Modeling of Bézier curves/surfaces

Part 1

– a CAGD approach based on OpenGL and C++ –

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Lecture 9 – May 2, 2022

Pierre Étienne Bézier

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A parabola construction

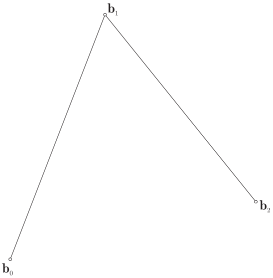

Fig. 1: A parabola construction

- Consider the non collinear points $\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2 \in \mathbb{R}^\delta, \delta \geq 2$.
- Let $u \in [0,1]$ be an arbitrarily fixed real number.
- Define the inner points

$$\begin{aligned} \mathbf{b}_0^1(u) &= (1-u)\mathbf{b}_0 + u\mathbf{b}_1, \\ \mathbf{b}_1^1(u) &= (1-u)\mathbf{b}_1 + u\mathbf{b}_2 \end{aligned}$$

  of the segments $\mathbf{b}_0\mathbf{b}_1$ and $\mathbf{b}_1\mathbf{b}_2$.
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- If u varies in $[0,1]$, then the orbit of the point $\mathbf{b}_0^2(u)$ is an arc of a parabola:

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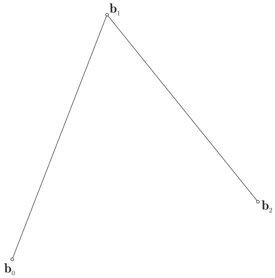

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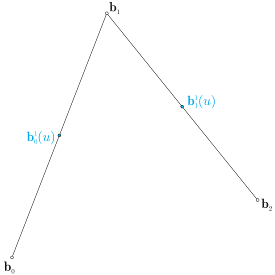

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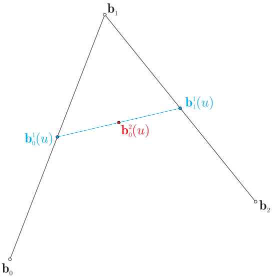

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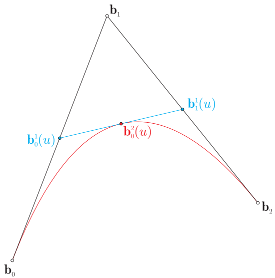

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A recurrence relation

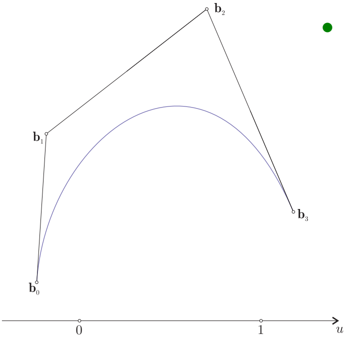

Fig. 2: The de Casteljau algorithm

- Consider the control polygon

$$[\mathbf{b}_i]_{i=0}^n \in \mathcal{M}_{1,n+1}(\mathbb{R}^\delta), \delta \geq 2, n \geq 1$$

and the recurrence relation

$$\mathbf{b}_i^r(u) = (1-u)\mathbf{b}_i^{r-1}(u) + u\mathbf{b}_{i+1}^{r-1}(u),\ u \in [0,1],$$

where

$$\begin{aligned} r &= 1,2,\ldots,n, \\ i &= 0,1,\ldots,n-r, \\ \mathbf{b}_i^0 &= \mathbf{b}_i,\ i = 0,1,\ldots,n. \end{aligned}$$

- The last point $\mathbf{b}_0^n(u)$, $u \in [0,1]$ of the recurrence defines the Bézier curve of degree n.

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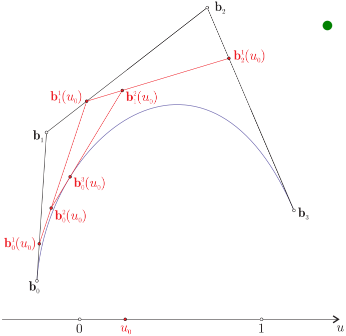

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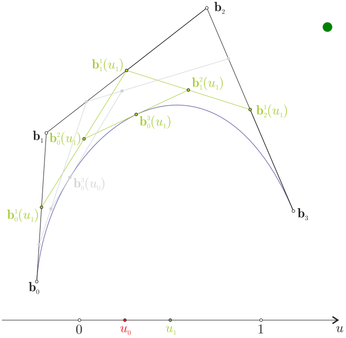

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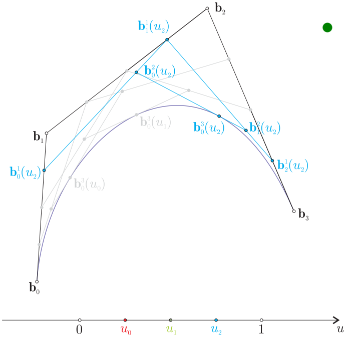

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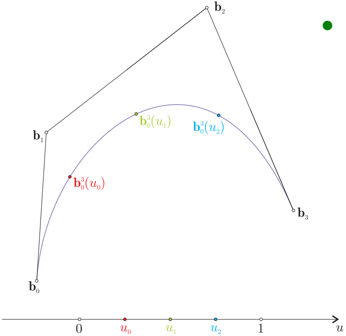

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Triangular scheme of the de Casteljau corner cutting algorithm

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\begin{array}{cccccccc}
\mathbf{b}_0 & = & \mathbf{b}_0^0(u) & & & & & \\
 & & & \mathbf{b}_0^1(u) & & & & \\
\mathbf{b}_1 & = & \mathbf{b}_1^0(u) & & \mathbf{b}_0^2(u) & & & \\
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 & & & & & & \mathbf{b}_0^{n-1} & \\
\vdots & & \vdots & \vdots & \vdots & \cdots & & \mathbf{b}_0^n(u) \\
 & & & & & & \mathbf{b}_1^{n-1} & \\
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\end{array}
$$

Triangular scheme of the de Casteljau corner cutting algorithm

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\begin{array}{cccccccc}
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 & & & \mathbf{b}_0^1(u) & & & & \\
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 & & & & & & \mathbf{b}_0^{n-1} & \\
\vdots & & \vdots & \vdots & \vdots & \cdots & & \mathbf{b}_0^n(u) \\
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Bernstein polynomials

Consider the function system

$$B = \left\{ B_i^n(u) = \binom{n}{i} u^i (1-u)^{n-i} : u \in [0,1] \right\}_{i=0}^{n},$$

where:

- the function $B_i^n(u), u \in [0,1]$ is the ith Bernstein polynomial of degree n;
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Recurrence property of Bernstein polynomials

The Bernstein polynomials of degree n fulfill the recurrence property

$$B_i^n(u) = (1-u)B_i^{n-1}(u) + uB_{i-1}^{n-1}(u),\ \forall u \in [0,1]$$

and the identities

$$B_0^0(u) \equiv 1,\ \forall u \in [0,1],\ \text{and}\ B_i^n(u) \equiv 0,\ \forall u \in [0,1],\ \text{if } i < 0 \text{ or } i > n.$$

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Partition of the unity

The system

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of Bernstein polynomials of degree n forms the partition of the unity, i.e.

$$\sum_{i=0}^{n} B_i^n(u) \equiv 1, \ \forall u \in [0,1].$$

Proof.

- By the binomial theorem of Newton:

$$\sum_{i=0}^{n} B_i^n(u) = \sum_{i=0}^{n} \binom{n}{i} u^i (1-u)^{n-i} = [u + (1-u)]^n \equiv 1, \ \forall u \in [0,1].$$

- Moreover, $B_i^n(u) \geq 0, \ \forall u \in [0,1]$.

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Maximum points of Bernstein polynomials of degree n

Fig. 3: Bernstein polynomials of degree 3.

The ith $(i = 0, 1, \ldots, n)$ Bernstein polynomial

$$B_i^n(u) = \binom{n}{i} u^i (1-u)^{n-i}, u \in [0,1]$$

of degree n attains its maximum at the point

$$\bar{u}_i = \frac{i}{n}.$$

Moreover, $B_i^n(u)$ is strictly increasing (resp. decreasing) on the interval $[0, \bar{u}_i]$ (resp. $[\bar{u}_i, 1]$).

