Real-time Fault Estimation for a Class of Discrete-Time Linear Parameter-Varying Systems

Chris van der Ploeg, Emilia Silvas, Nathan van de Wouw , Peyman Mohajerin Esfahani

***Abstract*— Estimating and detecting faults is crucial in ensuring safe and efficient automated systems. In the presence of disturbances, noise or varying system dynamics this estimation is even more difficult. To address this challenge, this article proposes a novel filter to estimate multiple fault signals for a class of discrete-time linear parameter-varying (LPV) systems. The design of such a filter is formulated as an optimization problem and solved recursively, while the system dynamics vary over time. To synthesize the filter, conditions for existence and detectability of the fault are introduced and the problem is formulated and solved using the quadratic programming framework. The strength of this method is that an analytical solution can be derived to approximate an optimal estimation filter at each time instance, which supports real-time implementation.**

The method is illustrated and validated on an automated vehicle lateral dynamics, which is often used as example for LPV systems. The results show that the estimation filter can well-estimate and decouple unknown disturbances and known or measurable parameter variations in the dynamics while detecting the unknown fault. Furthermore, it is shown that the proposed filter is applicable to realistic, noisy situations and can be adapted to maintain a good estimation performance.

***Index Terms*— Fault diagnosis, fault estimation, linear parameter-varying systems.**

I. INTRODUCTION

THE problem of fault diagnosis has been an extensively studied topic over the past decades. The detection and estimation of a fault can support an action of the system mitigating the effect of the fault, improving the safety of the system and potential users. In literature, various categories of fault diagnosis methods are elaborated upon, see [1], [2] and the references therein. In the scope of fault *detection* (i.e., detecting the presence of a fault) and *estimation* (i.e., determining the exact magnitude and shape of a fault), choosing between fault-sensitivity and attenuation/decoupling of disturbances and uncertainties is often the most difficult trade-off [3]. The task of *isolation* can be seen as a special case of detection and estimation, where not only disturbances and uncertainties are considered for attenuation/decoupling but also any other fault which is not of interest, though, the complexity of this problem highly depends on the condition of fault isolability [4].

The class of linear parameter-varying (LPV) systems is often considered in the scope of fault detection and estimation and is particularly suitable for treating non-linear systems with parameter variations as linear systems with time-varying and potentially measurable parameters. Finding an appropriate solution for the problem of fault sensitivity and disturbance/uncertainty insensitivity for LPV systems is a difficult trade-off, as it highly depends on the requirements (i.e., fault sensitivity versus disturbance/uncertainty attenuation [5]). A class of solutions is defined through the use of linear matrix inequalities (LMI) to robustly formulate the sensitivity problem in an optimization framework using Lyapunov functions. Related to this solution, parameter-independent Lyapunov functions [6] are used in a polytopic framework, which due to their time-independent nature could result in conservative solutions [7]. Other works consider the use of parameter-dependent Lyapunov functions for filter synthesis in either a polytopic framework [8] or in a linear fractional transformation framework [9]. These methods show to be able to incorporate disturbances, uncertainties and time-varying system parameters, although their computational burden is often high and may not guarantee the full decoupling of disturbances, hence being less suitable for the task of isolation/estimation.

A different solution to the LPV detection/estimation problem is the use of a geometric approach. By exploiting the known model-description, fault/disturbance directions that are not of interest can be projected in parameter-varying unobservable subspaces of the fault detector/estimator [10]. A nullspace approach, an application of the geometric approach, is proposed in [11], which has been extended by a robust formulation for non-linear systems [12]. This approach has also been applied for parameter-varying systems [13], albeit for continuous-time model descriptions.

Our contributions: In summary, there exist many approaches to the problem of fault detection and estimation for linear parameter-varying systems. Yet, there does not yet exist a solution which could guarantee the decoupling of disturbances, while isolating and estimating the fault of interest in real-time (i.e., having a *practically implementable* solution in the form of a discrete filter with *low computational burden*).

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As such, we define our contributions as follows.

1) **Discrete-time filter synthesis:** We propose a novel parameter-varying polynomial decomposition (similar to the Hurwitz stability matrix) for LPV dynamical systems, which allows the formulation of a convex optimization problem for isolation/estimation filter synthesis, at each time instance.
2) **Conditions of isolability:** At the base of fault isolation and estimation, lie the conditions for *existence* for such a filter. In Fact III.2 we offer these conditions, which, do not yet exist in literature, using the novel polynomial time-varying matrix construction from Lemma III.1. This could allow for *tractable* evaluation of isolability for LPV systems.
3) **Real-time implementable solution:** Due to the convex nature of the proposed LPV estimation filter synthesis, we can determine an analytical solution for determining a filter polynomial for the detection/estimation task. This analytical solution allows for an *implementable* real-time filter synthesis at each time-step using the valuable practical considerations on implementation and numerical well-posedness with regards to LPV systems.

