



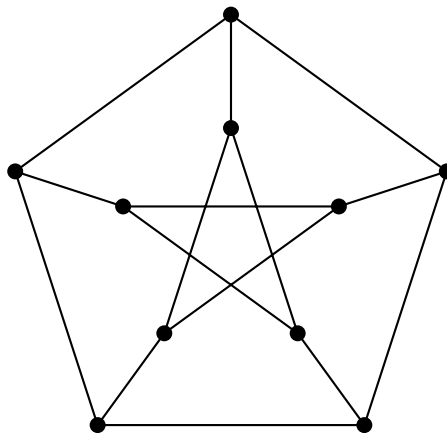
AUC

DEPARTMENT OF MATHEMATICS

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# Graph Theory

Lecture Notes



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*"An die Professorin, der ich meine Wertschätzung nicht  
zeigen konnte,  
und an die Professorin, der ich es niemals vergelten kann."*

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# Chapter 1

## Graphs

### 1.1 The Basics

#### 1.1.1 Recall

##### Definition 1.1 Set

A **set** is merely an accumulation of objects. These objects are called **elements** of the set. If an object  $x$  is an element of  $S$ , we write  $x \in S$ . The set of all elements with a certain property  $P$  is denoted via  $\{x \mid x \text{ has property } P\}$ .

##### Definition 1.2 Relation

An  $n$ -**ary relation**  $R$  on a set  $A$  is a subset of the power set of  $A^n$ , i.e.,  $R \subseteq \mathcal{P}(A^n)$ . If  $n = 2$ , we call the relation **binary**.

A binary relation  $R$  on a set  $A$  is called:

- (i) **symmetric** if  $R(a, b)$  implies  $R(b, a)$  for all  $a, b \in A$ .
- (ii) **asymmetric** if  $R(a, b)$  implies  $\neg R(b, a)$  for all  $a, b \in A$ .
- (iii) **antisymmetric** if  $R(a, b) \wedge R(b, a)$  implies  $a = b$  for all  $a, b \in A$ .
- (iv) **reflexive** if  $R(a, a)$  for all  $a \in A$ .
- (v) **irreflexive** if  $\neg R(a, a)$  for all  $a \in A$ .
- (vi) **transitive** if  $R(a, b) \wedge R(b, c)$  implies  $R(a, c)$  for all  $a, b, c \in A$ .

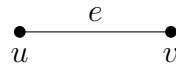
#### 1.1.2 Definition of a Graph

##### Definition 1.3 Graph

A **graph**  $G = (V, E)$  is a pair of sets  $V$  and  $E$  s.t.  $E$  consists of subsets of  $V$  of size two.  $V$  is called the set of **vertices** and  $E$  the set of **edges**. A graph  $G$  is called **finite** if  $V$  is a finite set. The **order**  $|G|$  of a graph  $G = (V, E)$  is the cardinality of its vertex set, so  $|G| = |V|$ . The **size**  $\|G\|$  of  $G$  is the cardinality of its edge set,  $\|G\| = |E|$ .

### 1.1.3 Visualisation

Let  $G = (V, E)$  be a graph. We visualise vertices  $u, v, \dots \in V$  by dots and edges  $e = \{u, v\} \in E$  by the diagram:

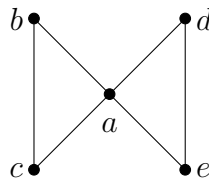


**Example 1..** *Bowtie Graph*

Let  $G = (V, E)$  be the graph with  $V = \{a, b, c, d, e\}$  and

$$E = \{\{a, b\}, \{a, c\}, \{a, d\}, \{a, e\}, \{b, c\}, \{d, e\}\}.$$

The graph  $G$  has order 5 and size 6. It can be visualized via:



This visualisation motivates its name: **bowtie graph**.

### 1.1.4 Notation

1. For a graph  $G = (V, E)$  we may denote its vertex set by  $V(G)$  or  $V_G$  for clarity.
2. Similarly, we often denote  $E$  by  $E(G)$  or  $E_G$ .
3. We denote an edge  $\{u, v\}$  simply by  $uv$ .
4. Edges are often called  $e, e_1, e_2, f, \dots$ , while vertices are called  $u, v, x, y, \dots$ .

### 1.1.5 Terminology

#### Definition 1.4 Adjacency and Incidence

Let  $G = (V, E)$  be a graph.

