Electromagnetic Fields

EPM 112

Course instructor

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Chapter (1)

Coulomb's Law and Electric Field Intensity

1. 1. Introduction

- **Stationary and moving electric charges produce forces on other stationary and moving electric charges. This is called "*force field*".**
- **This force field is called an "*electric field*", "*magnetic field*", *or* "*electromagnetic field*".**
- **Charges moving with constant velocity give rise to a magnetic field that is called a "*megnetostatic field*".**
- **Accelerated charge produce an "*electromagnetic field*" that is consists of related "time-varying electric and magnetic fields".**

- Electric and magnetic fields are also found in and about: all electric circuits, transistors, capacitors, inductors, electric motors, generators, relays, TV picture tubes, solar cells, etc.

- This chapter is concerning with the study of the stationary electric charges and the electrostatic force field that produced in free space (absence of material).

- Coulomb's force law will be used to develop the concept of electric field.

1. 2. Electric charges and electric charge densities

- The smallest unit of electric charges is found on the negatively charged electron and the positively charged proton.
- Thus, electric charge must exist in multiples of the magnitude of the charge found on the electron, i.e $q_e = \pm 1.602 \times 10^{-19}\, C$.
- A negative charge of $1\, C$ would represent about 6.24×10^{18} electrons.

- In the real world electric charges are found in one or combination of the following:

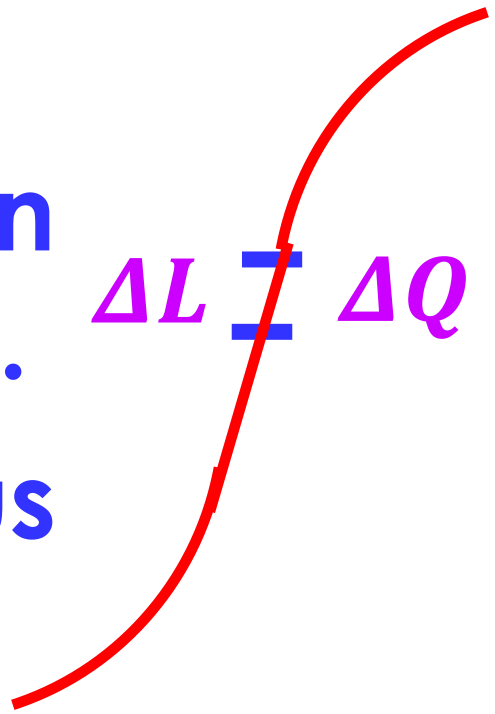
  a) *Concentrated in a point charge.*
  b) *Distributed on a line.*
  c) *Distributed on a surface.*
  d) *Distributed within a volume or as any combination of these distribution.*

a. Point charge

- If the dimensions of an electric charge distribution are very small as compared to the distance of neighboring charge, it is called a point charge.
- In general, we think of a point charge as occupying a very small physical space.

b. Line charge

- An electric charge aggregate along a thin line is referred to as a line charge as shown.



- A small length ΔL on the line would thus contain a charge ΔQ.
- A line charge density ρ_l along a line of charge is defined a

$$\rho_l = \lim_{\Delta L \to 0} \frac{\Delta Q}{\Delta L} \quad C/m$$

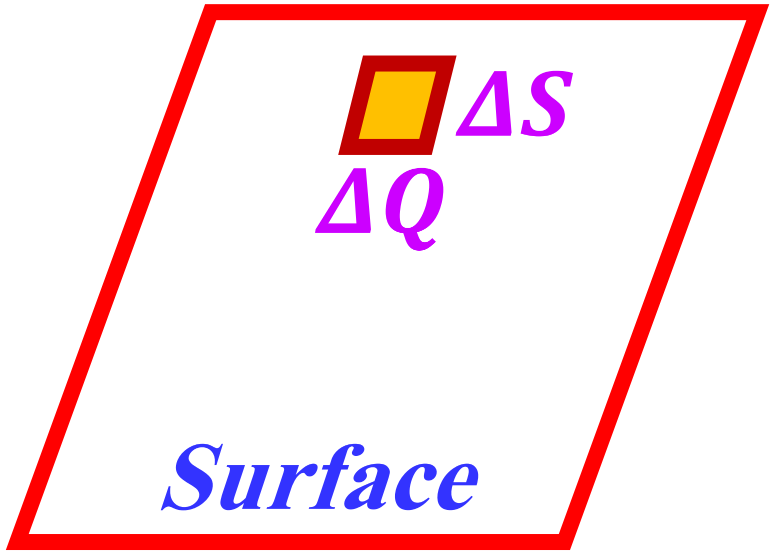
- Since ΔQ and ΔL are very small, then

$$\rho_L = \frac{dQ}{dL} \quad C/m \qquad \text{or} \qquad dQ = \rho_l \, dL \quad C$$

