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GRAPH

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## Some Important Theorems on Graph

**Theorem 1. The Handshaking theorem :** *The total degree of all the vertices of a graph is twice the number of edges.*

OR

*If $G = (V, E)$ be an undirected graph with e edges and n vertices, then*

$$\sum^{n} d(v_i) = 2e$$

**Proof :** Let $v_1, v_2, v_3, \ldots, v_n$ be the $n$ vertices of graph $G$. Since the degree of a vertex is the number of edges incident with that vertex, the sum of the degree counts the total number of times an edge is incident with a vertex. Since every edge is incident with exactly two vertices, each edge gets counted twice, once at each end.

Therefore, $\quad d(v_1) = 2,\ d(v_2) = 2, \ldots, d(v_n) = 2.$

Then $d(v_1) + d(v_2) + \ldots + d(v_n) = 2 + 2 + 2 + \ldots$

$+ 2$ ($n$ (times) $= 2n$.

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$$\Rightarrow \qquad \sum_{i=1}^{n} d(v_i) = 2e.$$

where $\qquad e$ = number of edges.

Theorem 2. *The number of vertices of odd degree in a graph is always even.*

**Proof :** Let $G = G(V, E)$ be a graph such that

$V = \{v_1, v_2, v_3, ..., v_n\}$ and $E \{e_1, e_2, ..., e_m\}$

with $n$ vertices and $m$ edges and we know that by **handshaking theorem.**

$$\sum_{i=1}^{n} d(v_i) = 2e = 2m. \qquad ...(1)$$

We can write the above equation such as

$$\sum_{i=1}^{n} d(v_i) = \sum_{i=even}^{n} d(v_i) + \sum_{i=odd}^{n} d(v_i)$$

$$\sum_{i=odd}^{n} d(v_i) = \sum_{i=1}^{n} d(v_i) - \sum_{i=even}^{n} d(v_i)$$

$= 2m$ – Even degree. $\qquad$ [from (1)]

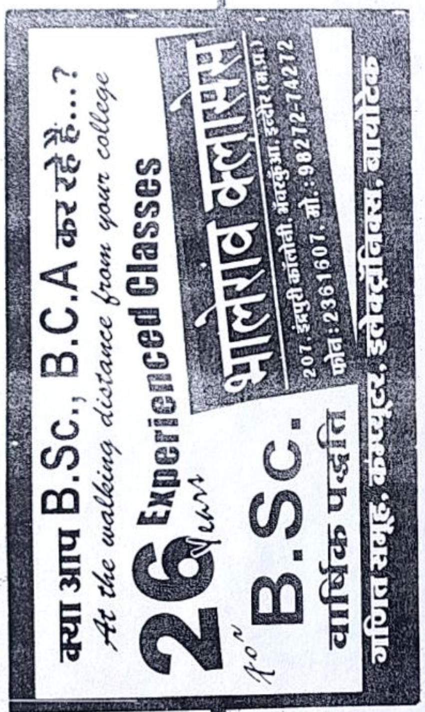
$\Rightarrow$ Even degree – Even degree.

Where $2m$ is always even when $m$ is an odd or even,

= Even degree.

[The subtraction of two even number is always even]

Hence, the number of vertices of odd degree in a graph is always even.




**Theorem 3.** ***Show that the maximum number of edges in a simple graph with n vertices is*** $\frac{n(n-1)}{2}$.

**Proof :** Let $G = G(V, E)$ be a simple graph (neither parallel edges nor self loop) such that

$V = \{v_1, v_2, ...., v_n\}$ and $E = \{e_1, e_2, ..., e_m\}$

with $n$-vertices and $e$ edges and we know that by handshaking theorem,

$$\sum_{i=1}^{n} d(v_i) = 2e$$

$$\Rightarrow \qquad d(v_1) + d(v_2) + ... + d(v_n) = 2e \qquad ...(1)$$

and we also know that the maximum degree of any vertex in a simple graph with $n$ vertex is

$= n - 1$

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Then from equation (1), we get

$(n-1)+(n-1)+\ldots+n$ times $=2e$

$$\Rightarrow \qquad n(n-1) = 2e$$

$$\Rightarrow \qquad e = \frac{n(n-1)}{2}$$

**Theorem 4.** ***Prove that a simple graph with n vertices must be connected if it has more than $\frac{(n-1)(n-2)}{2}$ edges.***

**Proof :** Consider a simple graph of $n$ vertices. Choose $n-1$ vertices $v_1, v_2, \ldots, v_{n-1}$ of $G$. We see that more than $\frac{(n-1)(n-2)}{2}$ number of edges can be drawn between these vertices. Thus, if we have more than $\frac{(n-1)(n-2)}{2}$ edges atleast one edge should be drawn between the $n$th vertices $v_n$ to some vertex $v_i$, $1 \le i \le n-1$ of $G$. Hence, $G$ must be connected.

**Theorem 5.** ***A simple graph with n vertices and k components can have at most $\frac{(n-k)(n-k+1)}{2}$ edges.***

**Proof :** Let $G = (V, E)$ be a simple graph with $n$-vertices and $k$ components, each of the components has $n_1, n_2, \ldots, n_k$ as number of vertices.

So that $\qquad n_1 + n_2 + \ldots + n_k = n$

$$\Rightarrow \qquad \sum_{i=1}^{k} n_i = n \qquad \ldots(1)$$

We know that

$$\sum_{i=1}^{k}(n_i - 1) = n-k \qquad \left[\because \sum_{i=1}^{k} 1 = k \text{ and from eqn. (1)}\right]$$

Squaring both sides,

$$\left[\sum_{i=1}^{k}(n_i - 1)\right]^2 = (n-k)^2$$

$$\Rightarrow \qquad \left[\sum_{i=1}^{k}(n_i - 1)\right]^2 = n^2 + k^2 - 2nk$$

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$$\Rightarrow \sum_{i=1}^{k}(n_i-1)^2+2\,(\text{some positive term}) = n^2+k^2-2nk$$

$$\Rightarrow \sum_{i=1}^{k}(n_i^2+1-2n_i) \le n^2+k^2-2nk.$$

[by the property of summation]

$$\Rightarrow \sum_{i=1}^{k}n_i^2+\sum_{i=1}^{k}1-2\sum_{i=1}^{k}n_i \le n^2+k^2-2nk$$

$$\Rightarrow \sum_{i=1}^{k}n_i^2+k-2n \le n^2+k^2-2nk \quad \text{[from eqn., (1)]}$$

$$\Rightarrow \sum_{i=1}^{k}n_i^2 \le n^2+k^2-2nk-k+2n. \quad \text{...(2)}$$

We know that the maximum number of edges in the $i$th component of $g$ which is a simple graph is $\frac{1}{2}n_i(n_i-1)$.

Thus, total number of edges

$$\frac{1}{2}\sum_{i=1}^{k}n_i(n_i-1)$$

$$= \frac{1}{2}\left[\sum_{i=1}^{k}n_i^2-n_i\right]$$

$$= \frac{1}{2}[n^2+k^2-2nk-k+2n-n] \quad \text{[from eqns. (2) and (1)]}$$

$$= \frac{1}{2}[(n-k)^2+(n-k)]$$

$$= \frac{(n-k)}{2}(n-k+1).$$