4.1 Divisibility

$a \mid b \leftrightarrow \exists c$ such that $b = ac$

+ Theorems

Let $a, b, c \in \mathbb{Z}$ with $a \neq 0$:

- If $a \mid b$ and $a \mid c$, then $a \mid (b + c)$
  - We also have $a \mid (mb + nc)\ \forall m, n \in \mathbb{Z}$
- If $a \mid b$, then $a \mid bc\ \ \forall c \in \mathbb{Z}$
- If $a \mid b$ and $b \mid c$, then $a \mid c$

4.2 Division Algorithm

Let $a, d \in \mathbb{Z}$ with $d > 0$
$\exists!\, q, r \in \mathbb{Z}$ such that $a = dq + r$ for $0 \leq r < d$

<u>Notation:</u> If $a = dq + r$ for $0 \leq r < d$, we write:

- $q = a \text{ div } d$
- $r = a \bmod d$

4.3 Modulo

Let $a, b, m \in \mathbb{Z}$ with $m \geq 2$. a is congruent to b modulo m if $m \mid (a - b)$, and we denote it as $a \equiv b \pmod{m}$

Note: $a \equiv b \pmod{m} \leftrightarrow b \equiv a \pmod{m}$

+ Theorems

- Let $a, b, c, d, m \in \mathbb{Z}$ with $m \geq 2$. If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then:
  - $a + c \equiv b + d \pmod{m}$
  - $ac \equiv bd \pmod{m}$

- $a + c \equiv b + c \pmod{m} \rightarrow a \equiv b \pmod{m}$
- $c \not\equiv 0 \pmod{m}, ac \equiv bc \pmod{m} \nrightarrow a \equiv b \pmod{m}$

4.4 Arithmetic Modulo

For $m \geq 2$, define $\mathbb{Z}_m = \{0,1,2, \dots, m-1\}$
$a +_m b = (a + b) \pmod{m}$
$a \cdot_m b = (a \cdot b) \pmod{m}$

5.1 Prime Numbers

$p \in \mathbb{Z}$ is prime if it has exactly two divisors, 1 and itself

5.2 Fundamental Theorem of Arithmetic

All integers greater than 1 can be written as a unique product of prime numbers

+ Theorems

- For $n \in \mathbb{Z}$ such that $n > 1$. If n is not prime, then n has a prime divisor p such that $p \leq \sqrt{n}$
- There exists an infinite number of prime numbers

5.3 Greatest Common Divisor

For $a, b \in \mathbb{Z}$ such that $a \neq 0$ or $b \neq 0$
The greatest integer d such that $d \mid a$ and $d \mid b$ is the GCD of a and b
a, b are coprime if $\text{GCD}(a, b) = 1$

5.4 Least Common Divisor

For $a, b \in \mathbb{Z}$ such that $a \neq 0$ and $b \neq 0$
The least integer m such that $a \mid m$ and $b \mid m$ is the LCM of a and b

+ Theorems

- For $n = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$ where p_i are prime numbers and $a_i > 0$ are integers and for $d \in \mathbb{Z}$
  Then $d \mid n \leftrightarrow d = p_1^{b_1} p_2^{b_2} \dots p_k^{b_k}$ where $0 \leq b_i \leq a_i$

- For $\begin{cases} a = p_1^{a_1} \dots p_k^{a_k} \\ b = p_1^{b_1} \dots p_k^{b_k} \end{cases}$ with p_i primes and $a_i, b_i \geq 0$:
  - $\text{GCD}(a, b) = p_1^{\min(a_1, b_1)} \dots p_k^{\min(a_k, b_k)}$
  - $\text{LCM}(a, b) = p_1^{\max(a_1, b_1)} \dots p_k^{\max(a_k, b_k)}$

- $\text{GCD}(a, b) \times \text{LCM}(a, b) = a \times b$

- For $a, b, q, r \in \mathbb{Z}$ such that $a = bq + r$, then $\text{GCD}(a, b) = \text{GCD}(b, r)$

