Fall 2022 - Analysis and Design of Algorithms

Lectures 7 and 8: Greedy Algorithms

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Greedy Algorithms

- Greedy Strategy:
    - At any step, when we have multiple options to choose from, choose the best option at the moment, i.e., the option that offers the highest *immediate* benefit.
    - This certainly does not lead to optimal solutions to all problems.
        - We have seen several examples in the dynamic programming lectures where the greedy strategy fails.
- In this lecture, we will see several examples where the greedy strategy works.

Outline

- Activity selection
- Fractional knapsack
- Huffman codes

Activity Selection [CLRS 16.1]

Given a set of *n* activities, S = {a_1, a_2, …, a_n}, where each activity a_i has a start time s_i and a finish time f_i, find a maximum-size subset of mutually compatible activities.

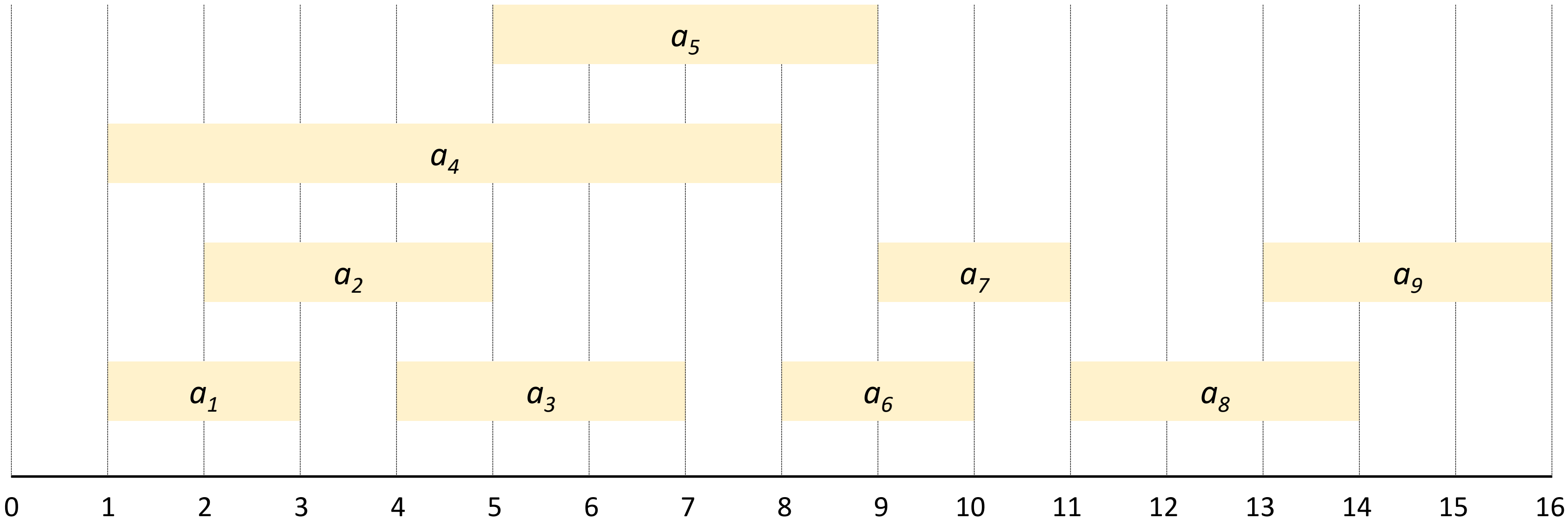
- Each activity a_i takes place during the half-open interval [s_i, f_i).
- Two activities a_i and a_j are compatible iff

  [s_i, f_i) and [s_j, f_j) do not overlap.
- Assume that the activities are already sorted by their finish times.

Activity Selection

- Example [CLRS notes]:

i	1	2	3	4	5	6	7	8	9
s_i	1	2	4	1	5	8	9	11	13
f_i	3	5	7	8	9	10	11	14	16

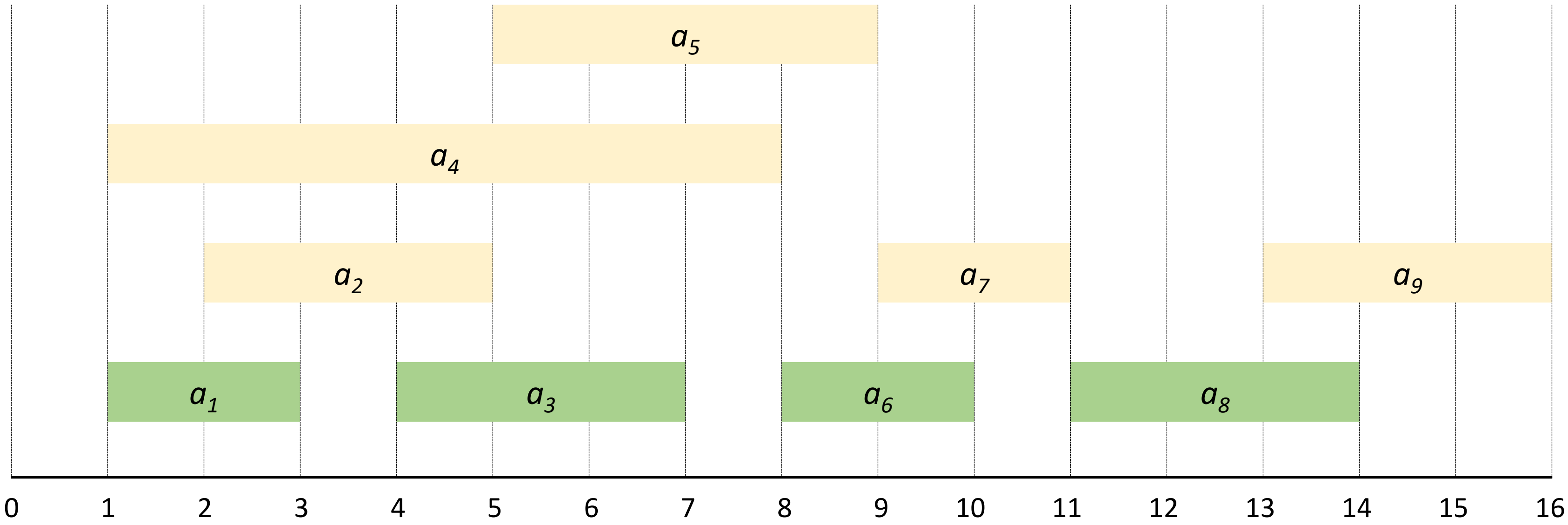


Activity Selection

- Example [CLRS notes]:

i	1	2	3	4	5	6	7	8	9
s_i	1	2	4	1	5	8	9	11	13
f_i	3	5	7	8	9	10	11	14	16

Solution 1

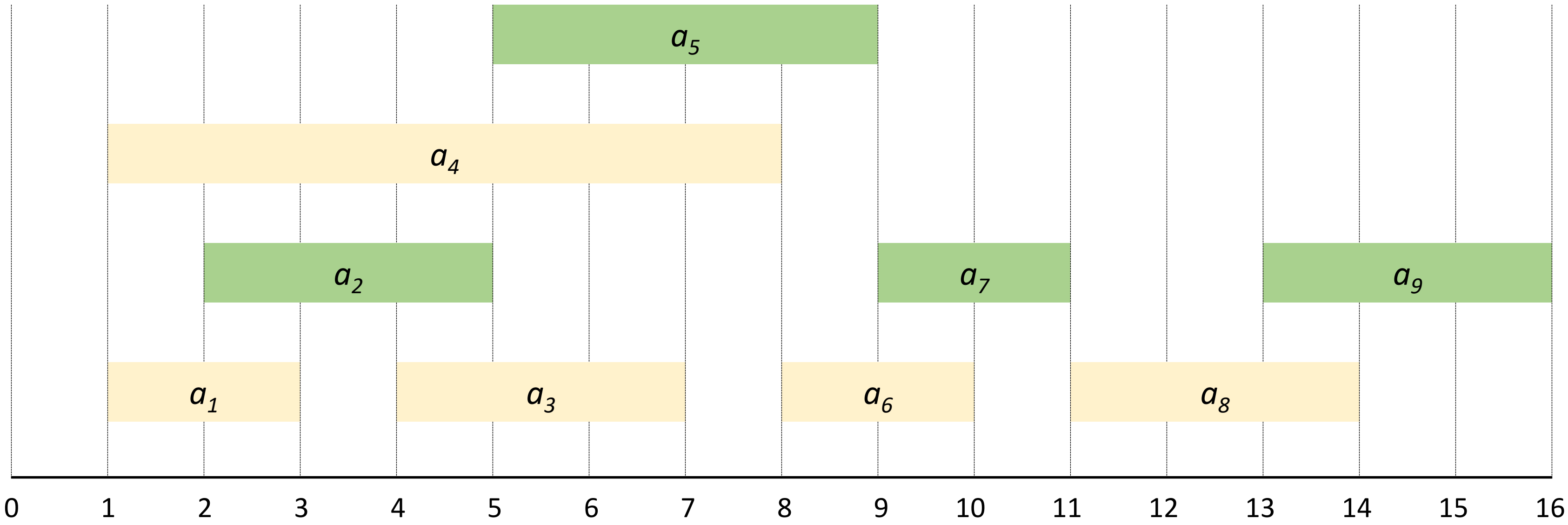


Activity Selection

- Example [CLRS notes]:

i	1	2	3	4	5	6	7	8	9
s_i	1	2	4	1	5	8	9	11	13
f_i	3	5	7	8	9	10	11	14	16

Solution 2



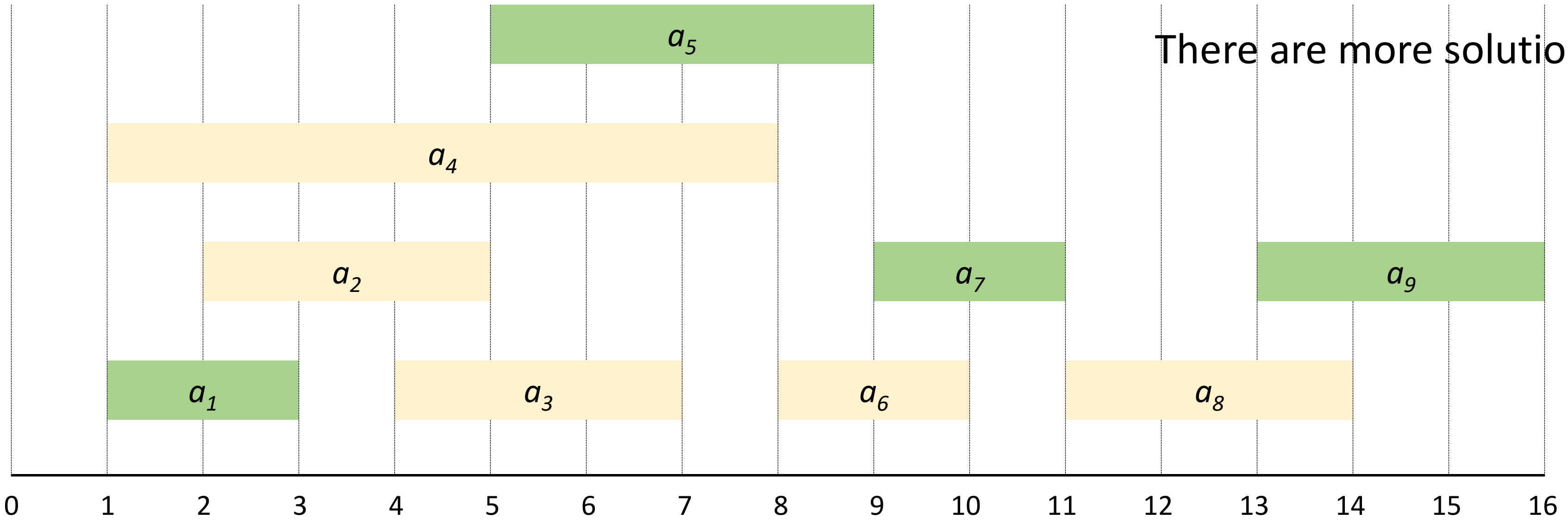
Activity Selection

- Example [CLRS notes]:

i	1	2	3	4	5	6	7	8	9
s_i	1	2	4	1	5	8	9	11	13
f_i	3	5	7	8	9	10	11	14	16

Solution 3

There are more solutions.

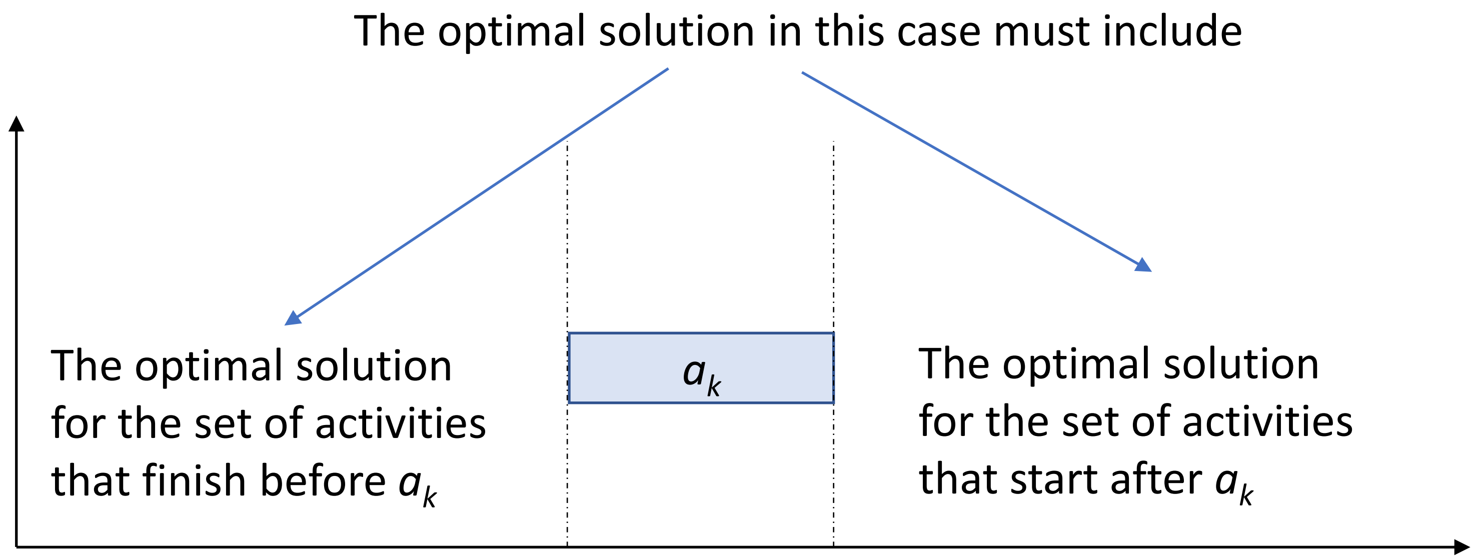


Activity Selection

- To solve the previous problem, we can use dynamic programming.
- However, we will discover a simpler *greedy* algorithm.
- We will start by the DP solution as a review, then we will discuss the greedy one.
  - Note: The DP solution presented next is **not** the most efficient DP solution to the problem.
  - The goal is to illustrate the difference between DP and greedy algorithms.
  - We use the formalization used in CLRS.

Activity Selection

- Examining the structure of the problem
    - Suppose some activity a_k is part of the optimal solution for the set S, i.e., a_k belongs to the maximum subset of mutually compatible activities.



The optimal solution in this case must include

The optimal solution for the set of activities that finish before a_k

a_k

The optimal solution for the set of activities that start after a_k

As in the examples covered previously, we don't know which a_k belong to the optimal solution, so we have to consider all options when writing the recursive definition.

Activity Selection

- The problem has optimal substructure.
  - The optimal solution of the original problem includes optimal solutions to the subproblems.

