Joining not necessarily polynomial arcs and surface patches defined by 4×1 and 4×4 information matrices, respectively

– brief description of possible projects –

– a CAGD approach based on OpenGL and C++ –

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Appendix to Lectures 5–6 – April 1, 2022

Arcs defined by 4×1 information matrices

All project types

Matrix representation

Consider the arc

$$\begin{cases} \mathbf{c} : [a, b] \to \mathbb{R}^3, \\ \mathbf{c}(u) = \begin{bmatrix} F_0(u) & F_1(u) & F_2(u) & F_3(u) \end{bmatrix} \begin{bmatrix} \mathbf{i}_0 \\ \mathbf{i}_1 \\ \mathbf{i}_2 \\ \mathbf{i}_3 \end{bmatrix}, \end{cases}$$

where:

- the function system $F = \{F_k : [a, b] \to \mathbb{R}\}_{k=0}^{3}$ is a linearly independent function system that fulfills the studied shape preserving properties (like convex hull, variation diminishing, endpoint interpolation and endpoint tangency);
- $M = [\mathbf{i}_k]_{k=0}^{3} \in \mathcal{M}_{4,1}\left(\mathbb{R}^3\right)$ is a user-defined information matrix (consisting of data like control points, first/second order partial derivatives).

Matrix representation

Consider the surface patch

$$\begin{cases} \mathbf{s} : [a,b] \times [c,d] \to \mathbb{R}^3 \\ \mathbf{s}(u,v) = \begin{bmatrix} F_0(u) & F_1(u) & F_2(u) & F_3(u) \end{bmatrix} \begin{bmatrix} \mathbf{i}_{0,0} & \mathbf{i}_{0,1} & \mathbf{i}_{0,2} & \mathbf{i}_{0,3} \\ \mathbf{i}_{1,0} & \mathbf{i}_{1,1} & \mathbf{i}_{1,2} & \mathbf{i}_{1,3} \\ \mathbf{i}_{2,0} & \mathbf{i}_{2,1} & \mathbf{i}_{2,2} & \mathbf{i}_{2,3} \\ \mathbf{i}_{3,0} & \mathbf{i}_{3,1} & \mathbf{i}_{3,2} & \mathbf{i}_{3,3} \end{bmatrix} \begin{bmatrix} G_0(v) \\ G_1(v) \\ G_2(v) \\ G_3(v) \end{bmatrix}, \end{cases}$$

where:

- function systems
$$F = \{F_k : [a,b] \to \mathbb{R}\}_{k=0}^{3}$$
and
$$G = \{G_\ell : [c,d] \to \mathbb{R}\}_{\ell=0}^{3}$$
are shape preserving bases of some vector spaces of functions (in most of the cases these function systems coincide);
- $M = \left[\mathbf{i}_{k,\ell}\right]_{k=0,\ell=0}^{3,3} \in \mathcal{M}_{4,4}\left(\mathbb{R}^3\right)$ is a user-defined information matrix (consisting of data like control points, first/second order (mixed) partial derivatives).

Settings

- $[a, b] \times [c, d] = [0, 1] \times [0, 1]$;
- function systems F and G coincide and they correspond to the cubic Bernstein polynomials

$$
\begin{aligned}
B &= \left\{ B_i^3(t) = \binom{3}{i} t^i (1-t)^{3-i} : t \in [0,1] \right\}_{i=0}^{3} \\
&= \left\{ (1-t)^3, 3t(1-t)^2, 3t^2(1-t), t^3 : t \in [0,1] \right\};
\end{aligned}
$$

- the information matrix $M = \left[\mathbf{p}_{k,\ell}\right]_{k=0,\ell=0}^{3,3} \in \mathcal{M}_{4,4}\left(\mathbb{R}^3\right)$ is a user-defined control net.

C^1-continuous bicubic Bézier patches

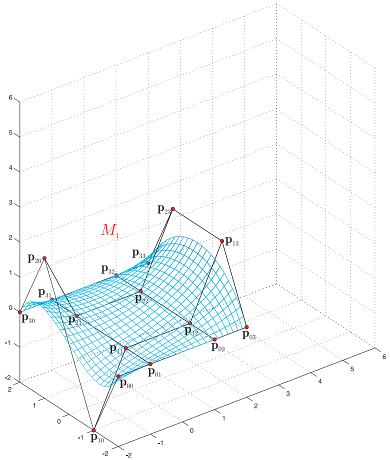


Fig. 1: A bicubic Bézier patch.

$$M_1 = \begin{bmatrix} \mathbf{p}_{0,0} & \mathbf{p}_{0,1} & \mathbf{p}_{0,2} & \mathbf{p}_{0,3} \\ \mathbf{p}_{1,0} & \mathbf{p}_{1,1} & \mathbf{p}_{1,2} & \mathbf{p}_{1,3} \\ \mathbf{p}_{2,0} & \mathbf{p}_{2,1} & \mathbf{p}_{2,2} & \mathbf{p}_{2,3} \\ \mathbf{p}_{3,0} & \mathbf{p}_{3,1} & \mathbf{p}_{3,2} & \mathbf{p}_{3,3} \end{bmatrix},$$

$$M_2 = \begin{bmatrix} \mathbf{q}_{0,0} & \mathbf{q}_{0,1} & \mathbf{q}_{0,2} & \mathbf{q}_{0,3} \\ \mathbf{q}_{1,0} & \mathbf{q}_{1,1} & \mathbf{q}_{1,2} & \mathbf{q}_{1,3} \\ \mathbf{q}_{2,0} & \mathbf{q}_{2,1} & \mathbf{q}_{2,2} & \mathbf{q}_{2,3} \\ \mathbf{q}_{3,0} & \mathbf{q}_{3,1} & \mathbf{q}_{3,2} & \mathbf{q}_{3,3} \end{bmatrix},$$

where

$$\begin{aligned} \mathbf{q}_{i,0} &= \mathbf{p}_{i,3}, \\ \mathbf{q}_{i,1} &= 2\mathbf{p}_{i,3} - \mathbf{p}_{i,2},\ i = 0,1,2,3. \end{aligned}$$

