# Probability and Stochastic Calculus Handbook

Xinhe Liu

2018-2-28

# Contents

# Part I

# Measure and Probability

# Chapter 1

# Probability

## 1.1 Probability Model

1. **Sample space**($\Omega$), the collection of all possible outcomes of an experiment

2. **Events** (a subset of $Omega$) or $\mathcal{F}$ sigma-algebra.

3. **Probability measure** (P) assigns probabilities to elements of $\mathcal{F}$. P must satisfy:

   (a) Nested $P(A) \geq 0$ for $\forall A \in \mathcal{F}$

   (b) $P(\Omega) = 1$

   (c) $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$ for adjoint events($A_i \bigcap A_j = \emptyset, i \neq j$).

**Corollary**

1. $P(A) + P(A^C) = 1$

2.

3.

**Inclusion-exclusion principle**

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{k=1}^{n}(-1)^{k+1}(\sum_{1\le i_1 \ldots \le i_k}) \, |A_{i_1} \cap \ldots \cap A_{i_k}|$$

$(\Omega, \mathcal{F}, P)$ is the **probability space**.

## 1.2 Conditional Probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Law of Total Probability

$$P(A) = \sum_{i}^{n} P(A|B_i)P(B_i)$$

Bayes Theorem

$$P(A|B) = \frac{P(A|B_i)P(B_i)}{\sum_j^n P(A|B_i)P(B_i)}$$

## 1.3 Independence

A and B are independent if

$$P(A \cap B) = P(A)P(B)$$

or

$$E(f(A)g(B)) = E(f(A))E(g(B), \forall f, g$$

$A_{i_1}, \ldots A_{i_n}$ is independent if for **every collection** $i_1, i_2, \ldots i_n$ we have

$$P(\bigcup_{i=1}^{n} A_{i_k}) = \prod_{j=1}^{k} A_{i_j}$$

Conditional Independence:

$$P(A \cap B|C) = P(A|C)P(B|C)$$

## 1.4 Random Variable

A Random Variable is a function assign a real-number to a event

$$X : \Omega \to X$$

A random variable is measurable if $\{\omega : X(\omega) \leq c\} \in \mathcal{F}, \forall c$

C.D.F is a function satisfies

1. $F_X(c)$ is nondecreasing on c
2. $\lim_{c\to\infty} F_X(c) = 1$
3. $\lim_{c\to-\infty} F_X(c) = 0$
4. **right-continous** : $\lim_{y\to c} F_X(y) = F_X(c)$
5. $P(X > c) = 1 - F_X(c)$
6. $P(X = c) = F_X(c) - \lim_{y\to c} F_X(y)$
7. $P(a < X \leq b) = F_X(b) - F_X(a)$

p.d.f satisfies

1. $f_X(x) > 0$
2. $\int_{-\infty}^{\infty} f_X(u)du = 1$
3. $\int_a^b f_X(u)du = P(a \leq X \leq b)$
4. $\int_{-\infty}^{\infty} uf_X(u)du = E(X)$

### 1.4.1 Expectation

$$E(X) = \sum k f_X(k)$$

(probability mass function)

For continuous RV, we need$\int_0^\infty f_X(u)du < \infty$ or $\int_{-\infty}^0 f_X(u)du > \infty$

**Theorem**

$$E(h(X)) = \sum h(k) f_X(k)$$
$$E(g(X)) = \int_{-\infty}^{\infty} g(u) f_X(u) du$$

## 1.5 Things to Remember

Discrete Distributions

1. Bernoulli(p) Distribution

$$E(X) = p, V(X) = p(1-p)$$

2. Binomial(n,p) Distribution

$$f_X(k) = \binom{N}{k} p_k (1-p)^{n-k}, E(X) = np, V(X) = np(1-p)$$

3. Geometric(p) Distribution - X is the trial prior to success

$$f_X(k) = p(1-p)^k, E(X) = (1-p)p, V(X) = (1-p)/p^2$$

4. Possion($\lambda$) Distribution - counts in a period of time

$$f_X(k) = \frac{e^{-\lambda}\lambda^k}{k!}, E(X) = \lambda, V(X) = \lambda$$

5.
6.
7.

Continuous Distributions

1. Bernoulli(p) Distribution

$$E(X) = p, V(X) = p(1-p)$$

2. Binomial(n,p) Distribution

$$f_X(k) = \binom{N}{k} p_k (1-p)^{n-k}, E(X) = np, V(X) = np(1-p)$$

3. Geometric(p) Distribution - X is the trial prior to success

$$f_X(k) = p(1-p)^k, E(X) = (1-p)p, V(X) = (1-p)/p^2$$

4. Possion($\lambda$) Distribution - counts in a period of time

$$f_X(k) = \frac{e^{-\lambda}\lambda^k}{k!}, E(X) = \lambda, V(X) = \lambda$$

   $X + Y \sim Poisson(\lambda_1 + \lambda_2)$

5.
6.
7.