- Following the DP paradigm of the previous lecture
  - Next step: find a recursive definition

Activity Selection

Some notation:

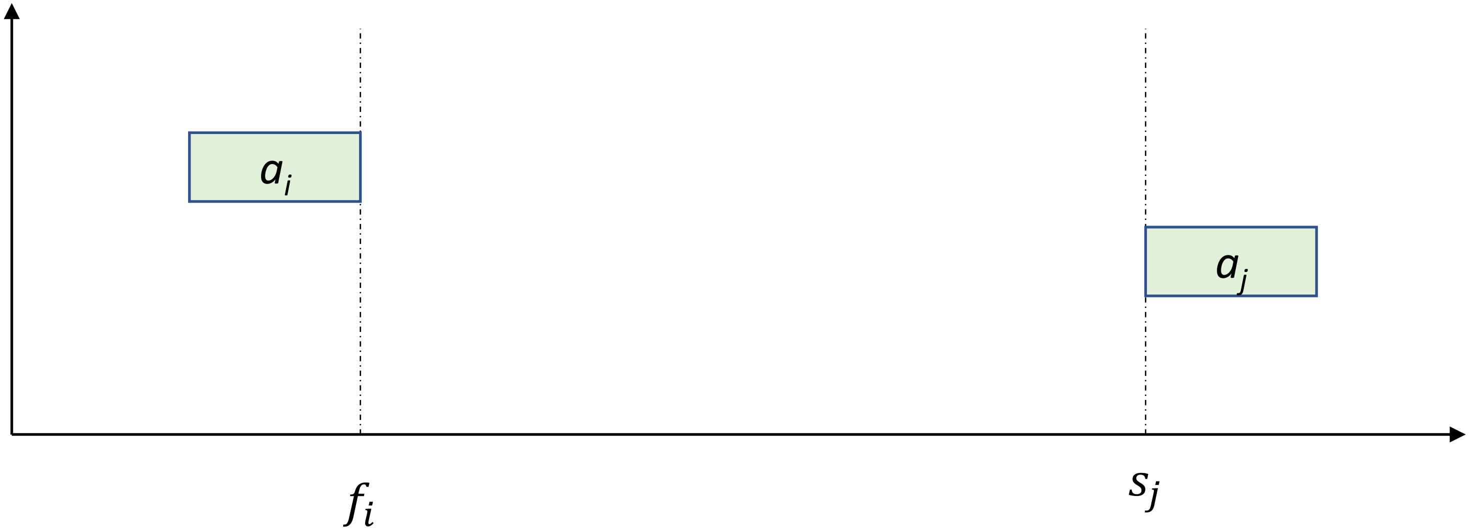
- S_{ij} is the set of activities that start after activity a_i finishes and that finish before activity a_j starts.

$$S_{ij} = \{a_k \in S : f_i \leq s_k < f_k \leq s_j\}$$

- A_{ij} is the maximum-size subset of mutually compatible activities in S_{ij}.

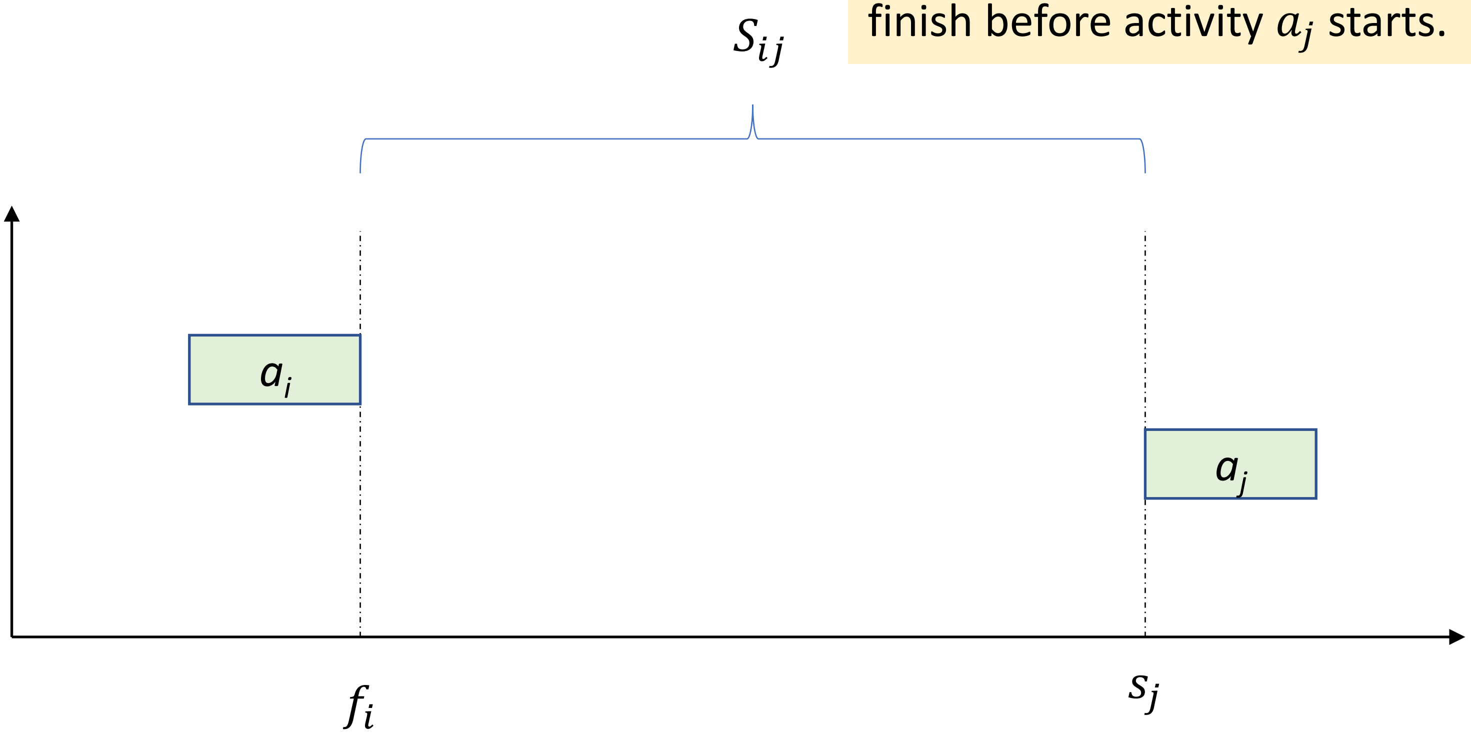
- $|A_{ij}|$ is the size of the set A_{ij}

Activity Selection



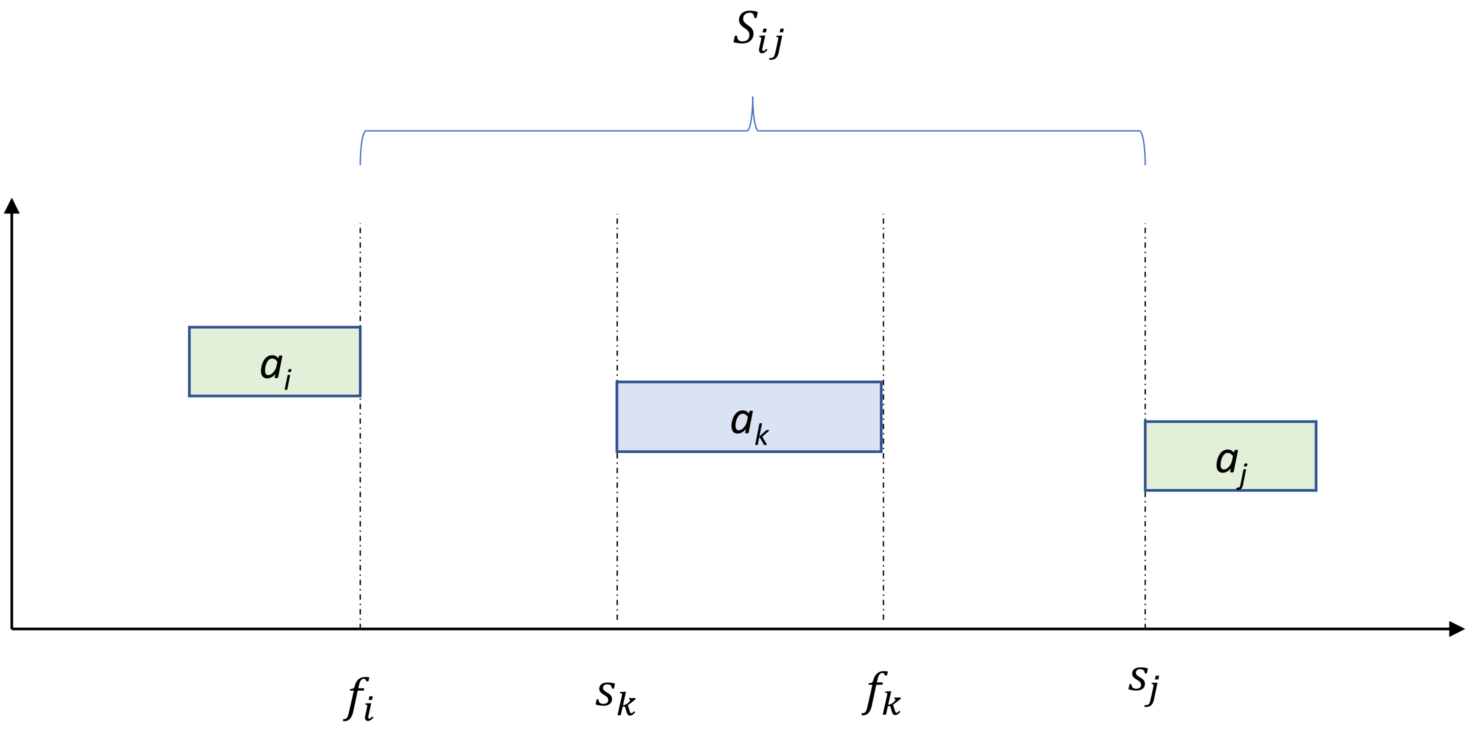
Activity Selection

S_{ij} is the set of activities that start after activity a_i finishes and that finish before activity a_j starts.



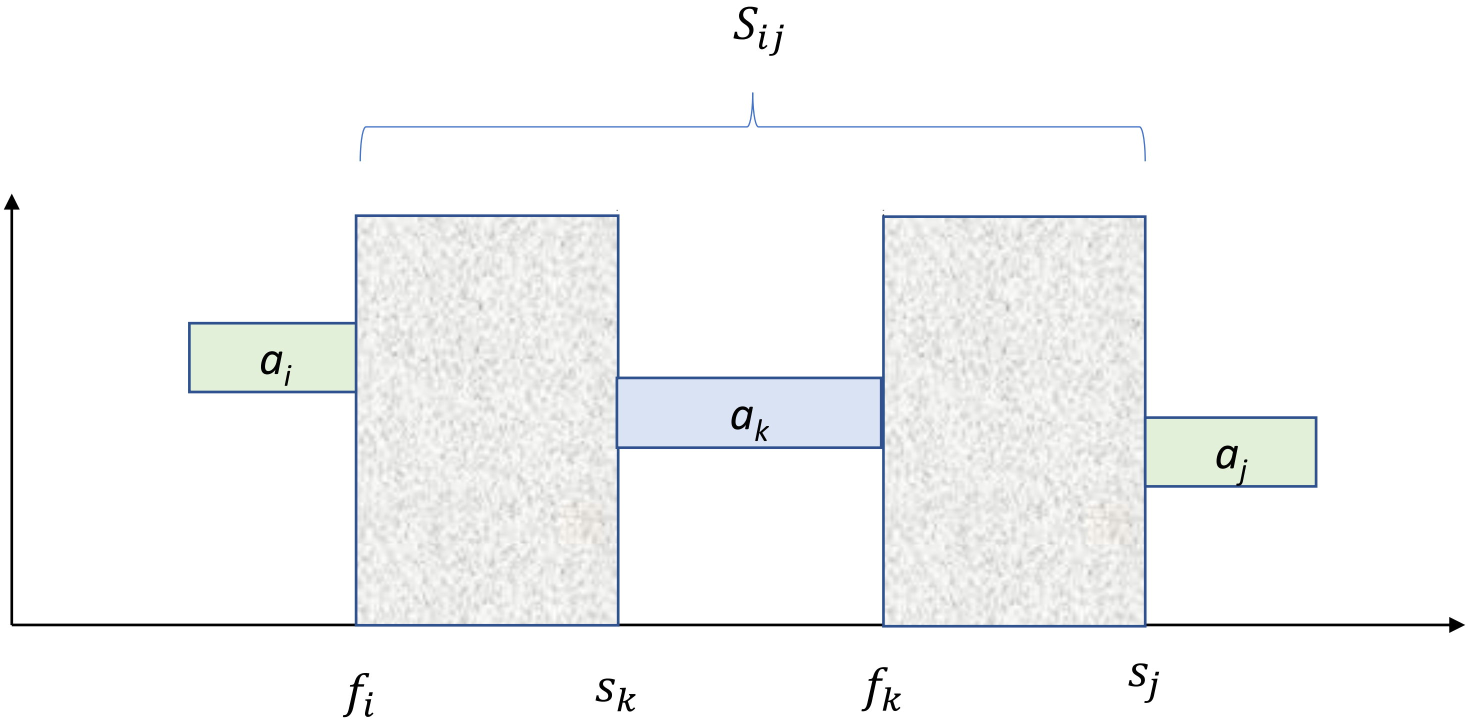
All activities that start and finish in this interval belong to S_{ij} .

Activity Selection



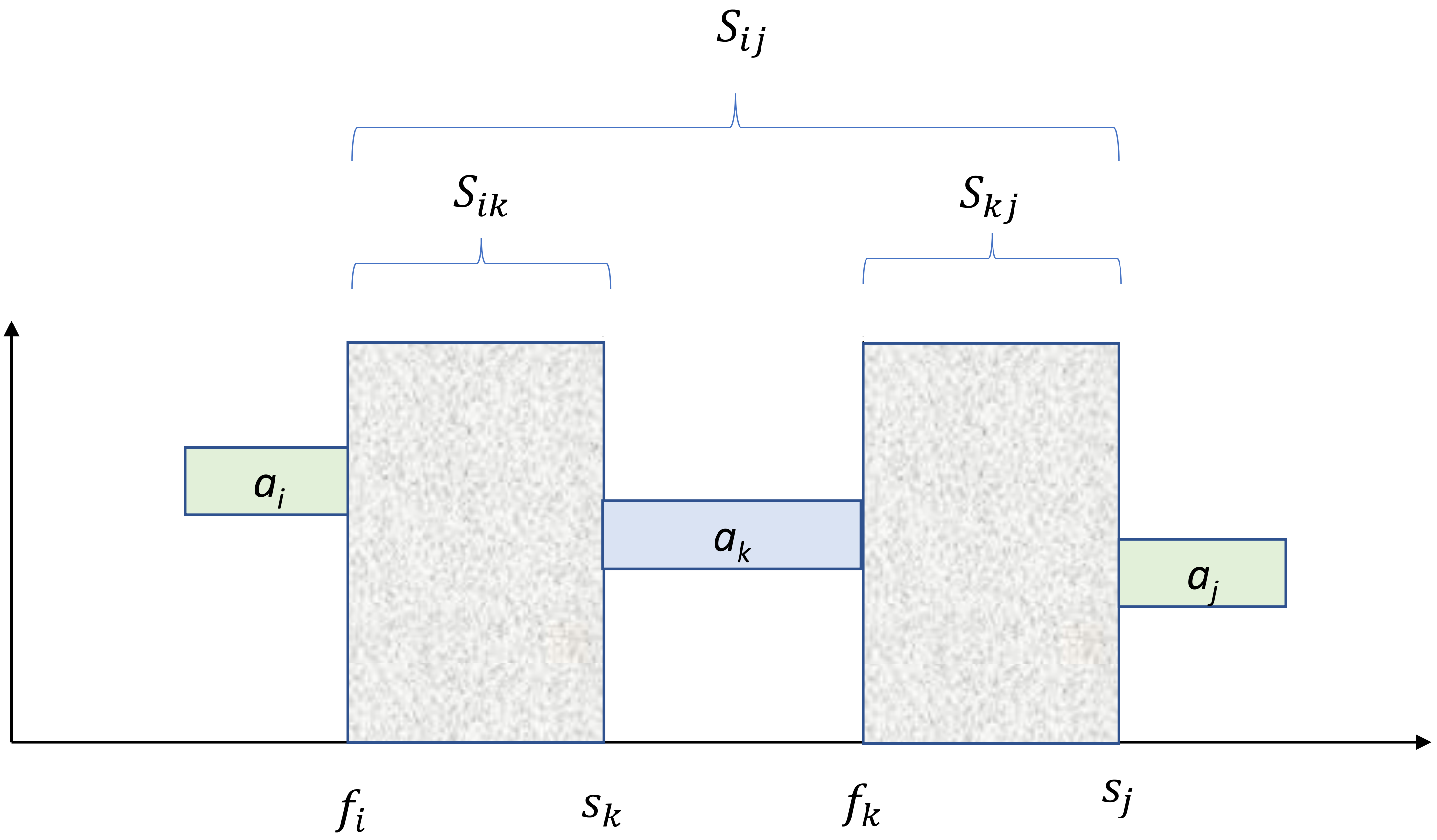
Suppose the optimal solution for S_{ij} includes activity a_k.

Activity Selection



Suppose the optimal solution for S_{ij} includes activity a_k, then it must include the optimal solutions for the shaded intervals as well.

