{\gamma_1 \quad \gamma_2 \quad \dots \quad \gamma_n \quad \alpha}{\beta \leftrightarrow \neg\beta}$$

where $\{\gamma_1, \gamma_2, \dots, \gamma_n\} \subseteq \Gamma$.

Then we build several partial proof trees.

$$\frac{\beta}{\frac{\frac{\gamma_1 \quad \gamma_2 \quad \dots \quad \gamma_n \quad \alpha}{\frac{\frac{\beta \leftrightarrow \neg\beta}{\beta \rightarrow \neg\beta} \leftrightarrow \neg E_1}{\neg\beta \rightarrow \neg E}} \neg E}{\beta \wedge \neg\beta} \wedge I}$$

Substituting β and $\neg\beta$ with each other, we can build a symmetric partial proof tree of $\Gamma \cup \{\alpha, \neg\beta\} \vdash \beta \wedge \neg\beta$.

$$\frac{\alpha \vee \neg\alpha}{\frac{\frac{\beta \vee \neg\beta}{\frac{\frac{\Gamma; [\alpha], [\beta]}{\beta \wedge \neg\beta} \vee E}{\frac{\Gamma; [\alpha], [\neg\beta]}{\beta \wedge \neg\beta} \vee E}}{\frac{\beta \wedge \neg\beta}{\frac{\neg\alpha}{\neg\alpha} \wedge E} \wedge E}{\frac{[\neg\alpha]}{\neg\alpha} \vee E} \vee E}$$

Then we have shown that $\Sigma \vdash \neg\alpha$.

Problem 4

4.a

From the proof tree we know that the conclusion is dependent only on the assumption $\alpha \vee \neg\alpha$.

4.b

From the proof tree we know that the conclusion is dependent only on the assumption $(\alpha \rightarrow \beta) \rightarrow (\neg\alpha \vee \beta)$.

$$\frac{\frac{(\alpha \rightarrow \alpha) \rightarrow (\neg \alpha \vee \alpha)}{\alpha \vee \neg \alpha} \quad \frac{[\alpha]}{\alpha \rightarrow \alpha} \rightarrow \neg I}{\neg E}$$

Problem 5

5.a

$$\begin{aligned}
 & (\neg A \vee \neg B) \rightarrow \neg(A \wedge B) \\
 \Leftrightarrow & ((A \rightarrow \perp) \vee (B \rightarrow \perp)) \rightarrow ((A \wedge B) \rightarrow \perp)
 \end{aligned}$$

5.b

$$\begin{aligned}
 & \frac{\frac{\frac{[(A \wedge B)]}{A \quad [A \rightarrow \perp]} \wedge E \quad \frac{[(A \wedge B)]}{B \quad [B \rightarrow \perp]} \wedge E}{\perp \rightarrow E \perp} \rightarrow E}{\perp} \rightarrow \perp - I \\
 & \frac{\perp}{(A \wedge B) \rightarrow \perp} \rightarrow \perp - I \\
 & \frac{(A \rightarrow \perp) \vee (B \rightarrow \perp)}{(A \wedge B) \rightarrow \perp} \rightarrow I \\
 & \text{① } (A \rightarrow \perp) \rightarrow V(B \rightarrow \perp) \\
 & \text{② } A \wedge B
 \end{aligned}$$