Curve and surface modeling

– a CAGD approach based on OpenGL and C++ –

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In CAGD the most widespread description form of curves is

$$\begin{cases} \mathbf{c} : [a, b] \to \mathbb{R}^{\delta},\ \delta \geq 2, \\ \mathbf{c}(u) = \sum_{i=0}^{n} \mathbf{p}_i\, F_i(u) \end{cases}$$

where $n \geq 1$, the vectors $\mathbf{p}_i \in \mathbb{R}^{\delta}$ are called control points forming a control polygon $P = [\mathbf{p}_i]_{i=0}^{n}$ and the continuous functions F_i are defined on the interval $[a, b]$.

- If the functions F_i are properly chosen the resulted curve mimics the shape of the control polygon, i.e. the control polygon provides an intuitive tool for the designer.
- The most well-known curves of this type are the Bézier, rational Bézier, B-spline and non-uniform rational B-spline (NURBS) curves.

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Closure for the affine transformation

If

$$\sum_{i=0}^{n} F_i(u) \equiv 1, \forall u \in [a, b]$$

(i.e. the functions form the partition of the unity), then the shape of the curve is invariant under the affine transformation (e.g. rotation, translation) of its control polygon.

Proof.

- The curve is, by construction, invariant under linear transformations, i.e. if $T : \mathbb{R}^\delta \to \mathbb{R}^\sigma$ is any linear transformation, then

$$T(\mathbf{c}(u)) = T\left(\sum_{i=0}^{n} \mathbf{p}_i F_i(u)\right) = \sum_{i=0}^{n} T(\mathbf{p}_i) F_i(u).$$

- Given any translation vector $\mathbf{t} \in \mathbb{R}^\delta$,

$$\sum_{i=0}^{n} (\mathbf{p}_i + \mathbf{t}) F_i(u) = \sum_{i=0}^{n} \mathbf{p}_i F_i(u) + \mathbf{t} \sum_{i=0}^{n} F_i(u) = \mathbf{c}(u) + \mathbf{t}.$$

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Convex hull property

If in addition to the partition of the unity,

$$F_i(u) \geq 0, \forall u \in [a, b], \ (i = 0, 1, \ldots, n)$$

(i.e. the blending functions are positive), then the resulted curve will be in the convex hull of its control polygon.

Proof.
In this case the sum

$$\sum_{i=0}^{n} \mathbf{p}_i F_i(u)$$

is a convex combination of control points $\mathbf{p}_i$ for all values of $u \in [a, b]$. □

Blending or normalized system

If the function system

$$F = \left\{ F_i : [a, b] \to \mathbb{R}^{\delta} \right\}_{i=0}^{n}$$

fulfills the convex hull property, then it is a blending or normalized system.

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Variation diminishing

This property is fulfilled, when no hyperplane can intersect the curve more times than its control polygon.

- In this case, the curve also preserves the convexity of its control polygon, i.e. if the control polygon of a plane curve is convex, then the curve forms the boundary of a convex domain in the plane.

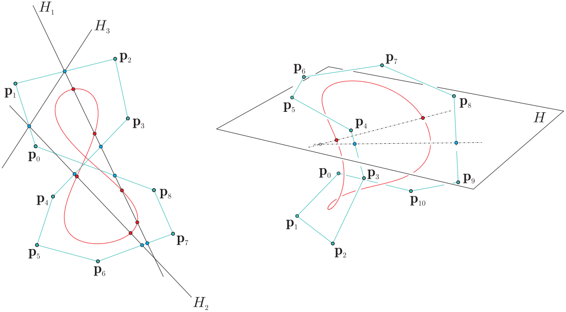

Fig. 1: Variation diminishing.

Fig. 2: Convexity preserving.

