THE AMERICAN
UNIVERSITY IN CAIRO

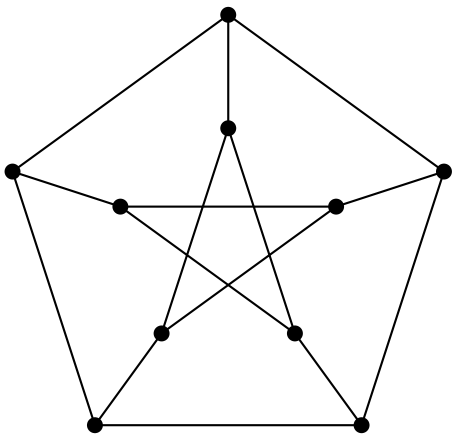
AUC

Department of Mathematics

Spring Term 2026

Graph Theory

Lecture Notes



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Term:
Spring 2026

"An die Professorin, der ich meine Wertschätzung nicht zeigen konnte,
und an die Professorin, der ich es niemals vergelten kann."

Contents

Chapter 1

Graphs

1. The Basics

1.1 Recall

1. A **set** is merely an accumulation of objects. These objects are called **elements** of the set. If an object x is an element of S, we write $x \in S$. The set of all elements with a certain property P is denoted via $\{x \mid x \text{ has property } P\}$.
2. An n**-ary relation** R on a set A is a subset of the power set of A^n, i.e., $R \subseteq \mathcal{P}(A^n)$. If $n = 2$, we call the relation **binary**.

   A binary relation R on a set A is called:

   (i) **symmetric** if $R(a, b)$ implies $R(b, a)$ for all $a, b \in A$.

   (ii) **asymmetric** if $R(a, b)$ implies $\neg R(b, a)$ for all $a, b \in A$.

   (iii) **antisymmetric** if $R(a, b) \wedge R(b, a)$ implies $a = b$ for all $a, b \in A$.

   (iv) **reflexive** if $R(a, a)$ for all $a \in A$.

   (v) **irreflexive** if $\neg R(a, a)$ for all $a \in A$.

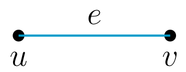
   (vi) **transitive** if $R(a, b) \wedge R(b, c)$ implies $R(a, c)$ for all $a, b, c \in A$.

Definition 1.2

A **graph** $G = (V, E)$ is a pair of sets V and E such that E consists of subsets of V of size two. V is called the set of **vertices** and E the set of **edges**. A graph G is called **finite** if V is a finite set. The **order** $|G|$ of a graph $G = (V, E)$ is the cardinality of its vertex set, so $|G| = |V|$. The **size** $\|G\|$ of G is the cardinality of its edge set, $\|G\| = |E|$.

1.3 Visualisation

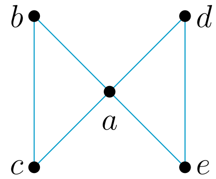
Let $G = (V, E)$ be a graph. We visualise vertices $u, v \dots \in V$ by dots and edges $e = \{u, v\} \in E$ by the diagram:



Example 1.4. *Bowtie Graph* Let $G = (V, E)$ be the graph with $V = \{a, b, c, d, e\}$ and

$$E = \{\{a, b\}, \{a, c\}, \{a, d\}, \{a, e\}, \{b, c\}, \{d, e\}\}.$$

The graph G has order 5 and size 6. It can be visualized via:



This visualisation motivates its name: **bowtie graph**.

1.5 Notation

1. For a graph $G = (V, E)$ we may denote its vertex set by $V(G)$ or V_G for clarity.
2. Similarly, we often denote E by $E(G)$ or E_G.
3. We denote an edge $\{u, v\}$ simply by uv.
4. Edges are often called $e, e_1, e_2, f \dots$, while vertices are called $u, v, x, y, \dots$.

Definition 1.6

Let $G = (V, E)$ be a graph.

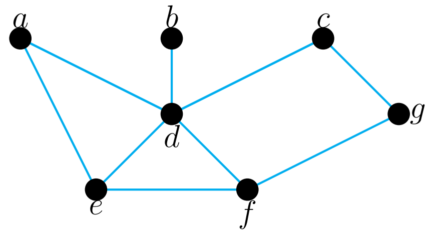
1. If $uv \in E$ is an edge, then we say that u and v are **adjacent** or **neighbours**. If $uv \notin E$, we call u and v **nonadjacent**.
2. If $e = uv \in E$, we say that u and v are the **end vertices** of e or that they are **incident** with e.
3. The **neighborhood** $N(v)$ of a vertex $v \in V$ is the set of all vertices adjacent to v, i.e., $N(v) = \{u \in V \mid uv \in E\}$. The **closed neighborhood** $N[v]$ of v is $N[v] := N(v) \cup \{v\}$.
4. The **neighborhood** $N(S)$ of a set of vertices is defined as $N(S) := \bigcup_{v \in S} N(v)$. Similarly, the **closed neighborhood** $N[S]$ is set to be $N[S] := N(S) \cup S (= \bigcup_{v \in S} N[v])$.
5. The **degree** $\deg(v)$ of $v \in V$ is the number of edges incident with v, i.e., $\deg(v) := |\{e \in E \mid v \in e\}| = |N(v)|$.
6. The **maximum degree** $\Delta(G)$ of G is defined as

$$\Delta(G) := \max\{\deg(v) \mid v \in V\}.$$

Similarly, $\delta(G) := \min\{\deg(v) \mid v \in V\}$ is the **minimum degree** of G.

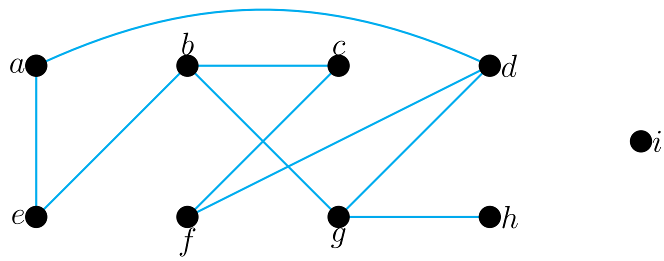
7. The **degree sequence** of a graph G is the sequence containing all degrees of the vertices of G (with repetition) in decreasing order.

Example 1.7. Consider G given by:



Then $\Delta(G) = 5$, $\delta(G) = 1$. $N(e) = \{a, d, f\}$, $N[b] = \{b, d\}$. $N[a, g] = \{a, c, d, e, f, g\}$ Order of G, size of G is 9. Degree sequence $(5, 3, 3, 2, 2, 2, 1)$.

Example 1.8. Consider G given via the diagram:



Then $V(G) = \{a, b, c, d, e, f, g, h, i\}$
$E(G) = \{\{a, d\}, \{a, e\}, \{b, c\}, \{b, e\}, \{b, g\}, \{c, f\}, \{d, f\}, \{d, g\}, \{g, h\}\}$

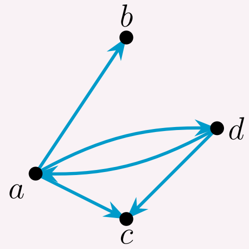
Order $|G| = 9$, size of G is 9, degree sequence $(3, 3, 3, 2, 2, 2, 2, 1, 0)$.
$N(f) = \{c, d\}$, $N[d, e] = \{a, b, c, d, e, f\}$, $\Delta(G) = 3$, $\delta(G) = 0$.

Remark 1.9. A graph can be considered as a set V together with a binary relation E on V which is symmetric and irreflexive.

Definition 1.10 Variants of Graphs

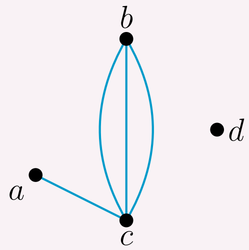
1. If $G = (V, E)$ and we replace E with a set of ordered pairs, then we call G a **directed graph** or **digraph**.

   Ex: $V(G) = \{a, b, c, d\}$, $E(G) = \{(a, b), (a, c), (a, d), (d, a), (d, c)\}$



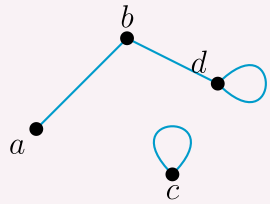
2. If $G = (V, E)$ and we replace E by a multiset (iterations of the same elements are distinguished), then we call G a **multigraph**.

   Ex: $E = [\{a, c\}, \{b, c\}, \{b, c\}, \{b, c\}]$



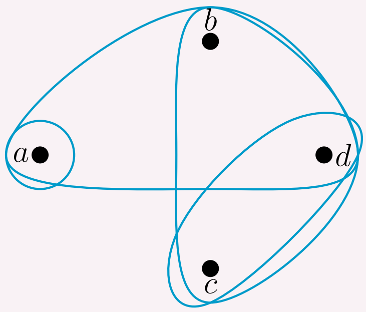
3. If $G = (V, E)$ and we extend E by allowing loops, we call G a **pseudograph**.

   Ex: $E = \{\{a, b\}, \{b, d\}, \{c, c\}, \{d, d\}\}$



4. If we allow edges to be arbitrary sets of vertices instead of 2-elementary ones, we call G a **hypergraph**.

   Ex: $E = \{\{a\}, \{a, b, d\}, \{b, c, d\}, \{c, d\}\}$



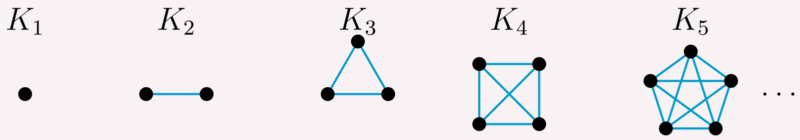
1.11 Setting

In this lecture, unless otherwise stated, by a graph we mean a finite, simple graph with $|V| \geq 1$.

Definition 1.12

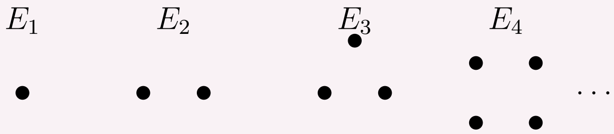
- The **complete graph** K_n for $n \geq 1$ is the graph consisting of n vertices such that any two vertices are adjacent.

  e.g.



- The **empty graph** E_n is the graph consisting of n vertices and no edges.

  e.g.



Theorem 1.13 The Handshaking Lemma

If $G = (V, E)$ is a graph, then

$$\sum_{v \in V} \deg(v) = 2|E|. \quad (*)$$

Proof. We proceed by induction on $n := |E|$.

<u>**n=0:**</u> If $|E| = 0$, then $\deg(v) = 0$ for any $v \in V$, whence clearly

$$0 = \sum_{v \in V} \deg(v) = 2|E| = 0.$$

<u>**n $\to$ n+1:**</u> Assume $(*)$ holds for any $G' = (V', E')$ with $|E'| = n$ (I.H.) and consider $G = (V, E)$ with $|E| = n + 1 (\geq 1)$ arbitrary. Let $e \in E$ arbitrary and consider $G' = (V, E \setminus \{e\})$. Then, if $e = uv$, we get $|E(G)| = |E(G')| + 1$ and

$$\deg_G(u) = \deg_{G'}(u) + 1 \quad \text{and} \quad \deg_G(v) = \deg_{G'}(v) + 1, \text{ whence}$$

$$\begin{aligned}
2|E(G)| &= 2|E(G')| + 2 \\
&\overset{\text{I.H.}}{=} \sum_{w \in V} \deg_{G'}(w) + 2 \\
&= \sum_{w \in V \setminus \{u,v\}} \deg_{G'}(w) + \deg_{G'}(u) + 1 + \deg_{G'}(v) + 1 \\
&= \sum_{w \in V} \deg_G(w), \quad \text{as desired.}
\end{aligned}$$

□

Corollary 1.14

Any graph G has an even number of vertices of odd degree.

Proof. Exercise. □

Corollary 1.15

For any graph $G = (V, E)$ we have

$$\delta(G) \leq 2\frac{|E|}{|V|} \leq \Delta(G).$$

Proof.

$$|V| \cdot \delta(G) = \sum_{v \in V} \delta(G) \leq \sum_{v \in V} \deg(v) \leq \sum_{v \in V} \Delta(G) = |V|\Delta(G)$$

Using Theorem 1.13, $\sum \deg(v) = 2|E|$. Dividing by $|V|$ yields the result. □

Lemma 1.16

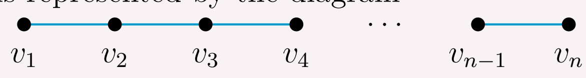
If $|G| \geq 2$, then G contains at least two vertices of the same degree.

Proof. If G has two vertices of degree 0, then we are done. Otherwise, we may assume that G has none. If $|G| = n$, and $v \in V$, then $1 \leq \deg(v) \leq n - 1$. Note that this leaves us with $n - 1$ choices of degrees for n many different vertices. Hence, at least two vertices must have the same degree. □

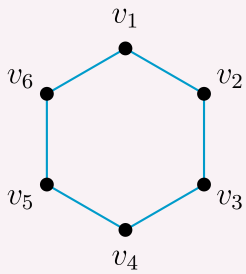
Remark 1.17. The above line of thought is called the **pigeon hole principle**. If there are n many pigeons wanting to fit into $n - 1$ many holes, then at least two of them have to cuddle up in the same hole.

Definition 1.18

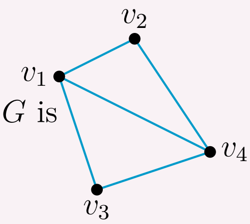
1) The **path** P_n is the graph on n vertices $v_1, \ldots, v_n$ with the edge set $E(P_n) = \{v_i v_{i+1} \mid 1 \leq i < n\}$, i.e. P_n is represented by the diagram

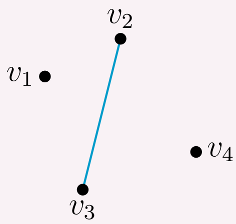


2) The **cycle** C_n is the graph on n vertices with edge set $E(C_n) = \{v_i v_{i+1} \mid 1 \leq i < n\} \cup \{v_n v_1\}$. E.g.

C_6 is 

3) Let $G = (V, E)$ be an arbitrary graph. The **complement** $\overline{G}$ of G is the graph $\overline{G} = (V, \overline{E})$, where $\overline{E} = \{uv \mid u, v \in V, uv \notin E\}$.

e.g. If G is 

, then $\overline{G}$ is 

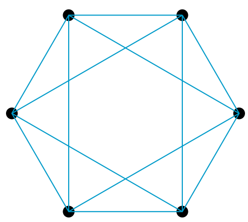
Definition 1.19

We call a graph G **regular** if any of its vertices has the same degree. If this degree is r, we say that G is r**-regular**.

Remark 1.20.

1. A graph G is regular iff $\delta(G) = \Delta(G)$.
2. K_n is $(n-1)$-regular and E_n is 0-regular.
3. An r-regular graph of order n has $\frac{1}{2}nr$ many edges.

Example 1.21. The graph below is 4-regular of order 6.



2. Subgraphs

There are two ways in which one graph can be part of another graph.

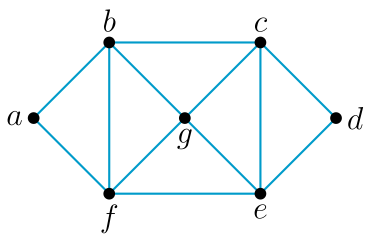
Definition 1.22

1. A graph H is called a **subgraph** of some graph G, written $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. We also say G **contains** H.
2. If $H \subseteq G$, we say that H is an **induced subgraph** of G, written $H \sqsubseteq G$, if $E(H) = \{uv \in E(G) \mid u, v \in V(H)\}$.

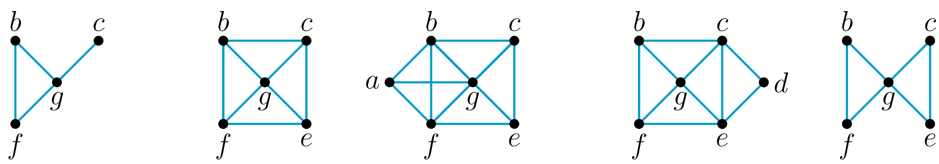
Remark 1.23.

1. $H \subseteq G$ is induced if for any two vertices in H we have: If they are adjacent in G, then they are adjacent in H.
2. Every induced subgraph is a subgraph but not vice versa.
3. If G is a graph and $S \subseteq V(G)$, then there is only one induced subgraph $H \sqsubseteq G$ with vertex set S, i.e. $V(H) = S$. We denote this graph by $\langle S \rangle$ and call it the subgraph of G induced by S.

Example 1.24. Consider G given as



Then



subgraph	✓	✓	✗	✓	✓
induced	✗	✓	✗	✓	✗

3. Walks in Graphs

Definition 1.25

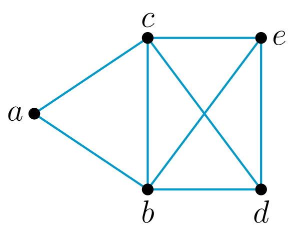
A (v_0, v_k)-**walk** in a graph is a sequence of vertices $(v_0, v_1, \ldots, v_k)$ s.t. any two consecutive vertices v_i and v_{i+1} are adjacent. We call the edges $\{v_0v_1, v_1v_2, \ldots, v_{k-1}v_k\}$ the **edges of the walk**. We say that the walk is **closed** if $v_0 = v_k$. The **length** of a walk is the number of edges in it (counting repetition).

Definition 1.26

We distinguish the following types of walks:

- A **trail** is a walk whose edges are pairwise distinct.
- A **circuit** is a closed walk whose edges are pairwise distinct.
- A **path** is a walk whose vertices are distinct.
- A **cycle** is a closed walk $(v_0, \ldots, v_k = v_0)$ with $k \geq 3$ and whose vertices $v_0, \ldots, v_{k-1}$ are pairwise distinct.

Example 1.27. Consider G via



Give examples for a:

- **walk** (d, b, c, d, b, a)
  da-walk, length 5
- **trail** (d, c, a, b, c, e)
  de-trail, length 5
- **path** (d, c, a, b, e)
  de-path, length 4
- **closed walk** (e, b, c, a, b, d, e)
  e-closed walk, length 6
- **circuit** (d, c, a, b, c, e, d)
  d-circuit, length 6
- **cycle** (d, c, a, b, e, d)
  d-cycle, length 5

Lemma 1.28

If $\delta(G) \geq 2$, then G contains a cycle as a subgraph.

Proof. Let $P = (v_0, \ldots, v_k)$ be a path in G of maximal length. This exists, as G is finite. Further, as $\delta(G) \geq 2$, we get $k \geq 2$. As $\deg(v_0) \geq \delta(G) \geq 2$, v_0 has at least two neighbors. One of them is v_1. Let us denote the other one by u. If $u \neq v_i$ for all $1 \leq i \leq k$, then $\tilde{P} = (u, v_0, v_1, \ldots, v_k)$ is still a path and of greater length than P, contradicting our assumptions. Hence, $u = v_i$ for some $1 \leq i \leq k$. But then the sequence $(v_0, v_1, \ldots, v_i = u, v_0)$ is the desired cycle subgraph of G. $\square$

Corollary 1.29 Contrapositive

If G does not contain any cycles, then $\delta(G) \leq 1$.

Theorem 1.30

Every uv-walk in a graph contains a uv-path.

Proof. We proceed by strong induction on the length $n \geq 1$ of the walk. **I.B. n=1.** If the uv-walk is of length one, then it is exactly (u, v), which is also a path. **I.S.** Assume every uv-walk of length at most $n \geq 1$ contains a uv-path (I.H.). Assume there is a uv-walk $W = (u = w_0, w_1, \ldots, w_n, w_{n+1} = v)$ of length $n + 1$. If W is already a path, we are done. Otherwise there are i, j s.t. $0 \leq i < j \leq n + 1$ and $w_i = w_j$. But then the walk $\tilde{W}$ which arises from W by deleting the vertices $w_{i+1}, \ldots, w_{j-1}, w_j$, i.e. $\tilde{W} = (u = w_0, \ldots, w_i, w_{j+1}, \ldots, w_{n+1} = v)$ is still a uv-walk, but of length at most n. Using I.H., we know that $\tilde{W}$ contains a uv-path, whence also W contains (the same) uv-path. ☐

4. Connectivity

Definition 1.31

A graph is **connected** if there exists an uv-path in G for any vertices $u, v \in V(G)$. Otherwise, it is called **disconnected**.

Intuition

A graph is connected if you could pick it up entirely by just lifting one vertex. If it is not connected, then the subgraph you lift that way is called a connected component.

Definition 1.32

A **connected component** of G is a maximal connected induced subgraph of G. i.e. $C \sqsubseteq G$ is a connected component iff (i) C is connected and (ii) for any $v \in V(G) \setminus V(C)$ the induced subgraph on $V(C) \cup \{v\}$ is **not** connected.

Remark 1.33. G is connected iff it has exactly one connected component.

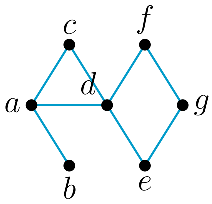
Even among connected graphs, there are different levels of being connected. E.g. the graph K_5  "feels" more connected than the graph . In order to properly describe this intuition, we need more notations.

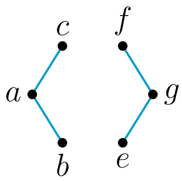
Definition 1.34 Vertex and Edge Deletion

Let G be a graph, $S \subseteq V_G$ and $T \subseteq E_G$.

1. By $G - S$ we denote the graph arising from G by removing from V_G all vertices in S and their incident edges.
2. If $S = \{v\}$, we write $G - v$.
3. By $G - T$ we denote the graph arising from G by removing only the edges in T, but no vertices.
4. If $T = \{e\}$, we write $G - e$.

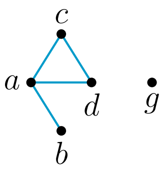
Example 1.35. Consider G as given below. Note that G only has one connected component.

$G =$ 

Then $G - d$ is 

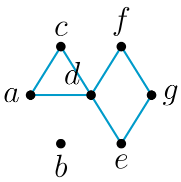
and has 2 connected components.

The vertex d is called a **cut vertex**.

Further, $G - \{e, f\}$ is 

. It also has 2 connected components.

The set $\{e, f\}$ is called a **cut set**.

Further, $G - ab$ is 

Again, it has 2 connected components.

We call the edge ab a **bridge**.

Definition 1.36

Let G be a graph.

1. We call $v \in V_G$ a **cut vertex** if $G - v$ has more connected components than G itself.
2. We call $e \in E_G$ a **bridge** if $G - e$ has more connected components than G itself.
3. We call $S \subseteq V_G$ a **cut set** if $G - S$ is disconnected.
4. A connected graph which does not contain any cut vertices is called **non-separable**.

1.37 Observation

1. If G is connected then v is a cut vertex of G iff $\{v\}$ is a cut set.
2. The vertex v is a cut vertex iff there are vertices u and w, different from v s.t. every uw-path uses v.
3. A graph has no cut sets iff it is a complete graph.

Definition 1.38

For a non-complete graph G, we define its **connectivity** $\kappa(G)$ as the minimal size of a cut set. For K_n, we set $\kappa(K_n) = n - 1$.

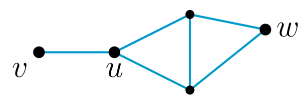
Lemma 1.39

If G is a nonseparable graph of order at least 3, then $\delta(G) \geq 2$ and every vertex of G is contained in a cycle.

Proof. Consider G nonseparable with $|G| \geq 3$. By definition, G is connected, i.e. $\delta(G) \geq 1$.

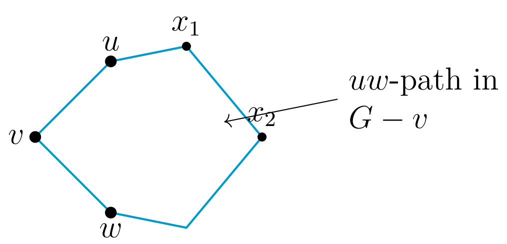
First we show that $\delta(G) \geq 2$. Otherwise, we have $\delta(G) = 1$, i.e. there is some vertex v s.t. $\deg(v) = 1$. Let u be the unique neighbor of v and w any other vertex of G (which exists as $|G| \geq 3$). Then clearly any vw-path must use the unique neighbor u of v, whence u is a cut vertex. This contradicts the fact that G is inseparable. Hence, $\delta(G) \geq 2$, as desired.

