Modeling of Bézier curves/surfaces

Part 2

– a CAGD approach based on OpenGL and C++ –

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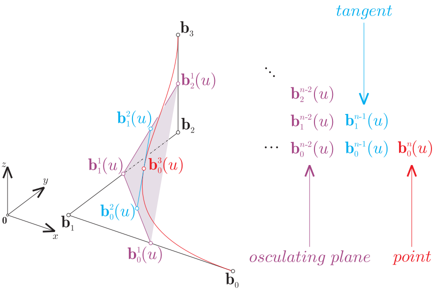

Fig. 1: The point, the tangent vector and the osculating plane of a Bézier curve of degree $n = 3$ at a parameter value $u \in [0, 1]$.

- We have proved $\forall u \in [0, 1]$ that

$$
\begin{aligned}
\frac{d^r}{du^r}\mathbf{b}(u) &= \frac{n!}{(n-r)!}\sum_{i=0}^{n-r}\Delta^r \mathbf{b}_i B_i^{n-r}(u) \\
&= \frac{n!}{(n-r)!}\Delta^r \sum_{i=0}^{n-r}\mathbf{b}_i B_i^{n-r}(u) \\
&= \frac{n!}{(n-r)!}\Delta^r \mathbf{b}_0^{n-r}(u),
\end{aligned}
$$

where the difference

$$\Delta^r \mathbf{b}_0^{n-r}(u)$$

is determined by the de Casteljau points

$$\mathbf{b}_0^{n-r}(u), \mathbf{b}_1^{n-r}(u), \ldots \mathbf{b}_r^{n-r}(u),$$

i.e. by the $(n-r)$th column of the triangular subdivision scheme.

Theorem (Subdivision of Bézier curves)

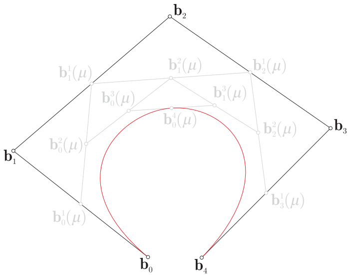

Fig. 2: Subdivision of a Bézier curve of degree 4 at a parameter value $\mu \in (0, 1)$.

Consider the Bézier curve

$$\mathbf{b}(u) = \sum_{i=0}^{n} \mathbf{b}_i B_i^n(u),\ u \in [0, 1]$$

of degree $n \geq 1$ and de Casteljau points

$$\begin{aligned}
\mathbf{b}_i^r(\mu) &= (1-\mu)\mathbf{b}_i^{r-1}(\mu) + \mu\mathbf{b}_{i+1}^{r-1}(\mu), \\
r &= 1, 2, \ldots, n, \\
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which correspond to a fixed parameter $\mu \in (0, 1)$. In this case the Bézier curve

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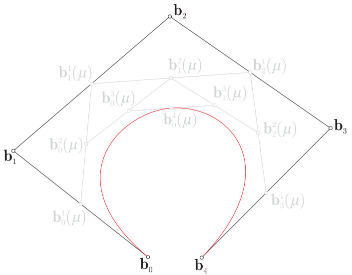

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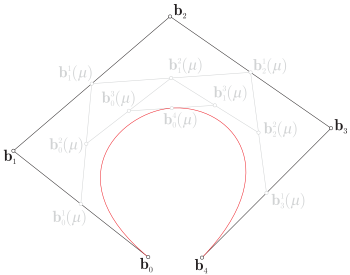

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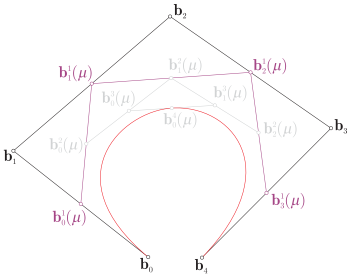

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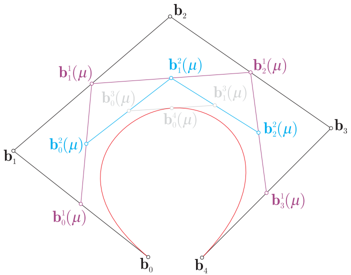

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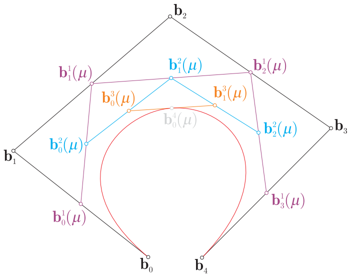

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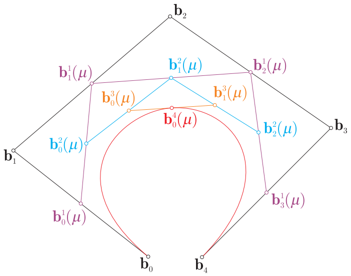

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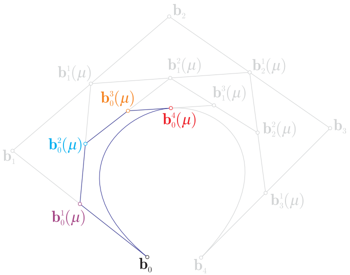

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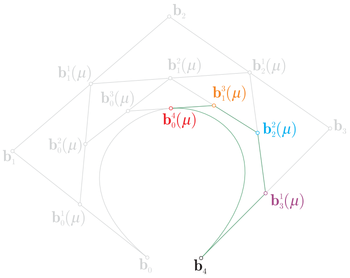

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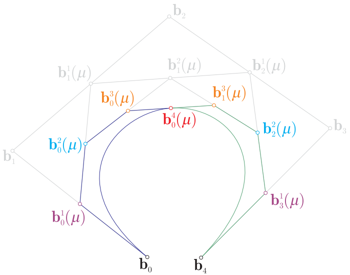

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Proof.

- Given the set of $n+1$ control points

$$\mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_n \in \mathbb{R}^\delta, \delta \geq 2$$

and a parameter value $\mu \in [0,1]$, we want to find two sets of $n+1$ control points

$$\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_n \in \mathbb{R}^\delta$$

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such that the Bézier curve defined by the control polygon $[\mathbf{p}_i]_{i=0}^n$ (resp. by $[\mathbf{q}_i]_{i=0}^n$) is the piece of the original Bézier curve on $[0,\mu]$ (resp. on $[\mu,1]$).

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Proof – continued.

