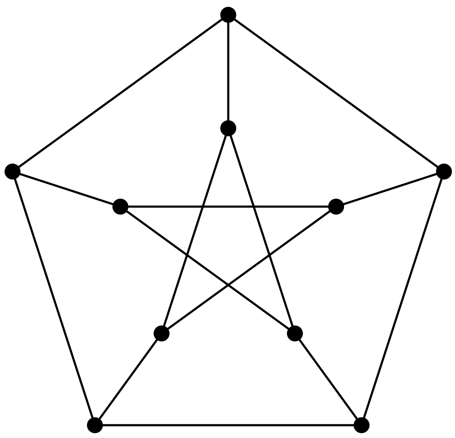
AUC

Department of Mathematics

Spring Term 2026

Graph Theory

Lecture Notes



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Term:
Spring 2026

"An die Professorin, der ich meine Wertschätzung nicht zeigen konnte,
und an die Professorin, der ich es niemals vergelten kann."

Contents

Chapter 1

Graphs

1. The Basics

1.1 Recall

1. A **set** is merely an accumulation of objects. These objects are called **elements** of the set. If an object x is an element of S, we write $x \in S$. The set of all elements with a certain property P is denoted via $\{x \mid x \text{ has property } P\}$.
2. An n**-ary relation** R on a set A is a subset of the power set of A^n, i.e., $R \subseteq \mathcal{P}(A^n)$. If $n = 2$, we call the relation **binary**.

   A binary relation R on a set A is called:

   (i) **symmetric** if $R(a, b)$ implies $R(b, a)$ for all $a, b \in A$.

   (ii) **asymmetric** if $R(a, b)$ implies $\neg R(b, a)$ for all $a, b \in A$.

   (iii) **antisymmetric** if $R(a, b) \wedge R(b, a)$ implies $a = b$ for all $a, b \in A$.

   (iv) **reflexive** if $R(a, a)$ for all $a \in A$.

   (v) **irreflexive** if $\neg R(a, a)$ for all $a \in A$.

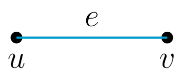
   (vi) **transitive** if $R(a, b) \wedge R(b, c)$ implies $R(a, c)$ for all $a, b, c \in A$.

Definition 1.2

A **graph** $G = (V, E)$ is a pair of sets V and E such that E consists of subsets of V of size two. V is called the set of **vertices** and E the set of **edges**. A graph G is called **finite** if V is a finite set. The **order** $|G|$ of a graph $G = (V, E)$ is the cardinality of its vertex set, so $|G| = |V|$. The **size** $\|G\|$ of G is the cardinality of its edge set, $\|G\| = |E|$.

1.3 Visualisation

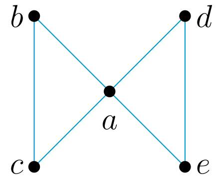
Let $G = (V, E)$ be a graph. We visualise vertices $u, v \dots \in V$ by dots and edges $e = \{u, v\} \in E$ by the diagram:



Example 1.4. *Bowtie Graph* Let $G = (V, E)$ be the graph with $V = \{a, b, c, d, e\}$ and

$$E = \{\{a, b\}, \{a, c\}, \{a, d\}, \{a, e\}, \{b, c\}, \{d, e\}\}.$$

The graph G has order 5 and size 6. It can be visualized via:



This visualisation motivates its name: **bowtie graph**.

1.5 Notation

1. For a graph $G = (V, E)$ we may denote its vertex set by $V(G)$ or V_G for clarity.
2. Similarly, we often denote E by $E(G)$ or E_G.
3. We denote an edge $\{u, v\}$ simply by uv.
4. Edges are often called $e, e_1, e_2, f \dots$, while vertices are called $u, v, x, y, \dots$.

Definition 1.6

Let $G = (V, E)$ be a graph.

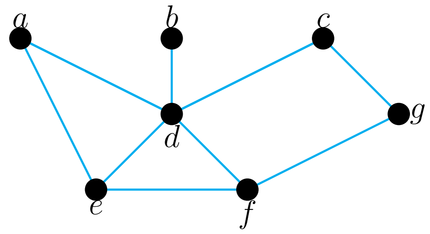
1. If $uv \in E$ is an edge, then we say that u and v are **adjacent** or **neighbours**. If $uv \notin E$, we call u and v **nonadjacent**.
2. If $e = uv \in E$, we say that u and v are the **end vertices** of e or that they are **incident** with e.
3. The **neighborhood** $N(v)$ of a vertex $v \in V$ is the set of all vertices adjacent to v, i.e., $N(v) = \{u \in V \mid uv \in E\}$. The **closed neighborhood** $N[v]$ of v is $N[v] := N(v) \cup \{v\}$.
4. The **neighborhood** $N(S)$ of a set of vertices is defined as $N(S) := \bigcup_{v \in S} N(v)$. Similarly, the **closed neighborhood** $N[S]$ is set to be $N[S] := N(S) \cup S (= \bigcup_{v \in S} N[v])$.
5. The **degree** $\deg(v)$ of $v \in V$ is the number of edges incident with v, i.e., $\deg(v) := |\{e \in E \mid v \in e\}| = |N(v)|$.
6. The **maximum degree** $\Delta(G)$ of G is defined as

$$\Delta(G) := \max\{\deg(v) \mid v \in V\}.$$

Similarly, $\delta(G) := \min\{\deg(v) \mid v \in V\}$ is the **minimum degree** of G.

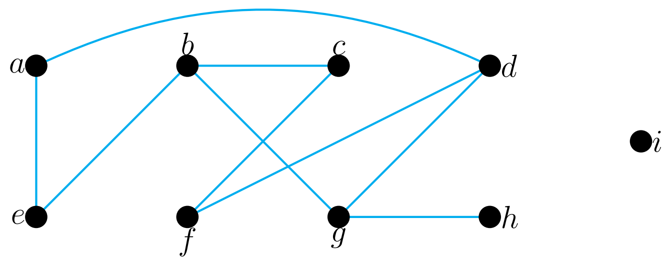
7. The **degree sequence** of a graph G is the sequence containing all degrees of the vertices of G (with repetition) in decreasing order.

Example 1.7. Consider G given by:



Then $\Delta(G) = 5$, $\delta(G) = 1$. $N(e) = \{a, d, f\}$, $N[b] = \{b, d\}$. $N[a, g] = \{a, c, d, e, f, g\}$
Order of G, size of G is 9. Degree sequence $(5, 3, 3, 2, 2, 2, 1)$.

