Ex1 Let $E = \mathbb{R}^2$, show that $E$ is not a vector over $\mathbb{R}$ under the following addition and scalar multiplication:

(i) $(a,b)+(c,d)=(a+d,b)$ and $\alpha(a,b)=(\alpha a,\alpha b)$

(ii) $(a,b)+(c,d)=(a+c,b+d)$ and $\alpha(a,b)=(\alpha a+2\alpha b,\alpha a+\alpha b)$

(iii) $(a,b)+(c,d)=(a+c,b+d)$ and $\alpha(a,b)=(\alpha^2 a,\alpha^2 b)$.

solution (i) Assume that $E$ is a vector space over $\mathbb{R}$

then $(1,1)+(1,1)=2(1,1)$

But $(1,1)+(1,1)=(1+1,1)$ and $2(1,1)=(2,2)$

hence $(2,1)=(2,2)$, which is impossible

therefore $E$ is not a vector space over $\mathbb{R}$.

(ii) Assume that $E$ is a vector space over $\mathbb{R}$.

then $(1,1)+(1,1)=2(1,1)$

But $(1,1)+(1,1)=(1+1,1+1)$ and $2(1,1)=(2+2,2+4)$

hence $(2,2)=(4,6)$, which is impossible

therefore $E$ is not a vector space over $\mathbb{R}$.

(iii) Assume that E is a vector space over $\mathbb{R}$, then

$$(1,1)+(1,1) = 2(1,1)$$

But $(1,1)+(1,1) = (1+1, 1+1)$ and $2(1,1) = (2,4)$ hence $(2,2) = (2,4)$ which is impossible

Therefore E is not a vector space over $\mathbb{R}$.

Ex2 Say if w is or is not a subspace of $\mathbb{R}^3$ over $\mathbb{R}$ in the following cases:

(i) $W = \{(a,b,c) \in \mathbb{R}^3 ; c \neq 0\}$

(ii) $W = \{(a,b,c) \in \mathbb{R}^3, a+b+5c = 0\}$

(iii) $W = \{(a,b,c) \in \mathbb{R}^3, 2a^2 - b^2 + c^2 = 0\}$

(iv) $W = \{(a,b,c) \in \mathbb{R}^3 ; b \geqslant 0\}$.

solution: (i) Assume that w is a subspace of $\mathbb{R}^3$ over $\mathbb{R}$, then $(0,0,0) \in W$, which is impossible hence w is not a subspace of $\mathbb{R}^3$ over $\mathbb{R}$.

(ii) We have $0+0+5\times 0 = 0$, hence $(0,0,0) \in W$ and so $W \neq \emptyset$.

Let $x = (a, b, c) \in W$ and $x' = (a', b', c') \in W$

and $\alpha \in K$, then $x + x' = (a+a', b+b', c+c')$

and $\alpha x = (\alpha a, \alpha b, \alpha c)$

But $a + b + 5c = 0$ and $a' + b' + 5c' = 0$

hence $(a+a') + (b+b') + 5(c+c') = a + b + 5c + a' + b' + 5c'$

$= 0 + 0 = 0$

and $\alpha a + \alpha b + 5(\alpha c) = \alpha(a + b + 5c) = \alpha 0 = 0.$

and so $x + x' \in W$ and $\alpha x \in W$

then $W$ is a subspace of $\mathbb{R}^3$ over $\mathbb{R}$

(iii) Assume that $W$ is a subspace of $\mathbb{R}^3$ over $\mathbb{R}$

then as $(0,1,1) \in W$ and $(1,1,0) \in W$ we obtain

that $(0,1,1) + (1,1,0) \in W$ hence $(1,2,1) \in W$ and

so $(1)^2 - (2)^2 + (1)^2 = 0$ which give $-2 = 0$, impossible

Thus $W$ is not a subspace of $\mathbb{R}^3$ over $\mathbb{R}$

(iv) Assume that $W$ is a subspace of $\mathbb{R}^3$ of $\mathbb{R}$

then as $(0,1,1) \in W$ and $-2 \in \mathbb{R}$, we obtain

that $-2(0,1,1) \in W$ hence $(0,-2,-2) \in W$ which give

that $-2 \geq 0$ impossible, Thus $W$ is not subspace of $\mathbb{R}^3$ over $\mathbb{R}$

