Index numbers

Index numbers are the indicators which reflect changes over a specified period of time in (i) prices of different commodities, (ii) industrial production, (iii) sales, (iv) imports and exports, (v) cost of living, etc. These indicators are of paramount importance to the management personnel or any government organisation or industrial concern for the purpose of reviewing position and planning action, if necessary; and in formulation of executive decisions. They reflect the pulse of an economy and serve as indicators of inflationary or deflationary tendencies. Just as in physics and chemistry barometer measures atmospheric pressure or at or pressure of gases, so in economics index numbers measures the pressure of economic behaviour and are rightly termed as 'economic barometers' or 'barometers of economic activity'. Since a look at some of the important indices like index numbers of wholesale prices, industrial production, agricultural production, etc, gives a fairly good idea as to what is happening to the economy of the country.

Definition. "Index numbers are statistical devices designed to measure the relative change in the level of phenomenon (variable or a group of variables) with respect to time geographical location or other characteristics such as income profession etc,." In other words, these are the numbers which express the value of a variable at any given date called the 'given period' as a percentage of the value of that variable at some standard date called the 'base period'. The variables may be:

(i) The price of a particular commodity, for example, silver, iron, etc., or a group of commodities like consumer goods, foodstuff, etc.
(ii) The volume of trade, exports and imports, agricultural or industrial production, sales in a departmental store, etc. and
(iii) The national income of a country or cost of living of persons belonging to a particular income group / profession etc.

Basic problems involved in the construction of index numbers the methods of the construction of Index Numbers

The methods of construction of index numbers want a careful study of the following problems

1. **purpose of index number**. An index number which is properly designed for a purpose can be most useful and powerful tool otherwise it can be equally misleading and dangerous. Moreover, please size statement of

the purpose usually settles some related problems for example if the purpose of index number is to measure the changes in the production of steel (say), the problem of selection of items of (commodities) is automatically settled.

2. **Selection of commodities**. Having define the purpose of index numbers, select only those commodities which are relevant to the index. For example, if the purpose of an index is to measure the cost of living of low-income group (poor families), we should select only those commodities or items which are consumed / utilised by persons belonging to this group and not include the goods / services which are ordinarily consumed by middle income or high – income group.
3. **Data for index numbers**. The data, usually the set of prices and of quantities consumed of the selected commodities for different periods, places, etc, constitute the raw material for the construction of index numbers. The data should be collected from reliable sources such as standard trade journals, official publications, periodical special reports from the producers, exporters, etc, or through field agency.
4. **Selection of base period**. The period with which the comparison of relative changes in the level of a phenomenon are made is termed as 'base period' and the index for this period is always taken as 100. The following are the basic criteria for the choice of a base period
   (i) 'The base period must be a normal period', that is a period free from all sorts of abnormalities of chance fluctuations such as economic boom or depression, labour strikes, wars, floods earthquakes etc.
   (ii) The base period should not be too distant from the given period. since index numbers are essential tools in business planning and in formulation of executive decisions, the base period should not be too far back in the past relative to the given period.
5. **Type of average to be used**. Since index numbers are specialised averages judicious choice of average to be used in their construction is of great importance. Usually the following averages are used (i) arithmetic mean (a.m.) simple or weighted, (ii) geometric mean (GM) simple or weighted, (iii) median.
6. **Selection of appropriate weights.** Generally, various items, commodities, say, wheat, rice, clothing, etc., included in the index are not of equal importance, proper weight should be attached to them to take into account their relative importance. Thes there are two types of indices
   (i) unweighted indices, in which no specific weights are attached to various commodities, and

(ii) waited indices, in which appropriate weights are assigned to various items.

CONSTRUCTION OF INDEX NUMBERS

We describe below some methods of constructing index numbers:

1. **Simple (Unweighted) Aggregate Method.** This method consists of expressing aggregate of prices in any year as a percentage of their aggregate in the base year. Thus price (or quantity) index for the i^th year $(i = 1, 2, \dots, k)$ as compared to the base year $(i = 0)$, is given by:
$$P_{0i} = \frac{\sum p_{ij}}{\sum p_{0j}} \times 100, \qquad and \qquad Q_{0i} = \frac{\sum q_{ij}}{\sum q_{0j}} \times 100$$
The drawbacks of this method are:
(i) The price of various commodities may be in different units, such as per litre, per metre, per quintal, etc.
(ii) The relative importance of various commodities is neglected.

2. **Weighted Aggregate Method.** This method provides for the different commodities to exert (बल लगाना) their influence in the index number by assigning appropriate weights to each. Usually the quantities consumed, sold or marketed in the base year, given year or some typical year are used as weights. If w_j is the weight associated with the j^th commodity the weighted aggregate price index is given by:
$$P_{0i} = \frac{\sum p_{ij}\ w_j}{\sum p_{0j}\, w_j} \times 100$$

By the use of different types of weights, a number of formulas have emerged for the construction of index numbers.

1. **Laspeyre's Price Index (or Base Year) Method**. If we take $w_j = q_{0j}$ in weighted aggregate index, or if the base year quantities are taken as weighs then the Laspeyre's aggregation Price Index or L – formula is given by:
$$P_{0i}^{La} = \frac{\sum p_{ij}\ q_{0j}}{\sum p_{0j}\, q_{0j}} \times 100$$

2. **Paasche's Price Index (or Given Year) Method**. By taking given year quantities as weights $\left(w_j = q_{ij}\right)$. We get Paasche's formula or P – formula as:

$$P_{0i}^{Pa} = \frac{\sum p_{ij}\ q_{ij}}{\sum p_{0j}\, q_{ij}} \times 100$$

3. **Marshall – Edgeworth Price Index (or Base and Given Year) Method.** If we take $w_j = \left(q_{0j} + q_{ij}\right)/2$ as weights that is if weights are the arithmetic mean of the base year quantities and the current year quantities, Marshall – Edgeworth (M.E.) formula is obtained as:

$$P_{0i}^{ME} = \frac{\sum p_{ij}\ \left(q_{0j} + q_{ij}\right)/2}{\sum p_{0j}\left(q_{0j} + q_{ij}\right)/2} \times 100$$

$$P_{0i}^{ME} = \frac{\sum p_{ij}\ \left(q_{0j} + q_{ij}\right)}{\sum p_{0j}\left(q_{0j} + q_{ij}\right)} \times 100$$