C^1-continuous bicubic Bézier patches

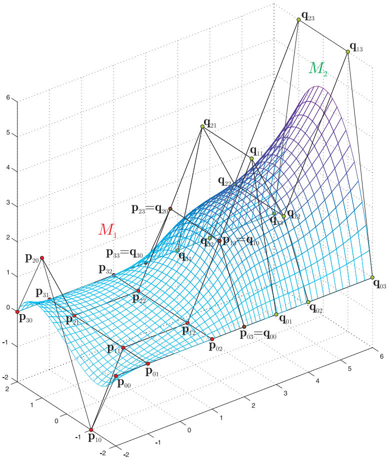


Fig. 1: C^1-continuous bicubic Bézier patches.

$$M_1 = \begin{bmatrix} \mathbf{p}_{0,0} & \mathbf{p}_{0,1} & \mathbf{p}_{0,2} & \mathbf{p}_{0,3} \\ \mathbf{p}_{1,0} & \mathbf{p}_{1,1} & \mathbf{p}_{1,2} & \mathbf{p}_{1,3} \\ \mathbf{p}_{2,0} & \mathbf{p}_{2,1} & \mathbf{p}_{2,2} & \mathbf{p}_{2,3} \\ \mathbf{p}_{3,0} & \mathbf{p}_{3,1} & \mathbf{p}_{3,2} & \mathbf{p}_{3,3} \end{bmatrix},$$

$$M_2 = \begin{bmatrix} \mathbf{q}_{0,0} & \mathbf{q}_{0,1} & \mathbf{q}_{0,2} & \mathbf{q}_{0,3} \\ \mathbf{q}_{1,0} & \mathbf{q}_{1,1} & \mathbf{q}_{1,2} & \mathbf{q}_{1,3} \\ \mathbf{q}_{2,0} & \mathbf{q}_{2,1} & \mathbf{q}_{2,2} & \mathbf{q}_{2,3} \\ \mathbf{q}_{3,0} & \mathbf{q}_{3,1} & \mathbf{q}_{3,2} & \mathbf{q}_{3,3} \end{bmatrix},$$

where

$$\begin{aligned} \mathbf{q}_{i,0} &= \mathbf{p}_{i,3}, \\ \mathbf{q}_{i,1} &= 2\mathbf{p}_{i,3} - \mathbf{p}_{i,2},\ i = 0, 1, 2, 3. \end{aligned}$$

Settings

- $[a,b]\times[c,d]=[0,1]\times[0,1]$;
- function systems F and G coincide and they correspond to the quartic polynomial basis functions

$$\begin{cases} F_0(t)=G_0(t)=F_3\left(1-t\right)=G_3\left(1-t\right), \\ F_1(t)=G_1(t)=F_2\left(1-t\right)=G_2\left(1-t\right), \\ F_2(t)=G_2(t)=4t^3\left(1-t\right)+3t^2\left(1-t\right)^2, \\ F_3(t)=G_3(t)=t^4; \end{cases}$$

- the information matrix $M=\left[\mathbf{p}_{k,\ell}\right]_{k=0,\ell=0}^{3,3}\in\mathcal{M}_{4,4}\left(\mathbb{R}^3\right)$ is a user-defined control net;
- the C^1 continuity geometric constraints are the same as in the case of polynomial bicubic Bézier patches.

Settings

- $[a, b] \times [c, d] = [0, \alpha] \times [0, \alpha]$, where $\alpha \in (0, \pi)$ is a user-defined shape (or tension) parameter;
- function systems F and G coincide and they correspond to the second order (i.e., quartic) trigonometric basis functions

$$\begin{cases} F_0(t) = G_0(t) = F_3\left(\alpha - t\right) = G_3\left(\alpha - t\right), \\ F_1(t) = G_1(t) = F_2\left(\alpha - t\right) = G_2\left(\alpha - t\right), \\ F_2(t) = G_2(t) = \frac{4\cos\left(\frac{\alpha}{2}\right)}{\sin^4\left(\frac{\alpha}{2}\right)} \sin\left(\frac{\alpha - t}{2}\right) \sin^3\left(\frac{t}{2}\right) + \frac{1+2\cos^2\left(\frac{\alpha}{2}\right)}{\sin^4\left(\frac{\alpha}{2}\right)} \sin^2\left(\frac{\alpha - t}{2}\right) \sin^2\left(\frac{t}{2}\right), \\ F_3(t) = G_3(t) = \frac{1}{\sin^4\left(\frac{\alpha}{2}\right)} \sin^4\left(\frac{t}{2}\right); \end{cases}$$

- the information matrix $M = \left[\mathbf{p}_{k,\ell}\right]_{k=0,\ell=0}^{3,3} \in \mathcal{M}_{4,4}\left(\mathbb{R}^3\right)$ is a user-defined control net;
- the C^1 continuity geometric constraints are the same as in the case of polynomial bicubic Bézier patches.

