Orbital angular momentum of general astigmatic modes

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We present an operator method to obtain complete sets of astigmatic Gaussian solutions of the paraxial wave equation. In case of general astigmatism, the astigmatic intensity and phase distribution of the fundamental mode differ in orientation. As a consequence, the fundamental mode has a nonzero orbital angular momentum, which is not due to phase singularities. Analogous to the operator method for the quantum harmonic oscillator, the corresponding astigmatic higher-order modes are obtained by repeated application of raising operators on the fundamental mode. The nature of the higher-order modes is characterized by a point on a sphere, in analogy with the representation of polarization on the Poincaré sphere. The north and south poles represent astigmatic Laguerre-Gaussian modes, similar to circular polarization on the Poincaré sphere, while astigmatic Hermite-Gaussian modes are associated with points on the equator, analogous to linear polarization. We discuss the propagation properties of the modes and their orbital angular momentum, which depends on the degree of astigmatism and on the location of the point on the sphere.

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I. INTRODUCTION

Optical vortices play an important role in the field of quantum information because they can be used to create higher-dimensional entangled states [1]. The advantage of higher-dimensional entanglement is that more information can be stored and that there is less sensitivity to decoherence [2]. A light beam containing optical vortices can be used to create twin photons in a nonlinear crystal. By changing the distribution of the vortices in the pump beam, the entangled state of the twin photons can be manipulated, such that it can be used for quantum communication protocols in higher dimensions [3]. Optical vortices in light beams can be created by using holograms [4], or spiral phase plates [5]. Upon propagation in free space, the vortices in a light beam display interesting behavior [6]. Vortices can be created and annihilated in pairs of opposite topological charge, and isolated vortices have a certain stability under propagation [7]. An important feature of an optical vortex is that it carries a density of orbital angular momentum (OAM). An example is the central vortex in a Laguerre-Gaussian (LG) mode. The value of the OAM per photon is $l\hbar$, with l the topological charge of the vortex [8]. The value of the OAM of a light beam can be modified when the beam is passed through astigmatic lenses [9]. An example of a beam without vortices that carries OAM, is an astigmatic Gaussian beam, where the orientation of astigmatism in the intensity and the phase are different [10]. This is called general astigmatism [11]. The OAM of astigmatic LG beams has also been discussed in simple cases [12,13].

The paraxial wave equation, which describes the propagation of light beams, is identical in form to the two-dimensional Schrödinger equation for a free particle. It is therefore convenient to use the operator language for the description of the propagation of a light beam. The "state" of the light beam in an arbitrary transverse plane z can then be obtained by applying the propagation operator to the state in the plane $z=0$, like it is done for the Schrödinger evolution. In a Heisenberg-like picture, a linear combination of the coordinate operator $\hat{R}$ and the momentum operator $\hat{P}$ will remain a linear combination of $\hat{R}$ and $\hat{P}$, since the propagation operator is Gaussian in the momentum operator $\hat{P}$. Also the propagation through optical elements, like thin lenses, can be described in terms of Gaussian operators in $\hat{R}$ and $\hat{P}$. This forms the basis of the operator method for the description of optical systems [9,14–16]. Nienhuis and Allen [17] define ladder operators that are linear combinations of $\hat{R}$ and $\hat{P}$, where the coefficients depend on the transverse plane z. Complete sets of solutions of the paraxial wave equation are obtained by repeated action of raising operators on the fundamental mode, in analogy to the operator method for the eigenstates of the quantum harmonic oscillator. This similarity is the reason that the familiar Hermite-Gaussian (HG) solutions of the paraxial wave equation are analogous to the eigenfunctions of the harmonic oscillator [18]. A similar operator method is discussed by Wünsche [19].

In this paper we generalize the operator method of Nienhuis and Allen [17] to include astigmatism. We derive complete sets of solutions of the paraxial wave equation that have OAM, both due to the presence of phase singularities, and due to the difference in the orientation of the astigmatic intensity and phase distributions. In Sec. II we discuss the properties of an astigmatic Gaussian beam. We distinguish between simple astigmatism and general astigmatism. For simple astigmatism, both the intensity and phase distributions are astigmatic, but with the same orientation, while for general astigmatism the orientations are different. Simple astigmatism also covers the case for which either the intensity or the phase distribution is astigmatic. In the case of general astigmatism the Gaussian beam has OAM. In Sec. III we obtain complete sets of higher-order modes by operating on the state of the Gaussian beam. In Sec. IV we discuss some properties of the modes, and characterize the nature of the basis sets of the modes by a point on a sphere, in analogy to the Poincaré-sphere representation of polarization. This gives a natural way to describe modes that are intermediate between LG and HG modes. In Sec. V we evaluate the OAM of the modes. The OAM is composed of three contributions, each with a distinct physical significance.

II. THE FUNDAMENTAL MODE

A. Gaussian beam

We consider a Gaussian beam that propagates in the positive z direction. When the Gaussian beam has cylinder symmetry its transverse profile can be described completely by a complex number α, which depends on z. The real part of α determines the width of the mode, while the radius of curvature of its wavefronts is determined by the imaginary part. When there is no cylinder symmetry the Gaussian beam is astigmatic. In its simplest form an astigmatic Gaussian beam is described by two complex numbers, one for each transverse dimension, and an angle which determines the orientation of astigmatism. The real parts of these complex numbers determine the shape of the intensity distribution of the beam, while their imaginary parts determine the shape of the curves of constant phase. The two complex numbers evolve independently upon propagation, so that the orientation of astigmatism does not vary with z. For astigmatism in its most general form the orientations of the intensity and the phase distributions of the beam differ. Upon propagation the orientation of both the phase and the intensity distribution changes and the beam tumbles [10,13].

In general, the Gaussian beam is determined by a complex symmetric 2×2 matrix α. We use Greek letters for 2x2 matrices, unless stated otherwise. We write $\alpha=\alpha_0-i\alpha_1$, where α_0 and α_1 are real and symmetric matrices. The normalized mode profile of the Gaussian beam in the transverse plane $z=0$ is written as

$$u_{00}(R,0)=\frac{1}{\sqrt{\pi}}[\det\,\alpha_0]^{1/4}\exp(-R\alpha R/2), \tag{1}$$

where R is the two-dimensional transverse coordinate vector. For notational convenience we write $R\alpha R\equiv R^T\cdot\alpha\cdot R$, where the dot is a matrix multiplication. For Gaussian beams with cylinder symmetry, α is a complex number times the unit matrix. When considering astigmatism in its simplest form, where astigmatism of the intensity and the phase distribution of the beam has the same orientation, the matrices α_0 and α_1 commute. For general astigmatism α_0 and α_1 do not commute.

The matrix α_0 determines the intensity distribution of the beam. The eigenvalues of α_0 are positive, so that the intensity distribution can be normalized. As a consequence, curves of constant intensity have an elliptical shape. The ellipticity or degree of astigmatism is determined by the ratio of the eigenvalues of α_0. On the other hand, there is no constraint on the determinant of the matrix α_1, which determines the phase distribution of the beam. For a positive determinant, curves of constant phase have an elliptical shape, while for a negative determinant, these curves are hyperbolas. The degree of astigmatism for the phase is determined by the ratio of the eigenvalues of α_1. For nonastigmatic Gaussian beams, α_1 vanishes in the focal plane. In the case of an astigmatic phase distribution the eigenvalues of α_1 are different and there are two focal planes. In a focal plane, one of the eigenvalues vanishes and the curves of constant phase are parallel lines.

