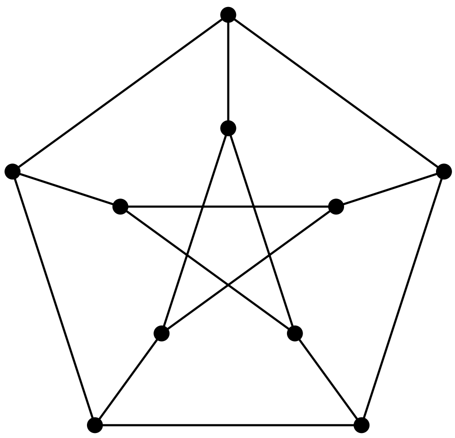
AUC

DEPARTMENT OF MATHEMATICS

SPRING TERM 2026

Graph Theory

Lecture Notes



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Term:
Spring 2026

"An die Professorin, der ich meine Wertschätzung nicht zeigen konnte,
und an die Professorin, der ich es niemals vergelten kann."

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Chapter 1

Graphs

1. The Basics

1.1 Recall

1. A **set** is merely an accumulation of objects. These objects are called **elements** of the set. If an object x is an element of S, we write $x \in S$. The set of all elements with a certain property P is denoted via $\{x \mid x \text{ has property } P\}$.
2. An n**-ary relation** R on a set A is a subset of the power set of A^n, i.e., $R \subseteq \mathcal{P}(A^n)$. If $n = 2$, we call the relation **binary**.

   A binary relation R on a set A is called:

   (i) **symmetric** if $R(a, b)$ implies $R(b, a)$ for all $a, b \in A$.

   (ii) **asymmetric** if $R(a, b)$ implies $\neg R(b, a)$ for all $a, b \in A$.

   (iii) **antisymmetric** if $R(a, b) \wedge R(b, a)$ implies $a = b$ for all $a, b \in A$.

   (iv) **reflexive** if $R(a, a)$ for all $a \in A$.

   (v) **irreflexive** if $\neg R(a, a)$ for all $a \in A$.

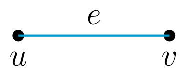
   (vi) **transitive** if $R(a, b) \wedge R(b, c)$ implies $R(a, c)$ for all $a, b, c \in A$.

Definition 1.2

A **graph** $G = (V, E)$ is a pair of sets V and E such that E consists of subsets of V of size two. V is called the set of **vertices** and E the set of **edges**. A graph G is called **finite** if V is a finite set. The **order** $|G|$ of a graph $G = (V, E)$ is the cardinality of its vertex set, so $|G| = |V|$. The **size** $\|G\|$ of G is the cardinality of its edge set, $\|G\| = |E|$.

1.3 Visualisation

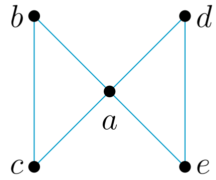
Let $G = (V, E)$ be a graph. We visualise vertices $u, v \cdots \in V$ by dots and edges $e = \{u, v\} \in E$ by the diagram:



Example 1.4. *Bowtie Graph* Let $G = (V, E)$ be the graph with $V = \{a, b, c, d, e\}$ and

$$E = \{\{a, b\}, \{a, c\}, \{a, d\}, \{a, e\}, \{b, c\}, \{d, e\}\}.$$

The graph G has order 5 and size 6. It can be visualized via:



This visualisation motivates its name: **bowtie graph**.

1.5 Notation

1. For a graph $G = (V, E)$ we may denote its vertex set by $V(G)$ or V_G for clarity.
2. Similarly, we often denote E by $E(G)$ or E_G.
3. We denote an edge $\{u, v\}$ simply by uv.
4. Edges are often called $e, e_1, e_2, f \ldots$, while vertices are called $u, v, x, y, \ldots$.

Definition 1.6

Let $G = (V, E)$ be a graph.