4. **Irving Fisher's 'Ideal' Index Number**. it is given by the geometric mean of Laspeyre's and Paasche's formulae. In other words:

$$P_{0i}^{F} = \left(P_{0i}^{La} \times P_{0i}^{Pa}\right)^{\frac{1}{2}}$$

$$P_{0i}^{F} = \left(\frac{\sum p_{ij}\ q_{0j}}{\sum p_{0j}\, q_{0j}} \times \frac{\sum p_{ij}\ q_{ij}}{\sum p_{0j}\, q_{ij}}\right)^{\frac{1}{2}} \times 100$$

Que 1. From the following data calculate price index numbers for 2005 with 1995 as base by :

(i) Laspeyre's, (ii) Paasche's, (iii) Marshall – Edgeworth, and (iv) Fisher's

Commodities	**1995**		**2005**	
	Price	**Quantity**	**Price**	**Quantity**
A	**20**	**8**	**40**	**6**
B	**50**	**10**	**60**	**5**
C	**40**	**15**	**50**	**15**
D	**20**	**20**	**20**	**25**

Solution:

Commodity	1995		2005		p_0q_0	p_0q_1	p_1q_0	p_1q_1
	p_0	q_0	p_1	q_1				
A	20	8	40	6	160	120	320	240
B	50	10	60	5	500	250	600	300
C	40	15	50	15	600	600	750	750
D	20	20	20	25	400	500	400	500
Total					**1660**	**1470**	**2070**	**1790**

(i) Laspeyre's Price Index

$$P_{0i}^{La} = \frac{\sum p_1 \ q_0}{\sum p_0 \, q_0} \times 100$$

$$P_{0i}^{La} = \frac{2070}{1660} \times 100$$

$$P_{0i}^{La} = 124.69$$

(ii) Paasche's Price Index

$$P_{0i}^{Pa} = \frac{\sum p_1 \ q_1}{\sum p_0 \, q_1} \times 100$$

$$P_{0i}^{Pa} = \frac{1790}{1470} \times 100$$

$$P_{0i}^{Pa} = 121.77$$

(iii) Marshall – Edgeworth Price Index

$$P_{0i}^{ME} = \frac{\sum p_1 \ (q_0 + q_1)}{\sum p_0 \, (q_0 + q_1)} \times 100$$

$$P_{0i}^{ME} = \frac{\sum p_1 q_0 + \ \sum p_1 \, q_1}{\sum p_0 \, q_0 + \sum p_0 \, q_1} \times 100$$

$$P_{0i}^{ME} = \frac{2070 + \ 1790}{1660 + 1470} \times 100$$

$$P_{0i}^{ME} = \frac{3860}{3130} \times 100$$

$$P_{0i}^{ME} = 123.32$$

(iv) Fisher's Price Index.

$$P_{0i}^{F} = \left(P_{0i}^{La} \times P_{0i}^{Pa}\right)^{\frac{1}{2}}$$

$$P_{0i}^{F} = (124.69 \times 121.77)^{\frac{1}{2}}$$

$$P_{0i}^{F} = 123.23$$

Que 2. Compute price index and quantity index numbers for the year 2005 with respect to 2000 as base year, using

(i) Laspeyre's Method
(ii) Paasche's Method, and
(iii) Fisher's Method.

Commodity	Quantity (units)		Expenditure (Rs.)	
	2000	2005	2000	2005
A	100	150	500	900
B	80	100	320	500
C	60	72	150	360
D	30	33	360	297

Solution:

We know that

$$Expenditure = price \times Quantity$$

$$price = \frac{Expenditure}{Quantity}$$

Laspeyre's Price Index

$$P_{0i}^{La} = \frac{\sum p_1 \; q_0}{\sum p_0 \, q_0} \times 100$$

$$P_{0i}^{La} = \frac{1570}{1330} \times 100$$

$$P_{0i}^{La} = 118.04$$

Laspeyre's Quantity Index

$$Q_{0i}^{La} = \frac{\sum q_1 \; p_0}{\sum p_0 \, q_0} \times 100$$

$$Q_{0i}^{La} = \frac{1726}{1330} \times 100$$

$$Q_{0i}^{La} = 129.77$$

Item	2000			2005			p_0q_0	p_0q_1	p_1q_0	p_1q_1
	q_0	e_0	p_0	q_1	e_1	p_1				
A	100	500	5	150	900	6	500	750	600	900
B	80	320	4	100	500	5	320	400	400	500
C	60	150	2.5	72	360	5	150	180	300	360
D	30	360	12	33	297	9	360	396	270	297
Total							**1330**	**1726**	**1570**	**2057**

Paasche's Price Index

$$P_{0i}^{Pa} = \frac{\sum p_1 \ q_1}{\sum p_0 \, q_1} \times 100$$

$$P_{0i}^{Pa} = \frac{2057}{1726} \times 100$$

$$P_{0i}^{Pa} = 119.17$$

Paasche's Quantity Index

$$Q_{0i}^{Pa} = \frac{\sum p_1 \ q_1}{\sum q_0 \, p_1} \times 100$$

$$Q_{0i}^{Pa} = \frac{2057}{1570} \times 100$$

$$Q_{0i}^{Pa} = 131.01$$

Fisher's Price Index.

$$P_{0i}^{F} = \left(P_{0i}^{La} \times P_{0i}^{Pa}\right)^{\frac{1}{2}}$$

$$P_{0i}^{F} = (118.04 \times 119.17)^{\frac{1}{2}}$$

$$P_{0i}^{F} = 118.60$$

Fisher's Quantity Index.

$$Q_{0i}^{F} = \left(Q_{0i}^{La} \times Q_{0i}^{Pa}\right)^{\frac{1}{2}}$$

$$Q_{0i}^{F} = (129.77 \times 131.01)^{\frac{1}{2}}$$

$$Q_{0i}^{F} = 130.38$$

Chain Base Method (Chain Indices).