B. Propagation

The electric field of a monochromatic light beam with frequency ω that propagates in the positive z direction is written as

$$E(\vec{r},t)=E_0\,\mathrm{Re}[u(R,z)\exp(ikz-i\omega t)], \tag{2}$$

where $\vec{r}=(R,z)$ is the position vector in three dimensions and $u(R,z)$ is the normalized transverse profile of the beam. When $|\partial u/\partial z|\ll ku$, the transverse profile varies only slowly with z, and the propagation of the light beam is well described by the paraxial wave equation

$$\left(\frac{\partial^2}{\partial R^2}+2ik\frac{\partial}{\partial z}\right)u(R,z)=0, \tag{3}$$

where $\partial^2/\partial R^2=\partial^2/\partial x^2+\partial^2/\partial y^2$. Well-known solutions of the paraxial equation are the HG modes, which form a complete set of solutions. The lowest-order of the HG modes is a Gaussian beam with cylinder symmetry. The HG modes resemble the eigenfunctions of the two-dimensional harmonic oscillator. As discussed in standard quantum mechanics textbooks, the excited eigenstates of the harmonic oscillator can be found by the repeated application of the raising operator on the ground state [20]. Nienhuis and Allen [17] have derived complete sets of solutions of the paraxial wave equation by a similar method. We generalize this operator method by including astigmatism. For the two-dimensional harmonic oscillator a lowering and a raising operator is obtained for each dimension. These operators satisfy the boson commutation rules. The general form of the mode profile $u_{00}(R,0)$ of a Gaussian beam in Eq. (1) corresponds to the ground-state wave function of the harmonic oscillator. Operating on this function with each of the lowering operators $\hat{a}_1$ and $\hat{a}_2$ separately must give zero. We write $\hat{a}_1$ and $\hat{a}_2$ in the form of the two-dimensional vector $\hat{A}$ and find from Eq. (1) that $\hat{A}u_{00}(R,0)=0$ is satisfied when

$$\hat{A}=\frac{1}{\sqrt{2}}\beta(\alpha\hat{R}+i\hat{P}), \tag{4}$$

where β is a proportionality matrix and $\hat{P}$ is the momentum operator, which takes the form $\hat{P}=-i\partial/\partial R$ in the coordinate representation. The corresponding vector of raising operators $\hat{a}_1^\dagger$ and $\hat{a}_2^\dagger$ is

$$\hat{A}^\dagger=\frac{1}{\sqrt{2}}\beta^*(\alpha^*\hat{R}-i\hat{P}). \tag{5}$$

The complex 2×2 matrix β must be chosen so that the lowering and raising operators satisfy the boson commutation rules. This will be discussed in Sec. III A.

The paraxial wave equation is identical in form to the two-dimensional Schrödinger equation for a free particle where the coordinate z replaces time. This suggests that the Dirac notation is convenient to use. For a general beam profile in the plane z we write $u(R,z)=\langle R|u(z)\rangle$, which is the mode function in the coordinate representation of the state of the beam $|u(z)\rangle$ in the plane z. In operator form the paraxial wave equation is given by

$$\left(\hat{P}^2 - 2ik\frac{\partial}{\partial z}\right)|u(z)\rangle = 0, \tag{6}$$

where $\hat{P}^2=\hat{p}_x^2+\hat{p}_y^2$. Similar to the solution of the Schrödinger equation for a free particle we have

$$|u(z)\rangle = \hat{U}(z)|u(0)\rangle, \tag{7}$$

with

$$\hat{U}(z) = \exp\left(-\frac{iz}{2k}\hat{P}^2\right). \tag{8}$$

For the lowering operator at $z=0$ we write $\hat{A}(0)$, and the Gaussian fundamental mode profile in the plane $z=0$ is determined by

$$\hat{A}(0)|u_{00}(0)\rangle=0. \tag{9}$$

For arbitrary plane z, the fundamental mode profile obeys

$$\hat{A}(z)|u_{00}(z)\rangle = 0. \tag{10}$$

It follows that the lowering operator in a plane with arbitrary z is related to the lowering operator in the plane $z=0$ by

$$\hat{A}(z) = \hat{U}(z)\hat{A}(0)\hat{U}(z)^{\dagger}. \tag{11}$$

We use the operator identity

$$\hat{U}(z)\hat{R}\hat{U}^{\dagger}(z) = \hat{R} - \frac{z}{k}\hat{P}, \tag{12}$$

and write $\hat{A}(0)$ as in Eq. (4). We find that the z-dependent lowering operator has the form

$$\hat{A}(z) = \frac{1}{\sqrt{2}}\beta(z)(\alpha(z)\hat{R} + i\hat{P}), \tag{13}$$

with z-dependent matrices α and β,

$$\alpha(z) = \left(1 + \frac{iz}{k}\alpha(0)\right)^{-1}\alpha(0), \quad \beta(z) = \beta(0)\left(1 + \frac{iz}{k}\alpha(0)\right). \tag{14}$$

The solution of (10) is a Gaussian determined by $\alpha(z)$. By taking the real and imaginary parts of $\alpha(z)$ we find the expressions for $\alpha_0(z)$ and $\alpha_1(z)$, which determine how the intensity and the phase distributions of the beam profile change upon propagation.

C. Gouy phase

The z-dependent lowering operator $\hat{A}(z)$ determines the expression for the fundamental mode profile in different planes within a phase factor $\exp(-i\chi(z))$. This factor $\chi(z)$ is a generalized form of the Gouy phase. The condition (10) does not determine the z-dependent phase $\chi(z)$. However, one may check that the solution of the paraxial wave equation (3), with the boundary condition (1), is

$$u_{00}(R,z) = \frac{1}{\sqrt{\pi}}[\det\,\alpha_0(0)]^{1/4}\sqrt{\frac{\det\,\alpha(z)}{\det\,\alpha(0)}}\,\exp\left[-\frac{1}{2}R\alpha(z)R\right]. \tag{15}$$

Here $\alpha(z)=\alpha_0(z)-i\alpha_1(z)$ is given in Eq. (14). This check requires the differentiation of $\det\,\alpha(z)$ with respect to z. For this we use the identity for a matrix M that depends on the parameter z:

$$\frac{1}{\det\,M(z)}\frac{d}{dz}\det\,M(z) = \mathrm{Tr}\frac{1}{M(z)}\frac{dM}{dz}. \tag{16}$$

This follows after taking the determinant of $M(z+dz) = M(z)[1+dz(1/M(z))(dM/dz)]$, and linearizing in dz. From the expression for $\alpha(z)$ in Eq. (14) we find $d\alpha(z)/dz = \alpha^2(z)/ik$, so that

$$\frac{d}{dz}\det\,\alpha(z) = \frac{1}{ik}\det\,\alpha(z)\mathrm{Tr}[\alpha(z)]. \tag{17}$$

Using this, it is straightforward to check that Eq. (15) is indeed a solution of the paraxial wave equation. The phase $\chi(z)$, which is an astigmatic generalization of the Gouy phase, is then given by

$$\chi(z) = -\arg\sqrt{\frac{\det\,\alpha(z)}{\det\,\alpha(0)}} = \frac{1}{2}\arg\det\left(1 + \frac{iz}{k}\alpha(0)\right). \tag{18}$$

Just as in the astigmatic case, χ increases by π between $z=-\infty$ and $z=\infty$.

D. Orbital angular momentum

When both the intensity and phase distribution of a Gaussian beam are astigmatic with different orientations, the beam carries OAM [10,13]. The beam tumbles upon propagation, and with this tumbling OAM is associated. The origin of the tumbling of the beam can be understood when the transverse momentum density is considered. For a normalized beam profile $u(R)$ the transverse momentum density per photon is given by [8]

$$P(R) = \frac{\hbar}{2i}\left(u^*(R)\frac{\partial}{\partial R}u(R) - u(R)\frac{\partial}{\partial R}u^*(R)\right), \tag{19}$$

which is in the direction of the transverse gradient of the phase of the beam. The OAM per photon in the propagation direction is then given by

$$L = \int dR\|R \times P(R)\|. \tag{20}$$

In Fig. 1 two ellipses of constant intensity and phase are depicted. On four points the direction of the transverse momentum density is represented by arrows. That the beam has

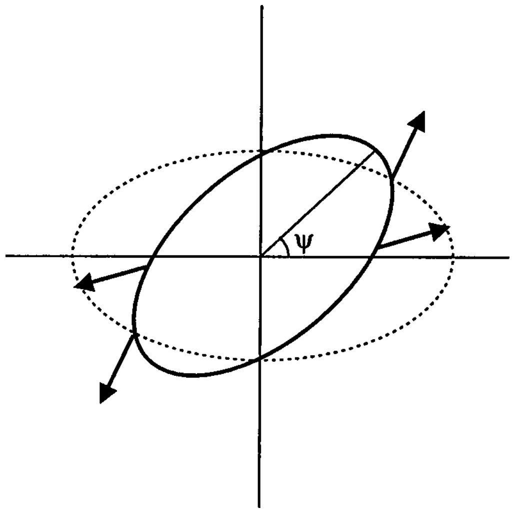

FIG. 1. Two ellipses of constant intensity and phase of a Gaussian mode with general astigmatism. On the solid ellipse the intensity is constant; on the dashed ellipse the phase is constant. The ellipse of constant intensity is rotated over an angle ψ with respect to the ellipse of constant phase. On four points on the ellipse of constant intensity the direction of the transverse momentum density is represented by arrows. These arrows point in the direction of the gradient of the phase, which is chosen outwards in this case. They represent the direction in which the intensity will move. Because the ellipses of constant intensity and phase are oriented differently, it can be seen from the arrows that upon propagation the beam will tumble, giving rise to orbital angular momentum.