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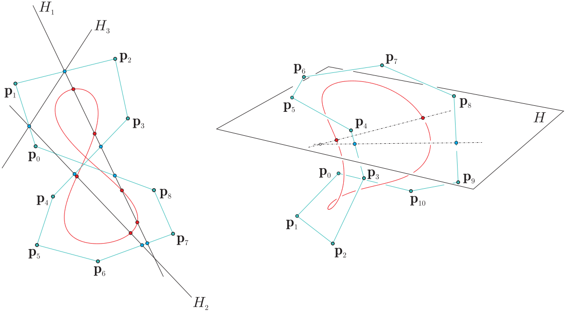

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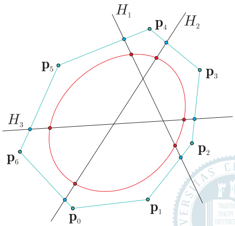

Fig. 2: Convexity preserving.

Length of a polygon and of a curve

Consider the length

$$L[Q] = L[\mathbf{q}_0, \mathbf{q}_1, \ldots, \mathbf{q}_n] = \sum_{i=1}^{n} \|\mathbf{q}_i - \mathbf{q}_{i-1}\|$$

of a polygon $Q = [\mathbf{q}_i]_{i=0}^{n}$, and the length of a curve $\mathbf{g} : [a, b] \to \mathbb{R}^\delta$ as the supremum of the lengths of all inscribed polygons

$$L[\mathbf{g}] = \sup_{a \le u_0 < u_1 < \cdots < u_m \le b;\ m \in \mathbb{N}} L[\mathbf{g}(u_0), \mathbf{g}(u_1), \ldots, \mathbf{g}(u_m)],$$

where $\|\cdot\|$ denotes a norm in $\mathbb{R}^\delta$.

Fig. 3: Inscribed polygon

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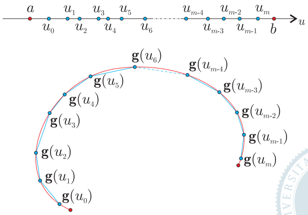

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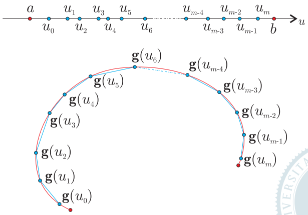

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Length diminishing

A system of functions

$$F = \{F_i : [a, b] \to \mathbb{R}\}_{i=0}^{n}$$

is length diminishing with respect to a norm $\|\cdot\|$ if for any control polygon $P = [\mathbf{p}_i]_{i=0}^{n}$ we have

$$L\left[\mathbf{c}(u) = \sum_{i=0}^{n} \mathbf{p}_i F_i(u)\right] \leq L\left[\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_n\right] = L\left[P\right].$$

Hodograph diminishing

- Assume that the function system

$$F = \{F_i : [a, b] \to \mathbb{R}\}_{i=0}^{n}$$

is differentiable in $[a, b]$.

- We may define the curve

$$\dot{\mathbf{c}}(u) = \sum_{i=0}^{n} \mathbf{p}_i \dot{F}_i(u),\ u \in [a, b]$$

of tangent directions also known as the hodograph of the curve

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- The hodograph of the control polygon $P = [\mathbf{p}_i]_{i=0}^{n}$ consists in the set of directions pointed by the vectors

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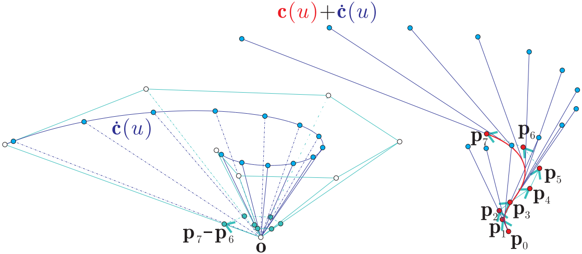
- We can form the convex closed cone generated by all derivatives $\dot{\mathbf{c}}(u)$:

$$\text{hhull}(\mathbf{c}) = \overline{\left\{ \sum_{j=1}^{m} \lambda_j \dot{\mathbf{c}}(u_j) : \lambda_1, \ldots, \lambda_m \geq 0,\ u_1, \ldots, u_m \in [a, b],\ m \in \mathbb{N} \right\}}$$

also known as the hodographic hull of the curve $\mathbf{c}$.