The various formula discussed so far assume that base period is fixed at some previous period. but the data for two periods being as homogeneous as possible and this is best attained by takin two adjacent periods. Hence instead of fixed base method we use chain base method which consists in calculating a series of index numbers (by a suitable formula) for each year with the preceding year as the base, for example, $P_{01}, P_{12}, P_{23}, \ldots\ldots, P_{(k-1)k}$ when P_{rs} represents the price index number with 'r' as base year and 's' as given year. And the basic index number is obtained by successive multiplications of the index numbers so obtained

$$P_{01} = first\ Link$$

$$P_{02} = P_{01} \times P_{12}$$

$$P_{03} = P_{01} \times P_{12} \times P_{23} = P_{02} \times P_{23}$$

$$\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots$$

$$P_{0k} = P_{0(k-1)} \times P_{(k-1)k}$$

In addition to homogeneity, the chain base method provides for the inclusion of new items and deletion of old ones in order to make the index more representative without affecting comparability and without requiring recalculation of index numbers which becomes necessary in the case of fixed base method.

Steps in the construction of chain indices:

1. Express the figures for each period as a percentage of the preceding period to obtain the link relatives (L.R.).
2. These link relatives are chained together by successive multiplication to get chain indices (C.I.) by the following formula:
$$Chain\ Index = \frac{Current\ year\ L.R. \times preceding\ year\ C.I.}{100}$$
3. Formula to convert Fixed Base Index (F.B.I.) into Chain Base Index (C.B.I) is given by
$$current\ year\ FBI = \frac{Current\ year\ CBI \times previous\ year\ FBI}{100}$$

Que 3. Show that for the following series of fixed base index numbers, the chain indices are same as fixed base index numbers:

Year :	**1995**	**1996**	**1997**	**1998**	**1999**	**2000**
Index Number	**100**	**120**	**122**	**116**	**120**	**120**

Year :	**2001**	**2002**	**2003**	**2004**	**2005**
Index Number	**137**	**136**	**149**	**156**	**137**

Que 4. From the chain base index numbers given below, obtain the fixed base index numbers:

Year:	2000	2001	2002	2003	2004	2005
Chain Indices	105	75	71	105	95	90

THE CRITERIA OF A GOOD INDEX NUMBER

mathematical tests. The components of errors in the construction of index numbers can be broadly classified as

(i) formula error, (ii) sampling error and (iii) homogeneity error.

Formula error arises due to the uses of different formulae none of which measures the 'price changes' or 'quantity changes' with perfection. Sampling error results from the sampling of the commodities to be included in the index for measuring the price changes or quantity changes. Faulty selection of the base for the two periods of comparison gives rise to homogeneity error.

As a measure for the formula error, a number of mathematical test, discussed below have been suggested.

1.Unit Test. This requires the index numbers to be independent of the units in which the prices and quantities of various commodities are quoted. This test is satisfied by all formulae of Laspeyre's, Paasche's, Marshall – Edgeworth's and Fisher's.

2.Time Reversal Test. This is one of the two very important tests. Any formula to be accurate must maintain time consistency by working both forward and backward with respect to time. "The test is that the formula for calculating an index number should be such that it will give the same ratio between one point of comparison and the other, no matter which of the two is taken as base". Or, putting it another way, the index number reckoned forward should be the reciprocal of that reckoned backward, except for a constant of proportionality.

Thus, if the time script (say, price) of any index formula be interchanged then the resulting index should be the reciprocal of the original index. Symbolically,

$$P_{ij} = \frac{1}{P_{ji}}, \qquad (i \neq j = 0,1,2,\dots,k)$$

For example, for the Laspeyre's formula,

$$P_{0i}^{La} = \frac{\sum p_{ij}\ q_{0j}}{\sum p_{0j}\, q_{0j}} \times 100 \qquad and \qquad P_{i0}^{La} = \frac{\sum p_{0j}\ q_{ij}}{\sum p_{ij}\, q_{ij}} \times 100$$

$$\therefore \qquad P_{0i}^{La} \times P_{i0}^{La} = \frac{\sum p_{ij}\ q_{0j}}{\sum p_{0j}\, q_{0j}} \times 100 \ \times \frac{\sum p_{0j}\ q_{ij}}{\sum p_{ij}\, q_{ij}} \times 100$$

$$P_{0i}^{La} \times P_{i0}^{La} \neq 1 \ (or\ any\ constant)$$

Hence, Laspeyre’s formula does not satisfy the time reversal test. similarly it can be seen that Paasche’s formula also does not satisfy this test.

For the Fisher's index formula,

$$P_{0i}^{F} = \left(\frac{\sum p_{ij}\ q_{0j}}{\sum p_{0j}\, q_{0j}} \times \frac{\sum p_{ij}\ q_{ij}}{\sum p_{0j}\, q_{ij}}\right)^{\frac{1}{2}} \times 100\,, \quad and$$

$$P_{i0}^{F} = \left(\frac{\sum p_{0j}\ q_{ij}}{\sum p_{ij}\, q_{ij}} \times \frac{\sum p_{0j}\ q_{0j}}{\sum p_{ij}\, q_{0j}}\right)^{\frac{1}{2}} \times 100$$

$$\therefore \ P_{0i}^{F} \ \times \ P_{i0}^{F} = 10000 \ (constant)$$

Hence, fisher's index formula satisfies time reversal test.

It can be easily verified that the index numbers based on

(i) The simple geometric mean of price relatives, and
(ii) Marshall – Edgeworth formula, also satisfy the time reversal test.