- The total charge will be

$$\boxed{Q = \int_{Line} \rho_l \, dL \quad C}$$

c. Surface charge



- An electric charge aggregate on a surface is called a surface charge as shown.
- The charge on a differential surface ΔS is ΔQ and can be viewed as a point charge. A surface charge density ρ_S is defined as:

$$\rho_S = \lim_{\Delta S \to 0} \frac{\Delta Q}{\Delta S} \qquad C/m^2$$

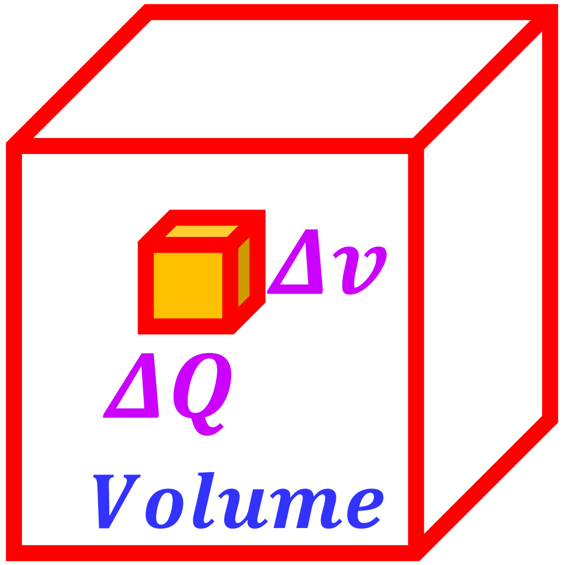
- But ΔQ and ΔS are very small values, then

$$\rho_S = \frac{dQ}{dS} \quad C/m^2 \qquad \text{or} \qquad dQ = \rho_S \, dS \quad C$$

- The total charge will be

$$Q = \int_{Surf} \rho_S \, dS \qquad C$$

d. Volume charge



- An electric charge aggregate within a volume is called a volume charge as shown.
- This volume charge can be viewed as a cloud of charged particles.
- A volume charge density ρ_V is defined as:

$$\rho_V = \lim_{\Delta V \to 0} \frac{\Delta Q}{\Delta V} \qquad C/m^3$$

- But ΔQ and ΔV are very small values, then

$$\rho_V = \frac{dQ}{dV} \quad C/m^3 \quad \text{or} \qquad dQ = \rho_V \, dV \quad C$$

- The total charge will be

$$Q = \int_{Vol} \rho_V \, dV \qquad C$$

Example (1-1)

Find the total charge on a line charge of infinite extent in the z-direction with the charge density of $\rho_L = \dfrac{\rho_o}{[1+(Z/a)^2]}$ C/m where ρ_o and a are constants.

Solution

$$dQ = \rho_L \, dL \qquad \text{where} \quad dL = dz \qquad (scalar)$$

$$Q = \int_{Line} \rho_L \, dL = \int_{-\infty}^{+\infty} \frac{\rho_o}{[1+(Z/a)^2]} dz$$

$$= \rho_o a \tan^{-1} \frac{z}{a}\Big|_{-\infty}^{+\infty} = \rho_o a \, \pi \qquad C$$

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Example (1-2)

Determine the total charge contained in the line charge distribution, $\rho_L(x, y, z) = 2x + 3y - 4z \;\; C/m$ which extends from *(2, 1, 5) to (4, 3, 6)*.

