Summary Quiz 6

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23.1 Cosets

If H is a multiplicative subgroup of G and for some fixed $g \in G$:

- The left coset of H in G is $gH = \{gh: h \in H\}$
- The right coset of H in G is $Hg = \{hg: h \in H\}$

For an additive subgroup H, we define the cosets as:

- The left coset: $g + H = \{g + h: h \in H\}$
- The right coset: $H + g = \{h + g: h \in H\}$

+ Theorems

- $\left.\begin{array}{c} G \text{ abelian } \rightarrow gH = Hg \;\; \forall g \in G \text{ and } H \leq G \\ H \leq Z(G) \rightarrow gH = Hg \; \forall g \in G \\ g \in Z(G) \rightarrow gH = Hg \;\; \forall H \leq G \end{array}\right\}$ but in general, $gH \neq Hg$
- $gH = Hg \leftrightarrow gHg^{-1} = H$
- $gH = Hg$ means $\forall h_1, \;\; \exists h_2 \; gh_1 = h_2 g$

23.2 Index

$[G: H] =$ Number of left cosets of H in G, is the index of H in G

+ Theorems

Let $H \leq G$ and $g_1, g_2 \in G$. The following are equivalent:

- Left Version:
  - $g_1 H = g_2 H$
  - $Hg_1^{-1} = Hg_2^{-1}$
  - $g_1 H \subseteq g_2 H, \qquad g_2 H \subseteq g_1 H$
  - $g_1 \in g_2 H, \qquad g_2 \in g_1 H$
  - $g_2^{-1} g_1 \in H, \qquad g_1^{-1} g_2 \in H$
- Right Version:
  - $Hg_1 = Hg_2$
  - $g_1^{-1} H = g_2^{-1} H$
  - $Hg_1 \subseteq Hg_2, \qquad Hg_2 \subseteq Hg_1$
  - $g_1 \in Hg_2, \qquad g_2 \in Hg_1$
  - $g_1 g_2^{-1} \in H, \qquad g_2 g_1^{-1} \in H$

25.1 Equivalence Relations in Cosets

Suppose $H \leq G$. Define $g_1 \sim g_2$ if $g_1 H = g_2 H$ or $Hg_1 = Hg_2$

This is an equivalence relation where the equivalence classes are the left, right cosets

$[g_1] = \{g_2: g_1 \sim g_2\} = \{g_2: g_1 H = g_2 H\} = \{g_1 h: h \in H\} = g_1 H$

+ Theorems

- Left (or right) cosets of H in G partition G
- $|\{gH: g \in G\}| = |\{Hg: g \in G\}|$
  The number of left cosets is equal to the number of right cosets
- For $H < G$ and for any $g \in G$, $\exists$ bijections: $\begin{cases} H \rightarrow gH \\ H \rightarrow Hg \end{cases}$
- $|H| = |gH| = |Hg|$

25.2 Lagrange

For G, a finite group, and H a subgroup of G, then $|G| = [G: H] \times |H|$

+ Theorems

- $|G|$ is prime $\rightarrow G = \langle a \rangle \;\; \forall a \neq \epsilon$
- If $K < H < G$, then $|G| = [G: H]\,[H: K]\,|K|, \quad [G: K] = [G: H]\,[H: K]$
- Suppose $[G: H] = 2$, then $gHg^{-1} = H \;\; \forall g \in G$
- For G cyclic and finite, if m divides $|G|$, then $\exists H$, a subgroup of G, such that $|H| = m$
- For G abelian and finite, if m divides $|G$, then $\exists H$, a subgroup of G, such that $|H| = m$
- In General:
  - $H < G \rightarrow |H|$ divides $|G|$
  - $g \in G \rightarrow |g|$ divides $|G|$
  - m divides $|G| \nrightarrow \exists H < G$ with $|H| = m$
  - m divides $|G| \nrightarrow \exists g \in G$ with $|g| = m$
  - $\left.\begin{array}{c} m \text{ divides } |G| \\ \exists H < G \text{ with } |H| = m \end{array}\right\} \nrightarrow \exists g \in G$ with $|g| = m$
- If G is finite and cyclic:
  - m divides $|G| \rightarrow \exists!\, H < G$ with $|H| = m$
  - m divides $|G| \rightarrow \exists g \in G$ with $|g| = m$
- If G is finite and abelian:
  - m divides $|G| \rightarrow \exists H < G$ with $|H| = m$
  - m divides $|G| \nrightarrow \exists g \in G$ withh $|g| = m$