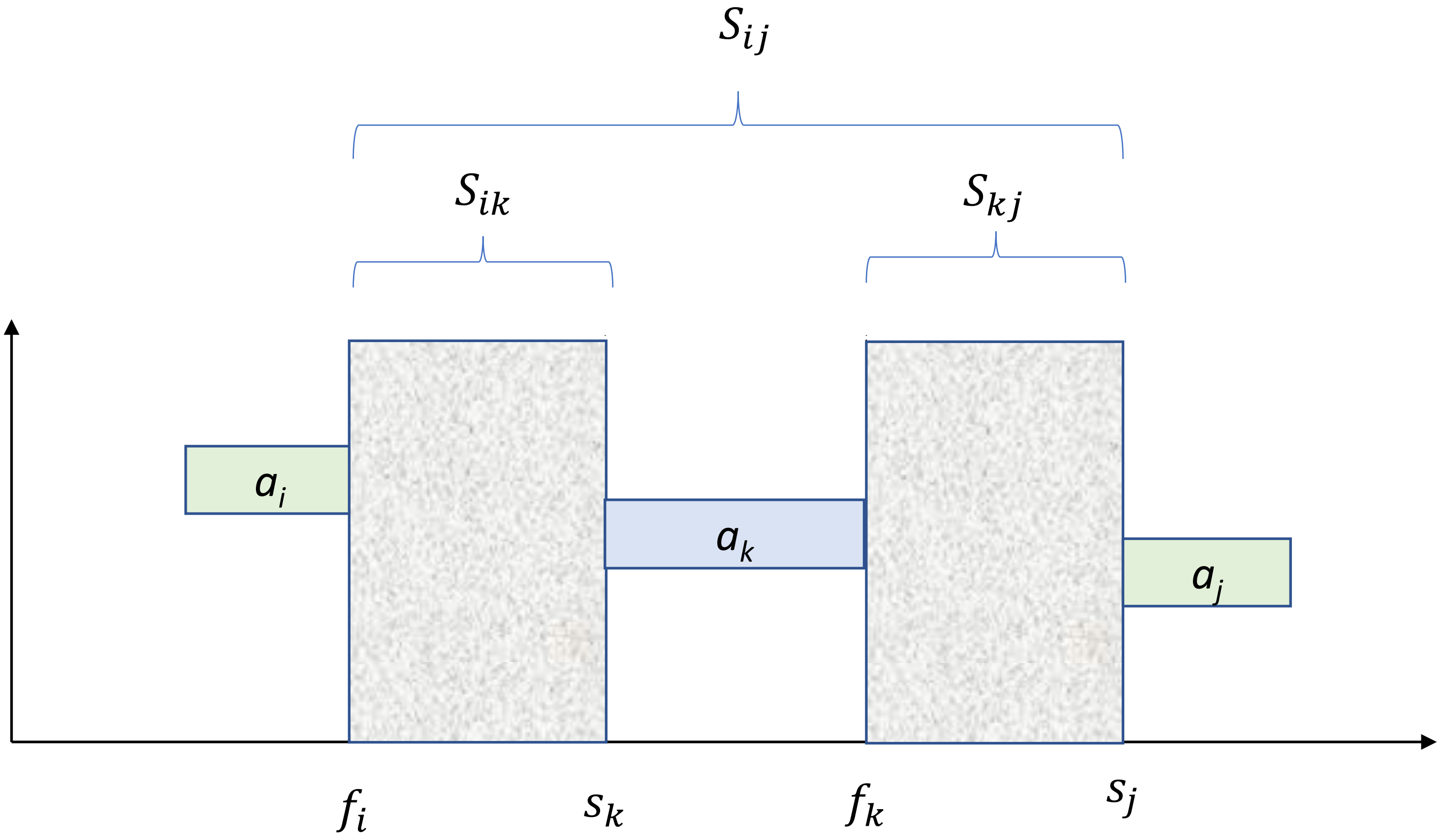
Activity Selection



Suppose the optimal solution for S_{ij} includes activity a_k, then it must include the optimal solutions for the shaded intervals as well.

Activity Selection

$|A_{ij}| = |A_{ik}| + |A_{kj}| + 1$



Suppose the optimal solution for S_{ij} includes activity a_k, then it must include the optimal solutions for the shaded intervals as well.

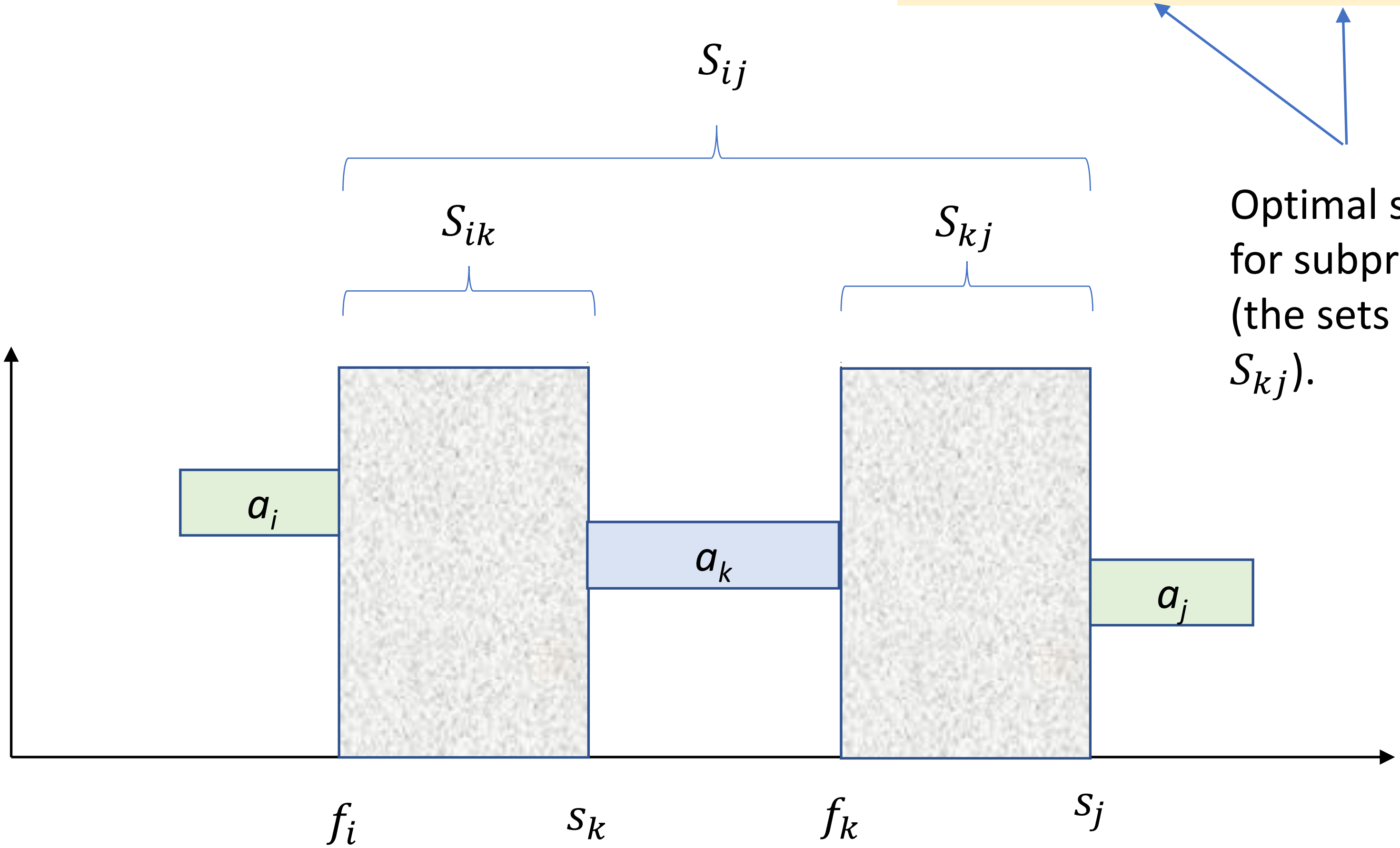
Activity Selection

Optimal solution for the set S_{ij}

Counting a_k

$$|A_{ij}| = |A_{ik}| + |A_{kj}| + 1$$

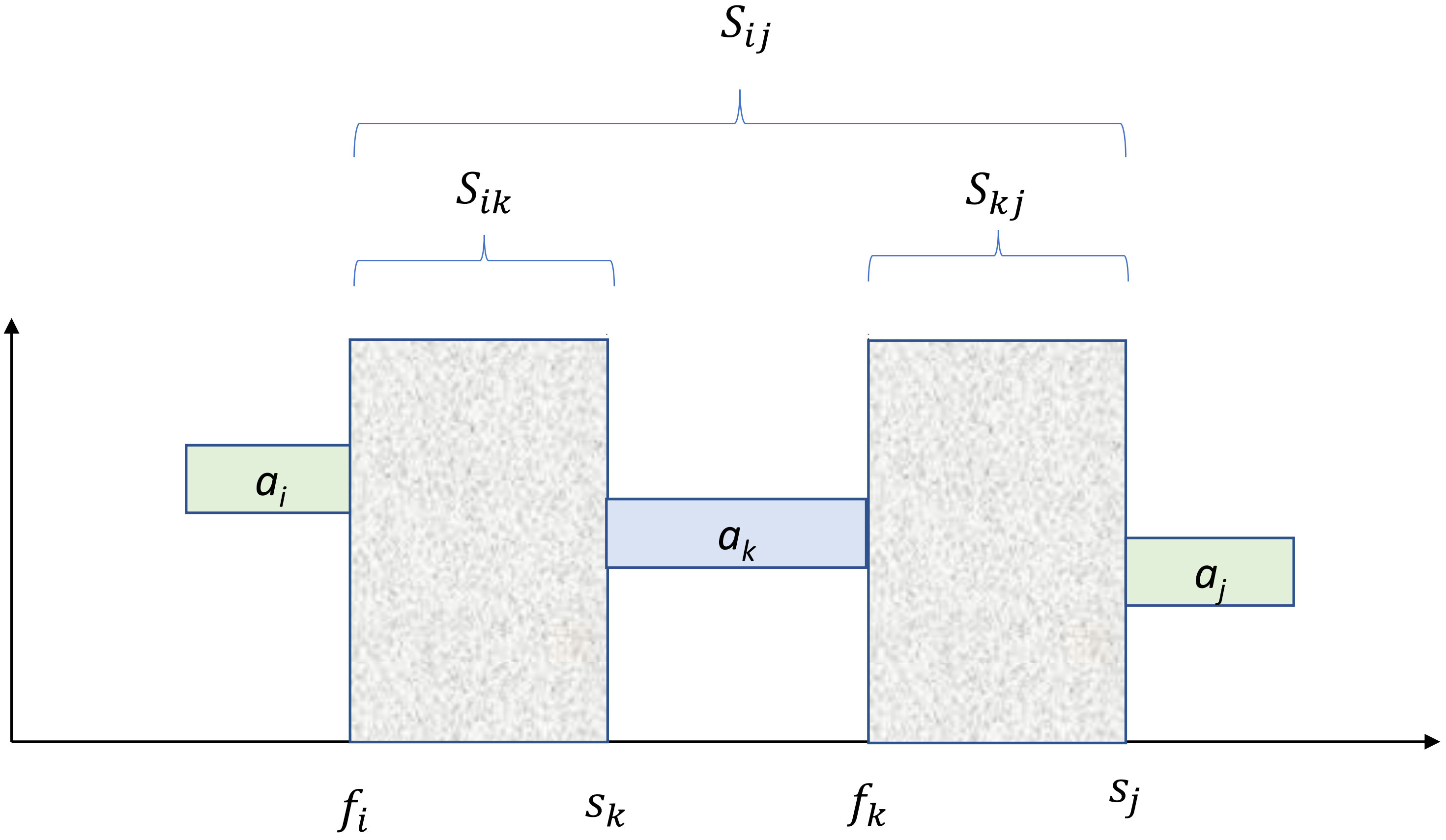
Optimal solution for subproblems (the sets S_{ik} and S_{kj}).



Suppose the optimal solution for S_{ij} includes activity a_k, then it must include the optimal solutions for the shaded intervals as well.

Activity Selection

Check your understanding
Is S_{ij} = S_{ik} ∪ S_{kj} ∪ {a_k} ?

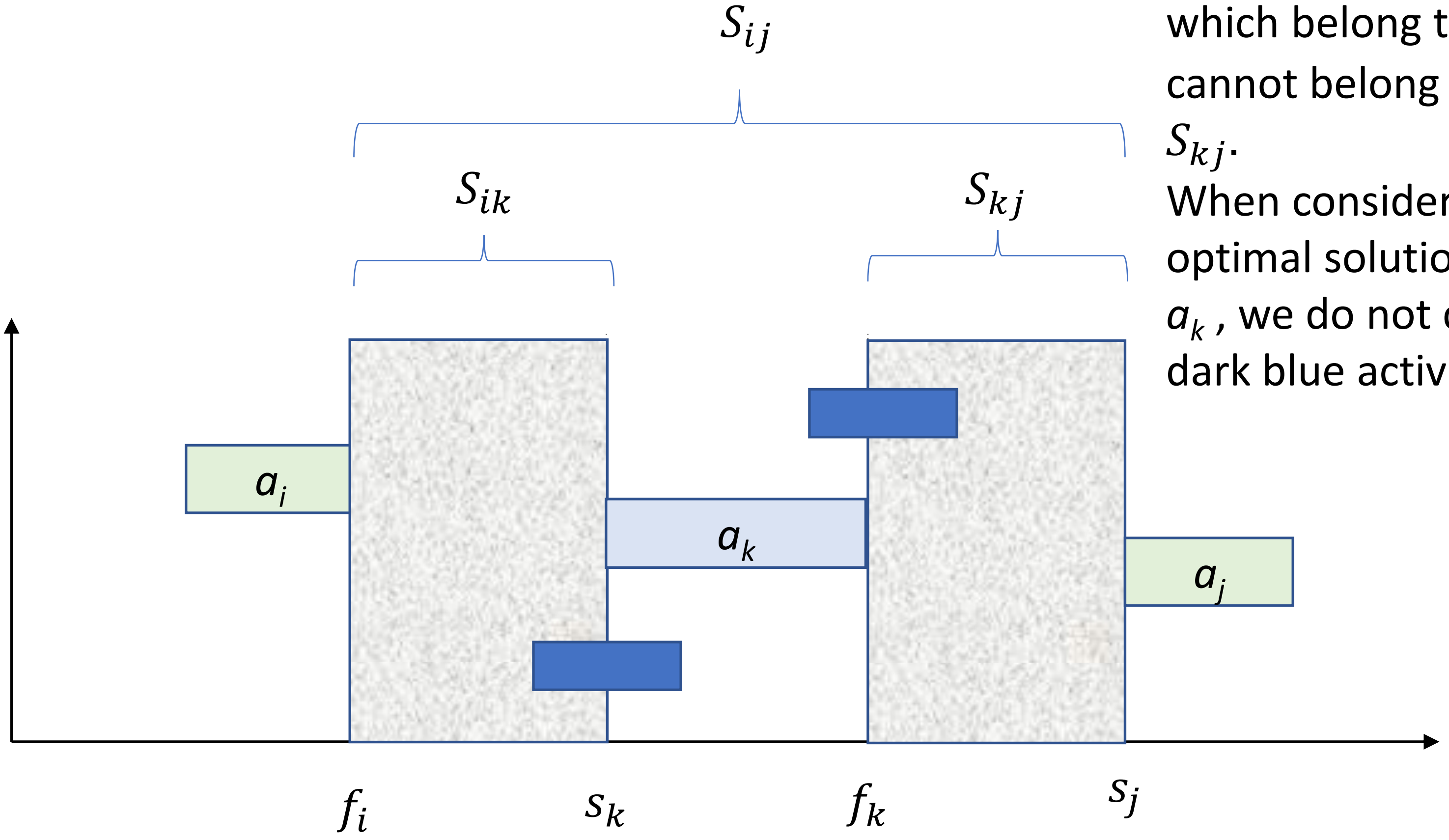


Suppose the optimal solution for S_{ij} includes activity a_k, then it must include the optimal solutions for the shaded intervals as well.

Activity Selection

Exercise:
Is S_{ij} = S_{ik} U S_{kj} U {a_k} ?

Not necessarily. There could be activities like the dark blue ones below which belong to S_{ij}, but cannot belong to S_{ik} U S_{kj}.
When considering an optimal solution that has a_k , we do not consider the dark blue activities.



Suppose the optimal solution for S_{ij} includes activity a_k, then it must include the optimal solutions for the shaded intervals as well.