- By evaluating the condition

$$\left.\frac{d^r}{du^r}\mathbf{b}^n(u)\right|_{u=0} = \left.\frac{d^r}{du^r}\mathbf{p}^n\left(\frac{u}{\mu}\right)\right|_{u=0}$$

  we get

$$\begin{aligned}
\frac{n!}{(n-r)!}\sum_{i=0}^{n-r}\Delta^r\mathbf{b}_i B_i^{n-r}(0) &= \frac{1}{\mu^r}\frac{n!}{(n-r)!}\sum_{i=0}^{n-r}\Delta^r\mathbf{p}_i B_i^{n-r}(0),\\
\frac{n!}{(n-r)!}\Delta^r\mathbf{b}_0 &= \frac{1}{\mu^r}\frac{n!}{(n-r)!}\Delta^r\mathbf{p}_0,\\
\Delta^r\mathbf{b}_0 &= \frac{1}{\mu^r}\Delta^r\mathbf{p}_0
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- As you can see, these derivatives depend only on points $[\mathbf{b}_j]_{j=0}^r$ and $[\mathbf{p}_j]_{j=0}^r$ which can be imagined as the control points of two Bézier curves of degree $r = 0, 1, \ldots, n$.
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Proof – continued.

- By evaluating the condition

$$\left.\frac{d^r}{du^r}\mathbf{b}^n(u)\right|_{u=0} = \left.\frac{d^r}{du^r}\mathbf{p}^n\left(\frac{u}{\mu}\right)\right|_{u=0}$$

we get

$$\begin{aligned}
\frac{n!}{(n-r)!}\sum_{i=0}^{n-r}\Delta^r\mathbf{b}_i B_i^{n-r}(0) &= \frac{1}{\mu^r}\frac{n!}{(n-r)!}\sum_{i=0}^{n-r}\Delta^r\mathbf{p}_i B_i^{n-r}(0),\\
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Subdivision of Bézier curves

Triangular scheme of the de Casteljau algorithm – revisited

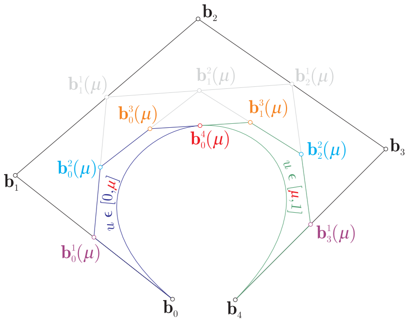

Fig. 3: Subdivision of a Bézier curve of degree 4 at a parameter value $\mu \in (0,1)$.

$$
\begin{array}{llllll}
\mathbf{b}_0 & & & & & \\
\mathbf{b}_1 & \mathbf{b}_0^1(\mu) & & & & \\
\mathbf{b}_2 & \mathbf{b}_1^1(\mu) & \mathbf{b}_0^2(\mu) & & & \\
\mathbf{b}_3 & \mathbf{b}_2^1(\mu) & \mathbf{b}_1^2(\mu) & \ddots & & \\
\vdots & \vdots & \vdots & & \mathbf{b}_0^{n-1}(\mu) & \\
\mathbf{b}_n & \mathbf{b}_{n-1}^1(\mu) & \mathbf{b}_{n-2}^2(\mu) & \cdots & \mathbf{b}_1^{n-1}(\mu) & \mathbf{b}_0^n(\mu) = \mathbf{b}_0^n(\mu)
\end{array}
$$

Subdivision of Bézier curves

- Points involved in the de Casteljau algorithm are in the convex hull of the original Bézier curve, thus, by subdividing the original curve, we get two Bézier curves the convex hulls of which are real subsets of the convex hull of the original curve.
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Theorem (Variation diminishing)

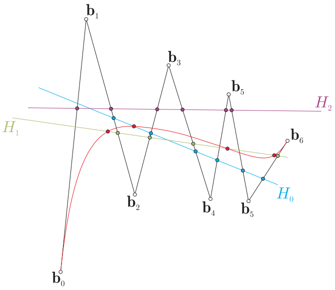

Fig. 4: Variation diminishing.

No hyperplane can intersect a Bézier curve more times than its control polygon.

Proof.

- The de Casteljau algorithm repeatedly cuts the "corners" of the control polygon.
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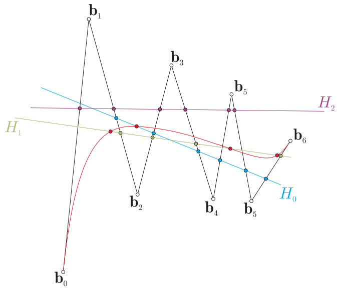

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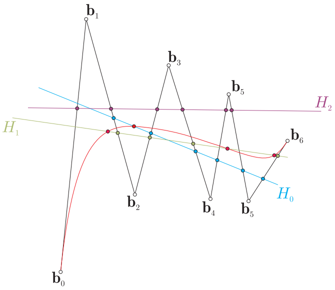

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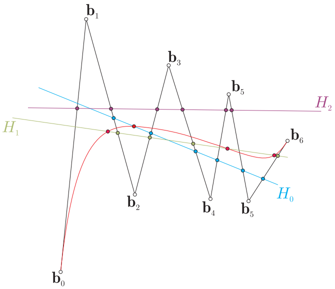

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Convex/concave Bézier curve

A Bézier curve it is convex, if the image of the curve and the segment that connects its endpoints determine a convex two-dimensional figure. A Bézier curve it is concave if it is not convex.

Theorem (Convexity preserving)

If the control polygon of a Bézier is convex, then the curve is also convex.

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A convex Bézier curve may have a concave control polygon!

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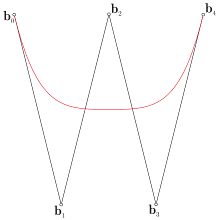

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A fast Bézier curve approximating algorithm

- *Input:* a control polygon $[\mathbf{b}_i]_{i=0}^n \in \mathcal{M}_{1,n+1}(\mathbb{R}^\delta)$, $\delta \geq 2$, $n \geq 1$ and an error level $\varepsilon \in (0, 1)$.
- *Output:* a binary tree the leafs of which contain segments; the union of these segments approximates (with the error level ε) the image of the Bézier curve determined by the given control polygon.

1. If $\mathbf{b}_0 = \mathbf{b}_n$ (i.e. if the Bézier curve is closed), then subdivide it at the parameter value $\mu = \frac{1}{2}$ and perform the next steps of the algorithm on the resulted two new Bézier curves of degree n.
2. Consider the segment $\mathbf{b}_0\mathbf{b}_n$ and determine the farthest control point $\mathbf{b}_i$ $(0 < i < n)$ from it. Let us denote by d_i the absolute value of the signed distance of the control point $\mathbf{b}_i$ from the segment $\mathbf{b}_0\mathbf{b}_n$.
3. If $d_i \leq \varepsilon$, then substitute the image of the current Bézier curve with the segment $\mathbf{b}_0\mathbf{b}_n$. Otherwise, subdivide the current Bézier curve at the parameter value $\mu = \frac{i}{n}$ into two new Bézier curves of degree n and apply Step 2 to both of them.

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Analysis of the approximating algorithm

- Let us denote by $\mathbf{c}$ the original Bézier curve of degree n, by $\mathbf{c}_0$ and $\mathbf{c}_1$ the first two Bézier curves of degree n resulted from the first subdivision step. Similarly, denote by $\mathbf{c}_{00}$ and $\mathbf{c}_{01}$ the Bézier curves of degree n which are the results of the potential subdivision of curve $\mathbf{c}_0$, and so on...

