A brief tutorial on Gomory cuts

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Integer programs (IPs) are solved in practice by branch-and-cut algorithms. These algorithms rely heavily on cutting planes, which are inequalities that are added to linear programming (LP) relaxations to cut off (bad) fractional points, but not the (good) integer feasible points. The classical cutting planes for solving IPs were developed by Ralph Gomory in the late 1950s, motivated in part by his time as a consultant for the US Navy (Gomory, 2010):

> *As the Navy had kept me on as a consultant I continued to work on Navy problems through monthly trips to Washington. On one of these trips a group presented a linear programming model of a Navy Task Force. One of the presenters remarked that it would be nice to have whole number answers as 1.3 aircraft carriers, for example, was not directly usable.*

Within the next few weeks, Gomory had developed his technique for generating cutting planes based on the simplex tableau. Soon after, he proved that his algorithm was finite and programmed it on the E101, allowing him to solve five-variable problems reliably(!). In this tutorial, we briefly discuss Gomory's fractional cuts and his subsequently developed mixed integer cuts, which are used by modern IP solvers to handle much-larger-than-five-variable problems reliably (Gleixner et al., 2019).

1 Gomory fractional cuts

Let's begin with the following IP.

$$\begin{aligned}
\max \quad & x_1 + x_2 && \text{(1a)}\\
& x_1 - 2x_2 \le 4 && \text{(1b)}\\
& x_1 + 3x_2 \le 11 && \text{(1c)}\\
& x_1,\ x_2 \ge 0 && \text{(1d)}\\
& x_1,\ x_2 \text{ integer.} && \text{(1e)}
\end{aligned}$$

The associated LP relaxation is shown in Figure 1. Adding (integer-valued) slack variables gives:

$$\begin{aligned}
\max \quad & x_1 + x_2 && \text{(2a)}\\
& x_1 - 2x_2 + x_3 = 4 && \text{(2b)}\\
& x_1 + 3x_2 + x_4 = 11 && \text{(2c)}\\
& x_1,\ x_2\ , x_3,\ x_4 \ge 0 && \text{(2d)}\\
& x_1,\ x_2\ , x_3,\ x_4 \text{ integer.} && \text{(2e)}
\end{aligned}$$

Solving the LP relaxation gives the following system, with objective $z = x_1 + x_2$.

$$\begin{aligned}
z \qquad\quad & + 0.4x_3 + 0.6x_4 = 8.2 && \text{(3a)}\\
x_1 \quad\ & + 0.6x_3 + 0.4x_4 = 6.8 && \text{(3b)}\\
x_2 & - 0.2x_3 + 0.2x_4 = 1.4 && \text{(3c)}\\
x_1,\ x_2\ , x_3,\ & x_4 \ge 0. && \text{(3d)}
\end{aligned}$$

This corresponds to the fractional point $(x_1, x_2, x_3, x_4) = (6.8, 1.4, 0, 0)$.

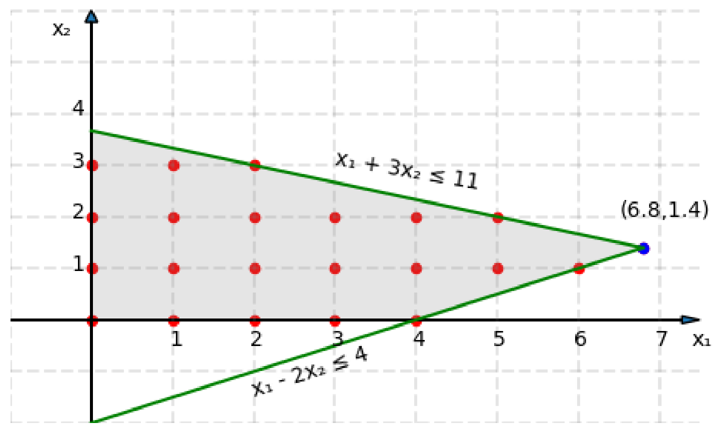

Figure 1: LP relaxation.

