# LINEAR ALGEBRA

## ASSIGNMENT - 5

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1) GIVEN :

V and W are vector spaces over field F. A linear transformation T from V to W is defined. V is finite dimensional.

TO PROVE :

$$\text{Rank}(T) + \text{Nullity}(T) = \text{Dim } V.$$

PROOF :

$$T : V \to W$$

Nullity : $N(T) = \{ \alpha \in V : T(\alpha) = 0' \in W \}$

Range : $R(T) = \{ \beta \in W : T(\alpha) = \beta \text{ for some } \alpha \in V \}$.

Let $\dim V = n$ (finite dimensional vector space)

Since $N(T)$ is a subspace of $V$ so it is also a finite dimensional vector space.

Let $S = \{ \alpha_1, \alpha_2, \ldots, \alpha_k \}$ = basis of $N(T)$.

i.e. $\dim(N(T)) = k$ $\quad --- \to (1)$.

By EXTENSION THEOREM, '$S'$' can be extended as basis of $V$.

Clearly,

Let $S' = \{\alpha_1, \alpha_2, \ldots, \alpha_k, \alpha_{k+1}, \ldots, \alpha_n\}$.
be the basis of $V$.

$\Rightarrow \quad \dim(V) = n \quad \text{---} \rightarrow (2)$

Clearly, $\{T(\alpha_1), T(\alpha_2), \ldots, T(\alpha_n)\} \subseteq R(T)$
Range

From (1) & (2), we need to show that $\dim(R(T)) = n - k$.

# We need to show that the elements $T(\alpha_{k+1}), T(\alpha_{k+2}), \ldots, T(\alpha_n)$ form the basis of $R(T)$.

CONDITION 1: SPANS THE VECTOR SPACE.

Let any $\beta \in R(T)$, then $\exists$ a $\alpha \in V$.

s.t. $\beta = T(\alpha)$.

Since $\alpha \in V$ so $\alpha = a_1\alpha_1 + \ldots + a_n\alpha_n$
$(\forall\, a_i \in F)$

$\Rightarrow \quad \beta = T(\alpha) = T(a_1\alpha_1 + \ldots + a_n\alpha_n)$

$\Rightarrow \beta = T(a_1\alpha_1 + \cdots + a_k\alpha_k + a_{k+1}\alpha_{k+1} + \cdots + a_n\alpha_n)$

$= a_1T(\alpha_1) + a_2T(\alpha_2) + \cdots + a_kT(\alpha_k)$
$+ a_{k+1}T(\alpha_{k+1}) + \cdots + a_nT(\alpha_n)$

(Since $T$ is a linear transformation)

Since $\alpha_1, \alpha_2 \cdots \alpha_k$ forms the basis of $N(T)$

$\Rightarrow T(\alpha_i) = 0' \quad \forall\ i \le k.$

$= a_1(0') + \cdots + a_k(0') + a_{k+1}T(\alpha_{k+1}) + \cdots + a_nT(\alpha_n)$

$\Rightarrow \beta = \underline{\underline{a_{k+1}T(\alpha_{k+1}) + \cdots + a_nT(\alpha_n)}}$

$\Rightarrow$ any $\beta \in R(T)$ can be expressed as linear combination of elements $T(\alpha_{k+1}), \cdots, T(\alpha_n)$.

$\Rightarrow T(\alpha_{k+1}) \cdots T(\alpha_n)$ spans $R(T)$.

$\therefore$ Condition 1 is satisfied.

CONDITION 2 : LINEARLY INDEPENDENT.

We need to show that $T(\alpha_{k+1}), \cdots T(\alpha_n)$ are all linearly independent.

Let $\exists\ a_{k+1}, \cdots a_n \in F$ st.

$\Rightarrow a_{k+1}\alpha_{k+1} + \cdots + a_n\alpha_n = 0'.$

$\Rightarrow T(a_{k+1}\cdot\alpha_{k+1} + \cdots + a_n\alpha_n) = 0'$

Since $T(\alpha) = 0.$

$\Rightarrow a_{k+1}\alpha_{k+1} + \cdots + a_n\alpha_n \in N(T)$.

Elements $a_{k+1}\alpha_{k+1} + \cdots + a_n\alpha_n$ can be expressed as a linear combination of basis of $N(T)$

$\Rightarrow a_{k+1}\alpha_{k+1} + \cdots + a_n\alpha_n$

$= c_1\alpha_1 + c_2\alpha_2 + \cdots + c_k\alpha_k$.

$\Rightarrow \underbrace{-c_1\alpha_1 - c_2\alpha_2 + \cdots - c_k\alpha_k + a_{k+1}\alpha_{k+1} + \cdots + a_n\alpha_n = 0}_{\text{Linear combination of } (\alpha_1, \alpha_2, \ldots, \alpha_n)}$.