Solution

The total line charge can be calculated as follows:

$Q = \int_{Line} \rho_L \; dL$ but $d\,\bar{L} = dx\,\bar{a}_x + dy\,\bar{a}_y + dz\,\bar{a}_z$ and its scalar is

$|dL| = \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$ therefore,

$$Q = \int_{Line} (2\,x + 3\,y - 4\,z)\,\sqrt{(dx)^2 + (dy)^2 + (dz)^2}$$

The above integration is very difficult to solve, but if we use the equations of the straight line (concept of the line integral) we will simplify the solution as follows:

$$\frac{x-2}{4-\ 2}=\frac{y-1}{3-\ 1}=\frac{z-5}{6-\ 5}$$

then

$$y=x-1 \qquad \text{and} \qquad dy=dx$$

$$z=0.5\,x+4 \qquad \text{and} \qquad dz=0.5\,dx$$

the total charge will be

$$Q=\int_{x=2}^{x=4}(2\,x+3\,x-3-2\,x-16)\sqrt{1+1\ +(0.5)^2}\ dx$$

$$=\ -30\ C$$

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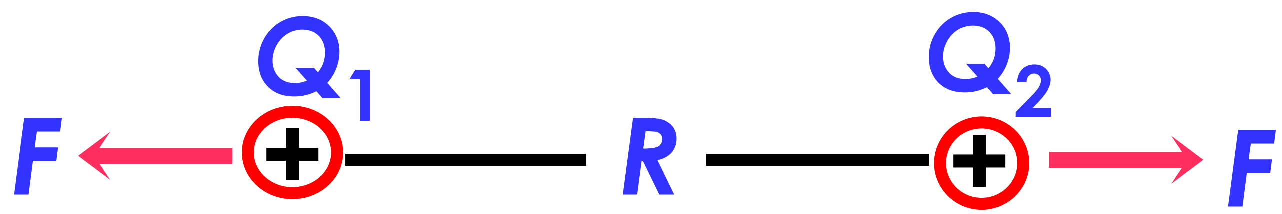
1. 3. Coulomb's law

Coulomb's law stated that the force between two stationary point charges (Q_1 and Q_2) separated in free space by a distance R which is large compared to their size is proportional to the product of the charge of the charges and inversely proportional to the distance between them. Therefore,



$$F = K\,\frac{Q_1\,Q_2}{R^2} \qquad N$$

- Force of repulsion, F, occurs when charges have the same sign.
- Charges attract when of opposite sign

Where

$$K = \frac{1}{4\,\pi\,\varepsilon_o}$$

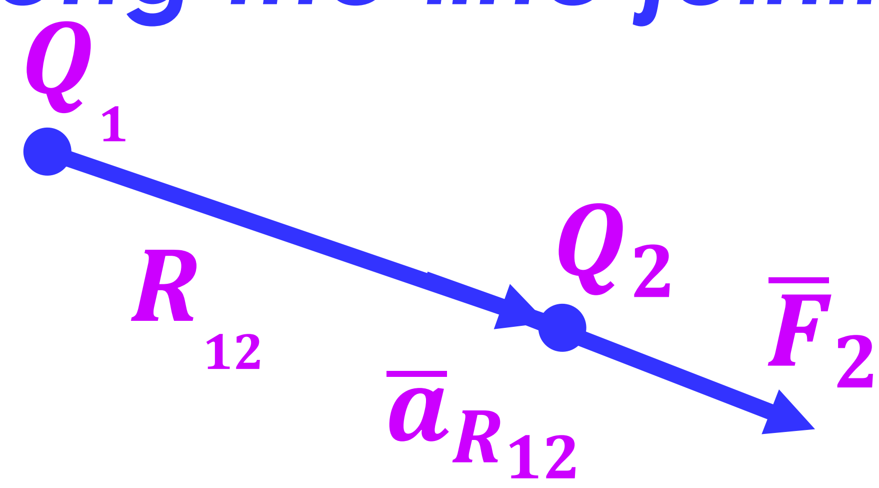
ε_o is the permittivity of free space $= 8.854 \times 10^{-12}\ F/m$

- Therefore, Coulomb's law is given by

$$F = \frac{Q_1\,Q_2}{4\,\pi\,\varepsilon_o\,R^2} \qquad N$$

<u>Any force can be defined by</u>:

- *Its magnitude which is given by the above equation.*
- *Its direction, the force acts along the line joining the two charges as shown.*