The LPV estimation filter is demonstrated on the lateral dynamics of an automated vehicle, a popular illustrative example for LPV fault detection/estimation techniques [14], [15], [16]. Herein, the estimation challenge is to detect an offset in the steering system, while the vehicle can have a time-varying yet measurable longitudinal velocity, representing the scheduling parameter.

The outline of this work is as follows. First, the problem formulation is provided in Section II. In Section III, the design of the LPV estimation filter is provided. Moreover, the problem is considered from a practical perspective, showing that the synthesis of such a estimation filter can be implemented by the use of generic computational tools, e.g., matrix inversion. In Section IV, the estimation filter is demonstrated by application to an example of the lateral dynamics of an automated vehicle. Finally, Section V draws conclusions and future work possibilities.

II. Model description and preliminaries

In this section, a class of LPV systems is introduced along with some basic definitions. The model is an LPV extension of the differential-algebraic equations (DAE) class of models introduced in [11] and is described as

$$H(w_k, \mathfrak{q})[x] + L(w_k, \mathfrak{q})[z] + F(w_k, \mathfrak{q})[f] = 0, \tag{1}$$

where $\mathfrak{q}$ represents the shift operator (i.e., $\mathfrak{q}[x(k)] = x(k+1)$), x, z, f, w represent discrete-time signals indexed by the discrete time counter k, taking values in $\mathbb{R}^{n_x}, \mathbb{R}^{n_z}, \mathbb{R}^{n_f}, \mathbb{R}^{n_w}$. The matrices $H(w_k, \mathfrak{q}), L(w_k, \mathfrak{q}), F(w_k, \mathfrak{q})$ are parameter-varying polynomial functions in the variable $\mathfrak{q}$, depending on the parameter signal w with n_r rows and n_x, n_z, n_f columns, respectively. The explicit relationship between the scheduling parameter signal w and time is assumed to be unknown, which renders the description of this system to be more suitable to the LPV model class as opposed to the linear time-variant model class [17]. Finally, w represents a scheduling parameter which is assumed to be unknown a-priori, but is measurable in real-time and takes values from a compact set $\mathcal{W} \subseteq \mathbb{R}^{n_w} \forall k$. In addition to the parameter signal w, we assume that the signal z is also measurable up to the current time k, while the signals x and f are unknown and represent the state of the system and the fault, respectively.

Remark II.1 (Non-measurable scheduling parameters or model uncertainty)**.** *Several suggestions exist in literature, in the scope of geometric nullspace-based estimation filters, which can be used in making these filters suitable for* non-measurable *scheduling parameters w [13, Section 3.3]. Notice, that the proposed approximation methods are directly suitable in the results from this manuscript.*

The model (1) encompasses a large class of parameter-varying dynamical systems, an example of which is a set of LPV state-space difference equations. This example will be used in the simulation study and can be derived from (1) by starting from the following parameter-varying difference equations:

$$\begin{cases} G(w_k)X(k+1) = A(w_k)X(k) + B_u(w_k)u(k) \\ \qquad +B_d(w_k)d(k) + B_f(w_k)f(k), \\ y(k) = C(w_k)X(k) + D_u(w_k)u(k) \\ \qquad +D_d(w_k)d(k) + D_f(w_k)f(k). \end{cases} \tag{2}$$

Herein, $u(k)$ represents the input signal, $d(k)$ the exogenous disturbance, $X(k)$ the internal state, $y(k)$ the state measurements and $f(k)$ the fault. By defining $z := [y; u]$, $x := [X; d]$ and the parameter-varying polynomial matrices

$$L(w_k, \mathfrak{q}) := \begin{bmatrix} 0 & B_u(w_k) \\ -I & D_u(w_k) \end{bmatrix}, \quad F(w_k, \mathfrak{q}) := \begin{bmatrix} B_f(w_k) \\ D_f(w_k) \end{bmatrix},$$
$$H(w_k, \mathfrak{q}) := \begin{bmatrix} -G(w_k)\mathfrak{q} + A(w_k) & B_d(w_k) \\ C(w_k) & D_d(w_k) \end{bmatrix},$$

in (1), it can be observed that (2) is an example of the model description (1).