1. If  $uv \in E$  is an edge, then we say that  $u$  and  $v$  are **adjacent** or **neighbours**. If  $uv \notin E$ , we call  $u$  and  $v$  **nonadjacent**.
2. If  $e = uv \in E$ , we say that  $u$  and  $v$  are the **end vertices** of  $e$  or that they are **incident** with  $e$ .
3. The **neighborhood**  $N(v)$  of a vertex  $v \in V$  is the set of all vertices adjacent to  $v$ , i.e.,  $N(v) = \{u \in V \mid uv \in E\}$ . The **closed neighborhood**  $N[v]$  of  $v$  is  $N[v] := N(v) \cup \{v\}$ .
4. The **neighborhood**  $N(S)$  of a set of vertices is defined as  $N(S) := \bigcup_{v \in S} N(v)$ . Similarly, the **closed neighborhood**  $N[S]$  is set to be  $N[S] := N(S) \cup S (= \bigcup_{v \in S} N[v])$ .
5. The **degree**  $\deg(v)$  of  $v \in V$  is the number of edges incident with  $v$ , i.e.,  $\deg(v) := |\{e \in E \mid v \in e\}| = |N(v)|$ .
6. The **maximum degree**  $\Delta(G)$  of  $G$  is defined as

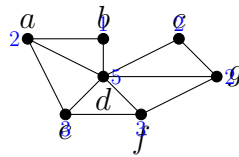
$$\Delta(G) := \max\{\deg(v) \mid v \in V\}.$$

Similarly,  $\delta(G) := \min\{\deg(v) \mid v \in V\}$  is the **minimum degree** of  $G$ .

7. The **degree sequence** of a graph  $G$  is the sequence containing all degrees of the vertices of  $G$  (with repetition) in decreasing order.

#### Example 1..

Consider  $G$  given by:



Then  $\Delta(G) = 5$ ,  $\delta(G) = 1$ .  $N(e) = \{a, d, f\}$ ,  $N[b] = \{b, d\}$ ,  $N[\{a, g\}] = \{a, c, d, e, f, g\}$ . Order of  $G$  is 7, size of  $G$  is 9. Degree sequence is  $(5, 3, 3, 3, 2, 2, 2)$ .

A graph can be considered as a set  $V$  together with a binary relation  $E$  on  $V$  which is symmetric and irreflexive.

### 1.1.6 Handshaking Lemma

#### Theorem 1.1 The Handshaking Lemma

If  $G = (V, E)$  is a graph, then

$$\sum_{v \in V} \deg(v) = 2|E|.$$

*Proof.* We proceed by induction on  $n := |E|$ . **n=0:** If  $|E| = 0$ , then  $\deg(v) = 0$  for any  $v \in V$ , whence clearly  $0 = \sum \deg(v) = 2|E| = 0$ .

**n → n+1:** Assume (\*) holds for any  $G' = (V', E')$  with  $|E'| = n$  (I.H.) and consider  $G = (V, E)$  with  $|E| = n + 1 (\geq 1)$  arbitrary. Let  $e \in E$  arbitrary and consider  $G' = (V, E \setminus \{e\})$ . Then, if  $e = uv$ , we get  $|E(G)| = |E(G')| + 1$  and

$$\deg_G(u) = \deg_{G'}(u) + 1 \quad \text{and} \quad \deg_G(v) = \deg_{G'}(v) + 1,$$

whence

$$\begin{aligned} 2|E(G)| &= 2|E(G')| + 2 \stackrel{\text{I.H.}}{=} \sum_{w \in V} \deg_{G'}(w) + 2 \\ &= \sum_{w \in V \setminus \{u, v\}} \deg_{G'}(w) + \deg_{G'}(u) + 1 + \deg_{G'}(v) + 1 \\ &= \sum_{w \in V} \deg_G(w), \text{ as desired.} \end{aligned}$$

□

#### Corollary 1.2 Odd Degrees

Any graph  $G$  has an even number of vertices of odd degree.

*Proof.* Exercise. □

#### Corollary 1.3 Bounds

For any graph  $G = (V, E)$  we have

$$\delta(G) \leq 2 \frac{|E|}{|V|} \leq \Delta(G).$$

*Proof.*

$$|V| \cdot \delta(G) = \sum_{v \in V} \delta(G) \leq \sum_{v \in V} \deg(v) \leq \sum_{v \in V} \Delta(G) = |V| \Delta(G)$$

Using Theorem 1.13,  $\sum \deg(v) = 2|E|$ . Dividing by  $|V|$  yields the result. □

#### Lemma 1.4 Pigeon Hole Principle for Graphs

If  $|G| \geq 2$ , then  $G$  contains at least two vertices of the same degree.

