

2.1 Equivalence Relations

Define $R = \{(x, y): x, y \in X, x \sim y\} \subseteq X \times X$

R : set of all pairs that are equivalent

\sim is an equivalence relation if it satisfies:

- Reflexive: $x \sim x \forall x \in X$
- Symmetric: $x \sim y \leftrightarrow y \sim x$
- Transitive: If $x \sim y$ and $y \sim z$, then $x \sim z$

2.2 Equivalence Classes

$$[x] = \{y \in X: y \sim x\}$$

3.1 Well-Defined Operations

An operation \cdot is well defined if:

$$\left. \begin{array}{l} x \sim y \\ w \sim z \end{array} \right\} \rightarrow (x \cdot w) \sim (y \cdot z)$$

Note: $x \sim y \leftrightarrow [x] = [y]$

+ Theorems

- ▶ Let X be a set with an equivalence relation, then $[x] \cap [y] \neq \emptyset \rightarrow [x] = [y]$
- ▶ Equivalence classes are either disjoint or equal
- ▶ Let X be a set with an equivalence relation, then the equivalence classes form a partition of X
- ▶ Let R_j ($j \in J$, for some index set J) form a partition of X . Say that $x \sim y$ means $x, y \in R_j$ for some j , then \sim is an equivalence relation on X

3.2 Number Theory

- ▶ Any non-empty set $S \subseteq \mathbb{N}$ has a unique $d \in S$ such that $\forall x \in S, d \leq x$
- ▶ For $a, b \in \mathbb{Z}, b > 0$, then $\exists! q, r \in \mathbb{Z}$ such that $a = bq + r, 0 \leq r < b$

3.3 Refinements

For two equivalence relations \approx and \sim , we say \approx is a refinement of \sim if each equivalence class of \approx is contained in an equivalence class of \sim

In other words, $a \approx b \rightarrow a \sim b$

5.1 Divisibility and Modulo

$m \mid n$ means $\exists x \in \mathbb{Z}$ such that $n = mx$

$$a \equiv b \pmod n \text{ means } n \mid (a - b) \rightarrow \frac{a - b}{n} \in \mathbb{Z}$$

+ Theorems

- ▶ Congruence modulo n is an equivalence relation
- ▶ If $a \equiv a' \pmod n$ and $b \equiv b' \pmod n$, then:
 - $a + b \equiv a' + b' \pmod n$
 - $ab \equiv a'b' \pmod n$
- ▶ $\mathbb{Z}_n = \{[k]_n: k \in \mathbb{Z}\}$, contains n elements
For $n = 5, [2] = \{\dots, -3, 2, 7, \dots\} \in \mathbb{Z}_5$
- ▶ $\mathbb{Z}_n^* = \{x \in \mathbb{Z}_n: x \text{ has a multiplicative inverse} \in \mathbb{Z}_n^*\}$
= The set of elements that are coprime with n

5.2 Prime and Irreducible

For $p \in \mathbb{Z}$ where $p > 1$:

- p is irreducible if the only divisors of p are 1 and p
- p is prime if whenever $p \mid ab$, then $p \mid a$ and $p \mid b$

+ Theorems

- ▶ p is prime $\leftrightarrow p$ is irreducible
- ▶ For $n > 1, \exists! \left\{ \begin{array}{l} p_1, \dots, p_s \text{ primes} \\ e_1, \dots, e_s \text{ positives} \end{array} \right. \text{ s.t. } n = p_1^{e_1} \times \dots \times p_s^{e_s}$

5.3 GCD and LCM

- ▶ $d = \text{GCD}(a, b)$ if and only if:
 - $d \mid a$ and $d \mid b$
 - If $c \mid a$ and $c \mid b$, then $c \mid d$
- ▶ $m = \text{LCM}(a, b)$ if and only if:
 - $a \mid m$ and $b \mid m$
 - If $a \mid n$ and $b \mid n$, then $m \mid n$

+ Theorems

- ▶ $\forall a, b \in \mathbb{Z}, \exists! \text{GCD } d \text{ and } \exists x, y \in \mathbb{Z} \text{ such that } d = ax + by$
- ▶ $\forall a, b \in \mathbb{Z}, \exists! \text{LCM } m$
- ▶ If $\text{GCD}(a, b) = 1$, then $\exists x, y$ such that $ax + by = 1$
- ▶ If $\text{GCD}(a, b) = d$, then $\{ax + by: x, y \in \mathbb{Z}\} = d \times \mathbb{Z}$
- ▶ $\text{GCD}(a, b) \times \text{LCM}(a, b) = |ab|$