Activity Selection – DP Solution

- Recursive definition:

  As before, we don't know which k would lead to the optimal solution, so we loop over all a_k in S_{ij} and select what leads to the maximum.

Let $c[i,j]$ be the size of the optimal solution for S_{ij}

$$c[i,j] = 0 \quad if \quad S_{ij} = \emptyset$$

$$c[i,j] = \max_{a_k \in S_{ij}} \{c[i,k] + c[k,j] + 1\} \quad if \quad S_{ij} \neq \emptyset$$

- As in the DP lecture, we could implement this by either a bottom-up approach or top-down approach with memoization.
- However, is this the best we can do?

Activity Selection – DP Solution

- Notes
  - The previous DP formalization can be simplified.
  - Furthermore, there is a more efficient DP solution than the previous one.

  - Exercise: Can you find a simplified recursive definition that won’t require solving two subproblems?

Activity Selection
A simpler solution

- Greedy strategy:
    - Instead of solving all the subproblems for each possible a_k, choose the activity a_k in a greedy way (before solving any of the subproblems!)

    - In this context, a possible greedy choice is to select an activity that would leave more space for the other activities.
        - The greedy choice we will use is based on the earliest finish time.
        - We will show other ways that do not work.

    - Note that the choice is made without considering the future choices, i.e., before solving any of the next subproblems.

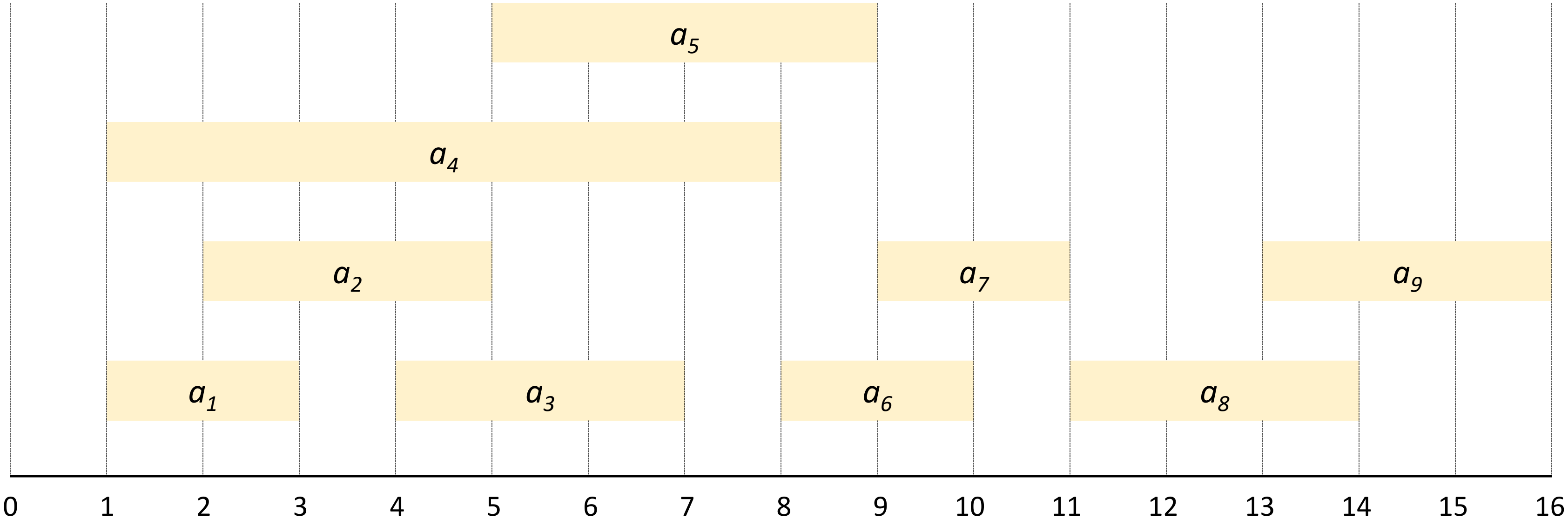
Activity Selection – Greedy Solution

- Greedy strategy:
  - **Choose the first activity to finish.**
  - As the activities are sorted by the finish time, this means that we select the first activity in the interval we are considering.

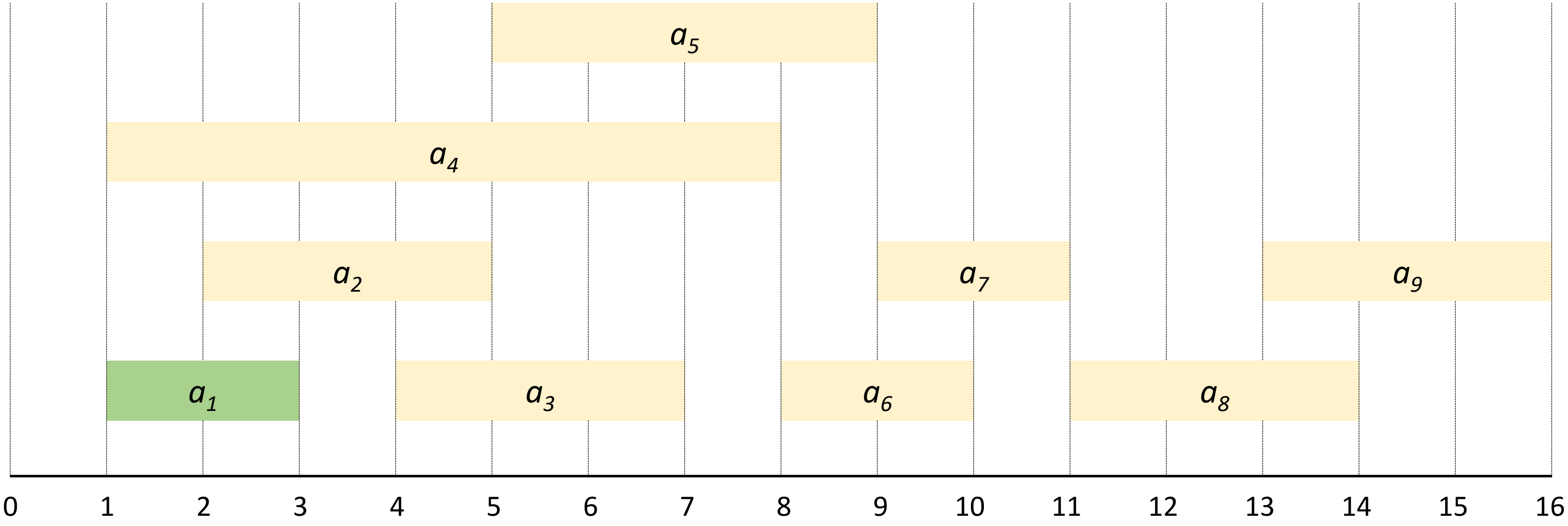
  - When the first activity is selected for the optimal solution, note that only one subproblem remains.

Activity Selection – Greedy Solution

Back to our example:



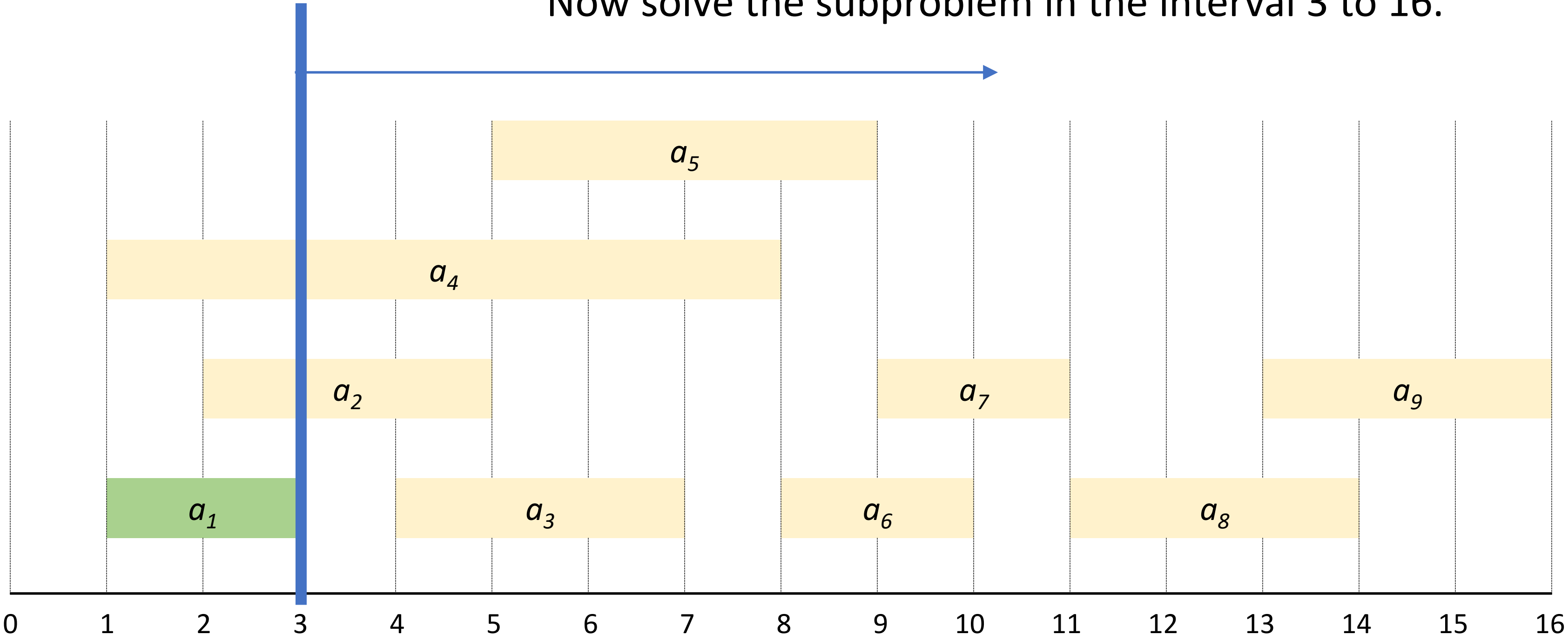
Activity Selection – Greedy Solution



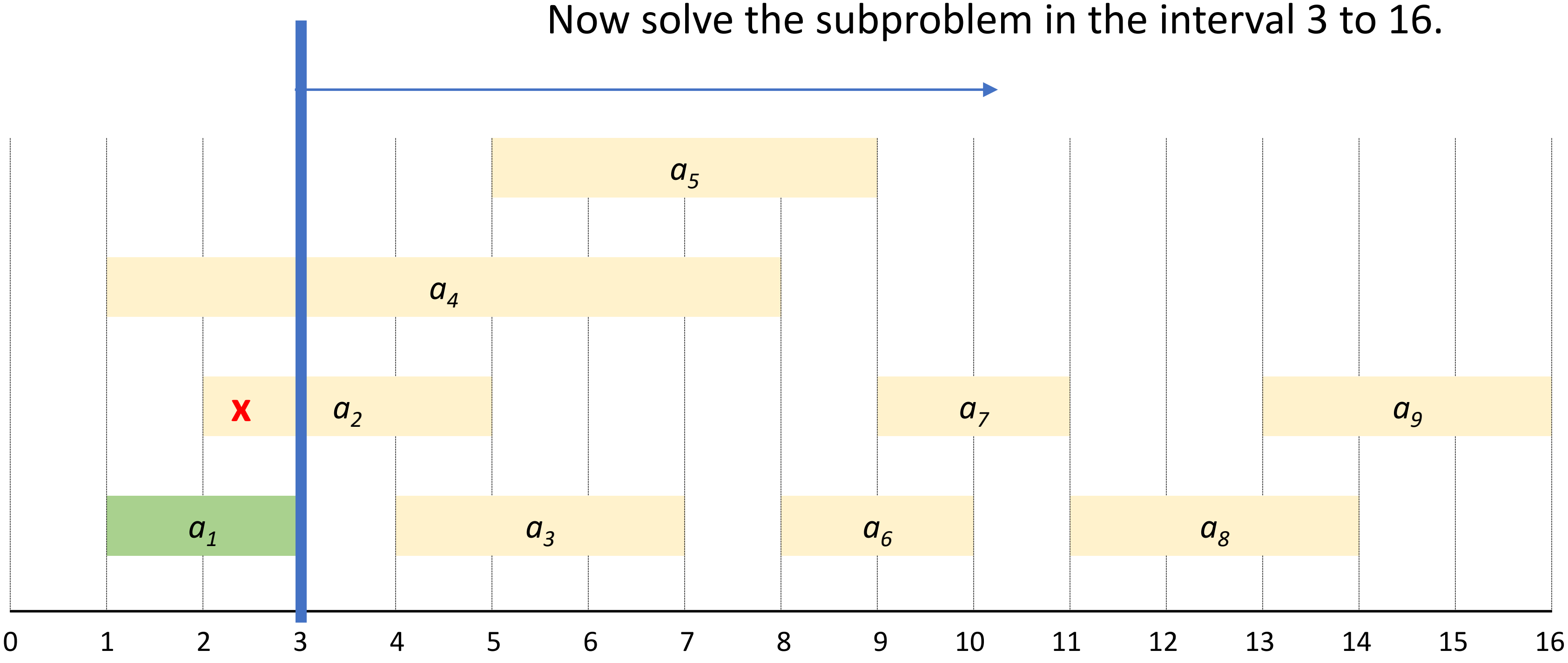
1- Select a_1 as it has the earliest finish time.

Activity Selection – Greedy Solution

Now solve the subproblem in the interval 3 to 16.