In the absence of a fault signal f, i.e., for $f = 0$, all possible z-trajectories of the system (1) can be denoted as

$$\mathcal{M}(w) := \{z : \mathbb{Z} \to \mathbb{R}^{n_z} \,|\, \exists x : \mathbb{Z} \to \mathbb{R}^{n_x} : \\ H(w_k, \mathfrak{q})[x] + L(w_k, \mathfrak{q})[z] = 0\}, \tag{3}$$

which is called the *healthy* behavior of the system. For fault detection, the primary objective is to identify whether the trajectory z belongs to this behavior. More specifically, the detection challenge is defined as making a proper LPV estimation filter which outputs zero when $z \in \mathcal{M}$ and gives a non-zero output when the fault signal f is non-zero.

III. Design of parameter-varying estimation filter

In [11], an LTI system, also known as a residual generator, is proposed via the use of an irreducible polynomial basis for the nullspace of $H(w_k, \mathfrak{q})$, denoted by $N_H(w, \mathfrak{q})$ [1]. In this work,

[1]In the remainder of this work, by not explicitly mentioning the time index "k" in w, we emphasize that the filter coefficients may depend on the parameter signal w in multiple time instances.

we take the problem a step further by finding an irreducible polynomial basis for the nullspace of $H(w_k, \mathfrak{q})$, denoted by $N_H(w, \mathfrak{q})$. Such a polynomial fully characterizes the healthy behavior of the system (1) as follows

$$\mathcal{M}(w) = \{z : \mathbb{Z} \to \mathbb{R}^{n_z} \mid N_H(w, \mathfrak{q}) L(w_k, \mathfrak{q})[z] = 0\}. \tag{4}$$

For the design of a estimation filter, it suffices to introduce a linear combination $N(w_k, \mathfrak{q}) = \gamma(w_k, \mathfrak{q}) N_H(w_k, \mathfrak{q})$, such that the following objectives for fault detection can be achieved:

$$\begin{cases} a^{-1}(\mathfrak{q}) N(w, \mathfrak{q}) H(w_k, \mathfrak{q}) = 0, & \forall w_k \in \mathcal{W}, \\ a^{-1}(\mathfrak{q}) N(w, \mathfrak{q}) F(w_k, \mathfrak{q}) \neq 0, & \forall w_k \in \mathcal{W}. \end{cases} \tag{5}$$

Here, the polynomial $a(\mathfrak{q})$ is intended to make the estimation filter proper. Moreover, it enables a form of noise attenuation, which is highly recommended towards experimental applications. The above conditions allow us to find a filter to decouple the residual from the time-varying behavior of the system. In fulfilling the requirements of (5), a proper LPV estimation filter of the following form can be created:

$$r := a^{-1}(\mathfrak{q}) N(w, \mathfrak{q}) L(w_k, \mathfrak{q})[z]. \tag{6}$$

Note, that the degree of $a(\mathfrak{q})$ is not less than the degree of $N(w_k, \mathfrak{q}) L(w_k, \mathfrak{q})$ and is stable and that the design of such polynomial is up to the user and can depend on various criteria (e.g., noise sensitivity). In the following lemma, a method to transform the conditions (5) into non-complex, scalar or vector equations is provided, forming a basis for the methodology proposed in the next section.

Lemma III.1. *Let $N(w, \mathfrak{q})$ be a feasible solution to* (5) *where the matrices $H(w_k, \mathfrak{q})$, $F(w_k, \mathfrak{q})$ are as in system* (1) *and have a particular form of*

$$H(w_k, \mathfrak{q}) = \sum_{i=0}^{d_H} H_i(w_k) \mathfrak{q}^i, \quad F(w_k, \mathfrak{q}) = \sum_{i=0}^{d_F} F_i(w_k) \mathfrak{q}^i,$$
$$N(w, \mathfrak{q}) = \sum_{i=0}^{d_N} N_i(w) \mathfrak{q}^i, \quad a(\mathfrak{q}) = \sum_{i=0}^{d_a} a_i \mathfrak{q}^i,$$

where d_H, d_F, d_N, d_a denote the degree of the respective polynomials. Given any parameter signal w, the conditions in (5) *can be equivalently rewritten as*

$$\bar{N}(w) \bar{H}(w) = 0, \tag{7a}$$
$$\bar{N}(w) \bar{F}(w) \neq 0, \tag{7b}$$

