UNIVERSITY OF PRETORIA
FACULTY OF NATURAL AND AGRICULTURAL SCIENCES
DEPARTMENT OF MATHEMATICS AND APPLIED MATHEMATICS

WTW 124 Mathematics

SEMESTER TEST 1
26 August 2025

TIME: 120 mins
MARKS: 38

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SURNAME:

FIRST NAMES:

STUDENT NUMBER:

SIGNATURE: Student-Worked Solutions

READ THE FOLLOWING INSTRUCTIONS

1. This paper consists of this cover page and **5 more pages** containing questions 1-4. Check whether your paper is complete.
2. The use of all electronic equipment is forbidden: No candidate is allowed to use any i-pad, cell phone, calculator, smart watch, etc. while writing this paper.
3. Do all scribbling on the facing page. It will not be marked.
4. If you need more than the available space for an answer, use the facing page and please indicate it clearly.
5. No pencil work or any work in red ink will be marked.
6. If you use correcting fluid (Tipp-Ex or similar), you lose the right to question the marking or claim that work has not been marked.

Please note that this is not an official memo.
If you have any questions, consult a lecturer.

Question 1.

a) Find two unit vectors perpendicular to both vectors $\bar{u} = \langle 1, 2, -1\rangle$ and $\bar{v} = \langle 3, 1, 2\rangle$. [3]

Any vector perpendicular to both $\bar{u}$ and $\bar{v}$ is perpendicular to the vector given by

$$\begin{aligned}\bar{u} \times \bar{v} &= \langle 1, 2, -1\rangle \times \langle 3, 1, 2\rangle \\ &= \langle 4+1, 2+3, -1-6\rangle \\ &= \langle 5, 5, -5\rangle.\end{aligned}$$

Then $\|\langle 5, 5, -5\rangle\| = \sqrt{5^2 + 5^2 + (-5)^2} = 5\sqrt{3}$, so two unit vectors perpendicular to both $\bar{u}$ and $\bar{v}$ are

$$\begin{aligned}&\frac{1}{5\sqrt{3}}\langle 5, 5, -5\rangle = \frac{1}{\sqrt{3}}\langle 1, 1, -1\rangle = \langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\rangle \\ \text{and } &-\frac{1}{5\sqrt{3}}\langle 5, 5, -5\rangle = -\frac{1}{\sqrt{3}}\langle 1, 1, -1\rangle = \langle -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\rangle.\end{aligned}$$

b) Let $\bar{u}$ and $\bar{v}$ be two non-zero vectors such that $\bar{u} \cdot \bar{v} = \|\bar{u} \times \bar{v}\|$. Find the magnitude of the angle between the rays determined by $\bar{u}$ and $\bar{v}$ [3]

We have that

$$\cos\theta = \frac{\bar{u} \cdot \bar{v}}{\|\bar{u}\|\|\bar{v}\|} \quad \text{and} \quad \sin\theta = \frac{\|\bar{u} \times \bar{v}\|}{\|\bar{u}\|\|\bar{v}\|},$$

where θ is the angle between the rays determined by $\bar{u}$ and $\bar{v}$.

Then

$$\cos\theta = \frac{\|\bar{u} \times \bar{v}\|}{\|\bar{u}\|\|\bar{v}\|} = \sin\theta \implies \tan\theta = 1 \implies \theta = \frac{\pi}{4}.$$

c) Let A, B and C be three matrices such that ABC exists, where A has size 3×3 and C has size 5×5. Describe the sizes B and ABC. [2]

By the associative property of matrix multiplication, we have that $ABC = (AB)C$.

Since ABC exists, then AB must have size 3×5 (because C has size 5×5).

This means that B must have size 3×5 (because A has size 3×3).

Therefore, ABC has size 3×5.

Question 2.

a) Let $\bar{u}$ and $\bar{v}$ be vectors in $\mathbb{R}^3$. Prove that if $\bar{u} + \bar{v} = \bar{0}$, then $\bar{u} \times \bar{v} = \bar{0}$. [3]

Assume $\bar{u} + \bar{v} = \bar{0}$. Then by the properties of vector addition, we have that $\bar{v} = (-\bar{u})$, such that

$$\bar{u} + \bar{v} = \bar{u} + (-\bar{u}) = \bar{0}.$$

Then $\bar{u} \times \bar{v} = \bar{u} \times (-\bar{u})$.

Since $\bar{u}$ and $\bar{v}$ are scalar multiples of each other, it follows from the properties of the cross product that

$$\bar{u} \times \bar{v} = \bar{u} \times (-\bar{u}) = \bar{0}.$$

b) Let A be a 2×2 matrix. If $A(A - I) = 0$, then $A = 0$ or $A = I$; prove or disprove. [3]

For instance, let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

Then $A \neq 0$ and $A \neq I$.

