



**Question 1.**

- a) Find two unit vectors perpendicular to both vectors  $\bar{u} = \langle 1, 2, -1 \rangle$  and  $\bar{v} = \langle 3, 1, 2 \rangle$ . [3]

Any vector perpendicular to both  $\bar{u}$  and  $\bar{v}$  is perpendicular to the vector given by

$$\begin{aligned}\bar{u} \times \bar{v} &= \langle 1, 2, -1 \rangle \times \langle 3, 1, 2 \rangle \\ &= \langle 4 + 1, 2 + 3, -1 - 6 \rangle \\ &= \langle 5, 5, -5 \rangle.\end{aligned}$$

Then  $\|\langle 5, 5, -5 \rangle\| = \sqrt{5^2 + 5^2 + (-5)^2} = 5\sqrt{3}$ , so two unit vectors perpendicular to both  $\bar{u}$  and  $\bar{v}$  are

$$\begin{aligned}\frac{1}{5\sqrt{3}}\langle 5, 5, -5 \rangle &= \frac{1}{\sqrt{3}}\langle 1, 1, -1 \rangle = \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right\rangle \\ \text{and } -\frac{1}{5\sqrt{3}}\langle 5, 5, -5 \rangle &= -\frac{1}{\sqrt{3}}\langle 1, 1, -1 \rangle = \left\langle -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle.\end{aligned}$$

- b) Let  $\bar{u}$  and  $\bar{v}$  be two non-zero vectors such that  $\bar{u} \cdot \bar{v} = \|\bar{u} \times \bar{v}\|$ . Find the magnitude of the angle between the rays determined by  $\bar{u}$  and  $\bar{v}$  [3]

We have that

$$\cos \theta = \frac{\bar{u} \cdot \bar{v}}{\|\bar{u}\| \|\bar{v}\|} \quad \text{and} \quad \sin \theta = \frac{\|\bar{u} \times \bar{v}\|}{\|\bar{u}\| \|\bar{v}\|},$$

where  $\theta$  is the angle between the rays determined by  $\bar{u}$  and  $\bar{v}$ .

Then

$$\cos \theta = \frac{\|\bar{u} \times \bar{v}\|}{\|\bar{u}\| \|\bar{v}\|} = \sin \theta \implies \tan \theta = 1 \implies \theta = \frac{\pi}{4}.$$

- c) Let  $A$ ,  $B$  and  $C$  be three matrices such that  $ABC$  exists, where  $A$  has size  $3 \times 3$  and  $C$  has size  $5 \times 5$ . Describe the sizes  $B$  and  $ABC$ . [2]

By the associative property of matrix multiplication, we have that  $ABC = (AB)C$ .

Since  $ABC$  exists, then  $AB$  must have size  $3 \times 5$  (because  $C$  has size  $5 \times 5$ ).

This means that  $B$  must have size  $3 \times 5$  (because  $A$  has size  $3 \times 3$ ).

Therefore,  $ABC$  has size  $3 \times 5$ .

**Question 2.**

- a) Let  $\bar{u}$  and  $\bar{v}$  be vectors in  $\mathbb{R}^3$ . Prove that if  $\bar{u} + \bar{v} = \bar{0}$ , then  $\bar{u} \times \bar{v} = \bar{0}$ . [3]

Assume  $\bar{u} + \bar{v} = \bar{0}$ . Then by the properties of vector addition, we have that  $\bar{v} = (-\bar{u})$ , such that

$$\bar{u} + \bar{v} = \bar{u} + (-\bar{u}) = \bar{0}.$$

Then  $\bar{u} \times \bar{v} = \bar{u} \times (-\bar{u})$ .

Since  $\bar{u}$  and  $\bar{v}$  are scalar multiples of each other, it follows from the properties of the cross product that

$$\bar{u} \times \bar{v} = \bar{u} \times (-\bar{u}) = \bar{0}.$$

- b) Let  $A$  be a  $2 \times 2$  matrix. If  $A(A - I) = 0$ , then  $A = 0$  or  $A = I$ ; prove or disprove. [3]

For instance, let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Then  $A \neq 0$  and  $A \neq I$ .

We have:

$$\begin{aligned} A(A - I) &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \left( \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \\ &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ &= 0. \quad \text{[By the properties of matrix multiplication]} \end{aligned}$$

Therefore the claim is false.

- c) Let  $A$  and  $B$  be  $n \times n$  matrices. Prove that if  $AB = BA$ , then  $A^T$  commutes with  $B^T$ . [3]

Assume that  $AB = BA$ . We show that  $A^T B^T = B^T A^T$

$$\begin{aligned} A^T B^T &= (BA)^T && \text{[Properties of the transpose]} \\ &= (AB)^T && \text{[By assumption]} \\ &= B^T A^T && \text{[Properties of the transpose]} \end{aligned}$$

**Question 3.**

- a) Use Gaussian elimination to find (if possible) conditions on real numbers  $a$  such that the following system of linear equations has no solution: [4]

$$\begin{aligned}x + ay - z &= 1 \\ -x + (a - 2)y + z &= -1 \\ 2x + 2y + (a - 2)z &= 1\end{aligned}$$