Example 1.8. Consider G given via the diagram:



Then $V(G) = \{a, b, c, d, e, f, g, h, i\}$
$E(G) = \{\{a, d\}, \{a, e\}, \{b, c\}, \{b, e\}, \{b, g\}, \{c, f\}, \{d, f\}, \{d, g\}, \{g, h\}\}$

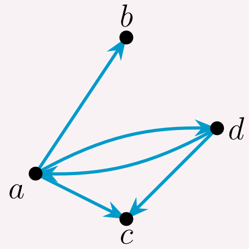
Order $|G| = 9$, size of G is 9, degree sequence $(3, 3, 3, 2, 2, 2, 2, 1, 0)$.
$N(f) = \{c, d\}$, $N[d, e] = \{a, b, c, d, e, f\}$, $\Delta(G) = 3$, $\delta(G) = 0$.

Remark 1.9. A graph can be considered as a set V together with a binary relation E on V which is symmetric and irreflexive.

Definition 1.10 Variants of Graphs

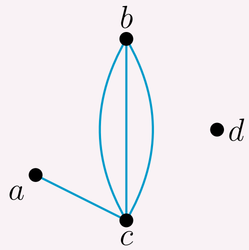
1. If $G = (V, E)$ and we replace E with a set of ordered pairs, then we call G a **directed graph** or **digraph**.

   Ex: $V(G) = \{a, b, c, d\}$, $E(G) = \{(a, b), (a, c), (a, d), (d, a), (d, c)\}$



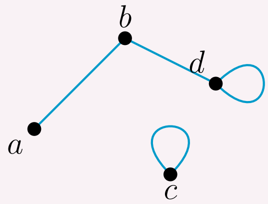
2. If $G = (V, E)$ and we replace E by a multiset (iterations of the same elements are distinguished), then we call G a **multigraph**.

   Ex: $E = [\{a, c\}, \{b, c\}, \{b, c\}, \{b, c\}]$



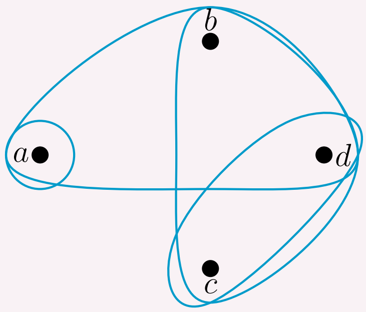
3. If $G = (V, E)$ and we extend E by allowing loops, we call G a **pseudograph**.

   Ex: $E = \{\{a, b\}, \{b, d\}, \{c, c\}, \{d, d\}\}$



4. If we allow edges to be arbitrary sets of vertices instead of 2-elementary ones, we call G a **hypergraph**.

   Ex: $E = \{\{a\}, \{a, b, d\}, \{b, c, d\}, \{c, d\}\}$



1.11 Setting

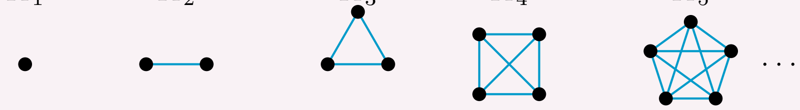
In this lecture, unless otherwise stated, by a graph we mean a finite, simple graph with $|V| \geq 1$.

Definition 1.12

- The **complete graph** K_n for $n \geq 1$ is the graph consisting of n vertices such that any two vertices are adjacent.

  e.g.

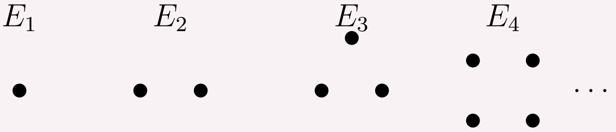
  K_1 K_2 K_3 K_4 K_5

- The **empty graph** E_n is the graph consisting of n vertices and no edges.

  e.g.

  E_1 E_2 E_3 E_4

Theorem 1.13 The Handshaking Lemma

If $G = (V, E)$ is a graph, then

$$\sum_{v \in V} \deg(v) = 2|E|. \quad (*)$$

Proof. We proceed by induction on $n := |E|$.

n=0: If $|E| = 0$, then $\deg(v) = 0$ for any $v \in V$, whence clearly

$$0 = \sum_{v \in V} \deg(v) = 2|E| = 0.$$

n → n+1: Assume $(*)$ holds for any $G' = (V', E')$ with $|E'| = n$ (I.H.) and consider $G = (V, E)$ with $|E| = n + 1 (\geq 1)$ arbitrary. Let $e \in E$ arbitrary and consider $G' = (V, E \setminus \{e\})$. Then, if $e = uv$, we get $|E(G)| = |E(G')| + 1$ and

$$\deg_G(u) = \deg_{G'}(u) + 1 \quad \text{and} \quad \deg_G(v) = \deg_{G'}(v) + 1, \text{ whence}$$

$$\begin{aligned} 2|E(G)| &= 2|E(G')| + 2 \\ &\overset{\text{I.H.}}{=} \sum_{w \in V} \deg_{G'}(w) + 2 \\ &= \sum_{w \in V \setminus \{u,v\}} \deg_{G'}(w) + \deg_{G'}(u) + 1 + \deg_{G'}(v) + 1 \\ &= \sum_{w \in V} \deg_G(w), \quad \text{as desired.} \end{aligned}$$

□

Corollary 1.14

Any graph G has an even number of vertices of odd degree.

Proof. Exercise. □

Corollary 1.15

For any graph $G = (V, E)$ we have

$$\delta(G) \leq 2\frac{|E|}{|V|} \leq \Delta(G).$$

Proof.

$$|V| \cdot \delta(G) = \sum_{v \in V} \delta(G) \leq \sum_{v \in V} \deg(v) \leq \sum_{v \in V} \Delta(G) = |V|\Delta(G)$$

Using Theorem 1.13, $\sum \deg(v) = 2|E|$. Dividing by $|V|$ yields the result. □

Lemma 1.16

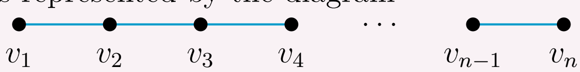
If $|G| \geq 2$, then G contains at least two vertices of the same degree.