3.Factor reversal test. Just as our formula should permit the interchange of two items without giving inconsistent results, so it ought to permit interchanging the prices and quantities without giving inconsistent results, That is, the two results multiplied together should give the true value ratio, except for a constant of proportionality. Symbolically, we should have

$$P_{oi} \times Q_{oi} = \frac{\sum V_{ij}}{\sum V_{oj}} = \frac{\sum p_{ij} q_{ij}}{\sum p_{oj} q_{0j}}$$

For example, for Fisher's index,

$$P_{0i}^{F} = \left(\frac{\sum p_{ij}\ q_{0j}}{\sum p_{0j}\, q_{0j}} \times \frac{\sum p_{ij}\ q_{ij}}{\sum p_{0j}\, q_{ij}}\right)^{\frac{1}{2}} \times 100\,, \quad and$$

$$Q_{0i}^{F} = \left(\frac{\sum q_{ij}\ p_{0j}}{\sum q_{0j}\, p_{0j}} \times \frac{\sum q_{ij}\ p_{ij}}{\sum q_{0j}\, p_{ij}}\right)^{\frac{1}{2}} \times 100$$

$$\therefore \; P_{0i}^{F} \times P_{i0}^{F} = \frac{\sum p_{ij} \; q_{ij}}{\sum p_{0j} \, q_{0j}} \times 10000$$

Where 10000, is constant of proportionality.

Hence fisher's index satisfied satisfy factor reversal test. It may be pointed out that none of the other formula satisfy the factor reversal test.

4.Circular test. this is another test for the adequacy of an index number. This test is based on the shiftability of the base and is an extension of the time reversal test. The test is that

$$P_{0i} \times P_{ij} \times P_{j0} = 1, \quad (i \neq j \neq 0) \;\; or$$

$$P_{ab} \times P_{bc} \times P_{ca} = 1 \qquad a \neq b \neq c$$

this test is satisfied only by the Indus is based on simple geometric mean of the price relatives.

Que 5. Why Fisher's index number is ideal index number?

Cost of Living (Consumer Price) Index Number.

Cost of living index numbers are constructed to study the effect of changes in the prices of a basket of goods and services on the purchasing power of a particular class of people during current period as compared with some base period. Change in the cost of living of an individual between two periods means that change in his money income which will be necessary for him to maintain the same standard of living in both periods. Thus, the cost of living index numbers are intended to measure the average increase in the cost of maintaining the same standard in a given year as in the base year.

Since the consumption habits of people differ widely from class to class and even within the same class from region to region, the changes in the level of prices affect different classes differently and consequently the general price index numbers usually fail to reflect the effects of changes in the general price level on the cost of living of different classes of people. Cost of living index numbers are, therefore compiled, to get a measure of the general price movement of the commodities consumed by different classes of people.

Construction of Cost of Living index. Cost of living index numbers are constructed by the following formula:

1. Aggregate expenditure (or Weighted Aggregate) Method. In this method weights to the assigned to various commodities are provided by the quantities consumed in the base year. Thus, in the usual notation

$$Cost\ of\ Living\ Index = \frac{\sum p_{ij}q_{0j}}{\sum p_{0j}q_{0j}} \times 100$$

this is nothing but Laspeyres' Index and is the most popular method of constructing cost of living index number.

2. Family Budget Method or the Method of Weighted Relatives. in this method, cost of living index is obtained on taking the weighted average of price relatives, the weights being the values of quantities consumed in the base year. Thus in the usual notation, if we write

$$Price\ relative = \ P_j = \frac{p_{ij}}{p_{0j}} \times 100$$

$$and \qquad w_j = p_{0j}q_{0j}$$

$$then, \qquad Cost\ of\ Living\ Index = \frac{\sum w_j P_j}{\sum w_j}$$

Que 6. For the given data construct the cost of living index for the year 2005 using method of weighted price relatives.

Item	**Unit**	**Price (in Rs.)**		**Weight**
		2001	**2005**	
A	**Kg.**	**50**	**75**	**10%**
B	**Litre**	**60**	**75**	**25%**
C	**Dozen**	**200**	**240**	**20%**
D	**Kg.**	**80**	**100**	**40%**
E	**One pair**	**160**	**200**	**5%**

Solution:

Item	Price (in Rs.)		Price relative $P = \frac{p_i}{p_0} \times 100$	Weight (w)	Pw
	2001 p_0	2005 p_i			
A	50	75	$\frac{75}{50} \times 100 = 150$	10%	1500
B	60	75	$\frac{75}{60} \times 100 = 125$	25%	3125
C	200	240	$\frac{240}{200} \times 100 = 120$	20%	2400
D	80	100	$\frac{100}{80} \times 100 = 125$	40%	5000

E	160	200	$\frac{200}{160} \times 100 = 125$	5%	625
Total				100	12650

$$Cost\ of\ Living\ Index = \frac{\sum Pw}{\sum w}$$

$$= \frac{12650}{100}$$

$$= 126.50$$

Que 7. From the data given below, calculate the cost of living index number for the current year by the aggregate expenditure method:

Article	Quantity Consumed in Base Year	Unit	Price (in '000 Rs.) per unit	
			Base year	Current year
Rice	5 quintals	Quintal	60	80
Millets	5 quintals	Quintal	40	50
Wheat	1 quintal	Quintal	50	100
Gram	1 quintal	Quintal	30	60
Arhar	½ quintal	Quintal	40	60
Other Pulses	2 quintals	Quintal	30	40
Ghee	4 kg.	Kg.	12.5	20
Gur	2 quintals	Quintal	25	50
Salt	12 ½ kg.	Kg.	40	50
Oil	24 kg.	Kg.	200	250
Clothing	40 metres	Metre	2.5	5
Firewood	10 quintals	Quintal	5	8
Kerosene	1 tin	Tin	40	60
House Rent	-	-	120	150

Solution: Answer = 128.9

STATISTICS IN PSYCHOLOGY AND EDUCATION

Psychometry had been developed as a branch of Psychology which deals with the measurement of psychological traits or the mental ability like intelligence, aptitude, opinion in correct, personality or scholastic achievement, etc. When individuals are ranked (or arranged) in an ordinal series according to their scholastic achievement, then the problem relates to the education. As such, the educational statistics may also be considered as a part of psychometry when individuals are arranged in a series with respect to some attributes (or trait) and these ranks will give us the serial position of the object in the group. In fact, the psychological and educational traits or characteristics are rather abstract in nature and they can be measured only with some approximation.