We write the system as an augmented matrix and apply Gaussian elimination:

$$\begin{aligned}& \left[ \begin{array}{ccc|c} 1 & a & -1 & 1 \\ -1 & a-2 & 1 & -1 \\ 2 & 2 & a-2 & 1 \end{array} \right] \sim \left[ \begin{array}{ccc|c} 1 & a & -1 & 1 \\ 0 & 2a-2 & 0 & 0 \\ 2 & 2 & a-2 & 1 \end{array} \right] \left( \begin{array}{l} R_2 + R_1 \rightarrow R_2 \\ R_3 - 2R_1 \rightarrow R_3 \end{array} \right) \\& \sim \left[ \begin{array}{ccc|c} 1 & a & -1 & 1 \\ 0 & 2a-2 & 0 & 0 \\ 2 & 2 & a-2 & 1 \end{array} \right] (R_3 + R_2 \rightarrow R_3)\end{aligned}$$

Now if  $a = 0$ , then the third row implies that  $0 = -1$ , which is false.

Therefore, by the theorem on the consistency of a system of linear equations, the system does not have a solution when  $a = 0$ .

- b) Consider the following matrices:

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 & -4 \\ -1 & -1 & 3 \\ -1 & -2 & 5 \end{bmatrix}.$$

Do  $A$  and  $B$  commute? Show all steps. [3]

Note that  $A$  and  $B$  both have size  $3 \times 3$ . Hence  $AB$  and  $BA$  both exist as  $3 \times 3$  matrices.

$$\begin{aligned}AB &= \begin{bmatrix} 1 & -2 & 2 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -4 \\ -1 & -1 & 3 \\ -1 & -2 & 5 \end{bmatrix} \\&= \begin{bmatrix} \langle 1, -2, 2 \rangle \cdot \langle 1, -1, -1 \rangle & \langle 1, -2, 2 \rangle \cdot \langle 2, -1, -2 \rangle & \langle 1, -2, 2 \rangle \cdot \langle -4, 3, 5 \rangle \\ \langle 2, 1, 1 \rangle \cdot \langle 1, -1, -1 \rangle & \langle 2, 1, 1 \rangle \cdot \langle 2, -1, -2 \rangle & \langle 2, 1, 1 \rangle \cdot \langle -4, 3, 5 \rangle \\ \langle 1, 0, 1 \rangle \cdot \langle 1, -1, -1 \rangle & \langle 1, 0, 1 \rangle \cdot \langle 2, -1, -2 \rangle & \langle 1, 0, 1 \rangle \cdot \langle -4, 3, 5 \rangle \end{bmatrix} \\&= \begin{bmatrix} 1+2-2 & 2+2-4 & -4-6+10 \\ 2-1-2 & 4-1-2 & -8+3+5 \\ 1-1 & 2-2 & -4+5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I\end{aligned}$$

$$\begin{aligned}
BA &= \begin{bmatrix} 1 & 2 & -4 \\ -1 & -1 & 3 \\ -1 & -2 & 5 \end{bmatrix} \begin{bmatrix} 1 & -2 & 2 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \langle 1, 2, -4 \rangle \cdot \langle 1, 2, 1 \rangle & \langle 1, 2, -4 \rangle \cdot \langle -2, 1, 0 \rangle & \langle 1, 2, -4 \rangle \cdot \langle 2, 1, 1 \rangle \\ \langle -1, -1, 3 \rangle \cdot \langle 1, 2, 1 \rangle & \langle -1, -1, 3 \rangle \cdot \langle -2, 1, 0 \rangle & \langle -1, -1, 3 \rangle \cdot \langle 2, 1, 1 \rangle \\ \langle -1, -2, 5 \rangle \cdot \langle 1, 2, 1 \rangle & \langle -1, -2, 5 \rangle \cdot \langle -2, 1, 0 \rangle & \langle -1, -2, 5 \rangle \cdot \langle 2, 1, 1 \rangle \end{bmatrix} \\
&= \begin{bmatrix} 1+4-4 & -2+2+0 & 2+2-4 \\ -1-2+3 & 2-1+0 & -2-1+3 \\ -1-4+5 & 2-2+0 & -2-2+5 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I
\end{aligned}$$

By the definition of matrix equality,  $AB = BA$ , and  $A$  therefore commutes  $B$ .

#### Question 4.

Consider the following two lines in  $\mathbb{R}^3$ :

$$L_1 = \{\langle 1, 2, -3 \rangle + t\langle 1, 2, -3 \rangle : t \in \mathbb{R}\} \text{ and } L_2 = \{\langle -3, 1, 0 \rangle + t\langle -2, 4, -6 \rangle : t \in \mathbb{R}\}.$$

- a) If  $P$  is a plane with equation  $2x - y - z = 3$ , then  $L_1 \subseteq P$ .

Is this statement true or false? Explain with full details.

[3]

$P$  has normal vector  $\bar{n} = \langle 2, -1, -1 \rangle$  and  $L_1$  has direction vector  $\bar{d} = \langle 1, 2, -3 \rangle$ .

