

Chapter 1

Spectral Lines

Observations of spectral lines provide most of the key information about the Universe: from the speed of objects to their abundance to their temperature and environment. Observations of the motions of spectral lines have led to a measurement of the size of the Universe to detect planets around other stars. So, how does something so fundamental to Astronomy actually form? What determines their positions? Intensity? Shape? Depth?

1.1 A Simple Model

Let's begin with a simple two level model for an atom. In this model, an electron will be sitting in one of two energy levels. Either an upper energy level or lower energy level as in Figure 1. For this model, three different situations are possible: Spontaneous Emission, Absorption, and Stimulate Emission. If the electron is in the upper states, there is some probability that the electron will decay to the lower state and on this decay, it will emit a photon with the energy proportional to the difference between the two states. This is spontaneous emission. If the electron is in the lower state, there is some probability that the electron will absorb the photon and move to the upper state. This is absorption. Finally, there is stimulated emission: If a photon with an energy equivalent to the energy difference between the levels passes by, there is some probability that it will 'convince' the electron to transition to the lower level.

1.2 Spontaneous Emission

The rate of decay for an electron between the upper level and the lower level is given by:

$$\frac{dn}{dt} = n_u A_{ul} \quad (1.1)$$

where n_u is the density of electrons in the upper state and A_{ul} is the Einstein coefficient for spontaneous emission and has units of $[s^{-1}]$. This can be calculated from first principles using

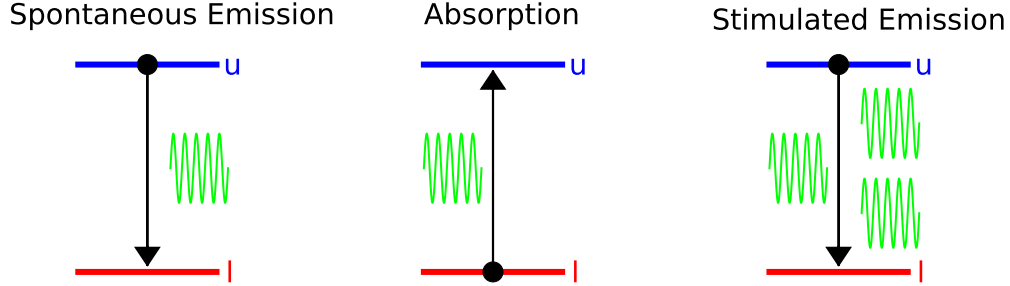


Figure 1.1: Diagram for the transition of an electron under spontaneous emission, absorption, and stimulated emission

quantum mechanics knowing the wave functions and dipole moment for the atom. When the electron transitions from the upper level to the lower level, it will emit a photon with energy equal to the energy difference between the levels. This will give a total amount of emission equal to:

$$j_\nu \rho = h\nu n_u A_{ul} \Psi(\nu) \quad (1.2)$$

In this equation, j_ν is the emissivity coefficient, ρ is the density of atoms, h is Planks constant, ν is the frequency of the emitted photon, and $\Psi(\nu)$ is the functional form of the line profile.

1.3 Absorption

The rate of electrons transitioning between the lower to the upper level due to absorption is going to be given by:

$$\frac{dn}{dt} = n_l B_{lu} I_\nu \quad (1.3)$$

where n_l is the density of electrons in the lower state, I_ν is the intensity of radiation, and B_{lu} is the Einstein coefficient for absorption. It gives the probability per unit time per unit intensity that a photon will be absorbed and it has units of $[sr \text{ cm}^2 \text{ ergs}^{-1} \text{ s}^{-1}]$. The absorption coefficient is then given by:

$$\kappa_\nu \rho I_\nu = h\nu n_l B_{lu} I_\nu \Phi(\nu) \quad (1.4)$$

In this equation, the line profile is given by $\Phi(\nu)$.

The value of the absorption coefficient is also typically given in terms of the oscillator strength. The oscillator strength can be calculated for the simplest systems analytical, but for more complex systems are typically measured in the laboratory. The relationship between the oscillatory strength and the B coefficient is given as:

$$h\nu B_{ij} = \frac{e^2\pi}{4\pi\epsilon_0 m_e} f_{ij} \quad (1.5)$$

where the oscillator strength has the property that $g_i f_{ij} = -g_j f_{ji}$.