Settings

- $[a,b]\times[c,d]=[0,\alpha]\times[0,\alpha]$, where $\alpha>0$ is a user-defined shape (or tension) parameter;
- function systems F and G coincide and they correspond to the second order (i.e., quartic) hyperbolic basis functions

$$\begin{cases} F_0(t)=G_0(t)=F_3(\alpha-t)=G_3(\alpha-t)\,, \\ F_1(t)=G_1(t)=F_2(\alpha-t)=G_2(\alpha-t)\,, \\ F_2(t)=G_2(t)=\frac{4\cosh\left(\frac{\alpha}{2}\right)}{\sinh^4\left(\frac{\alpha}{2}\right)}\sinh\left(\frac{\alpha-t}{2}\right)\sinh^3\left(\frac{t}{2}\right)+\frac{1+2\cosh^2\left(\frac{\alpha}{2}\right)}{\sinh^4\left(\frac{\alpha}{2}\right)}\sinh^2\left(\frac{\alpha-t}{2}\right)\sinh^2\left(\frac{t}{2}\right), \\ F_3(t)=G_3(t)=\frac{1}{\sinh^4\left(\frac{\alpha}{2}\right)}\sinh^4\left(\frac{t}{2}\right); \end{cases}$$

- the information matrix $M=\left[\mathbf{p}_{k,\ell}\right]_{k=0,\ell=0}^{3,3}\in\mathcal{M}_{4,4}\left(\mathbb{R}^3\right)$ is a user-defined control net;
- the C^1 continuity geometric constraints are the same as in the case of polynomial bicubic Bézier patches.

Settings

- $[a,b]\times[c,d]=[0,\alpha]\times[0,\alpha]$, where $\alpha\in(0,\pi)$ is a user-defined shape (or tension) parameter;
- function systems F and G coincide and they correspond to the first order algebraic-trigonometric basis functions

$$\begin{cases} F_0(t)=G_0(t)=F_3\left(\alpha-t\right)=G_3\left(\alpha-t\right), \\ F_1(t)=G_1(t)=F_2\left(\alpha-t\right)=G_2\left(\alpha-t\right), \\ F_2(t)=G_2(t)=\frac{(\alpha-t+\sin(\alpha-t)+\sin(t)-\sin(\alpha)+t\cos(\alpha)-\alpha\cos(t))\sin(\alpha)}{(\alpha-\sin(\alpha))(2\sin(\alpha)-\alpha-\alpha\cos(\alpha))}, \\ F_3(t)=G_3(t)=\frac{t-\sin(t)}{\alpha-\sin(\alpha)}; \end{cases}$$

- the information matrix $M=\left[\mathbf{p}_{k,\ell}\right]_{k=0,\ell=0}^{3,3}\in\mathcal{M}_{4,4}\left(\mathbb{R}^3\right)$ is a user-defined control net;
- the C^1 continuity geometric constraints are the same as in the case of polynomial bicubic Bézier patches.

Settings

- $[a,b] \times [c,d] = [0,\alpha] \times [0,\alpha]$, where $\alpha > 0$ is a user-defined shape (or tension) parameter;
- function systems F and G coincide and they correspond to the first order algebraic-hyperbolic basis functions

$$\begin{cases} F_0(t) = G_0(t) = F_3(\alpha - t) = G_3(\alpha - t), \\ F_1(t) = G_1(t) = F_2(\alpha - t) = G_2(\alpha - t), \\ F_2(t) = G_2(t) = \frac{(\alpha - t + \sinh(\alpha - t) + \sinh(t) - \sinh(\alpha) + t\cosh(\alpha) - \alpha\cosh(t))\sinh(\alpha)}{(\alpha - \sinh(\alpha))(2\sinh(\alpha) - \alpha - \alpha\cosh(\alpha))}, \\ F_3(t) = G_3(t) = \frac{t - \sinh(t)}{\alpha - \sinh(\alpha)}; \end{cases}$$

- the information matrix $M = \left[\mathbf{p}_{k,\ell}\right]_{k=0,\ell=0}^{3,3} \in \mathcal{M}_{4,4}\left(\mathbb{R}^3\right)$ is a user-defined control net;
- the C^1 continuity geometric constraints are the same as in the case of polynomial bicubic Bézier patches.

Settings

- $[a,b]\times[c,d]=[0,\alpha]\times[0,\alpha]$, where $\alpha\in(0,2\pi)$ is a user-defined shape (or tension) parameter;
- function systems F and G coincide and they correspond to the second order/quartic algebraic-trigonometric basis functions

$$\begin{cases} F_0(t)=G_0(t)=F_3(\alpha-t)=G_3(\alpha-t), \\ F_1(t)=G_1(t)=F_2(\alpha-t)=G_2(\alpha-t), \\ F_2(t)=G_2(t)=\frac{1}{2}H(t)+K(t), \\ F_3(t)=G_3(t)=L(t), \end{cases}$$

where

$$\begin{aligned} H(t)=c_2\Big(&2\alpha(\sin(t)-\sin(\alpha))-2\alpha(1-\cos(\alpha))t+\alpha^2+ \\ &+2\alpha\sin(\alpha-t)-\alpha^2\cos(\alpha-t)+ \\ &+\alpha^2(\cos(\alpha)-\cos(t))+2(1-\cos(\alpha))t^2+\alpha(\alpha-t)t\sin(\alpha)\Big), \end{aligned}$$