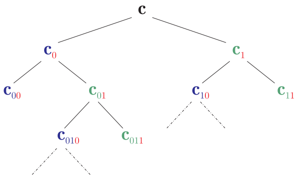

Fig. 6: Construction of a binary tree. By ordering the leaf nodes from left to right and by joining the first and last points of the control polygons stored in each leaf node, we can approximate the shape of the original Bézier curve within the given error level ε.

- In the 3rd step of the algorithm we could use any parameter value $\mu \in [0, 1]$ for the subdivision process, however, we can speed up the algorithm with the setting $\mu = \frac{i}{n}$, since the maximal effect on the shape of the curve generated by the current farthest control point $\mathbf{b}_i$ is attained at this parameter value.

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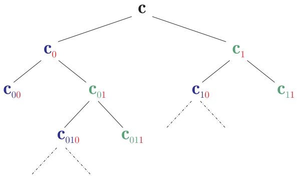

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Analysis of the approximating algorithm – continued

- Thus, with this setting the algorithm produces fewer segments in the leaf nodes, moreover, also follows the variation of the curvature of the original Bézier curve.
- The algorithm can also be used to approximate the length of the original Bézier curve:
  - let L_c be the length of the approximating path consisting of the segments that are determined by the first and last control points of the Bézier curves stored in the leaf nodes;
  - let L_p be the total length of the control polygons of the Bézier curves stored in the leaf nodes;
  - it can be proved, that from the set of all expressions which can be formed with L_c and L_p, the expression

$$L = \frac{2L_c + (n-1)L_p}{n+1}$$

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Theorem (Length diminishing)

The system

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of Bernstein polynomials of degree $n \geq 1$ is length diminishing with respect to any norm $\|\cdot\|$.

Proof.

Using a similar train of thought as in case of variation diminishing, is it simple to realize that

$$L\left[\mathbf{b}(u) = \sum_{i=0}^{n} \mathbf{b}_i B_i^n(u)\right] \leq L\left[\mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_n\right],$$

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$$\mathbf{b}(u) = \sum_{i=0}^{n} \mathbf{b}_i B_i^n(u) = \sum_{i=0}^{n} \mathbf{b}_i \binom{n}{i} u^i (1-u)^{n-i}, \ u \in [0,1]$$

  of degree $n \geq 1$ and a parameter value $\mu \in [0,1]$.

- Calculate in advance the binomial coefficients $\left\{\binom{n}{i}\right\}_{i=0}^{n}$ and also form the new points

$$[\mathbf{p}_i]_{i=0}^{n} = \left[\mathbf{b}_i \binom{n}{i}\right]_{i=0}^{n}.$$

- If $\mu < 1$, then we can form a new polynomial of degree n:

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Horner algorithm – continued

Fig. 7: Continuous parameter transformation.

- If $\mu > 0$, then we can also form a new polynomial of degree n:

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- The parameter transformation

$$f(\mu) = \begin{cases} s_\mu = \frac{\mu}{1-\mu}, \ \mu \in \left[0, \frac{1}{2}\right] \\ t_\mu = \frac{1-\mu}{\mu}, \ \mu \in \left[\frac{1}{2}, 1\right] \end{cases}$$

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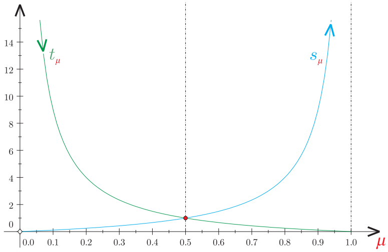

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- Thus, when $\mu \in \left[0, \frac{1}{2}\right]$ we can apply the Horner algorithm to the polynomial
$$\sum_{i=0}^{n} \mathbf{p}_i s_{\mu}^{i},$$
and when $\mu \in \left(\frac{1}{2}, 1\right]$ we use the Horner algorithm to evaluate the polynomial
$$\sum_{i=0}^{n} \mathbf{p}_{n-i} t_{\mu}^{i}.$$
- After evaluating the above polynomials, we multiply the result with the previously stored and corresponding power $(1-\mu)^n$ or μ^n.

Remark (Basis)

From the reparametrization used in the Horner algorithm also results, that the system
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- Thus, when $\mu \in \left[0, \frac{1}{2}\right]$ we can apply the Horner algorithm to the polynomial

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Degree elevation of Bézier curves

- Although Bézier curves of higher degree require longer time to process, they do have higher flexibility for designing shapes.
- Therefore, it would be very helpful to increase the degree of a Bézier curve (i.e. to introduce new control points) without changing its shape.

Theorem/algorithm – degree elevation

The control polygons

$$[\mathbf{b}_i]_{i=0}^{n} \in \mathcal{M}_{1,n+1}\left(\mathbb{R}^{\delta}\right),\ \delta \geq 2,\ n \geq 1$$

and

$$\left[\mathbf{b}_i^1\right]_{i=0}^{n+1} = \left[\mathbf{b}_0, \left[\mathbf{b}_i + \frac{i}{n+1}\left(\mathbf{b}_{i-1} - \mathbf{b}_i\right)\right]_{i=1}^{n}, \mathbf{b}_n\right] \in \mathcal{M}_{1,n+2}\left(\mathbb{R}^{\delta}\right)$$

generate exactly the same shape, i.e.

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Degree elevation of Bézier curves

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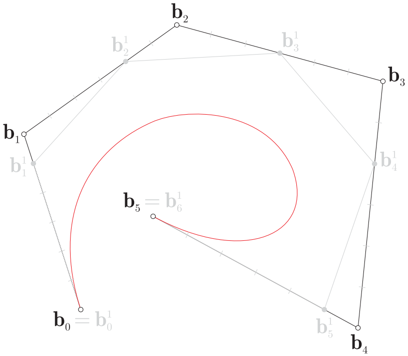
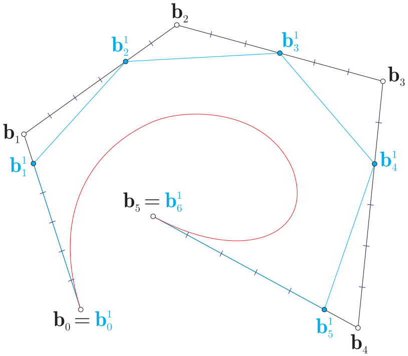


Fig. 8: Degree elevation of a Bézier curve of degree 5.



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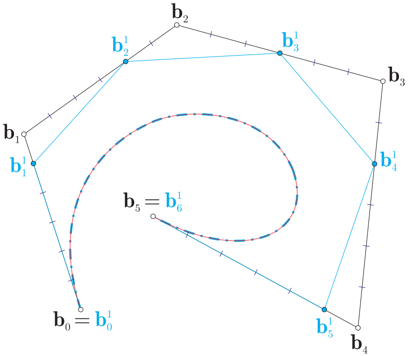

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Convergence

By repeated degree elevation we get a sequence

$$\left\{ \left[\mathbf{b}_i^r\right]_{i=0}^r \in \mathcal{M}_{1,r+1}\left(\mathbb{R}^\delta\right) \right\}_{r \geq n+1}$$

of control polygons that converges to the original Bézier curve determined by the original control polygon $\left[\mathbf{b}_i^r\right]_{i=0}^n$ as $r \to \infty$.
However, the rate of this convergence is much slower than of the sequence of control polygons obtained by subdivision.