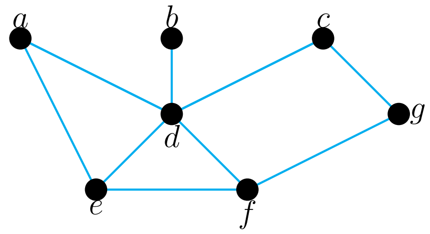
1. If $uv \in E$ is an edge, then we say that u and v are **adjacent** or **neighbours**. If $uv \notin E$, we call u and v **nonadjacent**.
2. If $e = uv \in E$, we say that u and v are the **end vertices** of e or that they are **incident** with e.
3. The **neighborhood** $N(v)$ of a vertex $v \in V$ is the set of all vertices adjacent to v, i.e., $N(v) = \{u \in V \mid uv \in E\}$. The **closed neighborhood** $N[v]$ of v is $N[v] := N(v) \cup \{v\}$.
4. The **neighborhood** $N(S)$ of a set of vertices is defined as $N(S) := \bigcup_{v \in S} N(v)$. Similarly, the **closed neighborhood** $N[S]$ is set to be $N[S] := N(S) \cup S (= \bigcup_{v \in S} N[v])$.
5. The **degree** $\deg(v)$ of $v \in V$ is the number of edges incident with v, i.e., $\deg(v) := |\{e \in E \mid v \in e\}| = |N(v)|$.
6. The **maximum degree** $\Delta(G)$ of G is defined as

$$\Delta(G) := \max\{\deg(v) \mid v \in V\}.$$

Similarly, $\delta(G) := \min\{\deg(v) \mid v \in V\}$ is the **minimum degree** of G.

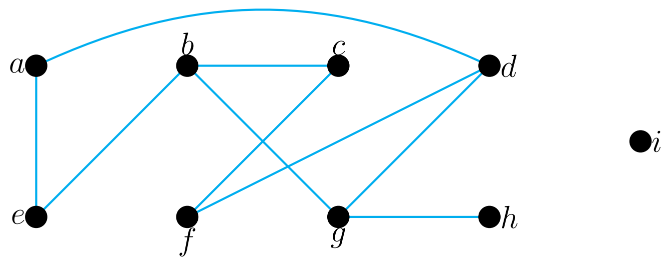
7. The **degree sequence** of a graph G is the sequence containing all degrees of the vertices of G (with repetition) in decreasing order.

Example 1.7. Consider G given by:



Then $\Delta(G) = 5$, $\delta(G) = 1$. $N(e) = \{a, d, f\}$, $N[b] = \{b, d\}$. $N[a, g] = \{a, c, d, e, f, g\}$ Order of G, size of G is 9. Degree sequence $(5, 3, 3, 2, 2, 2, 1)$.

Example 1.8. Consider G given via the diagram:



Then $V(G) = \{a, b, c, d, e, f, g, h, i\}$
$E(G) = \{\{a, d\}, \{a, e\}, \{b, c\}, \{b, e\}, \{b, g\}, \{c, f\}, \{d, f\}, \{d, g\}, \{g, h\}\}$

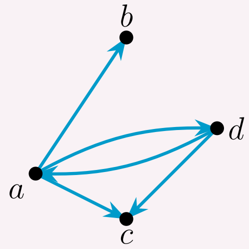
Order $|G| = 9$, size of G is 9, degree sequence $(3, 3, 3, 2, 2, 2, 2, 1, 0)$.
$N(f) = \{c, d\}$, $N[d, e] = \{a, b, c, d, e, f\}$, $\Delta(G) = 3$, $\delta(G) = 0$.

Remark 1.9. A graph can be considered as a set V together with a binary relation E on V which is symmetric and irreflexive.

Definition 1.10 Variants of Graphs

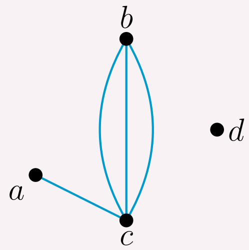
1. If $G = (V, E)$ and we replace E with a set of ordered pairs, then we call G a **directed graph** or **digraph**.

   Ex: $V(G) = \{a, b, c, d\}$, $E(G) = \{(a, b), (a, c), (a, d), (d, a), (d, c)\}$



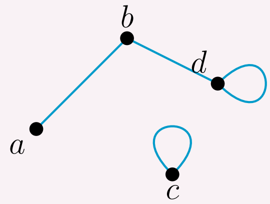
2. If $G = (V, E)$ and we replace E by a multiset (iterations of the same elements are distinguished), then we call G a **multigraph**.