A first cut. In any feasible solution to the IP, x_1 will take an integer value, so from equation (3b) we can write $0.6x_3 + 0.4x_4 = 0.8 + k$ for some integer k. Notice that the left side of this equation can only take nonnegative values, so the right side must as well, i.e., $0.8 + k \geq 0$ or $k \geq -0.8$. Since k is an integer, we know that $k \geq 0$. So, $0.6x_3 + 0.4x_4 = 0.8 + k \geq 0.8$. We have just argued for the Gomory fractional cut:

$$0.6x_3 + 0.4x_4 \geq 0.8.$$

By equations (2b) and (2c) we can express this inequality as:

$$x_1 \leq 6.$$

This inequality, shown in Figure 2, cuts off our fractional point $(6.8, 1.4, 0, 0)$.

A second cut. Now consider the equation (3c). As before, we can write $-0.2x_3 + 0.2x_4 = 0.4 + k$ for some integer k. However, the left side of this equation may not always take nonnegative values (due to the negative coefficient for x_3), so our previous argument will not work. However, if we add x_3 to both sides, we can write $0.8x_3 + 0.2x_4 = 0.4 + p$ for some (other) integer p. Using the same argument as before, we can generate another Gomory fractional cut:

$$0.8x_3 + 0.2x_4 \geq 0.4.$$

By equations (2b) and (2c) we can express this inequality as:

$$x_1 - x_2 \leq 5.$$

This inequality, also shown in Figure 2, cuts off our fractional point but is less helpful than the first cut.

The general case. We can write the Gomory fractional cut in more general terms as follows. Suppose that nonnegative integers $x_1, x_2, \ldots, x_n$ satisfy the equation $\sum_{i=1}^{n} a_i x_i = b$, where b is fractional, i.e., $b \notin \mathbb{Z}$. Think of this as a row of the simplex tableau (dictionary). The associated Gomory fractional cut is:

$$\sum_{i=1}^{n} (a_i - \lfloor a_i \rfloor) x_i \geq b - \lfloor b \rfloor. \tag{4}$$

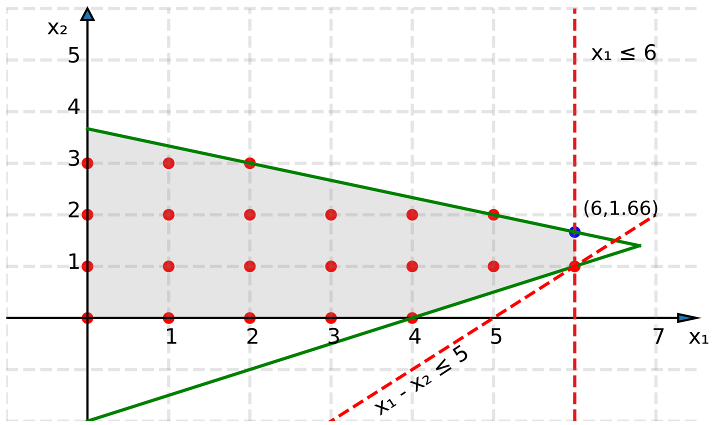

Figure 2: Gomory fractional cuts.

Proposition 1. *The Gomory fractional cut* (4) *is valid for the set* $X = \left\{x \in \mathbb{Z}^n_+ \mid \sum_{i=1}^n a_i x_i = b\right\}$.

Proof. Let $x^* \in X$. We are to show that $\sum_{i=1}^n (a_i - \lfloor a_i \rfloor) x_i^* \geq b - \lfloor b \rfloor$. By $x^* \in X$, the equality $\sum_{i=1}^n a_i x_i^* = b$ holds. Since each $\lfloor a_i \rfloor$ is integer, and since each x_i^* is integer, we can write $\sum_{i=1}^n (a_i - \lfloor a_i \rfloor) x_i^* = b - \lfloor b \rfloor + k$, for some integer k, Observe that the left side is nonnegative, so the right side is as well, $b - \lfloor b \rfloor + k \geq 0$ and thus $k \geq \lfloor b \rfloor - b > -1$. By $k \in \mathbb{Z}$, this implies $k \geq 0$. So, in conclusion,

$$\sum_{i=1}^n (a_i - \lfloor a_i \rfloor) x_i^* = b - \lfloor b \rfloor + k \geq b - \lfloor b \rfloor.$$

□

Two observations:

1. Subtracting $\lfloor a_i \rfloor$ from a_i in the Gomory fractional cut ensures that the left side is nonnegative, which was key to our arguments. We could have instead subtracted some other number $q < \lfloor a_i \rfloor$ from a_i and the resulting inequality would remain valid; however, this would weaken the inequality.