Picture:



Now, consider $v \in V_G$ arbitrary. We want to show that v is contained in a cycle in G. As $\delta(G) \geq 2$, v has at least 2 neighbors, say u and w. As G is nonseparable, $G - v$ is still connected. In particular, there is a uw-path $(u = x_0, x_1, x_2, \ldots, x_k = w)$ in $G - v$. But then the walk $(x_0 = u, x_1, \ldots, x_k = w, v, x_0 = u)$ is the desired cycle containing v.

Picture:



$\square$

Definition 1.40

We say that G is **k-connected** if $\kappa(G) \geq k$, i.e. if G is connected and $G - S$ is still connected for any $S \subseteq V_G$ with $|S| < k$.

Lemma 1.41

The following hold:

1) G is connected iff $\kappa(G) \geq 1$.
2) G is 1-connected iff G is connected.
3) G is 2-connected iff G is connected and has no cut vertices.
4) G is 2-connected iff G is non-separable.
5) If G is 2-connected, then it contains at least one cycle (for $|G| \geq 3$).
6) If G is k-connected, then G is j-connected for all $j \leq k$.
7) $|G| > \kappa(G)$.
8) $\kappa(G) \leq \delta(G)$.

Proof. 1)–6) are easy observations – verify them by yourselves.

7) If $G = K_n$, then $|G| = n > n-1 = \kappa(G)$. Otherwise, assume $\kappa(G) = k$, i.e. ex. $S \subseteq V_G$ s.t. $|S| = k$ and $G - S$ is disconnected. For $G - S$ to be disconnected, it must contain at least 2 vertices, whence

$$|G| \geq |S| + 2 = \kappa(G) + 2 > \kappa(G).$$

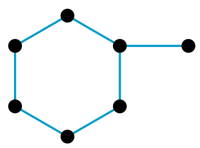
8) Assume $\kappa(G) > \delta(G)$ and let $v \in V_G$ s.t. $\deg(v) = \delta(G)$. Note that $|G| > \kappa(G) > \delta(G) = |N(v)|$, whence $G - N(v)$ contains at least one vertex besides v. But clearly, $G - N(v)$ is disconnected (as $\deg^{G-N(v)}(v) = 0$). Hence, $N(v)$ is a cut set and $\kappa(G) \leq |N(v)| = \delta(G)$, contradicting the assumptions. $\square$

5. Bipartite Graphs

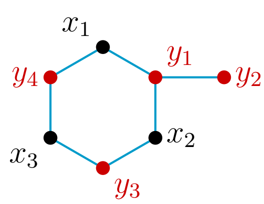
Definition 1.42

A graph G is called **bipartite** if we can partition the vertex set V_G into two disjoint sets $V_G = X \cup Y$ s.t. every edge of G has one end vertex in X and the other in Y.

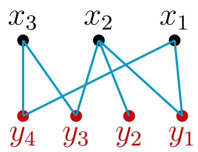
Example 1.43. Consider $G :=$



We can partition the vertices of G into two sets via $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3, y_4\}$.



Rearranging the position of the vertices makes it clear that G is bipartite:



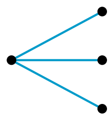
We see that there are no edges between any two vertices in X or in Y.

Remark 1.44. A graph G is bipartite if and only if we can color the vertices of G with two colors s.t. the end vertices of each edge have different colors.

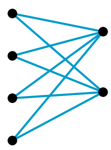
Definition 1.45

Let $m, n \in \mathbb{Z}_+$. The **complete bipartite graph** $K_{m,n}$ is the bipartite graph with $X = \{x_1, \ldots, x_m\}$, $Y = \{y_1, \ldots, y_n\}$, $V_G = X \cup Y$ and $E_G = \{xy \mid x \in X, y \in Y\}$.

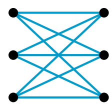
Example 1.46. Below are some examples of complete bipartite graphs.



$K_{1,3}$



$K_{4,2}$



$K_{3,3}$

The following theorem helps us decide whether or not a given graph is bipartite.

Theorem 1.47

A graph is bipartite iff it does not contain odd cycles.

Proof. "$\Rightarrow$": Assume G is bipartite and nevertheless there is a cycle of odd length, say $(x_0, x_1, \ldots, x_{2k}, x_{2k+1} = x_0)$. By Remark 1.44, we can color V_G in two colors, $C1$ and $C2$, s.t. adjacent vertices have different colors. Then, if x_0 has color $C1$, x_1 has color $C2$ whence x_2 has color $C1$. That way we see that the color of x_i is

$$\begin{cases} C1 \text{ if } i \text{ is even} \\ C2 \text{ if } i \text{ is odd} \end{cases}.$$

Following that logic, the vertex $x_0 = x_{2k+1}$ should have color $C1$ and color $C2$ at the same time, which is a contradiction.

"$\Leftarrow$": Now consider that G does not contain odd cycles. We will show that G is bipartite by providing a partition. We may assume that G is connected as otherwise we work component per component. Pick $v \in V_G$ arbitrary and define

$$X = \{w \in V_G \mid \text{the shortest } vw \text{ path has even length}\} \text{ and}$$

$$Y = \{w \in V_G \mid \text{the shortest } vw \text{ path has odd length}\}.$$

Clearly, X and Y are disjoint. We will show that there are no adjacent vertices in X or Y respectively. Note that $v \in X$.

Aiming for a contradiction, assume that there are vertices $w_1, w_2 \in X$ which are adjacent. Clearly, $w_1 \neq v$, as otherwise the shortest vw_2-path was exactly vw_2 of length 1. Similarly, $w_2 \neq v$. Let $P_1 = (v = x_0, x_1, \dots, x_{2k} = w_1)$ and $P_2 = (v = y_0, y_1, \dots, y_{2\ell} = w_2)$ be the shortest vw_1- and vw_2-paths. Suppose that $x_i = y_j$ for some $0 < i \leq 2k$ and $0 < j \leq 2\ell$. If $i < j$, then $(v = x_0, x_1, \dots, x_i, y_{j+1}, \dots, y_{2\ell} = w_2)$ is a vw_2 path shorter than P_2, a contradiction. Similarly, $j < i$ is impossible, whence $i = j$, whenever $x_i = y_j$.

Now, pick the largest i s.t. $x_i = y_i$. As $x_0 = v = y_0$, such an i always exists. Then we obtain the following cycle

$$C = (\underbrace{x_i, x_{i+1}, \dots, x_{2k}}_{2k-i} = w_1, \underbrace{w_2}_{1} = y_{2\ell}, \underbrace{y_{2\ell-1}, \dots, y_{i+1}}_{2\ell-i}, y_i = x_i).$$

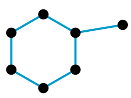
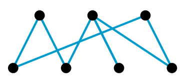
This is a cycle, as P_1 and P_2 were paths and i was maximal s.t. $x_i = y_i$. Further, the length of C is odd, as it equals

$$(2k - i) + 1 + (2\ell - i) = 2(k + \ell - i) + 1.$$

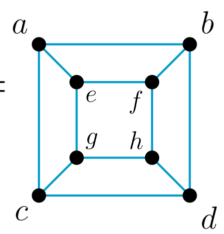
This contradicts our assumption that G does not contain odd cycles. We hence proved that no two vertices w_1 and w_2 from X can be adjacent. The arguments for $v_1, v_2 \in Y$ is analogous (try to write it down). This concludes the proof. $\square$

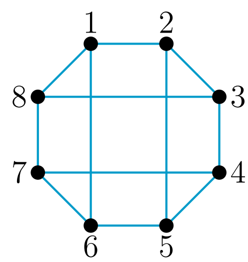
6. Graph Isomorphisms

In Example 1.43, we rearranged the given graph G_1 as G_2.

$G_1 =$  as $G_2 =$ 

We understand G_1 and G_2 as the same, even though on the first glance, they look very similar. Another example is given by

$G =$ 

$H =$ 

We can relabel the vertices of G via $a \mapsto 1, b \mapsto 2, c \mapsto 8, d \mapsto 3, e \mapsto 7, f \mapsto 4, g \mapsto 6$ and $h \mapsto 5$ and obtain H. The aim of this section is to formalise this concept.

Definition 1.48

We say that a graph G is **isomorphic** to a graph H if there exists a bijection $\varphi : V_G \to V_H$ s.t. for any $u, v \in V_G$ we have that $\{u, v\} \in E_G$ if and only if $\{\varphi(u), \varphi(v)\} \in E_H$. Then, the map φ is called an **isomorphism** and we write $G \cong H$.

Remark 1.49. Let $G \cong H$ via $\varphi : V_G \to V_H$. Then:

1) $|V_G| = |V_H|$ and $|E_G| = |E_H|$ and $\overline{G} \cong \overline{H}$.
2) The degree sequence of G equals the degree sequence of H.
3) G is connected iff H is connected.
4) $\deg_G(v) = \deg_H(\varphi(v))$ for all $v \in V_G$.

Chapter 2

Distance in Graphs

1. introduction

We have a natural understanding of the "distance" between two objects in our physical space. But there are many other ways of defining distances. E.g., the distance between people could be the positive difference of their birth years or the number of acquaintances you need to connect one to the other.

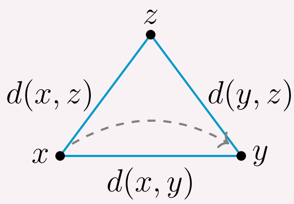
In this chapter we will introduce a notion of distance of vertices in a graph. But first let us note what are the characterising properties that make us call all these concepts "distances".

Definition 2.1

Let X be any set. We call a function $d : X \times X \to \mathbb{R}^{\geq 0} \cup \{\infty\}$ a **metric** if it satisfies for all $x, y, z \in X$:

1) $d(x, y) \geq 0$
2) $d(x, y) = 0$ iff $x = y$
3) $d(x, y) = d(y, x)$
4) $d(x, z) \leq d(x, y) + d(y, z)$ (**Triangle Inequality**)

We then call the pair (X, d) a **metric space**.



Example 2.2. Consider $X = \mathbb{R}$ and $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{\geq 0}$ via $d(x, y) := |x - y|$. Then $(\mathbb{R}, d)$ is a metric space.

Now we are ready to define a metric on an arbitrary graph.

Definition 2.3

Let G be any graph and $u, v \in V_G$. We define the **distance** $d(u, v)$ between u and v as the length of the shortest uv-path in G, i.e.

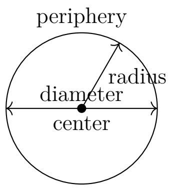
$$d(u, v) := \min\{\text{length}(P) \mid P \text{ is a } uv\text{-path}\}.$$

If there is no such path, we set $d(u, v) := \infty$.
2) If $d(u, v) = k$, then any uv-path of length k is called a **geodesic**.

Remark 2.4.

1) We may write $d_G(u, v)$ to emphasize that we consider the distance in G.
2) While in $(\mathbb{R}, d)$ geodesics are unique, in general this is not the case. Consider for example two opposite poles on a sphere.
3) $d(x, y) = \infty$ iff x and y are in different connected components.
4) (V_G, d) is a metric space for any connected graph G.

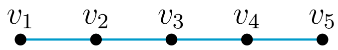
We call something eccentric if it is away from the usual. Similarly, in graphs we measure by eccentricity how far a vertex is from the center. Consider the following notions on a cycle:



Definition 2.5

1) The **eccentricity** $ecc(v)$ of a vertex v is its greatest distance to any other vertex, i.e. $ecc(v) = \max\{d(u, v) \mid u \in V_G\}$.
2) The **radius** $rad(G)$ is the smallest possible eccentricity and the **diameter** $diam(G)$ is the largest possible eccentricity.
3) The **center** $C(G)$ is the set $\{v \in V_G \mid ecc(v) = rad(G)\}$ and the **periphery** $P(G)$ is the set $\{v \in V_G \mid ecc(v) = diam(G)\}$.

Example 2.6. 1) Consider P_5, the path of length 4, i.e.

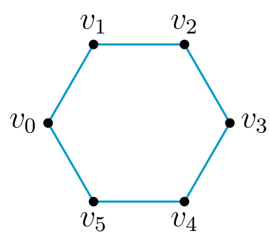


Then

$$\begin{aligned}
d(v_1, v_i) &= i - 1, \text{ whence } ecc(v_1) = \max\{0, 1, 2, 3, 4\} = 4.\\
d(v_2, v_i) &= |i - 2|, \text{ whence } ecc(v_2) = \max\{1, 0, 1, 2, 3\} = 3.\\
d(v_3, v_i) &= |i - 3|, \text{ whence } ecc(v_3) = \max\{2, 1, 0, 1, 2\} = 2.\\
d(v_4, v_i) &= |i - 4|, \text{ whence } ecc(v_4) = \max\{3, 2, 1, 0, 1\} = 3.\\
d(v_5, v_i) &= |i - 5|, \text{ whence } ecc(v_5) = \max\{4, 3, 2, 1, 0\} = 4.
\end{aligned}$$

Hence $rad(P_5) = \min\{ecc(v) \mid v \in V\} = \min\{4, 3, 2, 3, 4\} = 2$.
Also $C(P_5) = \{v \in V \mid ecc(v) = rad(P_5)\} = \{v_3\}$.
Further $diam(P_5) = \max\{ecc(v) \mid v \in V\} = \max\{4, 3, 2, 3, 4\} = 4$.
And $P(P_5) = \{v \in V \mid ecc(v) = diam(P_5)\} = \{v_1, v_5\}$.

2) Consider $G := C_6$, the cycle of length 6, i.e. $G =$



Then

$$\begin{aligned}
d(v_0, v_i) &= 3 - |3 - i|, & ecc(v_0) &= \max\{0, 1, 2, 3, 2, 1\} = 3.\\
d(v_1, v_i) &= |3 - |4 - i||, & ecc(v_1) &= \max\{1, 0, 1, 2, 3, 2\} = 3.\\
d(v_2, v_i) &= |3 - |5 - i||, & ecc(v_2) &= \max\{2, 1, 0, 1, 2, 3\} = 3.\\
d(v_3, v_i) &= |3 - i|, & ecc(v_3) &= \max\{3, 2, 1, 0, 1, 2\} = 3.\\
d(v_4, v_i) &= |3 - |1 - i||, & ecc(v_4) &= \max\{2, 3, 2, 1, 0, 1\} = 3.\\
d(v_5, v_i) &= |3 - |2 - i||, & ecc(v_5) &= \max\{1, 2, 3, 2, 1, 0\} = 3.
\end{aligned}$$

Hence, $rad(G) = \min\{ecc(v) \mid v \in V_G\} = \min\{3, 3, 3, 3, 3, 3\} = 3$
whence $C(G) = \{v \in V_G \mid ecc(v) = rad(G)\} = V_G$.
Further, $diam(G) = \max\{ecc(v) \mid v \in V_G\} = \max\{3, 3, 3, 3, 3, 3\} = 3$.
and $P(G) = \{v \in V_G \mid ecc(v) = diam(G)\} = V_G$.

Lemma 2.7

For any graph G we have $rad(G) \leq diam(G) \leq 2rad(G)$.

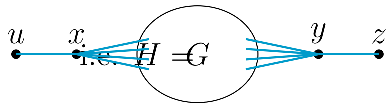
Proof. We have $rad(G) \leq diam(G)$ by definition. For the other inequality, pick $v \in C(G)$ arbitrary and consider $u, w \in V_G$ arbitrary s.t. $d(u, w) = diam(G)$. Then

$$d(u, w) \leq d(u, v) + d(v, w) \leq ecc(v) + ecc(v) = 2rad(G). \qquad \square$$

Theorem 2.8

Every graph G is isomorphic to the graph induced by the center of another graph H, i.e. ex. H s.t. $G \cong \langle C(H) \rangle$.

Proof. Let G be arbitrary. We build a new graph H which contains G as an induced subgraph via: $V_H = V_G \cup \{u, x, y, z\}$, i.e. adding 4 new vertices to G. Further, let $E_H = E_G \cup \{ux, yz\} \cup \{xv, vy \mid v \in V_G\}$.



Now $ecc(v) = 2$ for any $v \in V_G$. Nevertheless, $d(u, z) = 4$ and $d(x, z) = d(y, u) = 3$, whence $ecc(w) > 2$ for all $w \in V_H \setminus V_G$. Thus, $rad(H) = 2$ and $C(H) = V_G$, whence $\langle C(H) \rangle \cong G$. □

Lemma 2.9

A graph G is isomorphic to the graph induced by the periphery of another graph H iff either every vertex has eccentricity 1 or no vertex does.

Proof. "$\Rightarrow$" We use proof by contraposition. Assume ex. $u \in V_G$ s.t. $ecc(u) = 1 < diam(G)$. In particular, $G \neq P(G)$. Now, aiming for a contradiction, assume ex. H s.t. $G \leq H$ and $P(H) = V_G$. As $G \neq P(G)$, we know that $H \neq G$ and $diam(H) \geq 2$. As $u \in V_G = P(H)$, there is some $w \in V_H$ s.t. $d(u, w) = diam(H)$. But then, $w \in P(H) \cong V_G$, and as $ecc(u) = 1$, we also get $d(u, w) = 1 < diam(H)$. Hence, $P(H)$ cannot be V_G.

"$\Leftarrow$" If all vertices in G have eccentricity 1 or 0, then G is complete and $G \cong P(G)$. For the second case, assume $rad(G) > 1$. And consider H s.t. $V_H = V_G \cup \{v\}$ contains one new vertex which is connected to everyone else, i.e. $E_H = E_G \cup \{vx \mid x \in V_G\}$. Then, as $ecc(x) \geq 2$ for all $x \in V_G$,

$$ecc_H(x) = \begin{cases} 2 & \text{if } x \in V_G \\ 1 & \text{if } x = v \end{cases}.$$

Hence, $diam(H) = 2$ and $\langle P(H) \rangle = G$, as desired. □

2. Adjacency Matrices

We saw the visual benefits of studying graphs by their diagram. This is very useful to illustrate ideas and study small graphs. In applications on the other hand, when studying e.g. correlations of weather phenomena or social links, graphs tend to have thousands of vertices. Here, it is no longer practical to use neither the set- nor the diagram representation of graphs. The way computers store and analyze graphs is by using adjacency matrices.

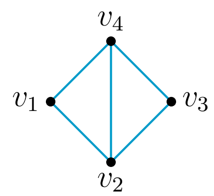
Definition 2.10

Let G be a graph of order n with vertices $V_G = \{v_1, v_2, \dots, v_n\}$. The **adjacency matrix** of G is the matrix $A_G = (a_{ij}) \in M_{n\times n}$ defined via

$$a_{ij} = \begin{cases} 1 & \text{if } v_i v_j \in E \\ 0 & \text{otherwise.} \end{cases}$$

We also write $A(i, j)$ for a_{ij}.

Example 2.11. Consider G given by



Then $A_G \in M_{4\times 4}$

$$A_G = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

is the adjacency matrix of G.

Remark 2.12. If $A_G = (a_{ij})$ is an adjacency matrix of a graph G, then

1) $a_{ii} = 0$ for all $1 \le i \le |G|$
2) A is symmetric.
3) $\sum_{j=1}^{|G|} a_{ij} = \deg(v_i)$ and thus $\sum_{i,j=1}^{|G|} a_{ij} = \sum_{i=1}^{|G|} \deg(v_i) = 2|E|$.
4) A_G is only unique up to reordering the vertices.

Example 2.13. Let revisit the graph G from 2.11. The fact that $A_G(2, 3) \neq 0$ means that v_2 and v_3 are adjacent. And $A(1, 3) = 0$ says that v_1 and v_3 are not. Now consider

$$A_G^2 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 2 & 1 \\ 1 & 3 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 3 \end{pmatrix}.$$

Let's interpret the values of A_G^2. Now, $A_G^2(1, 3) = 2$. How did we compute it? $A_G^2(1, 3) = \sum_{j=1}^{4} a_{1j} a_{j3}$. Now $a_{1j} a_{j3} = 1$ iff $v_1 v_j$ and $v_j v_3$ are edges iff (v_1, v_j, v_3) is a walk of length 2 from v_1 to v_3. Hence, $A_G^2(1,3) = \sum a_{1j}a_{j3}$ is the number of walks from v_1 to v_3 of length 2. This generalises and provides a strong tool to study graphs.

Theorem 2.14

Let G be a graph with $V_G = \{v_1, \dots, v_n\}$ and A_G the corresponding adjacency matrix. Then the entry $A_G^k(i,j)$ is the number of possible walks from v_i to v_j of length k.

Proof. We proceed by induction on the power k. (Note that $k = 0$ works too). $\underline{k = 1}$: We get that $A(i,j) = \begin{cases} 0 & \text{iff } v_iv_j \notin E_G \text{ iff there are } 0\ v_iv_j\text{-walks of length } 1 \\ 1 & \text{iff } v_iv_j \in E_G \text{ iff there is } 1\ v_iv_j\text{-walk of length } 1 \end{cases}$.

$\underline{k \to k+1}$: Assume that $A^k(i,j)$ gives exactly the number of v_iv_j-walks of length exactly k. Let's denote $A^k = (b_{ij})$ and $A = (a_{ij})$. Note that there is a v_iv_j-walk of length $k+1$ iff there ex. a vertex v_ℓ s.t. there is a v_iv_ℓ-walk of length k and an $v_\ell v_j$-walk of length one. Hence

$$\begin{aligned}
|\{v_iv_j\text{-walk of length } k+1\}| &= \sum_{\ell | v_\ell \in N(v_j)} |\{v_iv_\ell\text{-walk of length } k\}| \\
&\overset{\text{I.H.}}{=} \sum_{\ell | v_\ell \in N(v_j)} b_{i\ell} = \sum_{\ell=1}^{n} b_{i\ell}a_{\ell j} \\
&= \sum_{\ell=1}^{n} A^k(i,\ell) \cdot A(\ell,j) = A^{k+1}(i,j).
\end{aligned}$$

□

Corollary 2.15

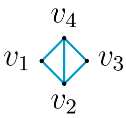
Let G be a graph with $V_G = \{v_1, \dots, v_n\}$ and A_G the adjacency matrix. Then $d(v_i, v_j) = \min\{k \mid A^k(i,j) \neq 0\}$. (Recall that $A_G^0 = I_n$).

Definition 2.16

Let G be a graph with adjacency matrix A. For every $k \in N$ we define the **Stoll matrix** S_k via

$$S_k = \sum_{i=0}^{k} A^k = I_n + A + A^2 + \dots + A^k.$$

Remark 2.17. As $S_k(i,j) = \sum_{i=0}^{k} A^i(i,j)$, we get that $S_k(i,j)$ is the number of v_iv_j-walks of length at most k.

Example 2.18. Recall the graph $G =$ 

with $A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$.

$A^2 = \begin{pmatrix} 2 & 1 & 2 & 1 \\ 1 & 3 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 3 \end{pmatrix}$ and $A^3 = \begin{pmatrix} 2 & 5 & 2 & 5 \\ 5 & 4 & 5 & 5 \\ 2 & 5 & 2 & 5 \\ 5 & 5 & 5 & 4 \end{pmatrix}$.

Then $S_0 = I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $S_1 = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$, $S_2 = \begin{pmatrix} 3 & 2 & 2 & 3 \\ 2 & 4 & 2 & 3 \\ 2 & 2 & 3 & 2 \\ 2 & 3 & 2 & 4 \end{pmatrix}$.

$S_3 = \begin{pmatrix} 5 & 7 & 4 & 8 \\ 7 & 8 & 7 & 8 \\ 4 & 7 & 5 & 7 \\ 7 & 8 & 7 & 8 \end{pmatrix}$. This means there are for example 4 v_1v_3 walks of length at most 3, namely (v_1, v_2, v_3), (v_1, v_4, v_3), (v_1, v_2, v_4, v_3) and (v_1, v_4, v_2, v_3).

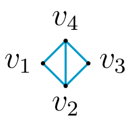
Theorem 2.19

Let G be a graph with $V_G = \{v_1, \dots, v_n\}$, adjacency matrix A and Stoll matrices S_k. Then the following hold.