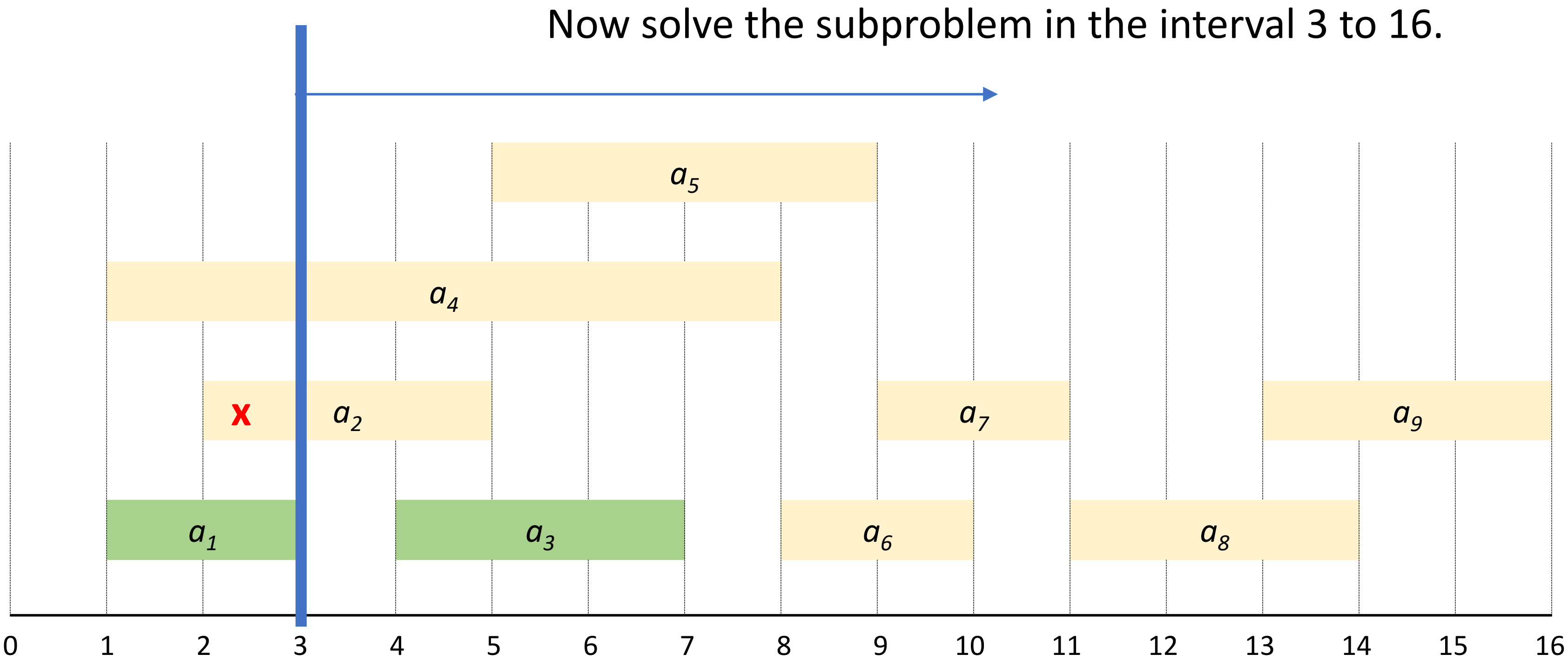


Activity Selection – Greedy Solution



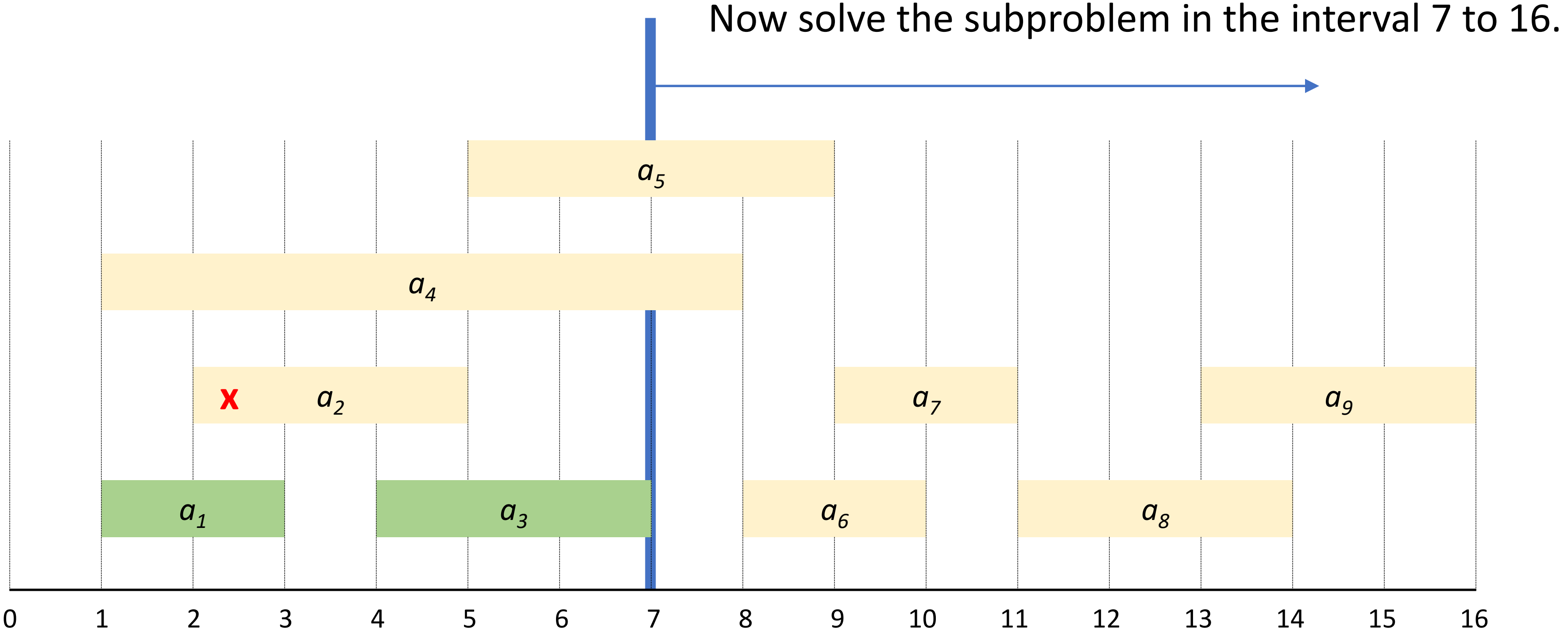
2- The next activity with earliest finish time is a_2, but we will skip it, as its start time < 3.

Activity Selection – Greedy Solution



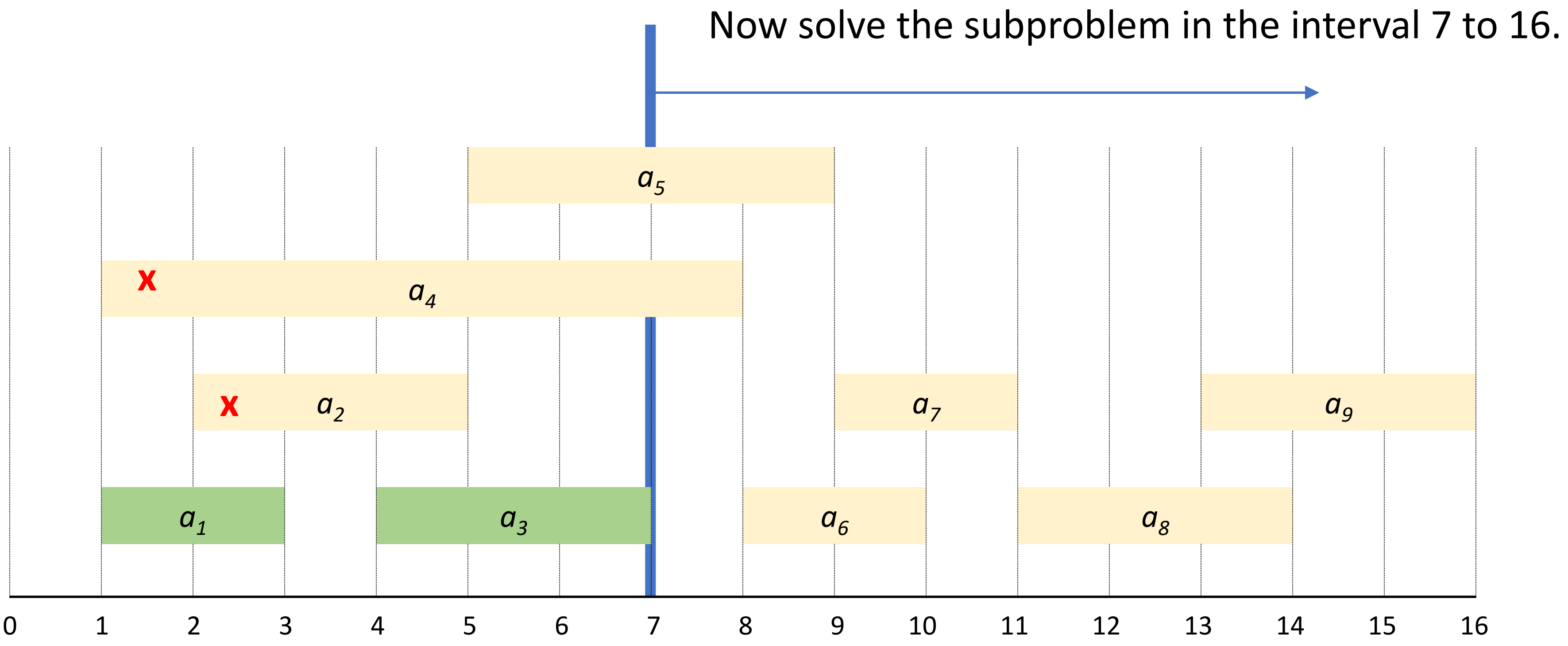
3- Select a_3 as it has the next earliest finish time, and it start time >= 3

Activity Selection – Greedy Solution



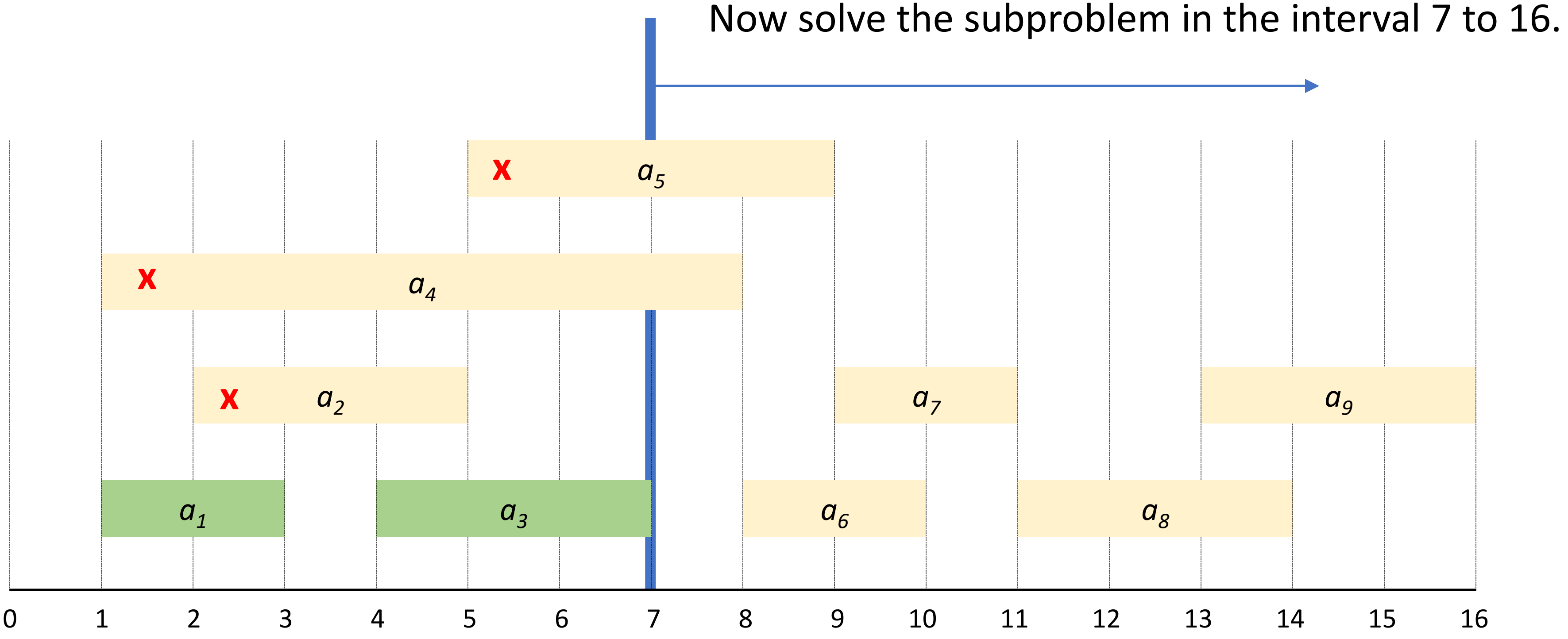
Now solve the subproblem in the interval 7 to 16.

Activity Selection – Greedy Solution



4- The next activity with earliest finish time is a_4, but we will skip it, as its start time < 7.

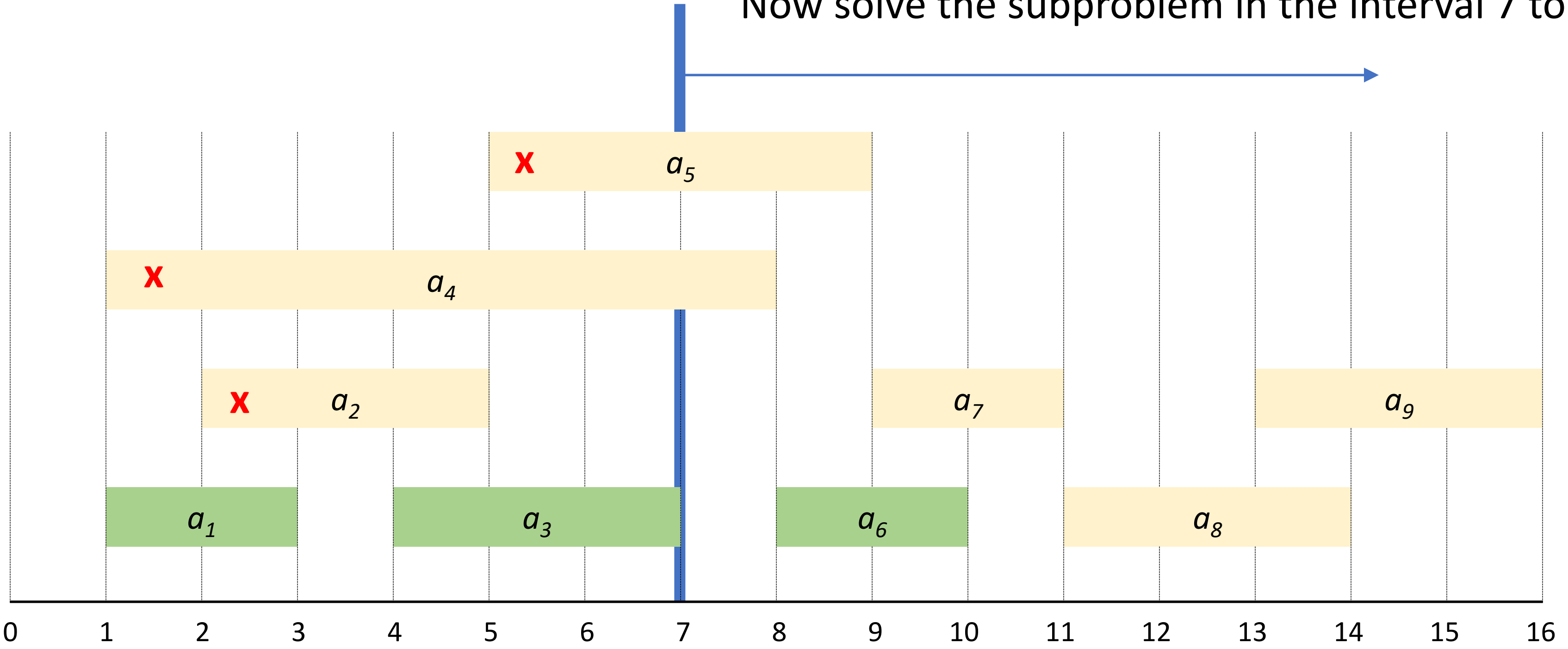
Activity Selection – Greedy Solution



5- The next activity with earliest finish time is a_5, but we will skip it, as its start time < 7.

Activity Selection – Greedy Solution

Now solve the subproblem in the interval 7 to 16.



6- Select a_6 as it has the next earliest finish time, and it start time >= 7

And so on.

Other options for the greedy choice?

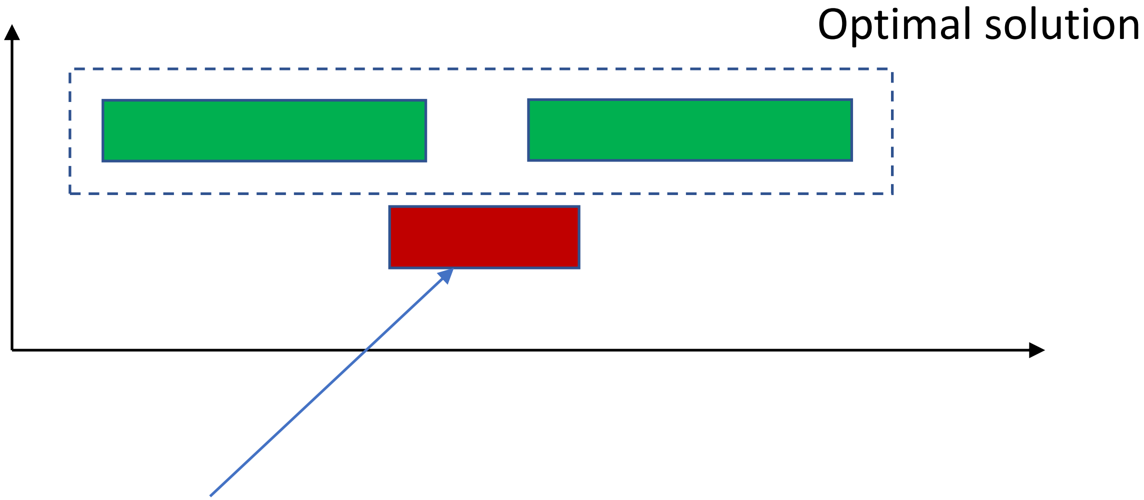
- In the previous example, we used "the earliest finish" criteria.
  - We will prove shortly that it will lead to an optimal solution.

- This does not mean that any greedy criteria will work. For example, think about the following criteria:
  - Choose the shortest activity first.
  - Choose the activity which has the minimum number of conflicts.

Both of them won't work.

Other Greedy Criteria
Choosing shortest activity first?