OAM can be understood from the expression for the OAM in Eq. (20) and the fact that the arrows have arms. In Fig. 1 the direction of the gradient of the phase is chosen to be outwards, corresponding to a situation where the beam has passed the two focal planes. We see that upon propagation the intensity ellipse will try to disalign with the phase ellipse. In case that the gradient of the phase is directed inwards, corresponding to a situation before the focal planes, the intensity ellipse tries to align with the phase ellipse.

By inserting the general expression for the fundamental mode in Eq. (15) in Eq. (20) we obtain the OAM of the fundamental mode. We find that

$$L_{00} = \frac{\hbar}{4}(\alpha_{1s} - \alpha_{1l})\left(\frac{1}{\alpha_{0l}} - \frac{1}{\alpha_{0s}}\right)\sin 2\psi, \tag{21}$$

where α_{0l} and α_{0s} are the eigenvalues of α_0, and where α_{1l} and α_{1s} are the eigenvalues of α_1. The angle ψ (this Greek letter is not a matrix) is the difference in orientation of astigmatism of the phase and the intensity. For nonzero OAM it is necessary that indeed both the intensity and the phase distribution are astigmatic, since the eigenvalues of both α_0 and α_1 must be different. Furthermore, it is also necessary that astigmatism of the intensity and phase distributions differ in orientation, because the angle ψ must be different from 0 or $\pi/2$. This corresponds to the case of general astigmatism, for which α_0 and α_1 do not commute. In Sec. V we derive the following matrix expression for the OAM of the fundamental mode:

$$L_{00} = \frac{\hbar}{2}\mathrm{Tr}\ \varepsilon\alpha_1\alpha_0^{-1}, \tag{22}$$

where

$$\varepsilon = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix} \tag{23}$$

is the two-dimensional antisymmetric tensor. This expression is equivalent to (21). We shall also see that the OAM is conserved indeed, which is not evident from Eq. (21), since α is z dependent.

III. OPERATOR METHOD

A. Coordinate picture

Recall that the excited states of the quantum harmonic oscillator are obtained by a repeated application of the raising operator on the ground state. In a similar fashion, we obtain astigmatic higher-order modes by operating on the state $|u_{00}\rangle$ of the astigmatic fundamental mode. This generalizes the operator method of Nienhuis and Allen [17]. In an arbitrary transverse plane z we define ladder operators that are linear combinations of the coordinate operator $\hat{R}=(\hat{x},\hat{y})$ and the momentum operator $\hat{P}=(\hat{p}_x,\hat{p}_y)$. In Eqs. (4) and (5), the vector $\hat{A}(z)$ of lowering operators $\hat{a}_1(z)$ and $\hat{a}_2(z)$, and the corresponding vector $\hat{A}^\dagger(z)$ of raising operators, are given, where the z-dependent matrices $\alpha(z)$ and $\beta(z)$ are given in Eq. (14). It follows from Eq. (11), that, when $\hat{a}_1(z)$, $\hat{a}_2(z)$, $\hat{a}_1^\dagger(z)$ or $\hat{a}_2^\dagger(z)$ is applied to a solution $|u(z)\rangle$ of the paraxial wave equation, another solution is obtained. The ladder operators must satisfy the boson commutation rules,

$$[\hat{a}_1(z),\hat{a}_1^\dagger(z)] = [\hat{a}_2(z),\hat{a}_2^\dagger(z)] = 1,$$

$$[\hat{a}_1(z),\hat{a}_2(z)] = [\hat{a}_1(z),\hat{a}_2^\dagger(z)] = 0. \tag{24}$$

The higher-order modes are obtained by operating with the raising operators $\hat{a}_1^\dagger(z)$ and $\hat{a}_2^\dagger(z)$ on the state of the fundamental mode $|u_{00}(z)\rangle$. We have

$$|u_{nm}(z)\rangle = \frac{1}{\sqrt{n!\,m!}}(\hat{a}_1^\dagger(z))^n(\hat{a}_2^\dagger(z))^m|u_{00}(z)\rangle, \tag{25}$$

where $|u_{nm}(z)\rangle$ is the state of the higher-order mode with mode numbers $n,m=0,1,2,\ldots$. The normalized profile of a higher-order mode is given by the wave function in the coordinate representation of its state $\langle R|u_{nm}(z)\rangle$. It follows from the commutation rules (24) and from Eq. (11), that the higher-order modes together with the fundamental mode constitute a complete set of solutions of the paraxial wave equation.

The commutation rules (24) are satisfied for all values of z as soon as they are satisfied for a single value of z, as follows from Eq. (11). From the commutation rules $[\hat{x},\hat{p}_x] = [\hat{y},\hat{p}_y]=i$ for $\hat{R}$ and $\hat{P}$, one finds that the commutation rules (24) require that

$$\beta\alpha_0\beta^\dagger = 1. \tag{26}$$

It follows, for all values of z, that $\beta^\dagger\beta=\alpha_0^{-1}$, and that β is invertible. Since α_0 is real, symmetric and positive definite, it uniquely defines a real, symmetric and positive-definite matrix γ, so that

$$\gamma^2 = \alpha_0^{-1} = \beta^\dagger\beta. \tag{27}$$

The real eigenvalues of γ are the two widths of the elliptic spot size of the intensity distribution of the fundamental mode profile $u_{00}(R)$. The requirement (27) is satisfied by writing

$$\beta = \sigma\gamma, \tag{28}$$

where σ is a unitary matrix. The z dependence of the matrices γ, α_1 and σ can be obtained by using Eq. (14). The fundamental mode is determined by choosing α, or, equivalently, γ and α_1, in one transverse plane. Higher-order modes are determined by the additional choice of the unitary matrix σ for one plane z.

B. Momentum picture

The ladder operators are linear combinations of the coordinate operator $\hat{R}$ and the momentum operator $\hat{P}$, which play a similar role. In the coordinate picture we focus on solutions $u(R,z)$ of the paraxial wave equation. The momentum picture of the solution is given by $\tilde{u}(P,z)$, which is the Fourier transform of $u(R,z)$. The vector (4) of lowering operators is rewritten as

$$\hat{A} = \frac{1}{\sqrt{2}}\kappa(\hat{R} + i\mu\hat{P}), \tag{29}$$

and we see that

$$\kappa = \beta\alpha, \quad \mu = \alpha^{-1}. \tag{30}$$

We write $\mu=\mu_0+i\mu_1$, where μ_0 and μ_1 are real and symmetric matrices. Moreover, μ_0 is positive definite. The commutation rules for the ladder operators are satisfied when $\kappa\mu_0\kappa^\dagger=1$, from which it follows that $\kappa^\dagger\kappa=\mu_0^{-1}$, so that κ is invertible. In full analogy to the decomposition (28) of β in a unitary and a real symmetric matrix, we write

$$\kappa = \tau\delta, \tag{31}$$

with τ unitary, and δ real, symmetric and positive definite, so that $\delta^2=\mu_0^{-1}$. The real eigenvalues of the matrix δ are the two widths of the elliptic momentum distribution. In the momentum picture the fundamental mode is determined by μ, or, equivalently, by δ and μ_1, while the higher-order modes also require τ.