Settings – continued

$$K(t) = c_3\Big(2(\alpha - t) + 2(\sin(t) - \sin(\alpha)) + 2(t\cos(\alpha) - \alpha\cos(t)) + 2\sin(\alpha - t) + + \alpha^2(t - \sin(t)) - (\alpha - \sin(\alpha))\,t^2\Big),$$

$$L(t) = c_4\left(2\cos(t) + t^2 - 2\right)$$

and

$$c_2 = \frac{4 - 4\cos(\alpha) - 2\alpha\sin(\alpha)}{\left(\alpha^2 - 4\cos(\alpha) - 4\alpha\sin(\alpha) + \alpha^2\cos(\alpha) + 4\right)^2},$$

$$c_3 = \frac{2(\alpha - \sin(\alpha))}{\left(2\cos(\alpha) + \alpha^2 - 2\right)\left(\alpha^2 - 4\cos(\alpha) - 4\alpha\sin(\alpha) + \alpha^2\cos(\alpha) + 4\right)},$$

$$c_4 = \frac{1}{2\cos(\alpha) + \alpha^2 - 2};$$

- the information matrix $M = \left[\mathbf{p}_{k,\ell}\right]_{k=0,\ell=0}^{3,3} \in \mathcal{M}_{4,4}\left(\mathbb{R}^3\right)$ is a user-defined control net;
- the C^1 continuity geometric constraints are the same as in the case of polynomial bicubic Bézier patches.

Settings

- $[a,b] \times [c,d] = [0,\alpha] \times [0,\alpha]$, where $\alpha > 0$ is a user-defined shape (or tension) parameter;
- function systems F and G coincide and they correspond to the second order/quartic algebraic-hyperbolic basis functions

$$
\begin{cases}
F_0(t) = G_0(t) = F_3(\alpha - t) = G_3(\alpha - t), \\
F_1(t) = G_1(t) = F_2(\alpha - t) = G_2(\alpha - t), \\
F_2(t) = G_2(t) = \frac{1}{2} H(t) + K(t), \\
F_3(t) = G_3(t) = L(t),
\end{cases}
$$

where

$$
\begin{aligned}
H(t) = c_2 \Big(& \alpha^2 \cosh(t) + 2t^2 \cosh(\alpha) + \alpha^2 \cosh(\alpha - t) + 2t\alpha - \alpha^2 - \alpha^2 \cosh(\alpha) \\
& - 2\alpha \sinh(t) - 2\alpha \sinh(\alpha - t) + 2\alpha \sinh(\alpha) - 2t^2 + t\alpha^2 \sinh(\alpha) \\
& - t^2 \alpha \sinh(\alpha) - 2t\alpha \cosh(\alpha) \Big),
\end{aligned}
$$

Settings – continued

$$K(t) = c_3\Big(2(\alpha - t) + 2\sinh(\alpha - t) + 2(\sinh(t) - \sinh(\alpha)) + \alpha^2(\sinh(t) - t) + $$
$$+ t^2(\alpha - \sinh(\alpha)) + 2(t\cosh(\alpha) - \alpha\cosh(t))\Big),$$

$$L(t) = c_4\left(2\cosh(t) - t^2 - 2\right)$$

and

$$c_2 = \frac{2\alpha\sinh(\alpha) - 4\cosh(\alpha) + 4}{(4\cosh(\alpha) + \alpha^2 + \alpha^2\cosh(\alpha) - 4\alpha\sinh(\alpha) - 4)^2},$$
$$c_3 = \frac{2(\sinh(\alpha) - \alpha)}{(4\cosh(\alpha) + \alpha^2 + \alpha^2\cosh(\alpha) - 4\alpha\sinh(\alpha) - 4)(\alpha^2 - 2\cosh(\alpha) + 2)},$$
$$c_4 = \frac{1}{2\cosh(\alpha) - \alpha^2 - 2};$$

- the information matrix $M = \left[\mathbf{p}_{k,\ell}\right]_{k=0,\ell=0}^{3,3} \in \mathcal{M}_{4,4}\left(\mathbb{R}^3\right)$ is a user-defined control net;
- the C^1 continuity geometric constraints are the same as in the case of polynomial bicubic Bézier patches.

Settings

- $[a, b] \times [c, d] = [0, 1] \times [0, 1]$;
- function systems F and G coincide and they correspond to the cubic, periodic and uniform B-spline basis functions, i.e. $F_i, G_i : [0, 1] \to [0, 1]$, $i = 0, 1, 2, 3$,

$$\begin{cases} F_0(t) & = & G_0(t) & = & \frac{(1-t)^3}{6}, \\ F_1(t) & = & G_1(t) & = & \frac{3t(1-t)^2+3(1-t)+1}{6}, \\ F_2(t) & = & G_2(t) & = & \frac{3t^2(1-t)+3t+1}{6}, \\ F_3(t) & = & G_3(t) & = & \frac{t^3}{6}; \end{cases}$$

- the information matrix $M = \left[\mathbf{p}_{k,\ell}\right]_{k=0,\ell=0}^{3,3} \in \mathcal{M}_{4,4}\left(\mathbb{R}^3\right)$ is a user-defined control net.