   Ex: $E = [\{a, c\}, \{b, c\}, \{b, c\}, \{b, c\}]$



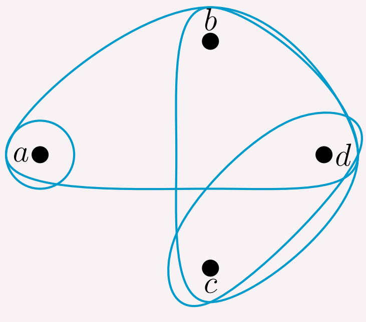
3. If $G = (V, E)$ and we extend E by allowing loops, we call G a **pseudograph**.

   Ex: $E = \{\{a, b\}, \{b, d\}, \{c, c\}, \{d, d\}\}$



4. If we allow edges to be arbitrary sets of vertices instead of 2-elementary ones, we call G a **hypergraph**.

   Ex: $E = \{\{a\}, \{a, b, d\}, \{b, c, d\}, \{c, d\}\}$



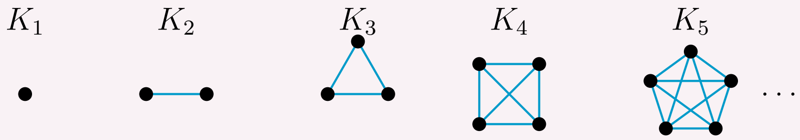
1.11 Setting

In this lecture, unless otherwise stated, by a graph we mean a finite, simple graph with $|V| \geq 1$.

Definition 1.12

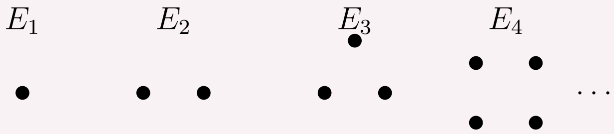
- The **complete graph** K_n for $n \geq 1$ is the graph consisting of n vertices such that any two vertices are adjacent.

  e.g.



- The **empty graph** E_n is the graph consisting of n vertices and no edges.

  e.g.



Theorem 1.13 The Handshaking Lemma

If $G = (V, E)$ is a graph, then

$$\sum_{v \in V} \deg(v) = 2|E|. \quad (*)$$

Proof. We proceed by induction on $n := |E|$.

<u>**n=0:**</u> If $|E| = 0$, then $\deg(v) = 0$ for any $v \in V$, whence clearly

$$0 = \sum_{v \in V} \deg(v) = 2|E| = 0.$$

<u>**n $\to$ n+1:**</u> Assume $(*)$ holds for any $G' = (V', E')$ with $|E'| = n$ (I.H.) and consider $G = (V, E)$ with $|E| = n + 1 (\geq 1)$ arbitrary. Let $e \in E$ arbitrary and consider $G' = (V, E \setminus \{e\})$. Then, if $e = uv$, we get $|E(G)| = |E(G')| + 1$ and

$$\deg_G(u) = \deg_{G'}(u) + 1 \quad \text{and} \quad \deg_G(v) = \deg_{G'}(v) + 1, \text{ whence}$$

$$\begin{aligned}
2|E(G)| &= 2|E(G')| + 2 \\
&\overset{\text{I.H.}}{=} \sum_{w \in V} \deg_{G'}(w) + 2 \\
&= \sum_{w \in V \setminus \{u,v\}} \deg_{G'}(w) + \deg_{G'}(u) + 1 + \deg_{G'}(v) + 1 \\
&= \sum_{w \in V} \deg_G(w), \quad \text{as desired.}
\end{aligned}$$

$\square$

Corollary 1.14

Any graph G has an even number of vertices of odd degree.

Proof. Exercise. $\square$

Corollary 1.15

For any graph $G = (V, E)$ we have

$$\delta(G) \leq 2\frac{|E|}{|V|} \leq \Delta(G).$$

Proof.

$$|V| \cdot \delta(G) = \sum_{v \in V} \delta(G) \leq \sum_{v \in V} \deg(v) \leq \sum_{v \in V} \Delta(G) = |V|\Delta(G)$$

Using Theorem 1.13, $\sum \deg(v) = 2|E|$. Dividing by $|V|$ yields the result. $\square$

Lemma 1.16

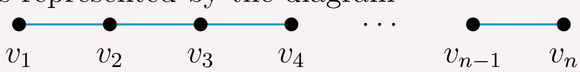
If $|G| \geq 2$, then G contains at least two vertices of the same degree.