For later use we want to express the matrices γ, α_1 and σ of the coordinate picture in their momentum-picture analogs δ, μ_1 and τ. The two picture are related by Eq. (30), from which it follows that

$$\beta = \kappa\mu, \tag{32}$$

and

$$\alpha_1\mu_0 = \alpha_0\mu_1, \quad \mu_0\alpha_1 = \mu_1\alpha_0. \tag{33}$$

Because $\gamma^2=\alpha_0^{-1}$ and $\delta^2=\mu_0^{-1}$, we find from (33) that

$$\gamma^2\alpha_1 = \mu_1\delta^2. \tag{34}$$

We use Eqs. (27), (32), and (31), to find that

$$\gamma^2 = \mu^\dagger\delta^2\mu. \tag{35}$$

From Eqs. (28), (32), and (31), we find that

$$\sigma = \tau\delta\mu\gamma^{-1}. \tag{36}$$

IV. PROPERTIES OF MODES

A. Propagation of the fundamental mode

In the coordinate picture it is convenient to write the fundamental mode profile as

$$u_{00}(R,z) = \frac{1}{\sqrt{\pi \det \beta}}\exp\left(-\frac{1}{2}R\alpha R\right). \tag{37}$$

From Eqs. (14) and (28), we see that this expression differs from Eq. (15) only by a constant phase $\sqrt{\det \sigma(0)}$. The fundamental mode profile in the momentum picture is the Fourier transform of (37),

$$\tilde{u}_{00}(P,z) = \frac{1}{\sqrt{\pi \det \kappa}}\exp\left(-\frac{1}{2}P\mu P\right). \tag{38}$$

From the relation between the coordinate and the momentum picture in Eq. (30), together with (14), the z dependence of μ is found as

$$\mu(z) = \mu(0) + \frac{iz}{k}, \tag{39}$$

or, equivalently,

$$\delta(z) = \delta(0), \quad \mu_1(z) = \mu_1(0) + \frac{z}{k}. \tag{40}$$

The fundamental mode is characterized by selecting δ and μ_1 in one transverse plane. Now we use Eqs. (35) and (34) to express $\gamma(z)$ and $\alpha_1(z)$ in terms of $\delta(0)$ and $\mu_1(0)$. We find that

$$\gamma^2(z) = \delta^{-2} + \left(\mu_1(0) + \frac{z}{k}\right)\delta^2\left(\mu_1(0) + \frac{z}{k}\right), \tag{41}$$

$$\gamma^2(z)\alpha_1(z) = \left(\mu_1(0) + \frac{z}{k}\right)\delta^2, \tag{42}$$

where we write δ for $\delta(0)$ since δ is independent of z. In each transverse plane the matrix γ determines the intensity distribution, while α_1 determines the phase distribution of the fundamental mode. In the limit of large z we find that

$$\gamma^2(z) \to \frac{z^2}{k^2}\delta^2, \quad \alpha_1(z) \to \frac{k}{z}. \tag{43}$$

The first identity is a generalization of the Fourier relation between the near field and the far field. The phase distribution, as determined by α_1, becomes circular for large z. This is understandable, because in the far field the beam profile in the focal region can be considered a point source, which produces circular phase fronts.

The propagation properties of the fundamental mode in the coordinate picture are determined by $\gamma(z)$ and $\alpha_1(z)$. In general, the fundamental mode has two focal planes, which are defined by the requirement that the determinant of α_1 vanishes. Then the curvature of the phase fronts vanishes in one transverse direction, and the curves of constant phase are parallel lines. When the determinant of α_1 vanishes, so does the determinant of μ_1, which follows from Eq. (33). It follows from Eq. (40) that det $\mu_1(z)=0$ for the two values of z that coincide with an eigenvalue of $\mu_1(0)$ times $-k$. We consider the evolution of ellipses of constant intensity during propagation. The ellipses are the curves of constant $R\alpha_0 R$. Since $\alpha_0=\gamma^{-2}$, the orientation and ellipticity of the ellipses are determined by the eigenvectors and eigenvalues of γ^2. We take the x and y axis to coincide with the eigenvectors of μ_1, and redefine z such that the plane $z=0$ is exactly in between the two focal planes. Then

$$\mu_1(0)=\begin{pmatrix} m_0 & 0 \\ 0 & -m_0 \end{pmatrix}, \tag{44}$$

with m_0 real. The focal planes are then the planes $z=\pm k m_0$.

In the case of simple astigmatism, δ commutes with $\mu_1(0)$, so that it is diagonal. The matrix $\gamma^2(z)$ in Eq. (41) is then diagonal for all z, and the diagonal elements have a hyperbolic dependence on z. In a focal plane, one of the hyperbolas attains its minimum value. When the diagonal elements of δ are different, the slopes of the asymptotes of the hyperbolas differ, so that for two values of z the intensity distribution is circular. Equation (43) shows that the ellipses of constant intensity at $z\to\pm\infty$ are identical. When m_0 vanishes, there is only one focal plane, and we see from Eq. (41) that $\gamma^2(z)=\gamma^2(-z)$. This symmetry is not present when $m_0 \neq 0$.

For general astigmatism δ is not diagonal. By using that $\alpha_0^{-1}=\gamma^2$ and $\mu_0^{-1}=\delta^2$ it follows from (33) that $[\gamma^2,\alpha_1]=[\mu_1,\delta^2]$. From Eq. (40) it follows that when δ^2 and μ_1 do not commute in one transverse plane, then they do not commute in any other plane. This proves that for general astigmatism the curves of constant intensity are never circular. When propagating from the plane $z=-\infty$ to $z=\infty$, the long axis of the ellipse rotates over 180°, and from Eq. (43) it follows that the ellipticity in the limits $z\to\pm\infty$ is identical. As an example, we assume that the basis in which δ is diagonal, is rotated over 45° with respect to the x and y direction, so that the orientations of δ and μ_1 are maximally different. Then also general astigmatism is maximal. We parametrize without loss of generality,

$$\delta=\frac{1}{2}\begin{pmatrix} d_+ + d_- & d_+ - d_- \\ d_+ - d_- & d_+ + d_- \end{pmatrix}, \tag{45}$$

where $d_+ > d_- > 0$ are the eigenvalues of δ. For the case $d_+ = \sqrt{3} d_-$ an ellipse of constant intensity is shown in Fig. 2 for different transverse planes. When propagating from $z=-\infty$ to $z=0$, the long axis of the ellipse rotates over 90°, and the ellipticity in the plane $z=0$ is the same as for $z\to\pm\infty$. This is confirmed by substituting (44) and (45) in Eq. (41), from which we find after explicity calculation that

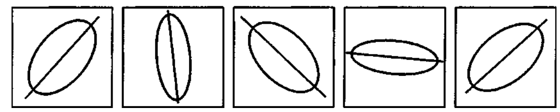

FIG. 2. An ellipse of constant intensity of the fundamental mode in different planes z for the case that general astigmatism is maximal. The ellipses in different planes are not to scale.

$$\gamma^2(0)=\delta^{-2}+\mu_1(0)\delta^2\mu_1(0)=(1+s^2)\delta^{-2}, \quad s=m_0 d_+ d_-. \tag{46}$$

From Eq. (43) it then follows that $\gamma^2(\pm\infty)\propto\gamma^{-2}(0)$. The inversion of γ corresponds to a rotation of the ellipse over 90°, where the ellipticity remains the same. We can see from Eqs. (41), (44), and (45), that replacing z by $-z$ is identical to interchanging the diagonal elements of γ^2. As a consequence, the ellipse in an arbitrary plane z is the image of the ellipse in the plane $-z$ as obtained by a reflection in the line $x=y$.

The curves of constant phase are defined by $R\alpha_1 R$ constant. In a focal plane one of the eigenvalues of α_1 is zero, and the curves of constant phase are lines parallel to the corresponding eigenvector. In the first and the second focal plane μ_1 takes the form

$$\mu_1=\begin{pmatrix} 0 & 0 \\ 0 & -2m_0 \end{pmatrix}, \quad \mu_1=\begin{pmatrix} 2m_0 & 0 \\ 0 & 0 \end{pmatrix}, \tag{47}$$

respectively. Writing $\vec{e}_x$ and $\vec{e}_y$ for the x and y direction, respectively, it follows from Eq. (33) that in the first focal plane $\alpha_1\mu_0\vec{e}_x=0$, while in the second focal plane $\alpha_1\mu_0\vec{e}_y=0$. Therefore, $\mu_0\vec{e}_x$ and $\mu_0\vec{e}_y$ determine the direction of the parallel lines of constant phase in the first and the second focal plane, respectively. For simple astigmatism $\delta^2=\mu_0^{-1}$ is diagonal, so that the angle between the column vectors is 90°. Then the parallel lines of constant phase in one focal plane are orthogonal to the parallel lines in the other focal plane. When δ is given by Eq. (45) we find that the angle ψ_0 between the lines in the two focal planes is given by

$$\cos\psi_0=\frac{d_+^4-d_-^4}{d_+^4+d_-^4}, \tag{48}$$

and we see that the lines in the two focal planes are not orthogonal. In the plane $z=0$, we can calculate $\alpha_1(0)$ by using (42) and (46), with the result

$$\alpha_1(0)=\frac{d_+^2 d_-^2}{1+s^2}\mu_1(0). \tag{49}$$

In this plane, the lines of constant phase are hyperbolas, with orthogonal asymptotes.