C^1-continuous uniform, periodic, bicubic B-spline patches

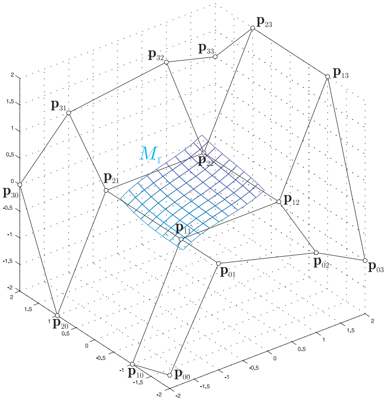


Fig. 2: A bicubic, uniform, periodic B-spline patch

$$M_1 = \begin{bmatrix} \mathbf{p}_{00} & \mathbf{p}_{01} & \mathbf{p}_{02} & \mathbf{p}_{03} \\ \mathbf{p}_{10} & \mathbf{p}_{11} & \mathbf{p}_{12} & \mathbf{p}_{13} \\ \mathbf{p}_{20} & \mathbf{p}_{21} & \mathbf{p}_{22} & \mathbf{p}_{23} \\ \mathbf{p}_{30} & \mathbf{p}_{31} & \mathbf{p}_{32} & \mathbf{p}_{33} \end{bmatrix}$$

C^1-continuous uniform, periodic, bicubic B-spline patches

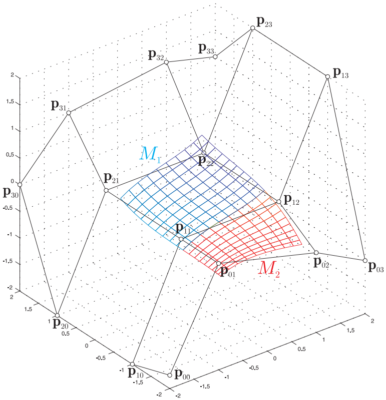


Fig. 2: Smooth bicubic, uniform, periodic B-spline patches

$$M_2 = \begin{bmatrix} \mathbf{p}_{00} & \mathbf{p}_{01} & \mathbf{p}_{02} & \mathbf{p}_{03} \\ \mathbf{p}_{00} & \mathbf{p}_{01} & \mathbf{p}_{02} & \mathbf{p}_{03} \\ \mathbf{p}_{10} & \mathbf{p}_{11} & \mathbf{p}_{12} & \mathbf{p}_{13} \\ \mathbf{p}_{20} & \mathbf{p}_{21} & \mathbf{p}_{22} & \mathbf{p}_{23} \end{bmatrix}$$

C^1-continuous uniform, periodic, bicubic B-spline patches

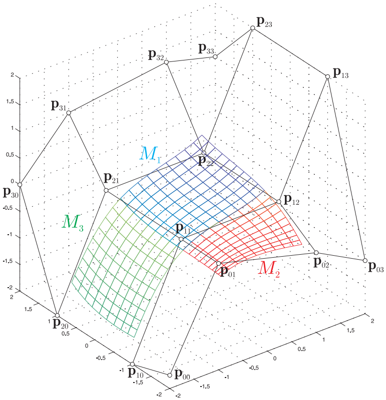


Fig. 2: Smooth bicubic, uniform, periodic B-spline patches

$$M_3 = \begin{bmatrix} \mathbf{p}_{00} & \mathbf{p}_{00} & \mathbf{p}_{01} & \mathbf{p}_{02} \\ \mathbf{p}_{10} & \mathbf{p}_{10} & \mathbf{p}_{11} & \mathbf{p}_{12} \\ \mathbf{p}_{20} & \mathbf{p}_{20} & \mathbf{p}_{21} & \mathbf{p}_{22} \\ \mathbf{p}_{30} & \mathbf{p}_{30} & \mathbf{p}_{31} & \mathbf{p}_{32} \end{bmatrix}$$

C^1-continuous uniform, periodic, bicubic B-spline patches

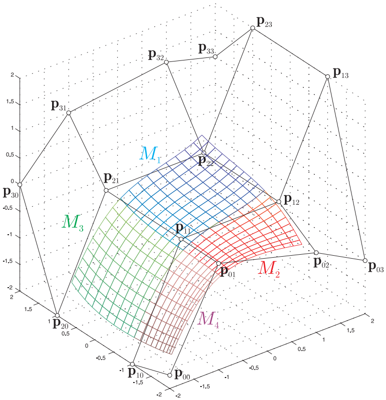


Fig. 2: Smooth bicubic, uniform, periodic B-spline patches

$$M_4 = \begin{bmatrix} \mathbf{p}_{00} & \mathbf{p}_{00} & \mathbf{p}_{01} & \mathbf{p}_{02} \\ \mathbf{p}_{00} & \mathbf{p}_{00} & \mathbf{p}_{01} & \mathbf{p}_{02} \\ \mathbf{p}_{10} & \mathbf{p}_{10} & \mathbf{p}_{11} & \mathbf{p}_{12} \\ \mathbf{p}_{20} & \mathbf{p}_{20} & \mathbf{p}_{21} & \mathbf{p}_{22} \end{bmatrix}$$

Extension to larger control nets

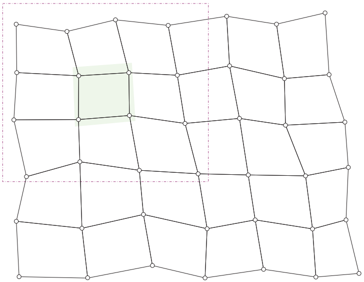
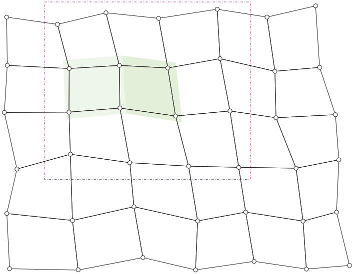


Fig. 3: Extracting smooth patches from a larger control net. Interior patches are C^2-continuous while patches along the boundary are only C^1-continuous. Patches along the boundary are obtained by duplicating/triplicating rows/columns/corners of the closest 4×4 logical control net.

