Question 1 [4 marks] (a) Suppose that T follows a t distribution with $r = 15$ degrees of freedom. Find the following quantities:

a) $P(|T| \le 0.691)$ b) $P(T > 1.753)$ c) $P(T \le 2.602)$ d) $t_{0.01}(15)$ e) $t_{0.10}(15)$

a) $P(|T| \le 0.691) = P(T \le 0.691) - P(T < -0.691) = P(T \le 0.691) - [1 - P(T \le 0.691)]$
$= 2P(T \le 0.691) - 1 = 2 \times 0.75 - 1 = 0.50$

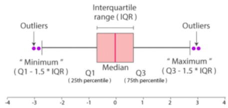
b)$P(T > 1.753) = 1 - P(T \le 1.753) = 1 - 0.95 = 0.05$

c) $P(T \le 2.602) = 0.99$

d) $t_{0.01}(15) = 2.602$

e) $t_{0.10}(15) = 1.341$

Boxplot:



Question 2 [4 marks]

Suppose that T follows a t distribution with $r = 15$ degrees of freedom. Find upper and lower bounds for the following probabilities:

a) $P(T > 1.3)$ b) $P(T < -2.75)$ c) $P(|T| > 1.6)$

a) $P(T > 1.3) = 1 - P(T \le 1.3)$, since $P(T < 0.691) < P(T < 1.3) < P(T < 1.341)$ and $0.75 < P(T < 1.3) < 0.9$, so $0.1 < 1 - P(T \le 1.3) < 0.25$.

b) $P(T < -2.75) = 1 - P(T \le 2.75)$,since $P(T < 2.602) < P(T \le 2.75) < P(T < 2.947)$ and $0.99 < P(T \le 2.75) < 0.995$, so $0.005 < P(T < -2.75) = 1 - P(T \le 2.75) < 0.01$.

c) $P(|T| > 1.6) = 1 - P(|T| \le 1.6) = 1 - [P(T \le 1.6) - P(T < -1.6)]$
$= 1 - [2P(T \le 1.6) - 1] = 2[1 - P(T \le 1.6)]$.
since $P(T < 1.341) < P(T \le 1.6) < P(T < 1.753)$ and $0.9 < P(T \le 1.6) < 0.95$,
so $0.10 = 2(1 - 0.95) < 2[1 - P(T \le 1.6)]) < 2(1 - 0.9) = 0.20$.

Question 3 [4 marks] Suppose that F follows an $F(4, 20)$ distribution. Find the following quantities.

a) $P(F \le 3.51)$ b) $P(F > 2.87)$ c) $P\left(\frac{1}{8.56} \le F \le 4.43\right)$

a) $P(F \le 3.51) = 0.975$

b) $P(F > 2.87) = 1 - P(F \le 2.87) = 1 - 0.95 = 0.05$

Histogram:

c) $P\left(\frac{1}{8.56} \le F \le 4.43\right) = P(F \le 4.43) - P\left(F(4,20) < \frac{1}{8.56}\right)$
$= 0.99 - P\left(8.56 < \frac{1}{F(4,20)}\right) = 0.99 - P(8.56 < F(20,4))$
$= 0.99 - [1 - P(F(20,4) < 8.56)] = 0.99 - [1 - 0.975] = 0.965$

Question 4 [4 marks]The following data gives the weight for 8 corn cobs which were produced using an organic corn fertilizer:

212 234 259 189 245 176 203 215

(a) For this sample of n = 8 observations compute the mean, the median, the interquartile range and the standard deviation.
First sort out data by $sort(x)$, we have

176 189 203 212 215 234 245 259

$$\bar{X} = 216.6, \tilde{x} = 213.5 = \frac{x_{(4)} + x_{(5)}}{2} = \frac{212 + 215}{2}, \; sd(x) = 28.09645$$

Forq_1 , $(n+1) \times \frac{1}{4} = 2 + 0.25$, so $q_1 = (1 - 0.25)x_{(2)} + 0.25x_{(3)} = 192.5$
For q_3 , $(n+1) \times \frac{3}{4} = 6 + 0.75$, so $q_3 = (1 - 0.75)x_{(6)} + 0.75x_{(7)} = 242.25$

For interquartile range = IQR $= q_3 - q_1 = 242.25 - 192.5 = 49.75$

(b) Among the four statistics, which are measures of central tendency and which are measures of dispersion.

mean and median are central tendency and interquartile range and standard deviation are measures of dispersion.