Proof. If G has two vertices of degree 0, then we are done. Otherwise, we may assume that G has none. If $|G| = n$, and $v \in V$, then $1 \leq \deg(v) \leq n - 1$. Note that this leaves us with $n - 1$ choices of degrees for n many different vertices. Hence, at least two vertices must have the same degree. $\square$

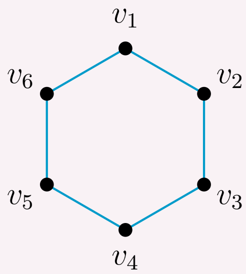
Remark 1.17. The above line of thought is called the **pigeon hole principle**. If there are n many pigeons wanting to fit into $n - 1$ many holes, then at least two of them have to cuddle up in the same hole.

Definition 1.18

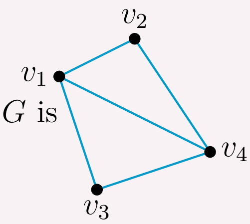
1) The **path** P_n is the graph on n vertices $v_1, \ldots, v_n$ with the edge set $E(P_n) = \{v_i v_{i+1} \mid 1 \leq i < n\}$, i.e. P_n is represented by the diagram

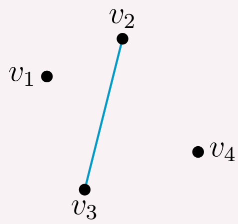


2) The **cycle** C_n is the graph on n vertices with edge set $E(C_n) = \{v_i v_{i+1} \mid 1 \leq i < n\} \cup \{v_n v_1\}$. E.g.

C_6 is 

3) Let $G = (V, E)$ be an arbitrary graph. The **complement** $\overline{G}$ of G is the graph $\overline{G} = (V, \overline{E})$, where $\overline{E} = \{uv \mid u, v \in V, uv \notin E\}$.

e.g. If G is 

, then $\overline{G}$ is 

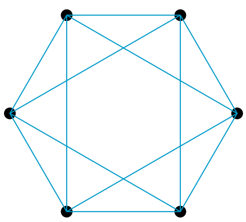
Definition 1.19

We call a graph G **regular** if any of its vertices has the same degree. If this degree is r, we say that G is r**-regular**.

Remark 1.20.

1. A graph G is regular iff $\delta(G) = \Delta(G)$.
2. K_n is $(n-1)$-regular and E_n is 0-regular.
3. An r-regular graph of order n has $\frac{1}{2}nr$ many edges.

Example 1.21. The graph below is 4-regular of order 6.



2. Subgraphs

There are two ways in which one graph can be part of another graph.

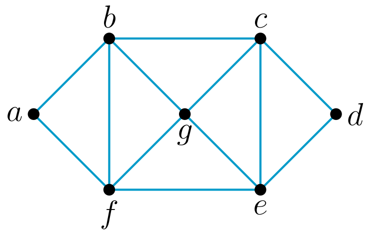
Definition 1.22

1. A graph H is called a **subgraph** of some graph G, written $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. We also say G **contains** H.
2. If $H \subseteq G$, we say that H is an **induced subgraph** of G, written $H \sqsubseteq G$, if $E(H) = \{uv \in E(G) \mid u, v \in V(H)\}$.

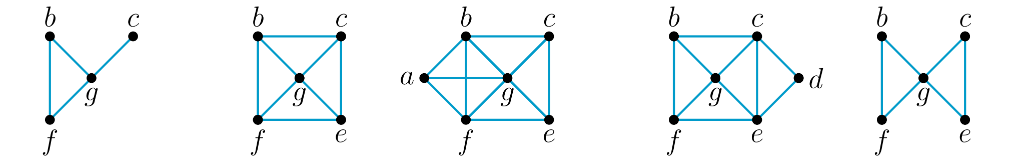
Remark 1.23.

1. $H \subseteq G$ is induced if for any two vertices in H we have: If they are adjacent in G, then they are adjacent in H.
2. Every induced subgraph is a subgraph but not vice versa.
3. If G is a graph and $S \subseteq V(G)$, then there is only one induced subgraph $H \sqsubseteq G$ with vertex set S, i.e. $V(H) = S$. We denote this graph by $\langle S \rangle$ and call it the subgraph of G induced by S.