Extension to larger control nets



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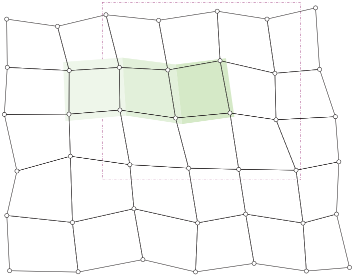

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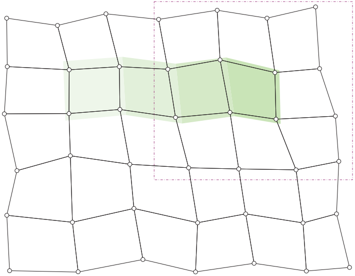

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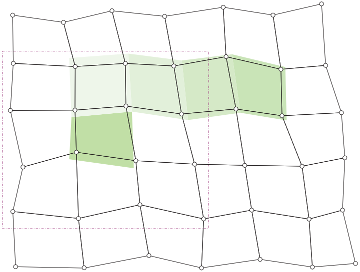

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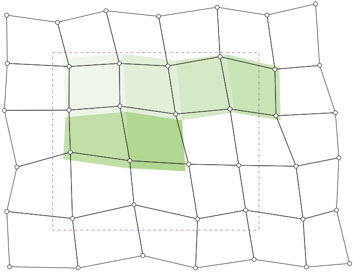

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Extension to larger control nets

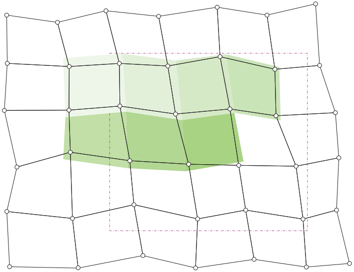
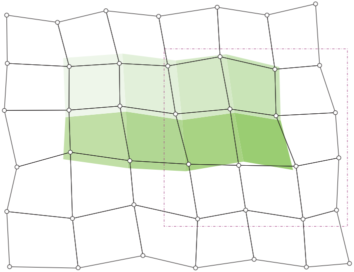


Fig. 3: Extracting smooth patches from a larger control net. Interior patches are C^2-continuous while patches along the boundary are only C^1-continuous. Patches along the boundary are obtained by duplicating/triplicating rows/columns/corners of the closest 4×4 logical control net.

Extension to larger control nets



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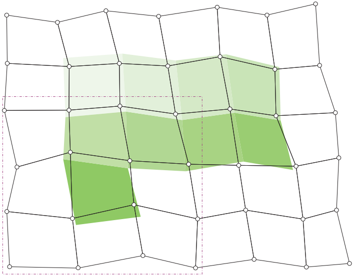

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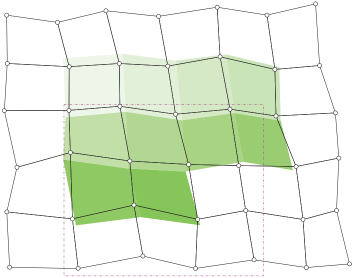

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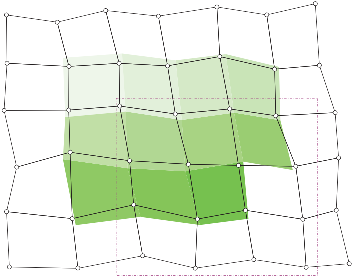
Extension to larger control nets



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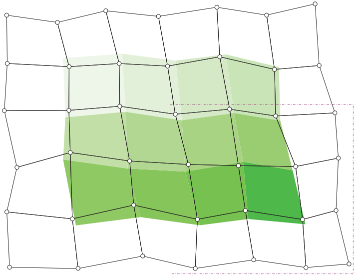

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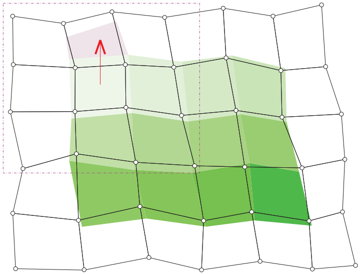

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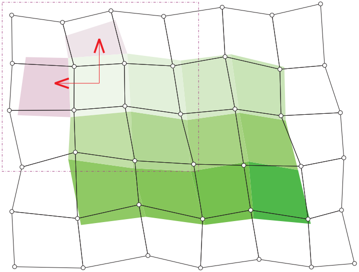

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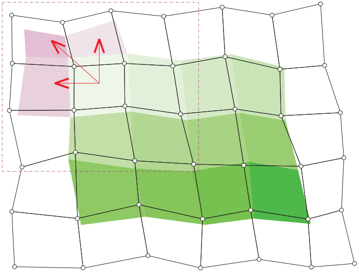

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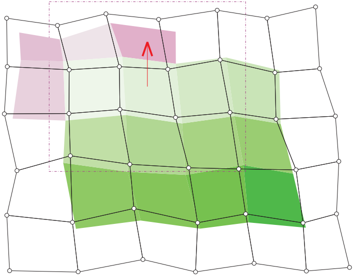