1.4 Stimulated Emission

Stimulated emission is the effect where a photon will induce an electron to transition from the upper to the lower level. The transition will depend on the Einstein coefficient, B_{ul} , that gives the probability that the decay will occur. The rate of stimulation emission is then given by:

$$\frac{dn}{dt} = n_u B_{ul} I_\nu \quad (1.6)$$

As this is very similar to absorption and depends on the incident radiation field, it is treated as an absorptive process and the absorption coefficient for stimulated emission is given as:

$$\kappa_\nu \rho I_\nu = -h\nu n_u B_{ul} I_\nu \Phi(\nu) \quad (1.7)$$

Notice that this absorption coefficient has a negative value! This process results in the net *increase* in the number of photons that are in the radiation field.

1.5 Transitions in Thermal Equilibrium

In thermal equilibrium, the number of photons transitioning down will be equal to the number of photons transitioning up. Hence,

$$A_{ul} n_u + B_{ul} n_u I_\nu = B_{lu} n_l I_\nu \quad (1.8)$$

This equation can be reformatted such that:

$$I_\nu = \frac{A_{ul}/B_{ul}}{\left(\frac{B_{lu} n_l}{B_{ul} n_u} - 1\right)} \quad (1.9)$$

Now, under conditions of thermal equilibrium, the ratio of one state to another will be given by the Boltzman equation,

$$\frac{n_i}{n_j} = \frac{g_i}{g_j} e^{-(E_i - E_j)/kT} \quad (1.10)$$

and I_ν will be given by the equation for blackbody radiation:

$$I_\nu = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1}. \quad (1.11)$$

Substituting in for both of these equations into Eq. 1.9 then gives:

$$\frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1} = \frac{A_{ul}/B_{ul}}{(\frac{B_{lu}g_l}{B_{ul}g_u}e^{h\nu/kT} - 1)} \quad (1.12)$$

In order for these equations to equal each other, then the following has to be true:

$$\frac{A_{ul}}{B_{ul}} = \frac{2h\nu^3}{c^2} \quad (1.13)$$

and

$$B_{ul}g_u = B_{lu}g_l \quad (1.14)$$

NB: Sometimes this derivation and the B Einstein coefficients are given in terms of the Energy Density and not the Intensity. In this case, there is an extra factor of $c/4\pi$ and the B coefficients have units of $[cm^3 erg^{-1} s^{-1}]$.

1.6 The Total Absorption Coefficient

The total Absorption coefficient is going to be given by the linear combination of the absorption and stimulated emission. It will have the following value:

$$\kappa_\nu \rho = h\nu \Phi(\nu) n_l B_{lu} (1 - \frac{n_u B_{ul}}{n_l B_{lu}}) \quad (1.15)$$

1.6.1 Example: The Absorption Coefficient in different limits

Since we now have a value for κ , we can take a look at the absorption coefficient in different limits. Under the conditions of thermal equilibria, we can substitute for the number density and Einstein coefficients to get:

$$\kappa \rho = h\nu B_{lu} n_l \Phi(\nu) (1 - e^{-h\nu/kT}) \quad (1.16)$$

If we take the following two limits:

$$h\nu \gg kT \rightarrow \kappa \rho = h\nu n B_{lu} \Phi(\nu)$$

$$h\nu \ll kT \rightarrow \kappa \rho = 0$$

In the first situation, we only have absorption. In the second situation, stimulated emission balances out absorption and there is effectively no total absorption.

However, what if we are not in thermal equilibrium? If there is something driving more particles into the upper state than into the lower state such that $n_u > n_l(g_u/g_l)$ then stimulated emission will be greater than the effect of absorption and the intensity of the radiation field will be amplified. This has been observed in astronomy as a type of objects known as masers: Microwave Amplification by Stimulated Emission of Radiation. At visible wavelengths, this population inversion and stimulated emission is the basic physics behind devices that are Light Amplification by Stimulated Emission of Radiation (or more commonly known as lasers).

1.7 The Source Function

The source function is given as the ration of the emissivity and absorption coefficients. By substituting in from Eq. 1.2 and 1.15, we find S_ν is:

$$S_\nu = \frac{j_\nu}{\kappa_\nu} = \frac{n_u A_{ul} \Psi(\nu)}{(n_l B_{lu} - n_u B_{ul}) \Phi(\nu)} \quad (1.17)$$

This equation can be simplified by making the substitutions for the Einstein coefficients found at the end of §1.5.

1.7.1 Example: The Source function in LTE

In local thermodynamic equilibrium, the line profiles will be equal to each other and making the substations for the Einstein coefficients found at the end of §1.5 will give the source function for line transitions as

$$S_\nu = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1},$$

which is the same as blackbody radiation. So the amount of emission coming from a transition in LTE is effectively a blackbody. So the height of a line or the depth of an absorption feature can be effectively thought of being given by the blackbody function.