*Proof.* If  $G$  has two vertices of degree 0, then we are done. Otherwise, we may assume that  $G$  has none. If  $|V| = n$ , and  $v \in V$ , then  $1 \leq \deg(v) \leq n - 1$ . Note that this leaves us with  $n - 1$  choices of degrees for  $n$  many different vertices. Hence, at least two vertices must have the same degree. □

The above line of thought is called the **pigeon hole principle**. If there are  $n$  many pigeons wanting to fit into  $n - 1$  many holes, then at least two of them have to cuddle up in the same hole.

### 1.1.7 Special Graphs

#### Definition 1.5 Paths

1. The **path**  $P_n$  is the graph on  $n$  vertices  $v_1, \dots, v_n$  with the edge set  $E(P_n) = \{v_i v_{i+1} \mid 1 \leq i < n\}$ .
2. The **cycle**  $C_n$  is the graph on  $n$  vertices with edge set  $E(C_n) = \{v_i v_{i+1} \mid 1 \leq i < n\} \cup \{v_n v_1\}$ .
3. Let  $G = (V, E)$  be an arbitrary graph. The **complement**  $\overline{G}$  of  $G$  is the graph  $\overline{G} = (V, \overline{E})$ , where  $\overline{E} = \{uv \mid u, v \in V, uv \notin E\}$ .

#### Definition 1.6 Regularity

We call a graph  $G$  **regular** if any of its vertices has the same degree. If this degree is  $r$ , we say that  $G$  is  **$r$ -regular**.

#### Definition 1.7 Complete Graph

The **complete graph**  $K_n$  for  $n \geq 1$  is the graph consisting of  $n$  vertices such that any two vertices are adjacent. The **empty graph**  $E_n$  is the graph consisting of  $n$  vertices and no edges.

### 1.1.8 Subgraphs

#### Definition 1.8 Subgraphs

1. A graph  $H$  is called a **subgraph** of some graph  $G$ , written  $H \subseteq G$ , if  $V(H) \subseteq V(G)$  and  $E(H) \subseteq E(G)$ .
2. If  $H \subseteq G$ , we say that  $H$  is an **induced subgraph** of  $G$ , written  $H \sqsubseteq G$ , if  $E(H) = \{uv \in E(G) \mid u, v \in V(H)\}$ .

If  $G$  is a graph and  $S \subseteq V(G)$ , then there is only one induced subgraph  $H \sqsubseteq G$  with vertex set  $S$ . We denote this graph by  $\langle S \rangle$  and call it the subgraph of  $G$  induced by  $S$ .

### 1.1.9 Walks in Graphs

#### Definition 1.9 Walk

A  $(v_0, v_k)$ -**walk** in a graph is a sequence of vertices  $(v_0, v_1, \dots, v_k)$  s.t. any two consecutive vertices  $v_i$  and  $v_{i+1}$  are adjacent. We call the edges  $\{v_0 v_1, v_1 v_2, \dots, v_{k-1} v_k\}$  the **edges of the walk**. We say that the walk is **closed** if  $v_0 = v_k$ . The **length** of a walk is the number of edges in it (counting repetition).

**Definition 1.10 Types of Walks**

We distinguish the following types of walks:

- A **trail** is a walk whose edges are pairwise distinct.
- A **circuit** is a closed walk whose edges are pairwise distinct.
- A **path** is a walk whose vertices are distinct.
- A **cycle** is a closed walk  $(v_0, \dots, v_k = v_0)$  with  $k \geq 3$  and whose vertices  $v_0, \dots, v_{k-1}$  are pairwise distinct.

**Lemma 1.5 Cycle Existence**

If  $\delta(G) \geq 2$ , then  $G$  contains a cycle as a subgraph.