6.1 Euclid's Algorithm

$$
\begin{aligned}
a &= q_1 \times b + r_1 \\
b &= q_2 \times r_1 + r_2 \\
r_1 &= q_3 \times r_2 + r_3 \\
&\vdots \\
r_{n-2} &= q_n r_{n-1} + r_n \\
r_{n-1} &= q_{n+1} r_n + r_{n+1}
\end{aligned}
$$

And $\text{GCD}(a, b) = r_n$ when $r_{n+1} = 0$

6.2 Bezout's Algorithm

For $a, b \in \mathbb{Z}$ and $a, b > 0$, then there exists $s, t \in \mathbb{Z}$ such that $sa + tb = \text{GCD}(a, b)$
We run Euclid's algorithm backwards:
$\text{GCD}(a, b) = r_n = r_{n-2} - q_{n-1} \times r_{n-2} = \cdots = sa + tb$

+ Example of Euclid and Bezout

Run Euclid first for GCD(662,414)
a: $662 = 1 \times 414 + 248$
b: $414 = 1 \times 248 + 166$
c: $248 = 1 \times 166 + 82$
d: $166 = 2 \times 82 + 2$
e: $82 = 41 \times 2 + 0$
So $\mathrm{GCD}(662, 414) = 2$

Run Bezout now:
$2 = 166 - 2 \times 82$ from d
$= 166 - 2(248 - 1 \times 166)$ from c
$= -2 \times 248 + 3 \times 166$ simple rearranging
$= -2 \times 248 + 3(414 - 1 \times 248)$ from b
$= 3 \times 414 - 5 \times 248$ simple rearranging
$= 3 \times 414 - 5(662 - 1 \times 414)$ from a
$= 8 \times 414 - 5 \times 662$ simple rearranging
$= -5 \times 662 + 8 \times 414$
So $-5 \times 662 + 8 \times 414 = 2 = \mathrm{GCD}(662, 414)$

+ Theorems

- If $\mathrm{GCD}(a, b) = 1$ and $a \mid (bc)$, then $a \mid c$
- $\exists s, t \in \mathbb{Z}$ such that $sa + tb = m \leftrightarrow \mathrm{GCD}(a, b) \mid m$
- For $m \geq 2$: If $ac \equiv bc \pmod{m}$, $\mathrm{GCD}(c, m) = 1$, then $a \equiv b \pmod{m}$
- For p, a prime number and $a_1, \dots, a_n \in \mathbb{Z}$
  If $p \mid (a_1 \times \cdots \times a_n)$, then $\exists\ 1 \leq i \leq n$ s.t $p \mid a_i$
- For $m \geq 2$ and $a \in \mathbb{Z}_m$
  The unique multiplicative inverse of $a \pmod{m}$ exists if and only if $\mathrm{GCD}(a, m) = 1$

9 Chinese Remainder Theorem

Let $m_1, m_2, \dots, m_r \in \mathbb{Z}$ be pairwise co-prime integers such that $m_i \geq 2$ $(1 \leq i \leq r)$
Let $a_1, a_2, \dots, a_r \in \mathbb{Z}$, then the system:

$$x \equiv a_1 \pmod{m_1}$$
$$\vdots$$
$$x \equiv a_r \pmod{m_r}$$

admits a unique solution

10 Fermat's Little Theorem

Let $p, a \in \mathbb{Z}$ such that p is prime. Then:

- $a^p \equiv a \pmod{p}$
- If $\gcd(a, p) = 1$, then $a^{p-1} \equiv 1 \pmod{p}$

+ Theorems

Let $p, q, M \in \mathbb{Z}$ such that p, q are two different primes, and $\gcd(M, pq) = 1$, then: $M^{(p-1)(q-1)} \equiv 1 \pmod{pq}$

13 Asymptomatic Notation

Let f, g be functions $N \to \mathbb{R}^+$ and let $c \in \mathbb{R}^+$ and $k \in \mathbb{N}$:

- Big O-Notation: Asymptotic Upper Bound
  $f = O(g)$ if $\exists c, k$ such that $f(n) \leq c \cdot g(n)\ \ \forall n \geq k$
- Big Ω-Notation: Asymptotic Lower Bound
  $f = \Omega(g)$ if $\exists c, k$ such that $f(n) \geq c \cdot g(n)\ \forall\, n \geq k$
- Big Θ-Notation: Asymptotic Tight Bound
  $f = \Theta(g)$ if $f = O(g)$ and $f = \Omega(g)$