B. Higher-order modes in one transverse plane

The higher-order modes are obtained by operating with the raising operator on the fundamental mode. The lowering operator in Eq. (4) is written in a more transparent way by using the operator identity

$$\exp\left(\frac{i}{2}\hat{R}\alpha_1\hat{R}\right)\hat{P}\exp\left(-\frac{i}{2}\hat{R}\alpha_1\hat{R}\right) = \hat{P} - \alpha_1\hat{R}. \quad (50)$$

In the coordinate picture we write

$$\hat{A} = \exp\left(\frac{i}{2}\hat{R}\alpha_1\hat{R}\right)\sigma\hat{B}\exp\left(-\frac{i}{2}\hat{R}\alpha_1\hat{R}\right), \quad (51)$$

where the real operators $\hat{b}_x$ and $\hat{b}_y$ that compose the vector

$$\hat{B} = \frac{1}{\sqrt{2}}\left(\frac{1}{\gamma}\hat{R} + i\gamma\hat{P}\right) \quad (52)$$

have the familiar form of the lowering operator of a quantum-mechanical harmonic oscillator. The corresponding expression for the raising operator is

$$\hat{A}^\dagger = \exp\left(\frac{i}{2}\hat{R}\alpha_1\hat{R}\right)\sigma^*\hat{B}^\dagger\exp\left(-\frac{i}{2}\hat{R}\alpha_1\hat{R}\right). \quad (53)$$

We consider the operation of the raising operator on the fundamental mode in the coordinate representation in an arbitrary transverse plane. We see that first the phase distribution $\exp(iR\alpha_1 R/2)$ in the expression for the fundamental mode in Eq. (1) is removed, so that the real operator $\hat{B}^\dagger$ operates on a real function. Finally the same phase factor $\exp(iR\alpha_1 R/2)$ is reinserted again, so that all the modes have this phase factor in common.

The unitary matrix σ expresses the freedom in the choice of the ladder operators. For nonastigmatic modes, σ determines whether the higher-order modes are HG or LG, or intermediate modes. In characterizing σ we point out an analogy with polarization. It is intuitively obvious that the HG and LG modes are analogous to linear and circular polarization, respectively, while the intermediate modes are analogous to elliptical polarization. It is customary to represent a normalized polarization vector $\vec{e}$ in the xy plane by a point on the unit sphere, which specifies a real unit vector,

$$\vec{u} = (\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta) \equiv (u_1, u_2, u_3), \quad (54)$$

where θ and ϕ are the spherical angles of the point on the sphere (thus the Greek letters θ and ϕ are not used for matrices). The vector $\vec{u}$ is the normalized Stokes vector [21], with components

$$u_1 = e_x^* e_x - e_y^* e_y, \quad u_2 = e_x^* e_y + e_x e_y^*, \quad u_3 = -ie_x^* e_y + ie_x e_y^*. \quad (55)$$

The first two components are the degrees of linear polarization parallel to the x and the y axis, and under 45°, and u_3 is the degree of circular polarization. Each point on the sphere uniquely represents a polarization apart from its overall phase. The sphere is called the Poincaré sphere. A point on the equator of the sphere ($\theta=\pi/2$) represents linear polarization, which is oriented in a direction that makes an angle $\phi/2$ with the x axis. The poles of the sphere ($\theta=0$, and $\theta=\pi$) represent circular polarization $\vec{e}=\vec{e}_\pm=(\vec{e}_x \pm i\vec{e}_y)\sqrt{2}$. For arbitrary values of the angles θ and ϕ, the point on the sphere represents the elliptical polarization,

$$\vec{e}_1(\theta,\phi) = v\vec{e}_+ + w\vec{e}_-, \quad (56)$$

with $v=\cos(\theta/2)\exp(-i\phi/2)$ and $w=\sin(\theta/2)\exp(i\phi/2)$. The polarization,

$$\vec{e}_2(\theta,\phi) = -w^*\vec{e}_+ + v^*\vec{e}_-, \quad (57)$$

which is orthogonal to $\vec{e}_1$, corresponds to the antipodal position on the Poincaré sphere.

In a similar way we can characterize the possible choices of a basis of transverse modes as a point on the sphere, which we shall call the Hermite-Laguerre sphere. For this we have to specify the matrix σ for each value of θ and ϕ. Equation (53) gives the two raising operators $\hat{a}_1^\dagger$ and $\hat{a}_2^\dagger$ as linear combinations of the operators $\hat{b}_x^\dagger$ and $\hat{b}_y^\dagger$, when we write the real operator (52) in terms of its components as $\hat{B}=(\hat{b}_x,\hat{b}_y)$. When we likewise decompose $\sigma\hat{B}$ as $(\hat{b}_1,\hat{b}_2)$, we simply require that the raising operators $\hat{b}_1^\dagger$ and $\hat{b}_2^\dagger$ are related to $\hat{b}_x^\dagger$ and $\hat{b}_y^\dagger$ by the same linear transformation that relates $\vec{e}_1$ and $\vec{e}_2$ to $\vec{e}_x$ and $\vec{e}_y$, apart from an overall phase factor. Because of the unitarity of σ, it is sufficient to specify only the first row of σ, which defines $\hat{a}_1$ as

$$\hat{a}_1 = \exp\left(\frac{i}{2}\hat{R}\alpha_1\hat{R}\right)(\sigma_{1x}\hat{b}_x + \sigma_{1y}\hat{b}_y)\exp\left(-\frac{i}{2}\hat{R}\alpha_1\hat{R}\right). \quad (58)$$

In analogy to Eq. (56), this corresponds to the point on the Hermite-Laguerre sphere specified by

$$\cos\phi\sin\theta = \sigma_{1x}\sigma_{1x}^* - \sigma_{1y}\sigma_{1y}^*,$$

$$\sin\phi\sin\theta = \sigma_{1x}\sigma_{1y}^* + \sigma_{1x}^*\sigma_{1y},$$

$$\cos\theta = -i\sigma_{1x}\sigma_{1y}^* + i\sigma_{1x}^*\sigma_{1y}. \quad (59)$$

Then the operator $\hat{a}_2$ corresponds automatically to the antipodal point on the sphere. Hence, strictly speaking, it is sufficient to consider only the matrices σ that correspond to points on the northern hemisphere. Since the phase constants of the modes are unimportant, the phases of the rows of σ are not relevant.

Choices of σ corresponding to a point on the equator of the Hermite-Laguerre sphere, give rise to HG-like modes, while the poles correspond to LG-like modes. For simplicity we take the x and y direction along the eigenvectors of γ. The diagonal elements of γ are γ_x and γ_y. For σ we take the unit matrix, for which $\theta=\pi/2$ and $\phi=0$, which is on the equator. This means that σ and γ have the same orientation. By using Eqs. (25) and (1) and we find that the normalized higher-order mode functions $\langle R|u_{nm}\rangle$ are given by

$$\langle R|u_{nm}\rangle = \frac{1}{\sqrt{n!\,m!\,\pi 2^{n+m}\gamma_x\gamma_y}} H_n\left(\frac{x}{\gamma_x}\right)H_m\left(\frac{y}{\gamma_y}\right) \times\exp\left(-\frac{1}{2}R\alpha R\right), \quad (60)$$

where $R=(x,y)$ and

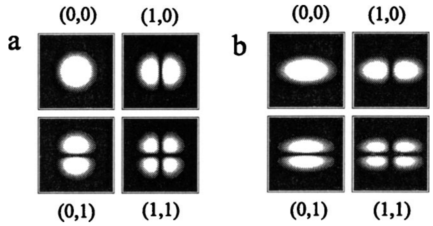

FIG. 3. Intensity profiles of Hermite-Gauss modes with mode numbers (n,m). In (a) the modes are nonastigmatic $(\gamma_x=\gamma_y)$, while in (b) the modes are astigmatic with $\gamma_x/\gamma_y=\sqrt{3}$.