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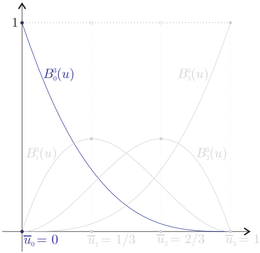

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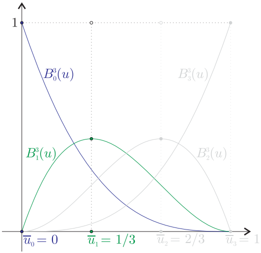

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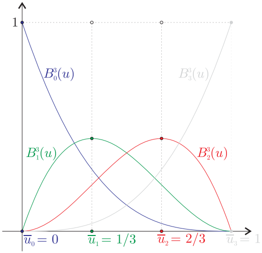

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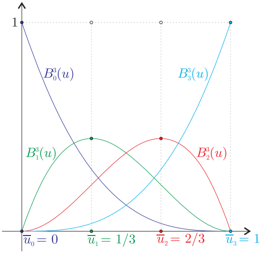

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&= n\binom{n-1}{i-1} u^{i-1}(1-u)^{n-i} - n\binom{n-1}{i} u^i (1-u)^{n-1-i} \\
&= n\left(B_{i-1}^{n-1}(u) - B_i^{n-1}(u)\right);
\end{aligned}$$

  - we need to solve the system

$$\begin{cases}
\frac{d}{du} B_i^n(u) = 0, \\
\frac{d^2}{du^2} B_i^n(u) < 0.
\end{cases}$$

Proof – continued.

- In order to simplify the equation

$$\frac{d}{du}B_i^n(u) = 0,\ i = 1, 2, \ldots, n-1,$$

we can successively write:

$$\begin{aligned} n\binom{n-1}{i-1}u^{i-1}(1-u)^{n-i} &= n\binom{n-1}{i}u^i(1-u)^{n-1-i} \\ \binom{n-1}{i-1}(1-u) &= \binom{n-1}{i}u \Big/ \cdot \frac{(i-1)!(n-1-i)!}{(n-1)!} \\ \frac{1-u}{n-i} &= \frac{u}{i}, \end{aligned}$$

the solution of which is

$$\bar{u}_i = \frac{i}{n}.$$

- It is easy to verify that $\frac{d^2}{du^2}B_i^n(u)\Big|_{u=\bar{u}_i} < 0$.

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Theorem (points of the de Casteljau algorithm by means of Bernstein polynomials)

The point $\mathbf{b}_i^r(u)$ ($r = 0, 1, \ldots, n$; $i = 0, 1, \ldots, n - r$; $u \in [0, 1]$) of the de Casteljau algorithm can be expressed as

$$\mathbf{b}_i^r(u) = \sum_{j=0}^{r} \mathbf{b}_{j+i} B_j^r(u)$$

by means of Bernstein polynomials of degree r.

Corollary (Bernstein/parametric representation of Bézier curves)

By substituting $i = 0$ and $r = n$ into the previous formula we get the

$$\mathbf{b}(u) := \mathbf{b}_0^n(u) = \sum_{j=0}^{n} \mathbf{b}_j B_j^n(u), \; u \in [0, 1]$$

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Proof.

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$$\sum_{j=0}^{r} \mathbf{b}_{j+i} B_j^r(u) = \sum_{j=i}^{i+r} \mathbf{b}_j B_{j-i}^r(u).$$

- Second, we prove the theorem by mathematical induction with respect to r.
  - The statement trivially holds for $r = 0$:

$$\mathbf{b}_i^0(u) = \sum_{j=i}^{i+0} \mathbf{b}_j B_{j-i}^0(u) = \mathbf{b}_i B_0^0(u) = \mathbf{b}_i, \ \forall u \in [0,1], \ i = 0, 1, \ldots, n.$$

  - We assume that the statement is valid for all $0 \leq k \leq r-1$ and we also prove it for $k = r$ $(1 \leq r \leq n)$.
  - Based on the recurrence relation of the de Casteljau algorithm, we have:

$$\mathbf{b}_i^r(u) = (1-u)\mathbf{b}_i^{r-1}(u) + u\mathbf{b}_{i+1}^{r-1}(u), \ \forall u \in [0,1].$$

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Closure for the affine transformations

The shape of a Bézier curve of degree $n \geq 1$ is invariant under the affine transformation of its control polygon.

Proof 1.
The de Casteljau algorithm consists of proportional subdivision points that are invariant under affine transformations. $\square$

Proof 2 – cf. Lecture 1.
The Bernstein polynomials of degree $n \geq 1$ form the partition of unity. $\square$

Convex hull property

Fig. 4: The convex hull of a plane Bézier curve of degree 3.

The shape of a Bézier curve of degree $n \geq 1$ lies in the convex hull of its control polygon.

Proof 1.
It follows automatically from the de Casteljau algorithm, since all subdivision points are on a segment of the convex hull of the control polygon. $\square$

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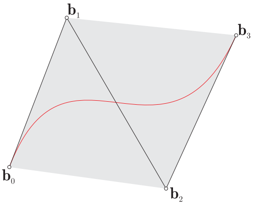

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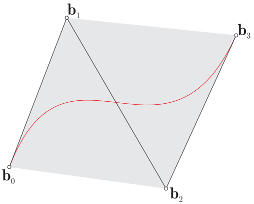

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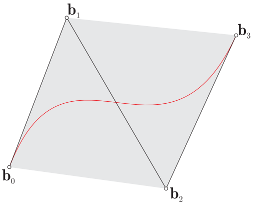

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Invariance under linear transformations of the parameter domain

The shape of the Bézier curve of degree $n \geq 1$ is invariant under affine transformations of the parameter domain.

Proof.

The proportional subdivision points of the de Casteljau algorithm do not change under the linear parameter transformation

$$u(t) = \frac{t-a}{b-a},\ t \in [a,b]. \qquad \square$$

Endpoint interpolation

The Bézier curve of degree $n \geq 1$ interpolates the first and last control point of its control polygon.

Proof.

$$\begin{aligned} \mathbf{b}(0) &= \sum_{i=0}^{n} \mathbf{b}_i B_i^n(0) = \mathbf{b}_0 B_0^n(0) + \mathbf{0} = \mathbf{b}_0, \\ \mathbf{b}(1) &= \sum_{i=0}^{n} \mathbf{b}_i B_i^n(1) = \mathbf{0} + \mathbf{b}_n B_n^n(1) = \mathbf{b}_n. \qquad \square \end{aligned}$$

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$$\begin{aligned} \mathbf{b}(0) &= \sum_{i=0}^{n} \mathbf{b}_i B_i^n(0) = \mathbf{b}_0 B_0^n(0) + \mathbf{0} = \mathbf{b}_0, \\ \mathbf{b}(1) &= \sum_{i=0}^{n} \mathbf{b}_i B_i^n(1) = \mathbf{0} + \mathbf{b}_n B_n^n(1) = \mathbf{b}_n. \quad \square \end{aligned}$$

Invariance under linear transformations of the parameter domain

The shape of the Bézier curve of degree $n \geq 1$ is invariant under affine transformations of the parameter domain.

Proof.

The proportional subdivision points of the de Casteljau algorithm do not change under the linear parameter transformation

$$u(t) = \frac{t-a}{b-a},\ t \in [a,b]. \quad \square$$

Endpoint interpolation

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Symmetry

The Bézier curve of degree $n \geq 1$ defined by the control polygon $[\mathbf{b}_i]_{i=0}^{n}$ is symmetric, i.e. the control polygon $[\mathbf{b}_{n-i}]_{i=0}^{n}$ generates the same Bézier curve (of course with different orientation).

Proof.

Since

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\begin{aligned}
B_i^n(u) &= \binom{n}{i} u^i (1-u)^{n-i} \\
&= \binom{n}{n-i} (1-u)^{n-i} (1-(1-u))^{n-(n-i)} \\
&= B_{n-i}^n(1-u), \ \forall u \in [0,1],
\end{aligned}
$$

it follows that

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\mathbf{b}(u) = \sum_{i=0}^{n} \mathbf{b}_i B_i^n(u) = \sum_{i=0}^{n} \mathbf{b}_{n-i} B_{n-i}^n(1-u), \ \forall u \in [0,1]. \qquad \square
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□

Properties of Bézier curves

Global/pseudo local controllability

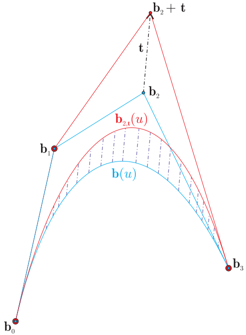

Fig. 5: Effect of a translated control point

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\begin{aligned}
\mathbf{b}_{i,\mathbf{t}}(u) &= \sum_{j=0, j\neq i}^{n} \mathbf{b}_j B_j^n(u) + (\mathbf{b}_i + \mathbf{t}) B_i^n(u) \\
&= \sum_{j=0}^{n} \mathbf{b}_j B_j^n(u) + \mathbf{t} B_i^n(u) \\
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Modification of a single control point $\mathbf{b}_i$ has a global effect on the shape of the Bézier curve

$$\mathbf{b}(u) = \sum_{j=0}^{n} \mathbf{b}_j B_j^n(u), \ u \in [0,1]$$

of degree $n \geq 1$.