For the measurement of psychological and educational characteristics and consequently for the scaling of psychological and educational data. Various devices, many of them based upon the use of normal probability curve, have been used. Here are the most practical consideration is that the scales for different tests should be comparable. Although arbitrary depending upon the choice of the investigator, the scale units should be equal, meaningful and stable and should provide compatibility of the means, of dispersions and form of the distribution.

Scaling Individual Test Items in Terms of Difficulty. In this case a number of problems or test items, say n are designed to test the same psychological trait or test, are administered to a large group of individuals. we are interested in arranging these items in order of difficulty, say, from very simple to very difficult. For this the set of problems is given to a group of individuals for solving them and for each problem the proportion of those who could solve it is obtained. Thus the proportion p_i of the individuals solving the i^{th} problem $(i = 1,2,..,n)$ is given by:

$$p_i = \frac{Number\ of\ individuals\ answering\ i^{th}\ problem\ correctly}{Number\ of\ individuals\ taken\ in\ the\ group}$$

and thus items can be arranged in order of 'percentage difficulty'. For example, an item answer the successfully by 80% of the individual is obviously much easier as compared to a problem solved correctly by only 45%. But comparison of percentage difficulty is only a crude method since these percentage do not successfully reflect the differences in difficulty.

In the construction of the difficulty scale, we assume that the ability of the trait (X) being measured is distributed normally about some mean μ and standard deviation σ. without loss of generality, we can assume $\mu = 0$. Under the assumption of the normality of the trait (X) the heterogeneity (or variability) of the group provides a better difficulty scale, known as σ – scale.

σ – scale is the minimum ability to answer an item correctly if ability is distributed normally $N(0, \sigma^2)$.

If p_i is the proportion of the individuals answering the i^{th} item successfully then its difficulty value is given by $\sigma > z_i$ where z_i is determined from the following relation:

$$P(Z > z_i) = \frac{1}{\sqrt{2\pi}} \int_{z_i}^{\infty} e^{-\frac{t^2}{2}} \, dt = p_i, (i = 1,2, \dots, n)$$

Where $Z \sim N(0, 1)$.

It will be seen that some of the $z_i{}'s$ are negative. To overcome this difficulty we shift the origin to a suitable constant θ.

Que 8. Five problems are solved by 15%, 34%, 50%, 62% and 80% respectively of a large unselected group. If the zero point in this test is taken to be at -3σ what is the $\sigma -$ value of each problem as measured from this point? Compare the difference in difficulty between A and B with the difference in difficulty between D and E.

Solu:

Problem	p	$P(Z > z) = p$	$P(0 < Z < z)$	$\sigma -$ value or distances from mean	σ distances from arbitrary zero.
A	0.15	$P(Z > z_1) = 0.15$	$P(0 < Z < z_1) = 0.35$	$z_1 = 1.04$	4.04
B	0.34	$P(Z > z_2) = 0.34$	$P(0 < Z < z_2) = 0.16$	$z_2 = 0.42$	3.42
C	0.50	$P(Z > z_3) = 0.50$	$P(0 < Z < z_3) = 0$	$z_3 = 0$	3
D	0.62	$P(Z > z_4) = 0.62$	$P(0 < Z < z_4) = 0.12$	$z_4 = -0.31$	2.69
E	0.80	$P(Z > z_5) = 0.80$	$P(0 < Z < z_1) = 0.30$	$z_4 = -0.84$	2.16

The difference in difficulty in A and B $= 1.04 - 0.42 = 0.62$

The difference in difficulty in D and E $= -0.31 + 0.84 = 0.53$

Thus

$$\frac{d_{A-B}}{d_{D-E}} = \frac{0.62}{0.53} = 1.2$$

Thus, difficulty of A relative to B is 1.2 times greater than the difficulty of D relative to E.

Que 9. Given a test question solved by 10% of a large unselected group, a second question solved by 20% of the same group and a third question solved by 30%, determine the relative difficulty to the questions assuming the capacity measured by the test questions to be distributed normally. Given that, with

$$\boldsymbol{f(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{\frac{x^2}{2}}\, dx},$$

$$\boldsymbol{f(0.52) = 0.7,\ \ f(0.84) = 0.8, \quad and\ \ f(1.28)\ = 0.9}$$

Solu: here

$$f(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{\frac{x^2}{2}}\, dx$$

$$f(a) = P(Z \le a) = 1 - P(Z > a)$$

The $\sigma -$ values are obtained as follows:

$$P(Z > z_1) = 0.10,\ \ P(Z > z_2) = \ 0.20 \quad and \ \ P(Z > z_3) = \ 0.30$$

Now

$$f(a) = 1 - P(Z > a)$$

$$f(z_1) = 1 - P(Z > z_1)$$

$$f(z_1) = 1 - 0.10$$

$$f(z_1) = 0.90$$

$$\Rightarrow \quad z_1 = 1.28$$

Similarly $z_2 = 0.84 \ \ and\ z_3 = 0.52$

Difference in difficulty between question 1 and 2 $= 1.28 - 0.84 = 0.44$

Difference in difficulty between question 2 and 3 $= 0.84 - 0.52 = 0.32$

Scaling of Scores on a Test. To make valid comparisons between the Raw scores, we need a common scale which is obtained under some assumption regarding the distribution of the trait being measured by the test. The standard scores and T – scores are such common scale. These scores are used to combine and compare scores.

Z (or σ) Scores. Deviations from the mean is expressed in terms of the standard deviation σ are called $\sigma -$ scores or Z – scores or reduced scores. For example, if the distribution of raw scores (X) in a test has a mean μ and standard deviation σ, i.e., if $E(X) = \mu \ and \ V(X) = \sigma$, then σ – score or Z – score corresponding to the raw score X is given by:

$$Z = \frac{X - \mu}{\sigma}$$

We have

$$E(Z) = E\left(\frac{X - \mu}{\sigma}\right) = \frac{1}{\sigma}\, E(X - \mu) = E(X) - \mu = 0.$$

And

$$V(Z) = V\left(\frac{X - \mu}{\sigma}\right) = \frac{1}{\sigma^2}\, V\,(X - \mu) = \frac{1}{\sigma^2}\, V\,(X) = 1.$$

Hence the mean of a set of σ – scores is always zero and its standard deviation is unity.