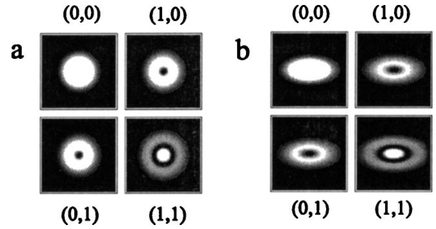

FIG. 5. Intensity profiles of Laguerre-Gauss modes with mode numbers (n,m). In (a) the modes are nonastigmatic, while in (b) the modes are astigmatic with $\gamma_x/\gamma_y=\sqrt{3}$.

$$H_n(\xi)=\exp(\xi^2/2)\left(\xi-\frac{\partial}{\partial\xi}\right)^n\exp(-\xi^2/2),\quad n=0,1,2,\ldots,\tag{61}$$

are the Hermite polynomials. In Fig. 3(a) the intensity profiles of the lowest-order modes are given in the nonastigmatic case $\gamma_x=\gamma_y$. These are the familiar HG modes. In the astigmatic case $\gamma_x\neq\gamma_y$, and the x and y direction are scaled differently compared with the nonastigmatic modes, as can seen in Fig. 3(b).

Now we assume that σ corresponds to the point on the equator with $\phi=\pi/2$. Then the angles of orientation of σ and γ differ by $\pi/4$. In Fig. 4(a) the intensity profiles of the modes are given for the nonastigmatic case. We see that the modes can be obtained from Fig. 3(a) by a rotation over an angle of $\pi/4$. The astigmatic case is given in Fig. 4(b). We see that again the x and the y direction are scaled differently, but compared with Fig. 3(b), the orientation of astigmatism is different from the orientation of the intensity profiles of the modes.

Now we consider the north pole $(\theta=0)$, which gives rise to LG-like modes. When we compare the nonastigmatic case in Fig. 5(a) with the astigmatic case in Fig. 5(b), we see that the x and the y direction are scaled differently. The nonastigmatic LG modes have circular symmetry, so that their orientation before rescaling is immaterial.

As an example of modes that are intermediate between the HG and LG modes we consider the modes where the point on the Hermite-Laguerre sphere is between the equator and the pole at azimuthal angle $\phi=0$. For simplicity we only consider the nonastigmatic case, where γ is a scalar quantity. The mode profiles of the modes contain phase singularities for each point between the pole and the equator, except at the equator itself. When the pole is approached, the phase singularities combine at the origin and give rise to the familiar azimuthal angle dependence $\exp[i(n-m)\phi]$ in the mode profiles of the LG modes, which are OAM eigenfunctions with eigenvalues $\hbar(n-m)$. Starting from a HG mode with $n\neq m$ at $\theta=\pi/2$, the intensity distribution becomes less anisotropic until at the north pole it is a LG mode, with circular symmetry. In Fig. 6 the intensity profiles of four modes are given at the polar angle $\theta=\pi/3$. We see that indeed the modes in Fig. 6 are intermediate between the HG and LG modes in Figs. 3(a) and 5(a), respectively. The profiles of the $(1,0)$ and $(0,1)$ mode contain a phase singularity at the origin, while the $(1,1)$ mode has four phase singularities, which are located in the holes of the intensity distribution. Modes that are intermediate between HG and LG modes have also been discussed elsewhere [22,23].

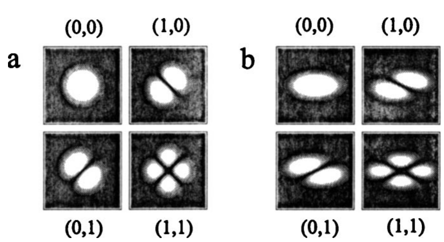

FIG. 4. Intensity profiles of Hermite-Gauss modes with mode numbers (n,m) that are rotated over $\pi/4$. In (a) the modes are nonastigmatic, while in (b) the modes are astigmatic with $\gamma_x/\gamma_y=\sqrt{3}$.

C. Propagation of the higher-order modes

The z dependence of the higher-order modes is determined by the matrix $\sigma(z)$, in addition to the matrices $\gamma(z)$ and $\alpha_1(z)$ discussed in Sec. IV A. In Sec. IV B we demonstrated that the matrix σ can be represented by two opposite points on the Hermite-Laguerre sphere. A description of the propagation of the higher-order modes requires the specification of these points on the sphere as a function of z. We obtain the z dependence of σ by using Eqs. (36) and (40). From Eqs. (30) and (14), it follows that κ is independent of

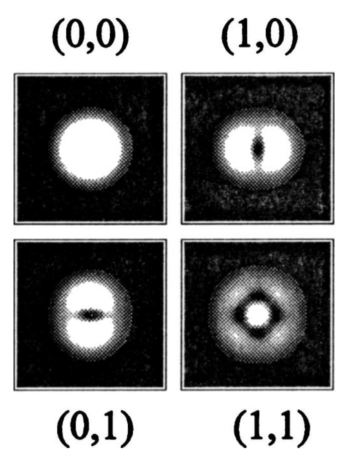

FIG. 6. Intensity profiles of nonastigmatic modes with mode numbers n and m, and which are represented by the point between the equator and the north pole with $\theta=\pi/3$ and $\phi=0$.

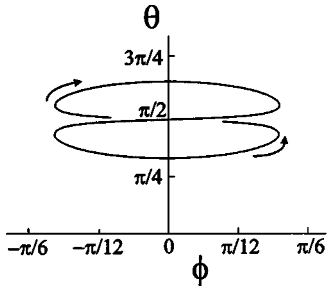

FIG. 7. The trajectory of the point on the Hermite-Laguerre sphere that represents the matrix σ. Between the focal planes at $z=0$, and at $z=\pm\infty$, the point is located at $\theta=\pi/2$ and $\phi=0$.

z. Then it follows from Eqs. (31) and (40), that the matrix τ, which determines the nature of the higher-order modes in the momentum picture, is independent of z. We find that the z dependence of σ is given by

$$\sigma(z) = \tau\delta\left(\delta^{-2} + i\mu_1(0) + \frac{iz}{k}\right)\gamma^{-1}(z). \qquad (62)$$

We see that upon propagation the nature of the higher-order modes changes. From Eq. (43) it follows that for $z\to\pm\infty$ we have $\sigma(z)\to\pm i\tau$. The factor i represents the far field Gouy phase. The representation of $\sigma(z)$ on the Hermite-Laguerre sphere is a trajectory $\vec{u}(z)$ of unit vectors. The limits $\vec{u}(\pm\infty)$ correspond to the Hermite-Laguerre representation of the unitary matrix τ. The choice of τ determines the nature of the higher-order modes in the far field.

Alternatively, we may choose σ for a single value of z. We specify σ by the angles $\theta=\pi/2$ and $\phi=0$, at the plane $z=0$, which is halfway between the focal planes. Then σ is basically the unit matrix in this plane. We consider the evolution of σ in the case that the angle of orientation between δ and $\mu_1(0)$ is maximally different, and write $\mu_1(0)$ and δ as in Eqs. (44) and (45), respectively. In Fig. 7 the trajectory of the point on the sphere is given for the case that $s=m_0d_+d_-=4\sqrt{3}$ and $d_+=\sqrt{3}d_-$. For $z=0$ the point is on the equator, while for negative and positive values of z the point is on the southern and northern hemisphere, respectively. For $z\to\pm\infty$ the point approaches the equator. This implies that, except for $z=0$ and $z=\pm\infty$, the modes contain phase singularities, as we discussed in Sec. IV B.