Example 1.24. Consider G given as



Then



subgraph	✓	✓	✗	✓	✓
induced	✗	✓	✗	✓	✗

3. Walks in Graphs

Definition 1.25

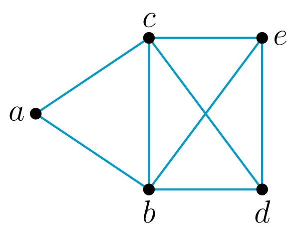
A (v_0, v_k)-**walk** in a graph is a sequence of vertices $(v_0, v_1, \dots, v_k)$ s.t. any two consecutive vertices v_i and v_{i+1} are adjacent. We call the edges $\{v_0v_1, v_1v_2, \dots, v_{k-1}v_k\}$ the **edges of the walk**. We say that the walk is **closed** if $v_0 = v_k$. The **length** of a walk is the number of edges in it (counting repetition).

Definition 1.26

We distinguish the following types of walks:

- A **trail** is a walk whose edges are pairwise distinct.
- A **circuit** is a closed walk whose edges are pairwise distinct.
- A **path** is a walk whose vertices are distinct.
- A **cycle** is a closed walk $(v_0, \dots, v_k = v_0)$ with $k \geq 3$ and whose vertices $v_0, \dots, v_{k-1}$ are pairwise distinct.

Example 1.27. Consider G via



Give examples for a:

- **walk** (d, b, c, d, b, a)
  da-walk, length 5
- **trail** (d, c, a, b, c, e)
  de-trail, length 5
- **path** (d, c, a, b, e)
  de-path, length 4
- **closed walk** (e, b, c, a, b, d, e)
  e-closed walk, length 6
- **circuit** (d, c, a, b, c, e, d)
  d-circuit, length 6
- **cycle** (d, c, a, b, e, d)
  d-cycle, length 5

Lemma 1.28

If $\delta(G) \geq 2$, then G contains a cycle as a subgraph.

Proof. Let $P = (v_0, \dots, v_k)$ be a path in G of maximal length. This exists, as G is finite. Further, as $\delta(G) \geq 2$, we get $k \geq 2$. As $\deg(v_0) \geq \delta(G) \geq 2$, v_0 has at least two neighbors. One of them is v_1. Let us denote the other one by u. If $u \neq v_i$ for all $1 \leq i \leq k$, then $\tilde{P} = (u, v_0, v_1, \dots, v_k)$ is still a path and of greater length than P, contradicting our assumptions. Hence, $u = v_i$ for some $1 \leq i \leq k$. But then the sequence $(v_0, v_1, \dots, v_i = u, v_0)$ is the desired cycle subgraph of G. $\square$

Corollary 1.29 Contrapositive

If G does not contain any cycles, then $\delta(G) \leq 1$.

Theorem 1.30

Every uv-walk in a graph contains a uv-path.

Proof. We proceed by strong induction on the length $n \geq 1$ of the walk. **I.B. n=1.** If the uv-walk is of length one, then it is exactly (u, v), which is also a path. **I.S.** Assume every uv-walk of length at most $n \geq 1$ contains a uv-path (I.H.). Assume there is a uv-walk $W = (u = w_0, w_1, \ldots, w_n, w_{n+1} = v)$ of length $n+1$. If W is already a path, we are done. Otherwise there are i, j s.t. $0 \leq i < j \leq n+1$ and $w_i = w_j$. But then the walk $\tilde{W}$ which arises from W by deleting the vertices $w_{i+1}, \ldots, w_{j-1}, w_j$, i.e. $\tilde{W} = (u = w_0, \ldots, w_i, w_{j+1}, \ldots, w_{n+1} = v)$ is still a uv-walk, but of length at most n. Using I.H., we know that $\tilde{W}$ contains a uv-path, whence also W contains (the same) uv-path. $\square$

4. Connectivity

Definition 1.31

A graph is **connected** if there exists an uv-path in G for any vertices $u, v \in V(G)$. Otherwise, it is called **disconnected**.

Intuition

A graph is connected if you could pick it up entirely by just lifting one vertex. If it is not connected, then the subgraph you lift that way is called a connected component.

Definition 1.32

A **connected component** of G is a maximal connected induced subgraph of G. i.e. $C \sqsubseteq G$ is a connected component iff (i) C is connected and (ii) for any $v \in V(G) \setminus V(C)$ the induced subgraph on $V(C) \cup \{v\}$ is **not** connected.

