Exercise 1: Rigid and perspective transformations in homogeneous coordinates

02504 Computer vision

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Homogeneous coordinates

Exercise 1.1

Consider the following four points in 2D

$$\boldsymbol{p}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \ \boldsymbol{p}_2 = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}, \ \boldsymbol{p}_3 = \begin{bmatrix} 6 \\ 4 \\ -1 \end{bmatrix}, \text{ and } \ \boldsymbol{p}_4 = \begin{bmatrix} 5 \\ 3 \\ 0.5 \end{bmatrix}, \tag{1}$$

written in their *homogeneous* form. What are their corresponding inhomogeneous coordinates $\boldsymbol{q}_i$?

Exercise 1.2

Consider now the following four points in 3D

$$\boldsymbol{P}_1 = \begin{bmatrix} 1 \\ 10 \\ -3 \\ 1 \end{bmatrix}, \ \boldsymbol{P}_2 = \begin{bmatrix} 2 \\ -4 \\ 1.1 \\ 2 \end{bmatrix}, \ \boldsymbol{P}_3 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 10 \end{bmatrix}, \text{ and } \ \boldsymbol{P}_4 = \begin{bmatrix} -15 \\ 3 \\ 6 \\ 3 \end{bmatrix}. \tag{3}$$

What are their corresponding inhomogeneous coordinates $\boldsymbol{Q}_i$?

Exercise 1.3

A 2D line is given as

$$x + 2y = 3\,. \tag{5}$$

Write this line in homogeneous form i.e. $\boldsymbol{l}^T\boldsymbol{p} = 0$.

Exercise 1.4

Using $\boldsymbol{l}$ from Ex. 1.3, which of the following 2D points are on this line?

$$\boldsymbol{p}_1 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \boldsymbol{p}_2 = \begin{bmatrix} 6 \\ 0 \\ 2 \end{bmatrix}, \boldsymbol{p}_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \boldsymbol{p}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \boldsymbol{p}_5 = \begin{bmatrix} 110 \\ -40 \\ 10 \end{bmatrix}, \text{ and } \boldsymbol{p}_6 = \begin{bmatrix} 11 \\ 4 \\ 1 \end{bmatrix}. \tag{7}$$

Exercise 1.5

Given the two lines

$$\boldsymbol{l}_0 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \text{ and } \boldsymbol{l}_1 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \tag{9}$$

what is their point of intersection $\boldsymbol{q}_\mathrm{I}$?

Exercise 1.6

Consider the following matrix

$$\boldsymbol{A} = \begin{bmatrix} 10 & 0 & 2 \\ 0 & 10 & -3 \\ 0 & 0 & 1 \end{bmatrix}. \tag{11}$$

What is the result of $\boldsymbol{A}\boldsymbol{p} = \boldsymbol{q}$, where $\boldsymbol{p}$ and $\boldsymbol{q}$ are 2D points in homogeneous coordinates? Explain what each of the (non-zero) coefficients in $\boldsymbol{A}$ does to the coordinates of $\boldsymbol{q}$.

Exercise 1.7

A new line is given

$$\boldsymbol{l} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ -1 \end{bmatrix}. \tag{12}$$

What is the (shortest) distance between the line $\boldsymbol{l}$ and the points

$$\boldsymbol{p}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \boldsymbol{p}_2 = \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \\ 1 \end{bmatrix}, \text{ and } \boldsymbol{p}_3 = \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \\ 4 \end{bmatrix}? \tag{13}$$

Exercise 1.8

Repeat Ex. 1.7 with

$$\boldsymbol{l} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}. \tag{16}$$

Programming exercises: Setting up

The programming exercises in this course will assume that you are programming in Python and using OpenCV. It will be possible to follow the course using a different language such as MATLAB, but you will be more on your own.

Exercise 1.9

Please install OpenCV 4.4.0 or later in your Python environment.

Tip: You can check which version you have installed by running

```
import cv2
print(cv2.__version__)
```

Exercise 1.10

In later weeks you will work on images your capture yourself. To prepare for this

- Capture an image with a camera
- Transfer it to your computer
- Load it with OpenCV
- Display it with Matplotlib

Tip: You can import Matplotlib as follows

```
import matplotlib.pyplot as plt
plt.imshow(im)
```

Tip: Do the colors of the image look weird? This is because OpenCV stores an image as (blue, green, red) while Matplotlib uses the more common (red, green blue). Flip the channels of the image after loading it in, to make the displayed colors normal.

Programming exercises: Pinhole camera

This time you will get familiar with manipulating points in programs. We will build a couple of helper functions, which in the end will let you project several 3D points into an image plane.

Exercise 1.11

First lets make some more 3D points to "photograph". Create a function that generates a number of 3D points in a recognizable shape like Fig. 1.

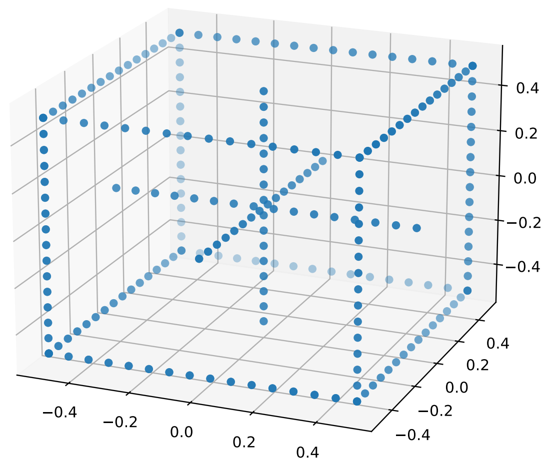

Figure 1: A 3D plot of a set of points generated by the `box3d` function. In this case $n = 16$.