1) $d(v_i, v_j)$ is the least k s.t. $S_k(i,j) \neq 0$.
2) $ecc(v_i)$ is the least k s.t. the i-th row of S_k has no zero entries.
3) $rad(G)$ is the least k s.t. S_k contains at least one row without zero entries (or ∞ otherwise).
4) $diam(G)$ is the least k s.t. S_k does not contain any zero entries.
5) G is disconnected iff S_{n-1} contains a zero.

Definition 2.20

Let G be a graph with $V_G = \{v_1, \dots, v_n\}$. The **distance matrix** of G is the matrix $D \in M_{n \times n}$ s.t. $D(i,j) = d(v_i, v_j)$.

Example 2.21. Back to our example $G =$ 

. Then the distance matrix D is

$$\begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 1 \\ 2 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

Example 2.22. Erdős Number Paul Erdős - Hungarian Mathematician, published over 1500 papers. Consider G with $V_G =$ all mathematicians, $E_G = \{xy \mid x \text{ and } y \text{ published together}\}$. Then $\deg(\text{Erdős}) > 500$ and the Erdős number of x is $d(\text{Erdős}, x)$.

Chapter 3

Trees

1. Introdction

The intuition for graph theoretic trees comes from actual trees in nature. Here, the stem splits into several branches that afterwards never rejoin.

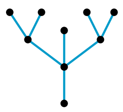
Definition 3.1

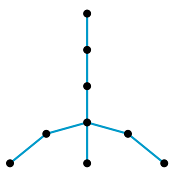
A graph which does not contain cycles is called **acyclic**. We call a graph G a **tree** if it is connected and acyclic. An arbitrary acyclic graph is called a **forest**. In a forest, any vertex of degree 1 is called a **leaf**.

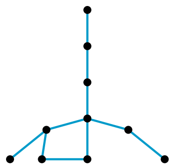
Remark 3.2.

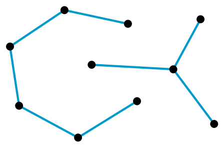
1) The graphs P_n, K_1, K_2 and $K_{1,n}$ are trees for any $n \in \mathbb{N}$.
2) Every tree is a forest.
3) Every connected component in a forest is a tree.
4) Every subgraph of a forest is a forest.

Example 3.3.

1)  **is a tree**.

2)  **is a tree**.

3)  **is not a tree**.

4)  **is not a tree**, but it **is a forest**.

Lemma 3.4

Any tree of order at least 2 has at least two leaves.

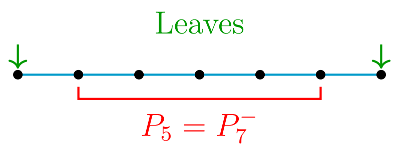
Proof. Let T be a tree with $|T| \geq 2$. In particular, T is connected. Consider a path of maximal length $P = (v_0, v_1, \ldots, v_n)$ in T. As $|T| \geq 2$, we know that $v_0 \neq v_n$. We claim that v_0 and v_n are leaves, i.e. $\deg(v_0) = \deg(v_n) = 1$. We execute the argument for v_0. As usual, we know that $N(v_0) \subseteq \{v_1, v_2, \ldots, v_n\}$. Let $u \in N(v_0)$ arbitrary, i.e. $u = v_i$ for some $i \geq 1$. But then $(v_0, v_1, \ldots, v_i, v_0)$ is a closed walk which is a cycle for all $i \geq 2$. As T does not contain cycles, we conclude that $i = 1$ and v_1 is the only neighbour of v_0. Hence $\deg(v_0) = 1$ and v_0 is a leaf. The argument for v_n is analogous. □

Definition 3.5 Tree Pruning

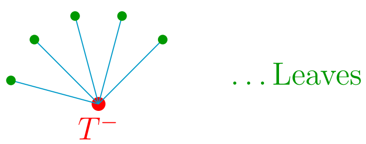
Let T be a tree of order at least 3. We denote by T^- the induced subgraph of T obtained by deleting all leaves of T.

Example 3.6.

1) If $T = P_7$ the path of length 6, then $T^- = P_7^- = P_5$ is the path of length 4:



2) If $T = K_{1,n}$ the complete bipartite graph, then $T^- = K_1 = E_1$ consists of one vertex only:



Lemma 3.7

Any tree of order n has exactly $n - 1$ edges.

Proof. We proceed by induction on $|T|$. If $|T| = 1$, then $T = K_1$, which has zero edges and the claim holds. Now assume we know that any tree of order n has exactly $n-1$ many edges and consider T of order $n+1$ arbitrary. By Lemma 3.4, T has a leaf u. Clearly, $T - u$ is still connected and of order n, whence $T - u$ has exactly $n - 1$ edges. As u was a leaf in T, T has exactly one edge more than $T - u$, whence

$$\|T\| = n = (n+1) - 1, \text{ as desired.} \qquad \square$$

Corollary 3.8

A forest of order n, consisting of k-many connected components, has exactly $n - k$ many edges.

We will now see that given a graph G is connected, Lemma 3.7 is not only a necessary, but even a sufficient condition for G to be a tree.

Theorem 3.9

A graph G of order n is a tree iff it is connected and has exactly $n - 1$ many edges.

Proof. "$\Rightarrow$" Clear by definition of a tree and Lemma 3.7.

"$\Leftarrow$" Assume G is connected of order n and contains exactly $n - 1$ many edges. If G contains a cycle, take any edge e_1 within the cycle and consider $G - e_1$. Then $G - e_1$ is still connected and of order n. If $G - e_1$ still contains a cycle, we proceed likewise and after $k \leq n-1$ many steps we obtain a graph $G - \{e_1, e_2, \ldots, e_k\}$ which is of order n, connected and without cycles, whence it is a tree. But $G - \{e_1, \ldots, e_k\}$ has $(n-1) - k < n - 1$ many edges, contradicting Lemma 3.7. $\square$

Theorem 3.10

A graph of order n is a tree iff it is acyclic and has $n - 1$ many edges.

Proof. "$\Rightarrow$" Clear.

"$\Leftarrow$" Assume G is of order n with $n-1$ many edges and acyclic, i.e. G is a forest. But by Corollary 3.8, if G has k-many connected components then $\|G\| = n - k = n - 1$, whence $k = 1$ and G is connected and hence a tree. $\square$

Corollary 3.11 Summary

Let G be a graph of order n. Then TFAE:

1) G is connected and acyclic (i.e. a tree).
2) G is connected and has $n - 1$ many edges.
3) G is acyclic and has $n - 1$ many edges.

3.12 Homework

Every edge in a tree is a bridge.

Lemma 3.13

For any two vertices $u, v \in V_T$ in a tree T, there is a unique uv-path.

Proof. As T is connected, there clearly is a uv-path for any $u, v \in V_T$. Now assume that $P_1 = (u = x_0, x_1, \ldots, x_k = v)$ and $P_2 = (u = y_0, y_1, \ldots, y_\ell = v)$ are two distinct uv-paths. Then $P_1 \cup P_2$ is again a tree. Let i be minimal s.t. $x_i \neq y_i$. Then $(P_1 \cup P_2) - y_i y_{i-1}$ is still connected, contradicting the fact that every edge in a tree is a bridge. $\square$

Corollary 3.14

Let T be a tree and $v \in V_T$. Then $ecc(v)$ is the length of the longest path starting from v.

Lemma 3.15

Let T be a tree of order at least 2. Consider $u, v \in V_T$ s.t. $ecc(v) = d(u, v)$. Then u is a leaf.

Proof. Let $P = (v = x_0, x_1, \ldots, x_k = u)$ be the unique vu path. If u were not a leaf, then it had at least one neighbour $w \notin P$. But then $(v = x_0, x_1, \ldots, x_k, w)$ would be a path starting in v and longer than P, contradicting Corollary 3.14. $\square$

Lemma 3.16

Let T be a tree of order at least 3. Then $C(T) = C(T^-)$.

Proof.

1) Show that $C(T) \subseteq T^-$, i.e. $C(T)$ contains no leaf. To this end, let u be a leaf and v its unique neighbour. As $|T| \geq 3$, v is not a leaf itself and $d(u, w) = d(v, w) + 1$ for any $w \in V_T \setminus \{u\}$, whence $ecc(u) > ecc(v)$ and hence $u \notin C(T)$.

2) Show that $ecc_{T^-}(v) = ecc_T(v) - 1$ for every non-leaf $v \in V_T$. To that end, consider an arbitrary non-leaf $v \in V_T$ and pick $u \in V_T$ s.t. $d(v, u) = ecc(v)$. By 3.15, u is a leaf. Let P be the unique vu-path in T and note that u is the only leaf on P. Hence only u will be deleted from P in T^-. As this holds for all paths in T starting in v of length $ecc(v)$, we obtain that $ecc_{T^-}(v) = ecc_T(v) - 1$, as desired.

3) We conclude from 1) + 2) that for any vertex $v \in T^-$, $ecc_{T^-}(v) = ecc_T(v) - 1$, whence $v \in C(T)$ iff $v \in C(T^-)$ (and $rad(T^-) = rad(T) - 1$).

$\square$

Lemma 3.17

Let T be a tree. Then $C(T)$ is either K_1 or K_2.

Proof. We do induction on $|T|$. If $|T| = 1$, then $T = K_1$ is its own center and we are done. Similarly for $|T| = 2$, where $T = K_2$. Now assume that the claim holds for all trees of order $n \geq 3$ and consider a tree T with $|T| = n + 1$ arbitrary. By 3.16, we know that $C(T) = C(T^-)$. By 3.4 we know that T contains at least two leaves, whence $|T^-| \leq |T| - 2 < n$. Hence, by I.H., $C(T) = C(T^-)$ is either K_2 or K_1 as desired. □

Lemma 3.18

Let T be a tree of order n and G an arbitrary graph s.t. $\delta(G) \geq n - 1$. Then G contains T as a subgraph.

Proof. We use induction on $|T|$. If $|T| = 1$, then $T = K_1$ is a subgraph of any graph G. Now assume we proved the claim for all trees of order at most n. Consider T with $|T| = n+1$ and G with $\delta(G) \geq n$ arbitrary. Let u be a leaf of T and denote by $T' := T - u$. Then $|T'| = n$, whence T' can be seen as a subgraph of G. Let v be the unique neighbour of u in T. Then $\deg_G(v) \geq \delta(G) \geq n$, but as $|T'| = n$ and v cannot be its own neighbour, there exist some $u' \in G$ adjacent to v and not contained in T'. Hence, the subgraph $(V_{T'} \cup \{u'\}, E_{T'} \cup \{vu'\})$ is the desired subgraph of G isomorphic to T. □

Summary

1) A tree of order n contains exactly $n - 1$ edges.
2) Any tree of order at least two contains at least two leaves.
3) A graph of order n is a tree iff it is connected of size $n - 1$.
4) A graph of order n is a tree iff it is acyclic and of size $n - 1$.
5) A graph is a tree iff for any vertices u, v there is a unique uv-path.
6) The centre of any tree is either K_1 or K_2.
7) Any graph G contains any tree of order at most $\delta(G) + 1$ as a subgraph.

2. Spanning Trees

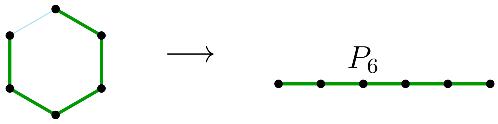
Definition 3.19

Let G be any graph. We call a subgraph $T \subseteq G$ a **spanning tree** for G if it is a tree and contains all vertices of G.

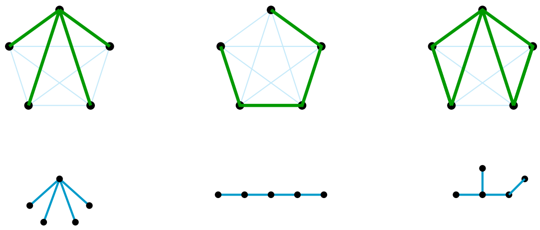
Remark 3.20. From the previous chapter it is clear that a spanning tree of a graph G of order n has n many vertices and $n - 1$ many edges.

Example 3.21. Consider the following graphs and spanning trees.

1) $G = C_6$, a possible spanning tree:



2) $G = K_5$, possible spanning trees:



$K_{1,4}$ P_5

Lemma 3.22

Every connected graph contains at least one spanning tree.

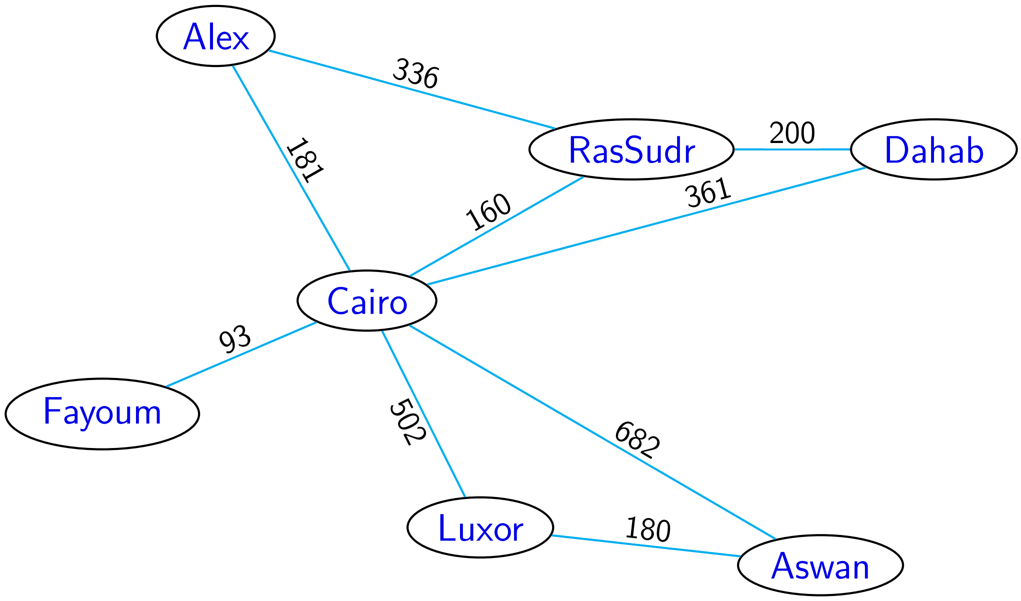
Proof. Assume G is connected and let T be a subgraph of G of maximal order s.t. T is a tree. We need to show that $V_T = V_G$. Otherwise, as G is connected, there is some vertex $u \in V_G \setminus V_T$ which is adjacent to some vertex $v \in V_T$. Now, consider the new subgraph $\hat{T} = (V_T \cup \{u\}, E_T \cup \{uv\})$. As $\deg_{\hat{T}}(u) = 1$, u is not contained in any cycles in $\hat{T}$, whence $\hat{T}$ is still a tree. As this contradicts maximality of $|T|$, we conclude that T must contain all vertices of G, whence it is a spanning tree for G. $\square$

Definition 3.23

A function $w : E_G \to \mathbb{R}$ is called a **weight function** on G. A graph G together with a weight function (i.e. the triple (V_G, E_G, w)) is called a **weighted graph**.

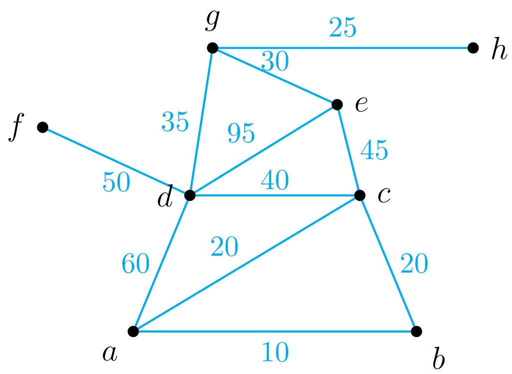
Example 3.24. *Visualisation*

We visualise the weighting of a graph by denoting the weight $w(e)$ on top of the edge e, e.g.

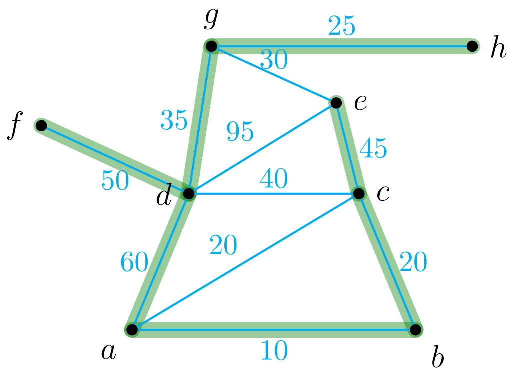


Here the weight function of an edge $e = uv$ is given by the (birds eye) distance between u and v.

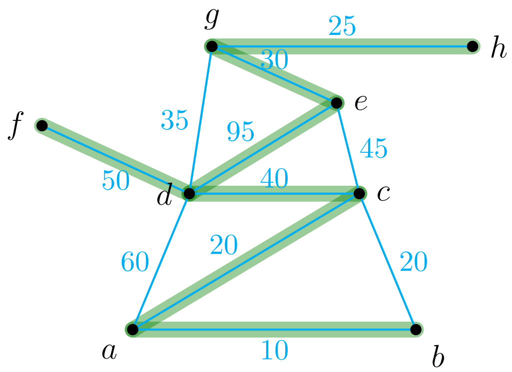
Example 3.25. Consider the following weighted graph.



We can find several spanning trees. Let's name some and compute their weight.



Total weight:
$45 + 20 + 10 + 60 + 50 + 35 + 25$
$= 245$



Total weight:
$10 + 20 + 40 + 50 + 95 + 30 + 25$
$= 270$

Definition 3.26

Let (G, w) be a connected weighted tree. A **minimum-weight spanning tree** T is a spanning tree of G s.t. the sum of the weights of its edges is minimal among all possible spanning trees of G, i.e. if $\tilde{T}$ is another spanning tree, then $\sum_{e \in E_T} w(e) \leq \sum_{e \in E_{\tilde{T}}} w(e)$.

Now how can we find a minimal spanning tree effectively? Consider the following algorithm:

3.27 Kruskal's Algorithm (1956)

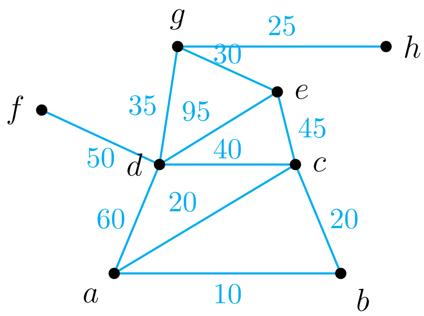
Consider the set of vertices as a forest $F = (V_G, \emptyset)$ where each vertex is a maximal subtree of F. Let $E := E_G$.

While (F is not a tree $\wedge$ $E \neq \emptyset$)

- Pick $e \in E$ of minimal weight. Let $E := E \setminus \{e\}$.
- If e connects two trees in F, let $E_F = E_F \cup \{e\}$.
- (i.e. $F + e$ is still acyclic)

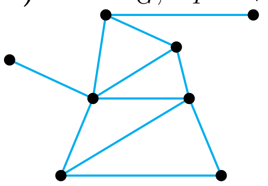
This algorithm stops after at most $|E_G|$ many repetitions.

Example 3.28. We apply the algorithm on the following weighted graph:

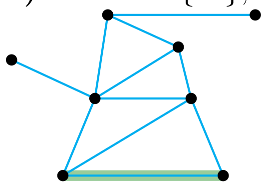


Let us mark edges we add to F green and the ones we disregard, red.

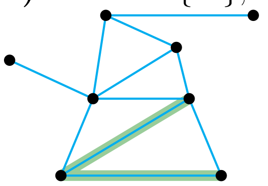
1) $E = E_G, E_F = \emptyset$



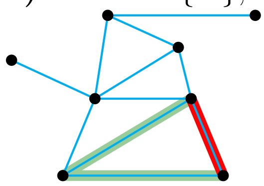
2) $E = E - \{ab\}, E_F = \{ab\}$



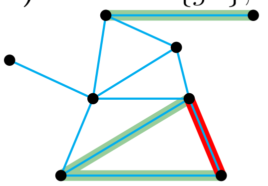
3) $E = E - \{ac\}, E_F = E_F \cup \{ac\}$



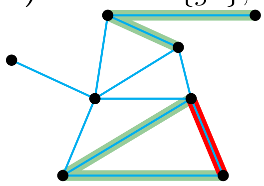
4) $E = E - \{bc\}, E_F = E_F$



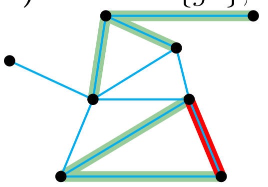
5) $E = E - \{gh\}, E_F = E_F \cup \{gh\}$



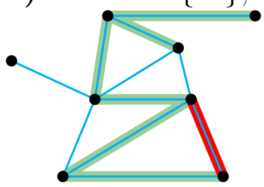
6) $E = E - \{ge\}, E_F = E_F \cup \{ge\}$



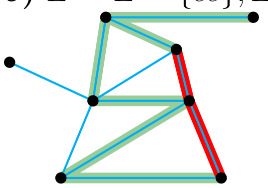
7) $E = E - \{gd\}, E_F = E_F \cup \{gd\}$



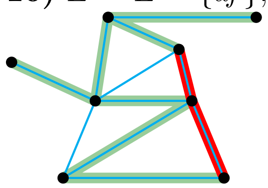
8) $E = E - \{cd\}, E_F = E_F \cup \{cd\}$



9) $E = E - \{ce\}, E_F = E_F$



10) $E = E - \{df\}, E_F = E_F \cup \{df\}$



Here the algorithm stops, as $F = (V_G, E_F)$ with $E_F = \{ab, ac, gh, eg, dg, cd, df\}$ is a single tree whence the conditions in the while loop are violated.
(The output is the spanning tree F. Note that the second condition in the while loop was still valid, as $E = \{ad, de\} \neq \emptyset$).

Theorem 3.29

Kruskal's algorithm is correct, i.e. it always terminates and its output is a minimum-weight spanning tree.

Proof. 1) Termination: As after $|E_G|$-many steps the condition $E \neq \emptyset$ is violated, the algorithm always terminates.

2) The output F is a spanning tree: As $V_F = V_G$, it clearly contains all vertices of G. Further, in each step the regarded edge e either connects two disconnected trees into one larger tree, or, if it would connect two vertices of the same subtree in F, is disregarded. Hence after each step, F is still a forest, i.e. acyclic. It remains to show that F is connected. If the algorithm stops because F is a tree, then it is clearly connected. If it stops because we went through all the edges, then any edge of G not contained in F would connect two vertices of the same connected component. Thus F has as many connected components as G, which is one, as G is connected.

3) F is a minimum-weight spanning tree. Aiming for a contradiction, assume this is not the case. Let $\{e_1, \ldots, e_{n-1}\}$ be all the edges in F, enumerated in the order they were added to F by the algorithm. Among all possible minimum-weight spanning trees, let T be one that agrees with F on the largest initial segment of $(e_1, \ldots, e_{n-1})$, i.e. if k is the smallest index s.t. $e_{k+1} \notin T$, then there is no minimum-weight spanning tree which contains $\{e_1, \ldots, e_{k+1}\}$. As by assumption F is not minimum-weight, we have $k < n - 1$. As T is a spanning tree which does not contain e_{k+1}, we know that $T + e_{k+1}$ contains a cycle C. As F did not contain cycles, there is one edge $e \in C \subseteq T$ which is not in F. Now $T + e_{k+1} - e$ is a connected graph of order n and size $n-1$, whence still a spanning tree. It contains the edges $\{e_1, \ldots, e_k, e_{k+1}\}$, hence it can no longer be of minimum weight. This means that $w(e_{k+1}) > w(e)$. But as $e \notin F$ and in particular $e \notin \{e_1, \ldots, e_k\}$ this means e was available at the step of the algorithm after we added e_k and of less weight than e_{k+1}. This contradicts the assumption that the algorithm chooses the edge of minimal weight which keeps F acyclic. ☐

Lemma 3.30

If G is a connected weighted graph s.t. distinct edges have distinct weights, then there is a unique minimum-weight spanning tree.