6.1 Groups

For some set S and an operation \cdot , (S, \cdot) is a group if:

- Closure: $ab \in S$
- Associativity: $(ab)c = a(bc)$
- Identity: $\exists e \in S$ such that $x e = e x = x$
- Inverses: $\forall x \in S, \exists y \in S$ such that $xy = yx = e$

9.1 Laws of Exponents

For a group G with some operation \cdot :

- $x^n = x \cdot x \cdot \dots \cdot x$ (n times)
- $x^{-n} = (x^{-1})^n = (x^n)^{-1}$
- $x^m \cdot x^n = x^{m+n}$
- $(x^m)^n = x^{mn}$

If $xy = yx \forall x, y \in G$, then G is abelian (commutative)

9.2 - 10.1 Properties of Groups

- The identity is unique
- The inverse of each element is unique
- $ax = b$ has a unique solution $x \forall a, b \in G$
- $ab = ac \rightarrow bc$
- $(ab)^{-1} = b^{-1}a^{-1}$
- $(a^{-1})^{-1} = a$
- If $xy = x$ for some $x, y \in G$, then $y = e$
- If $xy = e$ for some $x, y \in G$, then $y = x^{-1}$

8.1 Cayley Tables

\cdot	a	b	...
a	a	b	...
b	b	$a \cdot b$...
\vdots	\vdots	\vdots	\ddots

9.3 Properties of Cayley Tables

- ▶ Only one row and column matches the header completely and no other row or column matches the header in a single position
- ▶ Each row and column contains each element once

9.4 Product of Groups

For two groups G, H , their product is defined as:

$$G \times H = \{(g, h) : g \in G, h \in H\}$$

$$(x, a) \cdot_{G \times H} (y, b) = (x \cdot_G a, y \cdot_H b)$$

+ Theorems

- ▶ The product of groups is a group
- ▶ For $x = (a_1, \dots, a_t) \in G_1 \times \dots \times G_t$, then $|x| = \text{LCM}(|a_1|, \dots, |a_t|)$
- ▶ $G_1 \times \dots \times G_t$ is cyclic $\leftrightarrow \begin{cases} \text{Each } G_i \text{ is cyclic} \\ \text{GCD}(|G_i|, |G_j|) = 1 \quad \forall i \neq j \end{cases}$

9.5 Isomorphisms

If $\phi: G \rightarrow H$ is a bijection with $\phi(x \cdot_G y) = \phi(x) \cdot_H \phi(y)$
Then ϕ is an isomorphism, and G, H are isomorphic

If G, H are isomorphic, then permuting the Cayley Table of G gives the Cayley Table of H

9.6 Isomorphisms

If $\phi: G \rightarrow G$ is an isomorphism, then ϕ is an automorphism
 $\text{aut}(G)$ = The set of all automorphisms of G and it's a group

6.2 Symmetries

$S = \{\alpha, \beta, \dots\}$ is the set of symmetries of some object with the operation composition

▶ Example of Symmetries:

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

▶ Example of Composition:

$$\alpha \circ \beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 3 & 4 & 1 & 2 \end{pmatrix} \sim \beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} \sim \alpha$$

6.3 Properties of Symmetries

- $\alpha \circ \beta$ is a symmetry $\forall \alpha, \beta \in S$
- $(\alpha \circ \beta) \circ \gamma = \alpha \circ (\beta \circ \gamma) \quad \forall \alpha, \beta, \gamma \in S$
- $\exists \epsilon \in S$ such that $\epsilon \circ \alpha = \alpha \circ \epsilon = \alpha \quad \forall \alpha \in S$
- $\forall \alpha \in S, \exists \beta \in S$ such that $\alpha \circ \beta = \epsilon$

11.4 Generating Sets

If $\forall g \in S, g$ can be written with α, β , then $\{\alpha, \beta\}$ generates S

11.1 Subgroups

For a group G with operation \cdot , if $H \subseteq G$, and it's a group with the same operation \cdot , then H is a subgroup

Notation:

If H is a subgroup of G , we write $H \leq G, H < G$ (if $H \neq G$)

11.2 Subgroup Test

Suppose H is a subset of G , then if:

- $H \neq \emptyset$
- $x, y \in H \rightarrow x \cdot y \in H$
- $x \in H \rightarrow x^{-1} \in H$

