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LA Assignment - 5

Q1. Let V and W be vector spaces over the field F and let T be a linear transformation from V into W. Suppose that V is finite dimensional. Prove that rank (T) + nullity (T) = dim V

Proof: Let $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ be the set $\mathcal{B}$
Let $\mathcal{B}$ be ~~the~~ a basis for N, the null space of T.

$\because T\alpha_1, \ldots, T\alpha_n$ span the range of T
& $T\alpha_j = 0$ for $j \leq k$
$\therefore$ They belong to null space

$\Rightarrow T\alpha_{k+1}, \ldots T\alpha_n$ span the range

Consider $\sum_{i=k+1}^{n} c_i(T\alpha_i) = 0$

$$\Rightarrow T\left(\sum_{i=k+1}^{n} c_i\alpha_i\right) = 0$$

$\therefore \alpha = \sum_{i=k+1}^{n} c_i\alpha_i$ is in the null space of T.

$\because \alpha_1, \ldots \alpha_k$ form a basis for N, there must be scalars $b_1, \ldots, b_k$ such that

$$\alpha = \sum_{i=1}^{k} b_i\alpha_i$$

$$\Rightarrow \sum_{i=1}^{k} b_i\alpha_i - \sum_{j=k+1}^{n} c_j\alpha_j = 0$$

$\because$ $\alpha_1, \ldots, \alpha_n$ are linearly independent

$b_1 = \ldots = b_k = C_{k+1} = \ldots = C_n = 0$

$\Rightarrow$ If r is the rank of T, then

$T\alpha_{k+1}, \ldots T\alpha_n$ form a basis ~~of~~ for the range of T

$\Rightarrow r = n-k$

$\Rightarrow n = k+r$

$\Rightarrow \dim V = \text{nullity}(T) + \text{rank}(T)$

[Hence Proved]

Q2. Let V & W be vector spaces over the field F. Let T and U be linear transformation from V into W. The function $(T+U)$ defined by $(T+U)(\alpha) = T\alpha + U\alpha$ is a linear transformation from V into W. If c is any element of F, the function (cT) defined by $(cT)(\alpha) = c(T\alpha)$ is a linear transformation from V into W. The set of all linear transformations from V into W, together with the addition and scalar multiplication defined above, is a vector space over the field F.

Proof: Since T & U are linear transformations,

$$\begin{aligned}(T+U)(c\alpha+\beta) &= T(c\alpha+\beta) + U(c\alpha+\beta) \\ &= c(T\alpha) + T\beta + c(U\alpha) + U\beta \\ &= c(T\alpha + U\alpha) + (T\beta + U\beta) \\ &= c(T+U)(\alpha) + (T+U)(\beta)\end{aligned}$$

$\Rightarrow (T+U)$ is a linear transformation

Consider
$$\begin{aligned}(cT)(d\alpha+\beta) &= c[T(d\alpha+\beta)] \\ &= c[d(T\alpha) + T\beta] \\ &= cd(T\alpha) + c(T\beta) \\ &= d[c(T\alpha) + c(T\beta)] \\ &= d[(cT)\alpha] + (cT)\beta\end{aligned}$$

$\Rightarrow$ cT is a linear transformation.

Consider the set of linear Transformations $\{T_1, T_2, \ldots T_n\}$ from V into W

We need to show $cT_i + T_j$ lies in W to show that the above set is a vector space ; $i \neq j$

$\because$ We proved cT is a linear transformation cT_i lies in W and T_j lies in W by definition

$\because cT_i \in W$ & $T_j \in W$

Consider $cT_i = U_j$

$\cancel{\Rightarrow T_j}$ $\because T_j + U_j$ is a linear transformation as proved earlier

$T_j + U_j \in W$

$\Rightarrow cT_i + T_j \in W \quad \forall i \neq j$ &

$i, j = \{1, 2, \ldots n\}$

$\Rightarrow$ The set of linear ~~co~~ transformations of V into W ~~is~~ forms a vector space.

Q3. Let V be an n-dimensional vector space over the field F and let W be an W. m-dimensional vector space over F. Then the space $L(V, W)$ is finite dimensional and has dimension mn.

Proof: Let the ordered bases of V & W be $\mathcal{B}_1 = \{\alpha_1, \ldots \alpha_n\}$ and $\mathcal{B}_2' = \{\beta_1, \beta_2 \ldots \beta_m\}$ respectively

Consider the pair of integers (p, q) with $1 \le p \le m$ & $1 \le q \le n$, we define a linear transformation $E^{(p,q)}(V) \longrightarrow W$

$$E^{(p,q)}(\alpha_i) = \begin{cases} 0, & \text{if } i \neq q \\ \beta_p, & \text{if } i = q \end{cases}$$

As proved earlier, there $\exists$ a unique linear transformation from V into W satisfying the above conditions

Let T be a linear transformation from V into W

For each j, $1 \le j \le n$,

Let $A_{1j}, \dots, A_{mj}$ be the coordinates of the vector $T\alpha_j$ in the ordered basis $\mathcal{B}_2$

$$T\alpha_j = \sum_{p=1}^{m} A_{pj} \beta_p$$

Let U be the linear transformation

then for each j

$$U\alpha_j = \sum_p \sum_q A_{pq} E^{(p,q)}(\alpha_j)$$

$$= \sum_p \sum_q A_{pq} \delta_{jq} \beta_p$$

$$= \sum_{p=1}^{m} A_{pj} \beta_p$$

$$= T\alpha_j$$

$\Rightarrow U = T \quad \Rightarrow E^{(p,q)}$ span $L(V, W)$

If $U = \sum_p \sum_q A_{pq} E^{(p,q)}$ is the zero transformation

then $U\alpha_j = 0 \ \forall j$

$$\Rightarrow \sum_{p=1}^{m} A_{pj} \beta_p = 0$$

$\Rightarrow A_{pj} = 0 \quad \forall \ p \ \& \ j$

$\Rightarrow$ The mn transformations $E^{(p,q)}$ form a basis for $L(V, W)$ $\because T = U$

$\Rightarrow \dim(L(V, W)) = mn$

[Hence Proved]