- Choosing the shortest activity first will not necessarily lead to an optimal solution.
- Consider this counter example:

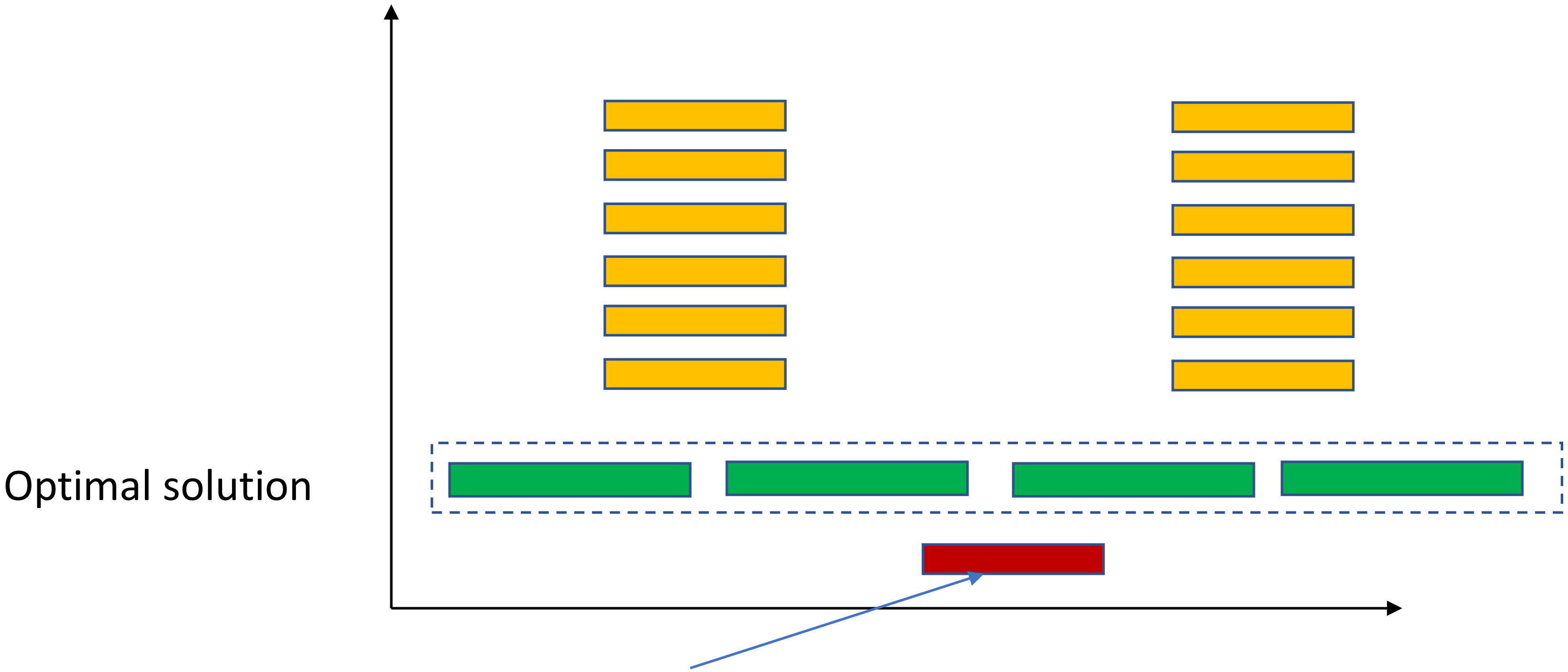


Choosing the activity with the shortest duration will not lead to an optimal solution.

Other Greedy Criteria

Choosing activity with minimum number of conflicts?

- This will not necessarily lead to an optimal solution as well. Counter example:



Choosing the activity with minimum number of conflicts will not lead to optimal solution.

Greedy Criteria

- This implies that not any criteria that just makes sense would work.

- We need to prove that the strategy will lead to an optimal solution.

- In greedy algorithms, we prove two properties:
    - The greedy-choice property
        - The greedy choice will lead to an optimal solution.
    - The optimal substructure property
        - Combining the greedy choice with the optimal solution of the subproblem will lead to an optimal solution of the problem.

Activity Selection – Greedy Solution

- As we now consider one subproblem only, the notation can be simplified.
- Let S_k be the set of activities that start after activity a_k finishes.

$$S_k = \{a_i \in S : s_i \geq f_k\}$$

- Optimal substructure property:
    - If a_1 is part of the optimal solution, then the optimal solution must contain the optimal solution of S_1. (This can be proven by a simple contradiction)
- Next, we will prove that the greedy choice will lead to an optimal solution.

Activity Selection – Greedy Solution Proof of Optimality [CLRS]

Theorem:

If S_k is nonempty and a_m has the earliest finish time in S_k, then a_m is included in some optimal solution for S_k.

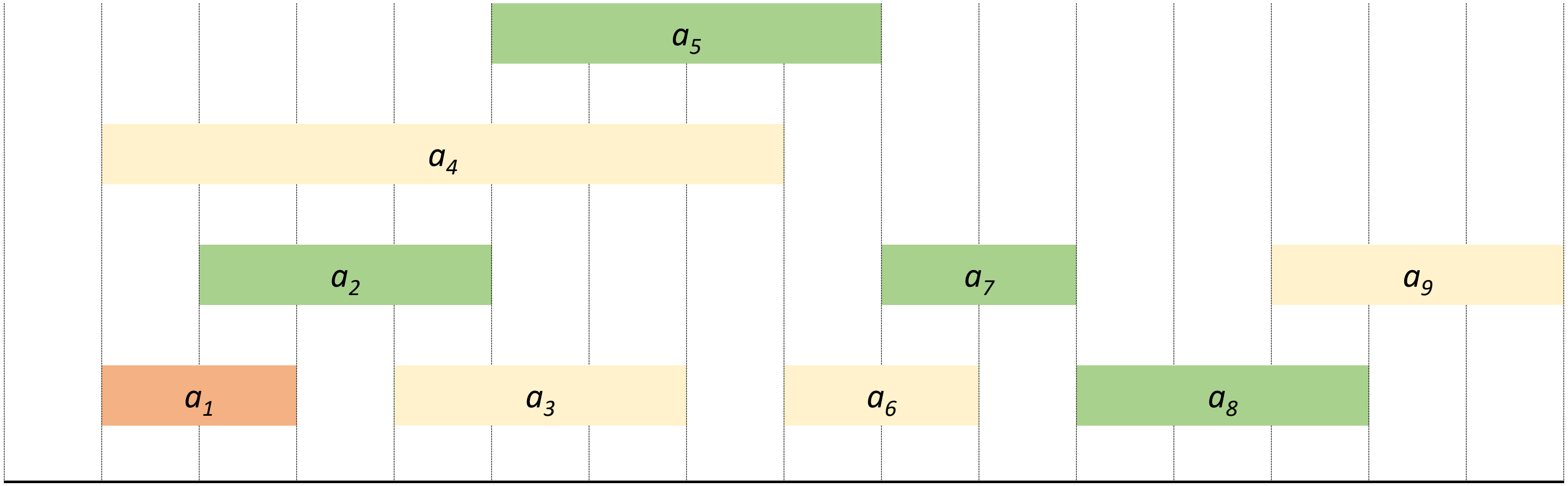
Proof:

- Let A_k be an optimal solution to S_k, i.e., A_k is a maximum-size subset of mutually compatible activities in S_k.
- Assume a_j is the activity with the earliest finish time in A_k.
- If $a_j = a_m$, done.
- If $a_j \neq a_m$
    - Let $A'_k = A_k - \{a_j\} \cup \{a_m\}$ // include a_m instead of a_j
    - The activities in A'_k must be non-overlapping, as the activities of the optimal solution A_k must be non-overlapping, and $f_m \leq f_j$.
    - Therefore $|A'_k| = |A_k|$ = the size of the maximum-size subset of mutually compatible activities in S_k, i.e, a_m is part of a maximum-size subset.

Activity Selection – Greedy Solution Proof of Optimality (Intuition)

Proof illustration via an example:

Note: you cannot use examples to prove a claim. This is for illustration.

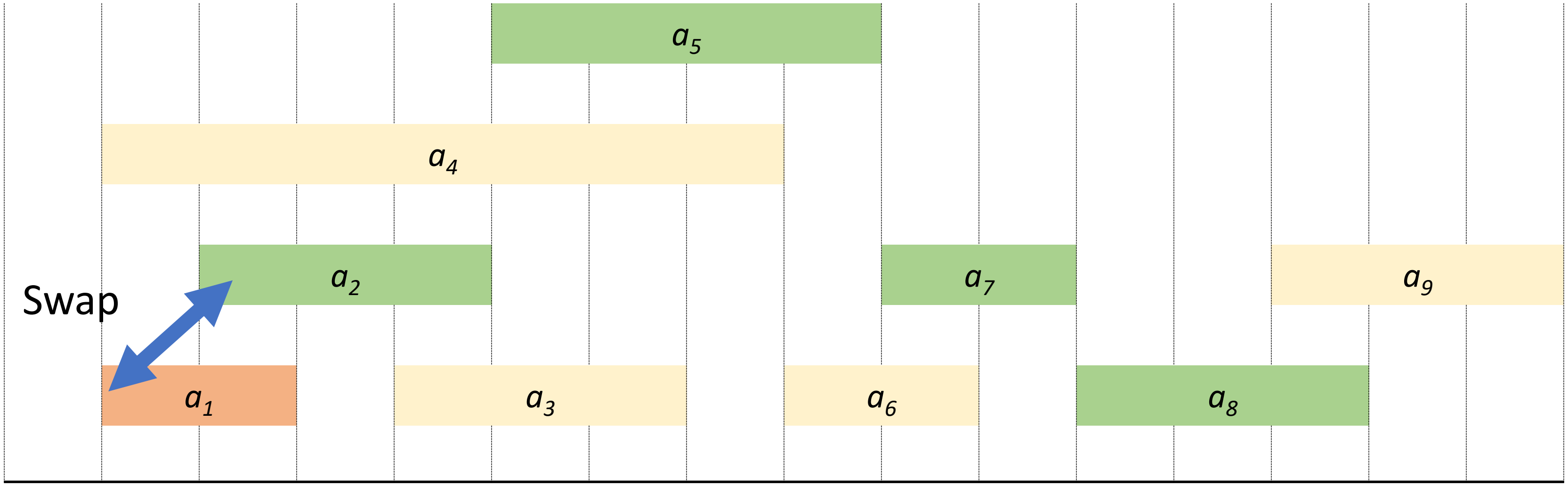


S = {a_1, …, a_9}

Given an optimal solution A = {a_2, a_5, a_7, a_8}, show that we can construct an optimal solution using the activity with the earliest time a_1

Activity Selection – Greedy Solution Proof of Optimality (Intuition)

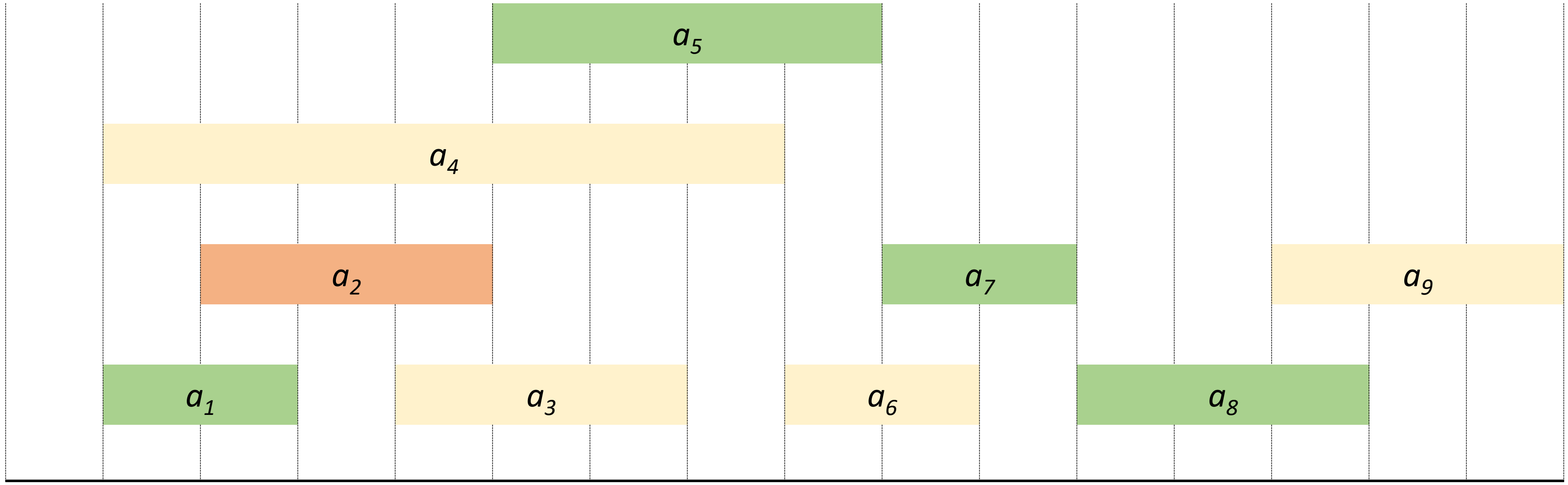
Proof illustration via an example:



S = {a_1, …, a_9}
Given an optimal solution A = {a_2, a_5, a_7, a_8}, show that we can construct an optimal solution using the activity with the earliest time a_1
As a_2 has the earliest time in A, let A'$_1$ = A$_1$ – {a_2} U {a_1} = {a_1, a_5, a_7, a_8}

Activity Selection – Greedy Solution Proof of Optimality (Intuition)

Proof illustration via an example:



A'$_1$ = {a_1, a_5, a_7, a_8} is an optimal solution as well.

Note that because a_1 finishes earliest, it was possible to swap it with a_2 without making overlaps with any other activity. **This is generalized in the proof.**

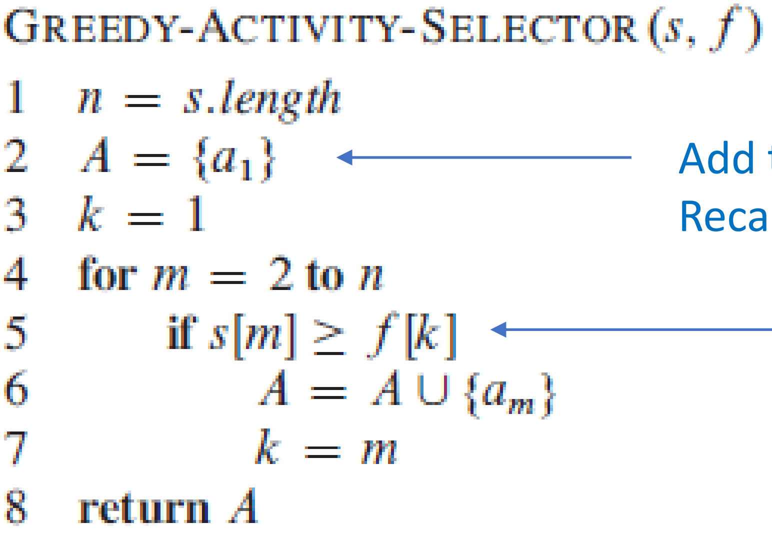
Activity Selection – Greedy Solution Implementation

- Assuming the activities are sorted by the finish times already, the running time of the greedy solution will be θ(n).

- If the activities are not sorted, the cost will be O(n lg n).

Activity Selection – Greedy Solution Implementation

- Iterative implementation [CLRS]



```
GREEDY-ACTIVITY-SELECTOR$(s, f)$
$n = s.length$
$A = \{a_1\}$
$k = 1$
for $m = 2$ to $n$
    if $s[m] \geq f[k]$
        $A = A \cup \{a_m\}$
        $k = m$
return $A$
```

Add the first activity a_1 to A
Recall that the activities are sorted by finish times.

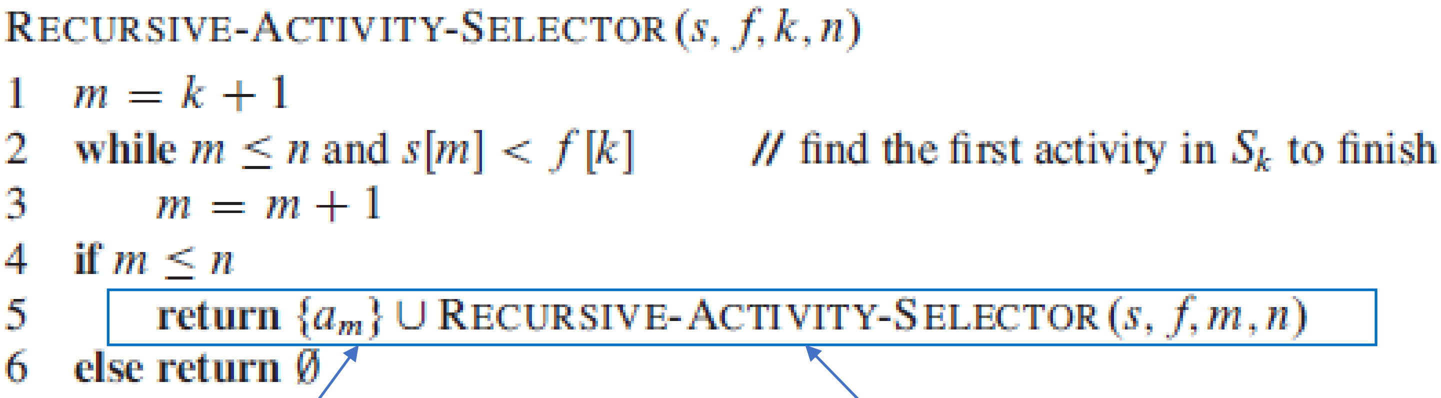
Find the first activity that starts after f[k]

Activity Selection – Greedy Solution Implementation

- Recursive implementation [CLRS]

  Assume having a dummy activity a_0 with $f_0 = 0$

  First call: Recursive-Activity-Selector(s, f, 0, n)



```
RECURSIVE-ACTIVITY-SELECTOR$(s, f, k, n)$
$m = k + 1$
while $m \leq n$ and $s[m] < f[k]$        // find the first activity in $S_k$ to finish
    $m = m + 1$
if $m \leq n$
    return $\{a_m\} \cup$ RECURSIVE-ACTIVITY-SELECTOR$(s, f, m, n)$
else return $\emptyset$
```

Greedy choice

One subproblem to solve

The Greedy Strategy – Summary [CLRS]

- Express the optimization problem as a problem in which we can make a choice, then solve one subproblem.

