

Summary Quiz 6

February 28, 2023 18:00

17.1 Permutations

S_Ω is the set of all bijections $\Omega \rightarrow \Omega$, S_Ω is a symmetric group
 S_Ω is denoted as S_n if $|\Omega| = n$

A subgroup of S_n is called a permutation group

If $\sigma \in S_n$, then $\sigma = \begin{pmatrix} 1 & 2 & \dots & n \\ \sigma(1) & \sigma(2) & \dots & \sigma(n) \end{pmatrix}$

+ Theorems

- ▶ S_n with the operation composition is a group
- ▶ $|S_n| = n!$

17.2 Cycles and Cycle Notation in S_n

$\sigma \in S_n$ is a cycle if $\exists a_1, \dots, a_k$ such that $\begin{cases} \sigma(a_j) = a_{j+1} \\ \sigma(a_k) = a_1 \\ \sigma(x) = x, \quad x \neq a_j \end{cases}$

17.3 Cycle Order

- A k -cycle has a_1, \dots, a_k terms based on the above definition
- All 1-cycles can be omitted
- 2-cycles are called transpositions

17.4 Cycle Notations

- ▶ Two-line notation:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 5 & 1 & 4 & 6 \end{pmatrix}$$

- ▶ One-Line Notation:

$$\sigma = (1 \ 3 \ 5 \ 4) (2) (6) = (1 \ 3 \ 5 \ 4) \\ \sigma^{-1} = (1 \ 4 \ 5 \ 3) = (4 \ 5 \ 3 \ 1), \text{ just } \sigma \text{ inverted}$$

17.5 Multiplying Cycles

For $\alpha = (1 \ 3 \ 4 \ 7)$ and $\beta = (2 \ 3 \ 5 \ 7)$, we perform multiplication:

$$\begin{array}{lcl} x & \beta(x) & \alpha(\beta(x)) \\ 1 & \beta(1) = 1 & \alpha(1) = 3 \\ 2 & \beta(2) = 3 & \alpha(3) = 4 \\ 3 & \beta(3) = 5 & \alpha(5) = 5 \\ 4 & \beta(4) = 4 & \alpha(4) = 7 \\ 5 & \beta(5) = 7 & \alpha(7) = 1 \\ 6 & \beta(6) = 6 & \alpha(6) = 6 \\ 7 & \beta(7) = 2 & \alpha(2) = 2 \end{array} \rightarrow \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 7 & 1 & 6 & 2 \end{pmatrix} \\ = (1 \ 3 \ 5) (2 \ 4 \ 7) (6) = (1 \ 3 \ 5) (2 \ 4 \ 7)$$

18.1 Supports

The support of a permutation π is $\{x: \pi(x) \neq x\}$

Two permutations are disjoint if their supports are disjoint
Example:

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 3 & 1 & 4 \end{pmatrix}, \quad \text{support}(\alpha) = \{1, 4, 5\}$$

18.2 Cycle Types

The cycle type of a permutation π is the list (with repetition) of the length of its disjoint cycles

+ Theorems

- ▶ Disjoint permutations commute: $\alpha(\beta(x)) = \beta(\alpha(x))$
- ▶ $x \in \text{support}(\pi) \rightarrow \pi(x), \pi(\pi(x)), \dots \in \text{support}(\pi)$
- ▶ Order of a permutation π is the LCM of the lengths of its disjoint cycles, so the LCM of its cycle type
- ▶ Every permutation π can be written as a product of disjoint cycles
- ▶ S_n is generated by the set of all cycles
- ▶ k -cycles can be written as product of $k - 1$ transpositions
- ▶ $(a_1 \ a_2 \ \dots \ a_k) = (a_1 \ a_k) (a_1 \ a_{k-1}) \dots (a_1 \ a_2) \\ = (a_1 \ a_2) (a_2 \ a_3) \dots (a_{k-1} \ a_k)$
- ▶ The set of all transpositions generates S_n ,
so $S_n = \langle \{(a \ b): 1 \leq a < b \leq n\} \rangle$
- ▶ The following are minimal generating sets for S_n :
 - $\{(1 \ a): 2 \leq a \leq n\}$
 - $\{(a \ a+1): 1 \leq a \leq n-1\}$
 - $\{(1 \ 2), (1 \ 2 \ \dots \ n)\}$

18.3 Dihedral Group

It's the symmetries of a regular n -gon with the following:

- ρ = rotation by $\frac{1}{n}$ circle = $(1 \ 2 \ \dots \ n)$

- μ = reflection through corner 1

$$= \begin{cases} (1) (2 \ 2m) (3 \ 2m-1) \dots (m \ m+2) (m+1), & n = 2m \\ (1) (2 \ 2m+1) (3 \ 2m) \dots (m+1 \ m+2), & n = 2m+1 \end{cases}$$

+ Theorems

D_n is a subgroup of S_n