Summary Midterm

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Equivalence Relations:
Define $R = \{(x, y): x, y \in X, x \sim y\} \subseteq X \times X$
R: set of all pairs that are equivalent

$\sim$ is an equivalence relation if it satisfies:
- Reflexive: $x \sim x \ \forall x \in X$
- Symmetric: $x \sim y \leftrightarrow y \sim x$
- Transitive: If $x \sim y$ and $y \sim z$, then $x \sim z$

Equivalence Classes:
$[x] = \{y \in X: y \sim x\}$

Well Defined Operations:
An operation $\cdot$ is well defined if $\left.\begin{matrix} x \sim y \\ w \sim z \end{matrix}\right\} \rightarrow (x \cdot w) \sim (y \cdot z)$

Theorems:
$[x] \cap [y] \neq \emptyset \rightarrow [x] = [y]$
Equivalence classes are either disjoint or equal

The equivalence classes form a partition of the set X

Number Theory:
For every set $S \subseteq \mathbb{N}$, $\exists d$ in S such that $\forall x \in S,\ d \leq x$
Let $a, b \in \mathbb{Z}, b > 0$, then $\exists!\, q, r \in \mathbb{Z}$ s.t $a = bq + r,\ 0 \leq r < b$

Refinements:
For two equivalence relations $\approx$ and $\sim$, we say $\approx$ is a refinement of $\sim$ if each equivalence class of $\approx$ is contained in an equivalence class of $\sim$

In other words, $a \approx b \rightarrow a \sim b$

Divisibility and Modulo:
$m \mid n$ means $\exists x \in \mathbb{Z}$ such that $n = mx$
$a \equiv b \pmod{n}$ means $n \mid (a - b) \rightarrow \dfrac{a-b}{n} \in \mathbb{Z}$

Theorems:
Congruence modulo n is an equivalence relation

If $a \equiv a' \pmod{n}$ and $b \equiv b' \pmod{n}$, then:
- $a + b \equiv a' + b' \pmod{n}$
- $ab \equiv a'b' \pmod{n}$

Prime and Irreducible:
For $p \in \mathbb{Z}$ where $p > 1$:
- p is irreducible if the only divisors of p are 1 and p
- p is prime if whenever $p \mid ab$, then $p \mid a$ and $p \mid b$

Theorems:
p is prime $\leftrightarrow$ p is irreducible
For any $n > 1, \exists! \begin{cases} p_1, \dots, p_s \text{ primes} \\ e_1, \dots, e_s \text{ positives} \end{cases}$ s.t $n = p_1^{e_1} \times \cdots \times p_s^{e_s}$

GCD and LCM:
$d = \text{GCD}(a, b)$ if and only if:
- $d \mid a$ and $d \mid b$
- If $c \mid a$ and $c \mid b$, then $c \mid d$

$m = \text{LCM}(a, b)$ if and only if:
- $a \mid m$ and $b \mid m$
- If $a \mid n$ and $b \mid n$, then $m \mid n$

Theorems:
- $\forall a, b \in \mathbb{Z},\ \exists!$ GCD d and $\exists x, y \in \mathbb{Z}$ such that $d = ax + by$
- $\forall a, b \in \mathbb{Z},\ \exists!$ LCM m
- If $\text{GCD}(a, b) = 1$, then $\exists x, y$ such that $ax + by = 1$
- If $\text{GCD}(a, b) = d$, then $\{ax + by: x, y \in \mathbb{Z}\} = d\mathbb{Z}$
- $\text{GCD}(a, b) \times \text{LCM}(a, b) = |ab|$

Groups:
For some set S and an operation $\cdot$, $(S, \cdot)$ is a group if:
- Closure: $ab \in S$
- Associativity: $(ab)c = a(bc)$
- Identity: $\exists \epsilon \in S$ such that $x\epsilon = \epsilon x = x$
- Inverses: $\forall x \in S, \exists y \in S$ such that $xy = yx = \epsilon$