2. The Gomory fractional cut remains valid when you add other constraints to the set X. This means you can still use it when your IP is more complicated.

2 Gomory mixed integer cuts (for pure IPs)

We have just seen Gomory fractional cuts. However, they are not used in practice. A primary reason is that they are subsumed by Gomory mixed integer (GMI) cuts, which Gomory introduced just two years later.

GMI cuts have two advantages over Gomory fractional cuts. First, they are more general in the sense that they apply when the problem has a mix of integer and continuous variables (MIPs), whereas Gomory fractional cuts only apply for problems in which all variables are integer (pure IPs). A second advantage is that even when dealing with pure IPs, GMI cuts are just as strong or stronger than Gomory fractional cuts.

For simplicity, we will stick with pure IPs in this tutorial. The interested reader is invited to consult the longer tutorial by Cornuéjols (2008) for the full version of GMI cuts.

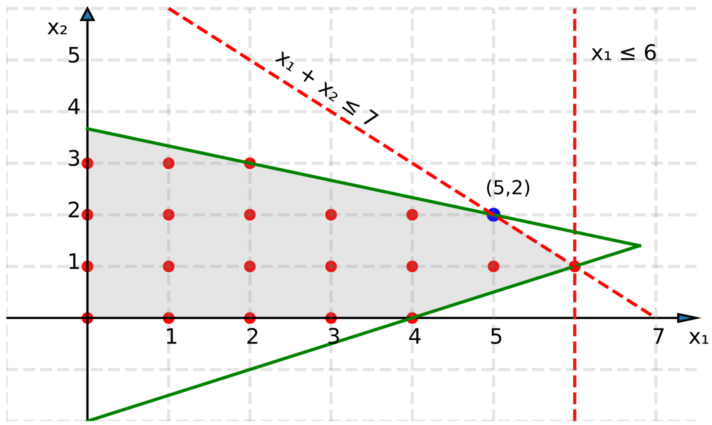

Figure 3: Gomory mixed integer cuts.

The general case. Again, suppose that nonnegative integers $x_1, x_2, \ldots, x_n$ satisfy the equation $\sum_{i=1}^{n} a_i x_i = b$, where b is fractional. Letting $I = \{1, \ldots, n\}$, the associated GMI cut is:

$$\sum_{i \in I: f_i \leq f} \frac{f_i}{f} x_i + \sum_{i \in I: f_i > f} \frac{1 - f_i}{1 - f} x_i \geq 1. \tag{5}$$

This inequality uses the fractional parts of b and a_i, which are denoted $f := b - \lfloor b \rfloor$ and $f_i := a_i - \lfloor a_i \rfloor$. Each is nonnegative. Notice that if $f_i \leq f$ for every i, then the resulting GMI cut is exactly the same as the Gomory fractional cut, which can be written in terms of the f notation as:

$$\sum_{i \in I} f_i x_i \geq f. \tag{6}$$

However, if at least one i satisfies $f_i > f$, then the GMI cut is stronger. This is because if $f_i > f$, then $\frac{1-f_i}{1-f} < \frac{f_i}{f}$, meaning the coefficient of x_i will be smaller.

To illustrate GMI cuts, recall the system identified previously:

$$z \qquad + 0.4x_3 + 0.6x_4 = 8.2 \tag{7a}$$
$$x_1 \quad + 0.6x_3 + 0.4x_4 = 6.8 \tag{7b}$$
$$x_2 - 0.2x_3 + 0.2x_4 = 1.4 \tag{7c}$$
$$x_1,\ x_2\ , x_3,\ x_4 \geq 0. \tag{7d}$$