The force on Q_2 due to Q_1 is $\overline{F}_2$

$$\overline{F}_2 = \frac{Q_1\, Q_2}{4\,\pi\,\varepsilon_o\, R_{12}^2}\,\overline{a}_{R_{12}} \qquad N$$

Where $\overline{a}_{R_{12}}$ is a unit vector in the direction of $\overline{R}_{12}$ and $\overline{a}_{R_{12}} = \frac{\overline{R}_{12}}{|R_{12}|}$, therefore the force on Q_1 due to Q_2 is $\overline{F}_1$ which can be expressed as:

$$\overline{F}_1 = \frac{Q_1\, Q_2}{4\,\pi\,\epsilon_o\, R_{21}^2}\,\overline{a}_{R_{21}} \qquad N$$

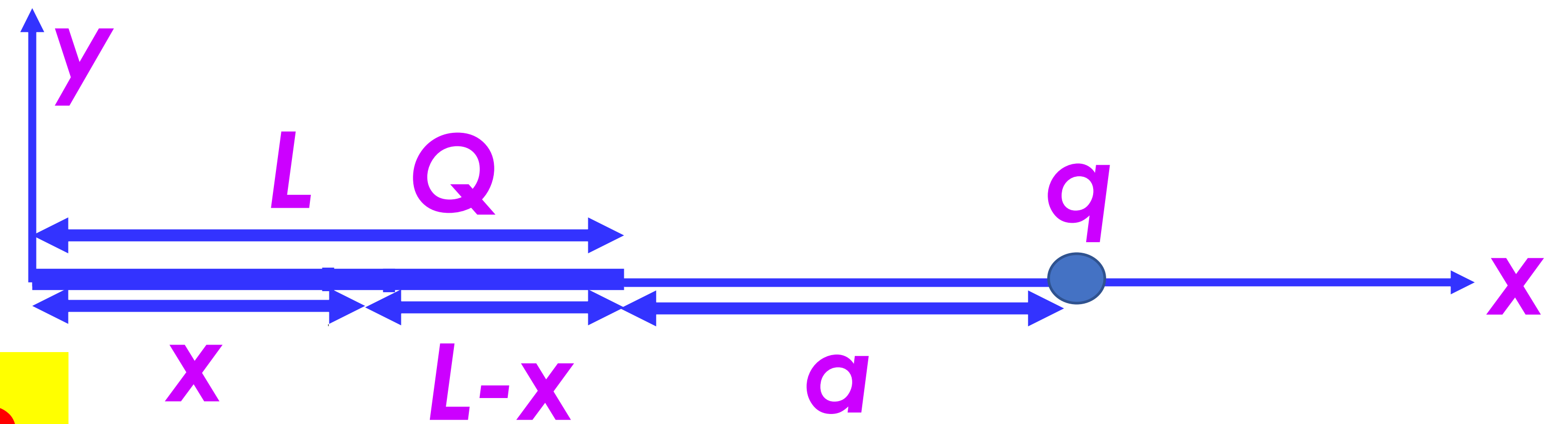
- The two charges experience a force of the same magnitude but have opposite direction. Then;

$$\overline{F}_1 = -\,\overline{F}_2 \qquad N$$

- The force is repulsive if the charges are like in sign and attractive if they are of opposite sign.

Example (1-3)

A long thin stick of length L is shown in figure with a uniform distribution of charge on it. Suppose the total charge on stick is Q. What will be the force on a charge q at a distance a from the end of stick along its axis.



Solution

The force on q due to an element of charge dQ will be

$$d\,\overline{F} = \frac{q\,dQ}{4\,\pi\,\varepsilon_o\,(L - x + a)^2}\,\overline{a}_x$$

Since the charge Q is distributed uniformly on the stick of length L, then $\rho_L = \frac{Q}{L}$ and incremental charge is $dQ = \rho_L\, dx = \frac{Q}{L}\, dx$. Therefore,

$$d\,\overline{F} = \frac{q\,Q\,dx}{4\,\pi\,\varepsilon_o\,L\,(L - x + a)^2}\,\overline{a}_x$$

then,

$$\overline{F} = \frac{q\,Q}{4\,\pi\,\varepsilon_o\,L}\int_0^L \frac{dx}{(L - x + a)^2}\,\overline{a}_x = \frac{q\,Q}{4\,\pi\,\varepsilon_o\,L}\left[\frac{1}{(L - x + a)}\right]_0^L \overline{a}_x$$

$$= \frac{q\,Q}{4\,\pi\,\varepsilon_o\,L}\left[\frac{1}{a} - \frac{1}{L + a}\right]\overline{a}_x = \frac{q\,Q}{4\,\pi\,\varepsilon_o}\left[\frac{1}{a\,(L + a)}\right]\overline{a}_x \quad N$$

