WTW 124 **CLASS TEST 2: Student-Worked Solutions**

Show your steps clearly and note that this is a closed book test.

1. Let $\bar{a}, \bar{b} \in \mathbb{R}^3$. Then $\bar{a} \times \bar{b} = \bar{b} \times \bar{a}$. True or False? Support your answer in detail. [2]

   Consider $\bar{a} = \langle 1, 0, 1 \rangle$ and $\bar{b} = \langle 2, 1, 2 \rangle$.

   Then $\bar{a} \times \bar{b} \quad = \langle 0-1, 2-2, 1-0 \rangle = \langle -1, 0, 1 \rangle$

   and $\bar{b} \times \bar{a} \quad = \langle 1-0, 2-2, 0-1 \rangle = \langle 1, 0, -1 \rangle$

   Therefore $\bar{a} \times \bar{b} = \bar{b} \times \bar{a}$. The claim is false.

2. Let

$$A = \begin{bmatrix} 1 & 4 & 3 \end{bmatrix}.$$

   Find, if possible, AA^T and A^TA. [3]

   A has size 1×3 and $A^T = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$ has size 3×1.

   Hence AA^T and A^TA exist as matrices with sizes 1×1 and 3×3 respectively.

$$\begin{aligned} AA^T &= \begin{bmatrix} 1 & 4 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} \langle 1,4,3 \rangle \cdot \langle 1,4,3 \rangle \end{bmatrix} \\ &= \begin{bmatrix} 1 + 16 + 9 \end{bmatrix} = \begin{bmatrix} 26 \end{bmatrix}. \end{aligned}$$

$$\begin{aligned} A^TA &= \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 4 & 3 \end{bmatrix} \\ &= \begin{bmatrix} \langle 1 \rangle \cdot \langle 1 \rangle & \langle 1 \rangle \cdot \langle 4 \rangle & \langle 1 \rangle \cdot \langle 3 \rangle \\ \langle 4 \rangle \cdot \langle 1 \rangle & \langle 4 \rangle \cdot \langle 4 \rangle & \langle 4 \rangle \cdot \langle 3 \rangle \\ \langle 3 \rangle \cdot \langle 1 \rangle & \langle 3 \rangle \cdot \langle 4 \rangle & \langle 3 \rangle \cdot \langle 3 \rangle \end{bmatrix} \\ &= \begin{bmatrix} 1 & 4 & 3 \\ 4 & 16 & 12 \\ 3 & 12 & 9 \end{bmatrix} \end{aligned}$$

3. Let A be a 2×2 matrix. Prove or disprove the following statement: if $A \neq 0$, then $A^2 \neq 0$. [2]

   Consider $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \neq 0$.

$$\begin{aligned} \text{Then } A^2 &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0. \end{aligned}$$

   The claim is false.

4. Find, if possible, conditions on $a, b \in \mathbb{R}$ such that the following system of linear equations has only one solution, by using Gaussian elimination: [3]

$$\begin{aligned} -x + 3y + 2z &= -8 \\ x + z &= 2 \\ 3x + 3y + az &= b. \end{aligned}$$

We interpret the system as an augmented matrix and apply Gaussian elimination:

$$\left[\begin{array}{ccc|c} -1 & 3 & 2 & -8 \\ 1 & 0 & 1 & 2 \\ 3 & 3 & a & b \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ -1 & 3 & 2 & -8 \\ 3 & 3 & a & b \end{array}\right] (R_1 \leftrightarrow R_2) \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 3 & 3 & -6 \\ 3 & 3 & a & b \end{array}\right] (R_2 + R_1 \to R_2)$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 3 & 3 & -6 \\ 0 & 3 & a-3 & b-6 \end{array}\right] (R_3 - 3R_1 \to R_3) \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 3 & 3 & -6 \\ 0 & 0 & a-6 & b-12 \end{array}\right] (R_3 - R_2 \to R_3)$$

Then by the theorem on the consistency of a linear system, the system has a unique solution if and only if

$$\forall b \in \mathbb{R}, a \neq 6$$

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