A first cut. In equation (7b), we have $f = 0.8$, $f_1 = f_2 = 0$, $f_3 = 0.6$, and $f_4 = 0.4$. Notice that $f_1, f_2, f_3, f_4 \leq f$, and the resulting inequality is exactly the same as the Gomory fractional cut:

$$\frac{0.6}{0.8} x_3 + \frac{0.4}{0.8} x_4 \geq 1,$$

which, by equations (2b) and (2c), can be expressed as:

$$x_1 \leq 6.$$

A second cut. Now consider equation (7c), which has $f = 0.4$, $f_1 = f_2 = 0$, $f_3 = 0.8$, and $f_4 = 0.2$. Notice that $f_1, f_2, f_4 \leq f$ and $f_3 \geq f$, and the resulting GMI cut is:

$$\frac{1-0.8}{1-0.4}x_3 + \frac{0.2}{0.4}x_4 \geq 1.$$

By equations (2b) and (2c) we can express this inequality as:

$$x_1 + x_2 \leq 7,$$

which is shown in Figure 3.

Re-solving the LP relaxation after adding GMI cuts gives the solution $(x_1, x_2) = (5, 2)$, which is optimal for the IP. Meanwhile, one round of Gomory fractional cuts still gives a fractional solution (see Figure 2).

Proposition 2. *The Gomory mixed integer cut* (5) *is valid for the set* $X = \left\{x \in \mathbb{Z}^n_+ \mid \sum_{i=1}^n a_i x_i = b\right\}$.

Proof. Our task is to show that every $x^* \in X$ satisfies inequality (5). So, let $x^* \in X$. This implies that

$$\sum_{i \in I} a_i x_i^* = b. \tag{8}$$

and $x^* \in \mathbb{Z}^n_+$. Since $\lfloor a_i \rfloor$ and $\lfloor b \rfloor$ and x_i^* are integers, and by equation (8), we can write

$$\sum_{i \in I : f_i \leq f} (a_i - \lfloor a_i \rfloor) x_i^* + \sum_{i \in I : f_i > f} (a_i - \lfloor a_i \rfloor - 1) x_i^* = b - \lfloor b \rfloor + k$$

for some integer k. In terms of our f notation, this is

$$\sum_{i \in I : f_i \leq f} f_i x_i^* + \sum_{i \in I : f_i > f} (f_i - 1) x_i^* = f + k. \tag{9}$$

We consider two cases. In the first case, suppose that $k \geq 0$, in which case

$$\begin{aligned}
\sum_{i \in I : f_i \leq f} \frac{f_i}{f} x_i^* + \sum_{i \in I : f_i > f} \frac{1 - f_i}{1 - f} x_i^* &\geq \sum_{i \in I : f_i \leq f} \frac{f_i}{f} x_i^* + 0 \\
&\geq \sum_{i \in I : f_i \leq f} \frac{f_i}{f} x_i^* + \sum_{i \in I : f_i > f} \frac{f_i - 1}{f} x_i^* \\
&= \frac{f + k}{f} \geq 1.
\end{aligned}$$

The last equation holds by (9). The last inequality holds by $k \geq 0$ and $f > 0$.

In the other case, suppose $k < 0$. Then, $k \leq -1$ since k is an integer, so

$$\begin{aligned}
\sum_{i \in I : f_i \leq f} \frac{f_i}{f} x_i^* + \sum_{i \in I : f_i > f} \frac{1 - f_i}{1 - f} x_i^* &= \sum_{i \in I : f_i \leq f} \frac{f_i}{f} x_i^* + \sum_{i \in I : f_i > f} \frac{f_i - 1}{f - 1} x_i^* \\
&\geq 0 + \sum_{i \in I : f_i > f} \frac{f_i - 1}{f - 1} x_i^* \\
&\geq \sum_{i \in I : f_i \leq f} \frac{f_i}{f - 1} x_i^* + \sum_{i \in I : f_i > f} \frac{f_i - 1}{f - 1} x_i^* \\
&= \frac{1}{f - 1}(f + k) \geq 1.
\end{aligned}$$

The last equation holds by (9). The last inequality holds by $k \leq -1$ and $f - 1 < 0$.

So, x^* satisfies the GMI inequality (5) in both cases, and so the GMI inequality is valid for X. $\square$

Further reading. Consult Cornuéjols (2008) for a definitive account of these and other valid inequalities for MIPs. Also see the video lectures by Krishnamoorthy (2019) and Peña (2016).

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