Since $(\alpha_1, \ldots, \alpha_n)$ are basis of $V$, they are linearly independent

$\Downarrow$

$c_1 = c_2 \cdots = c_k = 0$ &

$a_{k+1} = a_{k+2} = \cdots = a_n = 0$.

Condition 2 is also satisfied.

$\therefore \dim(R(T)) = n-k$.

Hence Proved.

2) Given :

V & W are vector spaces over field F.
T and U are linear transformations
from $V \to W$.

$(T+U)\alpha = T\alpha + U\alpha$ ; $(T+U) : V \to W$.
Also $(cT)\alpha = c(T\alpha)$ ; $(cT) : V \to W$.
$\forall c \in F$.

To Prove :

$(T+U)$ and $(cT)$ is a linear transformations from $V \to W$ together with the addition and scalar multiplication is a vector space over the field F.

T & U are linear Transformations.

$\Rightarrow T(c\alpha + \beta) = cT(\alpha) + T\beta$.

$\Rightarrow U(c\alpha + \beta) = cU(\alpha) + U\beta$.

For $(T+U)$ :

$$\Rightarrow (T+U)(c\alpha+\beta) = T(c\alpha+\beta) + U(c\alpha+\beta)$$
$$= cT(\alpha) + T\beta + cU(\alpha) + U\beta$$
$$= c(T\alpha + U\alpha) + \beta(T+U)$$
$$= c\alpha(T+U) + \beta(T+U)$$
$$= c(T+U)\alpha + (T+U)\beta$$

$\therefore (T+U)$ is a linear transformation

Also, $(T+U)0 = 0$.

For $(cT)$ : $c, d \in F$ (scalars)

$$cT(d\alpha + \beta) = c\,[T(d\alpha + \beta)]$$
$$= c\,[d\,T(\alpha) + T(\beta)]$$
$$= cd\,T(\alpha) + cT\beta$$
$$= d\,(cT)\alpha + (cT)\,\beta.$$

$\therefore$ $cT$ is a linear transformation.

$\Rightarrow$ $cT(0) = 0$.

Proving that $(T+U)(c\alpha+\beta)$ is a vector space :

A vector space is defined as follows :

1) a field $F$ of scalars.
2) a set $V$ of objects, called vectors.
3) an operation, called VECTOR ADDITION which associates with each pair of vectors $\alpha, \beta$ in $V$ a vector $\alpha+\beta$ in $V$, called the sum of $\alpha$ and $\beta$.

Vector space is a set of all linear transformations from $V$ into $W$ :

a) addition is COMMUTATIVE

Suppose $T$ and $U$ be two linear transformations :

then $(U+T)(c\alpha+\beta)$

$$= U(c\alpha+\beta) + T(c\alpha+\beta)$$
$$= c(U\alpha) + U\beta + c(T\alpha) + T\beta$$
$$= c(T\alpha) + T(\beta) + c(U\alpha) + U\beta$$
$$= T(c\alpha) + T(\beta) + U(c\alpha) + U\beta$$
$$= T(c\alpha+\beta) + U(c\alpha+\beta)$$
$$= (T+U)(c\alpha+\beta).$$

$\therefore$ It is commutative.

b) addition is ASSOCIATIVE :

Suppose $T, V$ and $Z$ are three linear transformations :

$$T(c\alpha+\beta) + (V+Z)(c\alpha+\beta)$$

$$= \underbrace{T(c\alpha+\beta) + V(c\alpha+\beta)} + Z(c\alpha+\beta)$$

$$\Rightarrow (T+V)(c\alpha+\beta) + Z(c\alpha+\beta).$$

Clearly, it is ASSOCIATIVE .

c) There is a unique transformation in $V$, called the ZERO TRANSFORMATION which takes in the vector and outputs a zero vector such that

$$T(\alpha) + \beta = 0 + \beta = \beta.$$

d) For each transformation in $V$, there is a unique transformation such that

$$T(\alpha) + V(\alpha) = 0$$

where $V(\alpha) = -T(\alpha)$.

$$\Rightarrow T(\alpha) + (-T(\alpha)) = T(\alpha) - T(\alpha)$$

$$= 0$$

1) Fourth property deals with scalar multiplication :

a) $1.(T(\alpha)) = T(\alpha)$

$\forall$ transformations $T$ in $V$.

(b) $(c_1 c_2)T(\alpha) = c_1(c_2(T)(\alpha))$

$= c_2(c_1 T(\alpha))$

$\forall \; c_1, c_2 \in F$

(c) $c(T+U)(d\alpha+\beta) = c\left[T(d\alpha+\beta) + U(d\alpha+\beta)\right]$

$= cT(d\alpha+\beta) + cU(d\alpha+\beta)$

$= (cT)(d\alpha+\beta) + (cU)(d\alpha+\beta).$

(d) $(c_1+c_2)(T(d\alpha+\beta))$

$= c_1\left[T(d\alpha+\beta)\right] + c_2\left[T(d\alpha+\beta)\right].$

$= (c_1T)(d\alpha+\beta) + (c_2T)(d\alpha+\beta)$

Since it satisfies all the properties

The set of all linear transformations from V to W, together with addition and scalar multiplication is a vector space over F.

