Basic Probability

$$\begin{aligned}Var(X) &= E(X-EX)^2\\ &= EX^2-(EX)^2\\ &= EX(X-1)+EX-(EX)^2\\ M_X(t) &= E(e^{tX})\\ Cov(X,Y) &= E(X-E(X))(Y-E(Y))\\ &= E(XY)-E(X)E(Y)\\ Var(aX+bY) &= a^2Var(X)+b^2Var(Y)\\ &\quad +2abCov(X,Y)\end{aligned}$$

Correlation: $\rho_{X,Y} = \dfrac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}$

$$f_{X_{(k)}}(x) = k\binom{n}{k}f(x)[F(x)]^{k-1}[1-F(x)]^{n-k}$$

Independence: $F_{X,Y}(x,y) = F_X(x)F_Y(y), \forall x,y$

$Cov(X,Y) = 0$, if X,Y independent

Discrete Unifrom (1, N)

$$p_X(x) = \frac{1}{N}, \text{ if } x = 1,2,...,N, \quad F_X(x) = \frac{x}{N}$$

$$E(X) = \frac{N+1}{2}, \quad Var(X) = \frac{N^2-1}{12}$$

$$M_X(t) = \frac{e^t - e^{(N+1)t}}{N(1-e^t)}$$

Discrete Uniform (N_0, N_1)

$$p_X(x) = \frac{1}{N_1-N_0+1}, \text{ if } x = N_0, N_0+1, ..., N_1$$

$$F_X(x) = \frac{x-N_0+1}{N_1-N_0+1}, \text{ if } x = N_0, N_0+1, ..., N_1$$

$$E(X) = \frac{N_1+N_0}{2}$$

$$Var(X) = \frac{(N_1-N_0)(N_1-N_0+2)}{12}$$

$$M_X(t) = \frac{e^{N_0 t} - e^{(N_1+1)t}}{(N_1-N_0+1)(1-e^t)}$$

Hypergeometric (N, M, n)

$$p_X(x) = \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}, \text{ if } x = 0,1,...,n$$

$$E(X) = \frac{nM}{N}, \quad Var(X) = \frac{nM}{N}\left(1-\frac{M}{N}\right)\left(\frac{N-n}{N-1}\right)$$

Bernoulli (p)

$$p_X(x) = p^x(1-p)^{1-x}, \text{ if } x = 0,1$$

$$E(X) = p, \quad Var(X) = p(1-P)$$

$$M_X(t) = (1-p)+pe^t$$

Binomial (n, p)

$$p_X(x) = \binom{n}{x}p^x(1-p)^{n-x}, \text{ if } x = 0,1,\ldots,n$$

$$E(X) = np, \quad Var(X) = np(1-P)$$

$$M_X(t) = (1-p+pe^t)^n$$

Geometric (p)

$$p_X(x) = p(1-p)^{x-1}, \text{ if } x = 1,2,\ldots$$

$$E(X) = \frac{1}{p}, \quad Var(X) = \frac{1-p}{p^2}$$

$$M_X(t) = \frac{pe^t}{1-(1-p)e^t}$$

Negative Binomial (r, p)

$$p_X(x) = \binom{x-1}{r-1}p^r(1-p)^{x-r}, \text{ if } x = r, r+1, \ldots$$

$$E(X) = \frac{r}{p}, \quad Var(X) = \frac{r(1-p)}{p^2}$$

Poisson (λ)

$$p_X(x) = \frac{e^{-\lambda}\lambda^x}{x!}, \text{ if } x = 0,1,2,\ldots$$

$$F_X(x) = e^{-\lambda}\sum_{i=0}^{x}\frac{\lambda^i}{i!}$$

$$E(X) = \lambda, \quad Var(X) = \lambda$$

$$M_X(t) = e^{\lambda(e^t-1)}$$

Continuous Uniform (a, b)

$$f_X(x) = \frac{1}{b-a}, \text{ if } a < x < b$$

$$E(X) = \frac{a+b}{2}, \quad Var(X) = \frac{(b-a)^2}{12}$$

$$M_X(t) = \frac{e^{tb}-e^{ta}}{t(b-a)}, 1 \text{ if } t = 0$$

Gamma (α, β)

$$f_X(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha}x^{\alpha-1}e^{-\frac{x}{\beta}}, \text{ if } x > 0$$