Then H is a subgroup

+ Theorems

- ▶ If $H \leq G$, then $\epsilon_G \in H$ and $\epsilon_H = \epsilon_G$
- ▶ If $H_1 \leq G$ and $H_2 \leq G$, then $H_1 \cap H_2 \leq G$
- ▶ If $K \leq H_1$ and $K \leq H_2$, then $K \leq H_1 \cap H_2$
- ▶ For $H_1 \leq G$ and $H_2 \leq G$:
If $H_1 \cup H_2 \leq G$, then $H_1 \leq H_2$ or $H_2 \leq H_1$

11.5 Product Set

If $S \subseteq G$, then $\langle S \rangle$ is the set of all possible products of elements in S and their inverses

+ Theorems

- ▶ $S \subseteq G \rightarrow \langle S \rangle \leq G$
- ▶ If $H_1 \leq K$ and $H_2 \leq K$, then $\langle H_1 \cup H_2 \rangle \leq K$

11.7 Greatest Lower Bound

If $\exists \alpha \in X$ such that $\begin{cases} \alpha \leq x, \alpha \leq y \\ x \leq \alpha \text{ and } y \leq \alpha \end{cases} \quad \forall x, y \in X$,
then α is the greatest lower bound of x, y , denoted $\text{glb}(x, y)$

11.8 Least Upper Bound

If $\exists \beta \in X$ such that $\begin{cases} x \leq \beta, y \leq \beta \\ x \leq \beta \text{ and } y \leq \beta \end{cases} \quad \forall x, y \in X$,
then β is the least upper bound of x, y , denoted $\text{lub}(x, y)$

11.6 Lattices

A lattice is the set X with operation \leq such that $\text{glb}(x, y)$ and $\text{lub}(x, y)$ exists $\forall x, y \in X$

It's a diagram of subgroups, where each line connecting H and K (with K vertically higher than H in the diagram) means $H \leq K$

Note: If $H \leq K$, and we have some subgroup F such that $H \leq F \leq K$, then $F = H$ or $F = K$

11.3 Symmetries of a Square Example

To show a square has at most 8 symmetries:

Let γ be some symmetry, then:

- $\gamma(1)$ (1st corner) has 4 options
- $\gamma(2)$ (2nd corner) is adjacent to $\gamma(1)$, so 2 options
- $\gamma(4)$ (4th corner) is adjacent to $\gamma(1)$, so 1 option left
- $\gamma(3)$ (3rd corner) has 1 option left

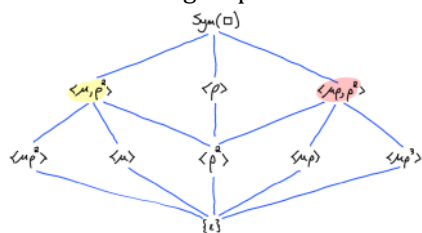
So $4 \times 2 \times 1 \times 1 = 8$ possibilities

To show a square has at least 8 symmetries, we show the above 8 symmetries are all possible, with matrices form

To find the subgroups of the symmetries of a square, we go through the product set of every subset of G , for instance:

$\langle \epsilon \rangle$, $\langle \mu \rangle$, $\langle \rho \rangle$, $\langle \mu, \rho \rangle$, ...

If the product set generates G , it's not a subgroup, otherwise, it is, and we can use the subgroups to draw the lattice:



14.1 Cyclic Groups

G is cyclic $\leftrightarrow \exists$ a generator $g \in G$ s.t $G = \langle g \rangle = \{g^k : k \in \mathbb{Z}\}$

The order of g is the smallest positive integer n with $g^n = \epsilon$

Notation:

- $|g|$ = Order of an element, $|g| = \infty \leftrightarrow g^k \neq \epsilon \forall k \in \mathbb{Z}$
- $|G|$ = Size of a group

The set $\{k : g^k = \epsilon\} = |g|\mathbb{Z}$, so $g^k = \epsilon \leftrightarrow |g|$ divides k

$|x| = |y|$ is equivalent to $x^k = \epsilon \leftrightarrow y^k = \epsilon$

If G is a group with n elements and $|g| = n < \infty$ then:

- $G = \langle g \rangle = \{g, g^2, \dots, g^n = \epsilon\}$
- $|G| = |g|$
- $|g^k| = \frac{n}{\text{GCD}(n, k)}$
- Generators of G are exactly $\{g^k : \text{GCD}(n, k) = 1\}$

To check if a group is cyclic or not, check all the generators, if the order of some generator g is the length of the group, then the group is cyclic

+ Theorems

- ▶ G is cyclic $\rightarrow G$ is abelian (commutative)
- ▶ G is cyclic \rightarrow All subgroups are cyclic
- ▶ G has no subgroups other than $\{\epsilon\}$ and G $\leftrightarrow G$ is cyclic of prime order $\leftrightarrow |G| = n$ is prime
- ▶ If G, H are both cyclic, then $G \cong H \leftrightarrow |G| = |H|$

