

# Summary Quiz 6

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## 17.1 Permutations

$S_\Omega$  is the set of all bijections  $\Omega \rightarrow \Omega$ ,  $S_\Omega$  is a symmetric group  
 $S_\Omega$  is denoted as  $S_n$  if  $|\Omega| = n$

A subgroup of  $S_n$  is called a permutation group

If  $\sigma \in S_n$ , then  $\sigma = \begin{pmatrix} 1 & 2 & \dots & n \\ \sigma(1) & \sigma(2) & \dots & \sigma(n) \end{pmatrix}$

### + Theorems

- ▶  $S_n$  with the operation composition is a group
- ▶  $|S_n| = n!$

## 17.2 Cycles and Cycle Notation in $S_n$

$\sigma \in S_n$  is a cycle if  $\exists a_1, \dots, a_k$  such that  $\begin{cases} \sigma(a_j) = a_{j+1} \\ \sigma(a_k) = a_1 \\ \sigma(x) = x, \quad x \neq a_j \end{cases}$

## 17.3 Cycle Order

- A  $k$ -cycle has  $a_1, \dots, a_k$  terms based on the above definition
- All 1-cycles can be omitted
- 2-cycles are called transpositions

## 17.4 Cycle Notations

- ▶ Two-line notation:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 5 & 1 & 4 & 6 \end{pmatrix}$$

- ▶ One-Line Notation:

$$\sigma = (1 \ 3 \ 5 \ 4) (2) (6) = (1 \ 3 \ 5 \ 4)$$

$$\sigma^{-1} = (1 \ 4 \ 5 \ 3) = (4 \ 5 \ 3 \ 1), \text{ just } \sigma \text{ inverted}$$

## 17.5 Multiplying Cycles

For  $\alpha = (1 \ 3 \ 4 \ 7)$  and  $\beta = (2 \ 3 \ 5 \ 7)$ , we perform multiplication:

$$\begin{array}{lll} x & \beta(x) & \alpha(\beta(x)) \\ 1 & \beta(1) = 1 & \alpha(1) = 3 \\ 2 & \beta(2) = 3 & \alpha(3) = 4 \\ 3 & \beta(3) = 5 & \alpha(5) = 5 \\ 4 & \beta(4) = 4 & \alpha(4) = 7 \\ 5 & \beta(5) = 7 & \alpha(7) = 1 \\ 6 & \beta(6) = 6 & \alpha(6) = 6 \\ 7 & \beta(7) = 2 & \alpha(2) = 2 \end{array} \rightarrow \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 7 & 1 & 6 & 2 \end{pmatrix}$$
$$= (1 \ 3 \ 5) (2 \ 4 \ 7) (6) = (1 \ 3 \ 5) (2 \ 4 \ 7)$$

## 18.1 Supports

The support of a permutation  $\pi$  is  $\{x: \pi(x) \neq x\}$

Two permutations are disjoint if their supports are disjoint  
Example:

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 3 & 1 & 4 \end{pmatrix}, \quad \text{support}(\alpha) = \{1, 4, 5\}$$

## 18.2 Cycle Types

The cycle type of a permutation  $\pi$  is the list (with repetition) of the length of its disjoint cycles

### + Theorems

- ▶ Disjoint permutations commute:  $\alpha(\beta(x)) = \beta(\alpha(x))$
- ▶  $x \in \text{support}(\pi) \rightarrow \pi(x), \pi(\pi(x)), \dots \in \text{support}(\pi)$
- ▶ Order of a permutation  $\pi$  is the LCM of the lengths of its disjoint cycles, so the LCM of its cycle type
- ▶ Every permutation  $\pi$  can be written as a product of disjoint cycles
- ▶  $S_n$  is generated by the set of all cycles
- ▶  $k$ -cycles can be written as product of  $k - 1$  transpositions
- ▶  $(a_1 \ a_2 \ \dots \ a_k) = (a_1 \ a_k) (a_1 \ a_{k-1}) \dots (a_1 \ a_2)$   
 $\quad \quad \quad = (a_1 \ a_2) (a_2 \ a_3) \dots (a_{k-1} \ a_k)$
- ▶ The set of all transpositions generates  $S_n$ ,  
so  $S_n = \langle \{(a \ b): 1 \leq a < b \leq n\} \rangle$
- ▶ The following are minimal generating sets for  $S_n$ :
  - $\{(1 \ a): 2 \leq a \leq n\}$
  - $\{(a \ a+1): 1 \leq a \leq n-1\}$
  - $\{(1 \ 2), (1 \ 2 \ \dots \ n)\}$

## 18.3 Dihedral Group

It's the symmetries of a regular  $n$ -gon with the following:

-  $\rho$  = rotation by  $\frac{1}{n}$  circle =  $(1 \ 2 \ \dots \ n)$

-  $\mu$  = reflection through corner 1

$$= \begin{cases} (1) (2 \ 2m) (3 \ 2m-1) \dots (m \ m+2) (m+1), & n = 2m \\ (1) (2 \ 2m+1) (3 \ 2m) \dots (m+1 \ m+2), & n = 2m+1 \end{cases}$$

### + Theorems

$D_n$  is a subgroup of  $S_n$