- Show that there is always an optimal solution that includes the greedy choice.

- Show that combining the optimal solution of the subproblem with the greedy choice leads to an optimal solution.

The Greedy Strategy

- Greedy-choice property:
  - A globally optimal solution is reached by making locally optimal (greedy) choices (without considering solutions to subproblems).
  - In dynamic programming, the situation was different. We make a choice after finding the solutions to subproblems.
    - This is why the solutions of DP can be built in a bottom-up manner.
  - The greedy approach usually works in a top-down manner, as the subproblem is solved after making a choice.

The Greedy Strategy

- Optimal substructure
    - Recall: A problem has optimal substructure if the optimal solution incorporates optimal solutions to subproblems.

    - In the context of greedy algorithms, we show that combining
        - the greedy choice
        - the optimal solution to the subproblem that we have to solve after making the greedy choice

  will lead to an optimal solution to the problem.

Knapsack Problem

- To see the difference between greedy algorithms and dynamic programming, we will revisit the knapsack problem covered previously.

- We discussed dynamic programming solutions to 0-1 knapsack and unbounded knapsack last time.

- In this lecture, we will see a variant of this problem that can be solved by a greedy algorithm.

Knapsack Problem

Given a knapsack (bag) that can hold a weight of at most W, and n items to pick from.

Each item has weight w_i kg and is worth v_i dollars.

How to choose items to put in the knapsack, such that the total value of the items in the knapsack is maximized?

Different versions of this problem:

- Knapsack with repetition (Unbounded Knapsack)
  - There is no limit on the quantity of each item. An item can appear 0, 1 or more times.
- Knapsack without repetition (0-1 Knapsack)
  - Each item can appear at most once.
- Fractional Knapsack
  - We can take a fraction of an item.

Knapsack Example

Example [CLRS]:

W = 50

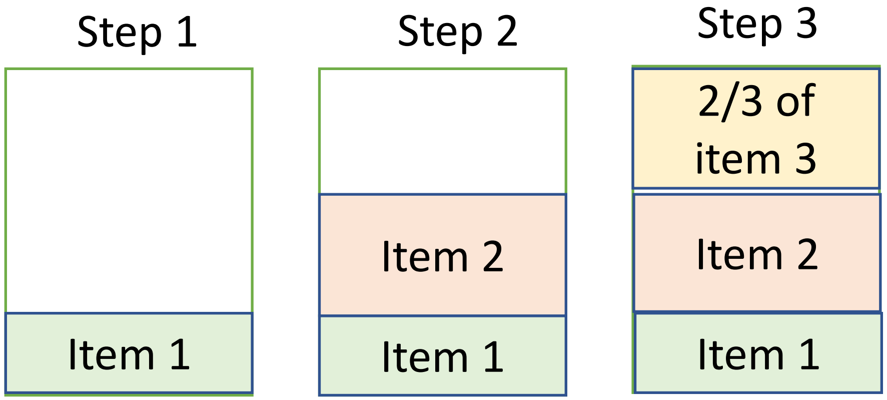
i	w_i	v_i
1	10	60
2	20	100
3	30	120

- For 0-1 knapsack, the optimal solution will be items 2 and 3.
  - The max total value will be 220.
- What if we are allowed to take fractions of items?
  - The optimal solution will be items 1 and 2, and $\frac{2}{3}$ of item 3.
  - The max total value will be 240

Fractional Knapsack

- Can be solved using a greedy strategy.
- Compute the value per kg of each item and sort the items accordingly.

i	w_i	v_i	v_i / w_i
1	10	60	6
2	20	100	5
3	30	120	4

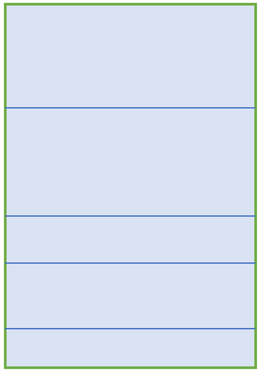


- Take as much as possible from the item with the highest value per kg, until its supply ends, or the knapsack is full.
- If there is room in the knapsack, move to the next item with the 2nd highest value per kg, and repeat.

Fractional Knapsack – Correctness Proof Sketch [informal]

- Prove that the greedy choice can lead to an optimal solution, then prove the optimal substructure.
- We will discuss the greedy choice first.
- Assume we have an optimal solution, in which we did not include as much as possible from item 1 (the item that has the highest v_i / w_i)

Assume this is the optimal solution with total value V



Knapsack

Item 1 has the highest value per kg, and some of it has not been included in the knapsack

Item 1

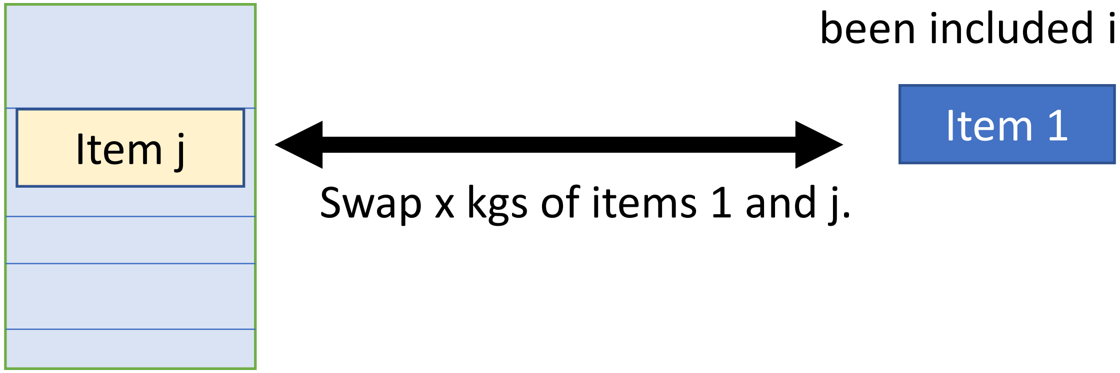
Consider the case, where items have distinct v_i/w_i

Fractional Knapsack – Correctness Proof Sketch [informal]

- If we replace x kgs of some item j in the optimal solution with x kgs of item 1 (j ≠ 1)

Assume this is the optimal solution with total value V

Item 1 has the highest value per kg, and some of it has not been included in the knapsack



Swap x kgs of items 1 and j.

Knapsack

New value of knapsack V' = V – x * $\frac{v_j}{w_j}$ + x * $\frac{v_1}{w_1}$

As $\frac{v_1}{w_1} > \frac{v_j}{w_j}$, then V' > V

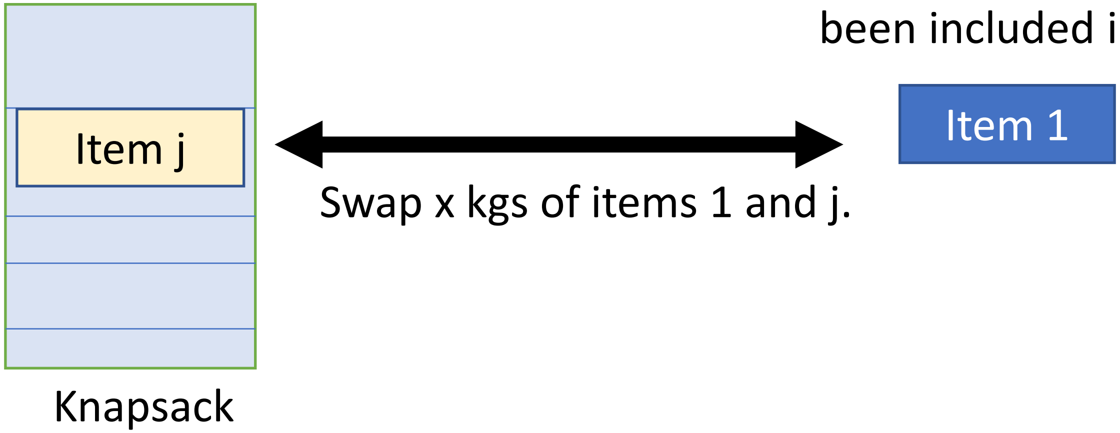
This is a contradiction. All of item 1 must be in the optimal solution.

Fractional Knapsack – Correctness Proof Sketch [informal]

- If we replace x kgs of some item j in the optimal solution with x kgs of item 1 (j ≠ 1)

Assume this is the optimal solution with total value V

Item 1 has the highest value per kg, and some of it has not been included in the knapsack



If we don't assume distinct v_i / w_i, then we can show that V' ≥ V. Including item 1 will never make the solution worse.

New value of knapsack V' = V – x * $\frac{v_j}{w_j}$ + x * $\frac{v_1}{w_1}$. As $\frac{v_1}{w_1} \geq \frac{v_j}{w_j}$, then V' ≥ V

Fractional Knapsack – Correctness Proof Sketch [informal]

Optimal substructure [CLRS]:

  - If the optimal solution for weight W contains (some of) item i, then if we remove x kgs of item i, what remains in the knapsack is the optimal solution for weight W – x using the other n – 1 items and w_i – x kgs from item i.

- The optimal substructure can be proven by contradiction.
  - If the optimal solution for weight W includes x kgs of item i and some non-optimal solution for weight W – x
  - Then, we can simply show that the solution of the problem for weight W cannot be optimal, because if we made the subproblem solution better, then we can use it to make the total value of the knapsack for weight W higher.

Fractional Knapsack – Correctness Proof Sketch [informal]

- Therefore, we can reach the optimal solution by combining the greedy choice and the optimal solution to the subproblem that we have to solve after making the greedy choice.

Greedy Strategy for 0-1 Knapsack?

- Note that the greedy approach won't work for the 0-1 knapsack.

i	$\mathbf{w_i}$	$\mathbf{v_i}$	$\mathbf{v_i / w_i}$
1	10	60	6
2	20	100	5
3	30	120	4

- The greedy strategy based on $\mathbf{v_i / w_i}$ for the 0-1 knapsack in the above example will lead to items 1 and 2 only, which is not optimal.

Huffman Codes

Data Compression

- Needed for many applications in practice
- Two categories:
  - **Lossless data compression**
    - Allows reconstructing the original data completely from the compressed version without any loss of information
    - Used when changes in the uncompressed data are not tolerable, e.g., text, programs, etc.
  - **Lossy data compression**
    - Allows reconstructing an approximation of the original data.
    - Used to compress audio, video and images.

Huffman Codes

- A technique for lossless data compression
- According to CLRS, it can achieve savings between 20% and 90%, depending on the characteristics of the input data.
- Uses a greedy method to find an optimal way for representing characters.

Designing a Binary Code

- How can characters be represented in binary?
    - Fixed-length codes
        - Each character is represented by a unique binary string (codeword) of a fixed length.
        - Example: ASCII, Unicode
    - Variable-length codes
        - The codewords representing the characters vary in their length.
        - Can be utilized in compression, by assigning short codewords to the characters that appear frequently.

Character	Fixed-length code	Variable-length code
A	00	0
B	01	111
C	10	110
D	11	10

Designing a Binary Code

- How to decode when using variable-length codes?
  - While encoding is straightforward, the decoding might not result into a single result if the code is not designed carefully.
  - Example

    Assume the codewords representing {A, B, C, D} are {1, 10, 110, 111}, how to decode the string 1110?

    Both AAB and AC are possible (ambiguity).

To avoid this, we use **prefix codes,** in which no codeword is a prefix of any other codeword.

These are known also as prefix-free codes.

Prefix Codes

- No codeword can be a prefix of any other codeword.
- Example codewords: {0, 10, 110, 111}
  - Given a string 001010111110, it's straightforward to decode it.

001010111110

001010111110

001010111110

001010111110

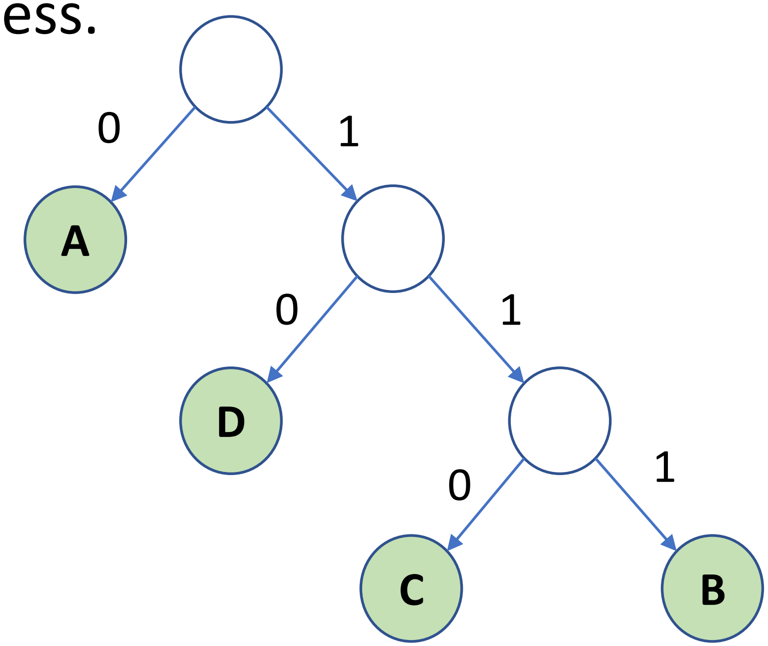
001010111110

001010111110

Prefix Codes

- Can be represented by a binary tree.
- Each path from the root to the leaves generates a codeword.
- Helps during the decoding process.

Character	Codeword
A	0
B	111
C	110
D	10



Example: 001010111110

Prefix Codes

Given a prefix code tree T, the number of bits needed to encode a file can be calculated as:

$$B(T) = \sum_{c \in C} d_T(c) \, * \, c.freq$$

Length of codeword representing c

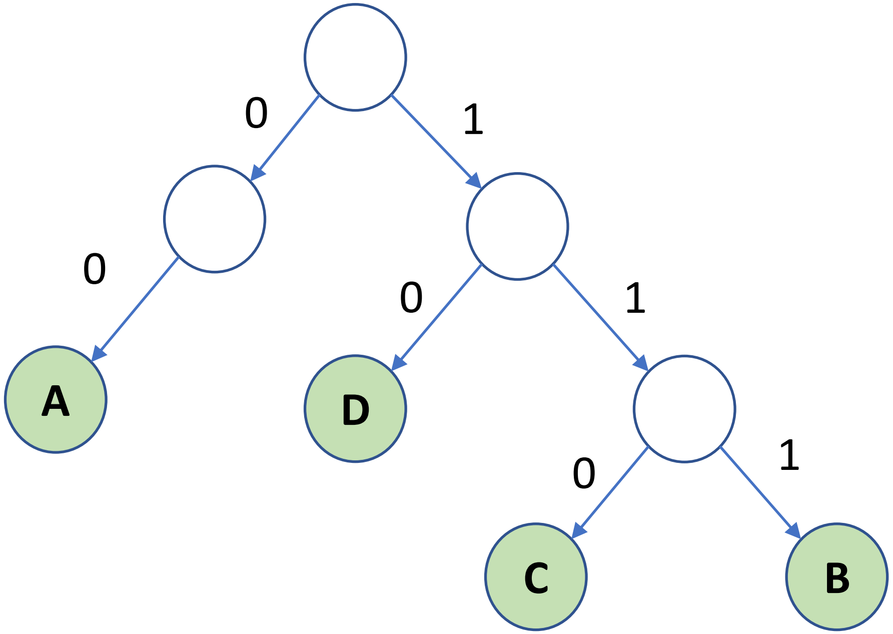
Frequency of character c

Goal: Find a prefix code that would minimize the above for a given file.