+ Theorems

- For $a, b \in \mathbb{R}$ s.t $a, b > 0$. We have $\log^a(x) = O\left(x^b\right)$
- $f(n) = O(g(n)) \leftrightarrow g(n) = \Omega(f(n))$
- $f(n) = \Theta(g(n)) \leftrightarrow g(n) = \Theta(f(n))$

14.1 Recursivity

Example: *Fibonacci Sequence*

$$F_0 = 0$$
$$F_1 = 1$$
$$F_n = F_{n-1} + F_{n-2} \quad (n \geq 2)$$

14.2 Characteristic Equation and Roots

Characteristic equation for a recursive function:

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \dots - c_k = 0$$

The solutions are called the *characteristic roots.*

Example: Fibonacci Sequence
Characteristic Equation: $r^2 - 1r^1 - 1 = 0$
$\to r^2 - r^1 - 1 = 0$

Using the quadratic formula, we get

$$r = 1 \pm \frac{\sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} = \frac{1 \pm \sqrt{5}}{2}$$

So $F_n = \alpha \left(\frac{1+\sqrt{5}}{2}\right)^n + \beta \left(\frac{1-\sqrt{5}}{2}\right)^n$
is a solution for any $\alpha, \beta \in \mathbb{R}$

Now we must find α, β such that the formula matches the base cases

$$F_0 = \alpha \left(\frac{1+\sqrt{5}}{2}\right)^0 + \beta \left(\frac{1-\sqrt{5}}{2}\right)^0 = 0$$
$$F_1 = \alpha \left(\frac{1+\sqrt{5}}{2}\right)^1 + \beta \left(\frac{1-\sqrt{5}}{2}\right)^1 = 1$$

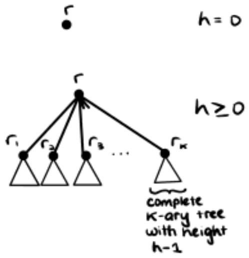
After solving, we get $\alpha = \frac{1}{\sqrt{5}}$ and $\beta = -\frac{1}{\sqrt{5}}$

+ Theorems

- (Lamé) Let $a, b \in \mathbb{Z}$ such that $a \geq b > 0$
  Euclid's algorithm takes $O(\log(b))$ steps

15 K-ary Trees

A complete k-ary tree with *height* h and *root* r is defined recursively as follows:



Recursive Definition for size:

$$n(h) = \begin{cases} 1 \text{ if } h = 0 \\ k \cdot n(h-1) + 1 \text{ otherwise} \end{cases}$$

Function Definition:

$$n(h) = \frac{k^{n+1} - 1}{k - 1}$$

16 Recursive Algorithms

Example: Number of strictly positives $NB(A[1, \dots, n])$

If $n = 1$: Base Case for size $n = 1$

  If $A[1] > 0$: Return 1

  Else: Return 0

Else:

  $m = \left\lfloor \frac{n}{2} \right\rfloor$ Split into 2 subproblems

  $a_1 = NB(A[1, \dots, m])$ Recursive Call

  $a_2 = NB(A[m+1, \dots, n])$ Recursive Call

  Return $a = a_1 + a_2$ Combining Step

Number of step in NB $\begin{cases} T(1) = 3, & (n = 1) \\ T(n) = 2 \times T\left(\frac{n}{2}\right) + 4, & (n > 1) \end{cases}$

- 2 is the number of recursive calls
- $T\left(\frac{n}{2}\right)$ is the size of the recusive step
- 4 is the merge step + extra work

17 Unfolding

Example: $T(n) = \begin{cases} 3, & n = 1 \\ 2T\left(\frac{n}{2}\right) + 4, & n \geq 2 \end{cases}$