- The hodographic hull of the control polygon $P = [\mathbf{p}_i]_{i=0}^{n}$ is the convex cone

$$\text{hhull}(\mathrm{P}) = \overline{\left\{ \sum_{i=1}^{n} \mu_i (\mathbf{p}_i - \mathbf{p}_{i-1}) : \mu_1, \ldots, \mu_n \geq 0 \right\}}.$$



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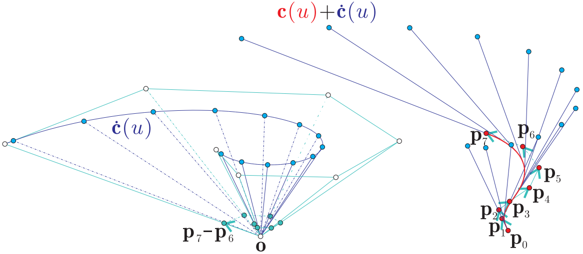

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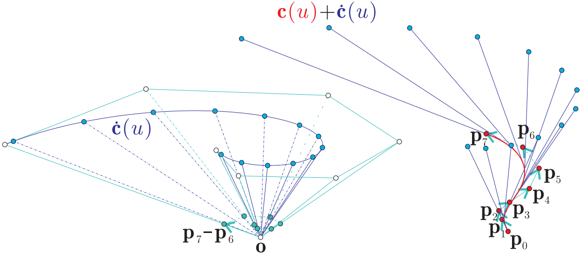

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- Observe that the hodographic hull of C^1 curves contains not only all tangent directions, but also all the directions formed by joining any two points of the curve.
- Moreover, $\mathrm{hhull}(\mathbf{c})$ coincides with the set

$$\overline{\left\{\sum_{j=1}^{m}\lambda_j\left(\mathbf{c}(u_j)-\mathbf{c}(v_j)\right):\ \lambda_j\geq 0,\ u_j<v_j,\ 1\leq j\leq m,\ m\in\mathbb{N}\right\}}.$$

- This allows us to define the hodographic hull of any curve even if it is not differentiable.
- The function system

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Preservation of monotonicity

In case of curve modeling it is required that the sense of the path tracing of the curve and the polygon agree.

- The function system

$$F = \{F_i : [a, b] \to \mathbb{R}\}_{i=0}^{n}$$

is monotonicity preserving if

$$\lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_n \Rightarrow \sum_{i=0}^{n} \lambda_i F_i(u),\ u \in [a, b]$$

is an increasing function.

- If the function

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is strictly increasing for any strictly increasing sequence of coefficients, then the system F is strict-monotonicity preserving.

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Endpoint interpolation property

The function system

$$F = \{F_i : [a, b] \to \mathbb{R}\}_{i=0}^{n}$$

fulfills the endpoint interpolation property if

$$\mathbf{c}(a) = \sum_{i=0}^{n} \mathbf{p}_i F_i(a) = \mathbf{p}_0 \text{ and } \mathbf{c}(b) = \sum_{i=0}^{n} \mathbf{p}_i F_i(b) = \mathbf{p}_n.$$

Fig. 5: Endpoint interpolation property.