Proof. Homework. ☐

Chapter 4

Euler and Hamilton

1. Euler

Imagine

A salesperson with their wagon wants to pass by every street in his neighbourhood to sell their goods. Of course, they want to minimize efforts, so they would like to avoid passing the same street twice. These type of problems are considered when discussing **Eulerian graphs**.

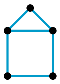
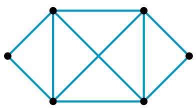
Then, as only few people buy, they switch to their car and only visit a central place in each city of the area. Again, to improve efficiency, they only want to visit each city once. This type of problem is studied when discussing **Hamiltonian graphs**.

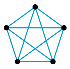
How do these two problems differ? Let's find out!

Definition 4.1

We call a trail in a graph G an **Eulerian trail** if it contains every edge of G. We call it an **Eulerian circuit** if it is a closed Eulerian trail. Finally, the graph G itself is called an **Eulerian graph** iff it contains an Eulerian circuit.

Example 4.2.

1)  **is not** Eulerian, but has an Eulerian trail.
2)  **is indeed** Eulerian.
3) Any cycle C_n is clearly Eulerian.
4) Any path of length $n \geq 1$ is **not** Eulerian but has an Eulerian trail.

5) 
K_5
is indeed Eulerian, but

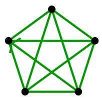
6) 
K_4
does not even contain an Eulerian trail.

7) Generally, every K_{2n+1} is Eulerian and every K_{2n+2} does not even contain an Eulerian trail for $n \geq 1$.

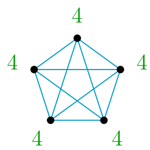
4.3 Observation

Consider the graph K_5. We observe the following properties:

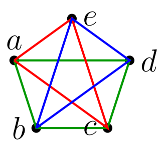
1) K_5 is Eulerian: $(a, b, c, d, e, a, d, b, e, c, a)$ is an Eulerian circuit.



2) Every vertex of K_5 has an even degree. (All 4).

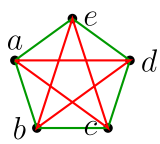


3) We can partition E_{K_5} into cycles (i.e. find mutually edge-disjoint cycles that together use all edges of K_5).



$C_1 = (a, b, c, d)$
$C_2 = (a, c, e)$
$C_3 = (b, d, e)$

or



$C_1 = (a, b, c, d, e, a)$
$C_2 = (a, c, e, b, d, a)$

These three properties do not appear together by coincidence. It turns out, they are equivalent to each other.

Lemma 4.4 Auxiliary Lemma

Let G be a connected graph with $|G| \geq 2$. If $\deg(v)$ is even for all $v \in V_G$, then G contains a cycle C. Moreover, $G - C$ still contains a cycle or is $E_{|G|}$.

Proof. Assume G is as above. If G would not contain a cycle, it was a tree. But then it had to contain a leaf v. But then $\deg(v) = 1$ is not even $\lightning$. For the "moreover" part, observe that

$$\deg^{G-C}(v) = \begin{cases} \deg^G(v) - 2 & \text{if } v \in V_C \\ \deg^G(v) & \text{else} \end{cases}$$

hence still even. Then each connected component of $G - C$ still contains a cycle (whence so does $G - C$), or is of order 1. □

Theorem 4.5 Euler-Hierholzer-Veblen

Let G be a connected graph. The following are equivalent:

1) G is Eulerian.
2) Every vertex of G is of even degree.
3) The edge set of G can be partitioned into a set of edge-disjoint cycles.

Corollary 4.6

A graph contains an Eulerian trail iff either each vertex has even degree or there are exactly two vertices of odd degree.

Proof. Proof of Theorem 4.5: As all three clearly hold for $|G| = 1$, we may assume that $|G| > 1$.

1) $\Rightarrow$ 2): Assume G is Eulerian. Let Q be an Eulerian circuit of G. Now consider $v \in V_G$ arbitrary. Without loss of generalisation, we may assume that Q does not start with v. Now, every appearence of v in Q corresponds to two distinct edges involving v, the one leading *into* v and the one leading *away* from v. As Q is Eulerian, it uses all edges incident with v whence in total there is an even number of edges incident with v and $\deg(v)$ is even. ✓

2) $\Rightarrow$ 3): Assume G only contains vertices of even degree. By Lemma 4.4, G contains at least one cycle C. We proceed by induction on the number n of cycles in C. **n=1**: If G contains only one cycle, then $G = C_{|G|}$ and hence the desired partition of edges is just the cycle G itself. **n $\rightarrow$ n+1**: Now assume every graph containing at most n-many cycles allows a partition into edge-disjoint cycles. Consider any connected G with $(n+1)$-many cycles. Pick an arbitrary cycle C in G. Then as in 4.4, in $H := (V_G, E_G - E_C)$, every vertex still has even degree. Now, every connected component of H contains at most n-many cycles. By induction hypothesis, we can partition each connected component of H, and hence H itself, into edge-disjoint cycles. Once we add C to this partition, we obtain the desired partition of G. ✓(Note that this gives you a cooking recipe of how to find cycles).

3) $\Rightarrow$ 1): Assume the edge set of G can be partitioned into k-many sets $S_1, S_2, \ldots, S_k$ s.t. the edges of each S_i form a cycle. Let Q be a circuit of maximal length in G s.t. the edges of Q equals the union of some sets S_i, i.e. such that there is $I \subseteq \{1, \ldots, k\}$ with $E_Q = \bigcup_{i \in I} S_i$. As the S_i are pairwise disjoint, we know that Q contains either no edge from S_i or all edges from S_i for every $i \leq k$. Now, if $E_Q = E_G$, then G is Eulerian and we are done. Otherwise, there is some edge not contained in Q, but incident with a vertex v in Q. The edge must be contained in exactly one S_ℓ with $\ell \notin I$. Note that Q and S_ℓ have no common edges, but they share the vertex v. Hence we may glue the circuit Q

and the cycle S_ℓ at v and obtain a new circuit Q' longer than Q with $E_{Q'} = \bigcup_{i \in I \cup \{\ell\}} S_i$, contradicting our choice of Q. Hence Q contained all edges of G and hence G is Eulerian. ✓ □

2. Hamilton

Sir William Rowan Hamilton (1805–1865)

- Irish pure mathematician
- Contributions to optics, mechanics and algebra.
- Also invented a game (The Icosian Game) build on graph theory (bought by Jaques and Son, huge failure).

Definition 4.7

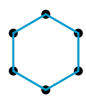
Let G be a graph. A **Hamiltonian path** is a path in G which uses all vertices of G. A **Hamiltonian cycle** is a cycle in G which uses all of V_G. We call G **traceable** if it contains a Hamiltonian path and we call it **Hamiltonian** if it contains a Hamiltonian cycle.

Remark 4.8.

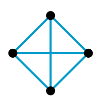
1) Every Hamiltonian graph is traceable but not vice versa.
2) Traceable graphs are connected.
3) If $|G| = n$, then G is Hamiltonian iff it contains C_n as a subgraph and it is traceable iff it contains P_n as a subgraph.

Example 4.9.

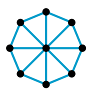
1. C_6 **is Hamiltonian** via $v_1 \dots v_6 v_1$.
   All vertices have even degree.



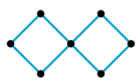
2. K_4 **is Hamiltonian** via $v_1v_2v_3v_4v_1$.
   All vertices have odd degree.



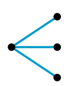
3. The graph G_1 **is Hamiltonian**.
   There are vertices of even and odd degree.



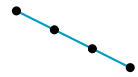
4. The graph G_2 is **not** Hamiltonian.
   Every vertex has even degree.



5. The graph $K_{1,3}$ is **not** Hamiltonian.
   All vertices have odd degree.



6. The path P_4 is **not** Hamiltonian.
   There are vertices of even and odd degree.



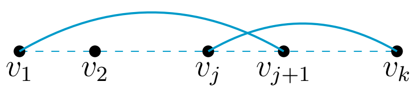
Remark 4.10. While it is rather easy to decide whether a graph is Eulerian (P-TIME, $O(|G|^2)$), it is surprisingly **hard** to do the same for Hamiltonian graphs. This problem is known to be **NP-complete** and still we did not manage to find an equivalent condition for Hamiltonianity (other than containing $C_{|G|}$ as a subgraph, which is basically the definition).

We hence see, even though the Eulerian graph problem and the Hamiltonian graph problem seem so similar, their resolution requires very different levels of efforts. The best we can do at the moment is give some **sufficient** criteria.

Theorem 4.11 Dirac

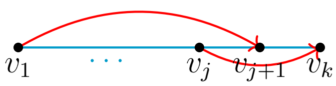
Let G be s.t. $|G| \geq 3$. If $\delta(G) \geq \frac{n}{2}$, then G is Hamiltonian.

Proof. Consider G arbitrary s.t. $|G| = n \geq 3$ and $\delta(G) \geq \frac{n}{2}$.
Then G is necessarily connected (think why).
Consider a path $P = (v_1, v_2, \ldots, v_k)$ of maximal length in G.
We claim that there is some $j < k$ s.t. $v_{j+1} \in N(v_1)$ and $v_j \in N(v_k)$. i.e.



is a subgraph of P. Note that as usual, as P is of maximal length, all neighbours of v_1 and v_k must be on P. As $\delta(G) \geq \frac{n}{2}$, v_k has at least $\frac{n}{2}$ many neighbours v_j in P. Aiming for a contradiction, assume for every neighbour $v_j \in N(v_k)$, $v_{j+1} \notin N(v_1)$. Then there are at least $\frac{n}{2}$ many vertices in P which are **not** neighbours of v_1. This now yields the desired contradiction, as all neighbours of v_1 are on P and thus $\deg(v_1) \leq (k-1) - \frac{n}{2} \leq (n-1) - \frac{n}{2} = \frac{n}{2} - 1 < \frac{n}{2}$, contradicting $\delta(G) \geq \frac{n}{2}$. Hence there is some j s.t. $v_1 v_{j+1}$ and $v_j v_k$ are edges, which leads to the existence of a cycle

$$C = (v_1, v_2, \ldots, v_j, v_k, v_{k-1}, \ldots, v_{j+1}, v_1).$$



Finally, we claim that C is indeed a Hamiltonian cycle, i.e. it contains all vertices of G. Otherwise, as G is connected, there is a vertex u in $G \setminus C$ which is adjacent to one vertex v_i in C. But as C is a cycle, we can form a new path starting in $u, v_i \ldots$ and then traveling through all $k-1$ many vertices of C. This path is longer than P, contradicting our choice of P. Hence, C indeed contains all vertices of G whence it is a Hamiltonian cycle and G is Hamiltonian. $\square$

Theorem 4.12 Fact - Ore, 1960

Let G be a graph of order $n \geq 3$. Suppose for every pair of non-adjacent vertices u, v we have that $\deg(u) + \deg(v) \geq n$. Then G is Hamiltonian.

Note that now Dirac's Theorem is a mere corollary of Ore's theorem.
We want to achieve yet another sufficient criterion for Hamiltonicity. This leads us to the so-called independence number.

Definition 4.13

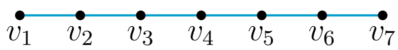
Let G be a graph. A set $S \subseteq V_G$ of vertices is called an **independent set** if any two vertices in S are nonadjacent. The **independence number** $\alpha(G)$ of G is the maximal size of an independent set.

Example 4.14.

- $\alpha(E_n) = n$, as V_{E_n} is an independent set.
- $\alpha(K_n) = 1$, as any two vertices are adjacent. Actually, the converse also holds, i.e. $\alpha(G) = 1$ iff G is complete.
- $\alpha(K_{n,m}) = \max\{n, m\}$, as any set is independent iff it is contained in one of the parts.

4.15 Notation

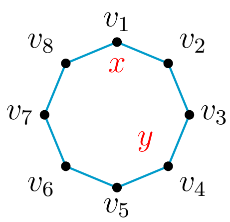
If P is a path and x, y are two vertices on P, then we denote by $P[x, y]$ the subpath on P from x to y. E.g. for



we have $P[v_6, v_3] = (v_6, v_5, v_4, v_3)$.

Similarly, if C is a cycle and $x, y \in C, x \neq y$, then we denote by $C^+[x, y]$ the xy-path on C in clockwise direction and by $C^-[x, y]$ the xy-path on C in counter-clockwise direction.

E.g. if C is



then $C^+[x, y] = (v_1, v_2, v_3, v_4)$ and $C^-[x, y] = (v_1, v_8, v_7, v_6, v_5, v_4)$.

Finally, for sequences $s = (x_1, \dots, x_\ell), t = (y_1, \dots, y_k)$ we define $s\hat{\ }t := (x_1, \dots, x_\ell, y_1, \dots, y_k)$ to be the concatenation of both.

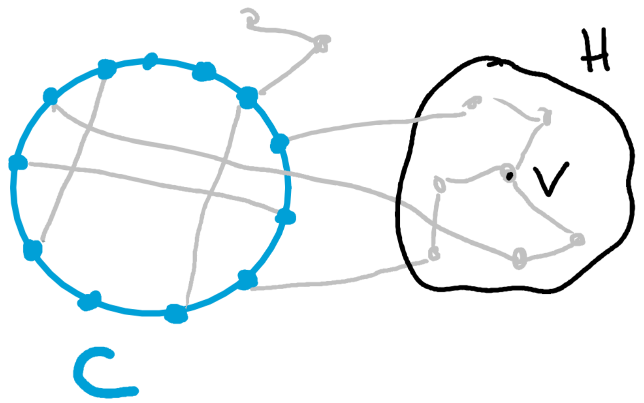
Theorem 4.16 Chvátal, Erdős, 1972

Let G be a graph of order at least 3. If $\kappa(G) \geq \alpha(G)$, then G is Hamiltonian.

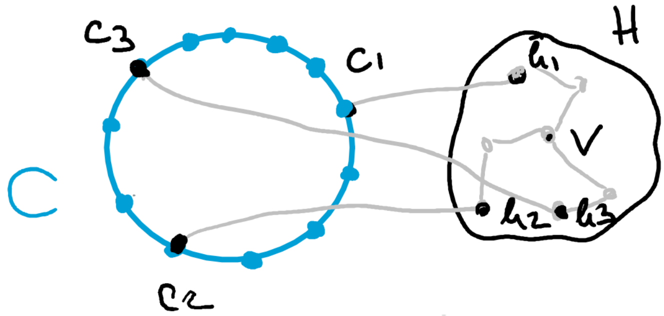
Proof. Let G be as above, i.e. $|G| \geq 3, \kappa(G) \geq 1, \kappa(G) \geq \alpha(G)$.

- First we argue that $\kappa(G) \geq 2$. Otherwise $\kappa(G) = \alpha(G) = 1$, whence G is a complete graph. As further $\kappa(K_n) = n - 1$, G would be K_2, contradicting $|G| \geq 3$.
- Hence now we know that $\kappa(G) \geq 2$. By 1.41(8), we know that $\delta(G) \geq \kappa(G) \geq 2$, whence by 1.39, G contains a cycle.

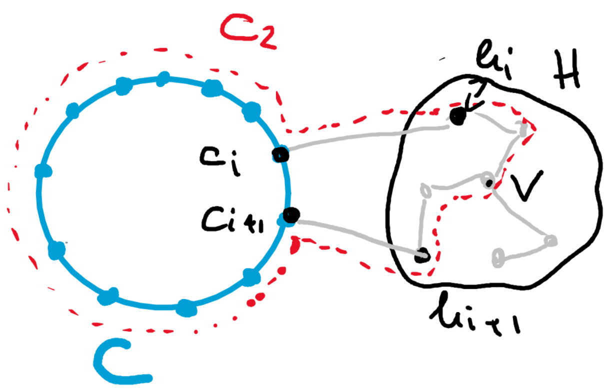
Now consider a cycle C of maximal length in C. We claim that C is Hamiltonian. Aiming for a contradiction, assume C is not Hamiltonian, i.e. there is some vertex $v \notin C$. Let H be the connected component of v in $G \setminus C$.



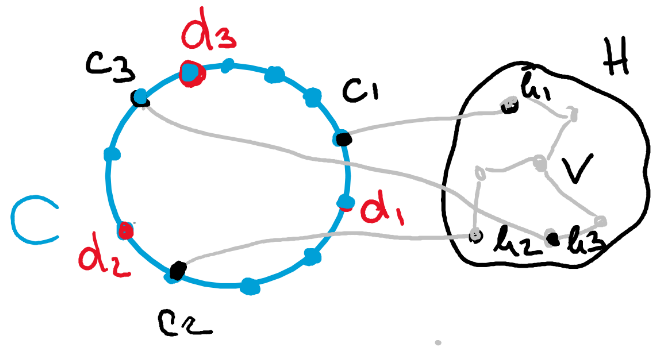
Now, we list all elements of C which are connected to some vertex in H in clockwise order: $\{c_1, c_2, \dots, c_r\}$ (s.t. $c_j \in C^+[c_{j-1}, c_{j+1}]$), i.e. where each c_i is adjacent to some $h_i \in H$.



Claim 1: No two c_i's are consecutive vertices in C. Proof: Otherwise assume there is an i s.t. c_{i+1} is the clockwise successor of c_i. Let P be a path from h_i to h_{i+1} in H. Then $C^+[c_{i+1}, c_i]\hat{\ }(c_i h_i)\hat{\ }P\hat{\ }(h_{i+1} c_{i+1})$ is a cycle strictly longer than C, contradicting our assumptions. $\lightning$

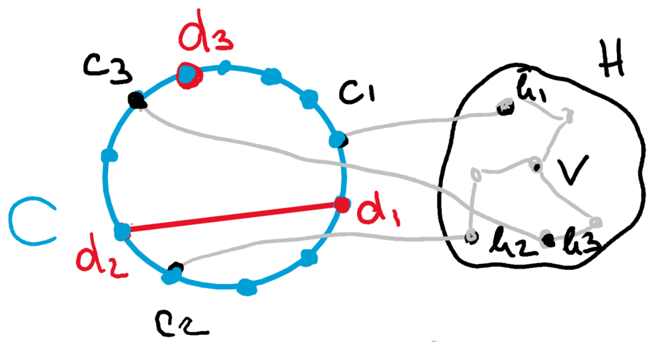


Now that no two c_i and c_j are clockwise successors, we can define the set $D = \{d_1, d_2, \ldots, d_r\}$ where each d_i is the clockwise successor of c_i in C and we get that $\{c_1, \ldots, c_r\} \cap \{d_1, \ldots, d_r\} = \emptyset$.

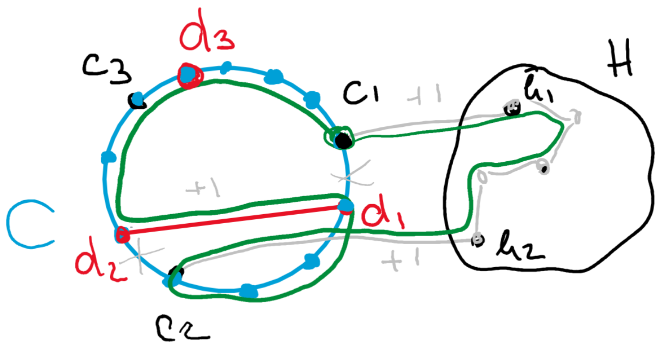


Claim 2: $\{c_1, \ldots, c_r\}$ is a cut set for G. Proof: This is clear as any path from v to a vertex in C has to pass through one of the vertices in $\{c_1, \ldots, c_r\}$, so $G - \{c_1, \ldots, c_r\}$ is disconnected. Consequently, as $\kappa(G)$ is the size of a smallest cut set, we obtain that $r \geq \kappa(G) \geq 2$.

Claim 3: There are d_i and d_j which are adjacent. Proof: Consider the set $X := \{d_1, d_2, \ldots, d_r, v\}$, and recall that there is no edge between any d_i and v. As $|X| = r + 1 \geq \kappa(G) + 1 > \alpha(G)$, X cannot be an independent set, whence at least one pair d_i, d_j must be adjacent.



Now we are ready for our final contradiction: We produce a cycle $\hat{C}$ longer than C. Assume $d_i d_j$ is an edge and $i < j$. Let $q_{h_i h_j}$ be a path in H from h_i to h_j. Now define $\hat{C} = (c_i)\hat{\ } q_{h_i h_j}\hat{\ }(c_j)\hat{\ } C^-[c_j, d_i]\hat{\ }(d_i d_j)\hat{\ } C^+[d_j, c_i]$.



Note that $\hat{C}$ uses all edges of C except the edges $c_i d_i$ and $c_j d_j$. Instead it uses at least the three additional edges $c_i h_i, h_j c_j$ and $d_i d_j$. Hence, $\hat{C}$ is strictly longer than the cycle C, which contradicts our choice of C. Conclusively, C must contain all vertices of G and hence is Hamiltonian. ☐

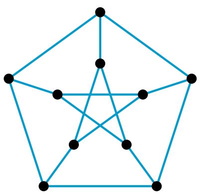
So far, we have encountered sufficient criteria for Hamiltonian graphs using the degree of vertices and the independence number. We will conclude the chapter by providing a last sufficient criterium using a new concept - **forbidden subgraphs**.

Definition 4.17

Let H and G be graphs. We say that G is ***H*-free** if H is not (isomorphic to) an induced subgraph of G. Moreover, if S is a collection of graphs, then we call G ***S*-free** iff G is H-free for any $H \in S$.

Example 4.18. The Petersen graph

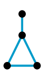
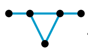
is **indeed** C_3-free
is **not** C_5-free
is **not** E_4-free
is **indeed** E_5-free

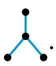
$G =$ 

Recall:
$\kappa(G) = 3$
$\alpha(G) = 4$

G is hence also $\{C_3, E_5\}$-free.

4.19 Notations

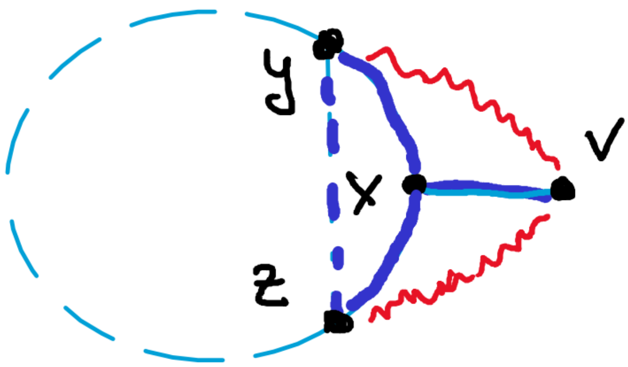
Let Z_1 be the graph $Z_1 =$  and $N =$ .

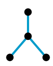
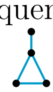
Further, we call the graph $K_{1,3}$ the **claw**, based on its shape: $K_{1,3} =$ , or also $K_{1,3} =$ .

Theorem 4.20 Goodman, Hedetniemi, 1974

Let G be 2-connected and $\{K_{1,3}, Z_1\}$-free, then G is Hamiltonian.

Proof. As $\delta(G) \geq \kappa(G) \geq 2$, we get that G contains a cycle. Consider such a cycle C of maximal length. We claim that C is Hamiltonian.
Otherwise, as G is connected, there was a vertex $v \in V_G$ not on C but adjacent to some vertex x on C, i.e.



Denote by y and z the neighbours of x on C.
Note that yv is not an edge as otherwise replacing the subsequence (y, x) in C by (y, v, x) would yield a cycle longer than C. Similarly, vz is not an edge. Consequently, the induced subgraph on $S = \{x, y, z, v\}$ is either $\langle S \rangle = K_{1,3}$  or $\langle S \rangle = Z_1$,  both of which contradict our assumptions on G. $\square$

The following final condition is now easy to verify:

Theorem 4.21 Duffus, Gould, Jacobson, 1980

Let G be a $\{K_{1,3}, N\}$-free graph.
i) If G is connected, it is traceable.
ii) If G is 2-connected, it is Hamiltonian.

Note that neither of these are necessary for G to be Hamiltonian. Indeed, for any graph H there is a Hamiltonian graph G which contains H as an induced subgraph.