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Variation diminishing can also be proved using the convergence property of control polygons obtained by repeated degree elevation.

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Remark

Degree reduction is the inverse operation of degree elevation.
In general, this operation is impossible. However, there are methods (e.g. least squares) using which we can approximate the shape of the original Bézier curve with another one of lower degree.

- We know that the system

$$M = \left\{ u^i : u \in \mathbb{R} \right\}_{i=0}^{n}$$

of the monomials and the system

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- In what follows, we will study the connection between these bases.

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Theorem (Basis transformation)

The connection between basis B and U of the vector space $\mathcal{P}_n$ can be expressed as

$$\begin{bmatrix} B_0^n(u) & B_1^n(u) & \cdots & B_n^n(u) \end{bmatrix} = \begin{bmatrix} u^n & u^{n-1} & \cdots & 1 \end{bmatrix} T,$$

where

$$T = \left[t_{ij}\right]_{i=0,j=0}^{n,n} = \left[(-1)^{n-j-i}\binom{n}{j}\binom{n-j}{i}\right]_{i=0,j=0}^{n,n}$$

is the transformation matrix.

Proof.

- We need to determine the elements t_{ij} such that the equalities

$$B_j^n(u) = \sum_{i=0}^{n} t_{ij} u^{n-i}$$

hold for all $j = 0, 1, \ldots, n$.

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Theorem (Basis transformation)

The connection between basis B and U of the vector space $\mathcal{P}_n$ can be expressed as

$$\begin{bmatrix} B_0^n(u) & B_1^n(u) & \cdots & B_n^n(u) \end{bmatrix} = \begin{bmatrix} u^n & u^{n-1} & \cdots & 1 \end{bmatrix} T,$$

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The matrix representation of a Bézier curve $\mathbf{b}(u) = \sum_{i=0}^{n} \mathbf{b}_i B_i^n(u)$ of degree $n \geq 1$ is

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Theorem (Inverse basis transformation)

The monomial u^i can be expressed as

$$u^i = \frac{1}{\binom{n}{i}} \sum_{k=0}^{n} \binom{k}{i} B_k^n(u).$$

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- We will use the identity

$$1 \equiv [u+(1-u)]^{n-i} = \sum_{j=0}^{n-i} \binom{n-i}{j} u^j (1-u)^{n-i-j}, \forall u \in [0,1].$$

- Thus, we can successively write:

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Application (Bernstein representation of a polynomial of finite degree)

The polynomial

$$\mathbf{p}(u) = \sum_{i=0}^{n} \mathbf{p}_i u^i,\ \mathbf{p}_i \in \mathbb{R}^\delta, \delta \geq 2, u \in [0,1]$$

can be rewritten as

$$\begin{aligned}
\sum_{i=0}^{n} \mathbf{p}_i u^i &= \sum_{i=0}^{n} \frac{\mathbf{p}_i}{\binom{n}{i}} \sum_{k=0}^{n} \binom{k}{i} B_k^n(u) \\
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Effect of multiple neighboring control points

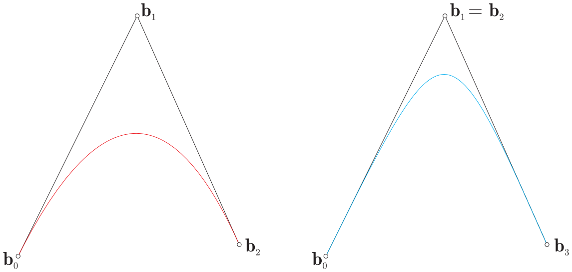

Fig. 9: Multiple neighboring control points attract the shape of the curve towards themeselves. This property follows either from the de Casteljau algorithm, or from the fact that such control point have greater weights/coefficients in the Bernstein representation of the Bézier curve.

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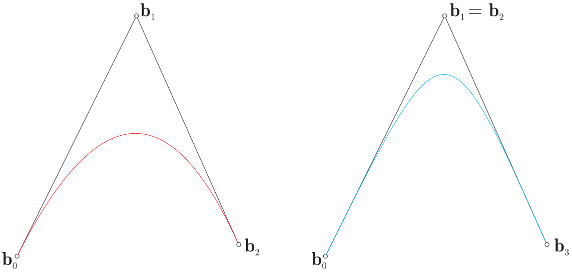

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Interpolation of data points by means of Bézier curves

- Consider the sequence of points

$$\mathbf{d}_j \in \mathbb{R}^\delta \ (j = 0, 1, \ldots, n)$$

called data points and associated parameter values

$$0 \leq u_0 < u_1 < \cdots < u_n \leq 1.$$

- We will refer to the sequence

$$\{(u_j, \mathbf{d}_j)\}_{j=0}^{n} \in \mathcal{M}_{1,n+1}\left([0,1] \times \mathbb{R}^\delta\right)$$

as nodes.

- The task is to find control points $\mathbf{b}_i \in \mathbb{R}^\delta$ $(i = 0, 1, \ldots, n)$ for which the interpolation conditions

$$\mathbf{b}(u_j) = \sum_{i=0}^{n} \mathbf{b}_i B_i^n(u_j) = \mathbf{d}_j, \ j = 0, 1, \ldots, n$$

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$$\underbrace{\begin{bmatrix} B_0^n(u_0) & B_1^n(u_0) & \cdots & B_n^n(u_0) \\ B_0^n(u_1) & B_1^n(u_1) & \cdots & B_n^n(u_1) \\ \vdots & \vdots & \ddots & \vdots \\ B_0^n(u_n) & B_1^n(u_n) & \cdots & B_n^n(u_n) \end{bmatrix}}_{\text{collocation matrix}} \begin{bmatrix} \mathbf{b}_0 \\ \mathbf{b}_1 \\ \vdots \\ \mathbf{b}_n \end{bmatrix} = \begin{bmatrix} \mathbf{d}_0 \\ \mathbf{d}_1 \\ \vdots \\ \mathbf{d}_n \end{bmatrix},$$

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Fig. 10: A Bézier curve of degree 5 that interpolates the nodes $\left(0, [-1, -1, -1]^T\right)$, $\left(\frac{1}{4}, [-1, -1, 1]^T\right)$, $\left(\frac{1}{2}, [-1, 1, 1]^T\right)$, $\left(\frac{3}{4}, [1, 1, 1]^T\right)$, and $\left(1, [1, -1, 1]^T\right)$.

General sweep

A surface may be defined by moving a curve in space such that during the motion we allow the shape modification of the moving curve. This description mode of surfaces is called general sweep.

