ST 554 Mathematical Statistics Formula Sheet

Combinatorics

Bonferroni Inequality: $P(A \cap B) \geq P(A) + P(B) - 1$

Boole's Inequality: $P(A \cup B) \leq P(A) + P(B)$

DeMorgan's Law: $(A \cup B)^C = A^C \cap B^C$

$$(A \cap B)^C = A^C \cup B^C$$

$$P(A) = 1 - P(A^C)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)},\ P(B) > 0$$

Baye's Rule: $P(A_j|B) = \dfrac{P(A_j)P(B|A_j)}{\sum_i P(A_i)P(B|A_i)}$

$P(A \cap B) = P(A)P(B)$, iff A, B independent

Basic Probability

$$F_Y(y) = P(Y \leq y) = P(2X \leq y) = P(X \leq \frac{y}{2})$$

$$f_Y(y) = f_X(g^{-1}(y))|\frac{dg^{-1}(y)}{dy}|,\ \text{if strict in/decrease}$$

Expectations, Moments, & MGFs

$$Eg(X) = \sum_X g(x)p_X(x)$$

$$Eg(X) = \int_X g(x)f_X(x)$$

$$E(ag(X) + b) = aEg(X) + b$$

$$Var(\alpha X + \beta) = \alpha^2 Var(X)$$

$$\text{rth Moment} = EX^r$$

$$\begin{aligned} Var(X) &= E(X - EX)^2 \\ &= EX^2 - (EX)^2 \\ &= EX(X-1) + EX - (EX)^2 \end{aligned}$$

$$M_X(t) = E(e^{tX})$$

Discrete Uniform $(1, N)$

$$p_X(x) = \frac{1}{N},\ \text{if } x = 1, 2, ..., N$$

$$F_X(x) = \frac{x}{N}$$

$$E(X) = \frac{N+1}{2}, \quad Var(X) = \frac{N^2 - 1}{12}$$

$$M_X(t) = \frac{e^t - e^{(N+1)t}}{N(1 - e^t)}$$

Discrete Uniform (N_0, N_1)

$$p_X(x) = \frac{1}{N_1 - N_0 + 1},\ \text{if } x = N_0, N_0 + 1, ..., N_1$$

$$F_X(x) = \frac{x - N_0 + 1}{N_1 - N_0 + 1},\ \text{if } x = N_0, N_0 + 1, ..., N_1$$

$$E(X) = \frac{N_1 + N_0}{2}$$

$$Var(X) = \frac{(N_1 - N_0)(N_1 - N_0 + 2)}{12}$$

$$M_X(t) = \frac{e^{N_0 t} - e^{(N_1+1)t}}{(N_1 - N_0 + 1)(1 - e^t)}$$

Hypergeometric (N, M, n)

$$p_X(x) = \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}},\ \text{if } x = 0, 1, ..., n$$

$$E(X) = \frac{nM}{N}, \quad Var(X) = \frac{nM}{N}\left(1 - \frac{M}{N}\right)\left(\frac{N-n}{N-1}\right)$$

Bernoulli (p)

$$p_X(x) = p^x(1-p)^{1-x},\ \text{if } x = 0, 1$$

$$E(X) = p, \quad Var(X) = p(1 - P)$$

Binomial (n, p)

$$p_X(x) = \binom{n}{x}p^x(1-p)^{n-x},\ \text{if } x = 0, 1, \ldots, n$$

$$E(X) = np, \quad Var(X) = np(1 - P)$$

Geometric (p)

$$p_X(x) = p(1-p)^{x-1},\ \text{if } x = 1, 2, \ldots$$

$$E(X) = \frac{1}{p}, \quad Var(X) = \frac{1-p}{p^2}$$

Negative Binomial (r, p)

$$p_X(x) = \binom{x-1}{r-1}p^r(1-p)^{x-r},\ \text{if } x = r, r+1, \ldots$$

$$X = W_1 + W_2 + \cdots + W_r = \sum_{i=1}^{r} W_i$$

$W_1, W_2, \ldots, W_r \sim$ Geometric(p)

$$E(X) = \frac{r}{p}, \quad Var(X) = \frac{r(1-p)}{p^2}$$

Poisson (λ)

$$p_X(x) = \frac{e^{-\lambda}\lambda^x}{x!},\ \text{if } x = 0, 1, 2, \ldots$$

$$F_X(x) = e^{-\lambda}\sum_{i=0}^{x}\frac{\lambda^i}{i!}$$

$$E(X) = \lambda, \quad Var(X) = \lambda$$

Continuous Uniform (a, b)

$$f_X(x) = \frac{1}{b-a},\ \text{if } a < x < b$$

$$E(X) = \frac{a+b}{2}, \quad Var(X) = \frac{(b-a)^2}{12}$$

Gamma (α, β)

$$f_X(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha}x^{\alpha-1}e^{-\frac{x}{\beta}},\ \text{if } x > 0$$

$$E(X) = \alpha\beta, \quad Var(X) = \alpha\beta^2$$

Exponential (β)

$$f_X(x) = \frac{1}{\beta^\alpha}e^{-\frac{x}{\beta}},\ \text{if } x > 0$$

$X \sim$Gamma$(1, \beta)$

$$E(X) = \beta, \quad Var(X) = \beta^2$$

Chi-Squared (p)

$$X \sim \text{Gamma}(\frac{p}{2}, 2)$$

Beta (α, β)

$$f_X(x) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \text{ if } x \in (0,1)$$

$$E(X) = \frac{\alpha}{\alpha+\beta}, \quad Var(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