1.8 Line Profiles

A number of different processes exist for the broadening of lines. The simplistic two level model for the atom is just that: simplistic. It does not include quantum mechanic effects like the uncertainty in the energy of a level or in proper relativistic treatment of the electron. Macroscopic effects like the temperature of the gas will also broaden lines as the Doppler shift of the emitting atoms relative to those that are absorbing with the same cloud. These effects are accounted for via the line profile function in the emissivity and absorption coefficients. In this section, we go a little deeper into looking at what these values are:

1.8.1 Natural

The natural broadening of the line is due to the uncertainty in the energy and life time of each level. It is described by a Lorentzian function, which have the general shape of a narrow peak and broad wings. The line profile from natural broadening is given by:

$$\Phi\nu = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\Gamma/4\pi}{(\nu - \nu_o)^2 + (\Gamma/4\pi)^2} d\nu \quad (1.18)$$

where Γ is the natural line broadening and is calculated by summing over all possible transitions between states.

1.8.2 Thermal

Thermal broadening is due to the thermal motions of the atoms in the object and the relative Doppler shift from the relative velocity of the emitting atom. The line profile for thermal broadening is:

$$\Phi\nu = \frac{\Delta\nu_D}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(\frac{\nu-\nu_o}{\Delta\nu_D})^2} d\nu \quad (1.19)$$

where $\nu_D = \frac{b}{c}\nu_o$ and $b = (\frac{2kT}{m})^{1/2}$ is the Doppler broadening.

1.8.3 Pressure

Pressure broadening is related to the presence of nearby particles. There are a number of different ways this can effect the line broadening, but in most circumstances is caused by collision Al effects between the particles. In general, it can be described by a Lorentzian function and may also introduce a shift in the center of the line.

1.8.4 Voigt

The Voigt line profile is a convolution of a Lorentzian and Gaussian line profiles. In the extremes, it will behave either as a Gaussian or a Lorentzian. The line profile for a Voigt distribution is given by:

$$\Phi(\nu) = \frac{\alpha/\pi^{3/2}}{\Delta\nu_D} \int_{-\infty}^{\infty} \frac{e^{-y^2} dy}{(V - y)^2 + \alpha^2} \quad (1.20)$$

where $y = (\nu' - \nu_o)/\Delta\nu_D$, $V = (\nu - \nu_o)/\Delta\nu_D$, and $\alpha = \Gamma/4\pi/\Delta\nu_D$.

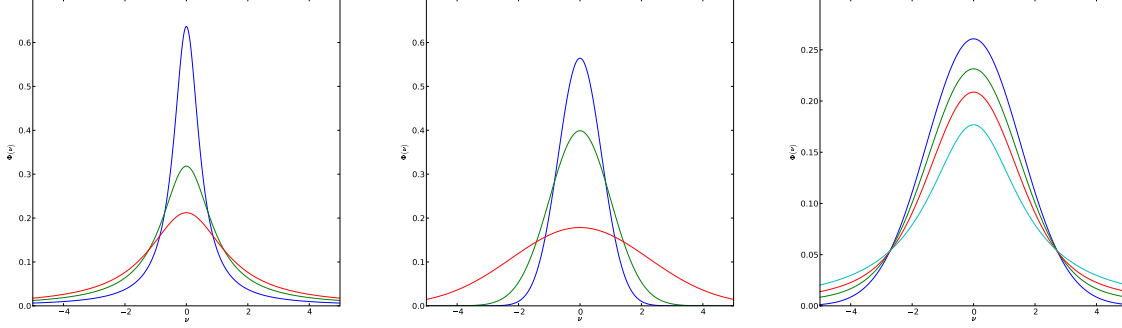


Figure 1.2: (left) A Lorentzian distribution for different values of Γ . (center) A thermal profile with different values for $\Delta\nu_D$. (right) A Voigt profile with differing values for Γ and $\Delta\nu_D$. The Lorentzian profile is evident from the extended wings in the Voigt profile whereas the thermal broadening creates a thicker profile.