Chapter 5

Planarity

1. Planar Graphs

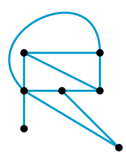
(lat. planaris - flat, level)

Definition 5.1

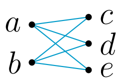
A **graph** $G = (V, E)$ is called **planar** if it can be drawn on a plane s.t. its edges at most intersect in their end vertices. Any such drawing is then called a **planar representation** or a **planar embedding**. If G is not planar, it is called **nonplanar**.

Example 5.2.

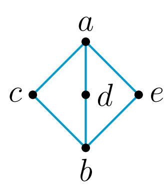
1) Clearly, trees, cycles and empty graphs are planar.
2) The graph



is planar and this is a planar representation:

3) The graph $K_{2,3}$ 

is planar, but this is not a planar representation. One planar representation is given by



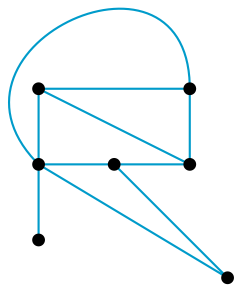
In order to prove that a graph is planar, we "just" have to provide one planar representation. But these can be very hard to find. In order to prove that a graph is not planar, we would have to check "all" possible representations. This is not feasible.

But we can help ourselves by throwing some math on the problem. To this end, we need some more terminology.

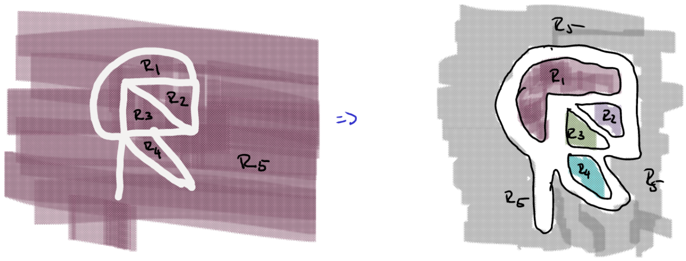
Definition 5.3

A **region** in a planar rep. is a maximal area of the plane s.t. any two points within can be connected by a curve which does not intersect or touch any part of the graph. Regions which are completely bounded (= surrounded) by graph edges are called **interior regions**. The unique non-interior region is called **exterior region**.

Example 5.4. Consider the planar representation



We can find the regions by considering this representation as a cookie cutter which we use to part the dough on our table surface. So the regions are:

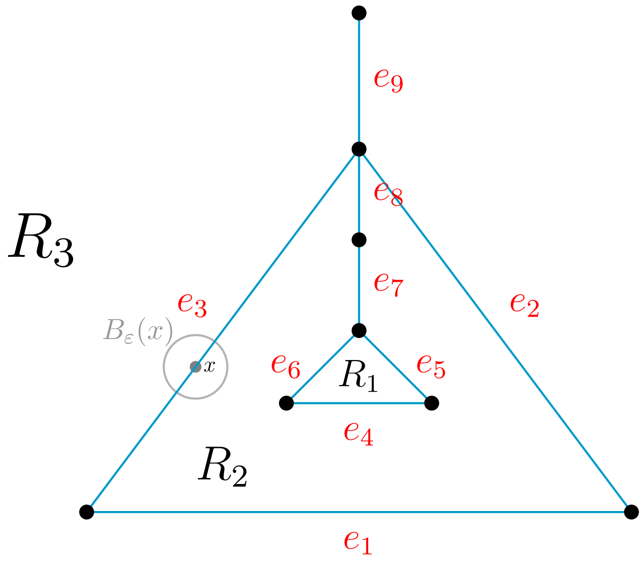


where $R_1 - R_4$ are interior and R_5 is the exterior region.

Definition 5.5

Let Γ be a planar representation of G in $\mathbb{R}^2$. We say that an edge e is **incident** with a region R iff every point x on the edge e and $\varepsilon > 0$ there is a point y in R s.t. $d(x, y) < \varepsilon$ (or $B_\varepsilon(x) \cap R \neq \emptyset$). We say that e is a **bound** for R, if it is incident with R and at least one other region. We denote the number of bounds for a given region R by $b(R)$ and call it its **boundary degree** while $B(R) = \{e \in E(G) \mid e \text{ is a bound for } R\}$ is called the **boundary** of R.

Example 5.6. Consider G given by its planar representation



Then e_6 is a bound of R_2, as it is incident with R_1 and R_2, but e_7 is not a bound of R_2, as it is only incident with R_2. Further, $B(R_2) = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ and $b(R_2) = 6$. Also, $B(R_3) = \{e_1, e_2, e_3\}$ and $B(R_1) = \{e_4, e_5, e_6\}$.

Remark 5.7.

- A very proper treatment needs a good fusion of real analysis with topology and exceeds our time frame (nice thesis topic).
- Any edge is incident with either 1 or 2 regions, whence

$$\sum b(R) = \sum |B(R)| = 2|\{e \mid e \text{ is a bound for some } R\}| \leq 2|E|.$$

- Any region either has no bounds or at least three.
- A region has no bound iff it is the only region and G is a forest.

5.8 Fact

Assume Γ is the planar representation of a graph G with regions $R_1, R_2, \ldots, R_r$. Then

1) Either G is a forest and $r = 1$ and $b(R_1) = 0$ or
2) G contains a cycle and $r \geq 2$ with

$$3r \leq \sum_{i=1}^{r} b(R_i) \leq 2|E|.$$

5.9 Observation

Let e be the edge of a planar graph G. Then e is a bound for some region R iff e is part of a cycle in G.

We now will prove that the relation discovered in the intermezzo holds for any planar representation.

Theorem 5.10 Euler's Formula

Let Γ be any planar representation of a connected graph G. Then for $|G| = n, \|G\| = m$ and r the number of regions in Γ, we have

$$\boxed{n - m + r = 2.}$$

Theorem 5.11 Corollaries

1) As consequently $r = \|G\| - |G| + 2$, we get that for a planar graph G, the number of regions is independent from the chosen planar representation. We can hence say that G has r-many regions.
2) If G is a planar graph on k-many connected components $C_1 \dots C_k$ then each C_i is planar. If C_i has r_i-many regions, note that G has $\sum_{i=1}^{k} r_i - (k-1)$ many regions as all components share the common exterior region in a joint embedding. Thus the number of regions r of G is

$$r = (\sum r_i) - (k-1) = \sum(\|C_i\| - |C_i| + 2) - (k-1)$$

$$= \|G\| - |G| + 2k + 1 = \|G\| - |G| + k + 1.$$

Hence, for arbitrary planar G with $|G| = n, \|G\| = m$ and $\rho(G) = r$, we get

$$\boxed{n - m + r = k + 1}$$

Proof of Theorem 5.11. Let G be a connected planar graph in any given planar representation with regions $R_1, R_2, \dots, R_r$. Let $|G| = n$. We prove the theorem by induction on $\|G\| = m$.

$m = 0$: If G has no edges, then $G = K_1$. Then clearly $n = 1, r = 1$ and thus $n - m + r = 1 - 0 + 1 = 2$, as desired.

$m - 1 \to m$: Assume we established the claim for all graphs with less than m many edges. Consider G with m many edges. If G is a tree, then we know that $n = m + 1$ and $r = 1$ (whence $n - m + r = m + 1 - m + 1 = 2$, as desired). Otherwise, G contains at least one cycle. Let $e \in E(G)$ be one edge on that cycle. Then by Observation 5.10, e is a bound for two regions R_i and R_j. Note that in $G - e$ the regions R_i and R_j merge together to one new region R', whence $G - e$ has one less region than G. Thus, by I.H. we get

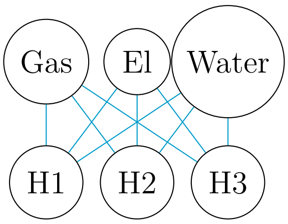
$$|G - e| - \|G - e\| + (r - 1) = n - (m - 1) + (r - 1) = n - m + r, \text{ as desired.} \qquad \square$$

Remark 5.12.

1) Every subgraph of a planar graph is planar.
2) If G is planar and R is any region, then in the induced planar representation for $B(R)$ (i.e. deleting all drawings apart from $B(R)$), R is still a region and $B(R)$ its boundary.
3) For any region we have $b(R) \in \{0\} \cup \mathbb{N}_{\geq 3}$. In particular, if $b(R) = 3$, then $B(R) \cong C_3$.

2. Nonplanar Graphs

Imagine you want to connect three houses each to gas, electricity and water, s.t. the pipes don't intersect. This question boils down to asking: Is the utility graph $K_{3,3}$ a planar graph?



(This is why $K_{3,3}$ is also called the **utility graph**.)

Theorem 5.13

The utility graph $K_{3,3}$ is not planar.

Proof. Aiming for a contradiction, assume $K_{3,3}$ was planar. Then it needed to have $r = \|K_{3,3}\| - |K_{3,3}| + 2 = 9 - 6 + 2 = 5$ regions. On the other hand, as $K_{3,3}$ is bipartite and thus does not contain odd cycles, every region has at least four bounds by Remark 5.13(3). Hence

$$\sum_{i=1}^{5} b(R_i) \geq 5 \cdot 4 = 20 > 2 \cdot 9 = 2 \cdot \|K_{3,3}\|,$$

contradicting Fact 5.9. Thus, $K_{3,3}$ cannot be planar. □

Theorem 5.14

Let G be planar of order at least 3. Then $\|G\| \leq 3(|G| - 2)$. Further, if equality holds then every region is bounded by exactly three edges.

Proof. Assume G consists of k connected components. The equation clearly holds for forests as then $\|G\| = |G| - k \leq 3|G| - 6$ iff $6 - k \leq 2|G|$. Otherwise, $r \geq 2$ and G contains a cycle. Recall that

$$(*) 3 \cdot r \leq \sum b(R_i) \leq 2\|G\|.$$

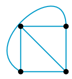
Hence, by Euler's generalised formula, we get $r = \|G\| - |G| + k + 1 \geq \|G\| - |G| + 2$. This yields

$$2\|G\| \geq 3r \geq 3(\|G\| - |G| + 2), \text{ whence } 3(|G| - 2) \geq \|G\|, \text{ as desired.}$$

Finally, for equality to hold we need in particular that $2\|G\| = 3r$, whence also $3r = \sum b(R_i)$. Thus, every region has boundary degree 3 as desired. $\square$

Corollary 5.15

The complete graph K_n is planar iff $n \leq 4$.

Proof. First note that $K_4 =$  is planar, and hence so are its subgraphs K_1, K_2 and K_3. Now, for K_5, by Lemma 5.15 we get $10 = \|K_5\| \leq 3(\|K_5\| - 2) = 3(3) = 9$, a contradiction. Hence, K_5 is not planar and so is neither of the K_n for $n \geq 5$, as they contain K_5 as a subgraph. $\square$

Theorem 5.16

If G is planar, then $\delta(G) \leq 5$.

Proof. Let G be a planar graph and set $n = |G|$ and $m = \|G\|$. Aiming for a contradiction, assume $\delta(G) > 5$. Then in particular, $n \geq 6$. Then

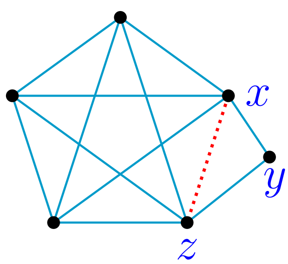
$$6 \cdot n \leq \sum_{v \in V_G} \deg(v) = 2m \overset{(5.15)}{\leq} 2(3(n-2)) = 6n - 12$$

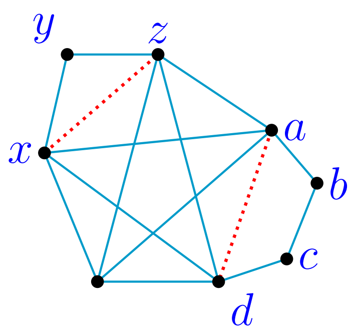
yields a contradiction. $\square$

3. Kuratowski's Theorem

We have seen that the graphs $K_{3,3}$ and K_5 are not planar. It turns out that these two graphs are the major obstruction for any graph to be planar. This section gives an introduction to Kuratowski's theorem. But first, we want to introduce a new notation.

Example 5.17. Consider

$G =$ 

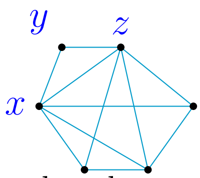
$H =$ 

Note that neither of G or H contains K_5 (or $K_{3,3}$) as a subgraph. Nevertheless, they both look so similar to K_5 that they should not be planar. Indeed, G arises from K_5 by replacing the edge xz by $\{xy, yz\}$ and H arises from G by replacing the edges $\{xz, ad\}$ by $\{xy, yz, ab, bc, cd\}$. This process is called subdivision.

Definition 5.18

1) Let G be a graph and $e = xy \in E(G)$. An **edge subdivision** of e is the replacement of e by a finite path of length ≥ 2 starting in x and ending in y.
2) Let G and H be graphs. Then H is called a **subdivision** of G iff H can be obtained through a finite sequence of edge subdivisions.

Example 5.19.

1) The graphs G, H from 5.18 are edge subdivisions of K_5.
2) The graph 

is **not** an edge subdivision of K_5, as the edges $\{xy, yz\}$ were added rather than replacing the edge xz.

Lemma 5.20

A graph G is planar iff every subdivision of G is planar.

Proof. "$\Rightarrow$" Consider a planar representation Γ of G. Let H be a subdivision of G where a sequence of n-many edge subdivisions were performed. We do induction on n.

If $n = 0$, then $H = G$ is clearly planar.

Now assume we proved the claim for n and consider H arising from G through $n+1$ many subdivisions. Let H_0 be the graph arising from G through the first n-many subdivisions. By I.H. H_0 is planar and H can be obtained from H_0 through exactly one edge-subdivision. Let Γ_0 be a planar drawing of H_0 and $e = xy$ be the edge which is subdivided to obtain H, say by replacing it by the path $P = (x = x_0, x_1, \ldots, x_k = y)$ for $k \geq 2$. Let $\vec{xy}$ be the geodesic from x to y in $\mathbb{R}^2$. Then we draw in the vertex x_i at the point $x + \frac{i}{k}\vec{xy}$ for any i. This yields a planar representation of H, as desired. $\square$

Corollary 5.21

If G contains a subdivision of $K_{3,3}$ or K_5 as a subgraph, then G is not planar. (Or: If G planar $\Rightarrow$ no subdivision of $K_5, K_{3,3} \subseteq G$)

Theorem 5.22 Kuratowski's Theorem

A graph is planar if and only if it does not contain a subdivision of $K_{3,3}$ or K_5 as a subgraph.

Remark 5.23. The left over, hard direction is the backwards direction, i.e. if G does not contain a subdivision of $K_{3,3}$ or K_5, then G is planar.

Chapter 6

Graph Colourings

Disclaimer

The title is actually somewhat misleading, as our first convention will be that we can understand different colours simply as different positive integers. Hence we will use the "colours" $1, 2, 3, \ldots$ instead of red, blue, brown etc. Visualisations can nevertheless reassociate these numbers with colours.

1. The Chromatic Number

Definition 6.1

Let G be a graph. A **k-colouring** is a function $K : V_G \to \{1, 2, \ldots, k\}$ for $k \in \mathbb{Z}_+$ s.t. $K(x) = K(y)$ implies $xy \notin E_G$, i.e. adjacent vertices receive distinct colours. We call $\{1, 2, \ldots, k\}$ the **colours** of K. We say that K is a **colouring** of G if it is a k-colouring for some k and we call G **k-colourable** if there is a k-colouring for G.

Remark 6.2. Let G be any graph.

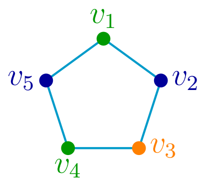
1) If G is k-colourable, then it is j-colourable for any $j \geq k$.

2) If K is a k-colouring of G, then $V_r := K^{-1}(r) = \{v \in V_G \mid K(v) = r\}$ is an independent subset of G for any $r \in \{1, 2, \ldots, k\}$.

3) Every graph of order n is clearly n-colourable as for $V_G = \{v_1, v_2, \ldots, v_n\}$ we can simply define the injective function $K : V_G \to \{1, 2, \ldots, n\}$ via $K(v_i) = i$. Any injective such function is clearly a colouring.

4) Remarks 2) and 3) indicate that the interesting question will be to find small k s.t. G is k-colourable.

Example 6.3. Consider the graph C_5. Let's try to find a minimal k s.t. G is k-colourable.



$\Rightarrow$ It seems we need at least 3-colours. C_5 is 3-colourable, as $K : V_G \to \{1,2,3\}$ via $K(v_1) = K(v_4) = 1$, $K(v_2) = K(v_5) = 2$ and $K(v_3) = 3$ is a 3-colouring for G. We also could have defined K via $V_1 = \{v_1, v_4\}$, $V_2 = \{v_2, v_5\}$ and $V_3 = \{v_3\}$.

Remark 6.4.

1) Note that every k-colouring K of G gives rise to a partition of V_G into sets $V_1, V_2, \ldots, V_k$ s.t. no two vertices in V_i are adjacent. G is hence k-partite. Note that V_i could be empty for some i.

2) As each V_i is an independent set, we get $k \geq \frac{|G|}{\alpha(G)}$ (HW).

Definition 6.5

The **chromatic number** $\chi(G)$ of a graph G is the smallest positive integer $k \in \mathbb{Z}_+$ s.t. G is k-colourable.

Example 6.6.

1) $\chi(G) = 1$ iff $G = E_n$ is the empty graph for some n.

2) $\chi(K_n) = n$, as all of the n-many vertices are pairwise adjacent.

3) $\chi(K_{n,m}) = 2$, as we can colour each part of the partition in one colour.

4) $\chi(P_n) = 2$ for $n \geq 2$.

5) $\chi(C_n) = \begin{cases} 2 & \text{if } n \text{ is even} \\ 3 & \text{otherwise} \end{cases}$ (HW).

Remark 6.7.

1) G is k-colourable iff $k \geq \chi(G)$.

2) $\frac{|G|}{\alpha(G)} \leq \chi(G) \leq |G|$ for any graph G.

3) G is 2-colourable iff G is bipartite (iff $\chi(G) \leq 2$).

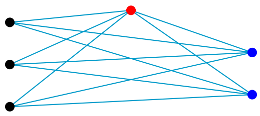
4) If F is a forest, then $\chi(F) = 2$.

5) If $H \subseteq G$ is a subgraph of G, then $\chi(H) \leq \chi(G)$.

Definition 6.8

Let $n_1, \ldots, n_k \in \mathbb{Z}_+$ be positive integers, $k \geq 1$. The **complete k-partite graph** $K_{n_1,n_2,\ldots,n_k}$ is the graph whose vertex set is the disjoint union of k-many pairwise disjoint sets $V_1, \ldots, V_k$ with $|V_i| = n_i$; and edge set $E = \{uv \mid u \in V_i, v \in V_j, i \neq j\}$. If we don't care about the specific value of k, we call $K_{n_1\ldots n_k}$ the **complete multipartite graph**.

Example 6.9. The complete 3-partite graph $K_{3,1,2}$ is



We see that even for small numbers, the graph is difficult to draw. For which $n_1 \ldots n_k$ is $K_{n_1\ldots n_k}$ planar?

Lemma 6.10

Let G be a graph and $k \geq 1$. Then G is k-colourable iff G is a subgraph of a complete k-partite graph.

Proof. G is k-colourable
iff ex. $K : V_G \to \{1, 2, \ldots, k\}$ s.t. $K(u) = K(v) \implies uv \notin E_G$
iff ex. $K : V_G \to \{1, 2, \ldots, k\}$ s.t. $K^{-1}(i)$ is an independent set (or empty) for all i
iff we can partition V_G into k-many independent sets $V_1, \ldots, V_k$ ($V_i = \emptyset$ allowed)
iff G is a subgraph of $K_{n_1\ldots n_k}$ for some $n_1 \ldots n_k \in \mathbb{Z}_+$. $\square$

Next we want to introduce a greedy algorithm for finding a vertex colouring of a given graph. This algorithm is more efficient than giving every vertex a different colour, but it does not always produce a $\chi(G)$-colouring.

6.11 The Greedy Algorithm

Let G be a graph of order n.

1) Label the vertices by $v_1, \ldots, v_n$.
2) Fix the set of available colours to be $\{1, 2, \ldots, n\}$.
3) Let $i = 1$.
4) While $i \leq n$

- Colour v_i with the smallest available colour not used on any of its previously coloured neighbours, i.e. set

$$K(v_i) = \min\{\{1, \ldots, n\} \setminus \{K(v_j) \mid j < i, v_j \in N(v_i)\}\}.$$

- Set $i \to i + 1$.

Lemma 6.12

Let G be a graph and K be a colouring function produced by the greedy algorithm applied to G. Then for all $v \in V_G$, we get

$$K(v) \leq \deg(v) + 1.$$

Proof. We do induction on $|G| = n$. For $n = 1$, clearly $G = K_1$, and $K(v_1) = 1 = \deg(v_1) + 1$. Now assume the claim holds for any graph of order at most n and consider G with $|G| = n + 1$. Run the greedy algorithm on G. After the first n rounds, we obtain a "greedy" colouring of $G - v_{n+1}$, whence $K(v_i) \leq \deg^{G-v_{n+1}}(v_i) + 1 \leq \deg^{G}(v_i) + 1$ for all $1 \leq i \leq n$. Now we run the last round and assign $K(v_{n+1}) = \min\{\{1, \ldots, n\} \setminus \{K(v_j) \mid v_j \in N(v_{n+1})\}\}$. At most $\deg(v_{n+1})$-many colours have been used on its neighbours, whence among $\{1, 2, \ldots, \deg(v_{n+1}), \deg(v_{n+1}) + 1\}$, there is at least one colour available. As the algorithm picks the smallest available colour, we conclude $K(v_{n+1}) \leq \deg(v_{n+1}) + 1$, as desired. $\square$

Corollary 6.13

For any graph G we have $\chi(G) \leq \Delta(G) + 1$.

Proof. Apply the greedy algorithm on G. Then for any $v \in V_G$, we get $K(v) \leq \deg(v) + 1 \leq \Delta(G) + 1$, whence $K : V_G \to \{1, 2, \ldots, \Delta(G) + 1\}$ and $\chi(G) \leq \Delta(G) + 1$. $\square$

Finding the chromatic number of a graph is a very important, but computationally hard problem. It is known to be NP-hard and its best runtime is in $\mathcal{O}(2^n)$. We hence need to establish good bounds to approach the problem effectively.

Remark 6.14.

- For any graph G we have $\chi(G) \leq \Delta(G) + 1$.
- The bound is sharp, as
  - $\chi(K_n) = n = (n - 1) + 1 = \Delta(K_n) + 1$ for all $n \in \mathbb{Z}_+$ and
  - $\chi(C_n) = 3 = 2 + 1 = \Delta(C_n) + 1$ for all odd $n \in \mathbb{Z}_+$.
- But are there more examples to witness that the bound is sharp (i.e. cannot be improved)? **No.**
- Note: What do C_n and K_n have in common? They are regular!

Goal of this lecture:

Theorem 6.15 Brooks' Theorem, 1941

If G is a connected graph which is neither complete, nor an odd cycle, then $\chi(G) \leq \Delta(G)$.

We will split the proof into several Lemmata. Set $\Delta(G) =: \Delta$.