*Proof.* Let  $P = (v_0, \dots, v_k)$  be a path in  $G$  of maximal length. This exists, as  $G$  is finite. Further, as  $\delta(G) \geq 2$ , we get  $k \geq 2$ . As  $\deg(v_0) \geq \delta(G) \geq 2$ ,  $v_0$  has at least two neighbors. One of them is  $v_1$ . Let us denote the other one by  $u$ . If  $u \neq v_i$  for all  $1 \leq i \leq k$ , then  $\tilde{P} = (u, v_0, v_1, \dots, v_k)$  is still a path and of greater length than  $P$ , contradicting our assumptions. Hence,  $u = v_i$  for some  $1 \leq i \leq k$ . But then the sequence  $(v_0, v_1, \dots, v_i = u, v_0)$  is the desired cycle subgraph of  $G$ .  $\square$

**Theorem 1.6 Walk-Path Theorem**

Every  $uv$ -walk in a graph contains a  $uv$ -path.

*Proof.* We proceed by strong induction on the length  $n \geq 1$  of the walk. **I.B.  $n=1$ .** If the  $uv$ -walk is of length one, then it is exactly  $(u, v)$ , which is also a path. **I.S.** Assume every  $uv$ -walk of length at most  $n \geq 1$  contains a  $uv$ -path (I.H.). Assume there is a  $uv$ -walk  $W = (u = w_0, w_1, \dots, w_n, w_{n+1} = v)$  of length  $n+1$ . If  $W$  is already a path, we are done. Otherwise there are  $i, j$  s.t.  $0 \leq i < j \leq n+1$  and  $w_i = w_j$ . But then the walk  $\tilde{W}$  which arises from  $W$  by deleting the vertices  $w_{i+1}, \dots, w_{j-1}, w_j$ , i.e.  $\tilde{W} = (u = w_0, \dots, w_i, w_{j+1}, \dots, w_{n+1} = v)$  is still a  $uv$ -walk, but of length at most  $n$ . Using I.H., we know that  $\tilde{W}$  contains a  $uv$ -path, whence also  $W$  contains (the same)  $uv$ -path.  $\square$

**1.1.10 Connectivity****Definition 1.11 Connected**

A graph is **connected** if there exists an  $uv$ -path in  $G$  for any vertices  $u, v \in V(G)$ . Otherwise, it is called **disconnected**.

**Definition 1.12 Connected Component**

A **connected component** of  $G$  is a maximal connected induced subgraph of  $G$ . i.e.  $C \subseteq G$  is a connected component iff (i)  $C$  is connected and (ii) for any  $v \in V(G) \setminus V(C)$  the induced subgraph on  $V(C) \cup \{v\}$  is **not** connected.

$G$  is connected iff it has exactly one connected component. Even among connected graphs, there are different levels of being connected. E.g. the graph  $K_5$  "feels" more connected than the graph consisting of two triangles joined by a single edge.

### Definition 1.13 Deletion

Let  $G$  be a graph,  $S \subseteq V_G$  and  $T \subseteq E_G$ .

1. By  $G - S$  we denote the graph arising from  $G$  by removing from  $V_G$  all vertices in  $S$  and their incident edges.
2. If  $S = \{v\}$ , we write  $G - v$ .
3. By  $G - T$  we denote the graph arising from  $G$  by removing only the edges in  $T$ , but no vertices.
4. If  $T = \{e\}$ , we write  $G - e$ .

### Definition 1.14 Cut Vertex and Bridge

Let  $G$  be a graph.

1. We call  $v \in V_G$  a **cut vertex** if  $G - v$  has more connected components than  $G$  itself.
2. We call  $e \in E_G$  a **bridge** if  $G - e$  has more connected components than  $G$  itself.
3. We call  $S \subseteq V_G$  a **cut set** if  $G - S$  is disconnected.
4. A connected graph which does not contain any cut vertices is called **non-separable**.

### Definition 1.15 Connectivity

For a non-complete graph  $G$ , we define its **connectivity**  $\kappa(G)$  as the minimal size of a cut set. For  $K_n$ , we set  $\kappa(K_n) = n - 1$ . We say that  $G$  is  **$k$ -connected** if  $\kappa(G) \geq k$ , i.e. if  $G$  is connected and  $G - S$  is still connected for any  $S \subseteq V_G$  with  $|S| < k$ .

### Lemma 1.7 Properties of Connectivity

The following hold:

1.  $G$  is connected iff  $\kappa(G) \geq 1$ .
2.  $G$  is 2-connected iff  $G$  is connected and has no cut vertices.
3.  $|G| > \kappa(G)$ .
4.  $\kappa(G) \leq \delta(G)$ .

