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1. Given:

Vector spaces V, W over field $\mathbb{F}$

A linear transformation $T: V \to W$.

V is finite dimensional.

To Prove:

$$\text{rank}(T) + \text{nullity}(T) = \dim V$$

Proof:

Let $\dim V = n$ and $\mathcal{B} = \{\bar{\alpha}_1, \bar{\alpha}_2, \dots, \bar{\alpha}_n\}$ be basis of V.

Let null space of T be N.

$\because$ null space of T is a subset of V

$\therefore \exists\ \mathcal{B}' \subseteq \mathcal{B}$ such that ~~$\mathcal{B}' \subseteq \mathcal{B}$~~. $\mathcal{B}'$ spans N.

By definition, $\text{nullity}(T) = |\mathcal{B}'|$

Now, $\forall \bar{\alpha} \in \mathcal{B}'$, $T\bar{\alpha} = \bar{0}_W$

$\therefore \forall \bar{\alpha} \in \mathcal{B}'$, $\alpha \in N$.

Without loss of generality, we can assume $|\mathcal{B}'| = k$ & $\mathcal{B}' = \{\bar{\alpha}_1, \bar{\alpha}_2, \dots, \bar{\alpha}_k\}$.

$\therefore \text{nullity}(T) = k$

Let $\bar{\beta} \in V$.

$\therefore \bar{\beta} = c_1\bar{\alpha}_1 + c_2\bar{\alpha}_2 + \dots + c_k\bar{\alpha}_k + c_{k+1}\bar{\alpha}_{k+1} + \dots + c_n\bar{\alpha}_n$

$\Rightarrow T\bar{\beta} = c_1T\bar{\alpha}_1 + c_2T\bar{\alpha}_2 + \dots + c_kT\bar{\alpha}_k + c_{k+1}T\bar{\alpha}_{k+1} + \dots + c_nT\bar{\alpha}_n$

$= c_{k+1}T\bar{\alpha}_{k+1} + \dots + c_nT\bar{\alpha}_n$

$[\because T\bar{\alpha}_i = 0,\ \forall i \le k]$

Let $S = \{T\bar{\alpha} \mid \bar{\alpha} \in \mathcal{B} - \mathcal{B}'\}$.
We see that S spans range(T).

Let $d_1, d_2, \dots d_{n-k} \in \mathbb{F}$ such that
$d_1 T\bar{\alpha}_{k+1} + d_2 T\bar{\alpha}_{k+2} + \dots + d_{n-k} T\bar{\alpha}_n = 0$
$\Rightarrow T(d_1\bar{\alpha}_{k+1} + d_2\bar{\alpha}_{k+2} + \dots + d_{n-k}\bar{\alpha}_n) = 0$

However $\mathcal{B} - \mathcal{B}'$ does not span any non-zero vector of null space

$\therefore d_1 = d_2 = \dots = d_{n-k} = 0$
$\therefore$ ~~$\mathcal{B} - \mathcal{B}'$~~ S forms basis for ~~nom~~ range (T).

$\therefore \text{rank}(T) = |S| = n - k$

$\therefore \text{rank}(T) + \text{nullity}(T) = (n-k) + k = n = \dim V$

2. Given:

Vector spaces V & W over field $\mathbb{F}$.

Linear transformations $T: V \to W$ & $U: V \to W$.

$(T+U)(\bar{\alpha}) = T\bar{\alpha} + U\bar{\alpha}$

$(cT)(\bar{\alpha}) = c(T\bar{\alpha})$

To Prove:

i) $T+U$ is a linear transformation.

ii) cT is a linear transformation.

iii) Set of all linear transformations from V to W, with given addition & scalar multiplication forms a vector space.

Proof:

i) Let $\bar{\alpha}, \bar{\beta} \in V$, & $c \in \mathbb{F}$.

$\therefore (T+U)(c\bar{\alpha}+\bar{\beta}) = T(c\bar{\alpha}+\bar{\beta}) + U(c\bar{\alpha}+\bar{\beta})$

$= cT\bar{\alpha} + T\bar{\beta} + cU\bar{\alpha} + cU\bar{\beta}$

[$\because$ T & U are linear]

$= c(T\bar{\alpha} + U\bar{\alpha}) + (T\bar{\beta} + U\bar{\beta})$

$= c\,(T+U)(\bar{\alpha}) + (T+U)\bar{\beta}$

$\therefore (T+U)$ is a linear transformation.

ii) Let $\bar{\alpha}, \bar{\beta} \in V$, & ~~$c \in \mathbb{F}$~~, $c_1, c_2 \in \mathbb{F}$.