Proof.
$B_j^n(u) > 0, \ \forall u \in (0,1), \ j = 0, \ldots, n.$ $\square$

However, we also say that the curve is pseudo locally controllable, because the maximal effect on the shape of the curve, generated by the modification of the control point $\mathbf{b}_i$, can be easily forecasted, since the unique maximal weight of the control point is $B_i^n(\bar{u}_i)$, where $\bar{u}_i =$

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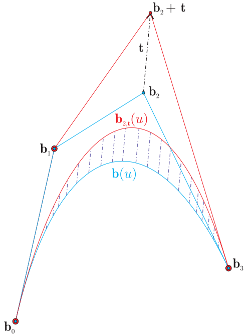

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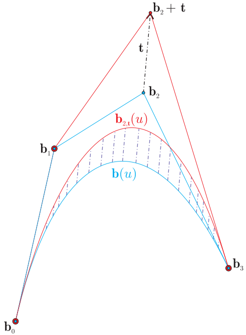
Properties of Bézier curves

Global/pseudo local controllability



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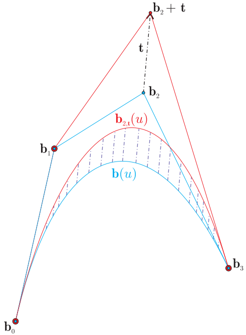

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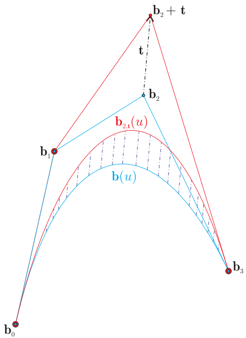

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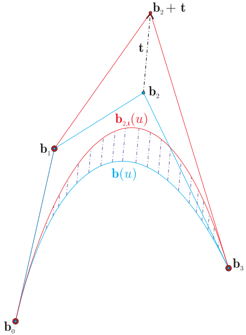

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Representation of lines

The Bézier curve is a line if and only if its control points are collinear.

Proof.

Immediate corollary of the de Casteljau algorithm. □

Linear precision

If the control points are uniformly distributed on a straight line, then the Bézier curve determined by them describes exactly the initial straight line. This property is equivalent to

$$\sum_{i=0}^{n} \frac{i}{n} B_i^n(u) \equiv u, \forall u \in [0,1]$$

and it is called linear precision property.

Proof.

$$\begin{aligned}\sum_{i=0}^{n} \frac{i}{n} B_i^n(u) &= \sum_{i=1}^{n} \frac{i}{n} B_i^n(u) = u \sum_{i=1}^{n} \frac{(n-1)!}{(i-1)!(n-i)!} u^{i-1}(1-u)^{n-i} \\ &\overset{i-1\to i}{=} u \sum_{i=0}^{n-1} \frac{(n-1)!}{i!(n-1-i)!} u^{i}(1-u)^{n-1-i} \equiv u,\ \forall u \in [0,1].\end{aligned}$$

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\end{aligned}$$

□

Hodograph of a Bézier curve of degree $n \geq 1$

We can successively write:

$$
\begin{aligned}
\frac{d}{du}\mathbf{b}(u) &= \frac{d}{du}\left(\sum_{i=0}^{n}\mathbf{b}_i B_i^n(u)\right) = \sum_{i=0}^{n}\mathbf{b}_i \frac{d}{du}B_i^n(u) \\
&= n\sum_{i=0}^{n}\mathbf{b}_i\left(B_{i-1}^{n-1}(u) - B_i^{n-1}(u)\right) \\
&= n\sum_{i=0}^{n}\mathbf{b}_i B_{i-1}^{n-1}(u) - n\sum_{i=0}^{n}\mathbf{b}_i B_i^{n-1}(u) \\
&= n\sum_{i=1}^{n}\mathbf{b}_i B_{i-1}^{n-1}(u) - n\sum_{i=0}^{n-1}\mathbf{b}_i B_i^{n-1}(u) \\
&\overset{i-1\to i}{=} n\sum_{i=0}^{n-1}\mathbf{b}_{i+1} B_i^{n-1}(u) - n\sum_{i=0}^{n-1}\mathbf{b}_i B_i^{n-1}(u) \\
&= n\sum_{i=0}^{n-1}(\mathbf{b}_{i+1} - \mathbf{b}_i) B_i^{n-1}(u) \\
&= \sum_{i=0}^{n-1}\Delta\mathbf{b}_i B_i^{n-1}(u),\ \forall u \in [0,1];\ \Delta\mathbf{b}_i = n\left(\mathbf{b}_{i+1} - \mathbf{b}_i\right),\ i = 0, 1, \ldots, n-1
\end{aligned}
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&= n\sum_{i=1}^{n}\mathbf{b}_i B_{i-1}^{n-1}(u) - n\sum_{i=0}^{n-1}\mathbf{b}_i B_i^{n-1}(u) \\
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&\overset{i-1\to i}{=} n\sum_{i=0}^{n-1}\mathbf{b}_{i+1} B_i^{n-1}(u) - n\sum_{i=0}^{n-1}\mathbf{b}_i B_i^{n-1}(u) \\
&= n\sum_{i=0}^{n-1}(\mathbf{b}_{i+1} - \mathbf{b}_i) B_i^{n-1}(u) \\
&= \sum_{i=0}^{n-1}\Delta\mathbf{b}_i B_i^{n-1}(u),\ \forall u \in [0,1];\ \Delta\mathbf{b}_i = n(\mathbf{b}_{i+1} - \mathbf{b}_i),\ i = 0, 1, \ldots, n-1
\end{aligned}
$$

Properties of Bézier curves

Hodograph of a Bézier curve of degree $n \geq 1$

We can successively write:

$$
\begin{aligned}
\frac{d}{du}\mathbf{b}(u) &= \frac{d}{du}\left(\sum_{i=0}^{n}\mathbf{b}_i B_i^n(u)\right) = \sum_{i=0}^{n}\mathbf{b}_i \frac{d}{du}B_i^n(u) \\
&= n\sum_{i=0}^{n}\mathbf{b}_i\left(B_{i-1}^{n-1}(u) - B_i^{n-1}(u)\right) \\
&= n\sum_{i=0}^{n}\mathbf{b}_i B_{i-1}^{n-1}(u) - n\sum_{i=0}^{n}\mathbf{b}_i B_i^{n-1}(u) \\
&= n\sum_{i=1}^{n}\mathbf{b}_i B_{i-1}^{n-1}(u) - n\sum_{i=0}^{n-1}\mathbf{b}_i B_i^{n-1}(u) \\
&\overset{i-1\to i}{=} n\sum_{i=0}^{n-1}\mathbf{b}_{i+1} B_i^{n-1}(u) - n\sum_{i=0}^{n-1}\mathbf{b}_i B_i^{n-1}(u) \\
&= n\sum_{i=0}^{n-1}(\mathbf{b}_{i+1} - \mathbf{b}_i) B_i^{n-1}(u) \\
&= \sum_{i=0}^{n-1}\Delta\mathbf{b}_i B_i^{n-1}(u),\ \forall u \in [0,1];\ \Delta\mathbf{b}_i = n(\mathbf{b}_{i+1} - \mathbf{b}_i),\ i = 0, 1, \ldots, n-1
\end{aligned}
$$