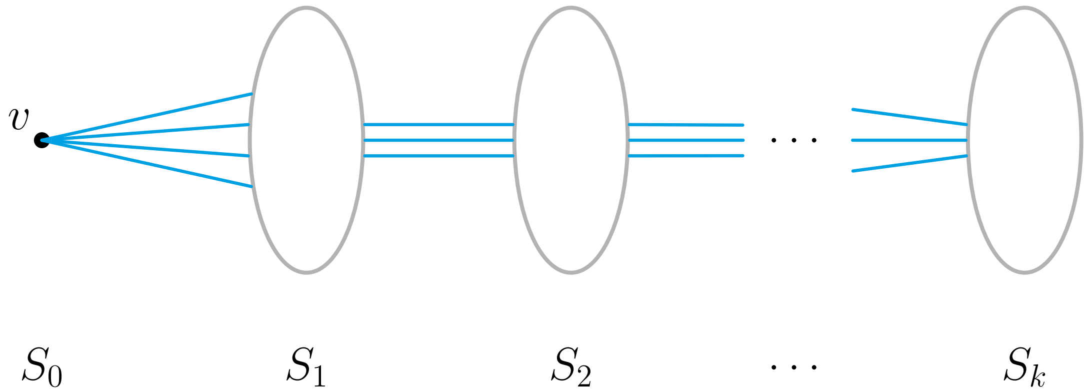
Lemma 6.16 Step 1 - Remark on $\Delta \leq 2$

Theorem 6.15 holds for $\Delta \leq 2$, as if $\Delta = 0$, then $G = K_1$, and if $\Delta = 1$, then $G = K_2$ is complete, which is excluded. For $\Delta = 2$, either $\delta(G) = \Delta = 2$ and G is an even cycle, whence $\chi(G) = 2 \leq \Delta(G)$ holds, or $\delta(G) = 1$ and G is a path of length at least 2, and again $\chi(G) = 2 \leq \Delta(G)$. Thus, from now on, consider such G which satisfy $\Delta(G) \geq 3$.

Lemma 6.17 Step 2 - Lemma on non-regular graphs

If G from 6.15 is not regular, then $\chi(G) \leq \Delta(G)$ (i.e. the theorem holds).

Proof. If G is non-regular, then ex. $v \in V_G$ s.t. $\deg(v) < \Delta(G)$. The idea is to introduce a smart colouring which uses at most $\Delta(G)$ many colours. Note that as G is (finite and) connected, $ecc(v) = k < \infty$. Let $S_i := \{u \in V_G \mid d(u,v) = i\}$. Hence, $S_0 = \{v\}$ and $S_k = \{u \in V_G \mid d(u,v) = ecc(v)\}$. Note further that for $1 \leq i \leq k$, every vertex $u \in S_i$ has at least one neighbour in S_{i-1} (i.e. the predecessor of u on a vu-path of shortest length).

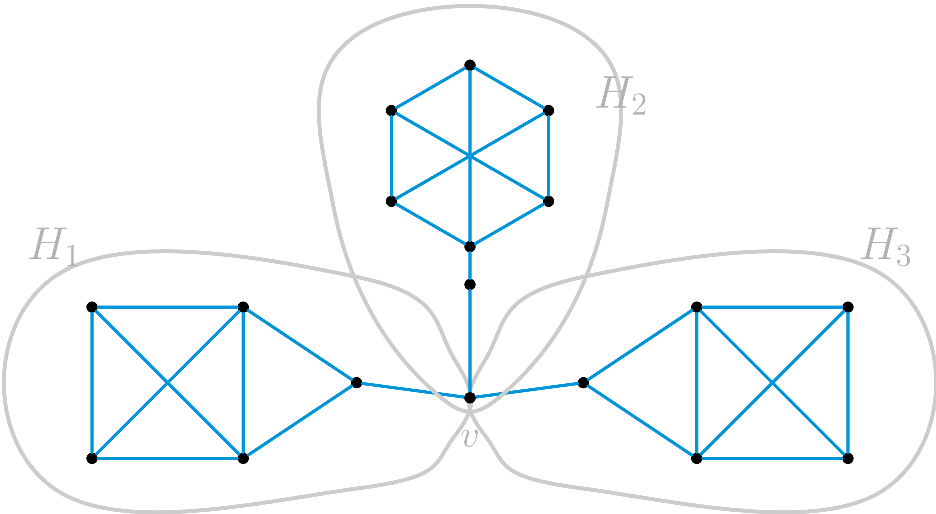


We want to apply the greedy algorithm after labelling the vertices of G in a smart way: We start by randomly labelling the vertices in S_k and <u>then</u> those in S_{k-1} and so on and so forth until the last vertex v gets labelled v_n where $n = |G|$. Hence, if $v_i \in S_\ell$ and $v_j \in S_m$ for $\ell < m$, then $i > j$. Now we run the greedy algorithm and observe which colours the vertices get. For $v_i \in S_\ell$ with $\ell \neq 0$ (i.e. $v_i \neq v_n = v$), note that v_i gets the smallest available colour not used on its <u>previously coloured</u> neighbours, i.e. on its neighbours in $S_\ell \cup S_{\ell+1} \cup \cdots \cup S_k$. As v_i has at least one neighbour in $S_{\ell-1}$, these are at most $\deg(v_i) - 1 \leq \Delta(G) - 1$ many. Hence, at least one of the colours $\{1, 2, \ldots, \Delta(G)\}$ is still available, whence $K(v_i) \leq \Delta(G)$ as desired. For $v_n = v$, it gets the smallest available colour not used on <u>any</u> of its neighbours, as it is the last to get coloured. But as by choice, $\deg(v_n) < \Delta(G)$, again one of the colours $\{1, 2, \ldots, \Delta(G)\}$ must be available and $K(v_n) \leq \Delta(G)$. Hence, $K(u) \leq \Delta$ for all $u \in V_G$, whence G is Δ-colourable and $\chi(G) \leq \Delta$, as desired. □

Lemma 6.18 Step 3 - Regular graphs with cut vertices

Let G be a connected, regular graph with $\Delta(G) \geq 3$. If G contains a cut vertex (i.e. $\kappa(G) = 1$), then $\chi(G) \leq \Delta(G)$.

Example 6.19. You might wonder if these graphs exist. Here is an example:



All H_i are non-regular.

Proof. of 6.18:
Let v be a cut vertex in G, i.e. $G - v$ is disconnected. Let $H_1, H_2, \ldots, H_k$ be the distinct connected components of $G - v$ with $k \geq 2$. Further, define $G_i := \langle V(H_i) \cup \{v\} \rangle$ to be the graph induced on H_i with v. Note that $\deg^{G_i}(v) < \deg^G(v) = \Delta$, while still $\deg(u) = \Delta$ for all other $u \in V(G_i)$. Hence $\Delta(G_i) = \Delta$. By Step 2, we can find Δ-colouring of G_i. Possibly after permuting the colours, we may assume that $K_i(v) = K_j(v)$ for all $1 \leq i, j \leq k$. Hence $K = K_1 \cup \cdots \cup K_k$ is the desired Δ-colouring of G. □ □

Lemma 6.20 Step 4

Suppose that G is regular and 2-connected with $\Delta(G) \geq 3$. Then there exist three vertices v, v_1, v_2 such that v is adjacent to both v_1, v_2 where v_1, v_2 are nonadjacent, and $G - \{v_1, v_2\}$ is connected.

Proof. As G is regular, every vertex has degree Δ. To show this case we will divide this case into two subcases.

Case a. Suppose that G is regular and 3-connected.
Since G is regular and not complete, it follows that $\Delta < n-1$. Let v_1 be any vertex of G and let A be the set of all vertices which are nonadjacent to v_1. As $\deg(v_1) = \Delta < n-1$ there must be a vertex nonadjacent to v_1, and so $A \neq \emptyset$. Suppose for the moment that no neighbour of v_1 is adjacent to some vertex in A. But then there will be no path in G from v_1 to any vertex in A contradicting that G is connected. Therefore, there exists some neighbour v of v_1 which is adjacent to some vertex $v_2 \in A$. As $\kappa(G) \geq 3$, we know that $G - \{v_1, v_2\}$ is connected. This completes Case a.

Case b. Suppose that G is regular and $\kappa(G) = 2$. (This means G is 2-connected but not 3-connected.)
This means that there are two vertices v, w which form a cut set. So $G - \{v, w\}$ is disconnected. Let $G_1, G_2, \dots, G_t$ where $t \geq 2$ be the connected components of $G - \{v, w\}$. Since $\Delta \geq 3$, each G_i must contain at least 2 vertices. We will establish two facts.

Claim 1. The vertex v has at least one neighbour in each G_i.
To see this, observe that w is not a cut vertex of G since $\kappa(G) = 2$. So $G - w$ is connected. Suppose that v is not adjacent to any vertex in G_i for some i. Let x be a vertex in G_i, as $G - w$ is connected, there is vx-path P in $G - w$. Since v is nonadjacent to all vertices in G_i, the successor of v in P must be some vertex z from G_j where $j \neq i$. But then $P[z, x]$ is a path which does not use either v or w. This shows that G_i and G_j are connected in $G - \{v, w\}$, a contradiction. Thus, the vertex v must be adjacent to at least one vertex in each G_i as required.

Claim 2. The vertex v has a neighbour v_1 in G_1 that is not a cut vertex of $G - v$.
To see this, by Claim 1 the vertex v has some neighbour in G_1. For the contrary, suppose that all neighbours of v in G_1 are cut vertices of $G - v$. Among all such neighbours choose a neighbour u of maximum distance $d(u, w)$. Let P be a shortest uw-path, so P has length $d(u, w)$ (such path is called geodesic), say

$$P = (u = u_0, u_1, u_2, u_3, \dots, u_{k-1}, u_k = w).$$

In fact, it is possible that $u_1 = w$ meaning that u, w are adjacent. Note that $u_0, u_1, \dots, u_{k-1}$ are all in G_1 because G_1 is a connected component of $G - \{v, w\}$. As $\deg_G(u) = \Delta \geq 3$, it must be that the degree of u in $G - v$ is at least 2, meaning that u has neighbours other than u_1. Moreover, since u is a cut vertex of $G - v$ there must exist a neighbour y of u such that $y \neq u_1$ and every path in $G - v$ from y to u_1 passes through u. This implies that

$$d(y, w) = d(y, u) + d(u, w) = 1 + d(u, w).$$

Since $\kappa(G) = 2$, we have that $G - u$ is connected, and so it must be that y is a neighbour of v. But $d(y, w) > d(u, w)$ which contradicts the choice of the vertex u. Thus, there exists a neighbour of v in G_1 which is not a cut vertex of $G - v$.

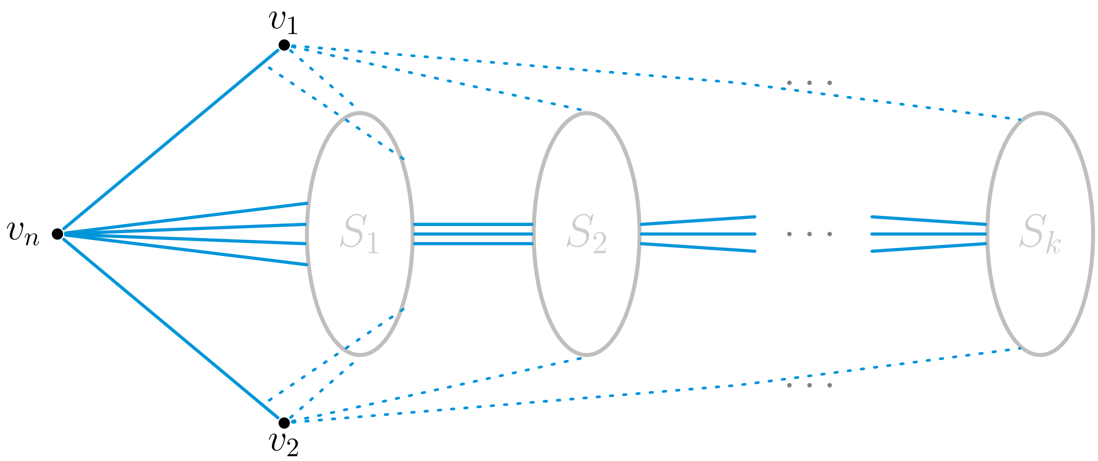
By Claim 2, the vertex v has a neighbour v_1 in G_1 that is not a cut vertex of $G - v$. By a similar argument, v also has a neighbour v_2 in G_2 that is not a cut vertex of $G - v$ as well. As they lie in different connected components of the graph $G - \{v, w\}$, vertices v_1 and v_2 are nonadjacent, and moreover, it must be that the graph $G - \{v_1, v_2\}$ is connected. This finishes Case b.

Therefore, in either subcase of Case 3 we identified three vertices v, v_1, v_2 in G where v is adjacent to both v_1, v_2, but v_1, v_2 are nonadjacent, and also $G - \{v_1, v_2\}$ is connected. $\square$

Lemma 6.21 Step 5 - Left overs

Let G be regular, 2-connected, non-complete with $\Delta(G) \geq 3$. Then $\chi(G) \leq \Delta(G)$.

Proof. By Fact 6.20, there exist vertices v_n, v_1, v_2 s.t. $v_1, v_2 \in N(v_n)$, $v_1 \notin N(v_2)$ and $G - \{v_1, v_2\}$ is connected. Let $H := G - \{v_1, v_2\}$ and as in step 1, partition V_H via $S_i = \{u \in H \mid d(v_n, u) = i\}$. Let $|G| = n$ and label all vertices as in Step 1, except that the labels of v_1, v_2 and v_n stay. So if $ecc^H(v_n) = k$, we start labeling the vertices in S_k with $v_3, v_4 \dots$ and so on, followed by the vertices in S_{k-1}, until we reach $S_0 = \{v_n\}$, which is already labeled. Now we get the picture:



We are ready to run the greedy algorithm. First, we get $K(v_1) = 1$ and then also $K(v_2) = 1$, as they are not adjacent. Now, any v_i with $3 \leq i < n$, there is at least one neighbour which is not coloured in step i. Precisely, if $v_i \in S_\ell$, then there is a neighbour $v_j \in S_{\ell-1}$ with $i < j$ which is not coloured. As $\deg(v_i) = \Delta$, at least one of the colours $\{1, 2, \dots, \Delta\}$ is still available, whence $K(v_i) \leq \Delta$. Finally, $\deg(v_n) = \Delta$ and all its neighbours have been coloured, but two of its neighbours, v_1 and v_2, have the same colour, whence once again, at least one of the colours $\{1, 2, \dots, \Delta\}$ is still available and also $K(v_n) \leq \Delta$. We hence proved that $\chi(G) \leq \Delta$, as desired. $\square$

This concludes the proof of Brooks' Theorem.

Corollary 6.22

Let G be a graph with connected components $H_1, H_2, \dots, H_k$.

1) Then $\Delta(G) = \max\{\Delta(H_i) \mid 1 \leq i \leq k\}$ and $\chi(G) = \max\{\chi(H_i) \mid 1 \leq i \leq k\}$.
2) $\chi(G) = \Delta(G) + 1$ if and only if ex. i s.t. H_i is either a complete graph or an odd cycle and $\Delta(H_i) = \Delta(G)$.

6.23 Homework

Let G be a graph of order n, then

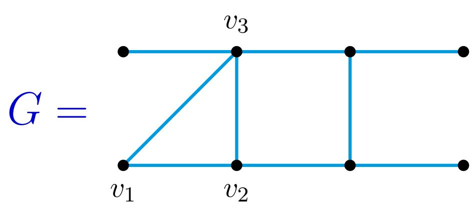
$$\frac{n}{\alpha(G)} \leq \chi(G) \leq n + 1 - \alpha(G).$$

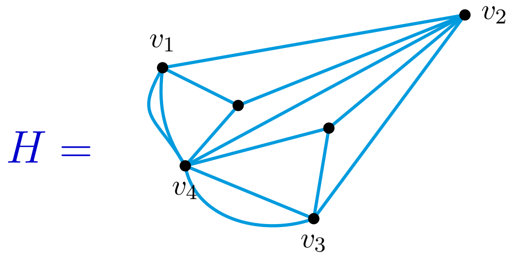
The rest of the lecture is devoted to give a last bound for the chromatic number of a graph.

Definition 6.24

The **clique number** of a graph G, denoted by $\omega(G)$, is the largest positive integer m s.t. G contains K_m as a subgraph.

Example 6.25.





Then $\omega(G) = 3$ witnessed by $\langle\{v_1, v_2, v_3\}\rangle \cong K_3$ and $\omega(H) = 4$ witnessed by $\langle\{v_1, v_2, v_3, v_4\}\rangle \cong K_4$, and H being planar.

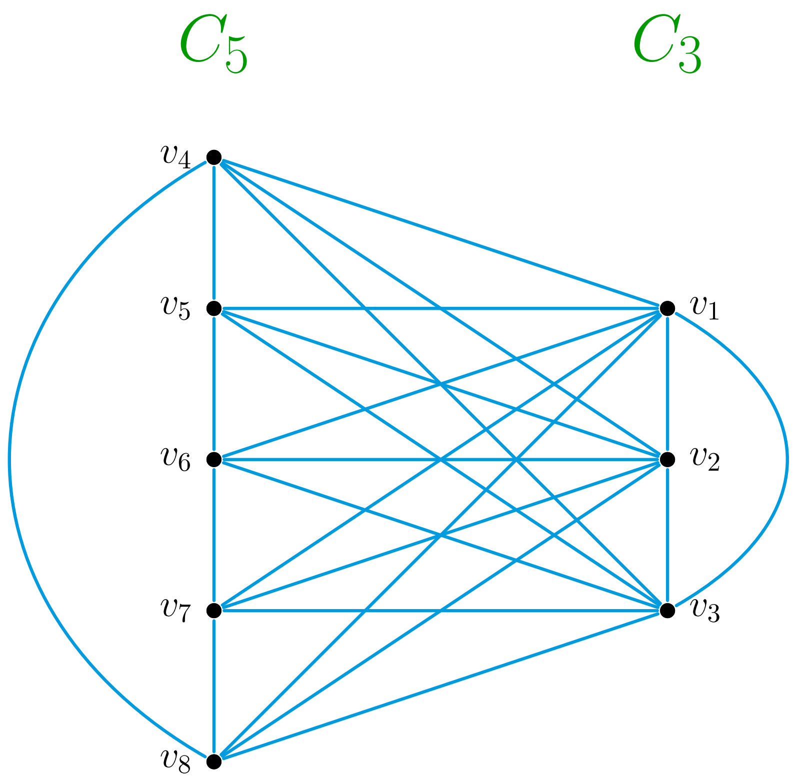
Lemma 6.26

For any graph G we get $\omega(G) \leq \chi(G)$.

$\rightarrow$ Let $\omega(G) = \ell$, then G contains K_ℓ as a subgraph which implies that we already need ℓ colours just to colour K_ℓ and $\ell = \omega(G) \leq \chi(G)$.

The question arises whether $\omega(G)$-many colours should always be enough to colour a graph G. This hope rather quickly fails, as for example $\omega(C_5) = 2$, but $\chi(C_5) = 3 > \omega(G)$. Another example is given below.

Example 6.27. Let G be the graph consisting of the disjoint union of C_5 and C_3 together with all edges between C_3 and C_5, i.e.



Then $\omega(G) = 5$, as e.g. $\langle\{v_1, v_2, v_3, v_4, v_5\}\rangle \cong K_5$ (so $\omega(G) \geq 5$), but any 6 vertices need to include at least three vertices from C_5, which cannot be mutually incident, so $\omega(G) < 6$, and thus $\omega(G) = 5$. Further, $\chi(G) = 6$: The cycle C_5 and C_3 each need at least 3 colours, and these colours have to be distinct as the vertices are mutually incident.

Let's summarize all bounds we have established.

6.28 Summary

Let G be any graph of order n.

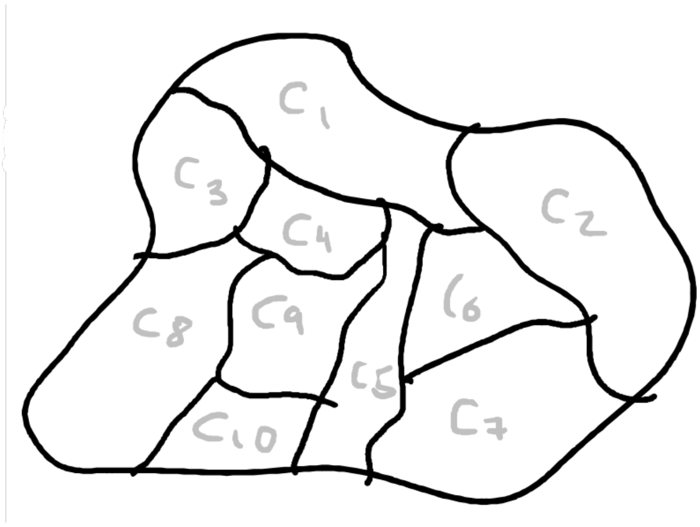
1) $\chi(G) \leq \Delta(G) + 1 \leq n$.
2) If G is connected then $\chi(G) \leq \Delta(G)$ iff G is neither complete nor an odd cycle.
3) $\omega(G) \leq \chi(G)$ and $\frac{n}{\alpha(G)} \leq \chi(G) \leq n + 1 - \alpha(G)$.

2. The 4-Colour Problem

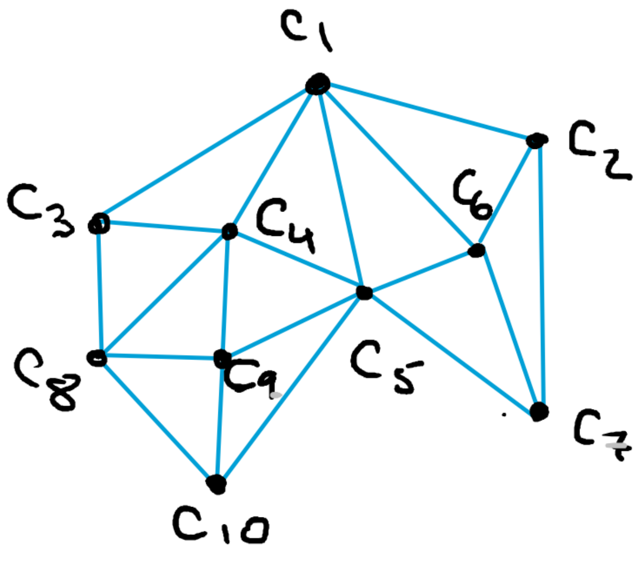
6.29 The Problem (Francis Guthrie, 1852)

Given any map in the plane - how many colours do we need to colour it in a way s.t. no countries who share part of their border (more than a point) have the same colour? We can rephrase this in graph theoretic terms. Note that we can represent the issue as a graph where the vertices represent countries and edges connect countries with touching borders. Such a graph will be planar.

Example 6.30. The map

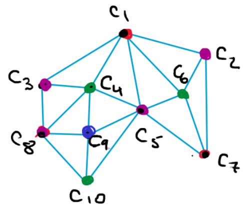


can be represented by the planar graph:



Now, a colouring of the map exactly corresponds to a vertex colouring of the associated graph, e.g.

yields



For this graph, we obtained a 4-colouring. We know that $\omega(G) \leq \chi(G)$ for any graph and $\omega(G) \leq 4$ for any planar graph. But are 4 colours always enough?

Theorem 6.31 The Four-Colour-Theorem (1976, Appel, Haken)

If G is planar, then $\chi(G) \leq 4$. I.e. every planar graph is 4-colourable.

6.32 Some History

- 1852 - Problem introduced by Francis Guthrie to De Morgan (his prof).
- 1852-1879 Brilliant minds, incl. Hamilton, Cayley, Peirce... tried to solve the problem without success.
- 1879 - Alfred Kempe announced a proof.
- 1890 - Fatal mistake was found in proof. Soon after - Heawood + Kempe prove that $\chi(G) \leq 5$.
- 1976 - After 124 years a proof was presented by Appel-Haken. $\rightarrow$ relies heavily on computers, not accepted by all mathematicians.
- 1996 - Easier proof presented, but still uses computers.
- $\rightarrow$ The search continues.

Maybe somewhat surprisingly, the proof for $\chi(G) \leq 5$ is much more accessible. Recall that if G is planar, we have $\delta(G) \leq 5$ (Thm 5.17).

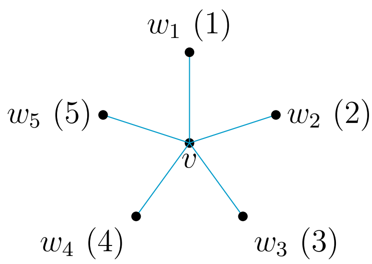
Theorem 6.33

If G is planar, then $\chi(G) \leq 5$, i.e. G is 5-colourable.

Proof. Let $|G| = n$. We proceed by induction on n.