- Suppose that we move in space the Bézier curve

$$\mathbf{a}(u) = \sum_{i=0}^{n} \mathbf{a}_i B_i^n(u),\ u \in [0,1]$$

of degree $n \geq 1$ that is defined by the control polygon

$$[\mathbf{a}_i]_{i=0}^{n} \in \mathcal{M}_{1,n+1}\left(\mathbb{R}^3\right).$$

- Assume that control points $\mathbf{a}_i$ $(i = 0, 1, \ldots, n)$ also move along Bézier curves of degree $m \geq 1$.
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Fig. 1: *u* and *v* isoparametric lines of a Bézier surface of degree $(3,3)$ at the point $(\tilde{u},\tilde{v}) \in [0,1]\times[0,1]$.

- Due to the endpoint interpolation property of Bézier curves, we have that

$$\mathbf{b}_{i0} = \mathbf{a}_i(0),\ i = 0,1,\ldots,n.$$

- Under these conditions we can define the surface

$$\begin{aligned}\mathbf{s}(u,v) &= \sum_{i=0}^{n} \mathbf{a}_i(v) B_i^n(u) \\ &= \sum_{i=0}^{n} \left(\sum_{j=0}^{m} \mathbf{b}_{ij} B_j^m(v) \right) B_i^n(u) \\ &= \sum_{i=0}^{n}\sum_{j=0}^{m} \mathbf{b}_{ij} B_i^n(u) B_j^m(v),\ (u,v) \in [0,1]\times[0,1].\end{aligned}$$

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Bézier surfaces

Tensor product form

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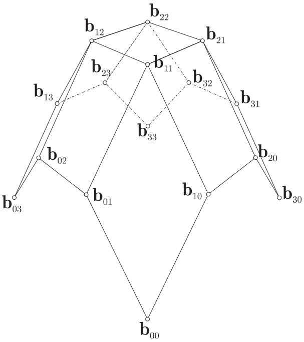

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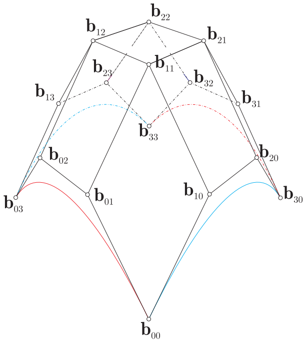

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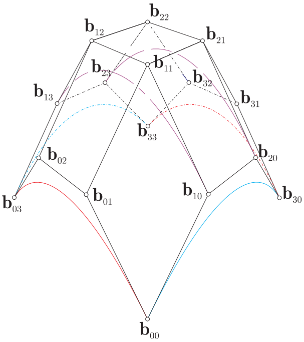

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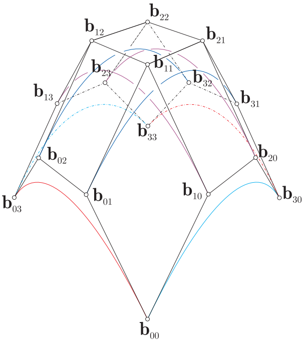

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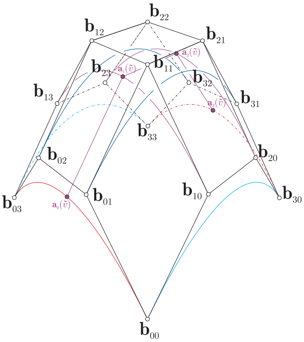

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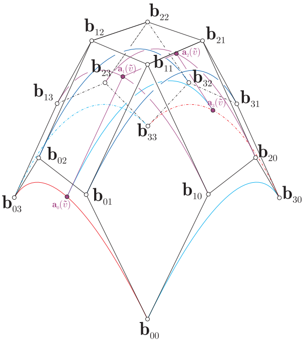

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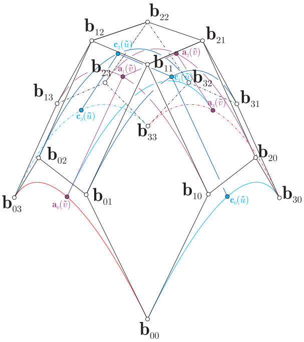

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Bézier surfaces

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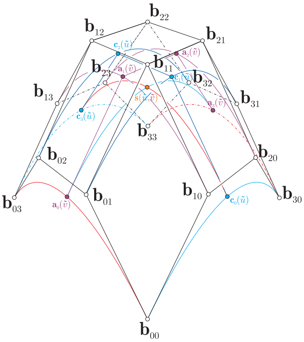

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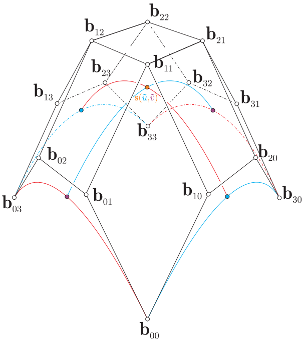

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Boundary curves and symmetry

The boundary curves of the Bézier patch **s** of degree (n, m) are polynomial curves. Their Bézier polygons are given by the boundary polygons of the control net. In particular, the four corners of the control net all lie on the patch. By moving the control points of the boundary curves along Bézier curves determined by the corresponding rows or columns of the control net, we generate the same Bézier surface/patch of degree (n, m).

Affine invariance

The direct de Casteljau algorithm consists of repeated bilinear and possibly subsequent repeated linear interpolation. All these operations are affinely invariant; hence, so is their composition. Another way to prove this property is to show that

$$\sum_{i=0}^{n} \sum_{j=0}^{m} B_i^n(u) B_j^m(v) \equiv 1 \,, \forall (u, v) \in [0, 1] \times [0, 1].$$

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Remark

Similarly to Bézier curves there is no projective invariance of Bezier surfaces! In particular, we cannot apply a perspective projection to the control net and then plot the surface that is determined by the resulting image. Such operations will be possible by means of rational Bezier surfaces.

Convex hull

A Bézier surface of degree (n, m) lies in the convex hull of its control net, since for all $0 \leq u, v \leq 1$ the factors $B_i^n(u)B_j^m(v)$ are nonnegative and their sum is equal to 1, i.e. the parametric representation of the surface is a convex combination of its control points.

Variation diminishing

This property is not inherited from the univariate case. In fact, it is not at all clear what the definition of variation diminution should be in the bivariate case.

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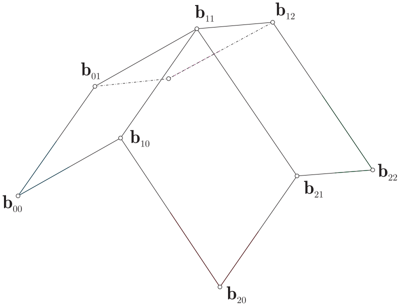

Fig. 2: Subdivision of a biquadratic Bézier surface.