(c) Are there any outliers in this sample? If so, which values are outliers?

Lower bound $q_1 - 1.5 * IQR = 192.5 - 1.5 * 49.75 = 117.875$

Upper bound $q_3 + 1.5 * IQR = 242.25 + 1.5 * 49.75 = 316.875$

so no outliers.

Problem 1 [5 marks] Let $X_1, X_2, \ldots, X_n$ be a random sample from a population with density

$$f(x;\theta) = (\theta+1)\, x^{\theta}, \quad 0 < x < 1, \quad \theta > -1.$$

a) Show that the maximum likelihood estimator for θ is

$$\widehat{\theta} = -n\left(\sum_{i=1}^{n} \ln(X_i)\right)^{-1} - 1.$$

Solution

$$L(\theta) = \prod_{i=1}^{n} f(x;\theta) = \prod_{i=1}^{n} (\theta+1)\, {x_i}^{\theta} = (\theta+1)^n (x_1 x_2 \cdots x_n)^{\theta}, \;\; 0 < x_i < 1,$$

Take ln we have

$$\ln L(\theta) = n\ln(\theta+1) + \theta\ln(x_1 x_2 \cdots x_n) = n\ln(\theta+1) + \theta\sum_{i=1}^{n}\ln(x_i)$$

take derivative :

$$\frac{d\ln L(\theta)}{d\theta} = \frac{n}{\theta+1} + \sum_{i=1}^{n}\ln(x_i) = 0$$

$$\widehat{\theta} = -\frac{n}{\sum_{i=1}^{n}\ln(x_i)} - 1$$

b) Obtain the population mean, i.e. $E[X]$.
Solution

$$E(X) = \int_0^1 x f(x) dx = \int_0^1 x(\theta+1)x^{\theta} d\,x = (\theta+1)\int_0^1 x^{\theta+1} d\,x = \frac{\theta+1}{\theta+2}$$

c) What is the method of moments estimator for θ.
Solution From

$$EX = \overline{X}$$

we have

$$\frac{\theta+1}{\theta+2} = \overline{X}$$

Solve it we have

$$\widehat{\theta} = \frac{1}{1-\overline{X}} - 2 = \frac{2\overline{X} - 1}{1 - \overline{X}}$$

d) For the following set of 9 observations from this distribution, calculate the values of the maximum likelihood estimate and the method of moments estimate for θ.

0.8058 0.1412 0.2814 0.8590 0.5150
0.6946 0.5310 0.9380 0.9346

Since we have $\overline{X} = 0.6334$ and $\sum_{i=1}^{n}\ln(x_i) = -5.386105$
We have the maximum likelihood estimate and the method of moments estimate as following:

$$\widehat{\theta} = -\frac{n}{\sum_{i=1}^{n}\ln(x_i)} - 1 = -\frac{9}{-5.386105} - 1 = 0.6709663$$

$$\widehat{\theta} = \frac{2\overline{X} - 1}{1 - \overline{X}} = \frac{2 \times 0.6334 - 1}{1 - 0.6334} = 0.7277687$$

Problem 2 [5 marks] Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the continuous uniform $U(-\sqrt{\theta}, \sqrt{\theta})$ distribution, i.e. from the population with p.d.f.

$$f(x;\theta) = \frac{1}{2\sqrt{\theta}}, \quad -\sqrt{\theta} < x < \sqrt{\theta}, \quad \theta > 0.$$

a) Determine the method of moments estimator for θ.
Solution : Since we have $EX = \frac{a+b}{2} = \frac{-\sqrt{\theta}+\sqrt{\theta}}{2} = 0$, therefore, we need to think second moments, we have

$$E(X^2) = \int_{-\sqrt{\theta}}^{\sqrt{\theta}} \frac{x^2}{2\sqrt{\theta}} dx = \frac{1}{\sqrt{\theta}}\int_0^{\sqrt{\theta}} x^2 dx = \frac{\theta}{3}$$

We have

$$\frac{\theta}{3} = E(X^2) = \frac{1}{n}\sum_{i=1}^{n} X_i^2$$

$$\widehat{\theta} = \frac{3}{n}\sum_{i=1}^{n} X_i^2$$

b) Show that the method of moments estimator for θ is unbiased.
We have

$$E[\widehat{\theta}] = E\frac{3}{n}\sum_{i=1}^{n} X_i^2 = \frac{3}{n}\sum_{i=1}^{n} E[X_i^2] = \frac{3}{n}\sum_{i=1}^{n}\frac{\theta}{3} = \theta$$