$\therefore (c_1T)(c_2\bar{\alpha}+\bar{\beta}) = c_1T(c_2\bar{\alpha}+\bar{\beta})$

$= c_1\{c_2T\bar{\alpha} + T\bar{\beta}\}$ [$\because$ T is linear]

$= c_1c_2T\bar{\alpha} + c_1T\bar{\beta}$

$= c_2(c_1T\bar{\alpha}) + c_1T\bar{\beta}$

$= c_2(c_1T)(\bar{\alpha}) + (c_1T)\bar{\beta}$

$\therefore c_1T$ is a linear transformation.

iii) Let ~~L(V,~~ L be the set of all linear transformations from V to W.

Consider arbitrary $T, U, P \in L$ & $\bar{\alpha} \in V$

Based ~~of~~ on part (i),

$T+U \in L$

Now, $(T+U)(\bar{\alpha}) = T\bar{\alpha} + U\bar{\alpha}$

$= U\bar{\alpha} + T\bar{\alpha}$ [$\because$ W is a vector space]

$= (U+T)(\bar{\alpha})$

$\therefore T+U = U+T$

$((T+U)+P)(\bar{\alpha}) = (T+U)(\bar{\alpha}) + P(\bar{\alpha})$

$= T\bar{\alpha} + U\bar{\alpha} + P(\bar{\alpha})$

$= T\bar{\alpha} + (U+P)(\bar{\alpha})$

$= (T+(U+P))(\bar{\alpha})$

$\therefore (T+U)+P = T+(U+P)$

Let $O_L : V \to W$ such that $O_L(\bar{\alpha}) = \bar{O}_W$ where $\bar{O}_W$ is the zero vector in W.

Then for $c \in F$ & $\bar{\alpha}, \bar{\beta} \in V$,

$O_L(c\bar{\alpha} + \beta) = \bar{O}_W$

$= c\bar{O}_W + \bar{O}_W$

$= c\, O_L(\bar{\alpha}) + O_L(\beta)$

$\therefore O_L$ is a linear transformation

$\therefore O_L \in L$.

For $T \in L$, $(T+O_L)(\bar{\alpha}) = T\bar{\alpha} + O_L(\bar{\alpha})$

$= T\bar{\alpha} + \bar{O}_W$

$= T\bar{\alpha}$

$\therefore \exists\, O_L \in L$ such that $T + O_L = T$, $\forall T \in L$

Consider arbitrary $T \in L$,

Let $(-T): V \to W$ such that $(-T)(\bar{\alpha}) = -(T\bar{\alpha})$.

Let $c \in \mathbb{F}$, $\bar{\alpha}, \bar{\beta} \in V$,

$(-T)(c\bar{\alpha} + \bar{\beta}) = -(T(c\bar{\alpha} + \bar{\beta}))$

$= -cT\bar{\alpha} - T\bar{\beta}$

~~$= -T(c$~~

$= c(-T\bar{\alpha}) + (-T\bar{\beta})$

$= c(-T)(\bar{\alpha}) + (-T)(\bar{\beta})$

$\therefore (-T)$ is a linear transformation $\Rightarrow (-T) \in L$.

Now $T\bar{\alpha} + (-T)(\bar{\alpha})$

$= T\bar{\alpha} - T\bar{\alpha}$

$= \bar{0}_W$

$\therefore T + (-T) = 0_L$

$\therefore \forall T \in L, \exists -T \in L$ such that $T + (-T) = 0_L$

Now, $\forall c \in \mathbb{F}$ & T

Consider $1 \in \mathbb{F}$,

$\therefore (1 \cdot T)(\bar{\alpha}) = 1(T\bar{\alpha}) = T\bar{\alpha}$

$\therefore (1 \cdot T) = T, \forall T \in L$

Let $c \in \mathbb{F}$ & $T, U \in L$, $cT \in L$ [Part (ii)]

Let $\bar{\alpha} \in V$,

$\therefore \{c(T+U)\}(\bar{\alpha}) = c\{(T+U)\bar{\alpha}\}$

$= c(T\bar{\alpha} + U\bar{\alpha})$

$= cT\bar{\alpha} + cU\bar{\alpha}$

$= (cT)(\bar{\alpha}) + (cU)(\bar{\alpha})$

$\therefore c(T+U) = cT + cU$

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Let $c_1, c_2 \in \mathbb{F}$,

$$((c_1+c_2)T)(\bar{\alpha}) = (c_1+c_2)(T(\bar{\alpha}))$$
$$= c_1 T\bar{\alpha} + c_2 T\bar{\alpha} \quad [\because W \text{ is a vector space}]$$
$$= (c_1 T)\bar{\alpha} + (c_2 T)\bar{\alpha}$$