Remark 1.33. G is connected iff it has exactly one connected component.

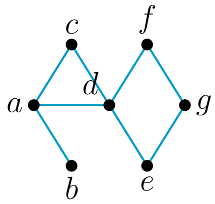
Even among connected graphs, there are different levels of being connected. E.g. the graph K_5  "feels" more connected than the graph . In order to properly describe this intuition, we need more notations.

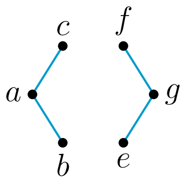
Definition 1.34 Vertex and Edge Deletion

Let G be a graph, $S \subseteq V_G$ and $T \subseteq E_G$.

1. By $G - S$ we denote the graph arising from G by removing from V_G all vertices in S and their incident edges.
2. If $S = \{v\}$, we write $G - v$.
3. By $G - T$ we denote the graph arising from G by removing only the edges in T, but no vertices.
4. If $T = \{e\}$, we write $G - e$.

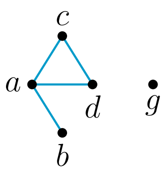
Example 1.35. Consider G as given below. Note that G only has one connected component.

$G =$ 

Then $G - d$ is 

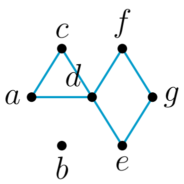
and has 2 connected components.

The vertex d is called a **cut vertex**.

Further, $G - \{e, f\}$ is 

. It also has 2 connected components.

The set $\{e, f\}$ is called a **cut set**.

Further, $G - ab$ is 

Again, it has 2 connected components.

We call the edge ab a **bridge**.

Definition 1.36

Let G be a graph.

1. We call $v \in V_G$ a **cut vertex** if $G - v$ has more connected components than G itself.
2. We call $e \in E_G$ a **bridge** if $G - e$ has more connected components than G itself.
3. We call $S \subseteq V_G$ a **cut set** if $G - S$ is disconnected.
4. A connected graph which does not contain any cut vertices is called **non-separable**.

1.37 Observation

1. If G is connected then v is a cut vertex of G iff $\{v\}$ is a cut set.
2. The vertex v is a cut vertex iff there are vertices u and w, different from v s.t. every uw-path uses v.
3. A graph has no cut sets iff it is a complete graph.

Definition 1.38

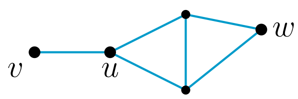
For a non-complete graph G, we define its **connectivity** $\kappa(G)$ as the minimal size of a cut set. For K_n, we set $\kappa(K_n) = n - 1$.

Lemma 1.39

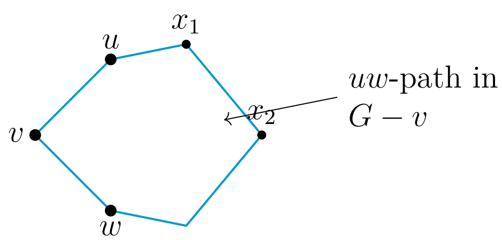
If G is a nonseparable graph of order at least 3, then $\delta(G) \geq 2$ and every vertex of G is contained in a cycle.

Proof. Consider G nonseparable with $|G| \geq 3$. By definition, G is connected, i.e. $\delta(G) \geq 1$.

First we show that $\delta(G) \geq 2$. Otherwise, we have $\delta(G) = 1$, i.e. there is some vertex v s.t. $\deg(v) = 1$. Let u be the unique neighbor of v and w any other vertex of G (which exists as $|G| \geq 3$). Then clearly any vw-path must use the unique neighbor u of v, whence u is a cut vertex. This contradicts the fact that G is inseparable. Hence, $\delta(G) \geq 2$, as desired.

Picture: 

Now, consider $v \in V_G$ arbitrary. We want to show that v is contained in a cycle in G. As $\delta(G) \geq 2$, v has at least 2 neighbors, say u and w. As G is nonseparable, $G - v$ is still connected. In particular, there is a uw-path $(u = x_0, x_1, x_2, \ldots, x_k = w)$ in $G - v$. But then the walk $(x_0 = u, x_1, \ldots, x_k = w, v, x_0 = u)$ is the desired cycle containing v.