*Proof.* 4) Assume  $\kappa(G) > \delta(G)$  and let  $v \in V_G$  s.t.  $\deg(v) = \delta(G)$ . Note that  $|G| > \kappa(G) > \delta(G) = |N(v)|$ , whence  $G - N(v)$  contains at least one vertex besides  $v$ . But

clearly,  $G - N(v)$  is disconnected (as  $\deg_{G-N(v)}(v) = 0$ ). Hence,  $N(v)$  is a cut set and  $\kappa(G) \leq |N(v)| = \delta(G)$ , contradicting the assumptions.  $\square$

## 1.2 Bipartite Graphs

### Definition 1.16 Bipartite

A graph  $G$  is called **bipartite** if we can partition the vertex set  $V_G$  into two disjoint sets  $V_G = X \cup Y$  s.t. every edge of  $G$  has one end vertex in  $X$  and the other in  $Y$ .

A graph  $G$  is bipartite if and only if we can color the vertices of  $G$  with two colors s.t. the end vertices of each edge have different colors.

### Definition 1.17 Complete Bipartite

Let  $m, n \in \mathbb{Z}_+$ . The **complete bipartite graph**  $K_{m,n}$  is the bipartite graph with  $X = \{x_1, \dots, x_m\}$ ,  $Y = \{y_1, \dots, y_n\}$ ,  $V_G = X \cup Y$  and  $E_G = \{xy \mid x \in X, y \in Y\}$ .

### Theorem 1.8 Characterization of Bipartite Graphs

A graph is bipartite iff it does not contain odd cycles.

*Proof.* " $\Rightarrow$ ": Assume  $G$  is bipartite and nevertheless there is a cycle of odd length, say  $(x_0, x_1, \dots, x_{2k}, x_{2k+1} = x_0)$ . By Remark 1.44, we can color  $V_G$  in two colors,  $C1$  and  $C2$ . If  $x_0$  has color  $C1$ ,  $x_1$  has color  $C2$ , whence  $x_2$  has color  $C1$ . That way we see that the color of  $x_i$  is  $C1$  if  $i$  is even and  $C2$  if  $i$  is odd. Following that logic,  $x_{2k+1}$  should have color  $C2$ , but  $x_{2k+1} = x_0$  has color  $C1$ , a contradiction.

" $\Leftarrow$ ": Now consider that  $G$  does not contain odd cycles. We will show that  $G$  is bipartite by providing a partition. We may assume that  $G$  is connected as otherwise we work component per component. Pick  $v \in V_G$  arbitrary and define

$$X = \{w \in V_G \mid \text{the shortest } vw \text{ path has even length}\}$$

$$Y = \{w \in V_G \mid \text{the shortest } vw \text{ path has odd length}\}.$$

Clearly  $X$  and  $Y$  are disjoint. We will show that there are no adjacent vertices in  $X$  or  $Y$  respectively. Note that  $v \in X$ . Aiming for a contradiction, assume there are vertices  $w_1, w_2 \in X$  which are adjacent. Let  $P_1$  and  $P_2$  be the shortest  $v - w_1$  and  $v - w_2$  paths. We construct a cycle using  $P_1$ , the edge  $w_1 w_2$ , and  $P_2$ . The length of this cycle is  $\text{len}(P_1) + 1 + \text{len}(P_2) = \text{even} + 1 + \text{even} = \text{odd}$ . This contradicts our assumption.  $\square$

## 1.3 Graph Isomorphisms

### Definition 1.18 Isomorphism

We say that a graph  $G$  is **isomorphic** to a graph  $H$  if there exists a bijection  $\varphi : V_G \rightarrow V_H$  s.t. for any  $u, v \in V_G$  we have that  $\{u, v\} \in E_G$  if and only if  $\{\varphi(u), \varphi(v)\} \in E_H$ . Then, the map  $\varphi$  is called an **isomorphism** and we write  $G \cong H$ .

Let  $G \cong H$  via  $\varphi : V_G \rightarrow V_H$ . Then:

1.  $|V_G| = |V_H|$  and  $|E_G| = |E_H|$ .
2. The degree sequence of  $G$  equals the degree sequence of  $H$ .
3.  $G$  is connected iff  $H$  is connected.
4.  $\deg_G(v) = \deg_H(\varphi(v))$  for all  $v \in V_G$ .