Then  $\bar{n} \cdot \bar{d} \neq 0$ , by the properties of the dot product. By definition of parallelism,  $\bar{n} \nparallel \bar{d}$ , and so by the theorem on the relationship between a plane and a line,  $L_1$  and  $P$  intersect in at least one point.

Let  $\bar{x} \in L_1$  be a point where  $t = 1$ .

Then  $\bar{x} = \langle 2, 4, -6 \rangle$ . We have  $2(4) - 4 + 6 = 10 \neq 3$ . Hence  $\bar{x} \notin P$ , and  $L_1 \not\subseteq P$ .

The claim is false.

- b) Give Cartesian equations for two parallel planes, each containing one of the lines above.

[4]

$L_1$  has direction vector  $\bar{d}_1 = \langle 1, 2, -3 \rangle$ , and  $L_2$  has direction vector  $\bar{d}_2 = \langle -2, 4, -6 \rangle$ .

Let  $P_1, P_2$  be planes containing  $L_1$  and  $L_2$  respectively.

We know that  $P_1 \parallel P_2$  if their normal vectors are parallel to each other.

So we need  $\bar{n}$  such that  $\bar{n} \perp \bar{d}_1$  and  $\bar{n} \perp \bar{d}_2$ .

[Sketch it if you need to!]

$$\begin{aligned}
\bar{n} &= \bar{d}_1 \times \bar{d}_2 \\
&= \langle -12 + 12, -6 - 6, 4 + 4 \rangle \\
&= \langle 0, -12, 8 \rangle
\end{aligned}$$

$P_1$  has equation

$$\begin{aligned}
&\langle 0, -12, 8 \rangle \cdot (\langle x, y, z \rangle - \langle 1, 2, -3 \rangle) = 0 \\
&\Rightarrow \langle 0, -12, 8 \rangle \cdot \langle x - 1, y - 2, z + 3 \rangle = 0 \\
&\Rightarrow -12y + 24 + 8z + 24 = 0 \Rightarrow -12y + 8z = -48 \Rightarrow -3y + 2z = -12.
\end{aligned}$$

$P_2$  has equation

$$\begin{aligned}
&\langle 0, -12, 8 \rangle \cdot (\langle x, y, z \rangle - \langle -3, 1, 0 \rangle) = 0 \\
&\Rightarrow \langle 0, -12, 8 \rangle \cdot \langle x + 3, y - 1, z \rangle = 0 \\
&\Rightarrow -12y + 12 + 8z = 0 \Rightarrow -12y + 8z = -12 \Rightarrow -3y + 2z = -3
\end{aligned}$$

- c) Let  $L$  be a line passing through the point  $\bar{p} = \langle 2, 0, 0 \rangle$  such that  $L$  is perpendicular to the line  $L_2$  at their intersection. Find a vector equation of the line  $L$ . [4]

$L_2$  has equation  $\bar{x} = \langle -3, 1, 0 \rangle + t\langle -2, 4, -6 \rangle$  for some  $t \in \mathbb{R}$ .

Recall that  $L_2$  has direction vector  $\bar{d}_2 = \langle -2, 4, -6 \rangle$ . We want to find  $\bar{d}_3 = \langle x, y, z \rangle$  as the direction vector of the unknown line  $L$ .

$$\begin{aligned} L_2 \perp L &\Rightarrow \bar{d}_2 \cdot \bar{d}_3 = 0 \\ &\Rightarrow \langle -2, 4, -6 \rangle \cdot \langle x, y, z \rangle = 0 \\ &\Rightarrow -2x + 4y - 6z = 0 \end{aligned}$$

$$\text{Pick } x = 1 \text{ and } y = 0 \Rightarrow -2 - 6z = 0 \Rightarrow -6z = 2 \Rightarrow z = -\frac{1}{3}$$

Then  $\bar{d}_3 = \langle 1, 0, -\frac{1}{3} \rangle$ .

Therefore  $L$  has vector equation

$$\bar{x} = \langle 2, 0, 0 \rangle + t\langle 1, 0, -\frac{1}{3} \rangle \text{ for some } t \in \mathbb{R}.$$

- d) Find the equation of the line through the point  $\bar{p} = \langle 2, -1, 4 \rangle$  and perpendicular to the plane  $3x - 2y - z = 0$ . [3]

Let  $Q$  be a plane with the equation  $3x - 2y - z = 0$ . Then  $Q$  has normal vector  $\bar{m} = \langle 3, -2, -1 \rangle$ .

Let  $L$  be the line through  $\bar{p}$  with direction vector  $\bar{d}$ .

$L, P$  must be such that  $L \perp P$ , so it must be the case that  $L \parallel \bar{n}$ .

Let  $\bar{d} = 2\bar{m}$ . Then by definition,  $\bar{d} \parallel \bar{m}$ , and furthermore  $L \perp Q$ !

Then an equation that satisfies this is:

$$L = \{ \langle 2, -1, 4 \rangle + t\langle 6, -4, -2 \rangle : t \in \mathbb{R} \}$$

#### Remark

These questions need a lot of geometric interpretation. If all fails in the test, **sketch a picture!**