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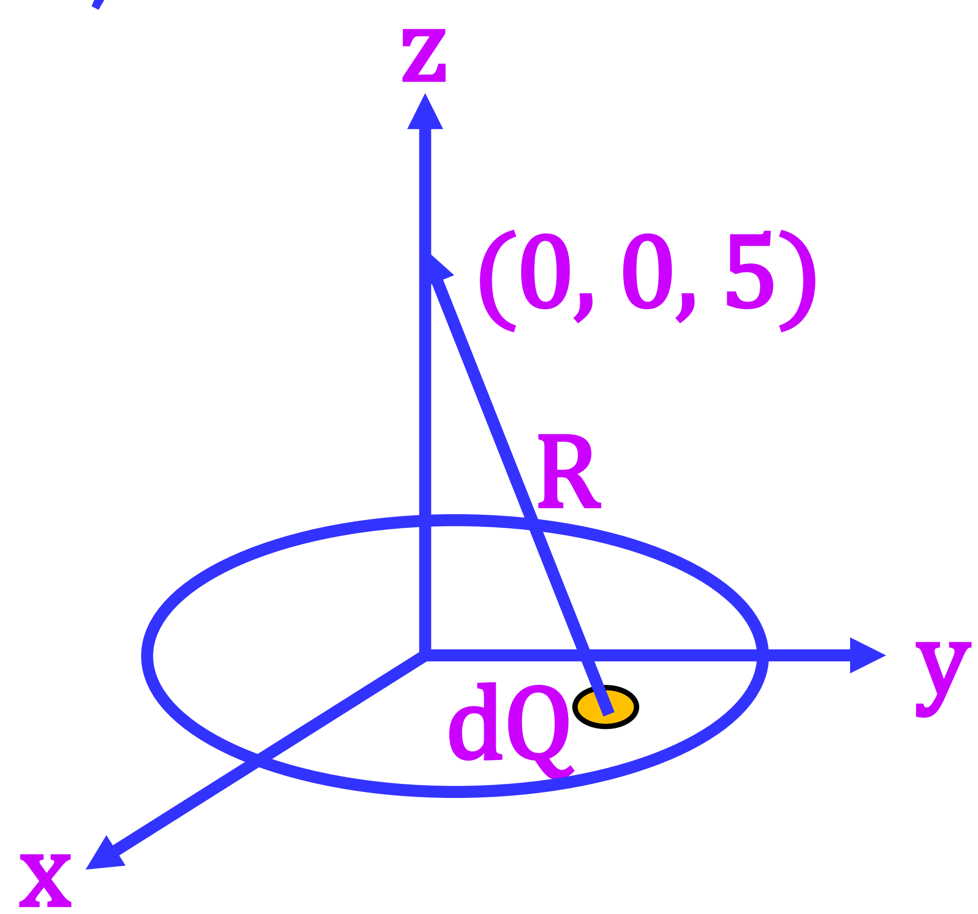
Example (1-4)

Find the force on a point charge of $50\,\mu C$ at $(0,0,5)$ m due to the charge of $500\,\pi\,\mu C$ that is uniformly distributed over the circular disk $r \le 5\,m,\ z = 0.$

Solution



The charge density is

$$\rho_S = \frac{Q}{S} = \frac{500\,\pi \times 10^{-6}}{\pi \times (5)^2} = 0.2 \times 10^{-4}\ C\,/\,m^2$$

In cylindrical coordinates,

$$\overline{R} = -r\,\overline{a}_r + 5\,\overline{a}_z$$

The differential charge is $Q = \rho_S\,dS = 0.2 \times 10^{-4}\,r\,dr\,d\varphi.$