Problem 3 [5 marks] Find the maximum likelihood estimates for μ and for σ^2 if a random sample of size 15 from a $N(\mu, \sigma^2)$ population yielded

31.0 36.9 33.2 30.1 33.8
35.0 28.9 34.2 30.5 34.8
31.6 36.7 35.8 34.5 32.8

Solution: More detail about MLE for Normal distribution (you can find from lecture notes):

$$L(\mu,\sigma^2) = \prod_{i=1}^{n} f(x_i,\mu,\sigma^2) = \prod_{i=1}^{n}\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x_i-\mu)^2}{2\sigma^2}}$$
$$= \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2} = \left(\frac{1}{\sqrt{2\pi}}\right)^n \left(\frac{1}{\sigma^2}\right)^{\frac{n}{2}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2}$$

Take ln we have following:

$$\ln L(\mu,\sigma^2) = n\ln\frac{1}{\sqrt{2\pi}} - \frac{n}{2}\ln\sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2 \tag{1}$$

Take the partial derivative and we have following

$$\frac{\partial\ln L(\mu,\sigma^2)}{\partial\mu} = -\frac{1}{2\sigma^2}\sum_{i=1}^{n}2(x_i-\mu)(-1) = 0$$
$$\frac{\partial\ln L(\mu,\sigma^2)}{\partial\sigma^2} = -\frac{n}{2}\frac{1}{\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{n}(x_i-\mu)^2 = 0$$

Simplify them , we have :

$$\sum_{i=1}^{n}2(x_i-\mu) = 0 \qquad \widehat{\mu} = \frac{1}{n}\sum_{i=1}^{n}x_i = \overline{x}$$
$$-n + \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2 = 0 \qquad \widehat{\sigma^2} = \frac{1}{n}\sum_{i=1}^{n}(x_i-\overline{x})^2$$

$$\widehat{\mu} = \frac{1}{n}\sum_{i=1}^{n}x_i = \overline{x} = 33.32$$

$$\widehat{\sigma^2} = \frac{1}{n}\sum_{i=1}^{n}(x_i-\overline{x})^2 = 5.618933$$

Problem 4 [5 marks] Let $Y_1 < Y_2 < Y_3 < \ldots < Y_n$ be the order statistics of n independent observations from a $U(0,1)$ population. The pdf for the population is

$$f(x) = 1, \quad 0 < x < 1.$$

(a) Give the pdf for $Y_1 = \min_i X_i$ and for $Y_n = \max_i X_i$.

Solution

$$F(x) = \int_0^x f(t)dt = \int_0^x dt = x, \quad 0 < x < 1.$$

We have :

$$f_1(y) = n[1 - F(y)]^{n-1} f(y) = n(1-y)^{n-1}, \quad 0 < y < 1.$$
$$f_n(y) = nF(y)^{n-1} f(y) = ny^{n-1}, \quad 0 < y < 1.$$

(b) Use the results of (a) to verify that $E(Y_1) = 1/(n+1)$ and $E[Y_n] = n/n+1$.

$$E(Y_1) = \int_0^1 y f_1(y) dy = \int_0^1 yn(1-y)^{n-1}dy = n\int_0^1 y(1-y)^{n-1}dy = \frac{1}{n+1}$$

Here:

$$\int y(1-y)^{n-1}dy = \int (1-t)t^{n-1}(-dt), \quad \text{let } 1-y=t, \;\; dy=-dt$$
$$= \int (t^n - t^{n-1})dt = \frac{t^{n+1}}{n+1} - \frac{t^n}{n} + C = \frac{(1-y)^{n+1}}{n+1} - \frac{(1-y)^n}{n} + C$$

We have:

$$\int_0^1 y(1-y)^{n-1}dy = \left[\frac{(1-y)^{n+1}}{n+1} - \frac{(1-y)^n}{n}\right]_0^1 = 0 - \left[\frac{1}{n+1} - \frac{1}{n}\right] = \frac{1}{n(n+1)}$$

Therefore

$$E(Y_1) = n\int_0^1 y(1-y)^{n-1}dy = n\frac{1}{n(n+1)} = \frac{1}{n+1}$$
$$E[Y_n] = \int_0^1 y f_n(y)dy = \int_0^1 yny^{n-1}dy = n\int_0^1 y^n dy = \frac{n}{n+1}$$