where $\bar{N}(w)$, $\bar{H}(w)$, $\bar{F}(w)$, $\bar{a}$ are defined as

$$\bar{N}(w) := \begin{bmatrix} N_0(w) & N_1(w) & \dots & N_{d_N}(w) \end{bmatrix},$$
$$\bar{H}(w) := \begin{bmatrix} H_0(w_{k-d_a}) & \dots & 0 \\ \vdots & \ddots & \vdots \\ H_{d_H}(w_{k-d_a}) & & H_0(w_{k-d_a+d_N}) \\ \vdots & \ddots & \vdots \\ 0 & \dots & H_{d_H}(w_{k-d_a+d_N}) \end{bmatrix}^{\mathsf{T}},$$
$$\bar{F}(w) := \begin{bmatrix} F_0(w_{k-d_a}) & \dots & 0 \\ \vdots & \ddots & \vdots \\ F_{d_F}(w_{k-d_a}) & & F_0(w_{k-d_a+d_N}) \\ \vdots & \ddots & \vdots \\ 0 & \dots & F_{d_F}(w_{k-d_a+d_N}) \end{bmatrix}^{\mathsf{T}},$$
$$\bar{a} := \begin{bmatrix} a_0 & a_1 & \dots & a_{d_a} \end{bmatrix}.$$

Proof. For proving the results of this lemma, observe that (6) can be rewritten as (using (1) and (5)):

$$\begin{aligned} r &= -a^{-1}(\mathfrak{q}) N(w, \mathfrak{q}) F(w_k, \mathfrak{q})[f] \\ -a(\mathfrak{q})[r] &= N(w, \mathfrak{q}) F(w_k, \mathfrak{q})[f], \\ -\sum_{h=0}^{d_a} a_h \mathfrak{q}^h [r] &= \sum_{i=0}^{d_N} N_i(w) \mathfrak{q}^i \sum_{j=0}^{d_F} F_j(w) \mathfrak{q}^j [f] \\ &= \sum_{i=0}^{d_N} \sum_{j=0}^{d_F} N_i(w) F_j(\mathfrak{q}^i[w]) \mathfrak{q}^{i+j} [f] \end{aligned}$$

Multiplication of both sides with $\mathfrak{q}^{-d_a}$, in order to time-shift the relation to result in a present time residual $r(k)$ as a function of previous faults f and residuals r, yields

$$\begin{aligned} -\sum_{h=0}^{d_a} a_h \mathfrak{q}^{h-d_a} [r] &= \sum_{i=0}^{d_N} N_i(w) \mathfrak{q}^{i-d_a} \sum_{j=0}^{d_F} F_j(w) \mathfrak{q}^j [f], \\ &= \sum_{i=0}^{d_N} \sum_{j=0}^{d_F} N_i(w) F_j(\mathfrak{q}^{i-d_a}[w]) \mathfrak{q}^{i-d_a+j} [f], \end{aligned}$$

for which the right-hand side can be rewritten as

$$\bar{N}(w) \bar{F}(w) \Big(\begin{bmatrix} \mathfrak{q}^{-d_a} I & \mathfrak{q}^{1-d_a} I & \dots & \mathfrak{q}^{d_N+d_F-d_a} I \end{bmatrix} [f] \Big).$$

This proves the equivalence of (7b) and (5). The same line of reasoning applies for proving the equivalence of (7a) and (5). □

It is worth noting that the matrices $\bar{N}(w)$, $\bar{H}(w)$, and $\bar{F}(w)$ defined in Lemma III.1 depend on the parameter signal w through $d_a + 1$ consecutive values. That is, at time instant "k" the filter coefficient $\bar{N}(w)$ depends on $\{w_{k-d_a}, \dots, w_k\} \in \mathcal{W}^{d_a+1}$. We, however, refrain from explicitly denoting this dependency and simply use the notation of the entire trajectory w, say $\bar{N}(w)$. In this light and using the result from Lemma III.1, the conditions for fault detectability can be defined as follows.

Fact III.2. *Given the parameter signal w, there exists a feasible solution $\bar{N}(w)$ to the conditions* (7) *if and only if*

$$\textit{Rank}\left(\begin{bmatrix} \bar{H}(w) & \bar{F}(w) \end{bmatrix} \right) > \textit{Rank}\left(\bar{H}(w) \right). \tag{8}$$

The proof is omitted as it is a straightforward adaption from [12, Fact 4.4]. Using the results from Lemma III.1, the main theorem for the LPV fault detection filter can be proposed.

Proposition III.3 (LPV fault detection)**.** *Let the matrices $\bar{H}(w)$ and $\bar{F}(w)$ be matrices as defined in Lemma III.1, then an LPV fault detection filter of the form* (6) *can be found at* *every time instance k, depending on w_k, that fulfills (5), by solving the following convex QP:*

$$\bar{N}^*(w) := \arg\min_{\bar{N}} -\|\bar{N}\bar{F}(w)\|_\infty + \|\bar{N}^\intercal\|_2^2, \tag{9a}$$

$$\text{s.t.} \quad \bar{N}\bar{H}(w) = 0, \tag{9b}$$

where $\|\cdot\|_\infty$ denotes the supremum norm.