Then each differential charge results in a differential force as:

$$d\overline{F} = \frac{q\,dQ}{4\,\pi\,\varepsilon_o R^2}\,\overline{a}_R = \frac{\left(50\,\pi\times 10^{-6}\right)\left(\,0.2\times 10^{-4}\,r\,dr\,d\varphi\,\right)}{4\,\pi\times 8.854\times 10^{-12}\times (r^2+25\,)^{3/2}}\left(-\,r\,\overline{a}_r + 5\,\overline{a}_z\,\right)$$

Before integrating, note that the radial components will cancel and that the unit vector $\overline{a}_z$ is constant. Hence,

$$\overline{F} = \int_0^{2\,\pi}\int_0^{5} \frac{\left(\,50\,\pi\,\times\,10^{-6}\,\right)\left(0.2\,\times\,10^{-4}\,\right)5\,r\,dr\,d\varphi}{4\,\pi\,\times 8.854\,\times\,10^{-12}\,\times\left(\,r^2+\,25\,\right)^{3/2}}\,\overline{a}_z$$

$$= 90\,\pi\int_0^{5}\frac{r\,dr}{(r^2+25\,)^{3/2}}\,\overline{a}_z = 90\,\pi\left[\frac{-1}{\sqrt{r^2+25}}\right]\overline{a}_z = 16.56\,\overline{a}_z\;\;N$$

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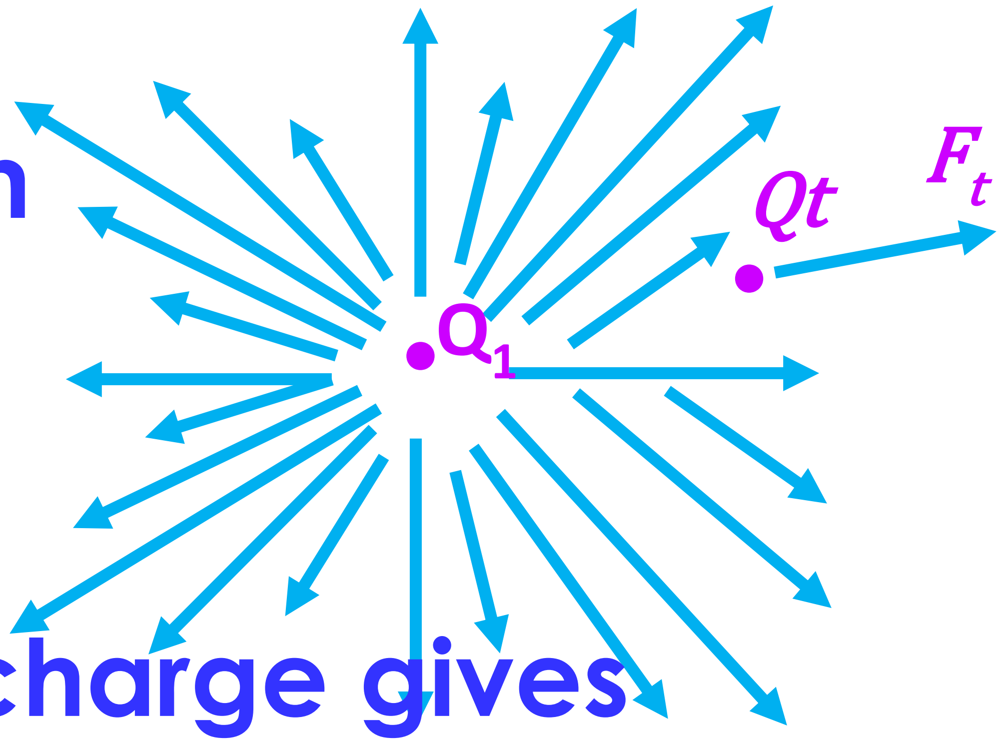
1. 4. Electric Field Intensity

- Consider one charge fixed in position, say Q_1, and move a second charge slowly around, there exists everywhere a force on this second charge, in other words, this second charge is displaying the existence of a force field as shown in figure. Call this second charge as a test charge Q_t.
- The force on the test charge is given by Coulomb's law,



$$\overline{F}_t = \frac{Q_1\, Q_t}{4\,\pi\,\varepsilon_o\, R_{1\,t}^2}\,\overline{a}_{1\,t} \qquad N$$