Descriptive Statistics:
QQ-Plots Normality:
If the theoretical data line and actual data points are close, then the QQ-Plot represents a normally distributed population

Random Sample:
A sample $X_1, \dots, X_n$ from a population with CDF F_X:
- $X_1, \dots, X_n$ are independent
- $X_1, \dots, X_n$ have the same distribution, same CDF F_X

A function of a sample, such as $h(X_1, \dots, X_n)$, is a statistic

Mean and Standard Deviation:
- $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$
- $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$

Linear Combination of RVs:
For $Y = a_1X_1 + \dots + a_nX_n$, we have:
$E[Y] = a_1E[X_1] + \dots + a_nE[X_n]$

If $X_1, \dots, X_n$ are independent, then:
$V[Y] = a_1^2V[X_1] + \dots + a_n^2V[X_n]$

Theorems:
- For $X_1, \dots, X_n$ independent such that $X_i \sim N(\mu_i, \sigma_i^2)$ and $Y = a_1X_1 + \dots + a_nX_n$, then:
  $$Y \sim N\left(\sum_{i=1}^{n} a_i\mu_i,\ \sum_{i=1}^{n} a_i^2\sigma_1^2\right)$$
- If $X_1, \dots, X_n$ is a sample with distribution $N(\mu, \sigma^2)$, then:
  $$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

T-Distribution:
For $Z \sim N(0,1)$ and $U \sim \chi^2(r)$ that are independent:
$$T = \frac{Z}{\sqrt{(U/r)}} \sim t(r)$$

Quantiles of the T-Distribution:
$P(T > t_\alpha(r)) = \alpha$ with $t_\alpha(r)$, the upper quantile of order α
- $t_{1-\alpha}(r) = -t_\alpha(r)$ since it's symmetric about $t = 0$
- As $r \to \infty$, the t distribution goes to $N(0,1)$

Standard Normal Distribution:
For $X \sim N(\mu, \sigma^2)$, the standard normal distribution is:
$$Z = \frac{X - \mu}{\sigma} \sim N(0,1)$$

For a sample from a population with distribution $N(\mu, \sigma^2)$:
$$\frac{\bar{X} - \mu}{\left(\frac{\sigma}{\sqrt{n}}\right)} \sim N(0,1), \quad \text{since } \bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

Theorems:
- If $Z \sim N(0,1)$, then $Z^2 \sim \chi^2(1)$
- For $X_1, \dots, X_n$ inpendent RVs such that $X_i \sim \chi^2(r_i)$: If $W = X_1 + \dots + X_n$, then $W \sim \chi^2(r_1 + \dots + r_n)$
- For $X_1, \dots, X_n$ a sale from a $N(\mu, \sigma^2)$ distribution:
  If $W = \sum_{i=1}^{n}\left(\frac{X_i - \mu}{\sigma}\right)^2$, then $W \sim \chi^2(n)$
- For $X_1, \dots, X_n$ a sale from a $N(\mu, \sigma^2)$ distribution:
  - $\bar{X}$ and S^2 are independent
  - $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$

Binomial Approximation with a Normal:
Define the following:
- $Y \sim binom(n, p)$ with CDF $F_Y(y)$
- $W = \frac{Y - np}{\sqrt{np(1-p)}}$, the standardized version of Y

For n where $np \geq 5$ and $np(1-p) \geq 5$, apply the theorem:
$F_Y(y) \approx \phi\left(\frac{y + 0.5 - np}{\sqrt{np(1-p)}}\right)$ where ϕ is the CDF of $N(0,1)$

F-Distribution:
For $U_1 \sim \chi^2(r_1)$ and $U_2 \sim \chi^2(r_2)$: $F = \frac{(U_1/r_1)}{(U_2/r_2)} \sim F(r_1, r_2)$

Quantiles of the F-Distribution:
$P\left(F > F_\alpha(r_1, r_2)\right) = \alpha$ with $F_\alpha(r_1, r_2)$, the upper quantile of order α
- $F \sim F(r_1, r_2) \to \frac{1}{F} \sim F(r_2, r_1)$
- In another form: $F_{1-\alpha}(r_1, r_2) = \frac{1}{F_\alpha(r_2, r_1)}$

Order Statistics:
Given $X_1, \dots, X_n$, a sample with CDF $F(x)$ and PDF $f(x)$:
Sorting gives the order statistics $Y_1, \dots, Y_n$ where $Y_1 \leq \dots \leq Y_n$