By the help of the de Casteljau algorithm we can cut a Bézier surface at a point

$$(\tilde{u}, \tilde{v}) \in (0,1) \times (0,1)$$

in four Bézier surfaces the degrees of which coincide with the degree of the original surface. Warning: in general, the computational costs in u and v directions are not the same!

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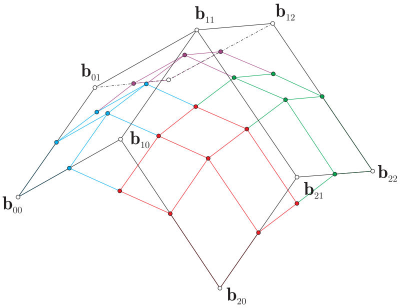

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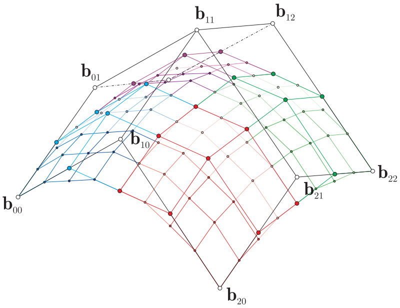

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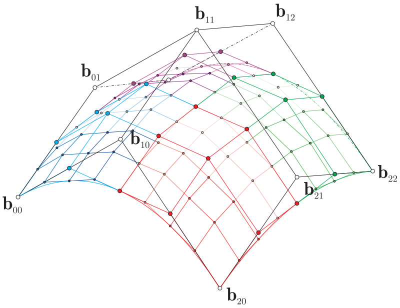

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Invariance under linear transformations of the parameter domain

The shape of a Bézier surface is invariant under affine transformations of the parameter domain, i.e. we can apply the parameter transformations

$$\frac{u-a}{b-a} \text{ and } \frac{v-c}{d-c}$$

from the rectangle $[a, b] \times [c, d]$ to the unit square $[0, 1] \times [0, 1]$ without changing the shape of the surface.

- Suppose you want to describe exactly the Bézier surface

$$\mathbf{s}(u,v) = \sum_{i=0}^{n}\sum_{j=0}^{m} \mathbf{b}_{ij} B_i^n(u) B_j^m(v),\ (u,v) \in [0,1]\times[0,1]$$

of degree (n, m) with another Bézier surface

$$\mathbf{s}^{0,1}(u,v) = \sum_{i=0}^{n}\left(\sum_{j=0}^{m+1} \mathbf{b}_{ij}^{0,1} B_j^{m+1}(v)\right) B_i^n(u),\ (u,v) \in [0,1]\times[0,1]$$

of degree $(n, m+1)$.

- This means that we have to elevate the degree of $n+1$ Bézier curves of degree m, i.e.

$$\mathbf{b}_{ij}^{0,1} = \left(1 - \frac{j}{m+1}\right)\mathbf{b}_{ij} + \frac{j}{m+1}\mathbf{b}_{i,j-1};\ j = 0,1,\ldots,m+1;\ i = 0,1,\ldots,n.$$

- If you need to raise the degree in both directions (e.g. first in v- and then in u-direction), then you can proceed similarly, i.e.

$$\mathbf{b}_{ij}^{1,1} = \left(1 - \frac{i}{n+1}\right)\mathbf{b}_{ij}^{0,1} + \frac{i}{n+1}\mathbf{b}_{i-1,j}^{0,1};\ i = 0,1,\ldots,n+1;\ j = 0,1,\ldots,m+1.$$

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- If you need to raise the degree in both directions (e.g. first in v- and then in u-direction), then you can proceed similarly, i.e.

$$\mathbf{b}_{ij}^{1,1} = \left(1 - \frac{i}{n+1}\right)\mathbf{b}_{ij}^{0,1} + \frac{i}{n+1}\mathbf{b}_{i-1,j}^{0,1};\ i = 0,1,\ldots,n+1;\ j = 0,1,\ldots,m+1.$$

- Notice that control points $\mathbf{b}_{ij}^{1,1}$ may be found in a one-step method:

$$\mathbf{b}_{ij}^{1,1} = \begin{bmatrix} \frac{i}{n+1} & 1 - \frac{i}{n+1} \end{bmatrix} \begin{bmatrix} \mathbf{b}_{i-1,j-1} & \mathbf{b}_{i-1,j} \\ \mathbf{b}_{i,j-1} & \mathbf{b}_{i,j} \end{bmatrix} \begin{bmatrix} \frac{j}{m+1} \\ 1 - \frac{j}{m+1} \end{bmatrix},$$

$i = 0, 1, \ldots, n+1$; $j = 0, 1, \ldots, m+1$.

Theorem (Partial derivatives)

The $(r+s)$th order mixed partial derivative of the Bézier surface

$$\mathbf{s}(u,v) = \sum_{i=0}^{n}\sum_{j=0}^{m} \mathbf{b}_{ij} B_i^n(u) B_j^m(v),\ (u,v) \in [0,1]\times[0,1]$$

is

$$\frac{\partial^{r+s}}{\partial u^r \partial v^s}\mathbf{s}(u,v) = \frac{n!m!}{(n-r)!(m-s)!}\sum_{i=0}^{n-r}\sum_{j=0}^{m-s} \Delta^{r,s}\mathbf{b}_{ij} B_i^{n-r}(u) B_j^{m-s}(v).$$

Remark

The first order partial derivative with respect to the parameter u is

$$\frac{\partial}{\partial u}\mathbf{s}(u,v) = n\sum_{i=0}^{n-1}\sum_{j=0}^{m} \Delta^{1,0}\mathbf{b}_{ij} B_i^{n-1}(u) B_j^m(v),$$

where

$$\Delta^{1,0}\mathbf{b}_{ij} = \mathbf{b}_{i+1,j} - \mathbf{b}_{ij}.$$

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Theorem (Joining with C^r-continuity)

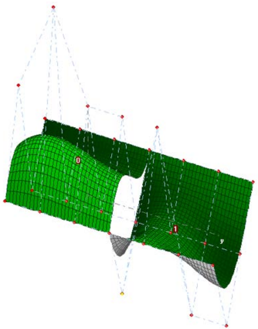

Fig. 3: Non smooth bicubic Bézier patches (both of them are defined on the unit square).

- Consider the Bézier surfaces

$$\begin{aligned} \mathbf{a}(u,v) &= \sum_{i=0}^{n}\sum_{j=0}^{m} \mathbf{a}_{ij} B_i^n\left(\frac{u-u_0}{u_1-u_0}\right) B_j^m\left(\frac{v-v_0}{v_1-v_0}\right), \\ (u,v) &\in [u_0,u_1]\times[v_0,v_1] \end{aligned}$$

and

$$\begin{aligned} \mathbf{b}(u,v) &= \sum_{i=0}^{n}\sum_{j=0}^{m} \mathbf{b}_{ij} B_i^n\left(\frac{u-u_1}{u_2-u_1}\right) B_j^m\left(\frac{v-v_1}{v_2-v_1}\right), \\ (u,v) &\in [u_1,u_2]\times[v_1,v_2]. \end{aligned}$$

- The condition of C^r-continuity along the parameter line $u = u_1$ is

$$\left(\frac{1}{\Delta u_0}\right)^i \Delta^{i,0}\mathbf{a}_{n-i,j} = \left(\frac{1}{\Delta u_1}\right)^i \Delta^{i,0}\mathbf{b}_{i,j},$$

where $\Delta u_k = u_{k+1} - u_k$, $i = 0,1,\ldots,r$, $j = 0,1,\ldots,m$.