Demerits. From practical point of view, they are less convenient to use than other measures due to the following reasons:

1. Since in general, about half of raw scores will lie below mean μ, that means half of the σ – scores will be negative in sign.
2. Another disadvantage is the very large unit viz., on standard deviation, making σ – scores small decimal fractions which are somewhat awkward to deal with in computation.

Standard Scores. The σ – scores transformed to the new mean μ' and s.d. σ' are called standard scores. Thus the standard score X' (with mean μ' and s.d. σ') corresponding to the raw score X with mean μ and s.d. σ is given by the relation:

$$\frac{X' - \mu'}{\sigma'} = \frac{X - \mu}{\sigma} \quad \ldots \ldots (*)$$

$$\Rightarrow \qquad X' = \mu' + \sigma' \left(\frac{X - \mu}{\sigma}\right)$$

$$X' = \mu' + \sigma' Z$$

Where $Z = \frac{X-\mu}{\sigma}$ is the σ – score corresponding to X.

A convenient form of (*) is

$$X' = \frac{\sigma'}{\sigma} X - \left[\left(\frac{\sigma'}{\sigma}\right) \mu - \mu'\right]$$

Hence, the relation for the conversion of raw score X to standard score X' is a straight line.

Que 10. The fifth grade norms for a reading examination are mean = 60, s.d. = 10, for an arithmetic examination mean = 26, and s.d. = 4. Ram scores 55 on the reading test and 24 on the arithmetic test. Compute his σ – scores. In which test he is better? Also compute his standard scores in a distribution with mean 100 and s.d. 20.

Solu. Ram's σ – score in reading $(\mu = 60,\ \sigma = 10,\ X = 55)$

$$z_1 = \frac{55-60}{10} = -0.5$$

Ram's σ – score in arithmetic $(\mu = 26,\ \sigma = 4,\ X = 24)$

$$z_2 = \frac{24-26}{4} = -0.5$$

Since, Ram's σ – scores are same in reading and arithmetic tests, he is equally good in both.

Now Ram's standard score for reading is given by $(\mu' = 100,\ \sigma' = 20,)$

$$X' = \mu' + \sigma' Z$$

$$X' = 100 + 20(-0.5)$$

$${X_1}' = 90$$

For arithmetic ${X_2}' = 90\ (since\ z_1 = z_2)$

Que 11.(a) A test is administered on 400 persons. It gave mean 60 and standard deviation 12. Compute the following table of equivalent raw scores.

Raw score:	**84**	**78**	**72**	**66**	**60**	**54**	**48**	**42**	**36**
σ – scores	**-**	**-**	**1**	**-**	**0**	**-**	**-**	**-**	**-**

$standard$ scores	-	-	-	-	-	45	-	-	-

(b) Convert the ten scores 1, 2, 3, …, 10 into standard scores with mean 50 and standard deviation 10.

Solution. (a) let X denote the raw scores. Then we are given $\mu = 60,\ \sigma = 12$

Then $Z = \frac{X-60}{12}$

And $\mu' = 50,\ \sigma' = 10$

$$X' = 50 + 10\,Z$$

Raw score (X)	σ – scores (Z)	Standard Scores (X')
84	2	70
78	1.5	65
72	1	60
66	0.5	55
60	0	50
54	-0.5	45
48	-1	40
42	-1.5	35
36	-2	30

(b)

Score (X)	X^2	$X-\mu$	$Z = \frac{X-\mu}{\sigma}$	$X' = 50 + 10\,Z$
1	1	-4.5	-1.57	34.3
2	4	-3.5	-1.25	37.5
3	9	-2.5	-0.87	41.3
4	16	-1.5	-0.52	44.8
5	25	-0.5	-0.17	48.3
6	36	0.5	0.17	51.7
7	49	1.5	0.52	55.2
8	64	2.5	0.87	58.7
9	81	3.5	1.25	62.5
10	100	4.5	1.57	65.7
$\sum X = 55$	$\sum X^2 = 385$			

Mean

$$\mu = \frac{1}{n}\sum X = \frac{1}{10}\ (55\ ) = 5.5$$

Variance

$$\sigma^2 = \frac{1}{n}\sum X^2 - \ \mu^2$$

$$\sigma^2 = \frac{385}{10} - \ (5.5)^2$$

$$\sigma^2 = 8.25$$

$$\sigma = 2.87$$

Normalised Scores. Here the scaling procedure is based on the assumption that the trait under consideration (X) is normally distributed with mean μ_x and standard deviation σ_x and the raw scores are converted into a system of normalised scores by transforming them into the equivalent points of a normal distribution.

Let p be the proportion of individuals getting scores below a score x. Then

$$p = P(X \le x) = P\left[Z \le \frac{x-\mu_x}{\sigma_x} = \ \xi(say)\right], where\ \ Z \sim N(0,1)$$

The number ξ given by

$$P(Z \le \xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\xi} e^{-\frac{u^2}{2}} du = p$$

$$\Rightarrow \quad \Phi(\xi) = p$$

Where $\Phi(\xi)$ is the distribution function of a standard normal variate, is called the normalised score corresponding to x. Like $\sigma -$ scores, normalised scores have their mean zero and variance unity and in all probability they almost lie in the range -3 to 3. Equivalent normalised scores represent the same level of the talent or achievement.

For practical convenience, normalised scores are transformed to new scale with mean μ (say), and standard deviation σ (say), by the relation:

$$\frac{\eta - \mu}{\sigma} = \xi \quad \Rightarrow \quad \eta = \ \mu + \sigma\xi,$$

Where μ and σ are pre assigned. $\eta's$ are called normalised standard scores.