CDF and PDF of Y_r for $1 \leq r \leq n$:
- $F_r(y) = \sum_{k=r}^{n}\binom{n}{k}\left(F(y)\right)^k\left(1 - F(y)\right)^{n-k}$
- $f_r(y) = \binom{n}{r-1, 1, n-r}\left(F(y)\right)^{r-1}\left(f(y)\right)\left(1 - F(y)\right)^{n-r}$

Binomial Formulas:
$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}, \quad \binom{n}{r-1,\ 5,\ n-r} = \frac{n!}{(r-1)!\,1!\,(n-r)!}$$

Sample Percentile:
For $0 < p < 1$, the $(100p)$th sample percentile denoted $\tilde{\pi}_p$ is:
- If $(n+1)p$ is an integer, then $\tilde{\pi}_p = y_{(n+1)p}$
- Otherwise, then: $\exists r, a, b$ such that $(n+1)p = r + \frac{a}{b}$
  So $\tilde{\pi}_p = \left(1 - \frac{a}{b}\right)y_r + \left(\frac{a}{b}\right)y_{r+1}$

Quartiles:
- $q_1 = \tilde{\pi}_{0.25}$
- $q_2 = \tilde{m} = \tilde{\pi}_{0.5}$, the median
- $q_3 = \tilde{\pi}_{0.75}$
- $IQR = q_3 - q_1$

Population Percentile:
The $(100p)$th population percentile, denoted π_p satsifies:
$P[X \leq \pi_p] = p$

Given the order statistics $Y_1, \dots, Y_n$, and the CDF of a binomial distribution $F(x)$:
$P[Y_i < \pi_p < Y_j] = F(j-1) - F(i-1)$

Central Tendencies:
- Sample Mean: $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$
- Sample Median: $\tilde{m} = \begin{cases} y_{(n+1)\times\frac{1}{2}}, & n \text{ is odd} \\ \frac{y_{\frac{n}{2}} + y_{\left(\frac{n}{2}\right)+1}}{2}, & n \text{ is even} \end{cases}$

Theorems:
- The sample mean $\bar{x}$ minimizes : $\sum_{i=1}^{n}(x_i - \theta)^2$
- The median of the sample $\tilde{m}$ minimizes: $\sum_{i=1}^{n}|x_i - \theta|$

Measures of Dispersion:
- Sample standard deviation: $S = \sqrt{S^2}$
- Sample Variance S^2
- Range: $Y_n - Y_1$
- $IQR = q_3 - q_1$

Point Estimation:
- Sample standard deviation: $S = \sqrt{S^2}$
- kth sample moment: $M_k = \frac{1}{n}\sum_{i=1}^{n} X_i^k$
- kth sample central moment: $\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^k$

Expected Value Formulas:
- $E[X] = \int_{-\infty}^{\infty} af_X(a)da$
- $E[X^2] = \int_{-\infty}^{\infty} a^2f_X(a)\,da$
- $E[X^k] = M_k$
- $E[X^2]$, denoted $V = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$

Statistics:
- If a statistic $h(X_1, \dots, X_n)$ estimates θ, then it's an estimator for θ, denoted by $\hat{\theta} = h(X_1, \dots, X_n)$
- A point estimate $\hat{\theta}$ for θ is $h(x_1, \dots, x_n)$ where $x_1, \dots, x_n$ are observed values from a sample
- An estimator $\hat{\theta}$ for θ is unbiased if $E[\hat{\theta}] = \theta$

Maximum Likelihood Estimator (MLE):
We want to maximize the likelihood functions:
- $L(\theta) = \prod_{i=1}^{n} f(x_i, \theta)$
- $\ln(L(\theta)) = \sum_{i=1}^{n}\ln\left(f(x_i, \theta)\right)$

We maximize the likelihood functions using either one:
- $\frac{\delta L(\theta)}{\delta\theta} = 0$
- $\frac{\delta \ln(L(\theta))}{\delta\theta} = 0$

Maximized at $(\theta_1, \dots, \theta_k) = \left(h_1(x_1, \dots, x_n), \dots, h_k(x_1, \dots, x_n)\right)$:
- The MLEs are $\hat{\theta}_i = h_i(X_1, \dots, X_n)$
- The maximum likelihood estimates are $\hat{\theta}_i = h_i(x_1, \dots, x_n)$

Method of Moments:
Given a random sample $X_1, \dots, X_k$ from a population with unknown θ_i, we need to estimate the unknown parameters

We get $E[X] = M_1$, if we can solve for θ_i, we stop otherwise:
We get $E[X^2] = M_2$, we get a system of equations, and if we can solve for θ_i, we stop, otherwise, we get $E[X^3] = M_3$ and so on...