Proposition III.3 lays the groundwork for creating a estimation filter for an LPV model with measurable scheduling parameters w. The term in the cost function of (9a), related to the fault polynomial $\bar{F}(w)$ ensures a maximised sensitivity for the fault, whereas the quadratic (regularization) term related to the filter polynomial $\bar{N}(w)$ ensures that the solution to the problem is bounded. The constraint in (9b) ensures that the effect of unknown system signals (i.e. unknown disturbances or states) is decoupled from the residual (this constraint is analogous to the desired filter requirement in (5)). At first glance, it can appear to be an unattractive solution to solve an optimization problem at each time-step, in order to obtain filter coefficients for the estimation filter. However, the problem proposed in (9) is a convex QP problem which, using simple mathematical tools such as matrix inversion, can be determined analytically, as explicated below.

Corollary III.4 (Analytical solution)**.** *Consider the convex QP optimization problem in (9). The solution to this optimization problem has an analytical solution given by the following polynomial:*

$$\begin{aligned} \bar{N}^*_\gamma(w) &= \frac{1}{2\gamma}\bar{F}^\intercal_{j^*}(w)(\gamma^{-1}I + \bar{H}(w)\bar{H}^\intercal(w))^{-1}, \qquad (10)\\ \textit{where} \quad j^* &= \arg\max_{j\le d_N} |\bar{N}^*_\gamma(w)\bar{F}_j(w)|, \end{aligned}$$

where j denotes the j-th column of the matrix $\bar{F}(w)$. Moreover, the approximation $\bar{N}^*_\gamma(w)$ in (10) converges to the optimal filter coefficient (9) as the parameter γ tends to ∞.

Proof. A dual program of (9) can be obtained by penalizing the equality constraint (9b) through a quadratic function as

$$\sup_{\gamma\ge 0} g(\gamma, w) = \lim_{\gamma\to\infty} g(\gamma, w), \tag{11}$$

where $\gamma \in \mathbb{R}_+$ represents the Lagrange multiplier and $g(\gamma)$ represents the dual function defined as

$$g(\gamma, w) := \inf_{\bar{N}} \gamma\|\bar{N}\bar{H}(w)\|_2^2 + \|\bar{N}^\intercal\|_2^2 - \|\bar{N}\bar{F}(w)\|_\infty.$$

Note, that the ∞-norm related to the fault sensitivity can temporarily be dropped by viewing the problem (9) and its dual problem (11) as a set of d_N different QPs; note that the matrix $\bar{F}$ has d_N columns. Hence, the set of dual functions is denoted as

$$\tilde{g}(\gamma, w) = \inf_{\bar{N}} \underbrace{\gamma\|\bar{N}\bar{H}(w)\|_2^2 + \|\bar{N}^\intercal\|_2^2 - \bar{N}\bar{F}(w)}_{\mathcal{L}(\bar{N},\gamma)}. \tag{12}$$

The solution to the convex quadratic dual problem can be found by first finding the partial derivative of the above Langrangian as follows:

$$\frac{\partial\mathcal{L}(\bar{N}^*_\gamma(w),\gamma)}{\partial\bar{N}} = 2\gamma\bar{N}^*_\gamma(w)\bar{H}(w)\bar{H}^\intercal(w) + 2\bar{N}^*_\gamma(w) - \bar{F}^\intercal(w).$$

Setting this partial derivative to zero, we arrive at

$$\bar{N}^*_\gamma(w) = \frac{1}{2\gamma}\bar{F}^\intercal(w)(\gamma^{-1}I + \bar{H}(w)\bar{H}^\intercal(w))^{-1}, \tag{13}$$

which provides d_N admissible solutions to the problem with dual functions $\tilde{g}(\gamma, w)$ (12). The optimal solution is found by choosing the column of $\bar{F}$, such that the fault sensitivity of the filter is maximal, i.e.,

$$\begin{aligned} \bar{N}^*_\gamma(w) &= \frac{1}{2\gamma}\bar{F}^\intercal_{j^*}(w)(\gamma^{-1}I + \bar{H}(w)\bar{H}^\intercal(w))^{-1},\\ \text{where} \quad j^* &= \arg\max_{j\le d_N} |\bar{N}^*_\gamma(w)\bar{F}_j(w)|, \end{aligned}$$

which proves equation (10). Substituting this solution back into the dual program (11) yields

$$\begin{cases} \max\limits_{\gamma} & -\frac{1}{4\gamma}\bar{F}^\intercal_{j^*}(w)(\gamma^{-1}I + \bar{H}(w)\bar{H}^\intercal(w))^{-1}\bar{F}_{j^*}(w),\\ \text{s.t.} & \gamma \ge 0. \end{cases}$$