Laws of Exponents:
For a group G with some operation $\cdot$:
- $x^n = x \cdot x \cdot \ldots \cdot x$ (n times)
- $x^{-n} = (x^{-1})^n = (x^n)^{-1}$
- $x^m \cdot x^n = x^{m+n}$
- $(x^m)^n = x^{mn}$

If $xy = yx\ \forall x, y \in G$, then G is abelian (commutative)

Properties of Groups:
- The identity is unique
- The inverse of each element is unique
- $ax = b$ has a unique solution $x\ \forall a, b \in G$
- $ab = ac \rightarrow bc$
- $(ab)^{-1} = b^{-1}a^{-1}$
- $(a^{-1})^{-1} = a$
- If $xy = x$ for some $x, y \in G$, then $y = \epsilon$
- If $xy = \epsilon$ for some $x, y \in G$, then $y = x^{-1}$

Cayley Tables:

$\cdot$	a	b	...
a	a	b	...
b	b	$a \cdot b$	...
$\vdots$	$\vdots$	$\vdots$	$\ddots$

Properties of Cayley Tables:
- Only one row and column matches the header completely and no other row or column matches the header in a single position
- Each row and column contains each element exactly once

Product of Groups:
For two groups G, H, their product is defined as:
$G \times H = \{(g,h) : g \in H,\ h \in H\}$
$(x,a)\ \cdot_{G\times H}\ (y,b) = (x\ \cdot_G\ a,\quad y\ \cdot_H\ b)$

Theorems:
- The product of groups is a group
- For $x = (a_1, \ldots, a_t) \in G_1 \times \cdots \times G_t$, then $|x| = \text{LCM}(|a_1|, \ldots, |a_t|)$
- $G_1 \times \cdots \times G_t$ is cyclic $\leftrightarrow \begin{cases} \text{Each } G_i \text{ is cyclic} \\ \text{GCD}\left(|G_i|, |G_j|\right) = 1\ \ \forall i \neq j \end{cases}$

Isomorphisms:
If $\phi: G \to H$ is a bijection with $\phi(x \cdot_G y) = \phi(x) \cdot_H \phi(y)$
Then ϕ is an isomorphism, and G, H are isomorphic

If G, H are isomorphic, then permuting the Cayley Table of G gives the Cayley Table of H

Automorphism:
If $\phi: G \to G$ is an isomorphism, then ϕ is an automorphism
$aut(G) =$ The set of all automorphisms of G and it's a group

Symmetries:
$S = \{\alpha, \beta, \ldots\}$ is the set of symmetries of some object with the operation composition

Example of Symmetries:
$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$

Example of Composition:
$\alpha \circ \beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 3 & 4 & 1 & 2 \end{pmatrix} \begin{matrix} \curvearrowright \beta \\ \curvearrowright \alpha \end{matrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$

Properties of Symmetries:
- $\alpha \circ \beta$ is a symmetry $\forall \alpha, \beta \in S$
- $(\alpha \circ \beta) \circ \gamma = \alpha \circ (\beta \circ \gamma)\ \ \forall \alpha, \beta, \gamma \in S$
- $\exists \epsilon \in S$ such that $\epsilon \circ \alpha = \alpha \circ \epsilon = \alpha\ \ \forall \alpha \in S$
- $\forall \alpha \in S,\ \exists \beta \in S$ such that $\alpha \circ \beta = \epsilon$

Generating Sets:
If $\forall g \in S$, g can be written with α, β, then $\{\alpha, \beta\}$ generates S

Subgroups:
For a group G with operation $\cdot$, if $H \subseteq G$, and it's a group with the same operation $\cdot$, then H is a subgroup

Notation:
If H is a subgroup of G, we write $H \leq G,\ \ H < G$ (if $H \neq G$)

Subgroup Test:
Suppose H is a subset of G, then if:
- $H \neq \emptyset$
- $x, y \in H \to x \cdot y \in H$
- $x \in H \to x^{-1} \in H$