Trees of Optimal Prefix Codes

- Binary trees corresponding to optimal prefix codes must be **full** binary trees.
- The number of leaves should be equal to the alphabet size |C| and the number of internal nodes should be
  |C| - 1.

- These properties can be proven formally.
- These are necessary conditions but not sufficient for optimality. We will use Huffman's algorithm to get an optimal solution.



For example, this cannot correspond to an optimal code. Why?

Huffman Codes

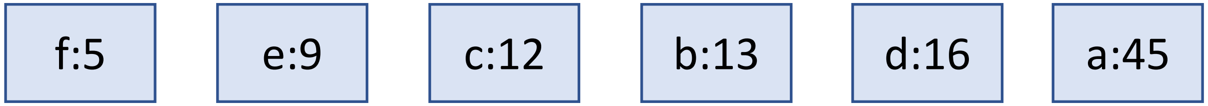
- Huffman codes: Optimal prefix codes
- To illustrate the algorithm, we will trace an example first.
- The following example is from CLRS

	a	**b**	**c**	**d**	**e**	**f**
Freq.	45	13	12	16	9	5

Huffman Codes

Example from CLRS

	a	**b**	**c**	**d**	**e**	**f**
Freq.	45	13	12	16	9	5

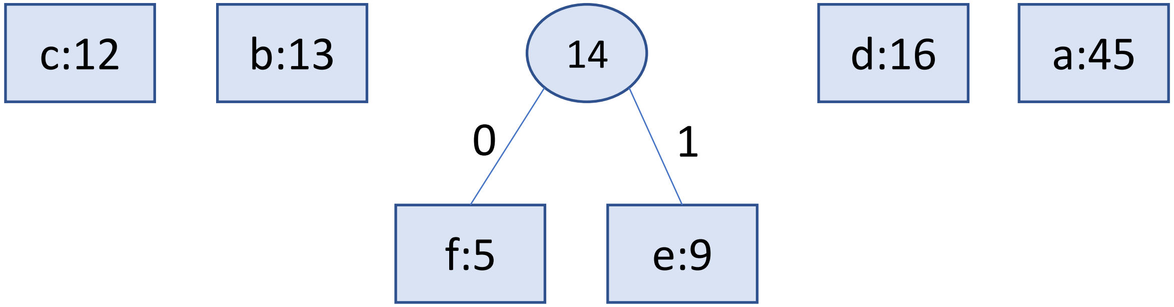


f:5 e:9 c:12 b:13 d:16 a:45

Huffman Codes

Example from CLRS

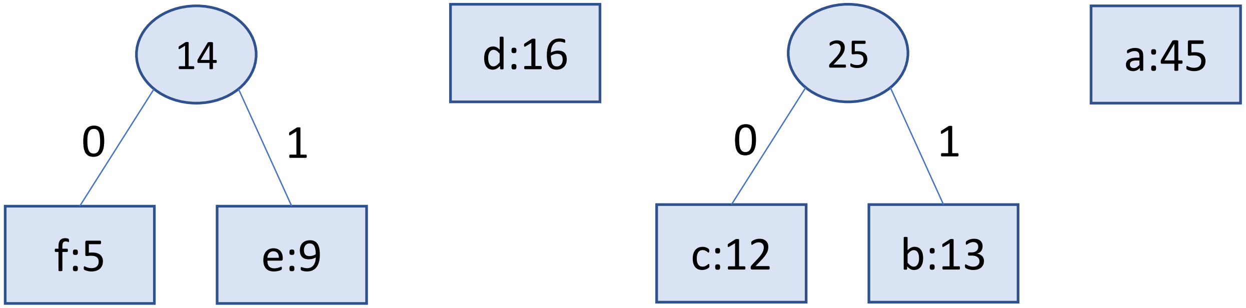
	a	b	c	d	e	f
Freq.	45	13	12	16	9	5



Huffman Codes

Example from CLRS

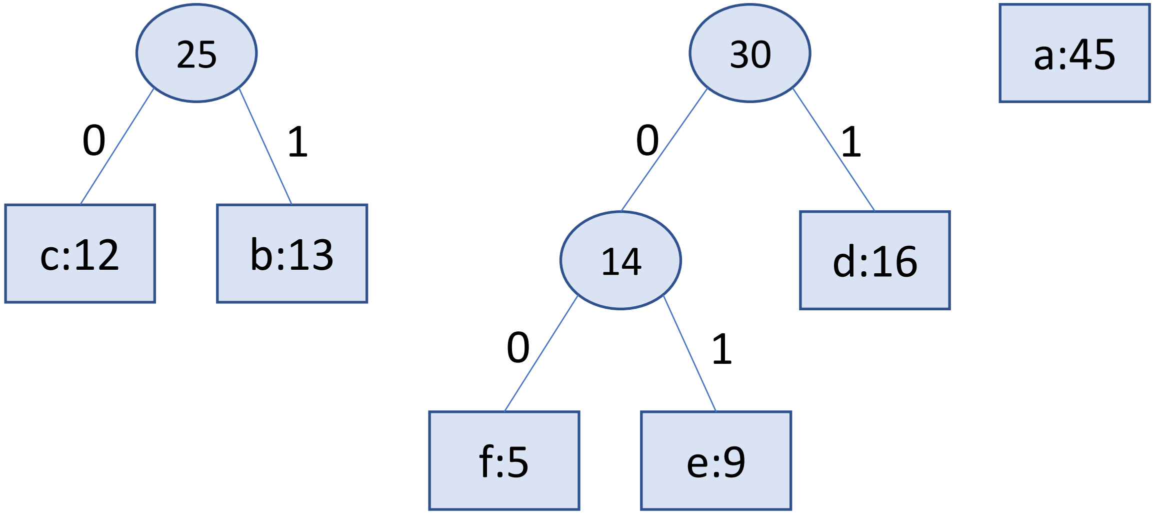
	a	**b**	**c**	**d**	**e**	**f**
Freq.	45	13	12	16	9	5



Huffman Codes

Example from CLRS

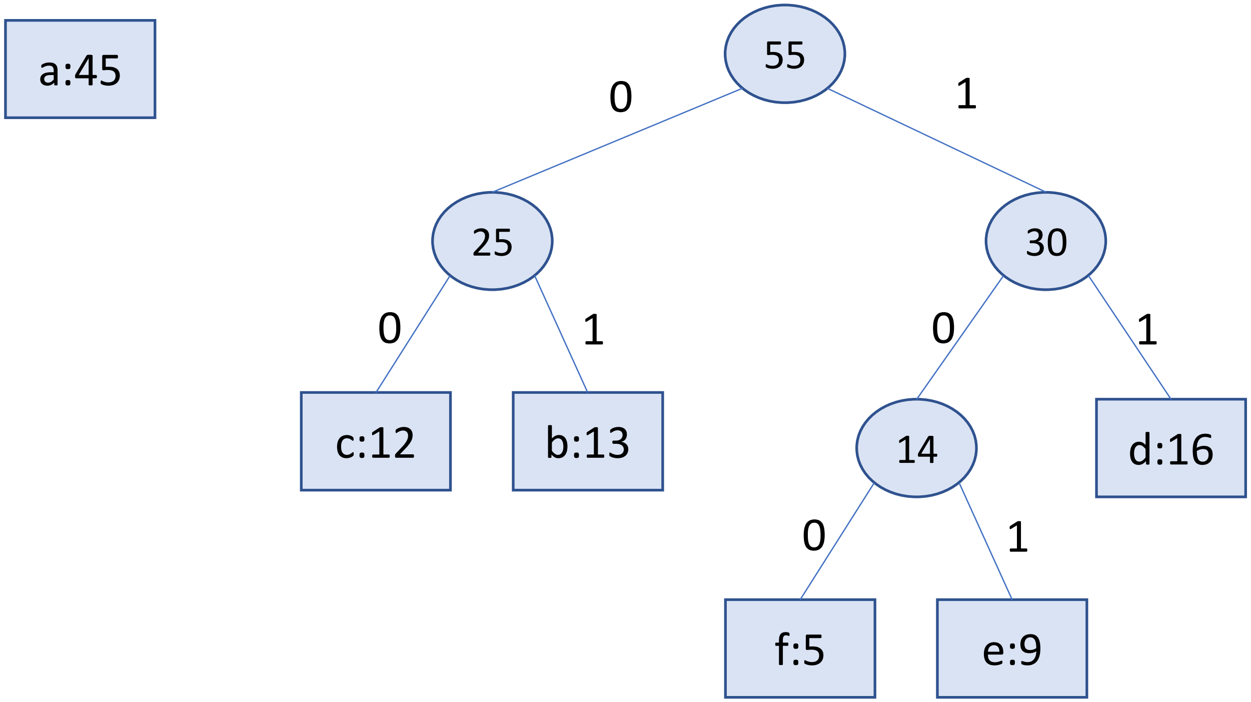
	a	**b**	**c**	**d**	**e**	**f**
Freq.	45	13	12	16	9	5



Huffman Codes

Example from CLRS

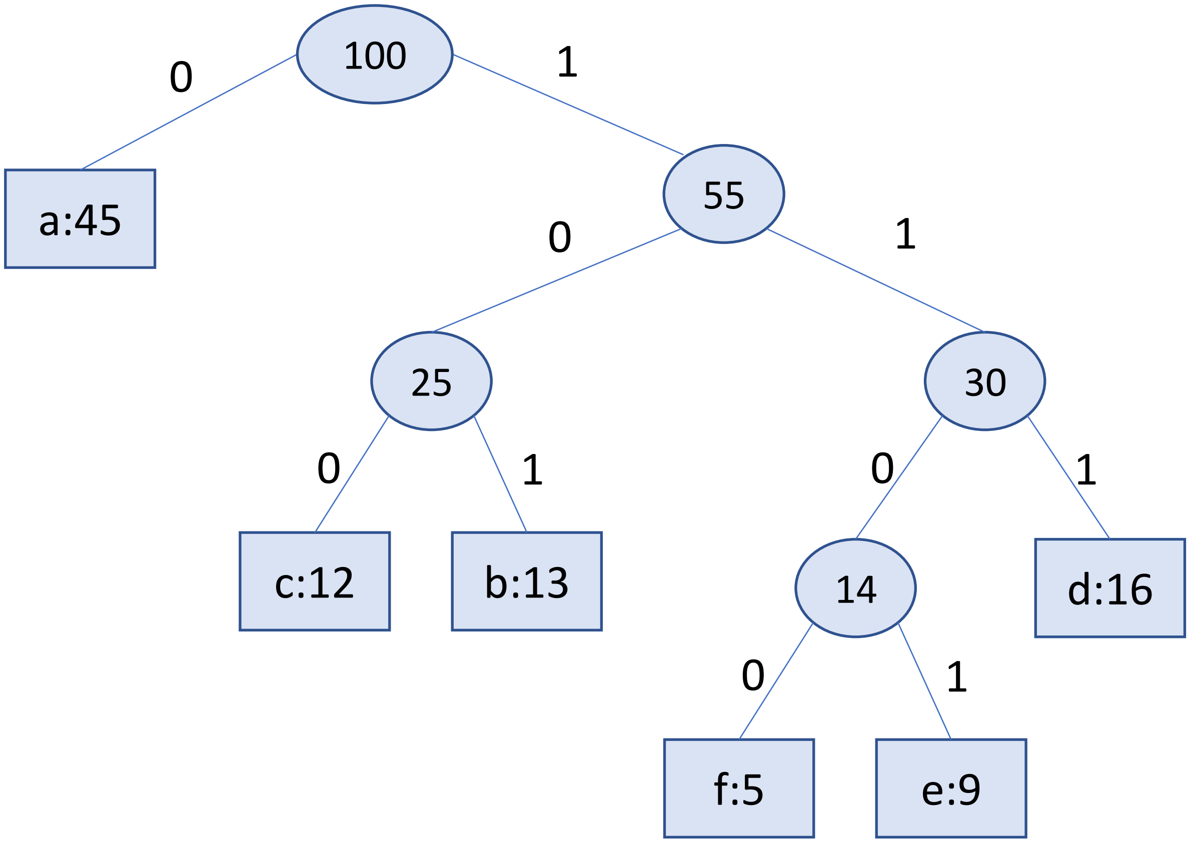
	a	b	c	d	e	f
Freq.	45	13	12	16	9	5



Huffman Codes

Example from CLRS

	a	**b**	**c**	**d**	**e**	**f**
Freq.	45	13	12	16	9	5

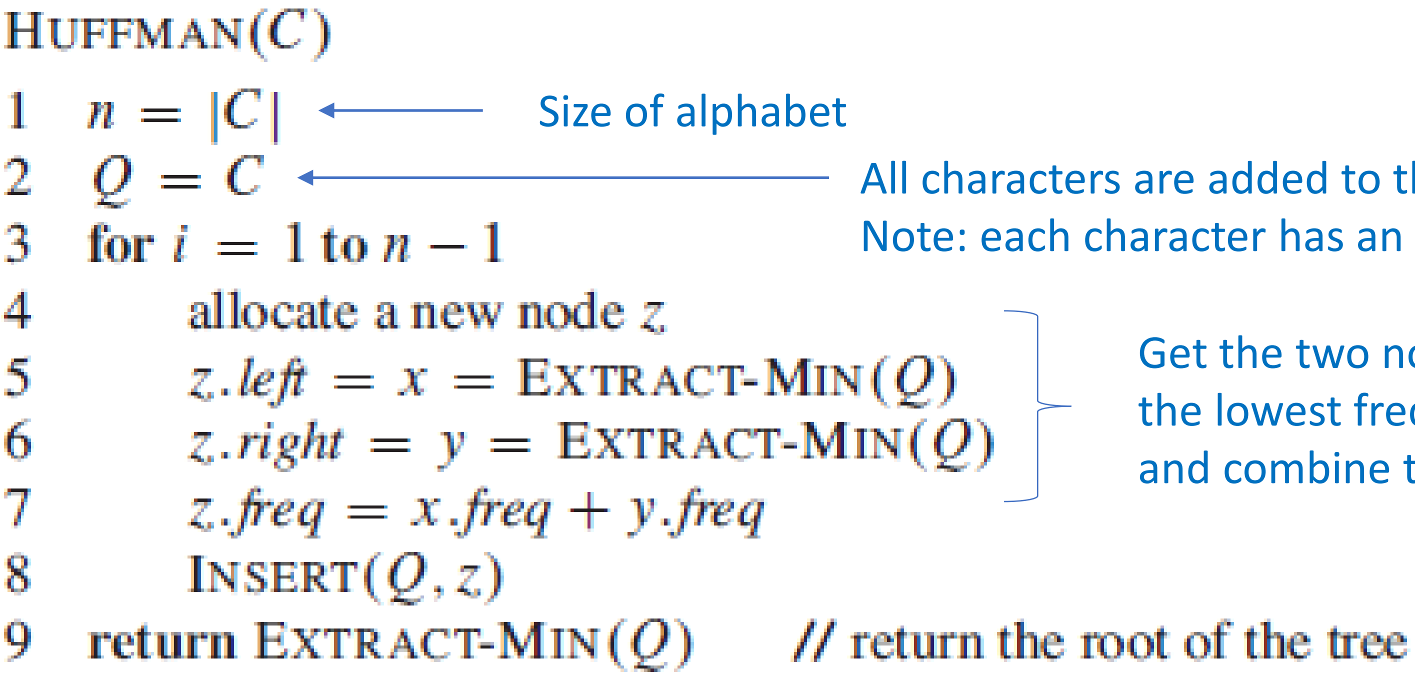


Huffman Codes

- How to implement the previous algorithm efficiently?
  - Need a data structure that supports minimum extraction and insertion.
  - Use a priority queue.
    - For simplicity, we will assume a binary min-heap is used for implementing the priority queue.
    - Advanced data structures could be used to achieve a better cost. See discussion in CLRS. This is extracurricular.

Huffman Codes

- Pseudocode [CLRS]:



```
HUFFMAN($C$)
$n = |C|$
$Q = C$
for $i = 1$ to $n - 1$
    allocate a new node $z$
    $z.left = x =$ EXTRACT-MIN($Q$)
    $z.right = y =$ EXTRACT-MIN($Q$)
    $z.freq = x.freq + y.freq$
    INSERT($Q, z$)
return EXTRACT-MIN($Q$)        // return the root of the tree
```

Size of alphabet

All characters are added to the queue.
Note: each character has an attribute freq

Get the two nodes with the lowest frequencies, and combine them.

Running time: O(n lg n)

Huffman Code – Correctness Proof

- Need to prove that the problem exhibits both the greedy choice and optimal substructure properties.