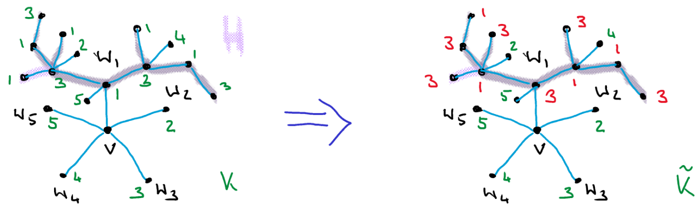
If $n \leq 5$, the theorem clearly holds as $\chi(G) \leq |G|$ for any G.

$\underline{n \rightarrow n+1}$: Assume any planar graph of order at most n is 5-colourable and consider G s.t. $|G| = n + 1$. As $\delta(G) \leq 5$, there exists $v \in V_G$ s.t. $\deg(v) \leq 5$. Consider the planar graph $G - v$. By I.H., $G - v$ is 5-colourable. Let K be that 5-colouring. Now, if among the neighbours of v at least one of $\{1, 2, 3, 4, 5\}$ was not used, we can use that colour for v and obtain the desired 5-colouring of G. Otherwise, $\deg(v) = 5$ and all neighbours $\{w_1, \ldots, w_5\}$ of v are coloured in a different colour, e.g. $K(w_i) = i$. We obtain:



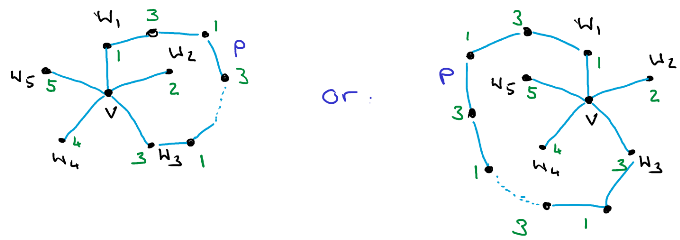
as a subgraph of G. Now, we make a case distinction.

Case 1: Assume there is no w_1, w_3-path that entirely uses vertices coloured with colours 1 and 3. Then let H be the subgraph of G containing all paths that start in w_1 and use only vertices of colour 1 and 3, i.e. $H = \bigcup\{P \mid P \text{ is a } w_1 u \text{ path } \& \ K(t) \in \{1,3\} \forall t \in P\}$. Note that $w_1 \in H$, but $w_3 \notin H$ by assumption.



Now, in H, exchange the colours 1 and 3 and observe that that new colouring $\tilde{K}$ is still a valid colouring for G, but now $\tilde{K}(w_1) = 3$. To see that it is valid, assume for contradiction that ex. x, y s.t. $\tilde{K}(x) = \tilde{K}(y)$ and $xy \in E_G$. Then wlog we may assume $x \in H, y \notin H$ and $\tilde{K}(x) = \tilde{K}(y) = 1$. But then $K(x) = 3, K(y) = 1$ and there is a $\{1,3\}$-coloured w_1, x-path P. Finally, $P^\frown(y)$ would be a $\{1,3\}$ coloured w_1, y-path, whence $y \in H$, contradicting our assumptions. Hence, $\tilde{K}$ is a 5-colouring of $G - v$, but now the neighbours of v only use colours $\{2,3,4,5\}$, whence we can set $\tilde{K}(v) = 1$ and obtain the desired 5-colouring of G.

Case 2: Assume there is a $\{1,3\}$-coloured $w_1 w_3$-path P in $G - v$. As G is planar, we have two options:



In either case, any $w_2 w_4$-path now would have to contain at least one vertex from P, as G is planar. But this means that there is no $w_2 w_4$-path which only uses colours 2 and 4. We can hence apply Case 1 to w_2 and w_4 and obtain the desired 5-colouring of G. $\square$

3. Chromatic Polynomials

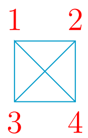
History

- Introduced 1912 by Georg David Birkhoff to tackle the 4-colour problem (he hoped for a negative answer).
- $P_G(k)$ shall describe the number of k-colourings of G.
- These will be polynomials in k of degree $|G| = n$, i.e. $P_G(k) = a_n k^n + \cdots + a_1 k + a_0$.
- Birkhoff hoped to use strong tools from analysis and algebra to find roots of these polynomials.
- In particular, he hoped to find a planar graph G which has $k = 4$ as a root, i.e. $P_G(4) = 0$, whence G has no 4-colouring, whence the 4-colour theorem would be wrong.
- Even though we know that he had no chance of success, he still developed many tools which are crucial in the area of algebraic graph theory.
- $\Rightarrow$ Sometimes truly the way is the goal.

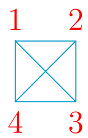
Definition 6.34

1) Let K_1 and K_2 be two colourings of the same graph G. We say that K_1 is **different** from K_2 ($K_1 \neq K_2$) if there is some $v \in V_G$ s.t. $K_1(v) \neq K_2(v)$.
2) We denote by $P_G(k)$ the number of different k-colourings of G.

Example 6.35. Consider the following colourings of K_4:



(K)



(K')

Then K and K' are different, as $K(v_3) = 3 \neq 4 = K'(v_3)$.

6.36 Discussion

- How many 4-colourings of K_4 are there? $\rightarrow$ We can choose any of the 4 colours to colour v_1, then any of the remaining 3 for v_2 and so on. In the end we obtain $4 \cdot 3 \cdot 2 \cdot 1 = 4!$ many 4-colourings.
- What about 6-colourings? $\rightarrow$ Following the same thoughts, we obtain $6 \cdot 5 \cdot 4 \cdot 3 = \frac{6!}{(6-4)!}$ many 6-colourings.
- And 3-colourings? $\rightarrow$ As $\chi(K_4) = 4$, there are no 3-colourings.

> **Definition 6.37**
>
> Let G be any graph. Then we denote by $P_G(k)$ the number of possible different colourings using at most the colours $\{1, 2, \dots, k\}$.

Remark 6.38.

- Generalising our discussion above, we obtain

$$P_{K_n}(k) = \begin{cases} 0 & \text{if } k < n \\ \frac{k!}{(k-n)!} & \text{if } n \leq k \end{cases}.$$

- Further check quickly that $P_{E_n}(k) = k^n$.

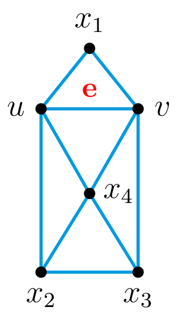
Remark 6.39. The following are equivalent for any graph G and $k \in \mathbb{Z}_+$:

1) $P_G(k) \geq 1$
2) $\chi(G) \leq k$
3) G is k-colourable.

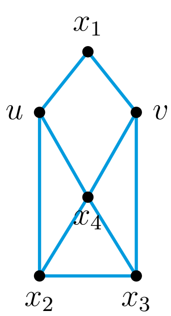
In order to show that $P_G(k)$ is a polynomial of degree k, we need an important observation which will allow us to use inductive arguments. To make sense of it, we need the following definition.

> **Definition 6.40 Edge Contraction**
>
> Let G be a graph and $e \in E(G)$, say $e = uv$. Then the **edge contraction** G/e is the graph obtained from G by the following:
> 1) Delete the edge e from G, i.e. construct $G - e$.
> 2) Identify the vertices u and v as one new vertex denoted $u \wedge v$.
> 3) Leaving only one copy of any resulting multi-edges.

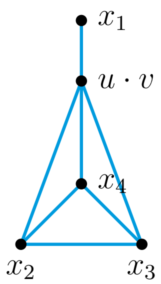
Example 6.41. Consider G and $e \in E(G)$ as follows:



Then $G - e$ is



and G/e is



Theorem 6.42 Birkhoff, Lewis, 1946

Let G be any non-empty graph and $e = uv \in E(G)$. Then

$$P_G(k) = P_{G-e}(k) - P_{G/e}(k).$$

Proof. Claim 1: $P_{G/e}(k)$ is equal to the number of k-colourings of $G - e$ which assign the same colour to u and v. $\to$ Let K be a k-colouring of $P_{G/e}$. Then we can create a k-colouring $\tilde{K}$ of $G - e$ via

$$\tilde{K}(x) = \begin{cases} K(u \wedge v) & \text{if } x = u \text{ or } x = v \\ K(x) & \text{otherwise.} \end{cases}$$

Hence, there are at least as many k-colourings of $G - e$ which assign the same colour to u and v as there are k-colourings of G/e. On the other hand, if $\tilde{K}$ is any k-colouring of $G - e$ which assigns the same colour to u and v, then we can define a new k-colouring K of G/e by setting

$$K(x) = \begin{cases} \tilde{K}(u) & \text{if } x = u \wedge v \\ \tilde{K}(x) & \text{else.} \end{cases}$$

Hence, there are at least as many k-colourings of G/e as there are k-colourings of $G - e$ assigning the same colour to u and v.

Claim 2: There are as many k-colourings of G as there are k-colourings of $G - e$ assigning different colours to u and v. $\to$ If $\tilde{K}$ is a k-colouring of G, then clearly it is a k-colouring of $G - e$ assigning different colours to u and v. And vice versa.

Now,

$$\begin{aligned} P_{G-e}(k) &= |\{K \mid K \text{ is a } k\text{-colouring of } G - e\}| \\ &= |\{K \mid K \ k\text{-colouring of } G - e \text{ with } K(u) = K(v)\}| \\ &+ |\{K \mid K \ k\text{-colouring of } G - e \text{ with } K(u) \neq K(v)\}| \\ &= P_{G/e}(k) + P_G(k). \end{aligned}$$

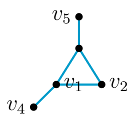
Thus, $P_G(k) = P_{G-e}(k) - P_{G/e}(k)$, as desired. $\square$

Remark 6.43. Combinatorics seems faster than Birkhoff-Lewis, so why bother?

1) We do not need to always reduce back to E_n, once we know the chromatic polynomial of other graphs $\to$ it becomes much faster.

2) It allows inductive arguments in proofs (see below).

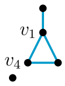
3) Combinatorial thoughts depend on the choice of vertices and hence do not always give the correct number!

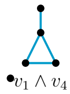
e.g. $G =$ 

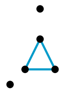
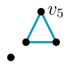
$P_G(3)\ldots$ there are 3 colours to pick from for v_5, also 3 for v_4, 2 for v_1, 2 for v_3 and either 1 or 0 for $v_2\ldots$ we need a case distinction!
$\rightsquigarrow$ Things get messy very quickly.

But: Applying Birkhoff-Lewis, we always get a solid solution!

Note: $P_{C_3}(3) = 3^3 - 3 \cdot 3^2 + 2 \cdot 3 = 6$. Now set $e_1 := v_1v_4$, $e_2 := v_2v_5$.

Then $G_0 := G - e_1$ is 

, $\qquad G_1 := G/e_1$ is 

Further $G_{00} := G_0 - e_2$ is 
, $\qquad G_{01} := G_0/e_2$ is 

and $G_{10} := G_1 - e_2$ is 
, $\qquad G_{11} := G_1/e_2$ is 

$$\begin{aligned} \text{Now } P_G(3) &= P_{G_0}(3) - P_{G_1}(3) = P_{G_{00}}(3) - P_{G_{01}}(3) - (P_{G_{10}}(3) + P_{G_{11}}(3)) \\ &= 3 \cdot 3 \cdot 6 - 3 \cdot 6 - 3 \cdot 6 + 6 \\ &= 6(9 - 3 - 3 + 1) = 24. \end{aligned}$$

Theorem 6.44

Let G be a graph of order n. The following hold.
1) $P_G(k)$ is a polynomial in k of degree n, $P_G(k) = a_nk^n + \cdots + a_1k + a_0$.
2) We always have $a_n = 1$ and $a_0 = 0$, i.e. $P_G(k) = k^n + a_{n-1}k^{n-1} + \cdots + a_1k$.
3) The a_i alternate in sign and $a_{n-1} = -|E(G)|$.

Chapter 7

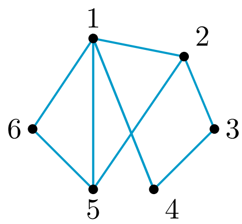
Matchings

1. Introduction

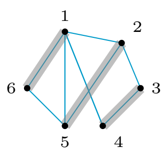
7.1 Motivation

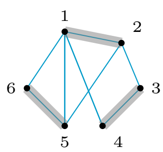
Imagine a group of people has to split up in pairs and take part in a competition. Every person has several people they'd be happy to compete with, and we assume this is symmetric. This situation can be modeled using a graph. How would a possible matching look like which would please everyone?

Example:

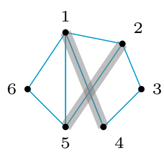


Possible matchings are



or


If we pair 2 with 5 3 and 4 with 1, this is still a matching.



But we have 2 happy teams and can't form another one. It is not perfect...

Observation: The goal will be to pick a set of edges which do not share end vertices.

Definition 7.2

Let G be any graph.

i) A **matching** for G is a set $M \subseteq E(G)$ of pairwise disjoint edges.

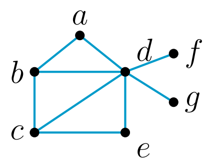
ii) $v \in V(G)$ is called **M-saturated** if exists $e \in M$ s.t. $v \in e$ (i.e. if it is the endpoint of some edge in M). Otherwise, we call v **M-unsaturated**.

iii) We call a matching M **maximal** iff $M \cup \{e\}$ is not a matching for any $e \in E(G) \setminus M$.

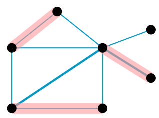
iv) We say that M is a **maximum matching** if it has the largest cardinality among all possible matchings.

v) Finally, we call M **perfect** if any $v \in V(G)$ is M-saturated.

Example 7.3. Consider the graph $G =$

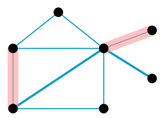


Then



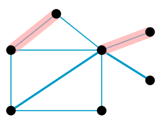
is a maximal matching and also a maximum matching.

Further,



is a maximal matching, but <u>not</u> a maximum matching.

Also,



is <u>not</u> a maximal matching, as $M \cup \{ce\}$ is still a matching.

Finally, there are no perfect matchings as $|V(G)|$ is odd.

Remark 7.4.

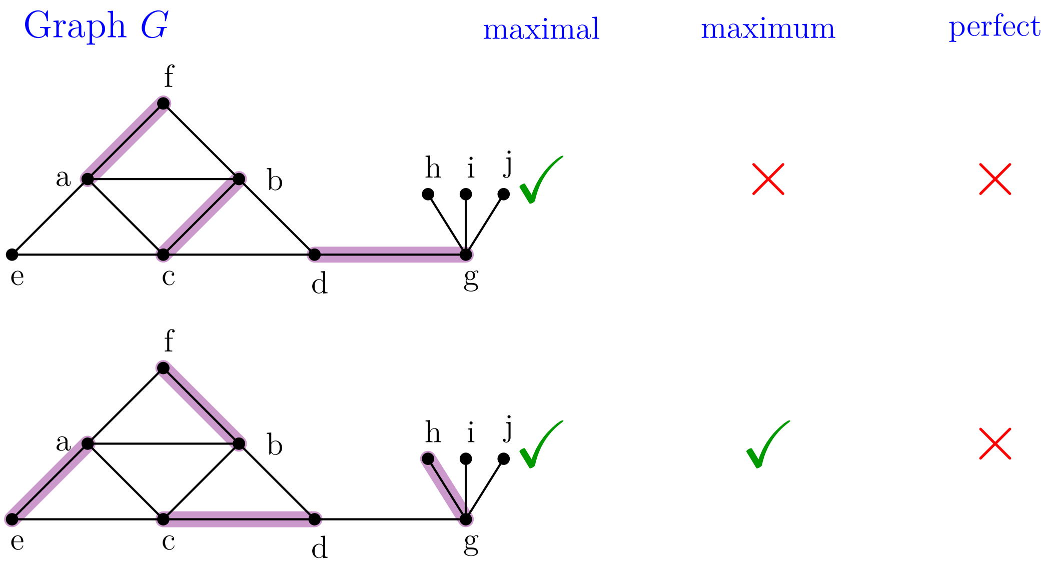
1) Any perfect matching is a maximum matching.

2) Any maximum matching is maximal.

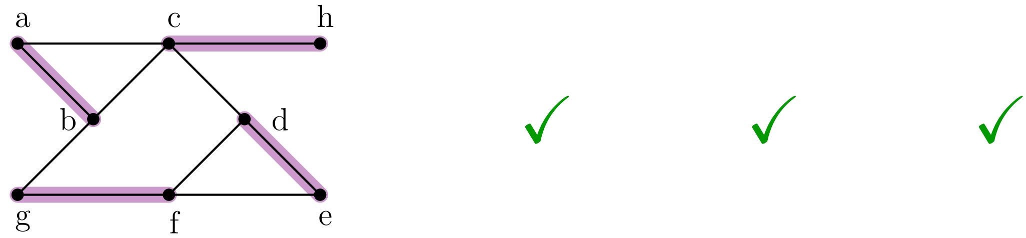
3) If G has a perfect matching, then $|G|$ is even.

Further, any matching M is perfect iff it is a maximum matching iff $|M| = \frac{|V|}{2}$.

Example 7.5.



Note that there can't be a perfect matching, even though $|G| = 10$ is even, as any matching uses at most one of hg, ig and jg.



Note that there can't be a perfect matching, even though $|G| = 10$ is even, as any matching uses at most one of hg, ig and jg.

We see that while it is rather easy to decide whether a matching is maximal or perfect, things are less clear for a maximum matching. Our next goal is to develop a criterion to help us decide that, called Berge's Theorem.

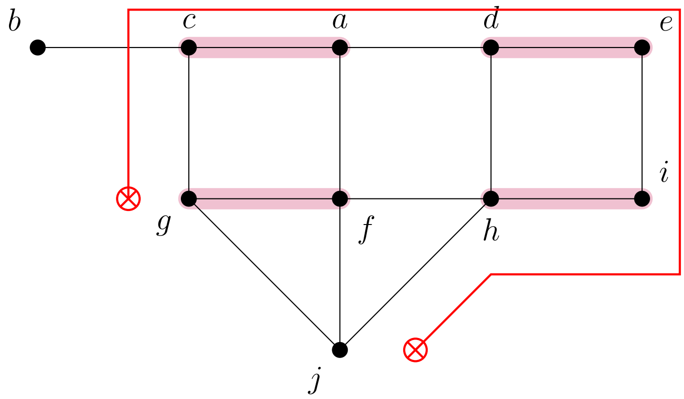
Definition 7.6

Let G be a graph and M a matching for G and p a path in G.

1) We say that p is **M-alternating** if its edges alternate between edges inside and outside M.
2) We call p **M-augmenting** if it is M-alternating and its start and end vertex are distinct and both <u>not</u> M-saturated.

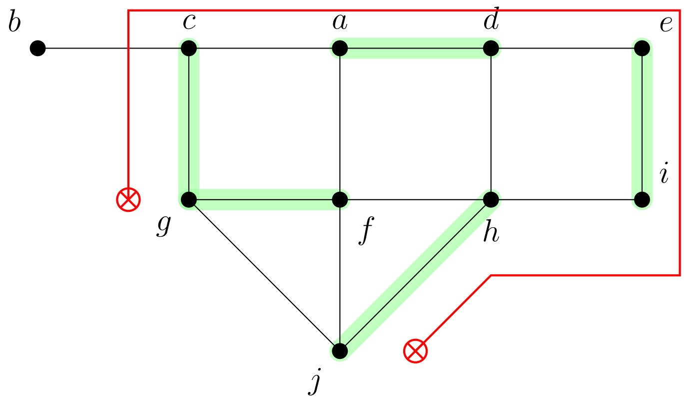
How does this relate to maximum matchings?

Example 7.7. Consider the graph G with matching M below.

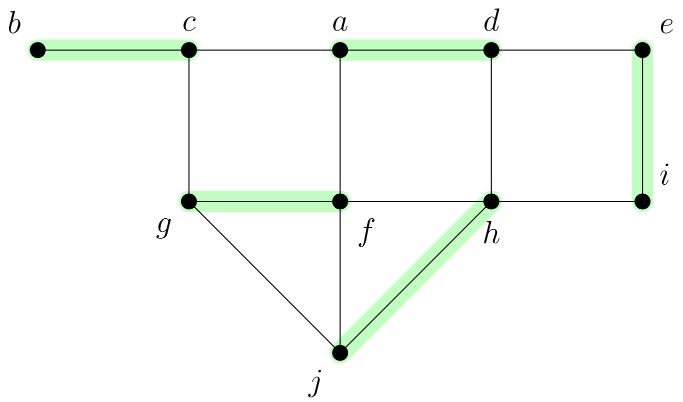


The path $p_1 = (g, c, a, d, e, i, h, j)$ is M-alternating, but not M-augmenting, its start vertex g is M-saturated.

On the other hand, the path $p_2 = (b, c, a, d, e, i, h, j)$ is M-augmenting. In particular, it is M-alternating. What happens if we define a new edge set M' by containing all edges of M outside of p and also exactly those edges of p which were not in M?



For $p_1 = (g, c, a, d, e, i, h, j)$ which was M-alternating, but not M-augmenting, we obtain an edge set which is <u>not</u> a matching.



For the M-augmenting path $p_2 = (b, c, a, d, e, i, h, j)$, on the other hand, the set M' is indeed a matching. Moreover, it contains more edges than M. Hence, M was not a maximum matching for G.

Theorem 7.8 Berge's Theorem, 1957

Let G be a graph and M a matching for G. Then M is a maximum matching iff there is no M-augmenting path in G.

Proof. "$\Rightarrow$" We proceed by contraposition. Assume M is a matching and $p = (x_1, \dots, x_n)$ an M-augmenting path. We will show that M is not maximal. If the edges of p are $e_1, e_2, \dots, e_{n-1}$, then as x_1 is not M-saturated, we get that $e_1 \notin M$. As p is M-alternating also $e_3, e_5, \dots$, i.e. all odd numbered edges are not in M, while all even numbered edges are in M. As also x_n is not M-saturated, $e_{n-1} \notin M$, whence $n-1$ is odd and n is even. Now define

$$M' := M \setminus \{e_2, e_4, \dots, e_{n-2}\} \cup \{e_1, e_3, \dots, e_{n-3}, e_{n-1}\}.$$

Note that $|M'| = |M| + 1$. We claim that M' is a matching, proving that M was not a maximum. But this is clear, as through the change of edges only x_1 and x_n are newly M'-saturated. As they were not M-saturated, x_1 is only the endpoint of e_1 in M' and x_n only of e_{n-1} in M'. Hence M' is again a matching, larger than M.

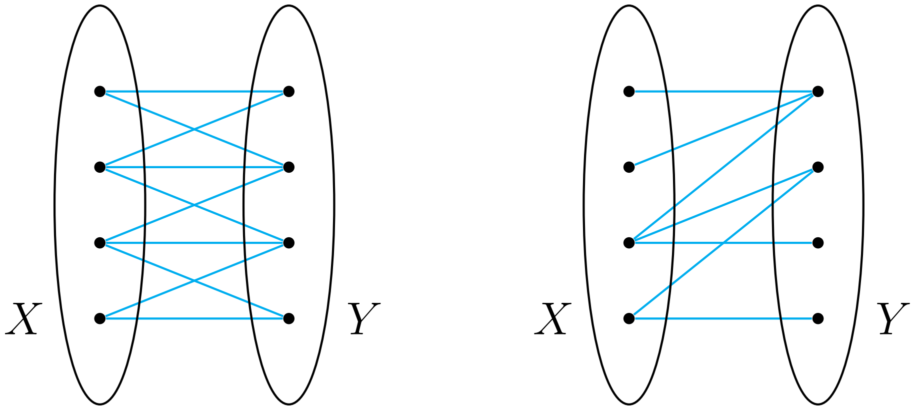
"$\Leftarrow$" We proceed again by contraposition. Assume M is a matching of G which is not a maximum, i.e. there is another matching M' of G with $|M'| > |M|$. We aim to find an M-augmenting path in G. To this end, we define a subgraph $H \subseteq G$ via $V(H) = V(G)$ and $E(H) = M' \Delta M (= M' \setminus M \cup M \setminus M')$.