1.9 Example: Curve of Growth

The curve of growth is a model for how the area of an absorption feature changes with changes in the oscillator strength, temperature, and density. The area of the line is a useful measurement as it is not effected by changes in resolution, whereas the shape of the profile may be blurred in lower resolution observations or even unresolved. The area of the feature is typically defined by the Equivalent Width, which is define as the width of an absorption feature if it had a unit height of the continuum level. In other words, if you could measure the area of the absorption feature, how wide would a rectangle with the same area and with one side equal to the height of the continuum intensity be? This is the equivalent width. It is typically given by the following equation:

$$W_\nu = \int \frac{I_c - I_\nu}{I_c} d\nu \quad (1.21)$$

where I_c is the intensity of the continuum. Assuming just absorption, the equivalent width is then given by:

$$W_\nu = \int (1 - e^{-\tau_\nu}) d\nu \quad (1.22)$$

And it is useful to note $W_\nu/\nu = W_\lambda/\lambda$. Equivalent width is typically given in units of either frequency or wavelength with W_λ is given in \AA .

We can make the following substitution for the optical depth and use Eq. 1.4 and 1.5 to substitute for the absorption coefficient to give:

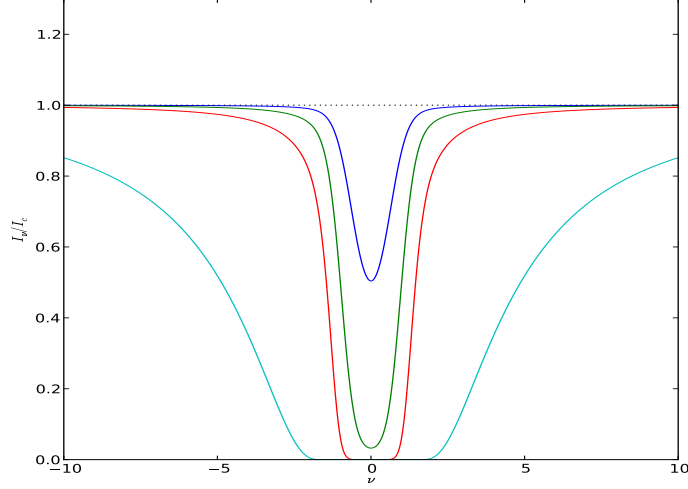


Figure 1.3: Example of lines as they range from optical thin, optically thick and dominated by thermal broadening, and very thick and affected by Natural broadening.

$$\tau_\nu = \int \kappa \rho ds = \int \left(\frac{\pi e^2}{m_e c} \right) f_{ij} \Phi(\nu) n ds$$

$$\tau_\nu = \frac{\pi e^2}{m_e c} f_{ij} \Phi(\nu) N \quad (1.23)$$

where $N = \int n ds$ and is the column density. It is the number of particles along the line of sight.

In the Figure below, we give an example of what the curve of growth looks like. It can typically be broken into three parts where the different parts are dominated by different physical phenomenon. The three parts are linear, flat, and sqrt and below we look into depth into these three regimes.

1.9.1 Optical Thin

When $\tau_\nu \ll 1$, then $W_\nu = \int \tau_\nu d\nu$. In this case, we can substitute Eq. 1.23 and only the line profile will depend on ν leaving:

$$W_\nu = N \frac{\pi e^2}{m_e c} f_{ij} \int \Phi(\nu) d\nu = 0.02654 \text{ cm}^2/s \ N f_{ij}$$

Or in terms of W_λ/λ ,

$$W_\lambda/\lambda = 8.85 \times 10^{-13} \text{ cm } N \lambda f_{ij} \quad (1.24)$$

This is known as the linear part of the curve of growth because $W_\lambda \sim N$. As the column density increases, the equivalent width also increases.

1.9.2 Thermally Dominated

In the flat part of the curve, the equivalent width is barely effected by a change in the number density and this is referred to as the flat part of the curve. During this part of the curve, the line profile is dominated by thermal broadening, and the optical depth can be approximated as:

$$\tau_\nu = \tau_o e^{-((\nu - \nu_o)/\Delta\nu_D)^2}$$

$$\tau_o = \frac{N f_{ij}}{\sqrt{\pi} \Delta\nu_D} \frac{\pi e^2}{m_e c} = 0.01497 N f_{ij} \lambda / b$$

In this case, the equivalent width is then going to be given by:

$$W_\nu = \int (1 - e^{-\tau_o e^{-((\nu - \nu_o)/\Delta\nu_D)^2}}) d\nu$$

,

which when evaluated, it will have the following form in terms of W_λ :

$$W_\lambda/\lambda = \frac{-2b}{c} F(\tau_o) \quad (1.25)$$

where $F(\tau_o) \approx \sqrt{\ln(N f_{ij} \lambda)}$. As can be seen, W_{λ}/λ only has a very weak depend on $N f_{ij} \lambda$. Hence the reason this is called the flat part of the curve.