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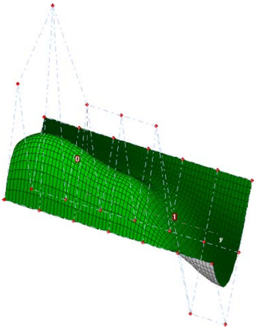

Fig. 3: C^1-continuous bicubic Bézier patches (both of them are defined on the unit square).

- Consider the Bézier surfaces

$$\begin{aligned} \mathbf{a}(u,v) &= \sum_{i=0}^{n}\sum_{j=0}^{m} \mathbf{a}_{ij} B_i^n\left(\frac{u-u_0}{u_1-u_0}\right) B_j^m\left(\frac{v-v_0}{v_1-v_0}\right), \\ (u,v) &\in [u_0,u_1]\times[v_0,v_1] \end{aligned}$$

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where $\Delta u_k = u_{k+1}-u_k$, $i=0,1,\ldots,r$, $j=0,1,\ldots,m$.

- The matrix representation of the Bézier surface

$$\mathbf{s}(u,v)=\sum_{i=0}^{n}\sum_{j=0}^{m}\mathbf{b}_{ij}B_i^n(u)B_j^m(v),\ (u,v)\in[0,1]\times[0,1]$$

is

$$\mathbf{s}(u,v)=\begin{bmatrix} B_0^n(u) & B_1^n(u) & \cdots & B_n^n(u)\end{bmatrix}\begin{bmatrix}\mathbf{b}_{00} & \mathbf{b}_{01} & \cdots & \mathbf{b}_{0m}\\ \mathbf{b}_{10} & \mathbf{b}_{11} & \cdots & \mathbf{b}_{1m}\\ \vdots & \vdots & \ddots & \vdots\\ \mathbf{b}_{n0} & \mathbf{b}_{n1} & \cdots & \mathbf{b}_{nm}\end{bmatrix}\begin{bmatrix}B_0^m(v)\\ B_1^m(v)\\ \vdots\\ B_m^m(v)\end{bmatrix}.$$

- The matrix representation above can be also written in the form

$$\mathbf{s}(u,v)=\begin{bmatrix} u^n & u^{n-1} & \cdots & 1\end{bmatrix} N \begin{bmatrix}\mathbf{b}_{00} & \mathbf{b}_{01} & \cdots & \mathbf{b}_{0m}\\ \mathbf{b}_{10} & \mathbf{b}_{11} & \cdots & \mathbf{b}_{1m}\\ \vdots & \vdots & \ddots & \vdots\\ \mathbf{b}_{n0} & \mathbf{b}_{n1} & \cdots & \mathbf{b}_{nm}\end{bmatrix} M^T \begin{bmatrix}v^m\\ v^{m-1}\\ \vdots\\ 1\end{bmatrix},$$

where

$$N = [n_{ij}]_{i=0,j=0}^{n,n} = \left[(-1)^{n-j-i}\binom{n}{j}\binom{n-j}{i}\right]_{i=0,j=0}^{n,n},$$

$$M = [m_{ij}]_{i=0,j=0}^{m,m} = \left[(-1)^{m-j-i}\binom{m}{j}\binom{m-j}{i}\right]_{i=0,j=0}^{m,m}.$$

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$$\mathbf{s}(u,v) = \sum_{i=0}^{n}\sum_{j=0}^{m} \mathbf{b}_{ij} B_i^n(u) B_j^m(v),\ (u,v) \in [0,1]\times[0,1]$$

is

$$\mathbf{s}(u,v) = \begin{bmatrix} B_0^n(u) & B_1^n(u) & \cdots & B_n^n(u) \end{bmatrix} \begin{bmatrix} \mathbf{b}_{00} & \mathbf{b}_{01} & \cdots & \mathbf{b}_{0m} \\ \mathbf{b}_{10} & \mathbf{b}_{11} & \cdots & \mathbf{b}_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{b}_{n0} & \mathbf{b}_{n1} & \cdots & \mathbf{b}_{nm} \end{bmatrix} \begin{bmatrix} B_0^m(v) \\ B_1^m(v) \\ \vdots \\ B_m^m(v) \end{bmatrix}.$$

- The matrix representation above can be also written in the form

$$\mathbf{s}(u,v) = \begin{bmatrix} u^n & u^{n-1} & \cdots & 1 \end{bmatrix} N \begin{bmatrix} \mathbf{b}_{00} & \mathbf{b}_{01} & \cdots & \mathbf{b}_{0m} \\ \mathbf{b}_{10} & \mathbf{b}_{11} & \cdots & \mathbf{b}_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{b}_{n0} & \mathbf{b}_{n1} & \cdots & \mathbf{b}_{nm} \end{bmatrix} M^T \begin{bmatrix} v^m \\ v^{m-1} \\ \vdots \\ 1 \end{bmatrix},$$

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where

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- Consider the knot vectors
$$U = \{u_i\}_{i=0}^{n}$$
and
$$V = \{v_j\}_{j=0}^{m}$$
which consist of strictly increasing subdivision points (also called knot values)
$$0 \leq u_0 < u_1 < \cdots < u_n \leq 1$$
and
$$0 \leq v_0 < v_1 < \cdots < v_m \leq 1,$$
respectively.
- Consider the Bézier surface
$$\mathbf{s}(u, v) = \sum_{i=0}^{n} \sum_{j=0}^{m} \mathbf{b}_{ij} B_i^n(u) B_j^m(v),\ (u, v) \in [0, 1] \times [0, 1]$$
that needs to fulfill the interpolation conditions
$$\mathbf{s}(u_i, v_j) = \mathbf{p}_{ij},\ i = 0, 1, \ldots, n;\ j = 0, 1, \ldots, m.$$

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that needs to fulfill the interpolation conditions
$$\mathbf{s}(u_i, v_j) = \mathbf{p}_{ij},\ i = 0, 1, \ldots, n;\ j = 0, 1, \ldots, m.$$

