WTW 124 CLASS TEST 1: Student-Worked Solutions

Show your steps clearly and note that this is a closed book test.

1. Consider the vectors $\bar{u} = \langle 1, 1, 2 \rangle$, $\bar{v} = \langle 0, 1, 0 \rangle$ and $\bar{w} = \langle 0, 1, 1 \rangle$ in $\mathbb{R}^3$. Find (if possible) the value of $\bar{u} + (\bar{v} \cdot \bar{w})$. [2]

   By the definition of the dot product, $\bar{v} \cdot \bar{w} \in \mathbb{R}$. Therefore, by the definition of vector addition $\bar{u} + (\bar{v} \cdot \bar{w})$ does not exist.

2. Let $\bar{u}$ and $\bar{v}$ be vectors in $\mathbb{R}^n$. Prove that if $\bar{u}$ and $\bar{v}$ are not parallel and $a\bar{u} + b\bar{v} = a_1\bar{u} + b_1\bar{v}$ for $a, a_1, b, b_1 \in \mathbb{R}$, then $a = a_1$ and $b = b_1$. [3]

   Assume that $\bar{u} \nparallel \bar{v}$ and that

   $$\begin{aligned} & a\bar{u} + b\bar{v} = a_1\bar{u} + b_1\bar{v} \\ & \implies a\bar{u} + b\bar{v} + (-a_1\bar{u}) + (-b_1\bar{v}) = a_1\bar{u} + b_1\bar{v} + (-a_1\bar{u}) + (-b_1\bar{v}) \quad \text{[Adding } (-a_1\bar{u}) \text{ and } (-b_1\bar{v}) \text{ on both sides]} \\ & \implies (a - a_1)\bar{u} + (b - b_1)\bar{v} = \bar{0}. \quad \text{[Additive inverse property]} \end{aligned}$$

   Since $\bar{u}$ and $\bar{v}$ are not parallel, they are not scalar multiples of each other.

   Hence the implication is true if and only if both $a - a_1 = 0$ and $b - b_1 = 0$, by the zero scalar multiplicative property.

   Therefore $a = a_1$ and $b = b_1$.

3. Find α such that the vectors $\bar{u} = \langle -3, \alpha, 2 - \alpha \rangle$ and $\bar{v} = \langle 2, \alpha, \alpha - 2 \rangle$ are orthogonal. [2]

   $\bar{u}$ and $\bar{v}$ are orthogonal if

   $$\begin{aligned} & \bar{u} \cdot \bar{v} = 0 \\ & \implies \langle -3, \alpha, 2 - \alpha \rangle \cdot \langle 2, \alpha, \alpha - 2 \rangle \\ & \implies -6 + \alpha^2 + 2\alpha - 4 - \alpha^2 + 2\alpha = 0 \\ & \implies 4\alpha = 10 \\ & \implies \alpha = \frac{5}{2} \end{aligned}$$

4. Consider the lines

   $$L_1 = \{\langle 1 + t, 2, 3 + 2t \rangle : t \in \mathbb{R}\} \text{ and } L_2 = \{\langle 1, 2, 2 + s \rangle : s \in \mathbb{R}\}.$$

   Show in details that $L_1 \neq L_2$. [3]

   $L_1 \nparallel L_2$, by the definition of parallellism, so therefore, L_1 and L_2 have at least one point of intersection, or none at all, by the theorem on the relationship between two lines.

   L_1 and L_2 intersect at a point $\bar{x}$ if and only if $\exists s, t \in \mathbb{R}$ such that

   $$\bar{x} = \langle 1 + t, 2, 3 + 2t \rangle = \langle 1, 2, 2 + s \rangle.$$

   By the definition of vector equality, $\bar{x} \in L_1 \cap L_2$ if and only if

   $$\begin{aligned} & 1 + t = 1 \implies t = 0 \\ & 2 = 2 \\ & 3 + 2t = 2 + s \implies 3 + 2(0) = 2 + s \implies s = 1 \end{aligned}$$

   Therefore $\bar{x} = \langle 1, 2, 3 \rangle \in L_1 \cap L_2$.

   Now for L_1 and L_2 to be equal, it must hold that L_1 and L_2 intersect at another point $\bar{y}$, by the theorem on the equality of lines.

   Pick $t = 1$. Then $\bar{y} = \langle 2, 2, 5 \rangle \in L_1$. Note that $\nexists s \in \mathbb{R}$ that satisfis the condition that $\bar{y} \in L_2$.

   Therefore $L_1 \neq L_2$.