Lines in this part of the curve are dense and optically thick in the centers. They are absorbing almost all of the light and will grow very slowly. Being dominated by thermal broadening, the sides are well described by a Gaussian function.

1.9.3 Very Optically Thick

For very strong lines, the natural dampening of the lines becomes the dominate effect in broadening the lines. In this case, the wings of the lines are then dominated by Lorentzian form of the Voigt profile. As such, the equivalent width is going to be:

$$W_\lambda/\lambda = \frac{1}{c} ((N f_{ij} \lambda)(\lambda) \frac{\Gamma}{\pi} (\frac{\pi e^2}{m_e c}))^{1/2} \quad (1.26)$$

In this case, the area of the curve increases by $N^{1/2}$ and is known as the square-root part of the curve.

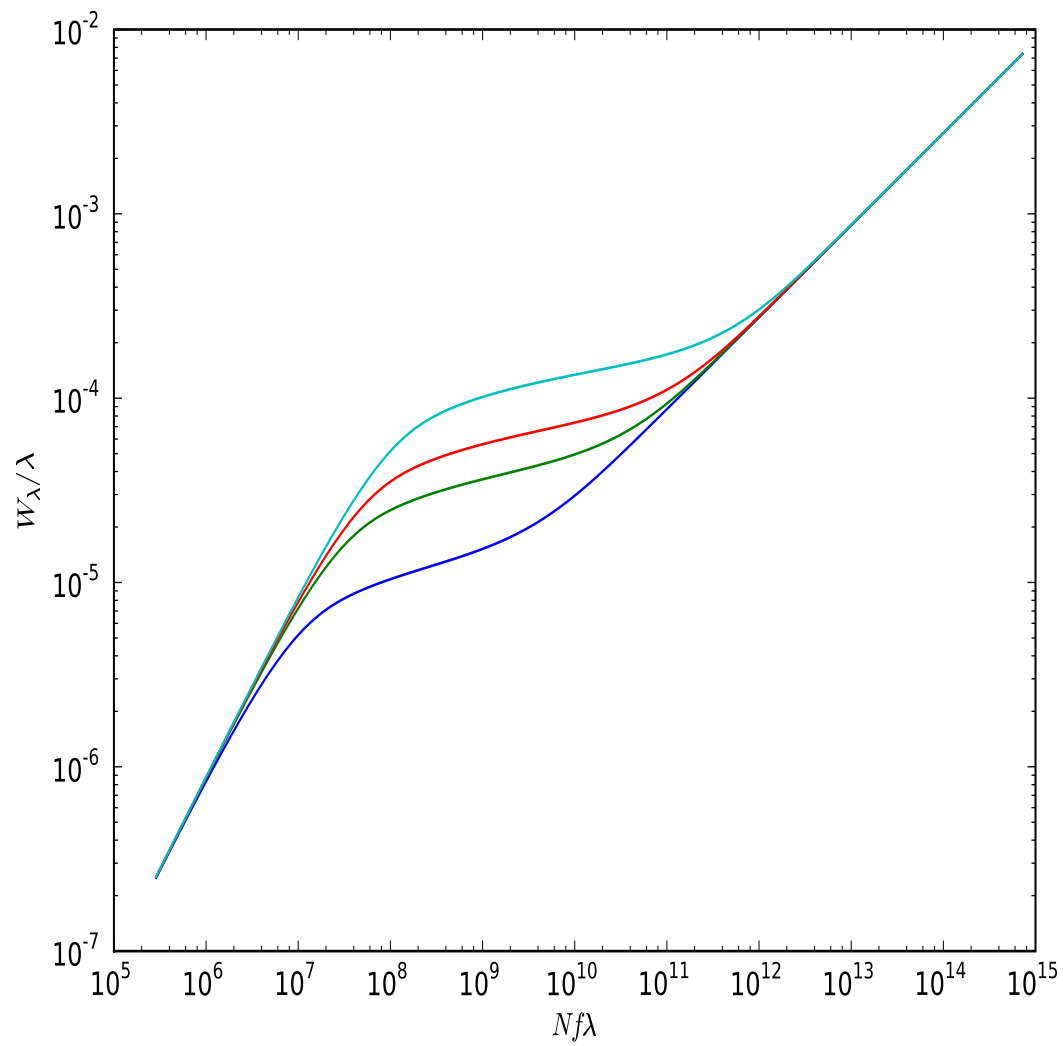


Figure 1.4: Example Curve of growth. These lines are for $b=1,3,5,10$ km/s respectively. Notice how the Equivalent Width behave linearly, then flat, and then rise again in the very optically thick regime.