Collocation matrix

Consider the subdivision points

$$a \leq u_0 < u_1 < \cdots < u_n \leq b$$

of the interval $[a, b]$. The collocation matrix of the function system

$$F = \{F_i : [a, b] \to \mathbb{R}\}_{i=0}^{n}$$

is

$$M \begin{pmatrix} F_0 & F_1 & \cdots & F_n \\ u_0 & u_1 & \cdots & u_n \end{pmatrix} = \begin{bmatrix} F_0(u_0) & F_1(u_0) & \cdots & F_n(u_0) \\ F_0(u_1) & F_1(u_1) & \cdots & F_n(u_1) \\ \vdots & \vdots & \ddots & \vdots \\ F_0(u_n) & F_1(u_n) & \cdots & F_n(u_n) \end{bmatrix}.$$

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Linear dependence

The function system

$$F = \{F_i : [a, b] \to \mathbb{R}\}_{i=0}^{n}$$

is called linearly dependent, if there exist real constants/scalars

$$\lambda_0, \lambda_1, \ldots, \lambda_n,$$

not all zero (i.e. $\sum_{i=0}^{n} \lambda_i^2 \neq 0$), such that

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If such scalars do not exist, then the function system F is linearly independent.
In this case the function system F forms a basis of a vector space of functions.

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is called linearly dependent, if there exist real constants/scalars

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Examples for linearly independent function systems

- The vector space of polynomials of degree at most n:

$$\{1, u, u^2, \ldots, u^n : u \in \mathbb{R}\}.$$

- The vector space of polynomials of degree at most n:

$$\left\{ \binom{n}{i} u^i (1-u)^{n-i} : u \in [0,1] \right\}_{i=0}^{n}.$$

- The vector space of trigonometric polynomials of order/degree at most $n/2n$:

$$\{1, \cos(u), \sin(u), \cos(2u), \sin(2u), \ldots, \cos(nu), \sin(nu) : u \in [0, 2\pi]\}.$$

- The vector space of trigonometric polynomials of order/degree at most $n/2n$:

$$\left\{ \frac{2^n}{\binom{2n}{n}(2n+1)} \left(1 + \cos\left(u - \frac{2i\pi}{2n+1}\right)\right)^n : u \in [0, 2\pi] \right\}_{i=0}^{2n}.$$

In case of a vector space of functions, which of its bases is the "best" for curve modeling?

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Theorem
If the determinant of the collocation matrix

$$M\begin{pmatrix} F_0 & F_1 & \cdots & F_n \\ u_0 & u_1 & \cdots & u_n \end{pmatrix}$$

is not zero for all subdivision points

$$a \leq u_0 < u_1 < \cdots < u_n \leq b$$

of the interval $[a, b]$, then the function system

$$F = \{F_i : [a, b] \to \mathbb{R}\}_{i=0}^{n}$$

is linearly independent.

Proof.

- By contradiction, suppose that F is linearly dependent. In this case $\exists \lambda_0, \lambda_1, \ldots, \lambda_n$ such that $\sum_{i=0}^{n} \lambda_i^2 \neq 0$ and $\sum_{i=0}^{n} \lambda_i F_i(u) = 0, \forall u \in [a, b]$.
- Consider the arbitrarily fixed subdivision points: $a \leq u_0 < u_1 < \cdots < u_n \leq b$.
- By substituting the subdivision points u_j $(j = 0, 1, \ldots, n)$ into the equation above, we get a system of linear equations, the matrix form of which is

$$\underbrace{\begin{bmatrix} F_0(u_0) & F_1(u_0) & \cdots & F_n(u_0) \\ F_0(u_1) & F_1(u_1) & \cdots & F_n(u_1) \\ \vdots & \vdots & \ddots & \vdots \\ F_0(u_n) & F_1(u_n) & \cdots & F_n(u_n) \end{bmatrix}}_{\text{collocation matrix} M} \begin{bmatrix} \lambda_0 \\ \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

- Since $\det M \neq 0$, it follows that only the trivial solution

$$\lambda_0 = \lambda_1 = \cdots = \lambda_n = 0$$

exists, that contradicts the linear dependence of the function system F.

