Chapter 1

Blackbody Radiation

Blackbody radiation has a long and interesting history in physical sciences. From our perspective, it gives, in terms of the most fundamental constants, a method for describing the intensity from an optically thick source in thermodynamical equilibrium.

Much of this material can also be found inn Chapter 1 from Rybicki & Lightman as well as Chapter 6 from Gray along with numerous other places.

1.1 Derivation of Blackbody radiation

First, consider a closed box in which photons can be created or destroyed. The number and energy of photons will adjust themselves such that they are in equilibrium. The intensity from an opening in the container is called B_ν and is related to the energy desnity by:

$$I_\nu = \frac{c}{4\pi} U_\nu \tag{1.1}$$

where U_ν is the energy density inside the container. To calculate the energy density, we must determine the average energy of each oscillator times the density of oscillators such that:

$$U_\nu d\nu = \bar{E}\ g(\nu) d\nu \tag{1.2}$$

where $\bar{E}$ is the average energy and $g(\nu)$ is the density of oscillators (or states).

1.1.1 Density of States

In equilibrium, a standing wave in a box will have modes at $x = 0, L$ and $L/\lambda = n_x/2$ where $n_x = 1, 2, 3...$ In terms of frequency, this will be:

$$\nu = \frac{c}{\lambda} = \frac{n_x c}{2L} \tag{1.3}$$

So to find the total number of modes between ν and $d\nu$, we would have $dN = dn_x dn_y dn_z$. Inserting the above, gives us $dN = (\frac{2L}{c})^3 d^3\nu$. Now, $d^3\nu = \frac{4\pi}{8}\nu^2 d\nu$. The factor of 1/8 comes from only caring about positive frequencies. To get the total number, we also have to include a factor of 2 to include the fact that light can have two modes of polarization, so that our total density of states will be:

$$g(\nu)d\nu = \frac{N}{V} = \frac{2 \times (2L/c)^3(4\pi/8)\nu^2 d\nu}{L^3} = \frac{8\pi}{c^3}\nu^2 d\nu \tag{1.4}$$

1.1.2 Average Energy

The energy for each state is given by $E_n = nh\nu$ where there are n photons in a given state. In thermal equilibrium, the probability of any given photon being in a given state is given by the partitiion function, where $P(n) \sim e^{-E_n/kT}$. So, to find the average energy per state, we have:

$$\bar{E} = \frac{\sum_0^\infty E_n e^{-E_n/kT}}{\sum_0^\infty e^{-E_n/kT}} \tag{1.5}$$

where:

$$\sum_0^\infty E_n e^{-E_n/kT} = \frac{h\nu e^{-h\nu/kT}}{(1 - e^{-h\nu/kT})^2} \tag{1.6}$$

and

$$\sum_0^\infty e^{-E_n/kT} = \frac{1}{1 - e^{-h\nu/kT}} \tag{1.7}$$

So our average energy per state will be:

$$\bar{E} = \frac{h\nu}{e^{h\nu/kT} - 1} \tag{1.8}$$

1.1.3 Blackbody Intensity

Combining together our equations for the average energy per state and the density of states, we can derive the energy desnity for blackbody radtion:

$$U_\nu d\nu = \frac{h\nu}{e^{h\nu/kT} - 1}\frac{8\pi\nu^2 d\nu}{c^3} = \frac{8\pi}{c^3}\frac{h\nu^3 d\nu}{e^{h\nu/kT} - 1} \tag{1.9}$$

This can then be converted to give the blackbody intensity , $I_\nu = \frac{c}{4\pi}U_\nu$ which is also called Plank's law:

$$B_\nu(T) = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/kT} - 1} \tag{1.10}$$

In terms of wavlength, it is defined as:

$$I_\lambda = \frac{2hc^2}{\lambda^5}\frac{1}{e^{hc/\lambda kT}-1} \tag{1.11}$$

1.2 Blackbody Relationships

We can take blackbody radiation in two regimes and see how it behaves.

The Rayleight Jeans Law When $h\nu << kT$, then Plank's law becomes:

$$B_\nu \approx \frac{2h\nu^3}{c^3}\frac{1}{1+\frac{h\nu}{kT}-1} = \frac{2kT\nu^2}{c^2} \tag{1.12}$$

The **Brightness Temperature** is given by assuming the source is a blackbody that would be giving that intensity at a given frequency. This is very applicable in radio astronomy, when you are often working at long wavelengths. In such cases, the brightness temperature is defined as:

$$T_b = \frac{c^2}{2k\nu^2}I_\nu \tag{1.13}$$

The Wein Limit When $h\nu >> kT$, then Plank's law becomes:

$$B_\nu \approx \frac{2h\nu^3}{c^3}e^{-h\nu/kT} \tag{1.14}$$

At higher frequency, the curve for blackbody radiation falls off very quickly.

1.2.1 Peak Intensity

The wavelength of peak intensity can be found by solving $dI_\nu/d\lambda = 0$. If we make the substitution that $y = hc/\lambda kT$, then it can be shown:

$$\frac{ye^y}{e^y-1} = 5$$

where solving numerically, it leaves the familiar Wien's Law:

$$\lambda_{max} = \frac{0.290\ cm\ deg}{T} \tag{1.15}$$

1.2.2 Total Power

The total power emitted by a blackbody per unit area can be determined by integrating $B_\nu(T)$ over the solid angle and frequency.

$$P(T) = \oint \int_0^\infty I_\nu d\nu cos\theta d\omega = \pi \int_0^\infty B_\nu d\nu = \frac{2\pi h}{c^2}(\frac{kT}{h})^4 \int_0^\infty \frac{x^3 dx}{e^x - 1}$$

where $x = h\nu/kT$. In the end, we have derived the Stefan-Boltzman constant, σ where

$$P(T) = (\frac{2\pi^5 k^4}{15h^3c^2})T^4 = \sigma T^4 = 5.67 \times 10^{-5} ergcm^{-2}K^{-4}s^{-1}\ T^4 \tag{1.16}$$