Proof. If G has two vertices of degree 0, then we are done. Otherwise, we may assume that G has none. If $|G| = n$, and $v \in V$, then $1 \leq \deg(v) \leq n - 1$. Note that this leaves us with $n - 1$ choices of degrees for n many different vertices. Hence, at least two vertices must have the same degree. □

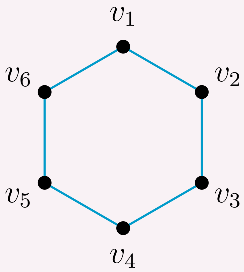
Remark 1.17. The above line of thought is called the **pigeon hole principle**. If there are n many pigeons wanting to fit into $n - 1$ many holes, then at least two of them have to cuddle up in the same hole.

Definition 1.18

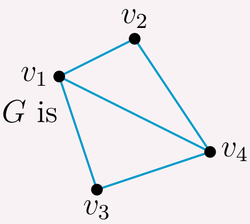
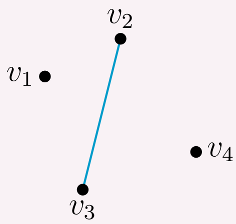
1) The **path** P_n is the graph on n vertices $v_1, \ldots, v_n$ with the edge set $E(P_n) = \{v_i v_{i+1} \mid 1 \leq i < n\}$, i.e. P_n is represented by the diagram



2) The **cycle** C_n is the graph on n vertices with edge set $E(C_n) = \{v_i v_{i+1} \mid 1 \leq i < n\} \cup \{v_n v_1\}$. E.g.



3) Let $G = (V, E)$ be an arbitrary graph. The **complement** $\overline{G}$ of G is the graph $\overline{G} = (V, \overline{E})$, where $\overline{E} = \{uv \mid u, v \in V, uv \notin E\}$.

e.g. If G is , then $\overline{G}$ is 

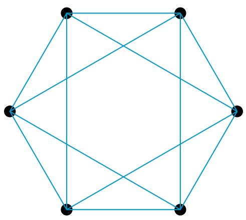
Definition 1.19

We call a graph G **regular** if any of its vertices has the same degree. If this degree is r, we say that G is r**-regular**.

Remark 1.20.

1. A graph G is regular iff $\delta(G) = \Delta(G)$.
2. K_n is $(n-1)$-regular and E_n is 0-regular.
3. An r-regular graph of order n has $\frac{1}{2}nr$ many edges.

Example 1.21. The graph below is 4-regular of order 6.



2. Subgraphs

There are two ways in which one graph can be part of another graph.

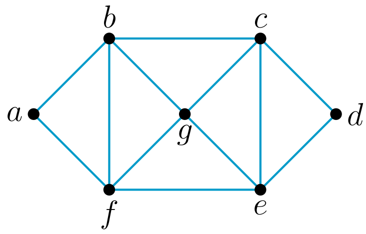
Definition 1.22

1. A graph H is called a **subgraph** of some graph G, written $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. We also say G **contains** H.
2. If $H \subseteq G$, we say that H is an **induced subgraph** of G, written $H \sqsubseteq G$, if $E(H) = \{uv \in E(G) \mid u, v \in V(H)\}$.

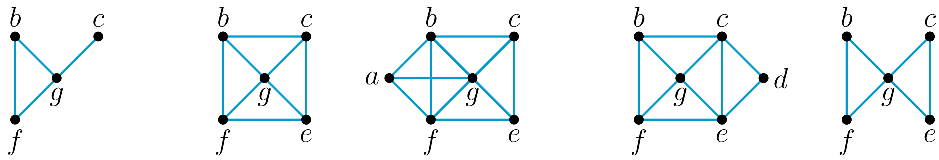
Remark 1.23.

1. $H \subseteq G$ is induced if for any two vertices in H we have: If they are adjacent in G, then they are adjacent in H.
2. Every induced subgraph is a subgraph but not vice versa.
3. If G is a graph and $S \subseteq V(G)$, then there is only one induced subgraph $H \sqsubseteq G$ with vertex set S, i.e. $V(H) = S$. We denote this graph by $\langle S \rangle$ and call it the subgraph of G induced by S.

Example 1.24. Consider G given as



Then



subgraph	✓	✓	✗	✓	✓
induced	✗	✓	✗	✓	✗

3. Walks in Graphs

Definition 1.25

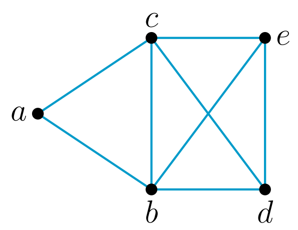
A (v_0, v_k)-**walk** in a graph is a sequence of vertices $(v_0, v_1, \ldots, v_k)$ s.t. any two consecutive vertices v_i and v_{i+1} are adjacent. We call the edges $\{v_0v_1, v_1v_2, \ldots, v_{k-1}v_k\}$ the **edges of the walk**. We say that the walk is **closed** if $v_0 = v_k$. The **length** of a walk is the number of edges in it (counting repetition).

Definition 1.26

We distinguish the following types of walks:

- A **trail** is a walk whose edges are pairwise distinct.
- A **circuit** is a closed walk whose edges are pairwise distinct.
- A **path** is a walk whose vertices are distinct.
- A **cycle** is a closed walk $(v_0, \ldots, v_k = v_0)$ with $k \geq 3$ and whose vertices $v_0, \ldots, v_{k-1}$ are pairwise distinct.

Example 1.27. Consider G via



Give examples for a:

- **walk** (d, b, c, d, b, a)
  da-walk, length 5
- **trail** (d, c, a, b, c, e)
  de-trail, length 5
- **path** (d, c, a, b, e)
  de-path, length 4
- **closed walk** (e, b, c, a, b, d, e)
  e-closed walk, length 6
- **circuit** (d, c, a, b, c, e, d)
  d-circuit, length 6
- **cycle** (d, c, a, b, e, d)
  d-cycle, length 5

Lemma 1.28

If $\delta(G) \geq 2$, then G contains a cycle as a subgraph.