Now we express the point on the Hermite-Laguerre sphere describing σ at $z=\pm\infty$ in terms of the point representing σ at $z=0$, for the case that the angles of orientation of $\mu_1(0)$ and δ are maximally different. Using the expressions for $\mu_1(0)$ and δ in Eqs. (44) and (45), respectively, we find that

$$\sigma(0) = \frac{\tau}{\sqrt{1+s^2}}\begin{pmatrix} 1+is & 0 \\ 0 & 1-is \end{pmatrix}. \qquad (63)$$

Since $\sigma(\pm\infty)=\pm i\tau$, this equation gives a relation between $\sigma(0)$ and $\sigma(\pm\infty)$. The point on the sphere can be given in terms of its Cartesian components as $\vec{u}=(u_1,u_2,u_3)$, as in Eq. (54). Writing $\vec{u}(0)$ and $\vec{u}(\infty)$ for the coordinates of the point on the sphere at $z=0$ and $z=\pm\infty$, respectively, we find that

$$u_1(\infty) = u_1(0),$$

$$u_2(\infty) = \frac{1-s^2}{1+s^2}u_2(0) + \frac{2s}{1+s^2}u_3(0),$$

$$u_3(\infty) = \frac{1-s^2}{1+s^2}u_3(0) - \frac{2s}{1+s^2}u_2(0), \qquad (64)$$

where $s=m_0d_+d_-$. The point at $z=\pm\infty$ is obtained from the point at $z=0$ by a rotation about the 1 axis over an angle 2 arctan s. We see that only when $\theta(0)=\pi/2$ and $\phi(0)=0$, as in Fig. 7, the location of the point on the sphere is the same for $z=0$ as for $z=\pm\infty$.

When, for $s=1$, the point is on the north pole at $z=0$, it approaches the equator at $\phi=\pi/2$ for $z\to\pm\infty$. In the present case, when the point is on the equator with $\phi=0$, the corresponding σ has the same orientation as μ_1, while for $\phi=\pi/2$ the angles of orientation differ by $\pi/4$. The angles of orientation of δ and μ_1 differ also by $\pi/4$ in the present case. Therefore the matrix $\sigma(\pm\infty)$ has the same orientation as δ, and also as $\gamma(0)$, which follows from Eq. (46). This means that at $z=0$ the modes are like the astigmatic LG modes in Fig. 5(b), while for $z\to\pm\infty$ they become like the astigmatic HG modes in Fig. 3(b).

V. ORBITAL ANGULAR MOMENTUM OF THE HIGHER-ORDER MODES

A. Orbital angular momentum for astigmatic modes

We calculate the OAM per photon of a higher-order general astigmatic beam with mode numbers (n,m). As indicated in Eqs. (19) and (20), and the OAM is easily expressed by employing the OAM operator, given by

$$\hat{l} = \hbar(\hat{x}\hat{p}_y - \hat{y}\hat{p}_x) = \hbar\hat{R}\varepsilon\hat{P}, \qquad (65)$$

where the antisymmetric matrix ε is given in Eq. (23). We express $\hat{R}$ and $\hat{P}$ in terms of $\hat{A}$ and $\hat{A}^\dagger$. We multiply Eq. (4) from the left by $\beta^\dagger$ and Eq. (5) by β^T. We add these expressions, and use Eq. (27) to find that

$$\hat{R} = \frac{1}{\sqrt{2}}(\beta^\dagger\hat{A} + \beta^T\hat{A}^\dagger). \qquad (66)$$

Similarly, using Eq. (29), we find that

$$\hat{P} = \frac{1}{i\sqrt{2}}(\kappa^\dagger\hat{A} - \kappa^T\hat{A}^\dagger). \qquad (67)$$

According to Eq. (20) the OAM is then found as

$$L_{nm} = \langle u_{nm}|\hat{l}|u_{nm}\rangle = \frac{\hbar}{4i}[(n+m+2)\mathrm{Tr}(\kappa\varepsilon\beta^\dagger) + (n-m)\mathrm{Tr}(\kappa\varepsilon\beta^\dagger\eta)] + \frac{\hbar}{4i}[(n+m)\mathrm{Tr}(\beta\varepsilon\kappa^\dagger) + (n-m)\mathrm{Tr}(\beta\varepsilon\kappa^\dagger\eta)], \qquad (68)$$

where

$$\eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{69}$$

We now use the momentum picture, and we express the expectation value in terms of the matrices δ, μ_1 and τ. By using Sec. III B, we find

$$\mathrm{Tr}(\kappa\varepsilon\beta^\dagger) = \mathrm{Tr}(\beta\varepsilon\kappa^\dagger) = i\mathrm{Tr}(\varepsilon\delta^2\mu_1), \tag{70}$$

and

$$\kappa\varepsilon\beta^\dagger + \beta\varepsilon\kappa^\dagger = \tau(\delta\varepsilon\delta^{-1} + \delta^{-1}\varepsilon\delta)\tau^\dagger + i\tau\delta[\mu_1,\varepsilon]\delta\tau^\dagger. \tag{71}$$

By using that δ is symmetric, we can show that

$$\delta\varepsilon\delta^{-1} + \delta^{-1}\varepsilon\delta = \varepsilon\mathrm{Tr}(\delta^2)/\det(\delta) \equiv 2r_+\varepsilon. \tag{72}$$

The parameter r_+, defined in Eq. (72), is a measure of the ellipticity of the momentum distribution. For an isotropic distribution corresponding to two equal eigenvalues of δ, r_+ attains the minimum value 1. After substitution of the quantities (70)–(72), in (68), we arrive at the general expression for the OAM:

$$L_{nm} = \frac{\hbar}{2}(n+m+1)\mathrm{Tr}(\varepsilon\delta^2\mu_1) + \frac{\hbar}{2i}(n-m)r_+\mathrm{Tr}(\tau\varepsilon\tau^\dagger\eta) + \frac{\hbar}{4}(n-m)\mathrm{Tr}(\tau\delta[\mu_1,\varepsilon]\delta\tau^\dagger\eta). \tag{73}$$

We see that the OAM in a higher-order astigmatic mode is separated in three contributions. Among the different matrices occurring in Eq. (73), only μ_1 depends on z, as is expressed by Eq. (40). It is easy to verify that this z dependence does not survive the trace, and we conclude that each of the three terms in (73) is invariant under free-space propagation.

The OAM in the mode $|u_{nm}\rangle$ can also be described in the coordinate picture, in terms of the matrices γ, α_1 and σ. By using (28) and (30) to express β and κ in these three matrices, we obtain from Eq. (68)

$$L_{nm} = \frac{\hbar}{2}(n+m+1)\mathrm{Tr}(\varepsilon\alpha_1\alpha_0^{-1}) + \frac{\hbar}{2i}(n-m)q_+\mathrm{Tr}(\sigma\varepsilon\sigma^\dagger\eta) - \frac{\hbar}{4}(n-m)\mathrm{Tr}(\sigma\gamma[\alpha_1,\varepsilon]\gamma\sigma^\dagger\eta), \tag{74}$$

with $q_+ = \mathrm{Tr}(\gamma^2)/2\det(\gamma)$. This result has the same structure as Eq. (73). The first term of Eqs. (73) and (74) are identical, and independent of z. However, the last two terms in (74) vary with z, although their sum is constant. Therefore, despite the similarity in form, these last two terms in (74) are not identical to the last two terms in (73).

B. Significance of the contributions to the angular momentum

When the mode indices n and m are equal, only the first term in (73) contributes. In the special case of the fundamental mode $n=m=0$, the result for L_{00} coincides with Eq. (22), which is easily checked by applying the identities (33), while recalling that $\delta^2 = \mu_0^{-1}$. The OAM per photon in the fundamental mode $(\hbar/2)\mathrm{Tr}\ \varepsilon\mu_0^{-1}\mu_1$ can be nonzero only when μ_0 and μ_1 do not commute. This is the case of general astigmatism, where the intensity and the phase distribution are astigmatic with a different orientation. The value of the OAM per photon can be many units $\hbar$ [10]. It is remarkable that the OAM increases by the same amount when one of the mode indices n or m is increased by 1. The choice of the unitary transformation τ [or $\sigma(\infty)$] corresponding to a position on the Hermite-Laguerre sphere, is immaterial for this contribution to the OAM.