Ex3 Let $a$ be a real positive number and $I = [-a, +a]$
Let $E$ be the set of all mappings of $I$ to $\mathbb{R}$
Say if $W$ is or is not a subspace of $E$ over $\mathbb{R}$
in the following cases:

(i) $W = \{f \in E \,;\, f(3) = 0\}$, with $a \geq 3$

(ii) $W = \{f \in E \,;\, f(2) = 0\}$ with $a \geq 2$

(iii) $W = \{f \in E \,,\, f(1) = f(3)\}$, with $a \geq 3$

Solution (i) We have $0_E(3) = 0$ hence $0_E \in W$ and so $W \neq \phi$.

Let $f, g \in W$ and $\alpha \in \mathbb{R}$, then $f(3) = 0$ and $g(3) = 0$
hence $(f+g)(3) = f(3) + g(3) = 0 + 0 = 0$
and $(\alpha f)(3) = \alpha\big(f(3)\big) = \alpha \cdot 0 = 0$
and so $f + g \in W$ and $\alpha f \in W$
whence $W$ is a subspace of $E$ over $\mathbb{R}$.

(ii) Assume that $W$ is a subspace of $E$ over $\mathbb{R}$
consider the mapping $f : I \longrightarrow \mathbb{R}$ defined by
$f(x) = 1$, then $f \in W$ hence $2f \in W$, and so

$(2f)(2) = 1$, whence $2(f(2)) = 1$ which gives $2 = 1$ impossible. thus $W$ is not a subspace of $E$ over $\mathbb{R}$.

(ii) We have $0_E(1) = 0 = 0_E(3)$ hence $0_E \in W$ and so $W \neq \phi$

Let $f, g \in W$ and $\alpha \in \mathbb{R}$ then $f(1) = f(3)$ and $g(1) = g(3)$ hence

$$(f+g)(1) = f(1) + g(1) = f(3) + g(3) = (f+g)(3)$$

and $(\alpha f)(1) = \alpha(f(1)) = \alpha(f(3)) = (\alpha f)(3)$

and so $f+g \in W$ and $\alpha f \in W$ then $W$ is a subspace of $E$ over $\mathbb{R}$.

Ex 4 Show that $W$ is a subspace of $M_n(K)$ over $K$ in the following cases:

(i) $W$ = the set of matrices of $M_n(K)$ of trace zero

(ii) $W$ = the set of symetric matrices of $M_n(K)$

(iii) $W$ = the set of upper triangular matrices of $M_n(K)$

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<u>Solution</u> (i) we have $tr(0)=0$, hence $0\in W$ and so $W\neq\phi$, Let $X,Y\in W$ and $a\in K$, then $tr(X)=0$ and $tr(Y)=0$ hence $tr(X+Y)=tr(X)+tr(Y)=0+0=0$

and $tr(aX)=a\,tr(X)=a.0=0$

and so $X+Y\in W$ and $aX\in W$ then $W$ is subspace of $M_n(K)$

(ii) we have $t_0=0$ hence $0\in W$, and so $W\neq\phi$

Let $X,Y\in W$ and $a\in K$ then $t_X=X$ and $t_Y=Y$

hence $t(X+Y)=t_X+t_Y=X+Y$

<u>or</u> $(X+Y)^t=X^t+Y^t=X+Y$

and $(aX)^t=aX^t=aX$

and so $X+Y\in W$ and $aX\in W$ whence $W$ is subspace of $M_n(K)$.