Assume $n = 2^k$ for some $k \in \mathbb{N}$, since n has to be divisible by 2

$$T(n) = 2T\left(\frac{n}{2}\right) + 4 = 2\left(2T\left(\frac{\left(\frac{n}{2}\right)}{2}\right) + 4\right) + 4$$

$$= 2^2 \times T\left(\frac{n}{2^2}\right) + 2(4) + 4$$

$$= 2^2\left(2T\left(\frac{\left(\frac{n}{2^2}\right)}{2}\right) + 2(4)\right) + 2(4) + 4$$

$$= 2^3 \times T\left(\frac{n}{2^3}\right) + 4(4) + 2(4) + 4$$

$$\vdots$$

$$= 2^k T\left(\frac{n}{2^k}\right) + \left(\sum_{i=0}^{k-1} 2^i\right)(4) = nT\left(\frac{2^k}{2^k}\right) + \left(\sum_{i=0}^{k-1} 2^i\right)4$$

$$= n \times T(1) + \left(2^{k-1+1} - 1\right)4 = 3n + (n-1)4$$

$$= 3n + 4n - 4 = 7n - 4 = O(n)$$

18.1 Graphs

A graph G is made of a non-empty set V of vertices (nodes) together with a set E of edges

Each edge in S is an unordered pair $\{u, v\} \subseteq V$ with $u \neq v$

We write $G = (V, E)$

- Loops aren't allowed so $\{u, u\} = \{u\}$ is not a pair
- Parallel edges $\{\{u, v\}, \{u, v\}\} = \{\{u, v\}\}$ aren't allowed
- Graphs with no loops and parallel edges are simple

+ Terminology

- **Adjacent:** u is adjacent to v if $\{u, v\}$ is an edge
- **Incident:** An edge e is incident to u if one of the two endpoints of e is u
- **Degree:** The degree of a vertex $v \in V$ is the number of edges incident to v

+ Theorems

- Handshaking Lemma: $\sum_{v \in V} \deg(v) = 2|E|$
- G has even number of vertices with an odd degree

18.2 Paths

A path is a sequence of distinct vertices $v_0, \dots, v_l$ such that $\{v_i, v_{i+1}\} \in E$ for $0 \leq i < l$

It can be described as $l - 1$ edges $\{v_0, v_1\}, \dots, \{v_{l-1}, v_l\}$

The vertices v_0 and v_l are the endpoints of the path and l it its length

If $\exists$ a path with endpoints $v, w \in V$, then v and w are connected

If all vertex-pairs are connected, then the graph is connected

20.1 Cycles

A cycle is a sequence of vertices $v_0, v_1, \dots, v_{l-1}, v_0$ s.t:

- $v_0, v_1, \dots, v_{l-1}$ is a path
- $\{v_0, v_1\}, \{v_1, v_2\}, \dots, \{v_{l-1}, v_0\}$ are distinct edges

The length of this cycle is l

Note: Cycles of length 0, 1 or 2 are not allowed

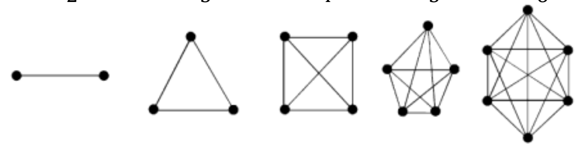
20.2 Walks

- A walk is a path where we allow repeated vertices
- A closed walk is a cycle where we allow repeated vertices

20.2 Families of Graphs

20.2.1 Complete Graphs K_n for $n \geq 1$

K_2 K_3 K_4 K_5 K_6

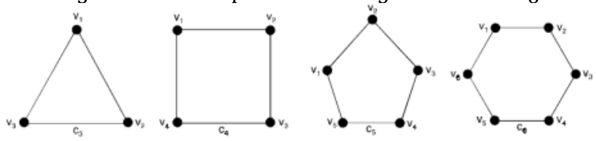


Every pair of vertices is connected by a unique edge
Each vertex is connected to $n-1$ other vertices

Number of edges: $|E| = \dfrac{n(n-1)}{2} = O(n^2)$

20.2.2 Cycles C_n for $n \geq 3$

C_3 C_4 C_5 C_6



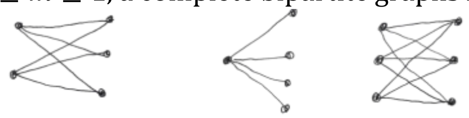
The whole graph is a single cycle with n vertices, the graph makes a closed chain