This quadratic negative (semi-)definite problem reaches its maximum when γ tends to infinity, which concludes the proof. □

Note, that the choice of γ in (10) is not trivial. Intuitively, a higher value of γ results in an increased fault-sensitivity of the filter. However, excessively increasing this gain will result in numerical problems. For the purpose of estimation, we are particularly interested in a unity zero-frequency gain. In the following corollary, it is shown how to incorporate this condition in the filter.

Corollary III.5 (Zero steady-state)**.** *Given a filter in the form of in (6), where the numerator coefficient $\bar{N}^*_\gamma(w)$ is a solution to the program (9) given analytically in (10). The steady-state relation of the mapping $(d, f) \mapsto r$, for any disturbance signal d and a constant fault f, is given by*

$$r = -\frac{\bar{N}^*_\gamma(w)\bar{F}(w)\mathbb{1}_{d_N\times d_F}}{\bar{a}\mathbb{1}_{d_a}} f,$$

where $\mathbb{1}_{d_N\times d_F}$, $\mathbb{1}_{d_a}$ denote a matrix and vector of ones of the dimensions $d_N \times d_F$ and d_a, respectively.

Proof. The model equation (1), multiplied with a filter $a^{-1}(\mathfrak{q})N(w, \mathfrak{q})$, satisfying the conditions (5), can be denoted as

$$\begin{aligned} a^{-1}(\mathfrak{q})N(w_k, \mathfrak{q})L(w_k, \mathfrak{q})[z] &= -\,a^{-1}(\mathfrak{q})N(w_k, \mathfrak{q})F(w_k, \mathfrak{q})[f],\\ \Rightarrow r &= -\,a^{-1}(\mathfrak{q})N(w_k, \mathfrak{q})F(w_k, \mathfrak{q})[f], \end{aligned}$$

where the last line is induced by (6). The steady-state behavior of this filter can be found by setting $\mathfrak{q} = 1$, resulting in

$$\begin{aligned} r &= -\,a^{-1}(1)N(w_k, 1)F(w_k, 1)[f],\\ &= -\,\frac{\bar{N}(w)\bar{F}(w)\mathbb{1}_{d_N\times d_F}}{\bar{a}\mathbb{1}_{d_a}} f, \end{aligned}$$

which provides the desired result, hence concluding the proof. □

The practical implementation of the proposed estimation filter to, e.g., an LPV minimal state-space realization can result in degraded performance of the estimation filter due to the effects of dynamic dependence [18]. As well as the decision on the Lagrangian parameter γ appears to be a non-trivial decision in a numerical setting. The following remark provides additional practical considerations for the implementation of the proposed estimation filter.

Remark III.6 (Practical considerations)**.** *For the estimation filter to function according to the objectives* (5) *(including unity DC gain for estimation), hence preventing any effects from dynamic dependencies [18], the filter can be implemented as an LPV Input-Output representation as follows:*

$$\begin{aligned} r(k) =& a_0^{-1}\bar{a}\mathbb{1}_{d_a} G(w)\begin{bmatrix} z(k-d_a) & \dots & z(k-d_a+d_N)\end{bmatrix}^\intercal \\ & - a_0^{-1}\sum_{i=1}^{d_a} a_i r(k-i), \\ G(w) =& \frac{\bar{N}(w)\bar{L}(w)}{\bar{N}(w)\bar{F}(w)\mathbb{1}_{d_N\times d_F}} \end{aligned}$$

where the matrix $\bar{L}(w)$ is defined as

$$\bar{L}(w) := \begin{bmatrix} L_0(w_{k-d_a}) & \dots & 0 \\ \vdots & \ddots & \vdots \\ L_{d_L}(w_{k-d_a}) & & L_0(w_{k-d_a+d_N}) \\ \vdots & \ddots & \vdots \\ 0 & \dots & L_{d_L}(w_{k-d_a+d_N}) \end{bmatrix}^\intercal .$$