Hodograph of a Bézier curve of degree $n \geq 1$

We can successively write:

$$
\begin{aligned}
\frac{d}{du}\mathbf{b}(u) &= \frac{d}{du}\left(\sum_{i=0}^{n}\mathbf{b}_i B_i^n(u)\right) = \sum_{i=0}^{n}\mathbf{b}_i \frac{d}{du}B_i^n(u) \\
&= n\sum_{i=0}^{n}\mathbf{b}_i\left(B_{i-1}^{n-1}(u) - B_i^{n-1}(u)\right) \\
&= n\sum_{i=0}^{n}\mathbf{b}_i B_{i-1}^{n-1}(u) - n\sum_{i=0}^{n}\mathbf{b}_i B_i^{n-1}(u) \\
&= n\sum_{i=1}^{n}\mathbf{b}_i B_{i-1}^{n-1}(u) - n\sum_{i=0}^{n-1}\mathbf{b}_i B_i^{n-1}(u) \\
&\overset{i-1\to i}{=} n\sum_{i=0}^{n-1}\mathbf{b}_{i+1} B_i^{n-1}(u) - n\sum_{i=0}^{n-1}\mathbf{b}_i B_i^{n-1}(u) \\
&= n\sum_{i=0}^{n-1}(\mathbf{b}_{i+1} - \mathbf{b}_i)B_i^{n-1}(u) \\
&= \sum_{i=0}^{n-1}\Delta\mathbf{b}_i B_i^{n-1}(u),\ \forall u \in [0,1];\ \Delta\mathbf{b}_i = n\left(\mathbf{b}_{i+1} - \mathbf{b}_i\right),\ i = 0, 1, \ldots, n-1
\end{aligned}
$$

Hodograph of a Bézier curve of degree $n \geq 1$

We can successively write:

$$
\begin{aligned}
\frac{d}{du}\mathbf{b}(u) &= \frac{d}{du}\left(\sum_{i=0}^{n}\mathbf{b}_i B_i^n(u)\right) = \sum_{i=0}^{n}\mathbf{b}_i \frac{d}{du}B_i^n(u) \\
&= n\sum_{i=0}^{n}\mathbf{b}_i\left(B_{i-1}^{n-1}(u) - B_i^{n-1}(u)\right) \\
&= n\sum_{i=0}^{n}\mathbf{b}_i B_{i-1}^{n-1}(u) - n\sum_{i=0}^{n}\mathbf{b}_i B_i^{n-1}(u) \\
&= n\sum_{i=1}^{n}\mathbf{b}_i B_{i-1}^{n-1}(u) - n\sum_{i=0}^{n-1}\mathbf{b}_i B_i^{n-1}(u) \\
&\overset{i-1\to i}{=} n\sum_{i=0}^{n-1}\mathbf{b}_{i+1} B_i^{n-1}(u) - n\sum_{i=0}^{n-1}\mathbf{b}_i B_i^{n-1}(u) \\
&= n\sum_{i=0}^{n-1}(\mathbf{b}_{i+1} - \mathbf{b}_i) B_i^{n-1}(u) \\
&= \sum_{i=0}^{n-1}\Delta\mathbf{b}_i B_i^{n-1}(u),\ \forall u \in [0,1];\ \Delta\mathbf{b}_i = n\left(\mathbf{b}_{i+1} - \mathbf{b}_i\right),\ i = 0, 1, \ldots, n-1
\end{aligned}
$$

Hodograph of a Bézier curve of degree $n \geq 1$

We can successively write:

$$
\begin{aligned}
\frac{d}{du}\mathbf{b}(u) &= \frac{d}{du}\left(\sum_{i=0}^{n} \mathbf{b}_i B_i^n(u)\right) = \sum_{i=0}^{n} \mathbf{b}_i \frac{d}{du} B_i^n(u) \\
&= n\sum_{i=0}^{n} \mathbf{b}_i \left(B_{i-1}^{n-1}(u) - B_i^{n-1}(u)\right) \\
&= n\sum_{i=0}^{n} \mathbf{b}_i B_{i-1}^{n-1}(u) - n\sum_{i=0}^{n} \mathbf{b}_i B_i^{n-1}(u) \\
&= n\sum_{i=1}^{n} \mathbf{b}_i B_{i-1}^{n-1}(u) - n\sum_{i=0}^{n-1} \mathbf{b}_i B_i^{n-1}(u) \\
&\overset{i-1\to i}{=} n\sum_{i=0}^{n-1} \mathbf{b}_{i+1} B_i^{n-1}(u) - n\sum_{i=0}^{n-1} \mathbf{b}_i B_i^{n-1}(u) \\
&= n\sum_{i=0}^{n-1} (\mathbf{b}_{i+1} - \mathbf{b}_i) B_i^{n-1}(u) \\
&= \sum_{i=0}^{n-1} \Delta\mathbf{b}_i B_i^{n-1}(u),\ \forall u \in [0,1];\ \Delta\mathbf{b}_i = n\left(\mathbf{b}_{i+1} - \mathbf{b}_i\right),\ i = 0, 1, \ldots, n-1.
\end{aligned}
$$

Corollary (Bézier curves are closed under derivation)

The hodograph (i.e. first order derivative) of the Bézier curve

$$\mathbf{b}(u) = \sum_{i=0}^{n} \mathbf{b}_i B_i^n(u),\ u \in [0,1]$$

of degree $n \geq 1$ is also a Bézier curve of degree $n-1$ that is determined by the control polygon

$$[\Delta \mathbf{b}_i]_{i=0}^{n-1} = [n(\mathbf{b}_{i+1} - \mathbf{b}_i)]_{i=0}^{n-1}.$$

Corollary (Hodograph diminishing)

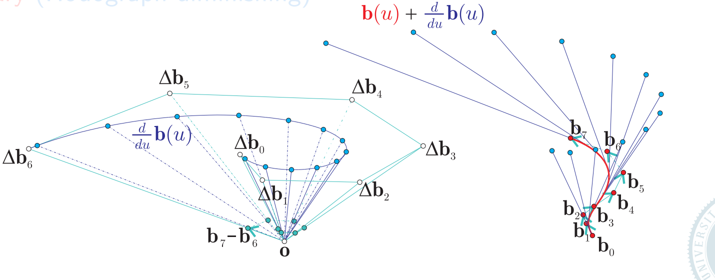

Fig. 6: Any Bézier curve fulfills the hodograph diminishing property

Corollary (Bézier curves are closed under derivation)

The hodograph (i.e. first order derivative) of the Bézier curve

$$\mathbf{b}(u) = \sum_{i=0}^{n} \mathbf{b}_i B_i^n(u),\ u \in [0,1]$$

of degree $n \geq 1$ is also a Bézier curve of degree $n-1$ that is determined by the control polygon

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Corollary (Hodograph diminishing)

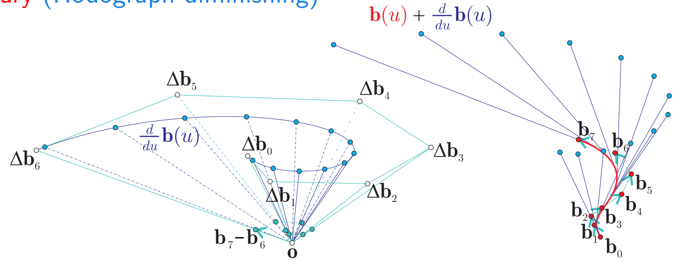


Fig. 6: Any Bézier curve fulfills the hodograph diminishing property

Inflection points

Fig. 7: If the hodograph has a tangent line that goes through the origin and its touching point is a regular point of the hodograph, then the Bézier curve has an inflection point.

Inflection points

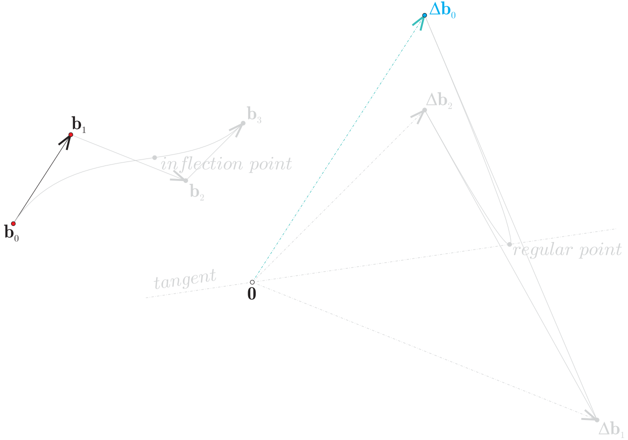


Fig. 7: If the hodograph has a tangent line that goes through the origin and its touching point is a regular point of the hodograph, then the Bézier curve has an inflection point.