Then H is a subgroup

Theorems:
- If $H \leq G$, then $\epsilon_G \in H$ and $\epsilon_H = \epsilon_G$
- If $H_1 \leq G$ and $H_2 \leq G$, then $H_1 \cap H_2 \leq G$
- If $K \leq H_1$ and $K \leq H_2$, then $K \leq H_1 \cap H_2$
- For $H_1 \leq G$ and $H_2 \leq G$:
  If $H_1 \cup H_2 \leq G$, then $H_1 \leq H_2$ or $H_2 \leq H_1$

Product Set:
If $S \subseteq G$, then $\langle S \rangle$ is the set of all possible products of elements in S and their inverses

Theorems:
- $S \subseteq G \to \langle S \rangle \leq G$
- If $H_1 \leq K$ and $H_2 \leq K$, then $\langle H_1 \cup H_2 \rangle \leq K$

Greatest Lower Bound:
If $\exists \alpha \in X$ such that $\begin{cases} \alpha \leq x,\ \alpha \leq y \\ z \leq x \text{ and } z \leq y \to z \leq \alpha \end{cases} \forall x, y \in X$,
then α is the greatest lower bound of x, y, denoted $glb(x,y)$

Least Upper Bound:
If $\exists \beta \in X$ such that $\begin{cases} x \leq \beta,\ y \leq \beta \\ x \leq z \text{ and } y \leq z \to \beta \leq z \end{cases} \forall x, y \in X$,
then β is the least upper bound of x, y, denoted $lub(x,y)$

Lattice:
A lattice is the set X with operation $\leq$ such that $glb(x,y)$ and $lub(x,y)$ exists $\forall x, y \in X$

It's a diagram of subgroups, where each line connecting H and K (with K vertically higher than H in the diagram) means $H \leq K$

Note: If $H \leq K$, and we have some subgroup F such that $H \leq F \leq K$, then $F = H$ or $F = K$

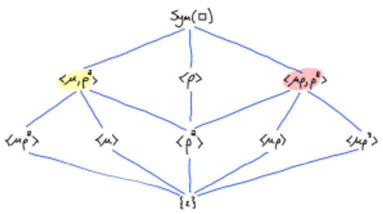
Symmetries of a Square Example:
To show a square has at most 8 symmetries:
Let γ be some symmetry, then:
- $\gamma(1)$ (1st corner) has 4 options
- $\gamma(2)$ (2nd corner) is adjacent to $\gamma(1)$, so 2 options
- $\gamma(4)$ (4th corner) is adjacent to $\gamma(1)$, so 1 option left
- $\gamma(3)$ (3rd corner) has 1 option left

So $4 \times 2 \times 1 \times 1 = 8$ possibilities

To show a square has at least 8 symmetries, we show the above 8 symmetries are all possible, with matrices form

To find the subgroups of the symmetries of a square, we go through the product set of every subset of G, for instance: $\langle\epsilon\rangle$, $\langle\mu\rangle$, $\langle\rho\rangle$, $\langle\mu,\rho\rangle$, ...

If the product set generates G, it's not a subgroup, otherwise, it is, and we can use the subgroups to draw the lattice:



Cyclic groups:
G is cyclic $\leftrightarrow \exists$ a generator $g \in G$ s.t $G = \langle g\rangle = \{g^k : k \in \mathbb{Z}\}$
The order of g is the smallest positive integer n with $g^n = \epsilon$

Notation:
- $|g|$ = Order of an element, $|g| = \infty \leftrightarrow g^k \neq \epsilon\ \forall k \in \mathbb{Z}$
- $|G|$ = Size of a group

The set $\{k : g^k = \epsilon\} = |g|\mathbb{Z}$, so $g^k = \epsilon \leftrightarrow |g|$ divides k

$|x| = |y|$ is equivalent to $x^k = \epsilon \leftrightarrow y^k = \epsilon$

If G is a group with n elements and $|g| = n < \infty$ for some $g \in G$ then:
- $G = \langle g\rangle = \{g, g^2, \ldots, g^n = \epsilon\}$
- $|G| = |g|$
- $|g^k| = \dfrac{n}{\mathrm{GCD}(n,k)}$
- Generators of G are exactly $\{g^k : GCD(n,k) = 1\}$