T – Scores. In particular if we take $\mu = 50$ and $\sigma = 10$, we get T – Scores. Thus T – Scores are normalised standard scores converted into a distribution with mean 50 and standard deviation 10 and are given by:

$$T = 50 + 10\,\xi$$

Calculation of T – Scores for a given frequency distribution. The procedure for obtaining T – Scores for any given frequency distribution is outlined in the following steps:

1. Arrange the test scores in descending order of magnitude (as is customary with most psychological educational data).
2. Obtain the cumulative frequency (c.f.) starting from the bottom of the distribution.
3. Obtain the c.f. below the mid value of each class interval under the assumption that the frequencies are uniformly distributed over the class intervals, i.e. find $\left[c.f. - \frac{f}{2}\right]$.
4. Express these cumulative frequencies as percentage or proportions $'p'$ of the total frequency N.
5. Obtain the normalised scores given by:
$$P(Z \le \xi) = \int_{-\infty}^{\xi} e^{-\frac{u^2}{2}} du = p$$
6. Finally, T – scores are obtained from normalised scores by $T = 50 + 10\,\xi$.

The steps outlined above can be elegantly displayed in the following table

Class Interval	f	$c.f.$	$c.f.$ below mid value of each class	Proportion $'p'$ of N	ξ	$T = 50 + 10\,\xi$
$x_1 - x_2$	f_1	N	$N - \frac{1}{2}f_1 = A_1$	$\frac{A_1}{N}$		
$x_2 - x_3$	f_2	$\sum_{i=2}^{n} f_i$	$\sum_{i=2}^{n} f_i - \frac{1}{2}f_2 = A_2$	$\frac{A_2}{N}$	$p = \int_{-\infty}^{\xi} e^{-\frac{u^2}{2}} du$	

$x_3 - x_4$	f_3	$\sum_{i=3}^{n} f_i$	$\sum_{i=3}^{n} f_i - \frac{1}{2} f_3 = A_3$	$\frac{A_3}{N}$		
….	….	….	….	….		
$x_{n-1} - x_n$	f_{n-1}	$f_n + f_{n-1}$	$f_n - \frac{1}{2} f_{n-1} = A_{n-1}$	$\frac{A_{n-1}}{N}$		
$x_n - x_{n+1}$	f_n	f_n	$\frac{1}{2} f_n = A_n$	$\frac{A_n}{N}$		

Que 12. Find the T – scores corresponding to the test scores X for the following frequency distribution:

X:	**1**	**2**	**3**	**4**	**5**	**6**	**7**
Y:	**5**	**10**	**20**	**5**	**4**	**4**	**2**

Solu: for the computation of the T – scores, we compute the table

x	f	$c.f.$	$c.f.$ below mid value of each score	$'p'$	ξ $P(Z \le \xi) = p$	$T = 50 + 10\,\xi$
7	2	50	49	0.98	2.05	71
6	4	48	46	0.92	1.41	64
5	4	44	42	0.84	0.995	60
4	5	40	37.5	0.75	0.675	57
3	20	35	25	0.50	0	50
2	10	15	10	0.10	- 0.840	42
1	5	5	2.5	0.05	-1.645	34

Percentile Scores. Percentile score of an individual is the percentage (to the nearest integer) of the frequency lying below his raw score x (say), assuming that score is a continuous variable. For any frequency distribution, the method of computing the percentile scores corresponding to any class interval (or mid value of the class) has incidentally been explained in column 5 while calculating the T scores. The percentile scores for different classes are thus given by:

$P = 100 \times$ cumulative frequency below mid – value of the class/N

INTELLIGENT QUOTIENT: I.Q. Is defined as "mental age (M.A.) in months expressed as percentage of the chronological age (C.A.) in months." Thus

$$I.Q. = \frac{Mental\ \ Age}{Chronological\ \ Age} \times 100$$

$$= \frac{M.A.}{C.A.} \times 100$$

This quotient will be an extremely useful index if the M.A. of individuals increases constantly and a grow older. But actually this is not so. The increase in mental age begins to slow down typically in the age of 13 year and stops by the age of 16 years. Thus after 16 years as an individual grows older and older, the numerator is practically remains constant while the denominator goes on increasing constantly. Consequently IQ computed from the formula will give distorted or bizarre picture of an adult's mental ability.

During the age of 13 years to 16 years, for every 3 months increase in chronological age, there is only 2 month increase in mental age.

IQ is regarded as an indicator of an individual's mental and intellectual development. Since M.A. of a child is always rising IQ reflects how relatively fast or slow his development is.

The following description is due to Terman and classifies an individual into various categories ranging from idiot to genius in terms of different levels of IQ.

Intelligence Quotient	Category
Below 20 – 25	Idiots
Upto 50	Imbeciles (मंदबुद्धि)
50 – 70	Morons
Below 70	Feeble minded (कमजोर)
70 – 80	Mental deficiency
80 – 90	Dull
90 – 110	Average Intelligence
110 – 120	Superior Intelligence
120 – 140	Very Superior Intelligence
Above 140	Genius
200	Super Genius

The classification due to Terman was revised in 1937 to make the more compact, less vivid (ज्वलंत) and less specific. The revised distribution is due to Merrill and is given by:

I.Q.	Category
140 +	Very Superior
120 – 139	Superior
110 – 119	High Average
90 – 109	Normal
80 – 89	Low average
70 – 79	Border line
69 and below	Feeble minded

Reliability of Test Scores

In modern test theory, "*every obtained score is thought of as being made up of Two parts, a component which is called the true score and a second component called the error score*." Symbolically, modern test theory can be expressed by the following linear model.

$$X_t = X_\infty + X_e$$

Where, $X_t =$ Obtained or Raw score or measurement

$X_\infty =$ True score or measurement

$X_e = X_t - X_\infty$, is the error score or measurement

a number of assumptions are made in the model

1. The true score (X_∞) is assumed to be the genuine value of the trait being measured, the value we expect on using a perfect instrument under ideal conditions. A true score cannot, of course, be determined experimentally.
2. The error component (X_e) of the score is that part which is attributed to such factors as temporary characteristics of an individual, like, health, fatigue (थकान), emotional upset, differences in motivation, etc. The factors which are beyond the control human hand. It is assumed that error components occur independently and at random such that
$$E(X_e) = 0,$$
That is the error components increase as often as they decrease a measurement.
$$E(X_\infty, X_e) = 0$$
$$E\left(X_{ei}, X_{ej}\right) = 0, i \neq j$$

that is error components are uncorrelated with the True values and the errors in other measurement.