The second term in (73) does not depend on μ_1, and it is completely determined by the ellipticity of the momentum distribution and by the position on the Hermite-Laguerre sphere of the far field. By using that $\sigma(\infty) = i\tau$, we find from Eqs. (54) and (59) that

$$\mathrm{Tr}\ \tau\varepsilon\tau^\dagger\eta = 2iu_3(\infty), \tag{75}$$

so that the second term in (73) is equal to $\hbar(n-m)r_+u_3(\infty)$. For nonastigmatic modes $(r_+=1)$, the similarity of OAM with the angular momentum due to photon polarization is clear when we recall that $u_3 = \cos\theta$. The polarization-related angular momentum per photon is $\hbar\cos\theta$, where θ is the polar angle of the position on the Poincaré sphere. On the other hand, the two mode indices correspond to opposite points on the sphere, and give an opposite contribution to the OAM. When $\theta=0$ (or $u_3=1$), the basis of LG modes is recovered. In this case, the well-known expressions [8] for the OAM are recovered. When $\theta \neq 0$, the point moves away from the pole of the sphere, and the basis sets of modes are intermediate between LG and HG modes. The unit of angular momentum is reduced by a factor $\cos\theta$. Astigmatism of the momentum distribution as characterized by a nonisotropic matrix δ, increases the unit of angular momentum by the factor $r_+ > 1$.

The third term in (73) depends on all three matrices δ, μ_1 and τ. For simple astigmatism with coinciding focal planes, μ_1 is isotropic, and this third term vanishes. For simple astigmatism with two different focal planes, and for general astigmatism this third term is nonzero in general.

C. Explicit examples

Now we consider the OAM for an astigmatic beam where the angles of orientation of $\mu_1(0)$ and δ are maximally different. The corresponding fundamental mode was discussed in Sec. IV A. The matrices $\mu_1(0)$ and δ are represented in Eqs. (44) and (45), respectively. We find from (73) that

$$\begin{aligned} L_{nm} &= \hbar(n+m+1)sr_- + \hbar(n-m)r_+[u_3(\infty) + su_2(\infty)] \\ &= \hbar(n+m+1)sr_- + \hbar(n-m)r_+[u_3(0) - su_2(0)], \end{aligned} \tag{76}$$

where $r_\pm = (d_+^2 \pm d_-^2)/2d_+d_-$ and $s = m_0d_+d_-$. The second equality follows from the relation (64), which expresses $\vec{u}(\infty)$ in terms of $\vec{u}(0)$, which are the coordinates of the point on the sphere that represents the unitary matrix $\sigma(0)$, in the plane halfway between the two focal planes. The fundamental mode in this plane is specified by $\gamma(0)$ as given in Eq. (46), which defines the elliptic intensity distribution, and $\alpha_1(0)$ as given in Eq. (49), which determines the phase distribution. In order to define the higher-order modes, we have to choose $\vec{u}(0)$, and we consider the three cases that $\vec{u}(0)$ is one of the three basis vectors.

When the unit vector $\vec{u}(0)$ is chosen at the north pole of the Hermite-Laguerre sphere, we have $u_3(0)=1$. The modes in the plane $z=0$ are astigmatic LG modes. When the momentum distribution is isotropic, so that $r_+=1$, $r_-=0$, the intensity profiles in the plane $z=0$ are the same as those of the standard nonastigmatic LG modes, which are shown in Fig. 5(a). The well-known result $\hbar(n-m)$ for the OAM of LG modes is recovered [8].

In the case that $u_2(0)=1$, the intensity pattern of the field in the plane $z=0$ is of the HG type, oriented in the directions $x=\pm y$. This orientation coincides with the orientation of δ and $\gamma(0)$, and the intensity profiles of the modes in the plane $z=0$ is as sketched in Fig. 3(b), apart from a rotation over $\pi/4$. The OAM (76) takes the attractively simple form

$$L_{nm} = \hbar s r_- - \hbar s\left(n\frac{d_-}{d_+} - m\frac{d_+}{d_-}\right). \tag{77}$$

This is nonzero only when the two focal planes are different, which implies that $m_0 \neq 0$. The first term is the OAM of the fundamental mode. Higher-order modes in the $x=y$ direction, corresponding to higher values of n, give a negative contribution to the OAM, while higher mode numbers m lead to a positive contribution. The anisotropy of the momentum distribution ($d_+ \neq d_-$) makes these contributions different in size.

Finally, we turn to the case that $u_1(0)=1$, which corresponds to intensity patterns in the plane $z=0$ of the HG type, oriented along the x and the y axis. Since the orientation differs from the orientation of $\gamma(0)$ by $\pi/4$, the patterns are as sketched in Fig. 4(b), apart from a rotation over $\pi/4$. It follows from Eq. (64) that $u_1(\infty)=1$ as well, so that the far field also has an HG intensity pattern. The intensity profiles of the higher-order modes have the same orientation as the phase distribution. This is a situation that does not give rise to OAM, like we saw for the fundamental mode. Therefore, the expression for the OAM in (76) does not contain $u_1(0)$. The OAM in Eq. (76) is simply given by $\hbar(n+m+1)sr_-$, which is again nonzero only when the focal planes are different.

VI. CONCLUSIONS AND DISCUSSION

We have presented an operator method to characterize in a systematic way the possible complete orthonormal sets of astigmatic Gaussian modes. The fundamental mode is specified by a Gaussian function in terms of a complex symmetric 2×2 matrix α. The real part of the matrix specifies the elliptic intensity distribution, and the imaginary part defines the phase distribution. The behavior of the mode under propagation, as determined by the paraxial wave equation, is given by Eqs. (15) and (14), in terms of a simple z-dependence of the matrix.

A general Gaussian function is also characterized by the requirement that it vanishes when a lowering operator is applied. In two dimensions, this defines two independent lowering operators. Higher-order modes can be constructed by repeated application of the corresponding raising operators $\hat{a}_1^\dagger$ and $\hat{a}_2^\dagger$, as expressed in Eq. (25). Since there are two independent raising operators, the choice of $\hat{a}_1^\dagger$ and $\hat{a}_2^\dagger$ as linear combinations of the basis set is an inherent degree of freedom, which determines the nature of the set of higher-order modes for a given fundamental mode. This freedom is used by selecting the unitary matrix σ, in one transverse plane. When the linear combinations are basically real, σ is equivalent to a rotation in two dimensions, and the set of modes has the nature of HG modes, which is reminiscent of linear polarization. In general the choice of the higher-order modes for a given fundamental mode can be represented by a point on the unit sphere. This Hermite-Laguerre sphere is analogous to the Poincaré sphere that represents polarization. The power of this description is that astigmatism and the Hermite-Laguerre nature of the modes as a function of the transverse plane is automatically accounted for. The z dependence of the symmetric matrix α (determining astigmatism) and the unitary matrix σ (which describes the Laguerre/Hermite nature of the modes) are studied in Sec. IV. The OAM of the various sets of modes is analyzed in Sec. V. It can be separated in a term that depends only on astigmatism and terms that depend also on the selection of the nature of the modes, as determined by the point on the Hermite-Laguerre sphere. In general, astigmatism has a tendency to enhance the OAM.

Simple astigmatism is naturally imposed on a nonastigmatic beam by sending it through an astigmatic lens (or, equivalently, by reflecting it by an astigmatic mirror). Simple astigmatism can be converted into general astigmatism by a second astigmatic lens, with an orientation different from the first. An astigmatic lens can be specified by a two-dimensional real symmetric matrix ζ, with eigenvalues $1/f_1$ and $1/f_2$ the inverse focal lengths, and with eigenvectors in the corresponding directions in the transverse plane. The effect of the lens is simply a multiplication of the mode by the factor $\exp(-ikR\zeta R/2)$. This is equivalent to a change of the matrix α that determines astigmatism. The effect of the lens is that α is replaced by $\alpha+ik\zeta$ after the lens, which implies that α_1 is replaced by $\alpha_1-k\zeta$. The matrix σ remains unaffected by the lens. Combined with the effect of free propagation, which is expressed in the simplest way by Eq. (39), this information is sufficient to evaluate the variation of all sets of modes as they propagate through a given lens system.

The paraxial wave equation is identical in form to the two-dimensional Schrödinger equation for a free particle. As a consequence, the complete sets of solutions that we derived in this paper are also complete sets of solutions of the Schrödinger equation for a free particle. The fundamental mode corresponds to a wave packet that tumbles upon propagation, and therefore has OAM. Also, the wave packet has a minimal size at two instants of time, corresponding to the two focal planes. In general, when the point on the Hermite-Laguerre sphere is not on the equator, the higher-order modes contain vortices, and the corresponding wave packets do as well. The operator method can easily be extended to three dimensions, so that solutions of the three-dimensional Schrödinger equation for a free particle are found.

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