Inflection points

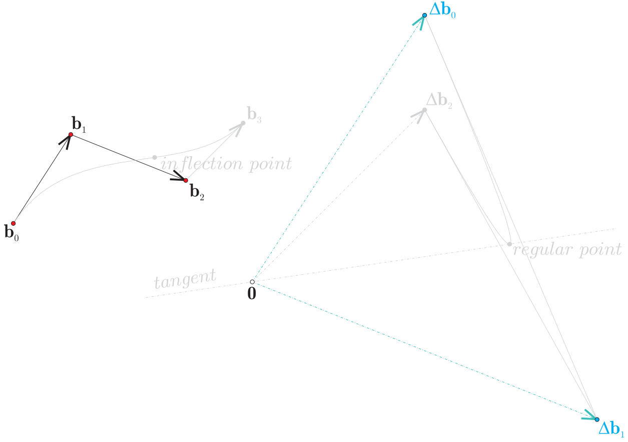


Fig. 7: If the hodograph has a tangent line that goes through the origin and its touching point is a regular point of the hodograph, then the Bézier curve has an inflection point.

Inflection points

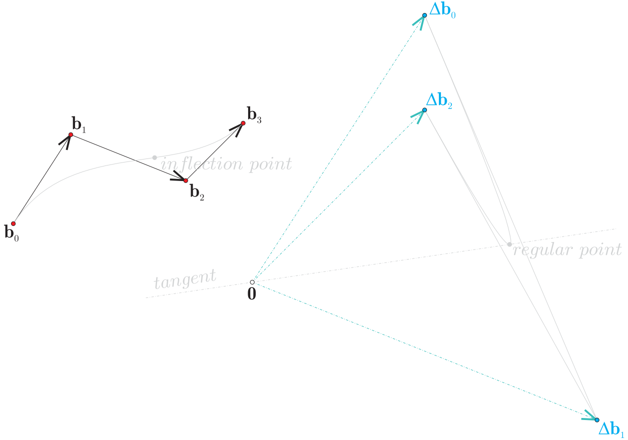


Fig. 7: If the hodograph has a tangent line that goes through the origin and its touching point is a regular point of the hodograph, then the Bézier curve has an inflection point.

Inflection points

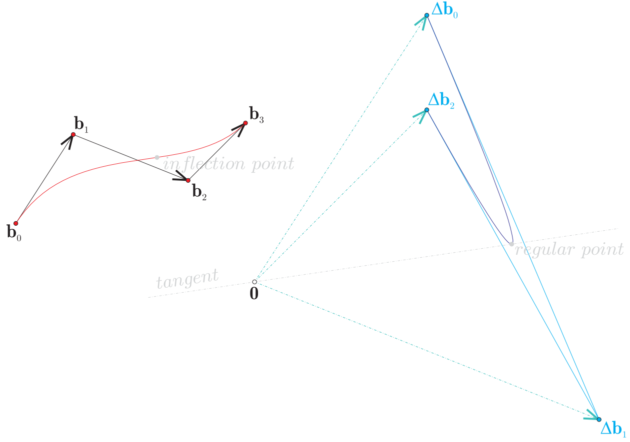


Fig. 7: If the hodograph has a tangent line that goes through the origin and its touching point is a regular point of the hodograph, then the Bézier curve has an inflection point.

Inflection points

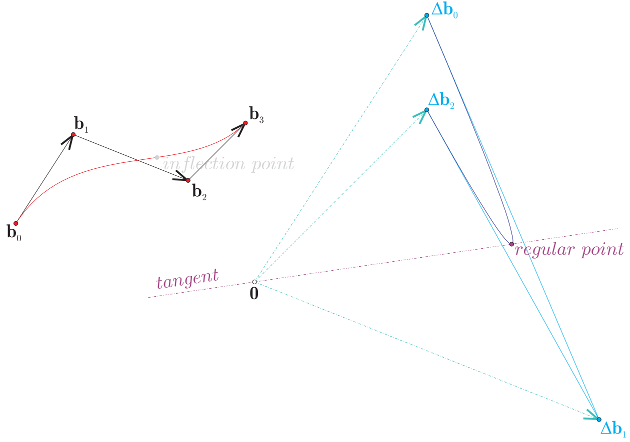


Fig. 7: If the hodograph has a tangent line that goes through the origin and its touching point is a regular point of the hodograph, then the Bézier curve has an inflection point.

Inflection points

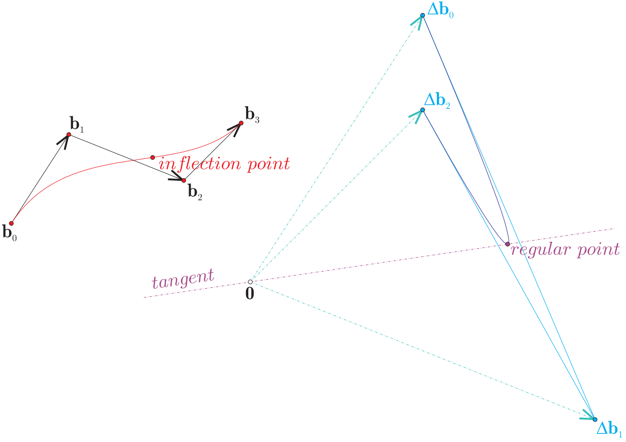


Fig. 7: If the hodograph has a tangent line that goes through the origin and its touching point is a regular point of the hodograph, then the Bézier curve has an inflection point.

Cusps

Fig. 8: If the hodograph goes through the origin, then the Bézier curve has a cusp.

Cusps

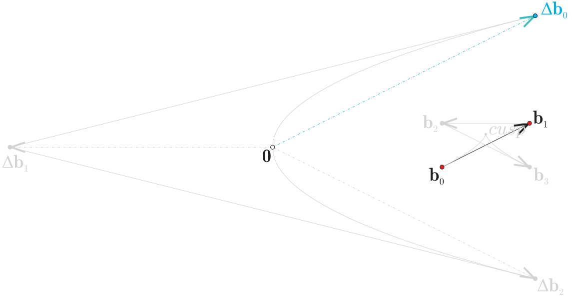


Fig. 8: If the hodograph goes through the origin, then the Bézier curve has a cusp.

Cusps

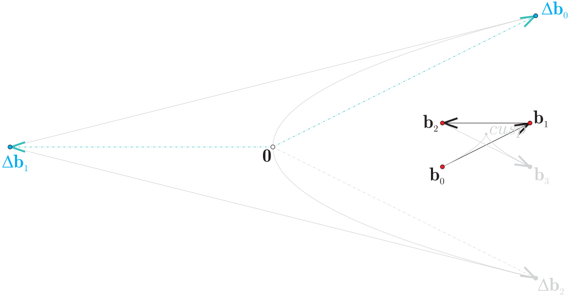


Fig. 8: If the hodograph goes through the origin, then the Bézier curve has a cusp.

Cusps

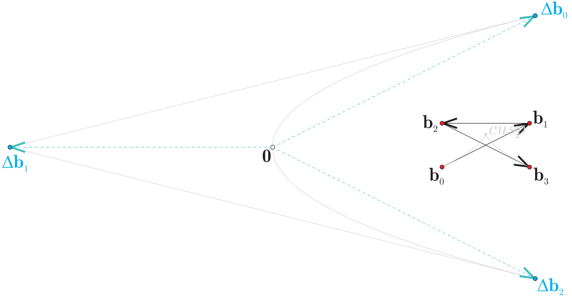
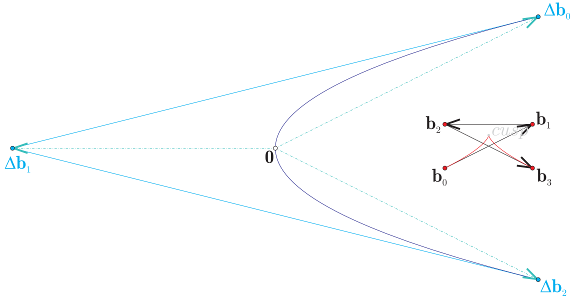


Fig. 8: If the hodograph goes through the origin, then the Bézier curve has a cusp.