- Writing this force as a force per unit charge gives

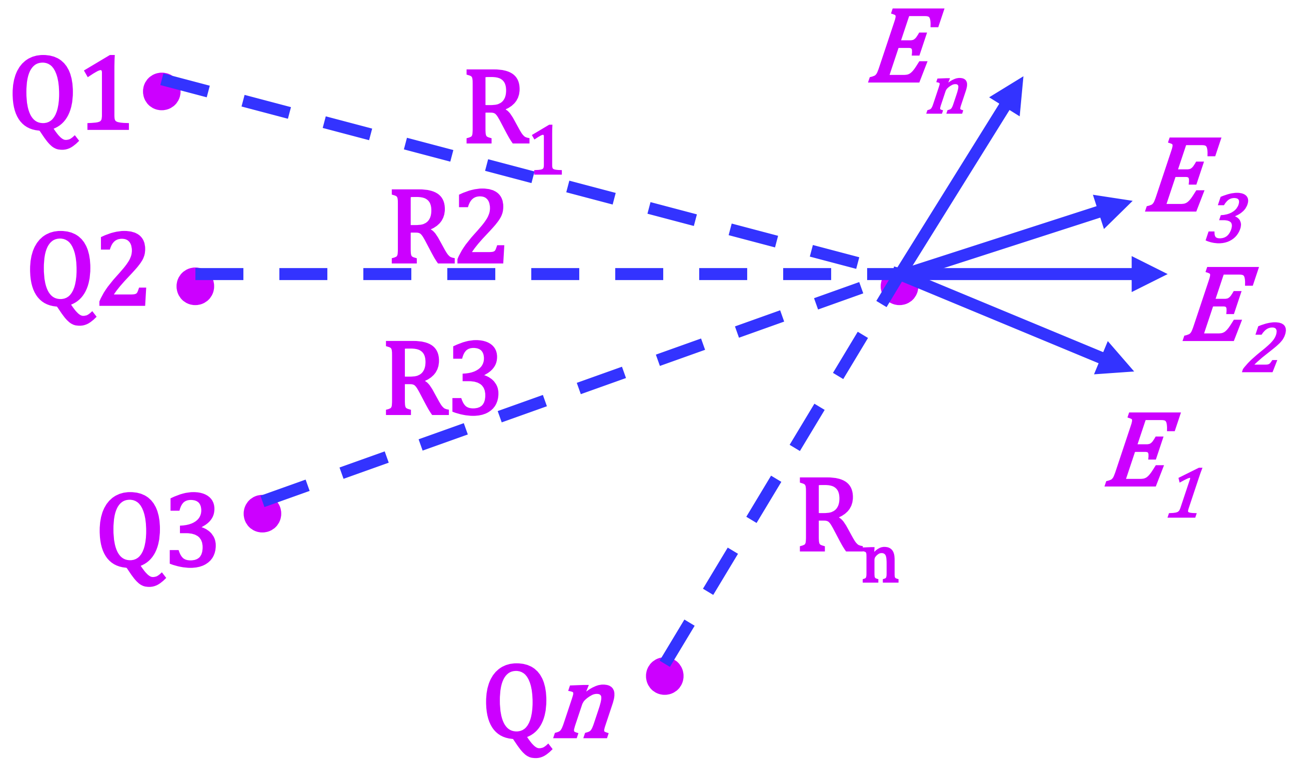
$$\frac{\overline{F}_t}{Q_t} = \frac{Q_1}{4\,\pi\,\varepsilon_o\, R_{1\,t}^2}\,\overline{a}_{1\,t} \qquad N/C$$

$$\overline{E} = \frac{\overline{F}_t}{Q_t} = \text{Electric field intensity } N/C \quad (or\ V/m), \qquad and$$

$$\overline{E} = \frac{Q_1}{4\,\pi\,\varepsilon_o\,R_{1\,t}^2}\,\overline{a}_{1\,t}$$

a) Electric Field of n Point Charge

- The electric field intensity due to n point charges, Q_1 at R_1, Q_2 at R_2, etc…, is the sum of the forces on the point P caused by Q_1, Q_2, …, Q_n acting alone, or:



$$\overline{E} = \frac{Q_1}{4\,\pi\,\varepsilon_o\,R_1^2}\,\overline{a}_{R\,1} + \frac{Q_2}{4\,\pi\,\varepsilon_o\,R_2^2}\,\overline{a}_{R\,2} + \ldots + \frac{Q_n}{4\,\pi\,\varepsilon_o\,R_n^2}\,\overline{a}_{R\,n}$$

where $\overline{a}_{R1}$, $\overline{a}_{R2}$, ..., and $\overline{a}_{Rn}$ are unit vectors in the direction of $\overline{R}_1$, $\overline{R}_2$, ..., and $\overline{R}_n$, respectively as shown in figure.