(iii) set $0=(b_{ij})$ $b_{ij}=0$ $\forall i>j$ then $0\in W$ and so $W\neq\phi$. Let $X,Y\in W$ and $a\in K$. set $X=(x_{ij})$, $Y=(y_{ij})$, $X+Y=(c_{ij})$ and $aX=(d_{ij})$

then $x_{ij}=0$ and $y_{ij}=0$ $\forall i>j$

hence $\forall i>j$ we have $c_{ij}=x_{ij}+y_{ij}=0+0=0$

and $d_{ij}=a\,x_{ij}=a.0=0$

and so $X+Y \in W$ and $aX \in W$, whence $W$ is a subspace of $M_n(K)$.

**Ex5** Find the real $a$ so the element $u=(a,-1,3) \in \mathbb{R}^3$ is a linear combination over $\mathbb{R}$ of the elements $x=(3,0,-2)$ and $y=(2,-1,-5)$

**Solution** We have $u=(a,-1,3)$ is a linear combination over $\mathbb{R}$ of the elements $x=(3,0,-2)$ and $y=(2,-1,5)$ so that there exist $b, c \in \mathbb{R}$ such that

$$u = bx + cy$$

But $u = bx + cy \Leftrightarrow (a,-1,3) = b(3,0,-2) + c(2,-1,-5)$

$\Leftrightarrow (a,-1,3) = (3b, 0, -2b) + (2c, -c, -5c)$

$\Leftrightarrow (a,-1,3) = (3b+2c, -c, -2b-5c)$

$\Leftrightarrow 3b+2c=a$ ; $-c=-1 \Leftrightarrow \boxed{c=1}$ $\quad -2b-5c=3$

$\quad -2b-5=3$

$\Leftrightarrow 3(-4)+2(1)=a \Rightarrow \boxed{a=-10}$ $\quad -2b=8 \Rightarrow \boxed{b=-4}$

hence $u$ is a linear combination of $x$ and $y$ over $\mathbb{R}$ if and only if $a=-10$.

Ex6 show that the element $x=(1,5,-1)$ of $\mathbb{R}^3$ is a linear combination over $\mathbb{R}$ of the elements $u=(3,2,1)$ $v=(0,1,-2)$ and $w=(0,0,1)$

Solution $x = au+bv+cw \Leftrightarrow (1,5,-1)=a(3,2,1)+b(0,1,-2)+c(0,0,1)$

$\Leftrightarrow (1,5,-1)=(3a,2a,a)+(0,b,-2b)+(0,0,c)$

$\Leftrightarrow (1,5,-1)=(3a,2a+b,a-2b+c)$

$\Leftrightarrow 3a=1 \Rightarrow \boxed{a=\frac{1}{3}}$ $\quad 2a+b=5 \Rightarrow b=5-\frac{2}{3}=\boxed{\frac{13}{3}}$

$c=-1-a+2b=\boxed{\frac{22}{3}}$

hence $x=\frac{1}{3}u+\frac{13}{3}v+\frac{22}{3}w$.

Ex7 write the element $u=(1,-2,5)$ of $\mathbb{R}^3$ as a linear combination over $\mathbb{R}$ of the elements $x_1$, $x_2$, and $x_3$ in the following cases.

(i) $x_1=(1,1,1)$ $\quad x_2=(0,1,1)$ and $x_3=(0,0,1)$.