Cusps



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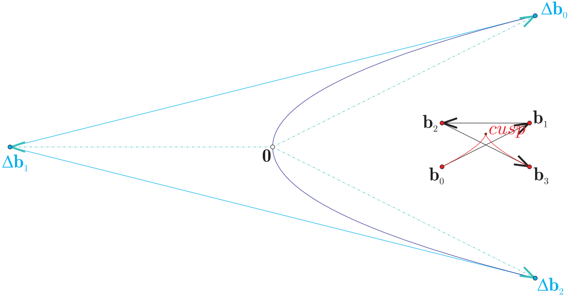
Cusps



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Null curvature points

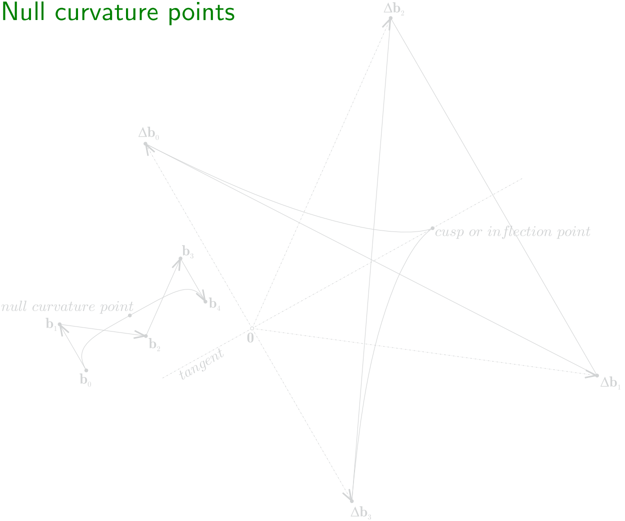
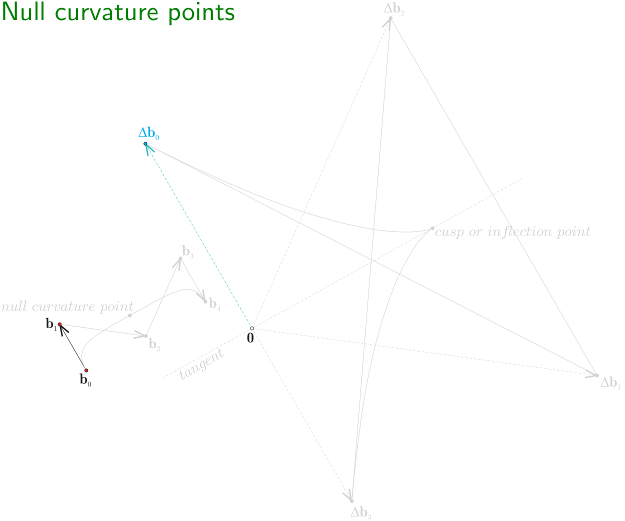


Fig. 9: If the hodograph has a tangent line that goes through the origin and its touching point is either a cusp or an inflection point of the hodograph, then the Bézier curve has a null curvature point.

Null curvature points



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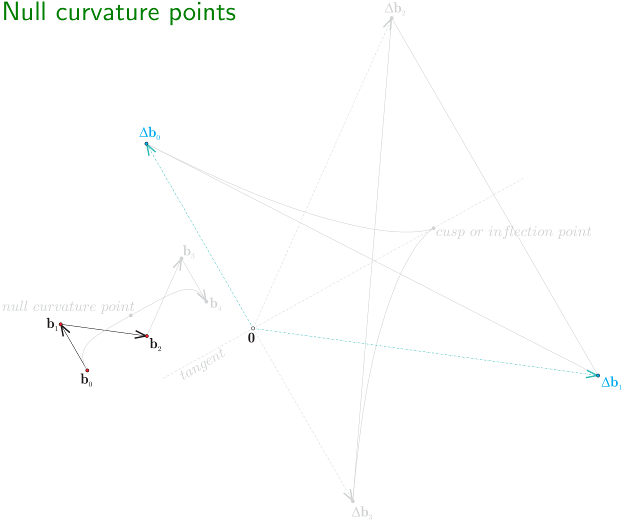
Null curvature points



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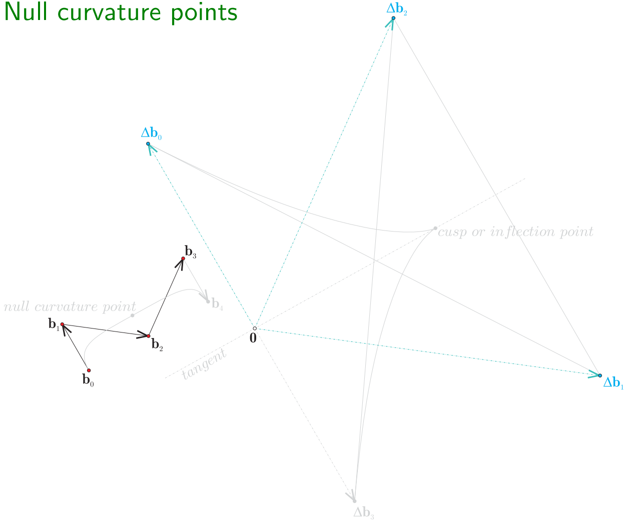

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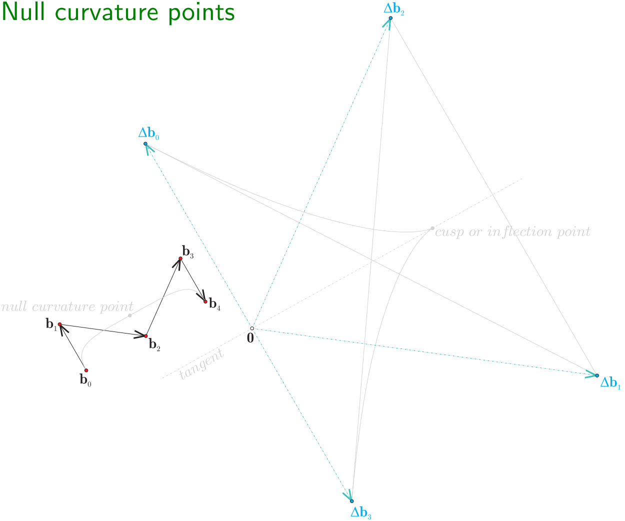
Null curvature points



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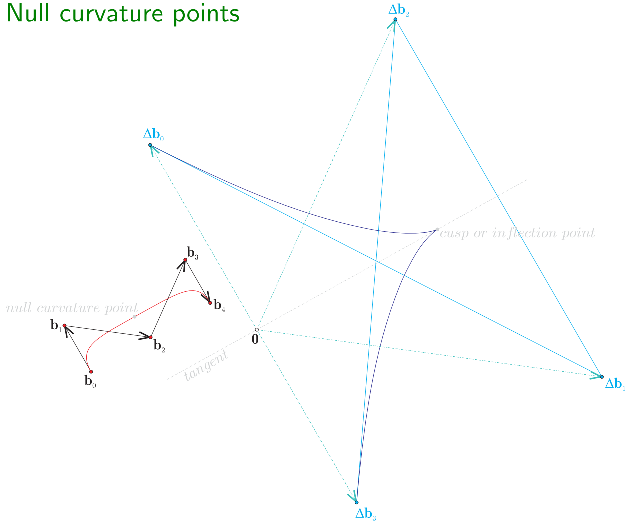
Null curvature points



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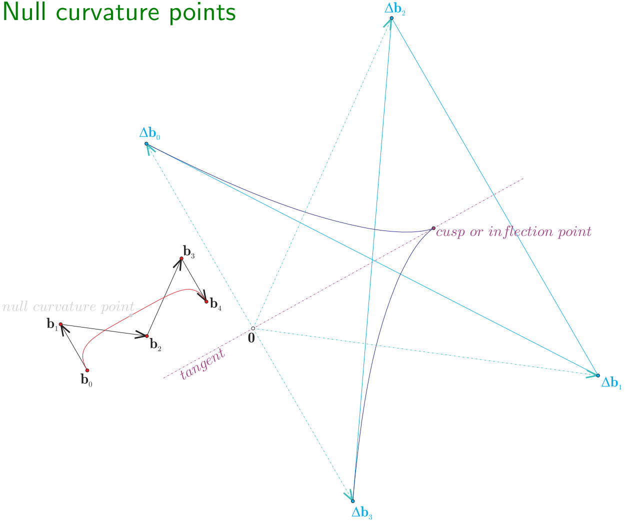
Null curvature points



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Null curvature points

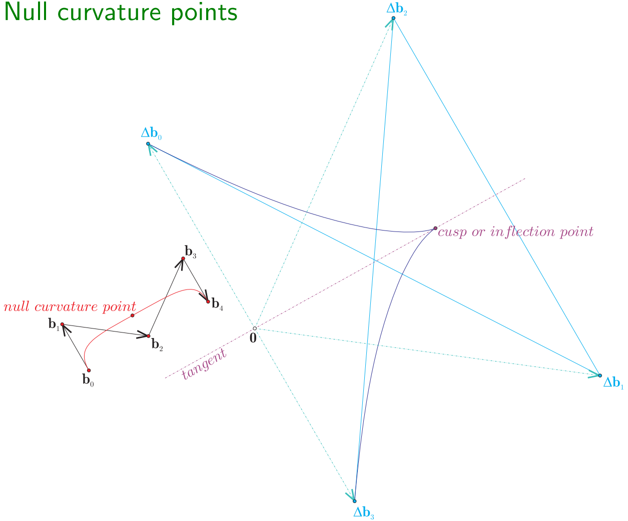

Fig. 9: If the hodograph has a tangent line that goes through the origin and its touching point is either a cusp or an inflection point of the hodograph, then the Bézier curve has a null curvature point.