Proof. Let $P = (v_0, \ldots, v_k)$ be a path in G of maximal length. This exists, as G is finite. Further, as $\delta(G) \geq 2$, we get $k \geq 2$. As $\deg(v_0) \geq \delta(G) \geq 2$, v_0 has at least two neighbors. One of them is v_1. Let us denote the other one by u. If $u \neq v_i$ for all $1 \leq i \leq k$, then $\tilde{P} = (u, v_0, v_1, \ldots, v_k)$ is still a path and of greater length than P, contradicting our assumptions. Hence, $u = v_i$ for some $1 \leq i \leq k$. But then the sequence $(v_0, v_1, \ldots, v_i = u, v_0)$ is the desired cycle subgraph of G. $\square$

Corollary 1.29 Contrapositive

If G does not contain any cycles, then $\delta(G) \leq 1$.

Theorem 1.30

Every uv-walk in a graph contains a uv-path.

Proof. We proceed by strong induction on the length $n \geq 1$ of the walk. **I.B. n=1.** If the uv-walk is of length one, then it is exactly (u, v), which is also a path. **I.S.** Assume every uv-walk of length at most $n \geq 1$ contains a uv-path (I.H.). Assume there is a uv-walk $W = (u = w_0, w_1, \ldots, w_n, w_{n+1} = v)$ of length $n+1$. If W is already a path, we are done. Otherwise there are i, j s.t. $0 \leq i < j \leq n+1$ and $w_i = w_j$. But then the walk $\tilde{W}$ which arises from W by deleting the vertices $w_{i+1}, \ldots, w_{j-1}, w_j$, i.e. $\tilde{W} = (u = w_0, \ldots, w_i, w_{j+1}, \ldots, w_{n+1} = v)$ is still a uv-walk, but of length at most n. Using I.H., we know that $\tilde{W}$ contains a uv-path, whence also W contains (the same) uv-path. □

4. Connectivity

Definition 1.31

A graph is **connected** if there exists an uv-path in G for any vertices $u, v \in V(G)$. Otherwise, it is called **disconnected**.

Intuition

A graph is connected if you could pick it up entirely by just lifting one vertex. If it is not connected, then the subgraph you lift that way is called a connected component.

Definition 1.32

A **connected component** of G is a maximal connected induced subgraph of G. i.e. $C \sqsubseteq G$ is a connected component iff (i) C is connected and (ii) for any $v \in V(G) \setminus V(C)$ the induced subgraph on $V(C) \cup \{v\}$ is **not** connected.

Remark 1.33. G is connected iff it has exactly one connected component.

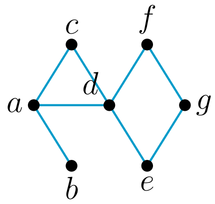
Even among connected graphs, there are different levels of being connected. E.g. the graph K_5  "feels" more connected than the graph . In order to properly describe this intuition, we need more notations.

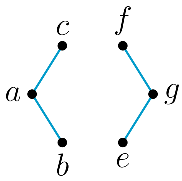
Definition 1.34 Vertex and Edge Deletion

Let G be a graph, $S \subseteq V_G$ and $T \subseteq E_G$.

1. By $G - S$ we denote the graph arising from G by removing from V_G all vertices in S and their incident edges.
2. If $S = \{v\}$, we write $G - v$.
3. By $G - T$ we denote the graph arising from G by removing only the edges in T, but no vertices.
4. If $T = \{e\}$, we write $G - e$.

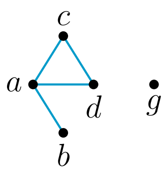
Example 1.35. Consider G as given below. Note that G only has one connected component.

$G =$ 

Then $G - d$ is 

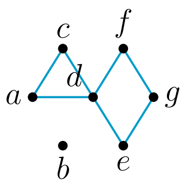
and has 2 connected components.

The vertex d is called a **cut vertex**.

Further, $G - \{e, f\}$ is 

. It also has 2 connected components.

The set $\{e, f\}$ is called a **cut set**.

Further, $G - ab$ is 

Again, it has 2 connected components.

We call the edge ab a **bridge**.

Definition 1.36

Let G be a graph.

1. We call $v \in V_G$ a **cut vertex** if $G - v$ has more connected components than G itself.
2. We call $e \in E_G$ a **bridge** if $G - e$ has more connected components than G itself.
3. We call $S \subseteq V_G$ a **cut set** if $G - S$ is disconnected.
4. A connected graph which does not contain any cut vertices is called **non-separable**.

1.37 Observation

1. If G is connected then v is a cut vertex of G iff $\{v\}$ is a cut set.
2. The vertex v is a cut vertex iff there are vertices u and w, different from v s.t. every uw-path uses v.
3. A graph has no cut sets iff it is a complete graph.

Definition 1.38

For a non-complete graph G, we define its **connectivity** $\kappa(G)$ as the minimal size of a cut set. For K_n, we set $\kappa(K_n) = n - 1$.

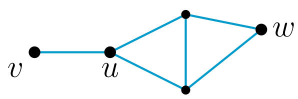
Lemma 1.39

If G is a nonseparable graph of order at least 3, then $\delta(G) \geq 2$ and every vertex of G is contained in a cycle.

Proof. Consider G nonseparable with $|G| \geq 3$. By definition, G is connected, i.e. $\delta(G) \geq 1$.

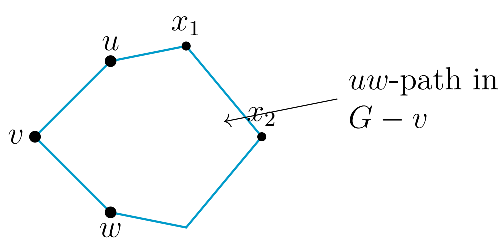
First we show that $\delta(G) \geq 2$. Otherwise, we have $\delta(G) = 1$, i.e. there is some vertex v s.t. $\deg(v) = 1$. Let u be the unique neighbor of v and w any other vertex of G (which exists as $|G| \geq 3$). Then clearly any vw-path must use the unique neighbor u of v, whence u is a cut vertex. This contradicts the fact that G is inseparable. Hence, $\delta(G) \geq 2$, as desired.

Picture:



Now, consider $v \in V_G$ arbitrary. We want to show that v is contained in a cycle in G. As $\delta(G) \geq 2$, v has at least 2 neighbors, say u and w. As G is nonseparable, $G - v$ is still connected. In particular, there is a uw-path $(u = x_0, x_1, x_2, \ldots, x_k = w)$ in $G - v$. But then the walk $(x_0 = u, x_1, \ldots, x_k = w, v, x_0 = u)$ is the desired cycle containing v.