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Toroid

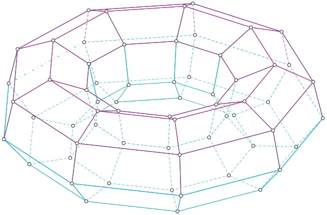

Fig. 4: A toroid

$$P = \begin{bmatrix} \mathbf{p}_{00} & \mathbf{p}_{01} & \mathbf{p}_{02} & \cdots & \mathbf{p}_{0m} & \mathbf{p}_{00} & \mathbf{p}_{01} & \mathbf{p}_{02} \\ \mathbf{p}_{10} & \mathbf{p}_{11} & \mathbf{p}_{12} & \cdots & \mathbf{p}_{1m} & \mathbf{p}_{10} & \mathbf{p}_{11} & \mathbf{p}_{12} \\ \mathbf{p}_{20} & \mathbf{p}_{21} & \mathbf{p}_{22} & \cdots & \mathbf{p}_{2m} & \mathbf{p}_{20} & \mathbf{p}_{21} & \mathbf{p}_{22} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{p}_{n0} & \mathbf{p}_{n1} & \mathbf{p}_{n2} & \cdots & \mathbf{p}_{nm} & \mathbf{p}_{n0} & \mathbf{p}_{n1} & \mathbf{p}_{n2} \\ \mathbf{p}_{00} & \mathbf{p}_{01} & \mathbf{p}_{02} & \cdots & \mathbf{p}_{0m} & \mathbf{p}_{00} & \mathbf{p}_{01} & \mathbf{p}_{02} \\ \mathbf{p}_{10} & \mathbf{p}_{11} & \mathbf{p}_{12} & \cdots & \mathbf{p}_{1m} & \mathbf{p}_{10} & \mathbf{p}_{11} & \mathbf{p}_{12} \\ \mathbf{p}_{20} & \mathbf{p}_{21} & \mathbf{p}_{22} & \cdots & \mathbf{p}_{2m} & \mathbf{p}_{20} & \mathbf{p}_{21} & \mathbf{p}_{22} \end{bmatrix},$$

where

$$\begin{aligned} \mathbf{p}_{ij} &= \begin{bmatrix} x(u_i, v_j) & y(u_i, v_j) & z(u_i, v_j) \end{bmatrix}^T, \\ u_i &= \frac{2i\pi}{n+1}, i = 0, 1, \ldots, n, \ n \geq 3, \\ v_j &= \frac{2j\pi}{m+1}, j = 0, 1, \ldots, m, \ m \geq 3, \\ x(u, v) &= (R + r\sin(u))\cos(v), \\ y(u, v) &= (R + r\sin(u))\sin(v), \\ z(u, v) &= r\cos(u), \ R \geq r > 0. \end{aligned}$$

Cylindrical

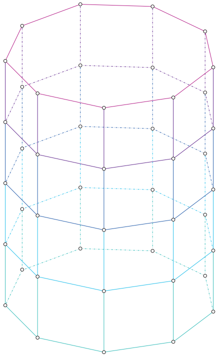

Fig. 5: A cylinder

$$P = \begin{bmatrix} \mathbf{p}_{00} & \mathbf{p}_{01} & \mathbf{p}_{02} & \cdots & \mathbf{p}_{0m} & \mathbf{p}_{00} & \mathbf{p}_{01} & \mathbf{p}_{02} \\ \mathbf{p}_{10} & \mathbf{p}_{11} & \mathbf{p}_{12} & \cdots & \mathbf{p}_{1m} & \mathbf{p}_{10} & \mathbf{p}_{11} & \mathbf{p}_{12} \\ \mathbf{p}_{20} & \mathbf{p}_{21} & \mathbf{p}_{22} & \cdots & \mathbf{p}_{2m} & \mathbf{p}_{20} & \mathbf{p}_{21} & \mathbf{p}_{22} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{p}_{n0} & \mathbf{p}_{n1} & \mathbf{p}_{n2} & \cdots & \mathbf{p}_{nm} & \mathbf{p}_{n0} & \mathbf{p}_{n1} & \mathbf{p}_{n2} \end{bmatrix},$$

where

$$\begin{aligned} \mathbf{p}_{ij} &= \begin{bmatrix} x(u_i, v_j) & y(u_i, v_j) & z(u_i, v_j) \end{bmatrix}^T, \\ u_i &= a + i\frac{b-a}{n+1}, i = 0, 1, \ldots, n,\ n \geq 3,\ a < b, \\ v_j &= \frac{2j\pi}{m+1}, j = 0, 1, \ldots, m,\ m \geq 3, \\ x(u, v) &= r\cos(v), \\ y(u, v) &= r\sin(v),\ r > 0, \\ z(u, v) &= u. \end{aligned}$$

Settings

- $[a, b] \times [c, d] = [0, 1] \times [0, 1]$;
- function systems F and G coincide and they correspond to the cubic Hermite basis functions, i.e. $F_i, G_i : [0, 1] \to \mathbb{R}$, $i = 0, 1, 2, 3$,

$$\begin{cases} F_0(t) = G_0(t) = 2t^3 - 3t^2 + 1, \\ F_1(t) = G_1(t) = -2t^3 + 3t^2, \\ F_2(t) = G_2(t) = t^3 - 2t^2 + t, \\ F_3(t) = G_3(t) = t^3 - t^2; \end{cases}$$

- the structure of the user-defined information matrix is

$$M = \begin{bmatrix} \mathbf{p}_{0,0} & \mathbf{p}_{0,1} & \mathbf{p}^v_{0,0} & \mathbf{p}^v_{0,1} \\ \mathbf{p}_{1,0} & \mathbf{p}_{1,1} & \mathbf{p}^v_{1,0} & \mathbf{p}^v_{1,1} \\ \mathbf{p}^u_{0,0} & \mathbf{p}^u_{0,1} & \mathbf{t}_{0,0} & \mathbf{t}_{0,1} \\ \mathbf{p}^u_{1,0} & \mathbf{p}^u_{1,1} & \mathbf{t}_{1,0} & \mathbf{t}_{1,1} \end{bmatrix},$$

  where:

  - $\mathbf{p}_{0,0}, \mathbf{p}_{0,1}, \mathbf{p}_{1,0}, \mathbf{p}_{1,1} \in \mathbb{R}^3$ represent the four corners of the patch;
  - $\mathbf{p}^u_{0,0}, \mathbf{p}^u_{0,1}, \mathbf{p}^u_{1,0}, \mathbf{p}^u_{1,1} \in \mathbb{R}^3$ are the first order partial derivatives in u-direction at the corners;
  - $\mathbf{p}^v_{0,0}, \mathbf{p}^v_{0,1}, \mathbf{p}^v_{1,0}, \mathbf{p}^v_{1,1} \in \mathbb{R}^3$ denote the first order partial derivatives in v-direction at the corners;
  - $\mathbf{t}_{0,0}, \mathbf{t}_{0,1}, \mathbf{t}_{1,0}, \mathbf{t}_{1,1} \in \mathbb{R}^3$ correspond to the second order mixed partial derivatives (or twist vectors) at the corners.