- Using the notations

$$\mathbf{P} = \begin{bmatrix} \mathbf{p}_{00} & \mathbf{p}_{01} & \cdots & \mathbf{p}_{0m} \\ \mathbf{p}_{10} & \mathbf{p}_{11} & \cdots & \mathbf{p}_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{p}_{n0} & \mathbf{p}_{n1} & \cdots & \mathbf{p}_{nm} \end{bmatrix}, \mathbf{B} = \begin{bmatrix} \mathbf{b}_{00} & \mathbf{b}_{01} & \cdots & \mathbf{b}_{0m} \\ \mathbf{b}_{10} & \mathbf{b}_{11} & \cdots & \mathbf{b}_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{b}_{n0} & \mathbf{b}_{n1} & \cdots & \mathbf{b}_{nm} \end{bmatrix},$$

$$U = \begin{bmatrix} B_0^n(u_0) & B_1^n(u_0) & \cdots & B_n^n(u_0) \\ B_0^n(u_1) & B_1^n(u_1) & \cdots & B_n^n(u_1) \\ \vdots & \vdots & \ddots & \vdots \\ B_0^n(u_n) & B_1^n(u_n) & \cdots & B_n^n(u_n) \end{bmatrix}, V = \begin{bmatrix} B_0^m(v_0) & B_1^m(v_0) & \cdots & B_m^m(v_0) \\ B_0^m(v_1) & B_1^m(v_1) & \cdots & B_m^m(v_1) \\ \vdots & \vdots & \ddots & \vdots \\ B_0^m(v_m) & B_1^m(v_m) & \cdots & B_m^m(v_m) \end{bmatrix},$$

we need to solve the system

$$\mathbf{P} = U\mathbf{B}V.$$

- The solution can be reduced to a set of curve interpolation problem in v- and u-directions:
  - first we solve the system

$$\mathbf{P} = \mathbf{C}V,$$

    where $\mathbf{C} = U\mathbf{B}$;
  - finally, we solve the system

$$\mathbf{C} = U\mathbf{B}.$$

- Using the notations

$$\mathbf{P} = \begin{bmatrix} \mathbf{p}_{00} & \mathbf{p}_{01} & \cdots & \mathbf{p}_{0m} \\ \mathbf{p}_{10} & \mathbf{p}_{11} & \cdots & \mathbf{p}_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{p}_{n0} & \mathbf{p}_{n1} & \cdots & \mathbf{p}_{nm} \end{bmatrix}, \mathbf{B} = \begin{bmatrix} \mathbf{b}_{00} & \mathbf{b}_{01} & \cdots & \mathbf{b}_{0m} \\ \mathbf{b}_{10} & \mathbf{b}_{11} & \cdots & \mathbf{b}_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{b}_{n0} & \mathbf{b}_{n1} & \cdots & \mathbf{b}_{nm} \end{bmatrix},$$

$$U = \begin{bmatrix} B_0^n(u_0) & B_1^n(u_0) & \cdots & B_n^n(u_0) \\ B_0^n(u_1) & B_1^n(u_1) & \cdots & B_n^n(u_1) \\ \vdots & \vdots & \ddots & \vdots \\ B_0^n(u_n) & B_1^n(u_n) & \cdots & B_n^n(u_n) \end{bmatrix}, V = \begin{bmatrix} B_0^m(v_0) & B_1^m(v_0) & \cdots & B_m^m(v_0) \\ B_0^m(v_1) & B_1^m(v_1) & \cdots & B_m^m(v_1) \\ \vdots & \vdots & \ddots & \vdots \\ B_0^m(v_m) & B_1^m(v_m) & \cdots & B_m^m(v_m) \end{bmatrix},$$

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- Using the notations

$$\mathbf{P} = \begin{bmatrix} \mathbf{p}_{00} & \mathbf{p}_{01} & \cdots & \mathbf{p}_{0m} \\ \mathbf{p}_{10} & \mathbf{p}_{11} & \cdots & \mathbf{p}_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{p}_{n0} & \mathbf{p}_{n1} & \cdots & \mathbf{p}_{nm} \end{bmatrix}, \mathbf{B} = \begin{bmatrix} \mathbf{b}_{00} & \mathbf{b}_{01} & \cdots & \mathbf{b}_{0m} \\ \mathbf{b}_{10} & \mathbf{b}_{11} & \cdots & \mathbf{b}_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{b}_{n0} & \mathbf{b}_{n1} & \cdots & \mathbf{b}_{nm} \end{bmatrix},$$

$$U = \begin{bmatrix} B_0^n(u_0) & B_1^n(u_0) & \cdots & B_n^n(u_0) \\ B_0^n(u_1) & B_1^n(u_1) & \cdots & B_n^n(u_1) \\ \vdots & \vdots & \ddots & \vdots \\ B_0^n(u_n) & B_1^n(u_n) & \cdots & B_n^n(u_n) \end{bmatrix}, V = \begin{bmatrix} B_0^m(v_0) & B_1^m(v_0) & \cdots & B_m^m(v_0) \\ B_0^m(v_1) & B_1^m(v_1) & \cdots & B_m^m(v_1) \\ \vdots & \vdots & \ddots & \vdots \\ B_0^m(v_m) & B_1^m(v_m) & \cdots & B_m^m(v_m) \end{bmatrix},$$

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- Using the notations

$$\mathbf{P} = \begin{bmatrix} \mathbf{p}_{00} & \mathbf{p}_{01} & \cdots & \mathbf{p}_{0m} \\ \mathbf{p}_{10} & \mathbf{p}_{11} & \cdots & \mathbf{p}_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{p}_{n0} & \mathbf{p}_{n1} & \cdots & \mathbf{p}_{nm} \end{bmatrix}, \mathbf{B} = \begin{bmatrix} \mathbf{b}_{00} & \mathbf{b}_{01} & \cdots & \mathbf{b}_{0m} \\ \mathbf{b}_{10} & \mathbf{b}_{11} & \cdots & \mathbf{b}_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{b}_{n0} & \mathbf{b}_{n1} & \cdots & \mathbf{b}_{nm} \end{bmatrix},$$

$$U = \begin{bmatrix} B_0^n(u_0) & B_1^n(u_0) & \cdots & B_n^n(u_0) \\ B_0^n(u_1) & B_1^n(u_1) & \cdots & B_n^n(u_1) \\ \vdots & \vdots & \ddots & \vdots \\ B_0^n(u_n) & B_1^n(u_n) & \cdots & B_n^n(u_n) \end{bmatrix}, V = \begin{bmatrix} B_0^m(v_0) & B_1^m(v_0) & \cdots & B_m^m(v_0) \\ B_0^m(v_1) & B_1^m(v_1) & \cdots & B_m^m(v_1) \\ \vdots & \vdots & \ddots & \vdots \\ B_0^m(v_m) & B_1^m(v_m) & \cdots & B_m^m(v_m) \end{bmatrix},$$

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Example

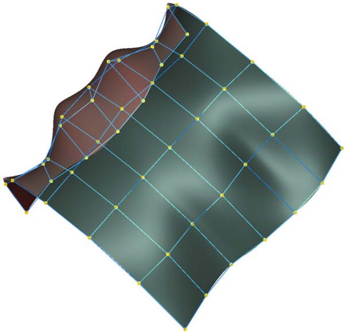

Fig. 4: An interpolating Bézier surface of degree $6, 6$.