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Interpolation of data points by means of approximating splines

- Consider the sequence of points

$$\mathbf{d}_j \in \mathbb{R}^\delta \ (j = 0, 1, \ldots, n)$$

called data points and associated parameter values

$$a \leq u_0 < u_1 < \cdots < u_n \leq b.$$

- We will refer to the sequence

$$\{(u_j, \mathbf{d}_j)\}_{j=0}^{n} \in \mathcal{M}_{1,n+1}\left([a, b] \times \mathbb{R}^\delta\right)$$

as nodes.

- The task is to find control points $\mathbf{p}_i \in \mathbb{R}^\delta$ $(i = 0, 1, \ldots, n)$ for which the interpolation conditions

$$\mathbf{c}(u_j) = \sum_{i=0}^{n} \mathbf{p}_i F_i(u_j) = \mathbf{d}_j, \ j = 0, 1, \ldots, n$$

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Example for interpolation

Fig. 6: Interpolation of data points

Totally positive matrix and function system

- A matrix is called totally positive if the matrix and all its minors have non-negative determinants.
- The function system

$$F = \{F_i : [a, b] \to \mathbb{R}\}_{i=0}^{n}$$

is totally positive if all collocation matrices

$$M \begin{pmatrix} F_0 & F_1 & \cdots & F_n \\ u_0 & u_1 & \cdots & u_n \end{pmatrix}$$

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Normalized totally positive basis

If the function system

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is totally positive, linearly independent and normalized, then we refer to system F as a normalized totally positive basis.

Theorem (whitout proof)

If the function system

$$F = \{F_i : [a, b] \to \mathbb{R}\}_{i=0}^{n}$$

is a normalized totally positive basis, then:

- the basis F satisfies the convex hull property;
- the basis F is variation diminishing;
- the basis F preserves the convexity;
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Remark

- The converse of the previous theorem is not true!
- It can be proved* that the normalized basis

$$C = \left\{ C_i(u) = \frac{2^n}{\binom{2n}{n}(2n+1)} \left(1 + \cos\left(u - \frac{2i\pi}{2n+1}\right)\right)^n : u \in [0, 2\pi] \right\}_{i=0}^{2n}$$

is not totally positive, but fulfills the (cyclic) variation diminishing property.

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In CAGD the general form of surface description is

$$\begin{cases} \mathbf{s} : [a,b] \times [c,d] \to \mathbb{R}^3, \\ \mathbf{s}(u,v) = \sum_{i=0}^{n} \sum_{j=0}^{m} \mathbf{p}_{i,j} F_i(u) G_j(v), \end{cases}$$

where vectors $\mathbf{p}_{i,j} \in \mathbb{R}^3$ are control points and they form a control net.

- Function systems

$$F = \{F_i : [a,b] \to \mathbb{R}\}_{i=0}^{n}$$

  and

$$G = \{G_j : [c,d] \to \mathbb{R}\}_{j=0}^{m}$$

  fulfill requirements detailed for curves – they can be of different types, however in practice they are almost always the same.
- This type of surfaces is called tensor product surface.

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$$\begin{cases} \mathbf{s} : [a, b] \times [c, d] \to \mathbb{R}^3, \\ \mathbf{s}(u, v) = \sum_{i=0}^{n} \sum_{j=0}^{m} \mathbf{p}_{i,j} F_i(u) G_j(v), \end{cases}$$

where vectors $\mathbf{p}_{i,j} \in \mathbb{R}^3$ are control points and they form a control net.

- Function systems

$$F = \{F_i : [a, b] \to \mathbb{R}\}_{i=0}^{n}$$

and

$$G = \{G_j : [c, d] \to \mathbb{R}\}_{j=0}^{m}$$

fulfill requirements detailed for curves – they can be of different types, however in practice they are almost always the same.

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(*a*) (*b*)

Fig. 7: *(a)* A torus. *(b)* A gyroscope.

(a) (b)

Fig. 8: *(a)* Another surface of revolution.
(b) A quartic non-orientable surface, also known as the Roman surface of Steiner.

Fig. 9: A NURBS surface – the de facto standard modeling tool in CAGD.