20.2.3 $(m \times n)$-grids for $n \geq m \geq 1$

$m=2$, $n=3$ $m=1$, $n=1$ $m=3$, $n=4$

22 Usual Family of Bipartite Graphs $K_{m,n}$

For $n \geq m \geq 1$, a complete bipartite graphs $K_{m,n}$



$K_{2,3}$ $K_{1,4}$ $K_{3,3}$

20.3 Subgraphs

$H = (V', E')$, where $V' \subseteq V$ and $E' \subseteq E$, then $H \subseteq G$

20.4 Connected Components

A connected component is a subgraph consisting of:

- All vertices that are connected to a given vertex
- Together with all edges incident to them

20.4 Forests, Trees and Leaves

- **Forest:** A forest is a graph that has no cycle
- **Tree:** A tree is a connected forest
- **Leaf:** A leaf in a forest is a vertex of degree 1

+ Theorems

- For $n = |V|,\ m = |E|$. If G is a forest, then $n > m$ and G has $n - m$ connected components
- For a tree G, $n = |V| = |E| + 1 = m + 1$

20.5 Spanning Trees

A spanning tree of a connected graph G is a subgraph of G that includes all vertices of G that is a tree
Every connected graph has a spanning tree

21.1 Partitions

Two sets S, T partition a set E if:

- $S \neq \emptyset, \quad S \cup T = E$
- $T \neq \emptyset, \quad S \cap T = \emptyset$

21.2 Bipartite Graphs

A graph is bipartite if V can be partitioned into A and B s.t each edge has one endpoint in A and one in B

+ Theorems

- For a bipartite graph with partition (A, B):
  $$\sum_{v \in A} \deg(v) = \sum_{v \in B} \deg(v)$$
- If a graph G has a closed walk of odd length, then G has a cycle of odd length
- A graph is bipartite $\leftrightarrow$ No odd-length cycles

23 Matching

- A matching is a graph with a subset $M \subseteq E$ where no pair of edges share a vertex
- A matching is <u>maximum</u> if it contains the greatest number of edges possible
- A matching is <u>perfect</u> if it matches all vertices

+ Theorems

- If a graph has an odd number of vertices, it cannot have a perfect matching
- If the set of edges is the union of two matchings s.t one isn't empty, then G is bipartite

23 Neighbour Set

Let $G = (V, E)$ be a graph and let $S \subseteq V$
The neighbour set of S (denoted $N(S)$) is the set of vertices having at least one neighbour in S
$N(S) = \{v \in V \mid \{v, s\} \text{ is an edge for some } s \in S\}$

+ Theorems (Hall's Theorem)

- For G, a bipartite graph with partition (A, B):
  ∃ a matching that matches all vertices in A $\leftrightarrow$ For every subset $S \subseteq A$ we have $|N(S)| \geq |S|$

12 RSA

12.1 Key Generation

- p, q: Two prime numbers
- $n = pq$, n is the modulo used. It's part of the public key
- $\lambda(n) = \text{lcm}(p-1, q-1) = \frac{|(p-1)(q-1)|}{\text{gcd}(p-1, q-1)}$, is kept a secret
- e, an integer such that $2 < e < \lambda(n)$ and $\text{GCD}(e, \lambda(n)) = 1$. It's part of the public key
- d: The private key. It's defined as $de \equiv 1 \left(\text{mod } (p-1)(q-1)\right)$, the multiplicative inverse of e mod $(p-1)(q-1)$
  Another formula for d: $de - k(p-1)(q-1) = 1$

12.2 Key Distribution

If Bob wants to send a text to Alice:
- Bob needs to know Alice's public key to encrypt the message
- Alice uses her private key to decrypt it

So Alice sends Bob her public key (n, e)

12.3 Encryption

Given the public key (n, e), we can encrypt the message M
We first turn the plaintext M into integers $m_1, \ldots, m_k$ such that $0 \leq m < n$

Then we compute the ciphertext of each m using the public key (n, e):
$c \equiv m^e \pmod{n}$ for each $m_1, \ldots, m_k$

Then we send the ciphertext values c to Alice

12.1 Decryption

Alice can decrypt the message c using the private key d:
$c^d \equiv m \pmod{n}$

Alice can then regroup all the integers m into the original message M