- Note that $|M' \setminus M| = |M'| - |M' \cap M| > |M| - |M' \cap M| = |M \setminus M'|$, whence H contains strictly more edges from M' than from M.
- Further, $\Delta(H) \leq 2$. To see that, consider $x \in V(H)$ arbitrary. Then there can be at most one edge from M containing x and also at most one other edge from M', as M and M' are matchings. Hence $\delta(x) \leq 2$, as desired.
- Now we know that every connected component of H either is a cycle of even length (using the same number of edges from M and M'), or a path. As $|M' \setminus M| > |M \setminus M'|$, there must be at least one connected component in H which is a path of odd length, starting and ending with an edge in M'. This yields the desired M-augmenting path in G. $\square$

2. Hall's Marriage Theorem

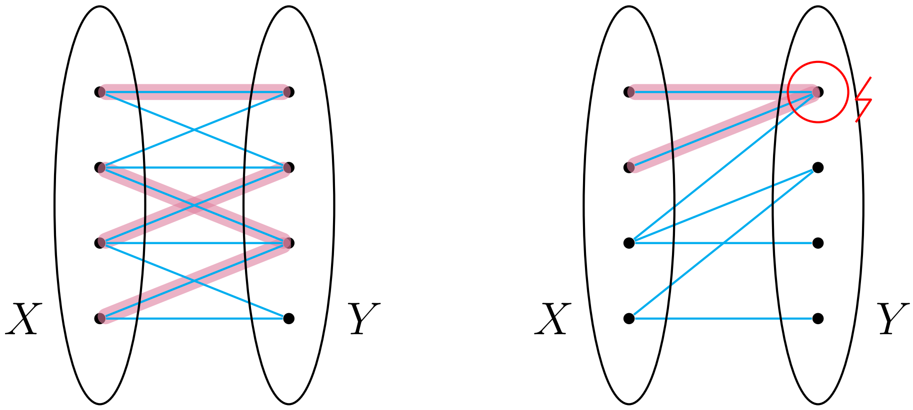
Finding matchings becomes of special interest in bipartite graphs. Historically, the questions were visualised by trying to match couples for marriage. As this does not actually give a bipartite graph, we will put ourselves into the holiday spirit and will discuss bipartite graphs, where one part represents a set of children and the other part a set of presents. We will try to help Santa and develop a criterion to decide whether we can make all the children happy.

Example 7.9. Consider the bipartite graphs below with $V(G) = X \cup Y$, where X represents a set of children and Y a set of presents. How does the problem above translate to graph theory?



We draw one edge between $x \in X$ and $y \in Y$ whenever child x would be happy. To make all children happy, we need to choose edges, such that any $x \in X$ appears as exactly one end vertex (every child gets exactly one present), but all edges are disjoint (children don't have to share a present). I.e. we need to find a matching M s.t. every $x \in X$ is M-saturated.

For our graphs we observe the following:



→ There is a matching which makes all children happy.

→ There are two children which both only want the first present. Hence, there is no appropriate matching.

How can we know from the graph when an appropriate matching exists? This is solved in Philip Hall's marriage theorem.

Definition 7.10

Let G be a bipartite graph with parts X and Y. We say that X is **matched into** Y if there is a matching M for G s.t. every $x \in X$ is M-saturated.

Remark 7.11. As above, X is matched into Y iff there is an injective function $f : X \to Y$ s.t. $\{x, f(x)\} \in E(G)$ for all $x \in X$.

Theorem 7.12 Hall's Marriage Theorem, 1935

Let G be a bipartite graph with parts X and Y. Then X is matched into Y iff for all sets $S \subseteq X$ we have $|S| \leq |N(S)|$.

Proof. "$\Rightarrow$" Assume X is matched into Y, say via $f : X \to Y$. Let $S \subseteq X$ be arbitrary. As $\{x, f(x)\} \in E(G)$ for all $x \in X$, we actually get that $f(S) \subseteq N(S)$. As further f is injective, by definition we get that $|S| \leq |N(S)|$, as desired.

"$\Leftarrow$" Assume now that for any $S \subseteq X$ we have $|S| \leq |N(S)|$. We want to show that there is a matching M which matches X into Y. Let M be any maximum matching. We claim that M matches X into Y. Aiming for a contradiction, assume not, i.e. there is some $u \in X$ which is not M-saturated. Define the set

$$A := \{v \in V(G) \mid \text{exists an } M\text{-alternating } uv\text{-path}\}.$$

We claim that $S := A \cap X$ violates the assumption, i.e. $|S| > |N(S)|$. We will split the proof int 2 parts. Let $T := A \cap Y$. Let's start with some observations. First note that as u is <u>not</u> M-saturated, for any M-alternating path $p = (u = u_1, u_2, \ldots, u_k)$ we have

$$u_i u_{i+1} \in M \text{ iff } i \in 2\mathbb{Z} \text{ iff } u_i \in T \text{ iff } u_{i+1} \in S.$$

Further, as M is maximal there are no M-augmenting paths. In particular, any $v \in A \backslash \{u\}$ must be M-saturated.

<u>Claim 1: $|S| - 1 = |T|$.</u> We define a function $f : S \setminus \{u\} \to T$ via the following: If $x \in S \setminus \{u\}$ then ex. $p_x = (u = u_1, u_2, \ldots, u_k = x)$ M-alternating. Then as $x \in S$, $u_{k-1}u_k \in M$ and M is a matching, u_{k-1} is the only neighbour of x s.t. $u_{k-1}x \in M$. Further $u_{k-1} \in T$ and we set $f(x) := u_{k-1}$. As M is a matching, f is injective. Further, for any $y \in T$ by definition there is an M-alternating uy-path $p_y = (u = y_1, y_2, \ldots, y_\ell = y)$. As $y \in Y$, $y_{\ell-1}y \notin M$ and y_ℓ is M-saturated, there must be a vertex $x \in X$ s.t. $y_\ell x \in M$. But then $p_y^{-1}(x) = (u, y_1, \ldots, y_\ell = y, x)$ is an M-alternating path whence $x \in S$. By definition of f, we get that $f(x) = y$, whence $y \in \text{range}(f)$. Thus, f is a bijection and $|S \setminus \{u\}| = |S| - 1 = |T|$.

<u>Claim 2: $N(S) = T$.</u> "$\supseteq$" Let $w \in N(S)$, i.e. exists $s \in S$ s.t. $sw \in E(G)$. As $s \in S \subseteq X$, the vertex w must be in Y. Further, let p_s be an M-alternating us-path. Then by adjoining w to p_s, we obtain an M-alternating uw-path, whence $w \in T$, as desired. "$\subseteq$" Let $w \in T$ be arbitrary and $s := f^{-1}(t)$ with f defined as in Claim 1. Then $s \in S \setminus \{u\}$ and $w \in N(s)$, as desired.

Conclusively, we have constructed a set $S \subseteq X$ s.t.

$$|S| = |T| + 1 = |N(S)| + 1 > |N(S)|,$$

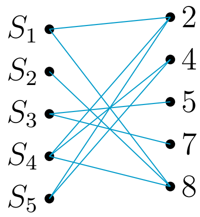
contradicting our assumptions. Thus, such a vertex u cannot exist, whence all vertices in X are M-saturated and M matches X into Y. $\square$

Application - System of Representatives

Definition 7.13

Let $\mathcal{F} = \{S_1, S_2, \ldots, S_k\}$ be a family of non-empty sets. A **system of distinct representatives** for $\mathcal{F}$ is a set $\{x_1, x_2, \ldots, x_k\}$ s.t. $x_i \in S_i$ and the x_i are pairwise distinct.

Example 7.14. Let $S_1 = \{2, 8\}, S_2 = \{8\}, S_3 = \{5, 7\}, S_4 = \{2, 4, 8\}, S_5 = \{2, 4\}$. Can we find a system of representatives for $\mathcal{F} = \{S_1, \ldots, S_5\}$? Let's visualise the problem:



With this visualisation the question translates into asking whether $\mathcal{F}$ is matched into $U = \cup S_i$. Let $S = \{S_1, S_2, S_4, S_5\}$. Then $|S| = 4$. On the other hand, $|N(S)| = |\{2, 4, 8\}| = 3$. So $|S| > |N(S)|$ and by Hall's marriage theorem, there is no system of distinct representatives for $\mathcal{F}$. If we consider $\mathcal{F}_0 := \{S_1, S_2, S_3, S_4\}$ however, then a system of distinct representatives is given by $\{2, 8, 5, 4\}$.

Theorem 7.15

Let $\mathcal{F} = \{S_1, S_2, \ldots, S_k\}$ be a family of nonempty sets. Then $\mathcal{F}$ has a system of representatives iff for any $I \subseteq \{1, 2, \ldots, k\}$ we have that $|I| \leq |\cup_{i \in I} S_i|$.

Proof. Exercise, easy consequence from Hall's marriage theorem. □

3. The König-Egerváry Theorem

We want to finish the chapter on matchings by relating them to yet another very important graph concept - the one of vertex covers.

Definition 7.16

Let G be a graph. A **vertex cover** for G is a vertex set $C \subseteq V(G)$ s.t. every edge of G has at least one endvertex in C, i.e. $\forall e \in E(G) \exists x \in C$ s.t. $x \in e$.
A vertex cover is called a **minimum vertex cover** if there is no vertex cover of smaller cardinality.

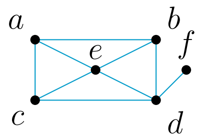
7.17 Application

Imagine a museum with many galleries. We want to position guards within the galleries that have all art pieces in sight. If we model galleries as edges and places where at least two galleries meet as vertices, then we obtain a graph. Now every vertex cover for G would provide an appropriate list of locations to place our guards. Of course, we want to minimize our spendings when usually we are interested in finding a minimum vertex cover.

Remark 7.18.

1) Every graph G has a vertex cover, namely V_G.
2) If C is a vertex cover for G and $C \subseteq D$, then D is a vertex cover for G.

Example 7.19. Consider G given by



Then one vertex cover is given by $C_1 := \{a, b, c, d\}$. But also, $C_2 := \{a, e, d\}$ is a vertex cover of smaller cardinality. It is not hard to see that there is no vertex cover containing only two vertices, whence C_2 is minimal.

Lemma 7.20

Let G be a graph, M any matching for G and C any vertex cover for G. Then $|M| \leq |C|$. In particular, a maximum matching contains at most as many edges as a minimal vertex cover contains vertices.

Proof. Let M be any matching for G, say $M = \{e_1, e_2, \ldots, e_k\}$, and C any vertex cover. For any $e_i \in M$ there is a vertex $x_i \in C$ incident with e_i. As all e_i's are disjoint, all the $x_1, x_2, \ldots, x_k$ are distinct. Thus, C contains at least as many elements as M, i.e. $|M| \leq |C|$. $\square$

Now, what changes if we restrict ourselves to bipartite graphs? Somehow we have more control over the relation between vertices and edges. Indeed, the Hungarian mathematicians Dénes König and Jenő Egerváry independently discovered the following in 1931.

Theorem 7.21 König-Egerváry

Let G be a bipartite graph. Then any maximum matching has the same cardinality as any minimum vertex cover.

Proof. Consider an arbitrary bipartite graph G and a maximum matching M. We will show that there exists a vertex cover C s.t. $|M| = |C|$. By Lemma 7.20, C then is a minimum vertex cover. Let X and Y form a bipartition of V_G.

If every $x \in X$ is M-saturated, then $|X| = |M|$. Clearly, as G is bipartite, the set $C := X$ is a vertex cover for G, whence $|C| = |X| = |M|$ is as desired.

Now, assume <u>not</u> every $x \in X$ is M saturated. Let $U := \{x \in X \mid x \text{ is not } M\text{-saturated}\}$. Then $|M| + |U| = |X|$. (*)

Similar to Hall's Lemma, set

$$A := \{v \in V(G) \mid \text{ex. an } M\text{-alternating } uv\text{-path for some } u \in U\}.$$

Further, set $S := A \cap X$ and $T := A \cap Y$. We claim that

$$C := (X \setminus S) \cup T \text{ is a vertex cover with } |C| = |M|.$$

Exactly as in the proof of 7.12, we can show that

1) Every vertex in $(S \setminus U) \cup T$ is M-saturated.
2) $|S \setminus U| = |T|$.
3) $T = N(S)$.

Thus, we immediately get that $|T| = |S| - |U|$ (**) and thus

$$\begin{aligned} |C| = |X \setminus S| \cup |T| &\overset{X \cap T = \emptyset}{=} |X \setminus S| + |T| \\ &\overset{S \subseteq X}{=} |X| - |S| + |T| \\ &\overset{(**)}{=} |X| - |S| + |S| - |U| \\ &= |X| - |U| \overset{(*)}{=} |M|, \text{ as desired.} \end{aligned}$$

It remains to show that C is a vertex cover for G. To this end, let $e = xy$ be an arbitrary edge with $x \in X, y \in Y$. We need to show that at least one of x or y is in $C = (X \setminus S) \cup T$. If $x \notin S$, i.e. $x \in X \setminus S$, we are done. Otherwise $x \in S$, whence $y \in N(S) = T$, as desired. This finishes the proof. $\square$

Chapter 8

Ramsey Theory

1. Introduction to Ramsey Theory

"Complete Disorder is impossible"

8.1 The party problem

Imagine you want to plan a party and you want to avoid that a few guests only will end up alone. You want to know how many guests you will have to invite such that either m many mutually know each other or n-many are mutual strangers. We will see that this will always happen eventually. That number is called the Ramsey number $R(m, n)$.

8.2 History

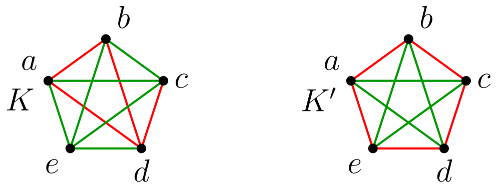
- Area based on a paper by Frank Ramsey "On a problem of formal logic" (1928).
- Ramsey (1903–1930), although dying very young, contributed to many different areas of research – Economy (Ramsey pricing), philosophy (Ramsey sentences), Logic (Ramsey expansions) and mathematics (Ramsey numbers).

Definition 8.3

Let G be a graph. An **edge k-colouring** of G is any function $K : E_G \to \{1, 2, \dots, k\}$.

In this course, we will mainly consider 2-colourings of edges and set the colours to be $\{red, green\}$.
Note that there are no conditions on the function except for the domain and codomain.

Example 8.4. Consider the following edge colourings of K_5.



Note that K produced a triangle on $\{a, b, d\}$ whose edges are all red, whereas with K' there is neither a red nor a green triangle.

Definition 8.5

1) Let G be a graph and $K : E_G \to \{1, \dots, k\}$ an edge colouring. We call a subgraph $H \subseteq G$ **monochromatic** if all edges of H have the same colour, i.e. $K|_H : E_H \to \{1, \dots, k\}$ has a range of size 1, $|K(E_H)| = |\{i\}| = 1$. We then call H a graph of colour i.
2) The **Ramsey number** $R(a, b)$ for $a, b \in \mathbb{Z}_+$ is the smallest integer n s.t. any edge 2-colouring of K_n either contains a red K_a or a green K_b as a subgraph.

Example 8.6.

1) First note that $R(a, b) = R(b, a)$ for all $a, b \in \mathbb{Z}_+$. (Exercise)
2) Let's observe $R(1, b)$ is the least $n \in \mathbb{Z}_+$ s.t. any edge 2-colouring contains either a red K_1 or a green K_b. As K_1 does not contain any edges, every edge colouring of any non-empty graph satisfies the condition, whence $R(1, b) = 1$.
3) $R(2, 2) = 2$, as any edge 2-colouring of K_2 either colours the unique edge red or green, whence we either obtain a red or green K_2.

Lemma 8.7

Let $r \geq R(a, b)$ for $a, b \in \mathbb{Z}_+$. Then any edge 2-colouring of K_r produces either a red K_a or a green K_b.

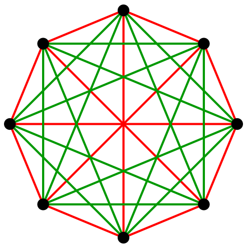
Proof. Let $k := R(a, b)$, $r \geq k$ and K be any edge 2-colouring of K_r. Clearly, K_r contains K_k as a subgraph. Then K_k, and hence K_r, either contains a red K_a or a green K_b, as desired. $\square$

Lemma 8.8

We have that $R(3, 4) = 9$.

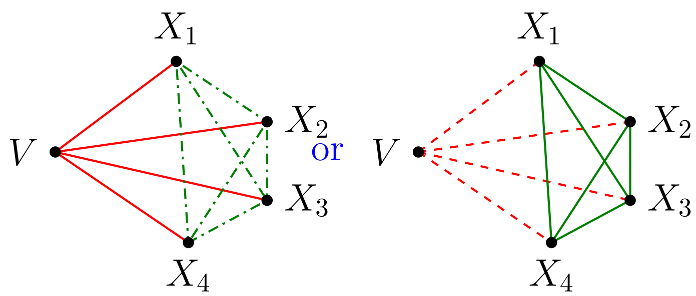
Proof. We have to show two things: 1) $R(3, 4) > 8$ and 2) $R(3, 4) \leq 9$.

1) $R(3, 4) > 8$, i.e. exists an edge 2-colouring of K_8 without a red K_3 and green K_4:



2) $R(3,4) \leq 9$, i.e. any edge 2-colouring of K_9 either produces a red K_3 or a green K_4.
<u>Claim:</u> There is a vertex v with either at least 4 red or at least 6 green edges incident. $\hookrightarrow$ Otherwise, as all vertices have degree 8, all vertices must be incident with exactly 3 red and 5 green edges. Consider then the subgraph $H \subseteq G$ consisting of all 9 vertices of G and all red edges. Then H is a graph which has odd many (i.e. 9) vertices of odd (i.e. 3) degree, contradicting the Handshaking Lemma.

► Now consider first the case that ex. $v \in V_G$ incident with 4 red edges, say $\{vx_1, vx_2, vx_3, vx_4\}$. If any of the edges x_ix_j is also red, we have a red K_3 on $\{v, x_i, x_j\}$. Otherwise, all these edges are blue [green] and we get a blue [green] K_4 on $\{x_1, x_2, x_3, x_4\}$, as desired.



► Now, in the leftover case, v has six green incident edges. Say $\{vx_1, \ldots, vx_6\}$. As $R(3,3) = 6$ (HW), we get that there is either a red K_3 contained in $\langle\{x_1, \ldots, x_6\}\rangle$, in which case we are done, or a green K_3, say on $\{x_i, x_j, x_k\}$. But then we find a green K_4 on $\{v, x_i, x_j, x_k\}$ and again our claim holds.

We thus saw that there is a 2-colouring of the edges of K_8 without a red K_3 or green K_5, however every 2-colouring of the edges of K_9 produces either a red K_3 or a green K_4. Thus, $R(3,4) = 9$. $\square$

8.9 Summary and more...

Let us sum up some of the known Ramsey numbers.

a	b	$R(a,b)$
1	k	1
2	k	k
3	3	6
3	4	9
3	5	14
3	6	18

a	b	$R(a,b)$
3	7	23
3	8	28
3	9	36
4	4	18
4	5	25
5	5	????

The problem gets very fast very complicated. Why should there even always be an answer?

Theorem 8.10 Ramsey

For any positive integers a, b the Ramsey number $R(a,b)$ exists, i.e. there is a positive integer n s.t. every edge 2-colouring of K_n either produces a red K_a or a green K_b.

Proof. We establish the proof by induction on $m := a+b$. $\underline{m=2}$: If $a+b=2$, then $a=b=1$ and we know that $R(1,1)=1$.

$\underline{m \to m+1}$: Assume for any $\tilde{a}, \tilde{b} \in \mathbb{Z}_+$ with $\tilde{a}+\tilde{b} \leq m$ we know that $R(\tilde{a}, \tilde{b})$ exists. Now assume $a, b \in \mathbb{Z}_+$ s.t. $a+b=m+1$. Then both $R(a-1,b)$ and $R(a,b-1)$ exist by I.H. Set $n := R(a-1,b)+R(a,b-1)$ and consider K_n. We claim that K_n does the job. To this end, pick an arbitrary edge 2-colouring K of K_n and choose $v \in V(K_n)$ arbitrary. Define two sets:

$$A := \{w \in V(K_n) \mid K(vw) = \text{red}\} \text{ and } B := \{w \in V(K_n) \mid K(vw) = \text{green}\}.$$

We know that $A \cup B \cup \{v\} = V(K_n)$ and A, B are disjoint. Thus, $|A|+|B| = |V(K_n)|-1 = n-1 = R(a-1,b)+R(a,b-1)-1$. Thus either (1) $|A| \geq R(a-1,b)$ or (2) $|B| \geq R(a,b-1)$.

If (1) $|A| \geq R(a-1,b)$, then there exists either a green K_b or a red K_{a-1} in A, say on $X \subseteq A$. But then either X is a green K_b in K_n or $X \cup \{v\}$ is a red K_a in K_n, as desired.

Similarly, if (2) $|B| \geq R(a,b-1)$, then there exists either a red K_a or a green K_{b-1} in B, say on $Y \subseteq B$. But then either Y is a red K_a in K_n or $Y \cup \{v\}$ is a green K_b in K_n, as desired.

Thus, in any case we find either a red K_a or a green K_b in K_n, whence the Ramsey number $R(a,b)$ exists and is bounded by

$$R(a,b) \leq R(a-1,b) + R(a,b-1). \qquad \square$$

So we saw that Ramsey numbers always exist, whence it should not be too hard to find them, right? WRONG. The list given in 8.9 actually lists <u>all</u> currently known Ramsey numbers. And even though there is continuous work in the area and bounds keep improving, the problem stays very hard. This is nicely illustrated by the following quote by Erdős: "Suppose aliens invade the earth and threaten to obliterate it in a year's time unless human beings can find the Ramsey number $R(5,5)$. We could marshal the world's best minds and fastest computers, and within a year we could probably calculate the value. If the aliens demand $R(6,6)$ however, we would have no choice but to launch a preemptive attack."

2. Excursion into Graph Ramsey Theory

Definition 8.11

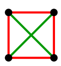
Let G, H be arbitrary graphs. Then **Ramsey number** $R(G,H)$ is the smallest $n \in \mathbb{Z}_+$ s.t. every edge 2-colouring of K_n contains either a red copy of G or a green copy of H as a subgraph.

Lemma 8.12

For any graphs G, H we have $R(G,H) \le R(|G|,|H|)$. In particular, the Ramsey number $R(G,H)$ always exists.

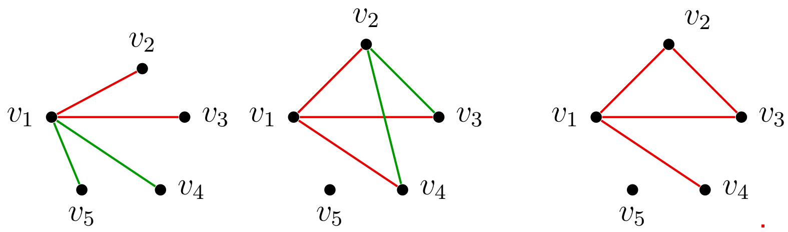
Proof. Let G, H be given and let $n = R(|G|,|H|)$. Consider any edge 2-colouring of K_n. Then it either contains a red copy of $K_{|G|}$ or a green copy of $K_{|H|}$. But as $G \subseteq K_{|G|}$ and $H \subseteq K_{|H|}$, we are done. $\square$

Example 8.13. We claim that $R(C_3, P_3) = 5$, where P_3 is the path on 3 vertices. First, note that $R(C_3, P_3) > 4$, as



is a colouring of K_4 which neither contains a red C_3 nor a green P_3.

Now consider an arbitrary edge 2-colouring of K_5. Pick $v_1 \in V(K_5)$ arbitrary. If two of the edges incident with v_1 are green, we obtain a green P_3. Otherwise, all but one edge must be red, say v_1v_2, v_1v_3 and v_1v_4 are red. Following the same argument, if both of v_2v_3 and v_2v_4 are green, we obtain a green P_3. Otherwise, at least one, say v_2v_3, must be red and we obtain a red C_3 on $\{v_1, v_2, v_3\}$.



$(v_4, v_1, v_5) = P_3$ $(v_3, v_2, v_4) = P_3$ $(v_1, v_2, v_3) = C_3$