A differential operator of order $r \geq 0$

$$\begin{cases} \Delta^0 \mathbf{b}_i = \mathbf{b}_i, \\ \Delta^r \mathbf{b}_i = \Delta^{r-1}\mathbf{b}_{i+1} - \Delta^{r-1}\mathbf{b}_i, \ r \geq 1. \end{cases}$$

Example $(r = 1, 2, 3)$

$$\begin{aligned} \Delta^1 \mathbf{b}_i &= \Delta^0 \mathbf{b}_{i+1} - \Delta^0 \mathbf{b}_i \\ &= \mathbf{b}_{i+1} - \mathbf{b}_i, \end{aligned}$$

$$\begin{aligned} \Delta^2 \mathbf{b}_i &= \Delta^1 \mathbf{b}_{i+1} - \Delta^1 \mathbf{b}_i \\ &= (\mathbf{b}_{i+2} - \mathbf{b}_{i+1}) - (\mathbf{b}_{i+1} - \mathbf{b}_i) \\ &= \mathbf{b}_{i+2} - 2\mathbf{b}_{i+1} + \mathbf{b}_i, \end{aligned}$$

$$\begin{aligned} \Delta^3 \mathbf{b}_i &= \Delta^2 \mathbf{b}_{i+1} - \Delta^2 \mathbf{b}_i \\ &= (\mathbf{b}_{i+3} - 2\mathbf{b}_{i+2} + \mathbf{b}_{i+1}) - (\mathbf{b}_{i+2} - 2\mathbf{b}_{i+1} + \mathbf{b}_i) \\ &= \mathbf{b}_{i+3} - 3\mathbf{b}_{i+2} + 3\mathbf{b}_{i+1} - \mathbf{b}_i. \end{aligned}$$

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$$\begin{aligned} \Delta^1 \mathbf{b}_i &= \Delta^0 \mathbf{b}_{i+1} - \Delta^0 \mathbf{b}_i \\ &= \mathbf{b}_{i+1} - \mathbf{b}_i, \end{aligned}$$

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Properties of Bézier curves

A differential operator of order $r \geq 0$

$$\begin{cases} \Delta^0 \mathbf{b}_i = \mathbf{b}_i, \\ \Delta^r \mathbf{b}_i = \Delta^{r-1}\mathbf{b}_{i+1} - \Delta^{r-1}\mathbf{b}_i, \ r \geq 1. \end{cases}$$

Example $(r = 1, 2, 3)$

$$\begin{aligned} \Delta^1 \mathbf{b}_i &= \Delta^0 \mathbf{b}_{i+1} - \Delta^0 \mathbf{b}_i \\ &= \mathbf{b}_{i+1} - \mathbf{b}_i, \end{aligned}$$

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Theorem (Closed formula of the differential operator $\Delta^r \mathbf{b}_i, r \geq 0$)

$$\Delta^r \mathbf{b}_i = \sum_{k=0}^{r} \binom{r}{k} (-1)^{r-k} \mathbf{b}_{i+k}.$$

Proof.

- We prove the theorem by mathematical induction with respect to r.
- The statement trivially holds for $r = 0$ and $r = 1$.
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- We can successively write:

$$\begin{aligned}
\Delta^r \mathbf{b}_i &= \Delta^{r-1} \mathbf{b}_{i+1} - \Delta^{r-1} \mathbf{b}_i \\
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&\overset{k \to k-1}{=} \sum_{k=1}^{r} \binom{r-1}{k-1} (-1)^{r-k} \mathbf{b}_{i+k} + \sum_{k=0}^{r-1} \binom{r-1}{k} (-1)^{r-k} \mathbf{b}_{i+k}
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Proof – continued.

- Since $\binom{r-1}{-1} = \binom{r-1}{r} = 0$ and $\binom{r}{k} = \binom{r-1}{k} + \binom{r-1}{k-1}$, it follows that

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Theorem (Higher order derivatives of Bézier curves)

The rth ($r \geq 0$) order derivative of a Bézier curve of degree $n \geq 1$ is

$$\frac{d^r}{du^r}\mathbf{b}(u) = \frac{n!}{(n-r)!}\sum_{i=0}^{n-r}\Delta^r \mathbf{b}_i B_i^{n-r}(u), \forall u \in [0,1].$$

Proof.

- The proof is based on mathematical induction with respect to $r \geq 0$.
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$$\begin{aligned}\frac{d}{du}\mathbf{b}(u) &= n\sum_{i=0}^{n-1}\left(\mathbf{b}_{i+1} - \mathbf{b}_i\right) B_i^{n-1}(u) \\ &= \frac{n!}{(n-1)!}\sum_{i=0}^{n-1}\Delta^1 \mathbf{b}_i B_i^{n-1}(u), \forall u \in [0,1].\end{aligned}$$

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Proof – continued.

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$$\begin{aligned}
\frac{d^r}{du^r}\mathbf{b}(u) &= \frac{d}{du}\left(\frac{d^{r-1}}{du^{r-1}}\mathbf{b}(u)\right)\\
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&= \frac{n!}{(n-r+1)!}\sum_{i=0}^{n-r+1}\Delta^{r-1}\mathbf{b}_i(n-r+1)\left(B_{i-1}^{n-r}(u)-B_i^{n-r}(u)\right)\\
&= \frac{n!}{(n-r)!}\left(\sum_{i=0}^{n-r+1}\Delta^{r-1}\mathbf{b}_i B_{i-1}^{n-r}(u)-\sum_{i=0}^{n-r+1}\Delta^{r-1}\mathbf{b}_i B_i^{n-r}(u)\right)\\
&= \frac{n!}{(n-r)!}\left(\sum_{i=1}^{n-r+1}\Delta^{r-1}\mathbf{b}_i B_{i-1}^{n-r}(u)-\sum_{i=0}^{n-r}\Delta^{r-1}\mathbf{b}_i B_i^{n-r}(u)\right)
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&= \frac{n!}{(n-r+1)!}\sum_{i=0}^{n-r+1}\Delta^{r-1}\mathbf{b}_i (n-r+1)\left(B_{i-1}^{n-r}(u) - B_i^{n-r}(u)\right) \\
&= \frac{n!}{(n-r)!}\left(\sum_{i=0}^{n-r+1}\Delta^{r-1}\mathbf{b}_i B_{i-1}^{n-r}(u) - \sum_{i=0}^{n-r+1}\Delta^{r-1}\mathbf{b}_i B_i^{n-r}(u)\right) \\
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\end{aligned}$$

Proof – continued.

- Based on the induction hypothesis:

$$\frac{d^{r-1}}{du^{r-1}}\mathbf{b}(u) = \frac{n!}{(n-r+1)!}\sum_{i=0}^{n-r+1}\Delta^{r-1}\mathbf{b}_i B_i^{n-r+1}(u), \forall u \in [0,1].$$

- We can successively write:

$$\begin{aligned}
\frac{d^r}{du^r}\mathbf{b}(u) &= \frac{d}{du}\left(\frac{d^{r-1}}{du^{r-1}}\mathbf{b}(u)\right) \\
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\end{aligned}$$

Proof – continued.

$$\overset{i-1\to i}{=} \quad \frac{n!}{(n-r)!}\left(\sum_{i=0}^{n-r}\Delta^{r-1}\mathbf{b}_{i+1}B_i^{n-r}(u)-\sum_{i=0}^{n-r}\Delta^{r-1}\mathbf{b}_iB_i^{n-r}(u)\right)$$

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Fig. 10: The direction vectors of the first and second order derivatives at the beginning of a Bézier curve of degree 4.