The matrix operation $G(w)$ ensures the isolation and estimation of the fault, while the terms related to the filter coefficients "a" ensure causality of the operation and reduced sensitivity to noise. Substituting the results from Proposition III.4, the matrix operation $G(w)$ is rewritten as:

$$G(w) = \frac{\bar{F}_{j^*}^\intercal(\gamma^{-1}I + \bar{H}(w)\bar{H}^\intercal(w)))^{-1}\bar{L}}{\bar{F}_{j^*}^\intercal(\gamma^{-1}I + \bar{H}(w)\bar{H}^\intercal(w)))^{-1}\bar{F}\mathbb{1}_{d_N\times d_F}},$$

from which it can be deduced that the term $\gamma^{-1}I$ solely ensures well-posedness of the involved inversion operations, since based on Proposition III.4 ideally γ tends to ∞. The filter is well-posed if and only if $\bar{H}(w)$ is of full rank. Alternatively, γ needs to be chosen large enough to ensure well-posedness, while retaining numerical boundedness for practical considerations.

Following the theoretical results from Theorem III.3 and the Lemma III.1, an LPV estimation filter is developed. The solution to the original filter synthesis problem from Theorem III.3 is approximated using simple computational tools, such as matrix inversion.

IV. Case study: automated driving

In this section, the proposed method for designing an LPV estimation filter is illustrated based on a fault estimation problem coming from the lateral dynamics of an automated passenger vehicle. A linear bicycle vehicle model is used as a benchmark model [19, Equation 1] which is controlled in *closed-loop* by the same PD control-law as proposed in [19]. In principle, this controller is able to mitigate the action of the fault, being an additive fault on the steering wheel actuator. However, within an automotive context it is undesired behavior to simply mitigate a fault without being aware of its magnitude. In fact, in the presence of substantial fault magnitudes, the vehicle is expected to transition to a safe state as opposed to resuming control while mitigating this fault. This shows the applicability of our proposed problem statement in this application context. First, the model as depicted in Fig. 1 can formulated as a set of continuous-time linear state-space equations as follows

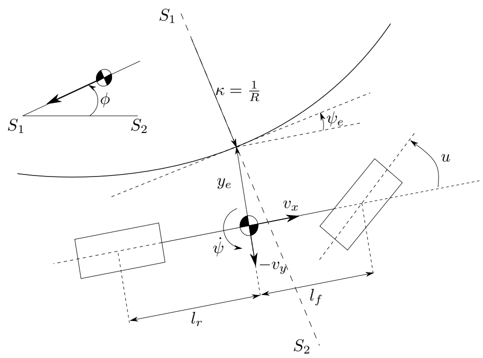

Fig. 1: Visual representation of the bicycle model.

$$\begin{aligned} \begin{bmatrix} \ddot{\psi} \\ \dot{v}_y \\ \dot{y}_e \\ \dot{\psi}_e \end{bmatrix} =& \begin{bmatrix} \frac{C_f+C_r}{v_x m} & \frac{l_f C_f - l_r C_r}{v_x m} & 0 & 0 \\ \frac{l_f C_f - l_r C_r}{v_x I} & \frac{l_f^2 C_f + l_r^2 C_r}{v_x I} & 0 & 0 \\ -1 & 0 & 0 & v_x \\ 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\psi} \\ v_y \\ y_e \\ \psi_e \end{bmatrix} + \begin{bmatrix} -\frac{C_f}{m} \\ -\frac{l_f C_f}{I} \\ 0 \\ 0 \end{bmatrix} \delta \\ & + \begin{bmatrix} -\frac{C_f}{m} \\ -\frac{l_f C_f}{I} \\ 0 \\ 0 \end{bmatrix} f + \begin{bmatrix} g & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & v_x \end{bmatrix} \begin{bmatrix} \sin(\phi) \\ \kappa \end{bmatrix} \qquad (14) \\ y =& \begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} \dot{\psi} & v_y & y_e & \psi_e \end{bmatrix}^\intercal , \end{aligned}$$

where $\dot{\psi}$ denotes the yaw-rate of the vehicle, v_y the lateral velocity of the vehicle, y_e the lateral deviation from the lane center, and, ψ_e the heading deviation from the lane center. The assumed fault, f, acts as an additive fault on the steering angle, δ. Possibly contributing to this fault, two disturbances are considered, where κ denotes the curvature of the road and ϕ denotes the banking angle of the road. The parameters $C_f = 1.50 \cdot 10^5 \text{ N} \cdot \text{rad}^{-1}$ and $C_r = 1.10 \cdot 10^5 \text{ N} \cdot \text{rad}^{-1}$ represent the lateral cornering stiffness of the front and rear tyres, respectively, $l_f = 1.3\,\text{m}$ and $l_r = 1.7\,\text{m}$ represent the distances from the front and rear axle to the center of gravity. It is furthermore assumed that $m = 1500\,\text{kg}$ represents the total mass of the vehicle, $I = 2600\,\text{kg} \cdot \text{m}^2$ represents the moment of inertia around the vertical axis of the vehicle and $g = 9.81\,\text{m} \cdot \text{s}^{-2}$ represents the gravitational acceleration. The parameter v_x represents the longitudinal velocity of the vehicle and acts as the scheduling parameter w_k from (1).