15.1 Complex Numbers

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$$

$$\mathbb{C} = \{re^{i\theta} : r, \theta \in \mathbb{R}\} \text{ where } r \geq 0 \text{ and } 0 \leq \theta < 2\pi$$

$$re^{i\theta} = r \cos \theta + ri \sin \theta \rightarrow e^{i\theta} = \cos \theta + i \sin \theta$$

For $z \in \mathbb{C}$:

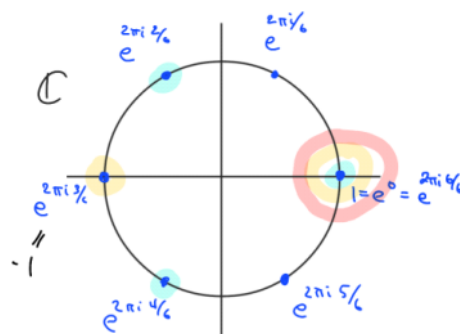
- $|z| = |a + bi| = \sqrt{a^2 + b^2} = r$
- $\frac{b}{a} = \tan \theta$

15.2 Roots of Unity

The n th root of unity is the solution to $z^n = 1$ for $z \in \mathbb{C}$

$$R_n = \left\{ e^{i2\pi \times \frac{1}{n}}, e^{i2\pi \times \frac{2}{n}}, \dots, e^{i2\pi \times \frac{n}{n}} \right\} = \left\langle e^{\frac{i2\pi}{n}} \right\rangle$$

Example: R_6



$$\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\} = \{e^{i\theta} : \theta \in \mathbb{R}\}$$

\mathbb{T} is a subgroup of \mathbb{C}^\times

R_n is a subgroup of \mathbb{T} (and of \mathbb{C}^\times)

$$\text{Let } R = \bigcup_{n=1}^{\infty} R_n = \left\{ e^{\frac{2\pi i j}{n}} : 0 \leq j < n, n \geq 1 \right\}$$

15.3 Subgroup Hierarchy

$$R_n < R < \mathbb{T} < \mathbb{C}^\times$$

15.4 Properties of R

- $|z|$ is finite $\forall z \in R$
- $|R|$ is infinite
- It's abelian but not cyclic
- Every finite subset is contained in a finite subgroup
- Every finite subgroup is cyclic
- Every infinite subgroup is not cyclic
- $R = \left\langle \left\{ e^{\frac{2\pi i}{n}} : n \geq 1 \right\} \right\rangle = \left\langle \left\{ e^{\frac{2\pi i}{n}} : n \geq k \right\} \right\rangle \forall k$

15.5 Subgroups of \mathbb{T}

- $R = \left\{ e^{\frac{2\pi i j}{n}} : 0 \leq j < n, n \geq 1 \right\}$
- $Z = \{e^{ik} : k \in \mathbb{Z}\}$

17.1 Permutations

S_Ω is the set of all bijections $\Omega \rightarrow \Omega$, S_Ω is a symmetric group

S_Ω is denoted as S_n if $|\Omega| = n$

A subgroup of S_n is called a permutation group

If $\sigma \in S_n$, then $\sigma = \begin{pmatrix} 1 & 2 & \dots & n \\ \sigma(1) & \sigma(2) & \dots & \sigma(n) \end{pmatrix}$

+ Theorems

- ▶ S_n with the operation composition is a group
- ▶ $|S_n| = n!$

17.2 Cycles and Cycle Notation in S_n

$\sigma \in S_n$ is a cycle if $\exists a_1, \dots, a_k$ such that
$$\begin{cases} \sigma(a_j) = a_{j+1} \\ \sigma(a_k) = a_1 \\ \sigma(x) = x, \quad x \neq a_j \end{cases}$$

17.3 Cycle Order

- A k -cycle has a_1, \dots, a_k terms based on the above definition
- All 1-cycles can be omitted
- 2-cycles are called transpositions

17.4 Cycle Notations

- ▶ Two-line notation:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 5 & 1 & 4 & 6 \end{pmatrix}$$

- ▶ One-Line Notation:

$$\sigma = (1 \ 3 \ 5 \ 4)(2)(6) = (1 \ 3 \ 5 \ 4)$$

$$\sigma^{-1} = (1 \ 4 \ 5 \ 3) = (4 \ 5 \ 3 \ 1), \text{ just } \sigma \text{ inverted}$$