Picture: 

$\square$

Definition 1.40

We say that G is **k-connected** if $\kappa(G) \geq k$, i.e. if G is connected and $G - S$ is still connected for any $S \subseteq V_G$ with $|S| < k$.

Lemma 1.41

The following hold:

1) G is connected iff $\kappa(G) \geq 1$.
2) G is 1-connected iff G is connected.
3) G is 2-connected iff G is connected and has no cut vertices.
4) G is 2-connected iff G is non-separable.
5) If G is 2-connected, then it contains at least one cycle (for $|G| \geq 3$).
6) If G is k-connected, then G is j-connected for all $j \leq k$.
7) $|G| > \kappa(G)$.
8) $\kappa(G) \leq \delta(G)$.

Proof. 1)–6) are easy observations – verify them by yourselves.

7) If $G = K_n$, then $|G| = n > n - 1 = \kappa(G)$. Otherwise, assume $\kappa(G) = k$, i.e. ex. $S \subseteq V_G$ s.t. $|S| = k$ and $G - S$ is disconnected. For $G - S$ to be disconnected, it must contain at least 2 vertices, whence

$$|G| \geq |S| + 2 = \kappa(G) + 2 > \kappa(G).$$

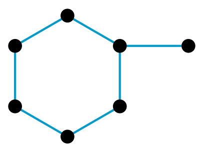
8) Assume $\kappa(G) > \delta(G)$ and let $v \in V_G$ s.t. $\deg(v) = \delta(G)$. Note that $|G| > \kappa(G) > \delta(G) = |N(v)|$, whence $G - N(v)$ contains at least one vertex besides v. But clearly, $G - N(v)$ is disconnected (as $\deg^{G-N(v)}(v) = 0$). Hence, $N(v)$ is a cut set and $\kappa(G) \leq |N(v)| = \delta(G)$, contradicting the assumptions. □

5. Bipartite Graphs

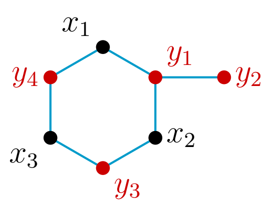
Definition 1.42

A graph G is called **bipartite** if we can partition the vertex set V_G into two disjoint sets $V_G = X \cup Y$ s.t. every edge of G has one end vertex in X and the other in Y.

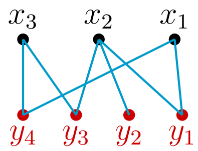
Example 1.43. Consider $G :=$



We can partition the vertices of G into two sets via $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3, y_4\}$.



Rearranging the position of the vertices makes it clear that G is bipartite:



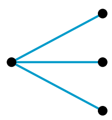
We see that there are no edges between any two vertices in X or in Y.

Remark 1.44. A graph G is bipartite if and only if we can color the vertices of G with two colors s.t. the end vertices of each edge have different colors.

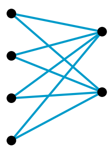
Definition 1.45

Let $m, n \in \mathbb{Z}_+$. The **complete bipartite graph** $K_{m,n}$ is the bipartite graph with $X = \{x_1, \ldots, x_m\}$, $Y = \{y_1, \ldots, y_n\}$, $V_G = X \cup Y$ and $E_G = \{xy \mid x \in X, y \in Y\}$.

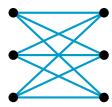
Example 1.46. Below are some examples of complete bipartite graphs.



$K_{1,3}$



$K_{4,2}$



$K_{3,3}$

The following theorem helps us decide whether or not a given graph is bipartite.

Theorem 1.47

A graph is bipartite iff it does not contain odd cycles.

Proof. "$\Rightarrow$": Assume G is bipartite and nevertheless there is a cycle of odd length, say $(x_0, x_1, \ldots, x_{2k}, x_{2k+1} = x_0)$. By Remark 1.44, we can color V_G in two colors, $C1$ and $C2$, s.t. adjacent vertices have different colors. Then, if x_0 has color $C1$, x_1 has color $C2$ whence x_2 has color $C1$. That way we see that the color of x_i is

$$\begin{cases} C1 \text{ if } i \text{ is even} \\ C2 \text{ if } i \text{ is odd} \end{cases}.$$

Following that logic, the vertex $x_0 = x_{2k+1}$ should have color $C1$ and color $C2$ at the same time, which is a contradiction.