In realistic traffic scenarios there is always some level of perturbation in the longitudinal velocity. To capture this, the scheduling parameter v_x is considered to vary in a range $[14 \dots 24]\ m/s$ at a frequency of $0.1\ Hz$. For application in the proposed estimation framework, the system matrices are applied to a discrete-time system using 0.01 seconds as exact discretization sampling interval. Fig. 2 depicts the simulation results of a 500 sample long scenario. With this simulation, the

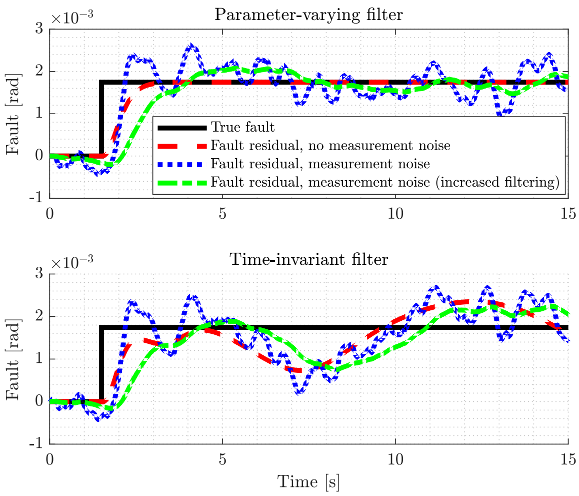

Fig. 2: The top and bottom graph depict the performance of the parameter-varying and time-invariant filter in the absence and presence of measurement noise.

effectiveness of the LPV estimation filter, using two different sets of filtering coefficients, is shown and compared to an LTI estimation filter (as used in [4], generated for a velocity $v_x = 19.5\,m/s$). Here, the fault to be estimated is simulated as a realistic abrupt steering wheel offset of magnitude $0.1\frac{\pi}{180}$ starting at time sample $k = 150$. Finally, in Fig. 2 the simulation results in two cases, with and without measurement noise are shown. We introduce realistic additive white sensor noises with standard deviations of $\sigma_{\dot{\psi}} = 8 \cdot 10^{-4}$rad/s, $\sigma_{y_e} = 5 \cdot 10^{-2}$m, $\sigma_{\psi_e} = 3 \cdot 10^{-3}$ rad. Note, that for the noisy simulations, two different filters are created and depicted. One of which with denominator $a(\mathfrak{q}) = (\mathfrak{q} + 0.95)^3$. For the other filter (denoted in the figure legend as "increased filtering"), $a(\mathfrak{q}) = (\mathfrak{q} + 0.98)^3$ is selected.

As shown in the results in Fig. 2 that the nominal LTI filter is not robust against the scheduling parameter variations, i.e., time-varying longitudinal velocity. Once the fault increases, the residual also increases but does not converge to the true fault. This is explained by the fact that the estimation filter is only designed to decouple unmeasured disturbances from the residual at a constant velocity, which in turn represents the mean of the actual, oscillating, velocity. Therefore, small effects of disturbances and unmeasured states appear in the residual. In the absence of measurement noise, the LPV filter estimates exactly the injected fault. In the presence of noise, it is clear that for both LTI and LPV filter the standard deviation to the true fault decreases if the poles of numerator $a(\mathfrak{q})$ are placed further towards the exterior of the unit circle.

V. Conclusion and future work

In this paper, a novel synthesis method for a fault estimation filter applicable a class of discrete-time LPV systems is introduced. The synthesis of such a filter is formulated by an optimization problem as a function of the known or measurable scheduling parameters, for which a solution exists given a set of conditions for such existence. Subsequently, it is shown that the solution of this problem can be approximated using simple tools, e.g., matrix inversion. Using the proposed LPV estimation filter, the modeled fault can be detected, estimated and therefore decoupled from unmeasurable exogenous disturbances. To show the benefits of the method, the proposed filter has been demonstrated on the lateral dynamics of an automated vehicle, showing that in several distinct cases the fault can be estimated. Future work will include the extension to unmeasurable uncertain dynamics and decoupling of non-linearities in the system dynamics.

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