We have:

$$\begin{aligned} A(A - I) &= \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \\ &= \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1-1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0. \end{aligned}$$

Therefore the claim is false.

c) Let A and B be $n \times n$ matrices. Prove that if $AB = BA$, then A^T commutes with B^T. [3]

Assume that $AB = BA$. We show that $A^T B^T = B^T A^T$

$$\begin{aligned} A^T B^T &= (BA)^T && \text{[Properties of the transpose]} \\ &= (AB)^T && \text{[By assumption]} \\ &= B^T A^T && \text{[Properties of the transpose]} \end{aligned}$$

Question 3.

a) Use Gaussian elimination to find (if possible) conditions on real numbers a such that the following system of linear equations has no solution: [4]

$$x + ay - z = 1$$
$$-x + (a-2)y + z = -1$$
$$2x + 2y + (a-2)z = 1$$

We write the system as an augmented matrix and apply Gaussian elimination:

$$\left[\begin{array}{ccc|c} 1 & a & -1 & 1 \\ -1 & a-2 & 1 & -1 \\ 2 & 2 & a-2 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & a & -1 & 1 \\ 0 & 2a-2 & 0 & 0 \\ 2 & 2 & a-2 & 1 \end{array}\right] \begin{pmatrix} R_2 + R_1 \to R_2 \\ R_3 - 2R_1 \to R_3 \end{pmatrix}$$
$$\sim \left[\begin{array}{ccc|c} 1 & a & -1 & 1 \\ 0 & 2a-2 & 0 & 0 \\ 2 & 2 & a-2 & 1 \end{array}\right] (R_3 + R_2 \to R_3)$$

Now if $a = 0$, then the third row implies that $0 = -1$, which is false.
Therefore, by the theorem on the consistency of a system of linear equations, the system does not have a solution when $a = 0$.

b) Consider the following matrices:

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 & -4 \\ -1 & -1 & 3 \\ -1 & -2 & 5 \end{bmatrix}.$$

Do A and B commute? Show all steps. [3]

Note that A and B both have size 3×3. Hence AB and BA both exist as 3×3 matrices.

$$AB = \begin{bmatrix} 1 & -2 & 2 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -4 \\ -1 & -1 & 3 \\ -1 & -2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} \langle 1,-2,2\rangle \cdot \langle 1,-1,-1\rangle & \langle 1,-2,2\rangle \cdot \langle 2,-1,-2\rangle & \langle 1,-2,2\rangle \cdot \langle -4,3,5\rangle \\ \langle 2,1,1\rangle \cdot \langle 1,-1,-1\rangle & \langle 2,1,1\rangle \cdot \langle 2,-1,-2\rangle & \langle 2,1,1\rangle \cdot \langle -4,3,5\rangle \\ \langle 1,0,1\rangle \cdot \langle 1,-1,-1\rangle & \langle 1,0,1\rangle \cdot \langle 2,-1,-2\rangle & \langle 1,0,1\rangle \cdot \langle -4,3,5\rangle \end{bmatrix}$$

$$= \begin{bmatrix} 1+2-2 & 2+2-4 & -4-6+10 \\ 2-1-2 & 4-1-2 & -8+3+5 \\ 1-1 & 2-2 & -4+5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$BA = \begin{bmatrix} 1 & 2 & -4 \\ -1 & -1 & 3 \\ -1 & -2 & 5 \end{bmatrix} \begin{bmatrix} 1 & -2 & 2 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \langle 1,2,-4\rangle \cdot \langle 1,2,1\rangle & \langle 1,2,-4\rangle \cdot \langle -2,1,0\rangle & \langle 1,2,-4\rangle \cdot \langle 2,1,1\rangle \\ \langle -1,-1,3\rangle \cdot \langle 1,2,1\rangle & \langle -1,-1,3\rangle \cdot \langle -2,1,0\rangle & \langle -1,-1,3\rangle \cdot \langle 2,1,1\rangle \\ \langle -1,-2,5\rangle \cdot \langle 1,2,1\rangle & \langle -1,-2,5\rangle \cdot \langle -2,1,0\rangle & \langle -1,-2,5\rangle \cdot \langle 2,1,1\rangle \end{bmatrix}$$

$$= \begin{bmatrix} 1+4-4 & -2+2+0 & 2+2-4 \\ -1-2+3 & 2-1+0 & -2-1+3 \\ -1-4+5 & 2-2+0 & -2-2+5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

By the definition of matrix equality, $AB = BA$, and A therefore commutes B.

Question 4.
Consider the following two lines in $\mathbb{R}^3$:

$$L_1 = \{\langle 1,2,-3\rangle + t\langle 1,2,-3\rangle : t \in \mathbb{R}\} \text{ and } L_2 = \{\langle -3,1,0\rangle + t\langle -2,4,-6\rangle : t \in \mathbb{R}\}.$$

a) If P is a plane with equation $2x - y - z = 3$, then $L_1 \subseteq P$.
Is this statement true or false? Explain with full details. [3]

P has normal vector $\bar{n} = \langle 2,-1,-1\rangle$ and L_1 has direction vector $\bar{d} = \langle 1,2,-3\rangle$.
Then $\bar{n} \cdot \bar{d} \neq 0$, by the properties of the dot product. By definition of parallelism, $\bar{n} \nparallel \bar{d}$, and so by the theorem on the relationship between a plane and a line, L_1 and P intesect in at least one point.

