Summary Quiz 4

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14.1 Cyclic Groups

G is cyclic $\leftrightarrow \exists$ a generator $g \in G$ such that $G = \langle g \rangle = \{g^k : k \in \mathbb{Z}\}$
The order of g is the smallest positive integer n with $g^n = \epsilon$
<u>Notation:</u>

- $|g|$ = Order of an element, $|g| = \infty \leftrightarrow g^k \neq \epsilon \; \forall k \in \mathbb{Z}$
- $|G|$ = Order of a group

The set $\{k : g^k = \epsilon\} = |g| \times \mathbb{Z}$, so $g^k = \epsilon \leftrightarrow |g|$ divides k
$|x| = |y|$ is equivalent to $x^k = \epsilon \leftrightarrow y^k = \epsilon$

If $|g| = n < \infty$ then:

- $G = \langle g \rangle = \{g, g^2, \ldots, g^n = \epsilon\}$
- $|G| = |g|$
- $|g^k| = \dfrac{n}{\mathrm{GCD}(n,k)}$
- Generators of G are exactly $\{g^k : GCD(n,k) = 1\}$

Note:
To check if a group is cyclic or not, check all the generators, if the order of some generator g is the length of the group, then the group is cyclic

+ Theorems

- G is cyclic $\rightarrow$ G is abelian (commutative)
- G is cylic $\rightarrow$ All subgroups are cyclic
- G has no subgroups other than $\{\epsilon\}$ and G
  $\leftrightarrow$ G is cyclic of prime order
  $\leftrightarrow |G| = n$ is prime
- If G, H are both cyclic, then $G \cong H \leftrightarrow |G| = |H|$

15.1 Complex Numbers

$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$
$\mathbb{C} = \{re^{i\theta} : r, \theta \in \mathbb{R}\}$ where $r \geq 0$ and $0 \leq \theta < 2\pi$
$re^{i\theta} = r\cos\theta + ri\sin\theta \rightarrow e^{i\theta} = \cos\theta + i\sin\theta$

For $z \in \mathbb{C}$:

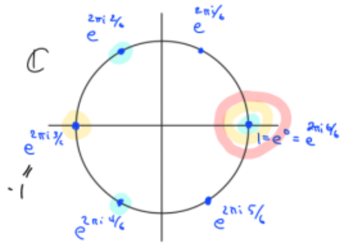
- $|z| = |a + bi| = \sqrt{a^2 + b^2} = r$
- $\dfrac{b}{a} = \tan\theta$

15.2 Roots of Unity

The nth root of unity is the solution to $z^n = 1$ for $z \in \mathbb{C}$
$R_n = \left\{e^{i2\pi\times\frac{1}{n}},\ e^{i2\pi\times\frac{2}{n}}, \ldots, e^{i2\pi\times\frac{n}{n}}\right\} = \left\langle e^{\frac{i2\pi}{n}} \right\rangle$

R_n forms an equilateral n-gon in $\mathbb{C}$

Example: R_6



$\mathbb{T} = \{z \in \mathbb{Z} : |z| = 1\} = \{e^{i\theta} : \theta \in \mathbb{R}\}$
$\mathbb{T}$ is a subgroup of $\mathbb{C}^\times$
R_n is a subgroup of $\mathbb{T}$ (and of $\mathbb{C}^\times$)

Let $R = \bigcup_{n=1}^{\infty} R_n = \left\{e^{2\pi i\times\frac{j}{n}} : 0 \leq j < n,\ n \geq 1\right\}$

<u>Subgroup Hierarchy:</u>
$R_n < R < \mathbb{T} < \mathbb{C}^\times$

15.3 Properties of R

- $R = \left\{e^{2\pi i\times\frac{j}{n}} : 0 \leq j < n,\ n \geq 1\right\}$
- $|z|$ is finite $\forall z \in R$
- $|R|$ is infinite
- It's abelian but not cyclic
- Every finite subset is contained in a finite subgroup
- Every finite subgroup is cyclic
- Every infinite subgroup is not cyclic
- $R = \left\langle \left\{e^{\frac{2\pi i}{n}} : n \geq 1\right\} \right\rangle = \left\langle \left\{e^{\frac{2\pi i}{n}} : n \geq k\right\} \right\rangle \; \forall k$