Derivatives at the endpoints of a Bézier curve of degree *n*

Since

$$\begin{aligned} \frac{d^r}{du^r}\mathbf{b}(0) &= \frac{n!}{(n-r)!}\sum_{i=0}^{n-r}\Delta^r\mathbf{b}_i B_i^{n-r}(0) \\ &= \frac{n!}{(n-r)!}\Delta^r\mathbf{b}_0 \end{aligned}$$

and

$$\begin{aligned} \frac{d^r}{du^r}\mathbf{b}(1) &= \frac{n!}{(n-r)!}\sum_{i=0}^{n-r}\Delta^r\mathbf{b}_i B_i^{n-r}(1) \\ &= \frac{n!}{(n-r)!}\Delta^r\mathbf{b}_{n-r}, \end{aligned}$$

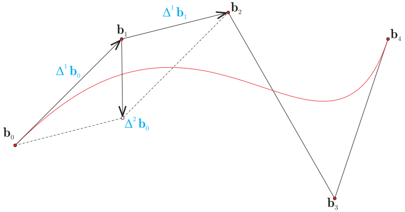
the rth order derivatives at the endpoints of a Bézier curve of degree n depend only on the first

$$\mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_r$$

and the last

$$\mathbf{b}_{n-r}, \mathbf{b}_{n-r+1}, \ldots, \mathbf{b}_n$$

$r+1$ control points.



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$$\frac{d^r}{du^r}\mathbf{b}(0) = \frac{n!}{(n-r)!}\sum_{i=0}^{n-r}\Delta^r\mathbf{b}_i B_i^{n-r}(0) = \frac{n!}{(n-r)!}\Delta^r\mathbf{b}_0$$

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Theorem (The relationship between derivatives and the de Casteljau-algorithm)

The rth ($r \geq 0$) derivative of a Bézier curve of degree $n \geq 1$ can also be represented as

$$\frac{d^r}{du^r}\mathbf{b}(u) = \frac{d^r}{du^r}\mathbf{b}_0^n(u) = \frac{n!}{(n-r)!}\Delta^r \mathbf{b}_0^{n-r}(u),\ \forall u \in [0,1],$$

where $\mathbf{b}_i^k$ are intermediate points of the de Casteljau-algorithm.

Proof.

- We will use the fact that the summation and difference operators are commutable, e.g.

$$\sum_{j=0}^{n-1} \Delta^1 \mathbf{b}_j = \sum_{j=0}^{n-1} (\mathbf{b}_{j+1} - \mathbf{b}_j) = \sum_{j=1}^{n} \mathbf{b}_j - \sum_{j=0}^{n-1} \mathbf{b}_j = \Delta^1 \sum_{j=0}^{n-1} \mathbf{b}_j.$$

- Thus, we can write successively

$$\begin{aligned}\frac{d^r}{du^r}\mathbf{b}(u) = \frac{d^r}{du^r}\mathbf{b}_0^n(u) &= \frac{n!}{(n-r)!}\sum_{j=0}^{n-r} \Delta^r \mathbf{b}_j B_j^{n-r}(u) = \frac{n!}{(n-r)!}\Delta^r \underbrace{\sum_{j=0}^{n-r} \mathbf{b}_j B_j^{n-r}(u)}_{\mathbf{b}_0^{n-r}(u)} \\ &= \frac{n!}{(n-r)!}\Delta^r \mathbf{b}_0^{n-r}(u),\ \forall u \in [0,1]. \quad \square\end{aligned}$$

Properties of Bézier curves

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Bézier curves joined with C^0 continuity

Consider two Bézier curves

$$\mathbf{a}(u) = \sum_{i=0}^{n} \mathbf{a}_i B_i^n \left(\frac{u - u_0}{u_1 - u_0} \right), \ u \in [u_0, u_1]$$

and

$$\mathbf{b}(u) = \sum_{i=0}^{n} \mathbf{b}_i B_i^n \left(\frac{u - u_1}{u_2 - u_1} \right), \ u \in [u_1, u_2]$$

of degree $n \geq 1$. The condition of C^0 continuity is

$$\mathbf{a}(u_1) = \mathbf{b}(u_1),$$

from which results that

$$\mathbf{a}_n = \mathbf{b}_0.$$

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of degree $n \geq 1$ are joined with C^1 continuity, if they are C^0 continuous (i.e. $\mathbf{a}_n = \mathbf{b}_0$) and

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$$\mathbf{a}(u),\ u \in [u_0, u_1],\ \Delta u_0 = u_1 - u_0,$$

and

$$\mathbf{b}(u),\ u \in [u_1, u_2],\ \Delta u_1 = u_2 - u_1$$

of degree $n \geq 1$ are joined with C^1 continuity, if they are C^0 continuous (i.e. $\mathbf{a}_n = \mathbf{b}_0$) and

$$\frac{d}{du}\mathbf{a}(u_1) = \frac{d}{du}\mathbf{b}(u_1),$$

from which follows that

$$\begin{aligned}
\frac{n!}{(n-1)!}\frac{1}{u_1 - u_0}\sum_{i=0}^{n-1}\Delta^1\mathbf{a}_i B_i^{n-1}(1) &= \frac{n!}{(n-1)!}\frac{1}{u_2 - u_1}\sum_{i=0}^{n-1}\Delta^1\mathbf{b}_i B_i^{n-1}(0),\\
n\frac{1}{\Delta u_0}(\mathbf{a}_n - \mathbf{a}_{n-1}) &= n\frac{1}{\Delta u_1}(\mathbf{b}_1 - \mathbf{b}_0),\\
\frac{\|\mathbf{a}_n - \mathbf{a}_{n-1}\|}{\Delta u_0} &= \frac{\|\mathbf{b}_1 - \mathbf{b}_0\|}{\Delta u_1},\\
\|\Delta^1\mathbf{a}_{n-1}\| : \|\Delta^1\mathbf{b}_0\| &= \Delta u_0 : \Delta u_1.
\end{aligned}$$

Example (Bézier curves joined with C^1 continuity)

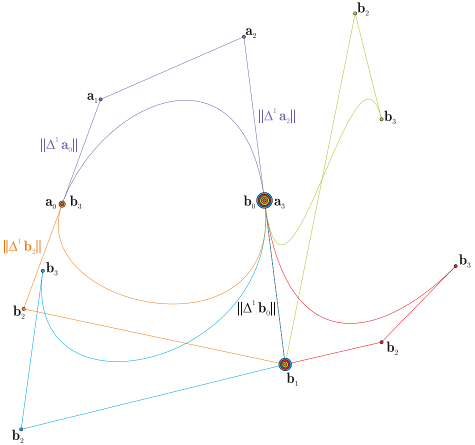

Fig. 11: Several cubic Bézier arcs are joined with C^1 continuity. For the sake of simplicity all cubic arcs are defined on the interval $[0, 1]$, i.e. the joint and its two adjacent control points are collinear and the segments defined by them are congruent.

Bézier curves joined with C^r $(r \geq 0)$ continuity

Bézier curves

$$\mathbf{a}(u) = \sum_{i=0}^{n} \mathbf{a}_i B_i^n\left(\frac{u-u_0}{u_1-u_0}\right), \; u \in [u_0, u_1], \; \Delta u_0 = u_1 - u_0,$$

and

$$\mathbf{b}(u) = \sum_{i=0}^{n} \mathbf{b}_i B_i^n\left(\frac{u-u_1}{u_2-u_1}\right), \; u \in [u_1, u_2], \; \Delta u_1 = u_2 - u_1$$

of degree $n \geq 1$ are joined with C^r $(r \geq 0)$ continuity, if the system of conditions

$$\left(\frac{1}{\Delta u_0}\right)^i \Delta^i \mathbf{a}_{n-i} = \left(\frac{1}{\Delta u_1}\right)^i \Delta^i \mathbf{b}_0, \; i = 0, 1, \ldots, r$$

is fulfilled.

Bézier curves joined with C^r $(r \geq 0)$ continuity

Bézier curves

$$\mathbf{a}(u) = \sum_{i=0}^{n} \mathbf{a}_i B_i^n\left(\frac{u-u_0}{u_1-u_0}\right),\ u \in [u_0, u_1],\ \Delta u_0 = u_1 - u_0,$$

and

$$\mathbf{b}(u) = \sum_{i=0}^{n} \mathbf{b}_i B_i^n\left(\frac{u-u_1}{u_2-u_1}\right),\ u \in [u_1, u_2],\ \Delta u_1 = u_2 - u_1$$

of degree $n \geq 1$ are joined with C^r $(r \geq 0)$ continuity, if the system of conditions

$$\left(\frac{1}{\Delta u_0}\right)^i \Delta^i \mathbf{a}_{n-i} = \left(\frac{1}{\Delta u_1}\right)^i \Delta^i \mathbf{b}_0,\ i = 0, 1, \ldots, r$$

is fulfilled.