Let $\bar{y} \in L_1$ be a point where $t = 0$.
Then $\bar{y} = \langle 1,2,-3\rangle$ We have $2 - 2 - (-3) = 3$. Therefore $\bar{y} \in P$.

Let $\bar{x} \in L_1$ be a point where $t = 1$.
Then $\bar{x} = \langle 2,4,-6\rangle$. We have $2(4) - 4 + 6 = 10 \neq 3$. Hence $\bar{x} \notin P$.

Therefore $L_1 \nsubseteq P$. The claim is false.

b) Give Cartesian equations for two parallel, each containing one of the lines above. [4]

L_1 has direction vector $\bar{d_1} = \langle 1,2,-3\rangle$, and L_2 direction vector $\bar{d_2} = \langle -2,4,-6\rangle$.

Let P_1, P_2 be planes containing L_1 and L_2 respectively.
We know that $P_1 \parallel P_2$ if their normal vectors are parallel to each other.

So we need $\bar{n}$ such that $\bar{n} \perp \bar{d_1}$ and $\bar{n} \perp \bar{d_2}$. [Sketch it if you need to!]

$$\begin{aligned} \bar{n} &= \bar{d_1} \times \bar{d_2} \\ &= \langle -12+12, -6-6, 4+4\rangle \\ &= \langle 0,-12,8\rangle \end{aligned}$$

P_1 has equation

$$\begin{aligned} &\langle 0,-12,8\rangle \cdot (\langle x,y,z\rangle - \langle 1,2,-3\rangle) = 0 \\ &\Rightarrow \langle 0,-12,8\rangle \cdot \langle x-1, y-2, z+3\rangle = 0 \\ &\Rightarrow -12y + 24 + 8z + 24 = 0 \Rightarrow -12y + 8z = -48 \Rightarrow -3y + 2z = -12. \end{aligned}$$

P_2 has equation

$$\begin{aligned}
&\langle 0, -12, 8\rangle \cdot (\langle x, y, z\rangle - \langle -3, 1, 0\rangle) = 0 \\
&\Rightarrow \langle 0, -12, 8\rangle \cdot \langle x+3, y-1, z\rangle = 0 \\
&\Rightarrow -12y + 12 + 8z = 0 \Rightarrow -12y + 8z = -12 \Rightarrow -3y + 2z = -3
\end{aligned}$$

c) Let L be a line passing through the point $\bar{p} = \langle 2, 0, 0\rangle$ such that L is perpendicular to the line L_2 at their intersection. Find a vector equation of the line L. [4]

L_2 has equation $\bar{x} = \langle -3, 1, 0\rangle + t\langle -2, 4, -6\rangle$ for some $t \in \mathbb{R}$.
Recall that L_2 has direction vector $\bar{d_2} = \langle -2, 4, -6\rangle$. We want to find $\bar{d_3} = \langle x, y, z\rangle$ as the direction vector of the unknown line L.

$$\begin{aligned}
L_2 \perp L &\Rightarrow \bar{d_2} \cdot \bar{d_3} = 0 \\
&\Rightarrow \langle -2, 4, -6\rangle \cdot \langle x, y, z\rangle = 0 \\
&\Rightarrow -2x + 4y - 6z = 0
\end{aligned}$$

$$\text{Pick } x = 1 \text{ and } y = 0 \implies -2 - 6z = 0 \implies -6z = 2 \implies z = -\frac{1}{3}$$

Then $\bar{d_3} = \langle 1, 0, -\frac{1}{3}\rangle$.
Therefore L has vector equation

$$\bar{x} = \langle 2, 0, 0\rangle + t\langle 1, 0, -\frac{1}{3}\rangle \text{ for some } t \in \mathbb{R}.$$

d) Find the equation of the line through the point $\bar{p} = \langle 2, -1, 4\rangle$ and perpendicular to the plane $3x - 2y - z = 0$. [3]

Let Q be a plane with the equation $3x - 2y - z = 0$. Then Q has normal vector $\bar{m} = \langle 3, -2, -1\rangle$.
Let L be the line through $\bar{p}$ with direction vector $\bar{d}$.

L, P must be such that $L \perp P$, so it must be the case that $\bar{d} \parallel \bar{n}$.
Let $\bar{d} = 2\bar{m}$. Then by definition, $\bar{d} \parallel \bar{m}$, and furthermore $L \perp Q$!

Then an equation that satisfies this is:

$$L = \{\langle 2, -1, 4\rangle + t\langle 6, -4, -2\rangle : t \in R\}$$

Remark

These questions need a lot of geometric interpretation. If all fails in the test, **sketch a picture!**