Picture:



∎

Definition 1.40

We say that G is **k-connected** if $\kappa(G) \geq k$, i.e. if G is connected and $G - S$ is still connected for any $S \subseteq V_G$ with $|S| < k$.

Lemma 1.41

The following hold:

1) G is connected iff $\kappa(G) \geq 1$.
2) G is 1-connected iff G is connected.
3) G is 2-connected iff G is connected and has no cut vertices.
4) G is 2-connected iff G is non-separable.
5) If G is 2-connected, then it contains at least one cycle (for $|G| \geq 3$).
6) If G is k-connected, then G is j-connected for all $j \leq k$.
7) $|G| > \kappa(G)$.
8) $\kappa(G) \leq \delta(G)$.

Proof. 1)–6) are easy observations – verify them by yourselves.

7) If $G = K_n$, then $|G| = n > n-1 = \kappa(G)$. Otherwise, assume $\kappa(G) = k$, i.e. ex. $S \subseteq V_G$ s.t. $|S| = k$ and $G - S$ is disconnected. For $G - S$ to be disconnected, it must contain at least 2 vertices, whence

$$|G| \geq |S| + 2 = \kappa(G) + 2 > \kappa(G).$$

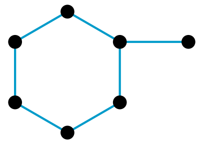
8) Assume $\kappa(G) > \delta(G)$ and let $v \in V_G$ s.t. $\deg(v) = \delta(G)$. Note that $|G| > \kappa(G) > \delta(G) = |N(v)|$, whence $G - N(v)$ contains at least one vertex besides v. But clearly, $G - N(v)$ is disconnected (as $\deg^{G-N(v)}(v) = 0$). Hence, $N(v)$ is a cut set and $\kappa(G) \leq |N(v)| = \delta(G)$, contradicting the assumptions. $\square$

5. Bipartite Graphs

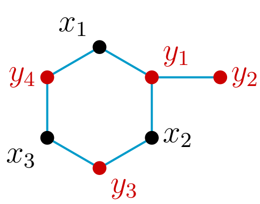
Definition 1.42

A graph G is called **bipartite** if we can partition the vertex set V_G into two disjoint sets $V_G = X \cup Y$ s.t. every edge of G has one end vertex in X and the other in Y.

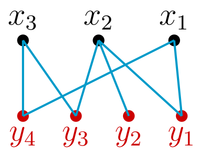
Example 1.43. Consider $G :=$



We can partition the vertices of G into two sets via $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3, y_4\}$.



Rearranging the position of the vertices makes it clear that G is bipartite:



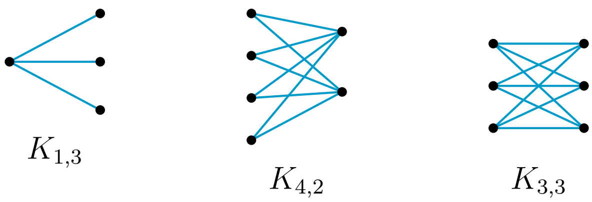
We see that there are no edges between any two vertices in X or in Y.

Remark 1.44. A graph G is bipartite if and only if we can color the vertices of G with two colors s.t. the end vertices of each edge have different colors.

Definition 1.45

Let $m, n \in \mathbb{Z}_+$. The **complete bipartite graph** $K_{m,n}$ is the bipartite graph with $X = \{x_1, \ldots, x_m\}$, $Y = \{y_1, \ldots, y_n\}$, $V_G = X \cup Y$ and $E_G = \{xy \mid x \in X, y \in Y\}$.

Example 1.46. Below are some examples of complete bipartite graphs.



The following theorem helps us decide whether or not a given graph is bipartite.

Theorem 1.47

A graph is bipartite iff it does not contain odd cycles.

Proof. "$\Rightarrow$": Assume G is bipartite and nevertheless there is a cycle of odd length, say $(x_0, x_1, \ldots, x_{2k}, x_{2k+1} = x_0)$. By Remark 1.44, we can color V_G in two colors, $C1$ and $C2$, s.t. adjacent vertices have different colors. Then, if x_0 has color $C1$, x_1 has color $C2$ whence x_2 has color $C1$. That way we see that the color of x_i is

$$\begin{cases} C1 \text{ if } i \text{ is even} \\ C2 \text{ if } i \text{ is odd} \end{cases}.$$

Following that logic, the vertex $x_0 = x_{2k+1}$ should have color $C1$ and color $C2$ at the same time, which is a contradiction.

"$\Leftarrow$": Now consider that G does not contain odd cycles. We will show that G is bipartite by providing a partition. We may assume that G is connected as otherwise we work component per component. Pick $v \in V_G$ arbitrary and define

$$X = \{w \in V_G \mid \text{the shortest } vw \text{ path has even length}\} \text{ and}$$

$$Y = \{w \in V_G \mid \text{the shortest } vw \text{ path has odd length}\}.$$

Clearly, X and Y are disjoint. We will show that there are no adjacent vertices in X or Y respectively. Note that $v \in X$.

Aiming for a contradiction, assume that there are vertices $w_1, w_2 \in X$ which are adjacent. Clearly, $w_1 \neq v$, as otherwise the shortest vw_2-path was exactly vw_2 of length 1. Similarly, $w_2 \neq v$. Let $P_1 = (v = x_0, x_1, \ldots, x_{2k} = w_1)$ and $P_2 = (v = y_0, y_1, \ldots, y_{2\ell} = w_2)$ be the shortest vw_1- and vw_2-paths. Suppose that $x_i = y_j$ for some $0 < i \leq 2k$ and $0 < j \leq 2\ell$. If $i < j$, then $(v = x_0, x_1, \ldots, x_i, y_{j+1}, \ldots, y_{2\ell} = w_2)$ is a vw_2 path shorter than P_2, a contradiction. Similarly, $j < i$ is impossible, whence $i = j$, whenever $x_i = y_j$.