Linear Regression:
Study the relationship between the independent variable x, the regressor, and the dependent variable Y, the response variable

Let $(x_1, Y_1), \dots, (x_n, Y_n)$ be the sample for the regression model

Linear Regression Assumptions:
- For the general regression model:
  - $E[Y] = \alpha_1 + \beta x$
  - $Y = \alpha_1 + \beta x + \epsilon$ where $\epsilon \sim N(0, \sigma^2)$ so $E[\epsilon] = 0$
- For the random sample:
  $$\left.\begin{aligned} &\alpha_1 = \alpha - \beta\bar{x} \\ &Y_i = \alpha + \beta(x_i - \bar{x}) + \epsilon_i \\ &\epsilon_i \sim (iid)\ N(0, \sigma^2) \\ &E[Y_i] = \alpha + \beta(x_i - \bar{x}) \\ &Var[Y_i] = \sigma^2 \end{aligned}\right\} Y_i \sim N(\alpha + \beta(x_i - \bar{x}), \sigma^2)$$

Notation for Linear Regression:
- $S_{xx} = \sum_{i=1}^{n}(x_i - \bar{x})^2 = \left(\sum_{i=1}^{n} x_i^2\right) - n\bar{x}^2$
- $S_{yy} = \sum_{i=1}^{n}(y_i - \bar{y})^2 = \left(\sum_{i=1}^{n} y_i^2\right) - n\bar{y}^2$
- $S_{xy} = \sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}) = \left(\sum_{i=1}^{n} y_i^2\right) - n\bar{x}\bar{y}$

MLEs for α, β, σ^2:
- $\hat{\alpha} = \bar{y}$
- $\hat{\beta} = \frac{S_{xy}}{S_{xx}}$
- $\hat{\sigma}^2 = \frac{S_{yy} - \hat{\beta}S_{xy}}{n}$

CDF $F_X(a)$ Formulas:
- $F_X(a) = P[X \leq a] = \int_{-\infty}^{a} f_X(a)\,da$
- $P[a < X \leq b] = F_X(b) - F_X(a) = \int_{a}^{b} f_X(a)\,da$

Bernoulli Distribution $X \sim bern(p)$:
- $P_X(a) = p^a(1-p)^{1-a}$
- $E[X] = p$
- $\hat{p} = 1 - \frac{V}{\bar{X}} = 1 - \frac{E[X^2]}{\bar{X}}$
- $Var[X] = p(1-p)$

Binomial Distribution $X \sim binom(n, p)$:
- $P_X(a) = \binom{n}{a}p^a(1-p)^{n-a}$
- $E[X] = np$
- $\hat{p} = \frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X}$
- $Var[X] = np(1-p)$

Geometric Distribution $X \sim geom(p)$:
- $P_X(a) = p(1-p)^{a-1}$
- $E[X] = \frac{1}{p}$
- $\hat{p} = \frac{n}{\sum_{i=1}^{n} X_i}$
- $Var[X] = \frac{1-p}{p^2}$

Poisson Distribution $X \sim pois(\lambda)$:
- $P_X(a) = \frac{\lambda^a e^{-\lambda}}{a!}$
- $E[X] = \lambda$
- $\hat{\lambda} = \frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X}$
- $\mathrm{Var}[X] = \lambda$

Exponential Distribution $X \sim \exp(\lambda)$:
- $f_X(a) = \lambda e^{-\lambda a}$
- $E[X] = \frac{1}{\lambda}$
- $F_X(a) = 1 - e^{-\lambda a}$
- $Var[X] = \frac{1}{\lambda^2}$
- $\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X} \to \hat{\lambda} = \frac{1}{\hat{\theta}} = \frac{1}{\bar{X}}$

Normal Distribution $X \sim N(\mu, \sigma^2)$:
- $f_X(a) = \frac{1}{\sigma\sqrt{2\pi}} \times e^{\frac{-(x-\mu)^2}{2\sigma^2}}$
- $E[X] = \mu$
- $F_X(a) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a} e^{\frac{y^2}{2}}dy$
- $Var[X] = \sigma^2$
- $\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X}$
- $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$