Poisson Process

## 1.6 Practices

1. Card Game
   (a) 2 kings from 52 cards : use Combination or conditional probability (step)
   (b) your second card is a King: use LFTP
   (c) * water, earth, wind, fire : win of water and earth, lose if fire (Hint: wind is irrelevant here)
2. Dice Game
   (a) sum of two dices are seven
3. Coin Game
4. Triangle* : probability of forming a triangle by two breaks on a length-1 stick
   - Way1: calcualte the opposite - one piece longer than 0.5 and use Inclusionexclusion principle
   - Way2:
5. Monty Hall* - Bayes
6. Can different random variables have the same distribution?
   yes! eg.) flipping coin twice: 1 for each head, -3 for two tails v.s. 1 for different, 2 for 2 heads, 3 for two tails
7. Can expectation be infinity?
   Yes! **Zeta Distribution**
8. Can expectation be not existed?
   Yes! **Chauchy Distribution**
9. The average number of claims in one year is 0.6, what's the probability of more than one claim (Poisson)

# Chapter 2

# Chapter Two

Stochastic Process M 1. easure-based probability1.1 Prob Space- $\Omega$(sample space), P(prob measure), F(sigma-algebra)Sigma Algebrainfinite (all inclusive) sigma-algebra**Kolmogorov Extension TheoremBorel SetsMeasure Countabily additive functionsLebesgue MeasureProbability Measure 1.2 Expectation 1. Measurability 2. Partial AveragingLebesgue Integralproperties: discrete form, comparison, linearity, Jensens InequalityRandom Variable (F measurable random variables)sigma algebra generated by R.V.distribution/ distribution measure/ density 1.3 Independence Probability definition, sigma-algebra definition expectation definition 1.4 Conditional Expectation A f-measurable random variable satisfies partial averaging identity

$$int_F E(Xf)dp = int_F X dx$$

Conditional Probabilities A F-measurable random variable, integral upto interaction probability

$$int_F P(A_i|f) = P(A_i and F)$$

Randon-Nikodym Theorem Properties 1. Linearity2 Taking out what is know3 iterated conditioning 2. independence property (like no information) 5. Jensens Inequality 1.5 Filtration and adapted process Filtration generated by X ( family of sigma algebras) Adapted Process (X(t) is t-measurable)

1.6 Martingale & Markov Process Martingale Stopping Times (function that the event is in filtration) Dobbs Optional Sampling Theorem Markov

Process Super martingale, Sub martingale 1. Browian MotionSymmetric Random WalkVariance, Expectation,First Variation (use Mean-Value Theorem when it is differentiable)Quadratic Variation (Attention: it is a Sum)Brownian MotioncontinuousStationaryIndependent increment,normal increments, variationLevys Theorem (Levys Criterion) 2. Ito IntegralMartingale PropertyIto IsometryQuadratic VariationIto-Doubelin formulaBrowian filtrationF(s)¡ F(t), Browian motion measurable at time t.Martingles 3. Black-Scholes 4. Multi-dimensional Joint Quadratic Variation Multi-dimensional Itos Formula Multi-dimensional Browian Motion Multi-Dimensional Levys Theorem One-dimensional, multi-dimensional 1. Risk-Neutral Pricing Random-Nikoym Theorem Girsanov Theorem Martingale Representation Theorem Futures and Forward 1. Relationship with P.D.E Transition Density Markov Process / Markov Property Feynman - Kac Theorem Kolmogrov Forward Equation Kolmogrov Backward Equation Dupires Equation - Volatility Delta Hedge under different rates with dividend (rate q

# Part II

# Stochastic Calculus

# Chapter 3

# Brownian Motion and Martingale

Browian Motion Levys Theorem

Reflection Principle

Prove the Martingales listed above (drift = 0) valuation

# Chapter 4

# Ito's Lemma and Black-Scholes

varaince swap

swaptoion

barrier options

quanto, exchange options

chooser, cap, forward-start option

What is transition density? Transition density of BM/GBM?

p (t,x,T,y)

Derive Kolmogorov Backward Equation/ Kolmogorov Forward Equation?

Derive Dupires Equation.

$$c_k = e^{-rt}P(S_t > K)c_{kk} = e^{-rt}p(0, S_0, T, K)$$

Martingale? Exponential Martingale? (- Moment Generating Functions)
Quadratic Variation?