"$\Leftarrow$": Now consider that G does not contain odd cycles. We will show that G is bipartite by providing a partition. We may assume that G is connected as otherwise we work component per component. Pick $v \in V_G$ arbitrary and define

$$X = \{w \in V_G \mid \text{the shortest } vw \text{ path has even length}\} \text{ and}$$

$$Y = \{w \in V_G \mid \text{the shortest } vw \text{ path has odd length}\}.$$

Clearly, X and Y are disjoint. We will show that there are no adjacent vertices in X or Y respectively. Note that $v \in X$.

Aiming for a contradiction, assume that there are vertices $w_1, w_2 \in X$ which are adjacent. Clearly, $w_1 \neq v$, as otherwise the shortest vw_2-path was exactly vw_2 of length 1. Similarly, $w_2 \neq v$. Let $P_1 = (v = x_0, x_1, \dots, x_{2k} = w_1)$ and $P_2 = (v = y_0, y_1, \dots, y_{2\ell} = w_2)$ be the shortest vw_1- and vw_2-paths. Suppose that $x_i = y_j$ for some $0 < i \leq 2k$ and $0 < j \leq 2\ell$. If $i < j$, then $(v = x_0, x_1, \dots, x_i, y_{j+1}, \dots, y_{2\ell} = w_2)$ is a vw_2 path shorter than P_2, a contradiction. Similarly, $j < i$ is impossible, whence $i = j$, whenever $x_i = y_j$.

Now, pick the largest i s.t. $x_i = y_i$. As $x_0 = v = y_0$, such an i always exists. Then we obtain the following cycle

$$C = (\underbrace{x_i, x_{i+1}, \dots, x_{2k}}_{2k-i} = w_1, \underbrace{w_2}_{1} = y_{2\ell}, \underbrace{y_{2\ell-1}, \dots, y_{i+1}}_{2\ell-i}, y_i = x_i).$$

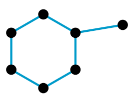
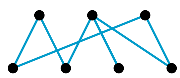
This is a cycle, as P_1 and P_2 were paths and i was maximal s.t. $x_i = y_i$. Further, the length of C is odd, as it equals

$$(2k - i) + 1 + (2\ell - i) = 2(k + \ell - i) + 1.$$

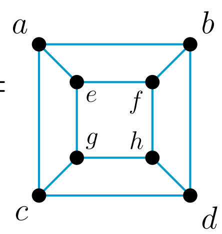
This contradicts our assumption that G does not contain odd cycles. We hence proved that no two vertices w_1 and w_2 from X can be adjacent. The arguments for $v_1, v_2 \in Y$ is analogous (try to write it down). This concludes the proof. $\square$

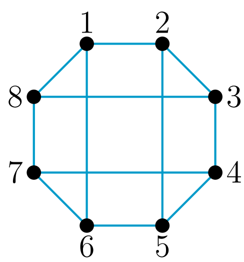
6. Graph Isomorphisms

In Example 1.43, we rearranged the given graph G_1 as G_2.

$G_1 =$  as $G_2 =$ 

We understand G_1 and G_2 as the same, even though on the first glance, they look very similar. Another example is given by

$G =$ 

$H =$ 

We can relabel the vertices of G via $a \mapsto 1, b \mapsto 2, c \mapsto 8, d \mapsto 3, e \mapsto 7, f \mapsto 4, g \mapsto 6$ and $h \mapsto 5$ and obtain H. The aim of this section is to formalise this concept.

Definition 1.48

We say that a graph G is **isomorphic** to a graph H if there exists a bijection $\varphi : V_G \to V_H$ s.t. for any $u, v \in V_G$ we have that $\{u, v\} \in E_G$ if and only if $\{\varphi(u), \varphi(v)\} \in E_H$. Then, the map φ is called an **isomorphism** and we write $G \cong H$.

Remark 1.49. Let $G \cong H$ via $\varphi : V_G \to V_H$. Then:

1) $|V_G| = |V_H|$ and $|E_G| = |E_H|$ and $\overline{G} \cong \overline{H}$.
2) The degree sequence of G equals the degree sequence of H.
3) G is connected iff H is connected.
4) $\deg_G(v) = \deg_H(\varphi(v))$ for all $v \in V_G$.