Definition of Reliability. The reliability of any set of measurement is defined as that part of the variance which is true variance.

If we write r_{tt} for the coefficient of Reliability of a test then, we have

$$r_{tt} = \frac{s_\infty^2}{s_t^2}$$

$$= 1 - \frac{s_e^2}{s_t^2} \quad \dots\dots\dots\dots (1)$$

$$= 1 - \frac{Error\ Variance}{Variance\ of\ raw\ scores}$$

Error Variance or Standard Error of Measurement. Solving equation for s_e we get

$$r_{tt} = 1 - \frac{s_e^2}{s_t^2}$$

$$r_{tt}\, s_t^2 = s_t^2 - s_e^2$$

$$s_e^2 = s_t^2 - r_{tt}\, s_t^2$$

$$s_e = \sqrt{s_t^2 - r_{tt}\, s_t^2}$$

This gives us the standard deviation of the error scores, also known as the standard error of the measurement.

Effect of Test Length on The Reliability of The Test. Increasing the length of a test tends to increase its reliability. This increased reliability is determined by Spearman Brown prophecy formula:

$$r_{nn} = \frac{n\ r_{11}}{1 + (n-1)\, r_{11}}$$

where r_{11} is the reliability of the given test of unit length,

r_{nn} is the correlation Coefficient between n forms of the given test and n alternate forms (or the mean of n forms against the mean of n other form), and

n is the number of times the length of a test is to be increased or decreased.

In particular, if we take $n = 2$, the reliability Coefficient for doubled test length becomes:

$$r_{22} = \frac{2\, r_{11}}{1 + \ r_{11}}$$

The test should not be lengthened more than six or seven times otherwise boredom, fatigue and laws of incentive, etc., may affect the results at adversely.

Que. 13. the reliability coefficient of a test is 50 items is 0.60.

- **(a) How much the test should be lengthened to rise the shelf correlation to 0.90?**
- **(b) What effect will the (i) doubling and (ii) Tripling the test length have upon the reliability Coefficient?**
- **(c) What is the reliability of a test having 125 comparable items?**

Solu: (a) here we are given $r_{11} = 0.60$ and $r_{nn} = 0.90$, $n = ?$

$$r_{nn} = \frac{n\ r_{11}}{1 + (n-1)\, r_{11}}$$

$$0.90 = \frac{n\ \times\ 0.60}{1 + (n-1)\ 0.60}$$

$$0.90 = \frac{n\ \times\ 0.60}{1 +\ 0.60\, n - 0.60}$$

$$0.90 = \frac{0.60\, n}{0.40 +\ 0.60\, n}$$

$$0.36\ + 0.54\, n\ = 0.60\, n$$

$$0.36\ = 0.60\, n -\ 0.54\, n$$

$$0.36\ = 0.06\, n$$

$$n = \frac{0.36}{0.06}\ = 6$$

Hence test should be six times as long to attain a reliability of 0.90. that is it should contain 50×6 = 300 items.

(b) on doubling the test length $r_{11} = 0.60,\ \ n = 2$

$$r_{nn} = \frac{n\ r_{11}}{1 + (n-1)\ r_{11}} = \frac{2\ \times\ 0.60}{1 + (2-1)\ 0.60} = \frac{1.20}{1 + 0.60} = \frac{1.20}{1.60} = 0.75$$

on tripling the test length $r_{11} = 0.60,\ \ n = 3$

$$r_{nn} = \frac{n\ r_{11}}{1 + (n-1)\ r_{11}} = \frac{3\ \times\ 0.60}{1 + (3-1)\ 0.60} = \frac{1.80}{1 + 1.20} = \frac{1.80}{2.20} = 0.81$$

(c) $r_{11} = 0.60,\ \ n = \frac{125}{50} = 2.5$

$$r_{nn} = \frac{n\ r_{11}}{1 + (n-1)\ r_{11}} = \frac{2.5\ \times\ 0.60}{1 + (2.5-1)\ 0.60} = \frac{1.50}{1 + 0.90} = \frac{1.50}{1.90} = 0.789$$

Que 14. Show that the reliability $\boldsymbol{\rho_k}$ of a test at length $\boldsymbol{k}$ in terms of its reliability at length $\boldsymbol{h}$ is given by:

$$\boldsymbol{\rho_k = \frac{k\rho_h}{1 + (k-h)\ \rho_h}}$$

Solu: let ρ_{11} be the reliability of the test of unit length

ρ_k : reliability of the test of length k

ρ_h : reliability of the test of length h.

Then

$$\rho_h = \frac{h\, r_{11}}{1 + (h-1)\, r_{11}}$$

$$\rho_h + (h-1)\rho_h r_{11} = h\, r_{11}$$

$$\rho_h = h\, r_{11} - (h-1)\rho_h r_{11}$$

$$\rho_h = r_{11}\,(h - h\,\rho_h + \rho_h)$$

$$r_{11} = \frac{\rho_h}{(h - h\,\rho_h + \rho_h)} \qquad \dots (1)$$

Also

$$\rho_k = \frac{k\, r_{11}}{1 + (k-1)\, r_{11}}$$

Putting value of r_{11} from equation (1)

$$\rho_k = \frac{k \ \dfrac{\rho_h}{(h - h\,\rho_h + \ \rho_h)}}{1 + (k-1)\ \dfrac{\rho_h}{(h - h\,\rho_h + \ \rho_h)}}$$

$$\rho_k = \frac{k\ \rho_h}{(h - h\,\rho_h + \ \rho_h) + (k-1)\ \rho_h}$$

$$\rho_k = \frac{k\ \rho_h}{h - h\,\rho_h + \ \rho_h + k\ \rho_h - \rho_h}$$

$$\rho_k = \frac{k\ \rho_h}{h - h\,\rho_h + k\ \rho_h}$$

$$\rho_k = \frac{k\ \rho_h}{h + (k-h)\ \rho_h}$$

Proved.