17.5 Multiplying Cycles

For $\alpha = (1 \ 3 \ 4 \ 7)$ and $\beta = (2 \ 3 \ 5 \ 7)$, we perform multiplication:

$$\begin{array}{lll} x & \beta(x) & \alpha(\beta(x)) \\ 1 & \beta(1) = 1 & \alpha(1) = 3 \\ 2 & \beta(2) = 3 & \alpha(3) = 4 \\ 3 & \beta(3) = 5 & \alpha(5) = 5 \\ 4 & \beta(4) = 4 & \alpha(4) = 7 \\ 5 & \beta(5) = 7 & \alpha(7) = 1 \\ 6 & \beta(6) = 6 & \alpha(6) = 6 \\ 7 & \beta(7) = 2 & \alpha(2) = 2 \end{array} \rightarrow \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 7 & 1 & 6 & 2 \end{pmatrix}$$

$$= (1 \ 3 \ 5)(2 \ 4 \ 7)(6) = (1 \ 3 \ 5)(2 \ 4 \ 7)$$

18.1 Supports

The support of a permutation π is $\{x: \pi(x) \neq x\}$

Two permutations are disjoint if their supports are disjoint

Example:

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 3 & 1 & 4 \end{pmatrix}, \quad \text{support}(\alpha) = \{1, 4, 5\}$$

18.2 Cycle Types

The cycle type of a permutation π is the list (with repetition) of the length of its disjoint cycles

+ Theorems

- ▶ Disjoint permutations commute: $\alpha(\beta(x)) = \beta(\alpha(x))$
- ▶ $x \in \text{support}(\pi) \rightarrow \pi(x), \pi(\pi(x)), \dots \in \text{support}(\pi)$
- ▶ Order of a permutation π is the LCM of the lengths of its disjoint cycles, so the LCM of its cycle type
- ▶ Every permutation π can be written as a product of disjoint cycles
- ▶ S_n is generated by the set of all cycles
- ▶ k -cycles can be written as product of $k - 1$ transpositions
- ▶ $(a_1 \ a_2 \ \dots \ a_k) = (a_1 \ a_k)(a_1 \ a_{k-1}) \dots (a_1 \ a_2)$
 $= (a_1 \ a_2)(a_2 \ a_3) \dots (a_{k-1} \ a_k)$
- ▶ The set of all transpositions generates S_n ,
so $S_n = \langle \{(a \ b): 1 \leq a < b \leq n\} \rangle$
- ▶ The following are minimal generating sets for S_n :
 - $\{(1 \ a): 2 \leq a \leq n\}$
 - $\{(a \ a+1): 1 \leq a \leq n-1\}$
 - $\{(1 \ 2), (1 \ 2 \ \dots \ n)\}$

18.3 Dihedral Group

It's the symmetries of a regular n -gon with the following:

- ρ = rotation by $\frac{1}{n}$ circle = $(1 \ 2 \ \dots \ n)$

- μ = reflection through corner 1

$$= \begin{cases} (1)(2 \ 2m)(3 \ 2m-1) \dots (m \ m+2)(m+1), & n = 2m \\ (1)(2 \ 2m+1)(3 \ 2m) \dots (m+1 \ m+2), & n = 2m+1 \end{cases}$$

$$D_n = \{\mu^i \rho^j\} = \{\rho^j \mu^i\}, \quad 0 \leq i \leq 1, \quad 0 \leq j \leq n-1$$

$$D_n = \{\mu, \rho: \mu^2 = \epsilon, \rho^2 = \epsilon, \rho\mu = \mu\rho^{-1}\}$$

+ Theorems

D_n is a subgroup of S_n

20.1 Conjugation

$\sigma\pi\sigma^{-1}$ is the conjugation of π by σ

$$\pi(i) = j \leftrightarrow (\sigma\pi\sigma^{-1})(\sigma(i)) = \sigma(j)$$

Conjugation Example:

$$\begin{array}{ll} \sigma = (1 \ 3 \ 4 \ 5)(7 \ 9) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 2 & 4 & 5 & 1 & 6 & 9 & 8 & 7 \end{pmatrix} \\ \pi = (1 \ 7 \ 3)(4 \ 6 \ 9)(8 \ 2) \\ \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \text{ by } \sigma \\ \sigma\pi\sigma^{-1} = (3 \ 9 \ 4)(5 \ 6 \ 7)(8 \ 2) \end{array}$$

+ Theorems

$\alpha, \beta \in S_n$ have the same cycle type $\leftrightarrow \beta = \sigma\alpha\sigma^{-1}$ for $\alpha \in S_n$