$$E(X) = \alpha\beta, \quad Var(X) = \alpha\beta^2$$

$$M_X(t) = (1-\beta t)^{-\alpha}$$

Exponential (β)

$$f_X(x) = \frac{1}{\beta^\alpha}e^{-\frac{x}{\beta}}, \text{ if } x > 0$$

$$X \sim \text{Gamma}(1, \beta)$$

$$E(X) = \beta, \quad Var(X) = \beta^2$$

$$M_X(t) = (1-\beta t)^{-1}$$

Chi-Squared (p)

$$X \sim \text{Gamma}(\frac{p}{2}, 2)$$

Beta (α, β)

$$f_X(x) = \frac{1}{B(\alpha,\beta)}x^{\alpha-1}(1-x)^{\beta-1}, \text{ if } x \in (0,1)$$

$$E(X) = \frac{\alpha}{\alpha+\beta}, \quad Var(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

$$M_X(t) = \sum_{n=0}^{\infty}\frac{B(\alpha+n,\beta)}{B(\alpha,\beta)}\frac{t^n}{n!}$$

Normal (μ, σ)

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \text{ if } x \in \mathbb{R}$$

$$M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

General Equations

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt, \quad \alpha > 0$$

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha) \quad \Gamma(\alpha) = (\alpha-1)! \quad \Gamma(\frac{1}{2}) = \sqrt{\pi}$$

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx$$

$$\sum_{x=1}^{N} x = \frac{N(N+1)}{2} \qquad \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n}{r}\binom{n-1}{r-1}$$

$$\lim_{n\to\infty}\left(1-\frac{c}{n}\right)^{-n} = e^c \qquad \lim_{n\to\infty}\left(1-\frac{a}{n}\right)^{n} = e^a$$

$$\prod_{i=1}^{n} a^{x_i} = a^{\sum_{i=1}^{n} x_i} \qquad \prod_{i=1}^{n} cx_i = c^n \prod_{i=1}^{n} x_i$$

Unit 1 Material

$$X \sim N_n(\mu,\Sigma) \& Y = AX + b$$

$$\implies Y \sim N_m(A\mu + b, A\Sigma A^T)$$

$$M_{X_n}(t) = M(t) \text{ for } |b| \le h_1 \implies X_n \xrightarrow{D} X$$

$$\lim_{n\to\infty} P[|X_n - X| \ge \epsilon] = 0 \implies X_n \xrightarrow{P} X$$

$$\text{CLT: } \frac{\sqrt{n}(\bar{x}_n - \mu)}{\sigma}$$

$$\Delta \text{ Theorem: } \sqrt{n}(g(\bar{X}) - g(\mu)) \sim N(0, g'(\mu)^2)\sigma^2$$

$$\text{MLE: } L(\theta) = \prod_{i=1}^{n} f(x_i;\theta)$$

$$\frac{\delta \log(L(\theta))}{\delta\theta} = 0 \text{ gives } \hat{\theta}_{mle}$$

$$\text{MME: } EX, \text{ solve for } \theta$$

Unbiased Estimator of θ

$$E(X) = r(\theta)$$

Fisher Number

$$I_n(\theta) = E_\theta\left(\frac{d\ln f(x;\theta)}{d\theta}\right)^2$$

$$= -E_\theta\left(\frac{d^2\ln f(x;\theta)}{d\theta^2}\right)$$

Cramer-Rao Lower Bound

$$\text{CRLB} = \frac{1}{n\theta}$$

Efficient estimator if $Var(Y) = \text{CRLB}$

Asymptotic Efficiency

$$c(\hat{\theta}_{1n}) = \frac{1/I(\theta_0)}{\sigma^2_{\hat{\theta}_1 n}} = \frac{\text{CRLB}}{Var(Y)}$$

Asymptotic Relative Efficiency

$$\text{ARE: } e(\hat{\theta}_{1n}, \hat{\theta}_{2n}) = \frac{\sigma^2_{\hat{\theta}_{2n}}}{\sigma^2_{\hat{\theta}_{1n}}}$$

Asymptotic Distribution

$$\sqrt{n}(\theta_{mle} - \theta) \xrightarrow{D} N(0, \frac{1}{I(\theta)})$$

Risk Function

$$R(\theta,\delta) = \int_{-\infty}^{\infty} L[\theta,\delta] f_Y(y;\theta)dy$$

$$= E(L[\theta,\delta])$$