Summary Quiz 6

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17.1 Permutations

S_Ω is the set of all bijections $\Omega \to \Omega$, S_Ω is a symmetric group
S_Ω is denoted as S_n if $|\Omega| = n$

A subgroup of S_n is called a permutation group

If $\sigma \in S_n$, then $\sigma = \begin{pmatrix} 1 & 2 & \dots & n \\ \sigma(1) & \sigma(2) & \dots & \sigma(n) \end{pmatrix}$

+ Theorems

- S_Ω with the operation composition is a group
- $|S_n| = n!$

17.2 Cycles and Cycle Notation in S_n

$\sigma \in S_n$ is a cycle if $\exists a_1, \dots, a_k$ such that $\begin{cases} \sigma(a_j) = a_{j+1} \\ \sigma(a_k) = a_1 \\ \sigma(x) = x, \quad x \neq a_j \end{cases}$

17.3 Cycle Order

- A k-cycle has $a_1, \dots, a_k$ terms based on the above defition
- All 1-cycles can be omitted
- 2-cycles are called transpositions

17.4 Cycle Notations

- Two-line notation:
  $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 5 & 1 & 4 & 6 \end{pmatrix}$
- One-Line Notation:
  $\sigma = (1 \quad 3 \quad 5 \quad 4)\,(2)\,(6) = (1 \quad 3 \quad 5 \quad 4)$
  $\sigma^{-1} = (1 \quad 4 \quad 5 \quad 3) = (4 \quad 5 \quad 3 \quad 1)$, just σ inverted

17.5 Multiplying Cycles

For $\alpha = (1 \quad 3 \quad 4 \quad 7)$ and $\beta = (2 \quad 3 \quad 5 \quad 7)$, we perform multiplication:

x	$\beta(x)$	$\alpha(\beta(x))$
1	$\beta(1) = 1$	$\alpha(1) = 3$
2	$\beta(2) = 3$	$\alpha(3) = 4$
3	$\beta(3) = 5$	$\alpha(5) = 5$
4	$\beta(4) = 4$	$\alpha(4) = 7$
5	$\beta(5) = 7$	$\alpha(7) = 1$
6	$\beta(6) = 6$	$\alpha(6) = 6$
7	$\beta(7) = 2$	$\alpha(2) = 2$

$\to \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 7 & 1 & 6 & 2 \end{pmatrix}$

$= (1 \quad 3 \quad 5)\,(2 \quad 4 \quad 7)\,(6) = (1 \quad 3 \quad 5)\,(2 \quad 4 \quad 7)$

18.1 Supports

The support of a permutation π is $\{x : \pi(x) \neq x\}$
Two permutations are disjoint if their supports are disjoint
Example:
$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 3 & 1 & 4 \end{pmatrix}$, $\quad$ support$(\alpha) = \{1,4,5\}$

18.2 Cycle Types

The cycle type of a permutation π is the list (with repetition) of the length of its disjoint cycles

+ Theorems

- Disjoint permutations commute: $\alpha(\beta(x)) = \beta(\alpha(x))$
- $x \in \text{support}(\pi) \to \pi(x), \quad \pi(\pi(x)), \dots \in \text{support}(\pi)$
- Order of a permutation π is the LCM of the lengths of its disjoint cycles, so the LCM of its cycle type
- Every permutation π can be written as a product of disjoint cycles
- S_n is generated by the set of all cycles
- k-cycles can be written as product of $k-1$ transpositions
- $(a_1 \quad a_2 \quad \dots \quad a_k) = (a_1 \quad a_k)\,(a_1 \quad a_{k-1}) \dots (a_1 \quad a_2)$
  $= (a_1 \quad a_2)\,(a_2 \quad a_3) \dots (a_{k-1} \quad a_k)$
- The set of all transpositions generates S_n,
  so $S_n = \langle \{ (a \quad b) : 1 \leq a < b \leq n \} \rangle$
- The following are minimal generating sets for S_n:
  - $\{(1 \quad a) : \ 2 \leq a \leq n\}$
  - $\{(a \quad a+1) : \ 1 \leq a \leq n-1\}$
  - $\{(1 \quad 2), \ (1 \quad 2 \quad \dots \quad n)\}$

18.3 Dihedral Group

It's the symmetries of a regular n-gon with the following:
- ρ = rotation by $\frac{1}{n}$ circle $= (1 \quad 2 \quad \dots \quad n)$
- μ = reflection through corner 1
$= \begin{cases} (1)\,(2 \quad 2m)\,(3 \quad 2m-1) \dots (m \quad m+2)\,(m+1), & n = 2m \\ (1)\,(2 \quad 2m+1)\,(3 \quad 2m) \dots (m+1 \quad m+2), & n = 2m+1 \end{cases}$

+ Theorems

D_n is a subgroup of S_n