Note:
To check if a group is cyclic or not, check all the generators, if the order of some generator g is the length of the group, then the group is cyclic

Theorems:
- G is cyclic $\rightarrow$ G is abelian (commutative)
- G is cylic $\rightarrow$ All subgroups are cyclic
- G has no subgroups other than $\{\epsilon\}$ and G
  $\leftrightarrow$ G is cyclic of prime order
  $\leftrightarrow |G| = n$ is prime
- If G, H are both cyclic, then $G \cong H \leftrightarrow |G| = |H|$

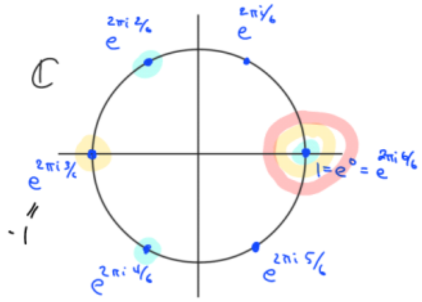
Complex Numbers:
$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$
$\mathbb{C} = \{re^{i\theta} : r, \theta \in \mathbb{R}\}$ where $r \geq 0$ and $0 \leq \theta < 2\pi$

$re^{i\theta} = r\cos\theta + ri\sin\theta \rightarrow e^{i\theta} = \cos\theta + i\sin\theta$

For $z \in \mathbb{C}$:
- $|z| = |a + bi| = \sqrt{a^2 + b^2} = r$
- $\dfrac{b}{a} = \tan\theta$

Roots of Unity:
The nth root of unity is the solution to $z^n = 1$ for $z \in \mathbb{C}$
$R_n = \left\{e^{i2\pi\times\frac{1}{n}},\ e^{i2\pi\times\frac{2}{n}}, \ldots, e^{i2\pi\times\frac{n}{n}}\right\} = \left\langle e^{\frac{i2\pi}{n}}\right\rangle$

Example: R_6



$\mathbb{T} = \{z \in \mathbb{Z} : |z| = 1\} = \{e^{i\theta} : \theta \in \mathbb{R}\}$

$\mathbb{T}$ is a subgroup of $\mathbb{C}^\times$

R_n is a subgroup of $\mathbb{T}$ (and of $\mathbb{C}^\times$)

Let $R = \bigcup_{n=1}^{\infty} R_n = \left\{e^{\frac{2\pi i j}{n}} : 0 \leq j < n,\ \ n \geq 1\right\}$

Subgroup Hierarchy:
$R_n < R < \mathbb{T} < \mathbb{C}^\times$

Properties of R:
- $|z|$ is finite $\forall z \in R$
- $|R|$ is infinite
- It's abelian but not cyclic
- Every finite subset is contained in a finite subgroup
- Every finite subgroup is cyclic
- Every infinite subgroup is not cyclic
- $R = \left\langle\left\{e^{\frac{2\pi i}{n}} : n \geq 1\right\}\right\rangle = \left\langle\left\{e^{\frac{2\pi i}{n}} : n \geq k\right\}\right\rangle\ \forall k$

Subgroups of $\mathbb{T}$:
- $R = \left\{e^{\frac{2\pi i j}{n}} :\ \ 0 \leq j < n,\ \ n \geq 1\right\}$
- $Z = \{e^{ik} :\ \ k \in \mathbb{Z}\}$

Permutations:

S_Ω is the set of all bijections $\Omega \to \Omega$ for some set Ω, a symmetric group

S_Ω is denoted as S_n if $|\Omega| = n$

A subgroup of S_n is called a permutation group

If $\sigma \in S_n$, then $\sigma = \begin{pmatrix} 1 & 2 & \dots & n \\ \sigma(1) & \sigma(2) & \dots & \sigma(n) \end{pmatrix}$

Theorems:

S_Ω with the operation composition is a group

$|S_n| = n!$

Cycles and Cycle Notation in S_n:

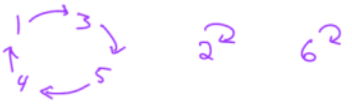
$\sigma \in S_n$ is a cycle if $\exists a_1, \dots, a_k$ such that $\begin{cases} \sigma(a_j) = a_{j+1} \\ \sigma(a_k) = a_1 \\ \sigma(x) = x, \quad x \neq a_j \end{cases}$

Cycle Order:

- A k-cycle has $a_1, \dots, a_k$ terms based on the above defition
- All 1-cycles can be omitted
- 2-cycles are called transpositions

Example: Two-line notation: $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 5 & 1 & 4 & 6 \end{pmatrix} \in S_n$

Graphical Representation:



Cycle One-Line Notation:

$\sigma = (1 \quad 3 \quad 5 \quad 4)\,(2)\,(6) = (1 \quad 3 \quad 5 \quad 4)$

$\sigma^{-1} = (1 \quad 4 \quad 5 \quad 3) = (4 \quad 5 \quad 3 \quad 1)$

Multiplying Cycles:

For $\alpha = (1 \quad 3 \quad 4 \quad 7)$ and $\beta = (2 \quad 3 \quad 5 \quad 7)$, we perform multiplication:

x	$\beta(x)$	$\alpha(\beta(x))$
1	$\beta(1) = 1$	$\alpha(1) = 3$
2	$\beta(2) = 3$	$\alpha(3) = 4$
3	$\beta(3) = 5$	$\alpha(5) = 5$
4	$\beta(4) = 4$	$\alpha(4) = 7$
5	$\beta(5) = 7$	$\alpha(7) = 1$
6	$\beta(6) = 6$	$\alpha(6) = 6$
7	$\beta(7) = 2$	$\alpha(2) = 2$

$\to \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 7 & 1 & 6 & 2 \end{pmatrix}$

$= (1 \quad 3 \quad 5)\,(2 \quad 4 \quad 7)\,(6) = (1 \quad 3 \quad 5)\,(2 \quad 4 \quad 7)$

Supports:

The support of a permutation π is $\{x : \pi(x) \neq x\}$

Two permutations are disjoint if their supports are disjoint

Example:

$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 3 & 1 & 4 \end{pmatrix}, \qquad \text{support}(\alpha) = \{1,4,5\}$

Cycle Types:

The cycle type of a permutation π is the list (with repetition) of the length of its disjoint cycles

Theorems:

- Disjoint permutations commute: $\alpha(\beta(x)) = \beta(\alpha(x))$
- $x \in \text{support}(\pi) \to \pi(x), \qquad \pi(\pi(x)), \dots \in \text{support}(\pi)$
- Order of a permutation π is the LCM of the lengths of its disjoint cycles, so the LCM of its cycle type
- Every permutation π can be written as a product of disjoint cycles
- S_n is generated by the set of all cycles
- k-cycles can be written as product of $k - 1$ transpositions
- $(a_1 \quad a_2 \quad \dots \quad a_k) = (a_1 \quad a_k)\,(a_1 \quad a_{k-1}) \dots (a_1 \quad a_2)$
  $= (a_1 \quad a_2)\,(a_2 \quad a_3) \dots (a_{k-1} \quad a_k)$
- The set of all transpositions generates S_n, so $S_n = \langle \{ (a \quad b) : 1 \le a < b \le n \} \rangle$
- The following are minimal generating sets for S_n:
  - $\{(1 \quad a) : \ 2 \le a \le n\}$
  - $\{(a \quad a+1) : \ 1 \le a \le n-1\}$
  - $\{(1 \quad 2),\ (1 \quad 2 \quad \dots \quad n)\}$

Dihedral Group:

It's the symmetries of a regular n-gon with the following:

- ρ = rotation by $\frac{1}{n}$ circle $= (1 \quad 2 \quad \dots \quad n)$
- μ = reflection through corner 1

$= \begin{cases} (1)\,(2 \quad 2m)\,(3 \quad 2m-1) \dots (m \quad m+2)\,(m+1), & n = 2m \\ (1)\,(2 \quad 2m+1)\,(3 \quad 2m) \dots (m+1 \quad m+2), & n = 2m+1 \end{cases}$

Theorems:

D_n is a subgroup of S_n