Now, pick the largest i s.t. $x_i = y_i$. As $x_0 = v = y_0$, such an i always exists. Then we obtain the following cycle

$$C = (\underbrace{x_i, x_{i+1}, \ldots, x_{2k}}_{2k-i} = w_1, \underbrace{w_2}_{1} = y_{2\ell}, \underbrace{y_{2\ell-1}, \ldots, y_{i+1}}_{2\ell-i}, y_i = x_i).$$

This is a cycle, as P_1 and P_2 were paths and i was maximal s.t. $x_i = y_i$. Further, the length of C is odd, as it equals

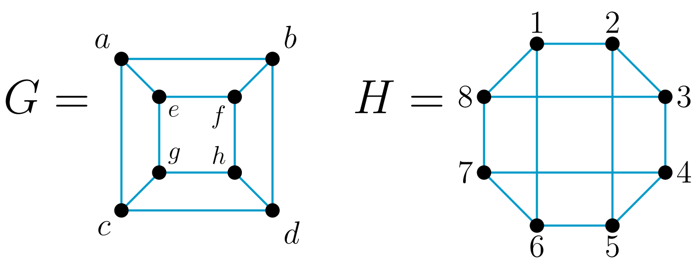
$$(2k - i) + 1 + (2\ell - i) = 2(k + \ell - i) + 1.$$

This contradicts our assumption that G does not contain odd cycles. We hence proved that no two vertices w_1 and w_2 from X can be adjacent. The arguments for $v_1, v_2 \in Y$ is analogous (try to write it down). This concludes the proof. $\square$

6. Graph Isomorphisms

In Example 1.43, we rearranged the given graph G_1 as G_2.

We understand G_1 and G_2 as the same, even though on the first glance, they look very similar. Another example is given by



We can relabel the vertices of G via $a \mapsto 1, b \mapsto 2, c \mapsto 8, d \mapsto 3, e \mapsto 7, f \mapsto 4, g \mapsto 6$ and $h \mapsto 5$ and obtain H. The aim of this section is to formalise this concept.

Definition 1.48

We say that a graph G is **isomorphic** to a graph H if there exists a bijection $\varphi : V_G \to V_H$ s.t. for any $u, v \in V_G$ we have that $\{u, v\} \in E_G$ if and only if $\{\varphi(u), \varphi(v)\} \in E_H$. Then, the map φ is called an **isomorphism** and we write $G \cong H$.

Remark 1.49. Let $G \cong H$ via $\varphi : V_G \to V_H$. Then:

1) $|V_G| = |V_H|$ and $|E_G| = |E_H|$ and $\overline{G} \cong \overline{H}$.
2) The degree sequence of G equals the degree sequence of H.
3) G is connected iff H is connected.
4) $\deg_G(v) = \deg_H(\varphi(v))$ for all $v \in V_G$.

Chapter 2

Distance in Graphs

1. Introduction

We have a natural understanding of the “distance” between two objects in our physical space. But there are many other ways of defining distances. E.g., the distance between people could be the positive difference of their birth years or the number of acquaintances you need to connect one to the other.

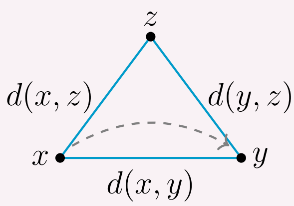
In this chapter we will introduce a notion of distance of vertices in a graph. But first let us note what are the characterising properties that make us call all these concepts “distances”.

Definition 2.1

Let X be any set. We call a function $d : X \times X \to \mathbb{R}^{\geq 0} \cup \{\infty\}$ a **metric** if it satisfies for all $x, y, z \in X$:

1) $d(x, y) \geq 0$
2) $d(x, y) = 0$ iff $x = y$
3) $d(x, y) = d(y, x)$
4) $d(x, z) \leq d(x, y) + d(y, z)$ (**Triangle Inequality**)

We then call the pair (X, d) a **metric space**.



Example 2.2. Consider $X = \mathbb{R}$ and $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{\geq 0}$ via $d(x, y) := |x - y|$. Then $(\mathbb{R}, d)$ is a metric space.

Now we are ready to define a metric on an arbitrary graph.

Definition 2.3

Let G be any graph and $u, v \in V_G$. We define the **distance** $d(u,v)$ between u and v as the length of the shortest uv-path in G, i.e.

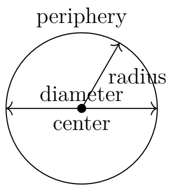
$$d(u,v) := \min\{\text{length}(P) \mid P \text{ is a } uv\text{-path}\}.$$

If there is no such path, we set $d(u,v) := \infty$.
2) If $d(u,v) = k$, then any uv-path of length k is called a **geodesic**.

Remark 2.4.

1) We may write $d_G(u,v)$ to emphasize that we consider the distance in G.
2) While in $(\mathbb{R}, d)$ geodesics are unique, in general this is not the case. Consider for example two opposite poles on a sphere.
3) $d(x,y) = \infty$ iff x and y are in different connected components.
4) (V_G, d) is a metric space for any connected graph G.

We call something eccentric if it is away from the usual. Similarly, in graphs we measure by eccentricity how far a vertex is from the center. Consider the following notions on a cycle:



Definition 2.5

1) The **eccentricity** $ecc(v)$ of a vertex v is its greatest distance to any other vertex, i.e. $ecc(v) = \max\{d(u,v) \mid u \in V_G\}$.
2) The **radius** $rad(G)$ is the smallest possible eccentricity and the **diameter** $diam(G)$ is the largest possible eccentricity.
3) The **center** $C(G)$ is the set $\{v \in V_G \mid ecc(v) = rad(G)\}$ and the **periphery** $P(G)$ is the set $\{v \in V_G \mid ecc(v) = diam(G)\}$.

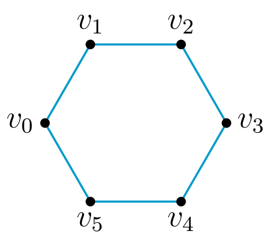
Example 2.6. 1) Consider P_5, the path of length 4, i.e.

Then

$$\begin{aligned}
d(v_1, v_i) &= i - 1, \text{ whence } ecc(v_1) = \max\{0, 1, 2, 3, 4\} = 4.\\
d(v_2, v_i) &= |i - 2|, \text{ whence } ecc(v_2) = \max\{1, 0, 1, 2, 3\} = 3.\\
d(v_3, v_i) &= |i - 3|, \text{ whence } ecc(v_3) = \max\{2, 1, 0, 1, 2\} = 2.\\
d(v_4, v_i) &= |i - 4|, \text{ whence } ecc(v_4) = \max\{3, 2, 1, 0, 1\} = 3.\\
d(v_5, v_i) &= |i - 5|, \text{ whence } ecc(v_5) = \max\{4, 3, 2, 1, 0\} = 4.
\end{aligned}$$

Hence $rad(P_5) = \min\{ecc(v) \mid v \in V\} = \min\{4, 3, 2, 3, 4\} = 2$.
Also $C(P_5) = \{v \in V \mid ecc(v) = rad(P_5)\} = \{v_3\}$.
Further $diam(P_5) = \max\{ecc(v) \mid v \in V\} = \max\{4, 3, 2, 3, 4\} = 4$.
And $P(P_5) = \{v \in V \mid ecc(v) = diam(P_5)\} = \{v_1, v_5\}$.