$\therefore (c_1+c_2)T = c_1 T + c_2 T$

And, $((c_1 c_2)T)(\bar{\alpha}) = c_1 c_2 (T\bar{\alpha})$
$$= (c_2 c_1) T(\bar{\alpha}) \quad [\because T\bar{\alpha} \in W \text{ \& } W \text{ is a vector space}]$$
$$= ((c_2 c_1)T)(\bar{\alpha})$$

$\therefore (c_1 c_2)T = (c_2 c_1)T$

$\therefore$ All vector space axioms are satisfied

$\therefore L$ over $\mathbb{F}$ along with defined addition & scalar multiplication forms a ~~F~~ vector space

3. Given:

~~An~~ An n-dimensional vector space V, over $\mathbb{F}$

An m-dimensional vector space W over $\mathbb{F}$

To Prove:

$L(V, W)$ is finite dimensional

& $\dim L(V, W) = mn$

Proof:

Let $B_V = \{\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_n\}$ be the basis of V.

Let $B_W = \{\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_m\}$ be the basis of W

Let $T_{ij} \in L(V, W)$ such that,

$T_{ij} \cancel{T_j}(\bar{\alpha}_i) = \bar{\beta}_j$ & $\forall \alpha \in B_V - \{\bar{\alpha}_i\}$

$T_{ij}(\bar{\alpha}) = \cancel{\bar{0}_W}$ $\bar{0}_W$

Let $B = \{T_{ij} \mid i, j \in \mathbb{N} \;\&\; i \leq n \;\&\; j \leq m\}$

Every linear transformation can be defined by the ~~value~~ value of the basis vectors after transformation.

Let $T \in L(V, W)$.

Let $T(\bar{\alpha}_i) = \cancel{E_i} \sum_{j=1}^{m} c_{ij} \bar{\beta}_j$

~~$\therefore \forall \bar{\alpha} \in V, \; T(\alpha) = \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} \bar{\beta}_j$~~

$\therefore T = \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} T_{ij}$

$\therefore B$ spans $L(V, W)$

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Let O_L be the ~~θ~~ linear transformation, such that $O_L = \bar{O}_W$

$\therefore \exists\ C_{ij}$ such that,

$$O_L = \sum_{i=1}^{n} \sum_{j=1}^{m} C_{ij} \cdot T_{ij}$$

Let $\bar{\alpha} \in V$,

$$O_L(\bar{\alpha}) = \sum_{i=1}^{n} \sum_{j=1}^{m} C_{ij} \cdot T_{ij}(\bar{\alpha})$$

Let $\bar{\alpha} = d_1 \bar{\alpha}_1 + d_2 \bar{\alpha}_2 + \dots + d_n \bar{\alpha}_n$

$$\therefore O_L(\bar{\alpha}) = \sum_{i=1}^{n} \sum_{j=1}^{m} C_{ij} \cdot d_i \bar{\beta}_j$$

$$\Rightarrow \bar{O}_W = (C_{11} \cdot d_1 + C_{21} d_2 + \dots + C_{n1} d_n) \bar{\beta}_1$$
$$+ (C_{12} \cdot d_1 + C_{22} \cdot d_2 + \dots + C_{n2} \cdot d_n) \bar{\beta}_2$$
$$+ \dots +$$
$$+ (C_{1m} d_1 + C_{2m} \cdot d_2 + \dots + C_{nm} \cdot d_n) \bar{\beta}_m$$

$\because \bar{\beta}_1, \bar{\beta}_2, \dots, \bar{\beta}_m$ are linearly independent

$\therefore \forall\ d_1, d_2, \dots, d_n$, ~~$\Sigma$~~ & for some $j \leq m$,

$$\sum_i C_{ij}\, d_i = 0 \quad \text{— (i)}$$

Consider arbitrary $k \in \mathbb{N}$ & $k \leq n$,

$$\sum_{i=1}^{k-1} C_{ij} d_i + C_{kj}(d_k + 1) + \sum_{i=k+1}^{n} C_{ij} d_i = 0 \quad \text{— (ii)}$$

[$\because$ (i) is valid for all $d_i \in \mathbb{F}$]

$(ii) - (i)$,

$C_{kj} = 0$

$\because$ choice of k & j is arbitrary

$\therefore \forall\, i, j \in \mathbb{N}$ such that $i \leq n$ & $j \leq m$,

$C_{ij} = 0$

$\therefore \mathcal{B}$ is linearly ~~indepeden~~ independent.

$\therefore \mathcal{B}$ is basis of $L(V, W)$.

& $|\mathcal{B}| = mn$

$\therefore L(V,W)$ is finite dimensional

& $\dim L(V, W) = mn$