(ii) $x_1=(1,0,2)$ $\quad x_2=(2,1,3)$ and $x_3=(0,1,1)$

solution Let's find $a,b,c \in \mathbb{R}$ such that $u=ax_1+bx_2+cx_3$

$\Leftrightarrow (1,-2,5)=a(1,1,1)+b(0,1,1)+c(0,0,1)$

$\Leftrightarrow (1,-2,5)=(a,a+b,a+b+c)$

$\Leftrightarrow a=1$ $\quad b=-2-a=-3$ $\quad [c=7) \Leftrightarrow u=x_1-3x_2+7x_3$

(ii) $u = ax_1 + bx_2 + cx_3 \iff (1,-2,5) = a(1,0,2) + b(2,1,3) + c(0,1,1)$

$\iff (1,-2,5) = (a+2b,\ b+c,\ 2a+3b+c)$

$\iff a+2b=1 \qquad b+c=-2 \qquad 2a+3b+c=5$

the augmented matrix of this system is:

$$A = \left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & -2 \\ 2 & 3 & 1 & 5 \end{array}\right] \xrightarrow[\text{REF}]{} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 2 & 1 \end{array}\right]$$

$$a+2b=1 \qquad b+c=-2 \qquad 2c=1$$

$$\Rightarrow c = \frac{1}{2} \qquad b = -\frac{5}{2} \qquad a = 6$$

So $\quad u = 6x_1 - \frac{5}{2}x_2 + \frac{1}{2}x_3$..

Ex8 Let $U = \{(a,b,c) \in \mathbb{R}^3;\ a+b=0\}$
and $V = \{(a,b,c) \in \mathbb{R}^3;\ a+b+2c=0 \text{ and } b=c\}$

1) show that $U$ and $V$ are two subspaces of $\mathbb{R}^3$ over $\mathbb{R}$

2) Find a systems of generators of $U$, $U+V$ and $U \cap V$.

3) show that the system of generators of $U+V$ is a system of generators of $\mathbb{R}^3$

4) deduce $\mathbb{R}^3 = U \oplus V$.

Ex 8 solution

$U = \{(a,b,c) \in \mathbb{R}^3 \; ; a+b=c\}$
$V = \{(a,b,c) \in \mathbb{R}^3 \; ; a+b+2c=0 \text{ and } b=c\}$

1) $U$ subspace ??

- $0 = (0,0,0)$
  $\Rightarrow 0+0=0 \Rightarrow 0 \in S \Rightarrow S \neq \{\emptyset\}$

- Let $\vec{u} = (a,b,c) \in U$ with $a+b=0$
  $\vec{v} = (a',b',c') \in U$ with $a'+b'=0$

$\vec{u}+\vec{v} = (a+a', b+b', c+c')$

$(a+a') + (b+b') = 0$
$a+b+a'+b' = 0$
$0 + 0 = 0$
$\Rightarrow \vec{u}+\vec{v} \in U$

. Let $K$ a scalar

$K\vec{u} = (Ka, Kb, Kc)$

$Ka + Kb = 0$

$K(a+b) = 0$
$K0 = 0$
$\Rightarrow K\vec{u} \in U$

So $U$ is a subspace of $\mathbb{R}^3$

$V = (a, b, c) \in \mathbb{R}^3 \quad a + b + 2c = 0 \;, b = c$

- $0 = (0, 0, 0)$

$0 + 0 + 20 = 0$ and $0 = 0$
$\Rightarrow \quad 0 \in V \quad \Rightarrow \quad V \neq \{\emptyset\}$

- Let $\vec{u} = (a, b, c)$ with $a + b + 2c = 0 \; ; b = c$
$\vec{v} = (a', b', c')$ with $a' + b' + 2c' = 0 \; ; b' = c'$

$u + v = (a + a', b + b', c + c')$

$b + b' = c + c'$ ✓

$(a + a') + (b + b') + 2(c + c') = 0$
$(a + b + 2c) + (a' + b' + 2c') = 0$
$0 + 0 = 0$ ✓

Let $K$ scalar

$K\vec{u} = (Ka, Kb, Kc)$

$Kb = Kc$ ✓
$Ka + Kb + K2c = 0$
$K(a + b + 2c) = 0$
$K0 = 0$ ✓

$\Rightarrow V$ is subspace of $\mathbb{R}^3$

Systeme of generator of U:

$U \{ (a, b, c) \in \mathbb{R}^3 ; a + b = 0$

$\Rightarrow a = -b$

$\Rightarrow U = \boxed{(-b, b, c)}$

$(-b, b, c) = b \underbrace{(-1, 1, 0)} + c \underbrace{(0, 0, 1)}$

Sys of gen of V:

$V = \{ (a, b, c) \in \mathbb{R}^3 ; a + b + 2c = 0$

$b = c$

$\Rightarrow a + c + 2c = 0$

$a + 3c = 0$

$a = -3c$

$\Rightarrow V \boxed{(-3c, c, c)}$

$(-3c, c, c) = c \underbrace{(-3, 1, 1)}$

Sys of gen of $U \cap V$

$U \cap V = (a, b, c) \in \mathbb{R}^3$ with

$$a + b = 0$$
$$a + b + 2c = 0$$
$$b - c = 0$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$R_2 - R_1 \to R_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$R_2 \longleftrightarrow R_3$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right]$$

$R_3 / 2$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$R_2 - R_3 \to R_2$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$R_1 - R_2 \to R_1$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$a = b = c = 0$

sys of gen $(0, 0, 0)$

Sys of gen of $U+V$:

$\{(-1,1,0),(0,0,1);(-3,1,1)\}$

3) $\{(-1,1,0),(0,0,1),(-3,1,1)\}$

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 \\ -3 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad R_2+3R_1 \to R_2 \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_2/_{-2} \quad \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & \frac{1}{-2} \\ 0 & 0 & 1 \end{bmatrix}$$

$r=3$ = nb of vectors :

So the sys of gen of $U+V$ is

a sys of gen of $\mathbb{R}^3$

4) Let $\forall z = (a, b, c) \in \mathbb{R}3$

1) $U + V = R^3$ (because sys of gen of $U+V$ = sy of gen of $\mathbb{R}$

2) $U \cap W = (0, 0, 0) = 0$

So $\Rightarrow \mathbb{R}^3 = U \oplus V$

Ex 9 Let $E = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R}) \; ; \; a = c = 0 \right\}$

$F = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R}) , \; b = d = 0 \right\}$

1) Show that E and F are a subspace of $M_2(\mathbb{R})$

2) Show that $M_2(\mathbb{R}) = E \oplus F$.

Solution 1) E subspace : We have $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in E$

hence $E \neq \emptyset$ . Let $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in E$ and

$x' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in E$ and $\alpha \in \mathbb{R}$ then $a = c = 0$ and $a' = c' = 0$

We have $X + X' = \begin{pmatrix} a+a' & b+b' \\ c+c' & d+d' \end{pmatrix}$ and $\alpha X = \begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{pmatrix}$

then $X + X' \in E$ and $\alpha X \in E$ , hence E is a subspace of $M_2(\mathbb{R})$ over $\mathbb{R}$.

F subspace : We have $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in F$ hence $F \neq \emptyset$

let $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in F$ and $Y = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in F$ and $\alpha \in \mathbb{R}$

then $b = d = 0$ and $b' = d' = 0$

We have $X + X' = \begin{pmatrix} a+a' & b+b' \\ c+c' & d+d' \end{pmatrix}$ and $\alpha X = \begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{pmatrix}$

$b + b' = d + d' = 0 + 0 = 0$ and $\alpha b = \alpha d = \alpha . 0 = 0$

then $X+X' \in F$ and $\alpha X \in F$ hence $F$ is a subspace of $M_2(\mathbb{R})$ over $\mathbb{R}$

2) Let $X \in M_2(\mathbb{R})$ then there exist $a, b, c, d \in \mathbb{R}$ such that $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we have

$X = \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix} + \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix} \in E$ and $\begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} \in F$

hence $M_2(\mathbb{R}) = E + F$

Let $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in E \cap F$ then $a = c = 0$ and $b = d = 0$

hence $x = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and so $E \cap F = \{0\}$.

whence $M_2(\mathbb{R}) = E \oplus F$.