Normal (μ, σ)

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \text{ if } x \in \mathbb{R}$$

Marginal Distributions

$$p_X(x) = \sum_y p_{X,Y}(x,y)$$

$$p_Y(y) = \sum_x p_{X,Y}(x,y)$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dx$$

Joint Expectations

$$Eh(X,Y) = \sum_{(x,y)} h(x,y) p_{X,Y}(x,y)$$

$$Eh(X,Y) = \iint_{\mathbb{R}^2} h(x,y) f_{X,Y}(x,y) dy dx$$

Covariance & Correlation

$$\begin{aligned} Cov(X,Y) &= E(X-E(X))(Y-E(Y)) \\ &= E(XY) - E(X)E(Y) \\ Var(aX+bY) &= a^2 Var(X) + b^2 Var(Y) \\ &+ 2ab Cov(X,Y) \end{aligned}$$

$$\text{Correlation: } \rho_{X,Y} = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}$$

Conditional Distributions

$$\begin{aligned} p_{Y|X}(y|x) &= \frac{p_{X,Y}(x,y)}{p_X(x)} \\ f_{Y|X}(y|x) &= \frac{f_{X,Y}(x,y)}{f_X(x)} \\ E[h(X,Y)|X=x] &= \sum_y h(X,Y)p(Y|X) \\ E[h(X,Y)|X=x] &= \int h(X,Y)p(Y|X)dy \\ E[ah(X,Y)+b|X] &= aE[h(X,Y)]+b \\ E[h(X,Y)+g(X,Y)|X] &= E[h|X]+E[g|X] \\ E[g(X)h(X,Y)|X] &= g(X)E[h(X,Y)|X] \\ EY &= E[E(Y|X)] \\ Var(Y) &= E[Var(Y|X)] + Var[E(Y|X)] \end{aligned}$$

Independence

$$\begin{aligned} \text{Independence: } F_{X,Y}(x,y) &= F_X(x)F_Y(y), \forall x,y \\ p_{X,Y}(x,y) &= p_X(x)p_Y(y), \forall x,y \\ f_{X,Y}(x,y) &= f_X(x)f_Y(y), \forall x,y \end{aligned}$$

$$Cov(X,Y) = 0, \text{ if } X, Y \text{ independent}$$

Convolution

$$F_{X+Y}(t) = P(X+Y \le t) = P(Y \le t - X)$$

MGF Approach

$$M_{X+Y}(t) = M_X(t)M_Y(t), \text{ if } X, Y \text{ independent}$$

Jacobians

$$\begin{aligned} y_1 &= h_1(x_1, x_2) \\ y_2 &= h_2(x_1, x_2) \\ f_{Y_1,Y_2}(y_1,y_2) &= f_{X_1,X_2}(h^{-1}(y_1,y_2))|\det(J_{h^{-1}}(y_1,y_2))| \end{aligned}$$

$$J = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_2}{\partial x_1} & \frac{\partial h_3}{\partial x_1} \\ \frac{\partial h_1}{\partial x_2} & \frac{\partial h_2}{\partial x_2} & \frac{\partial h_3}{\partial x_2} \\ \frac{\partial h_1}{\partial x_3} & \frac{\partial h_2}{\partial x_3} & \frac{\partial h_3}{\partial x_3} \end{bmatrix}$$

Generalities

$$\begin{aligned} X &= X_1, X_2, \ldots, X_n \\ F_X(x_1,\ldots,x_n) &= P(X_1 \le x_1, \ldots, X_n \le x_n) \\ p_X(x_1,\ldots,x_n) &= P(X_1 = x_1, \ldots, X_n = x_n) \\ f_X(x_1,\ldots,x_n) & \end{aligned}$$

Order Statistics

$$X_1, X_2, \ldots, X_n \overset{\text{iid}}{\sim} f$$

$$f_{X_{(k)}}(x) = k\binom{n}{k} f(x)[F(x)]^{k-1}[1-F(x)]^{n-k}$$

$$\begin{aligned} f_{X_{(j)},X_{(k)}}(x,y) = & \frac{n!}{(j-1)!(k-j-1)!(n-k)!} \\ & [F(x)]^{j-1}[F(x)]^{j-1}[F(y)-F(x)]^{k-j-1} \\ & [1-F(y)]^{n-k} f(x) f(y) \end{aligned}$$

Gamma & Beta Functions

$$\begin{aligned} \Gamma(\alpha) &= \int_0^\infty t^{\alpha-1}e^{-t}dt, \quad \alpha > 0 \\ \Gamma(\alpha+1) &= \alpha\Gamma(\alpha) \\ \Gamma(\alpha) &= (\alpha-1)! \\ \Gamma\left(\frac{1}{2}\right) &= \sqrt{\pi} \\ B(\alpha,\beta) &= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx \end{aligned}$$

Sums

$$\sum_{x=1}^{N} x = \frac{N(N+1)}{2}$$

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

Combinations & Permutations

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

$${}_nP_r = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = \frac{n}{r}\binom{n-1}{r-1}$$