Fields, Bounds, and Absolute Values

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1.1 The Real Numbers:

Field Axioms 1.1.1:
- F1 Commutativity: $a + b = b + a$
- F2 Associativity: $(a + b) + c = a + (b + c)$
- F3 Additive Identity: $\exists 0 \in \mathbb{F}$ such that $0 + a = a$
- F4 Additive Inverse: $\exists(-a) \in \mathbb{F}$ such that $a + (-a) = 0$
- F5 Commutativity: $a \cdot b = b \cdot a$
- F6 Associativity: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
- F7 Multiplicative Identity: $\exists 1 \in \mathbb{F}$ such that $1 \cdot a = a$
- F8 Multiplicative Inverse: $\exists a^{-1} \in \mathbb{F}$ such that $a \cdot a^{-1} = 1$ for $a \neq 0$
- F9 Distributivity: $a \cdot (b + c) = a \cdot b + a \cdot c$

Ordered Fields 1.2.1:
An ordered field is a field $\mathbb{F}$ along with a relation $<$ satsifying:
- O1 Transitivity: If $a < b$ and $b < c$, then $a < c$
- O2 ($<$ is a total order): Exactly one holds:
    - $a < b$
    - $a = b$
    - $b < a$
- O3: If $a < b$, then $a + c < b + c$
- O4: If $a < b$ and $0 < c$, then $a \cdot c < b \cdot c$

Bounds 1.3.1:
For $S \subseteq \mathbb{F}$ be a subset, and $a \in \mathbb{F}$:
- a is an upper bound for S if $x \leq a \ \ \forall x \in S$
- a is a lower bound for S if $x \geq a \ \ \forall x \in S$

S is bounded above if there exists an upper bound for S
S is bounded below if there exists a lower bound for S
S is bounded if it's bounded above and bounded below

If a set contains its bound, we can write:
- $a = \max(S)$ if $a \in S$ and a is an upper bound for S
- $a = \min(S)$ if $a \in S$ and a is a lower bound for S

Supremum and Infimum 1.3.5:
For $S \subseteq \mathbb{F}$ a non-empty subset and $a \in \mathbb{F}$:
a is a least upper bound (or supremum) for S if:
- a is an upper bound for S
- For any upper bound b of S, we have $b \geq a$

In this case, we write $a = \sup(S)$

a is a greatest lower bound (or infimum) for S if:
- a is a lower bound for S
- For any lower bound b of S, we have $b \leq a$

In this case, we write $a = \inf(S)$

Note:
- $\sup(S) = \infty$ if S is not bounded above
- $\inf(S) = -\infty$ if S is not bounded below
- $\sup(\emptyset) = -\infty$ and $\inf(\emptyset) = \infty$

Field Completeness 1.3.8:
For an ordered field $\mathbb{F}$, we say that $\mathbb{F}$ is complete if:
For any $S \subseteq \mathbb{F}$ a nonempty and bounded above set, $\sup(S)$ exists

Proposition 1.3.11:
For any $S \subseteq \mathbb{R}$ nonempty and bounded below, $\inf(S)$ exists

Theorem 1.3.13: Archimedean Property
The set $N_{\geq 1}$ is not bounded above

Absolute Value 1.4:
For some $a \in \mathbb{R}$, we define the absolute value:
$|a| = \begin{cases} a, & a \geq 0 \\ -a, & a < 0 \end{cases}, \qquad |a| = \max\{a, -a\}$

Properties of Absolute Values 1.4.1:
- $|-x| = |x|$
- $-|x| \leq x \leq |x|$
- $|xy| = |x| \times |y|$
- $|x + y| \leq |x| + |y|$ Triangle Inequality
- $\big||x| - |y|\big| \leq |x - y|$

Distance 1.4.2:
We define the distance between x and y as:
$d(x, y) = |x - y|$

Properties of Distances 1.4.3:
- $d(x, y) = d(y, x)$
- $d(x, y) = 0 \leftrightarrow x = y$
- $d(x, z) \leq d(x, y) + d(y, z)$