Let us define a function `box3d(n)`, that generates a list of coordinates (a $3 \times n$ array) in a box shaped like Fig. 1. The box is made of points along the 12 edges and in addition, we insert a cross through the middle. Each line has exactly `n` points between -0.5 and 0.5.

Exercise 1.12

Now lets us make our "camera". Create a function `projectpoints`, that takes as inputs:

- the camera matrix $\boldsymbol{K}$
- the pose of the camera ($\boldsymbol{R}$, $\boldsymbol{t}$)
- a $3 \times n$ matrix ($\boldsymbol{Q}$), representing n points in 3D to be projected into the camera.

The function should return the projected 2D points as a $2 \times n$ matrix.

Test your function with $\boldsymbol{Q} = \texttt{box3d}(16)$,

$$\boldsymbol{K} = \boldsymbol{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ \boldsymbol{t} = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} \tag{18}$$

Tip: You can do matrix multiplication in Numpy using @, for example `A@b`.

Exercise 1.13

Try instead with $\quad \boldsymbol{R} = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$

where $\theta = 30°$.

What is the effect?

Exercise 1.14

Play around with changing $\boldsymbol{R}$ and $\boldsymbol{t}$.

What is the relationship between the position of the camera and $\boldsymbol{t}$?

Solutions

Answer of exercise 1.1

The four 2D points are given in standard coordinates

$$\boldsymbol{q}_1 = \begin{bmatrix}1\\2\end{bmatrix}\,,\ \boldsymbol{q}_2 = \begin{bmatrix}2\\1\end{bmatrix}\,,\ \boldsymbol{q}_3 = \begin{bmatrix}-6\\-4\end{bmatrix}\,,\ \text{and}\ \ \boldsymbol{q}_4 = \begin{bmatrix}10\\6\end{bmatrix}\,. \tag{2}$$

Answer of exercise 1.2

The four 2D points are given in standard coordinates

$$\boldsymbol{Q}_1 = \begin{bmatrix}1\\10\\-3\end{bmatrix}\,,\ \boldsymbol{Q}_2 = \begin{bmatrix}1\\-2\\0.55\end{bmatrix}\,,\ \boldsymbol{Q}_3 = \begin{bmatrix}0\\0\\-0.1\end{bmatrix}\,,\ \text{and}\ \ \boldsymbol{Q}_4 = \begin{bmatrix}-5\\1\\2\end{bmatrix}\,. \tag{4}$$

Answer of exercise 1.3

The 2D line $\boldsymbol{l}$ where $\boldsymbol{l}^{\mathrm{T}}\boldsymbol{p} = 0$ is

$$\boldsymbol{l} = \begin{bmatrix}1\\2\\-3\end{bmatrix} \tag{6}$$

Answer of exercise 1.4

Any point on the line must obey $\boldsymbol{l}^{\mathrm{T}}\boldsymbol{p} = 0$. Using the same line $\boldsymbol{l}$ as defined in answer of Ex. 1.3 we find

$$\boldsymbol{l}^{\mathrm{T}}\boldsymbol{p}_1 = 0\,,\ \ \boldsymbol{l}^{\mathrm{T}}\boldsymbol{p}_2 = 0\,,\ \ \boldsymbol{l}^{\mathrm{T}}\boldsymbol{p}_3 = -3\,,\ \ \boldsymbol{l}^{\mathrm{T}}\boldsymbol{p}_4 = 0\,,\ \ \boldsymbol{l}^{\mathrm{T}}\boldsymbol{p}_5 = 0\,,\ \text{and}\ \ \boldsymbol{l}^{\mathrm{T}}\boldsymbol{p}_6 = 16\,. \tag{8}$$

In other words, points $\boldsymbol{p}_1$, $\boldsymbol{p}_2$, $\boldsymbol{p}_4$, and $\boldsymbol{p}_5$ are all on the line $\boldsymbol{l}$; the points $\boldsymbol{p}_3$ and $\boldsymbol{p}_6$ are not.

Answer of exercise 1.5

The intersection of two lines is the cross product of the homogeneous coordinate definitions.

$$\boldsymbol{l}_0 \times \boldsymbol{l}_1 = \begin{bmatrix}-2\\4\\2\end{bmatrix}\,. \tag{10}$$

Answer of exercise 1.6

The direct solution is $\boldsymbol{Ap} = [10x + 2, 10y - 3, 1]$. The two 10's scale the x- and y-coordinates. If the last diagonal factor was 10 and not 1, this would have no effect on the inhomogeneous coordinates. The two remaining factors are 2D translations. Finally, also notice that the translations happened after scaling.

Answer of exercise 1.7

The shortest distance d between a line $\boldsymbol{l} = [l_x, l_y, l_w]^{\mathrm{T}}$ and a point $\boldsymbol{p}$ is given

$$d = \frac{\left|\boldsymbol{l}^{\mathrm{T}}\boldsymbol{p}\right|}{|p_w|\sqrt{l_x^2 + l_y^2}}\,. \tag{14}$$

The solutions are

$$d_1 = 1\,,\ d_2 = 1\,,\ \text{and}\ \ d_3 = 1/2\,. \tag{15}$$

Answer of exercise 1.8

The definition of the shortest distance d is given in answer of Ex. 1.7, and the new solutions are

$$d_0 = \frac{1}{\sqrt{8}} = 0.3536\,,\ d_1 = \frac{4\sqrt{2} - 1}{\sqrt{8}} = 1.6464\,,\ \text{and}\ \ d_2 = \frac{\sqrt{2} - 1}{\sqrt{8}} = 0.1464\,. \tag{17}$$