2) Consider $G := C_6$, the cycle of length 6, i.e. $G =$



Then

$$\begin{aligned}
d(v_0, v_i) &= 3 - |3 - i|, & ecc(v_0) &= \max\{0, 1, 2, 3, 2, 1\} = 3.\\
d(v_1, v_i) &= |3 - |4 - i||, & ecc(v_1) &= \max\{1, 0, 1, 2, 3, 2\} = 3.\\
d(v_2, v_i) &= |3 - |5 - i||, & ecc(v_2) &= \max\{2, 1, 0, 1, 2, 3\} = 3.\\
d(v_3, v_i) &= |3 - i|, & ecc(v_3) &= \max\{3, 2, 1, 0, 1, 2\} = 3.\\
d(v_4, v_i) &= |3 - |1 - i||, & ecc(v_4) &= \max\{2, 3, 2, 1, 0, 1\} = 3.\\
d(v_5, v_i) &= |3 - |2 - i||, & ecc(v_5) &= \max\{1, 2, 3, 2, 1, 0\} = 3.
\end{aligned}$$

Hence, $rad(G) = \min\{ecc(v) \mid v \in V_G\} = \min\{3, 3, 3, 3, 3, 3\} = 3$
whence $C(G) = \{v \in V_G \mid ecc(v) = rad(G)\} = V_G$.
Further, $diam(G) = \max\{ecc(v) \mid v \in V_G\} = \max\{3, 3, 3, 3, 3, 3\} = 3$.
and $P(G) = \{v \in V_G \mid ecc(v) = diam(G)\} = V_G$.

Lemma 2.7

For any graph G we have $rad(G) \leq diam(G) \leq 2rad(G)$.

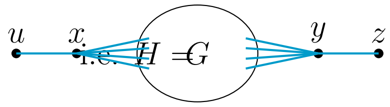
Proof. We have $rad(G) \leq diam(G)$ by definition. For the other inequality, pick $v \in C(G)$ arbitrary and consider $u, w \in V_G$ arbitrary s.t. $d(u, w) = diam(G)$. Then

$$d(u, w) \leq d(u, v) + d(v, w) \leq ecc(v) + ecc(v) = 2rad(G). \qquad \square$$

Theorem 2.8

Every graph G is isomorphic to the graph induced by the center of another graph H, i.e. ex. H s.t. $G \cong \langle C(H) \rangle$.

Proof. Let G be arbitrary. We build a new graph H which contains G as an induced subgraph via: $V_H = V_G \cup \{u, x, y, z\}$, i.e. adding 4 new vertices to G. Further, let $E_H = E_G \cup \{ux, yz\} \cup \{xv, vy \mid v \in V_G\}$.



Now $ecc(v) = 2$ for any $v \in V_G$. Nevertheless, $d(u, z) = 4$ and $d(x, z) = d(y, u) = 3$, whence $ecc(w) > 2$ for all $w \in V_H \setminus V_G$. Thus, $rad(H) = 2$ and $C(H) = V_G$, whence $\langle C(H) \rangle \cong G$. □

Lemma 2.9

A graph G is isomorphic to the graph induced by the periphery of another graph H iff either every vertex has eccentricity 1 or no vertex does.

Proof. "$\Rightarrow$" We use proof by contraposition. Assume ex. $u \in V_G$ s.t. $ecc(u) = 1 < diam(G)$. In particular, $G \neq P(G)$. Now, aiming for a contradiction, assume ex. H s.t. $G \leq H$ and $P(H) = V_G$. As $G \neq P(G)$, we know that $H \neq G$ and $diam(H) \geq 2$. As $u \in V_G = P(H)$, there is some $w \in V_H$ s.t. $d(u, w) = diam(H)$. But then, $w \in P(H) \cong V_G$, and as $ecc(u) = 1$, we also get $d(u, w) = 1 < diam(H)$. Hence, $P(H)$ cannot be V_G.

"$\Leftarrow$" If all vertices in G have eccentricity 1 or 0, then G is complete and $G \cong P(G)$. For the second case, assume $rad(G) > 1$. And consider H s.t. $V_H = V_G \cup \{v\}$ contains one new vertex which is connected to everyone else, i.e. $E_H = E_G \cup \{vx \mid x \in V_G\}$. Then, as $ecc(x) \geq 2$ for all $x \in V_G$,

$$ecc_H(x) = \begin{cases} 2 & \text{if } x \in V_G \\ 1 & \text{if } x = v \end{cases}.$$

Hence, $diam(H) = 2$ and $\langle P(H) \rangle = G$, as desired. □

2. Adjacency Matrices

We saw the visual benefits of studying graphs by their diagram. This is very useful to illustrate ideas and study small graphs. In applications on the other hand, when studying e.g. correlations of weather phenomena or social links, graphs tend to have thousands of vertices. Here, it is no longer practical to use neither the set- nor the diagram representation of graphs. The way computers store and analyze graphs is by using adjacency matrices.

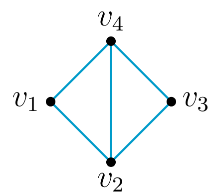
Definition 2.10

Let G be a graph of order n with vertices $V_G = \{v_1, v_2, \ldots, v_n\}$. The **adjacency matrix** of G is the matrix $A_G = (a_{ij}) \in M_{n\times n}$ defined via

$$a_{ij} = \begin{cases} 1 & \text{if } v_iv_j \in E \\ 0 & \text{otherwise.} \end{cases}$$

We also write $A(i,j)$ for a_{ij}.

Example 2.11. Consider G given by



Then $A_G \in M_{4\times 4}$

$$A_G = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

is the adjacency matrix of G.