3) GIVEN :

$V$ is an $n$-dimensional vector space over $F$.
$W$ is a $m$-dimensional vector space over field $F$.

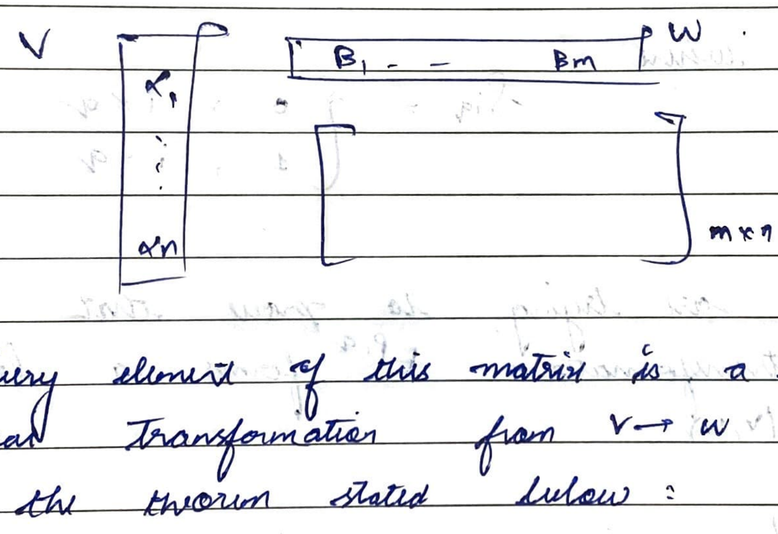
TO PROVE :

$L(V, W)$ space is finite dimensional and has dimension $mn$.

PROOF :

Let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \rightarrow$ ordered basis for $V$.

$\{\beta_1, \beta_2, \ldots, \beta_m\} \rightarrow$ ordered basis for $W$.




Every element of this matrix is a unique linear Transformation from $V \rightarrow W$ from the theorem stated below :

# Let $V$ be a finite dimensional vector space over the field $F$ and let $\{\alpha_1 \ldots \alpha_n\}$ be an ordered basis for $V$. Let $W$ be a vector space over the same field $F$ and let $\beta_1 \ldots \beta_n$ be any vectors in $W$. Then there is precisely one linear Transformation $T$ from $V \rightarrow W$ st.

$$\boxed{T\alpha_j = \beta_j}, \quad j = 1, 2 \ldots n.$$

Now, we represent the matrix of linear transformation as :

for each pair of integers $(p,q)$ with $1 \leq p \leq m$ and $1 \leq q \leq n$.

Now, we define a linear transformation

$E^{p,q} : V \longrightarrow W$ s.t. $1 \leq p \leq m$ & $1 \leq q \leq n$

$$E^{p,q}(\alpha_i) = \begin{cases} 0 , & i \neq q . \\ \beta_p , & i = q \end{cases}$$

$$= \delta_{iq} \beta_p .$$

where

$$\delta_{iq} = \begin{cases} 0 , & i \neq q \\ 1 , & i = q \end{cases}$$

We are trying to prove that $mn$ transformations $E^{p,q}$ forms a basis for $L(V,W)$

Assume $T$ be a linear Transformation from $V$ into $W$.

then for each $j$, $1 \leq j \leq n$

Let $A_{1j} \ldots A_{mj}$ be co-ordinates of vector $T\alpha_j$ in the ordered basis of $W$.

$$\Rightarrow \boxed{T\alpha_j = \sum_{p=1}^{m} A_{pj} \beta_p}$$

Now, for each $j$:

$$V\alpha_j = \sum_p \sum_q A_{pq} E^{p,q} \alpha_j$$

$$= \sum_p \sum_q A_{pq} \delta_{jq} \beta_p$$

$$= \sum_{p=1}^{m} A_{pj} \beta_p$$

$$= T\alpha_j$$

$\Rightarrow V = T$

(where $\sum \sum A_{pq} E^{p,q}$ has been proved equal to $T\alpha_j$).

Clearly, we can say that $E^{p,q}$ spans $L(V, W)$

To prove that they are INDEPENDENT:

If $V = \sum_p \sum_q A_{pq} E^{p,q}$ is the zero transformation

$\Rightarrow V\alpha_j = 0 \quad \forall j$.

$$\Rightarrow \sum_{p=1}^{m} A_{pj} \beta_p = 0.$$

& since $\beta_p$ is independent:

$\Rightarrow A_{pj} = 0 \quad \forall p \,\&\, j \quad 1 \leq p \leq m, \; 1 \leq j \leq n.$

Hence Proved.