Derive B-S P.D.E

Delta Hedge with futures/ options/ other instruments?

Self-financing condition?

Hedging Portfolio with Multiple Rates/ Collaterals

Stochastic Vol/ Interest Rate P.D.E

Asian Option/ Corridor Option (path dependent option) P.D.E

Derive B-S formula - Risk Neutral Pricing

Two Stock Options: Spread option: max(S1- S2,0)

Futures and Forward

derive Futures price (prove it is a martingale)

derive forward price

Forward and Futures Spread ( negatively correlated underlying and discount process, futures¿ forward

# Chapter 5

# Asset Pricing Theory

No Arbitrage pricing e.g. forward exchange rate put-call parity

Risk neutral probability e.g. Binomial model

State Price Arrow-Debreu securities

Adapted Process

$$(X(\omega) = X(\omega_n, , \omega_1))$$

- known $\omega$ at time n)

State Process

$$Y_{n+1} = f(Y_n, omega_n + 1)$$

(can be multidimensional) Theorem

$$r_n = g_n(Y_n)$$

for some $g_n$ $Y_n$ is markov -¿ then $Y_n$ is a state process

Stoping Time 1. $\tau \in 1, 2, N$ 2. $\tau(\omega) < n$ is determined by $(omega_1, omega_2, omega_n)$ ($F_n$ measurable)

Martingale Theorem (proof- indicator function and tower property) - optional sampling martingale stopped at stopping time is also a martingale $E(x_\tau) = X_0$

Markov Process $f_n(Y_n) = E_n(f_m(Y_m))$ for some $f_n = f_n(y), n \leq m$

Theorem $Y_{n+1} = g_{n+1}(X_{n+1}, Y_n$for some $g_n, X_n$ be IRVs and $Y_n$ are adapted - then Y is a markov process

$F_n$ independence X and all $F_n$ measurable $Z_n$are independent

Supermartingale and Submartingale $E(X_n + 1) \leq X_n$ supermartingale Theorem Doobs decomposition (use to prove optional sampling)

$$X_n = M_n - A_n$$

Fundamental Theorem of Optimal Investment (one-period)

$$U\hat{X_1}) = \lambda \frac{\widetilde{P}}{P}$$

Multiperiod-Asset Pricing problems

Barrier Options up-and-in call 1. state process - $M_n, S_n n^2$ computation 2. state process $-Z_n = 1M_n > u, S_n 2(n+1)$

Asian Option 1. state process $A_n, S_n$ solve 2-dimensional P.D.E. 2. state process $Z_n = A_n/S_n, S_n$ solve 1-dimensional P.D.E. ($f_N = Smax(A/S - 1)$ )

Think conditional(stopping time) Down-and-Rebate Option use indicator function 1down + 1up (Value to Continue) American Options max(value to continue, intrinsic value)

pricing models

Binomial model calibrate match vol crr

other staff independence

$$E(f(x)g(y)) = E(f(x))E(g(y))\forall f, g$$

OR

$$P(X)P(Y) = P(XY)$$

Markov Process not martingale non constant sequence of real numbers

Martingale not markov

$$X_0 = 0, X_1 = \epsilon)1$$

$$X_n = \epsilon_1(1 + \epsilon_2 + \ + \epsilon_n)$$

# Chapter 6

# S.D.E and P.D.E

Martingale - SDE

connected by change of variables

Markov- PDE

Standard S.D.E Solve GBM

Solve O-U process ( Vasicek model)

# Chapter 7

# Fixed Income

interest rate futures, forward interest rate Forward LIBOR

Derive HJM condition

# Chapter 8

# Change of Measure and Foreign Exchange

## 8.1 Tricks

Tower Property

For(t,t) = St , f(t,t) = Rt ( the underlying asset price)

Change Measure A-B-C by Zt = *numeraire B/ Numeraire A (with normalization)

exponential martingale Black - Scholes Form/ Blacks Form

MGF ()

Conditional Expectation- Tower Property

Itos Formula

Important Martingales

$$M^2 - [M, M]_t$$

$$MN - [M, N]$$

$$exp(M^2 - [M, M]t/2)$$

$$Itof - \int_0^t (df/dt + d^2 f/dx^2)$$

Martingale stopped at stopping time

partition patrial averaging

decomposition

martingale

(explain) Change of Measure- Change of Numeraire- Girsanov Theorem (Change Drift)

Foreign Exchange

Change of Measure - foreign exchange measure, foreign exchange measure

price a Quanto

## 8.2 A Subheading

The following sectioning commands are available:

part
chapter
section
subsection
subsubsection

paragraph
subparagraph

But note that—unlike the `book` and `report` classes—the `article` class does not have a “chapter” command.