Remark 2.12. If $A_G = (a_{ij})$ is an adjacency matrix of a graph G, then

1) $a_{ii} = 0$ for all $1 \le i \le |G|$
2) A is symmetric.
3) $\sum_{j=1}^{|G|} a_{ij} = \deg(v_i)$ and thus $\sum_{i,j=1}^{|G|} a_{ij} = \sum_{i=1}^{|G|} \deg(v_i) = 2|E|$.
4) A_G is only unique up to reordering the vertices.

Example 2.13. Let revisit the graph G from 2.11. The fact that $A_G(2,3) \neq 0$ means that v_2 and v_3 are adjacent. And $A(1,3) = 0$ says that v_1 and v_3 are not. Now consider

$$A_G^2 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 2 & 1 \\ 1 & 3 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 3 \end{pmatrix}.$$

Let's interpret the values of A_G^2. Now, $A_G^2(1,3) = 2$. How did we compute it? $A_G^2(1,3) = \sum_{j=1}^{4} a_{1j}a_{j3}$. Now $a_{1j}a_{j3} = 1$ iff v_1v_j and v_jv_3 are edges iff (v_1, v_j, v_3) is a walk of length 2 from v_1 to v_3. Hence, $A_G^2(1,3) = \sum a_{1j}a_{j3}$ is the number of walks from v_1 to v_3 of length 2. This generalises and provides a strong tool to study graphs.

Theorem 2.14

Let G be a graph with $V_G = \{v_1, \dots, v_n\}$ and A_G the corresponding adjacency matrix. Then the entry $A_G^k(i,j)$ is the number of possible walks from v_i to v_j of length k.

Proof. We proceed by induction on the power k. (Note that $k = 0$ works too). $\underline{k = 1}$: We get that $A(i,j) = \begin{cases} 0 & \text{iff } v_iv_j \notin E_G \text{ iff there are } 0\ v_iv_j\text{-walks of length } 1 \\ 1 & \text{iff } v_iv_j \in E_G \text{ iff there is } 1\ v_iv_j\text{-walk of length } 1 \end{cases}$.

$\underline{k \to k+1}$: Assume that $A^k(i,j)$ gives exactly the number of v_iv_j-walks of length exactly k. Let's denote $A^k = (b_{ij})$ and $A = (a_{ij})$. Note that there is a v_iv_j-walk of length $k+1$ iff there ex. a vertex v_ℓ s.t. there is a v_iv_ℓ-walk of length k and an $v_\ell v_j$-walk of length one. Hence

$$\begin{aligned} |\{v_iv_j\text{-walk of length } k+1\}| &= \sum_{\ell | v_\ell \in N(v_j)} |\{v_iv_\ell\text{-walk of length } k\}| \\ &\overset{\text{I.H.}}{=} \sum_{\ell | v_\ell \in N(v_j)} b_{i\ell} = \sum_{\ell=1}^{n} b_{i\ell}a_{\ell j} \\ &= \sum_{\ell=1}^{n} A^k(i,\ell) \cdot A(\ell,j) = A^{k+1}(i,j). \end{aligned}$$

□

Corollary 2.15

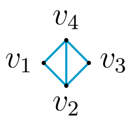
Let G be a graph with $V_G = \{v_1, \dots, v_n\}$ and A_G the adjacency matrix. Then $d(v_i,v_j) = \min\{k \mid A^k(i,j) \neq 0\}$. (Recall that $A_G^0 = I_n$).

Definition 2.16

Let G be a graph with adjacency matrix A. For every $k \in N$ we define the **Stoll matrix** S_k via

$$S_k = \sum_{i=0}^{k} A^k = I_n + A + A^2 + \cdots + A^k.$$

Remark 2.17. As $S_k(i,j) = \sum_{i=0}^{k} A^i(i,j)$, we get that $S_k(i,j)$ is the number of v_iv_j-walks of length at most k.

Example 2.18. Recall the graph $G =$ 

with $A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$.

$A^2 = \begin{pmatrix} 2 & 1 & 2 & 1 \\ 1 & 3 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 3 \end{pmatrix}$ and $A^3 = \begin{pmatrix} 2 & 5 & 2 & 5 \\ 5 & 4 & 5 & 5 \\ 2 & 5 & 2 & 5 \\ 5 & 5 & 5 & 4 \end{pmatrix}$.

Then $S_0 = I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $S_1 = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$, $S_2 = \begin{pmatrix} 3 & 2 & 2 & 3 \\ 2 & 4 & 2 & 3 \\ 2 & 2 & 3 & 2 \\ 2 & 3 & 2 & 4 \end{pmatrix}$.

$S_3 = \begin{pmatrix} 5 & 7 & 4 & 8 \\ 7 & 8 & 7 & 8 \\ 4 & 7 & 5 & 7 \\ 7 & 8 & 7 & 8 \end{pmatrix}$. This means there are for example 4 v_1v_3 walks of length at most 3, namely (v_1, v_2, v_3), (v_1, v_4, v_3), (v_1, v_2, v_4, v_3) and (v_1, v_4, v_2, v_3).

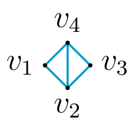
Theorem 2.19

Let G be a graph with $V_G = \{v_1, \dots, v_n\}$, adjacency matrix A and Stoll matrices S_k. Then the following hold.

1) $d(v_i, v_j)$ is the least k s.t. $S_k(i,j) \neq 0$.
2) $ecc(v_i)$ is the least k s.t. the i-th row of S_k has no zero entries.
3) $rad(G)$ is the least k s.t. S_k contains at least one row without zero entries (or ∞ otherwise).
4) $diam(G)$ is the least k s.t. S_k does not contain any zero entries.
5) G is disconnected iff S_{n-1} contains a zero.

Definition 2.20

Let G be a graph with $V_G = \{v_1, \dots, v_n\}$. The **distance matrix** of G is the matrix $D \in M_{n \times n}$ s.t. $D(i,j) = d(v_i, v_j)$.

Example 2.21. Back to our example $G =$ 

. Then the distance matrix D is

$$\begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 1 \\ 2 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

Example 2.22. **Erdős Number** Paul Erdős - Hungarian Mathematician, published over 1500 papers. Consider G with $V_G =$ all mathematicians, $E_G = \{xy \mid x \text{ and } y \text{ published together}\}$. Then $\deg(\text{Erdős}) > 500$ and the Erdős number of x is $d(\text{Erdős}, x)$.