Example

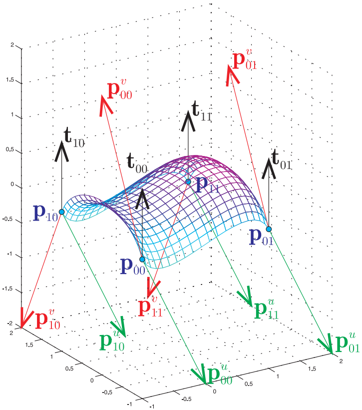

Fig. 6: A bicubic Hermite patch

C^1-continuous bicubic Hermite patches

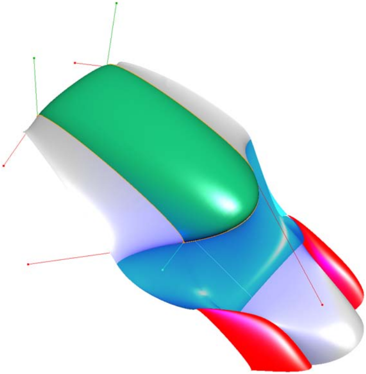

Fig. 7: A model that consists of 9 bicubic Hermite patches.

- In order to join two bicubic Hermite patches with C^1-continuity, all information [corners, first and second order (mixed) partial derivatives] must be the same along the common boundary curve.

5

- Evaluation and interactive rendering of an isolated arc: points, first and second order derivatives (information vectors and possible shape parameters have to be modifiable).
- Evaluation and interactive rendering of an isolated patch: points, first order partial derivatives, unit normal vectors, u- and v-directional isoparametric curves (information vectors and possible shape parameters have to be modifiable).
- In case of cubic Bézier, quartic polynomial, second order trigonometric/hyperbolic, first and second order algebraic-trigonometric/hyperbolic and cubic Hermite arcs: interactive joining/merging/extension of arcs with C^1-continuity for at least 2 of the 4 pairs of directions (left–right, right–left, left–left, right–right). In case of cubic, uniform and periodic B-spline arcs: interactive modeling of an **open composite B-spline curve** generated by a control polygon of arbitrary dimension $n \geq 4$.
- In case of bicubic Bézier, biquartic polynomial, second order trigonometric/hyperbolic, first and second order algebraic-trigonometric/hyperbolic and bicubic Hermite patches: interactive joining/merging/extension of patches with C^1-continuity at least in 3 from 8 directions. In case of bicubic, uniform and periodic B-spline patches: interactive modeling of open composite B-spline surfaces generated by an arbitrary control net of dimension $n \times m$ $(n, m \geq 4)$.
- In case of surface patches: applying materials and lighting effects.

10

- Evaluation and interactive rendering of an isolated arc: points, first and second order derivatives (information vectors and possible shape parameters have to be modifiable).
- Evaluation and interactive rendering of an isolated patch: points, first order partial derivatives, unit normal vectors, u- and v-directional isoparametric curves (information vectors and possible shape parameters have to be modifiable).
- Rendering, modifying and labeling of selected patch data (e.g. control nets, partial derivatives, twist vectors).
- In case of cubic Bézier, quartic polynomial, second order trigonometric/hyperbolic, first and second order algebraic-trigonometric/hyperbolic and cubic Hermite arcs: interactive joining/merging/extension of arcs with C^1-continuity along all four direction pairs. In case of cubic, uniform and periodic B-spline arcs: interactive modeling of both **open and closed composite B-spline curves** generated by control polygons of arbitrary dimension $n \geq 4$.
- In case of bicubic Bézier, biquartic polynomial, second order trigonometric/hyperbolic, first and second order algebraic-trigonometric/hyperbolic and bicubic Hermite patches: interactive joining/merging/extension of patches with C^1-continuity in all 8 possible directions. In case of bicubic, uniform and periodic B-spline patches: interactive modeling of open, toroidal, cylindrical, spherical composite B-spline surfaces generated by an arbitrary control net of dimension $n \times m$ $(n, m \geq 4)$.

10 – continued

- In general, updating an existing arc/patch affects neighboring arcs/patches: in case of each arc/patch use a list that stores the pointers and direction pairs of neighboring arcs/patches.
- Applying user-defined colors and materials. Using interactively all three possible light sources.
- Applying 2-dimensional textures.
- Use of vertex and fragment shaders, and communication with them.
- Input and output for later use.
- Highly customized and interactive graphical user interface for shape modification* (tool bars, icons, dialogs, handling mouse events, use of transparency/blending for update preview, highlighting affected arcs and patches, interactive specification of colors, materials, textures and shaders, etc.).

*Independently of minimal and maximal requirements, you should be able: to select the arcs/patches of composite spline curves/surfaces; to select the control polygon/net of the arcs/patches; to modify the position of the control points or the direction of (partial) derivatives; to update other attributes that are associated with the arcs/patches; to automatically preserve the order of continuity at/along the common joints/boundary curves.