- The electric field due to all charges is the vectorial sum of each electric field intensity due to each charge acting alone, i.e,

$$\overline{E} = \overline{E}_1 + \overline{E}_2 + \overline{E}_3 + \cdots + \overline{E}_n$$

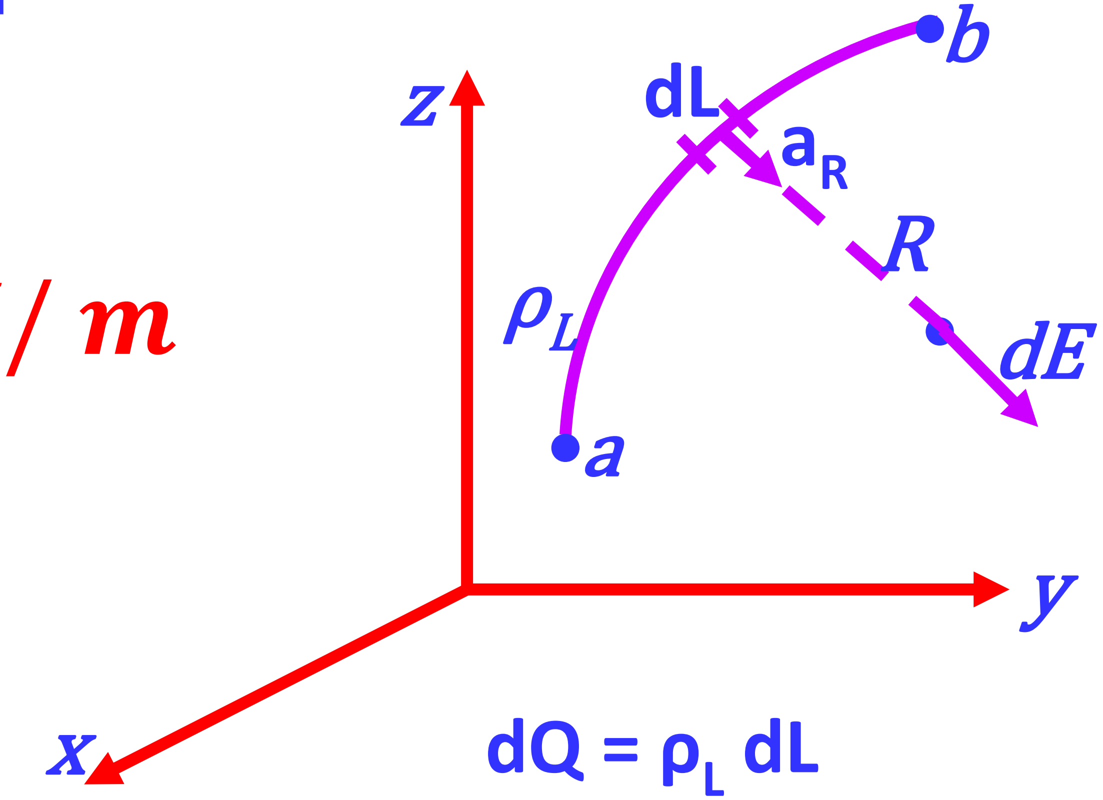
or

$$\overline{E} = \sum_{K=1}^{n} \frac{Q_K}{4\,\pi\,\varepsilon_o\,R_K^2}\,\overline{a}_{R_K}$$

b) Electric Field due to a line charge

- The differential electric field intensity $d\overline{E}$ become

$$d\overline{E} = \frac{dQ}{4\,\pi\,\varepsilon_o\,R^2}\,\overline{a}_R = \frac{\rho_L\,dL}{4\,\pi\,\varepsilon_o\,R^2}\,\overline{a}_R \qquad V/\,m$$



- Summing all contributions from a to b, we form a vector integral to obtain $\overline{E}$; thus,

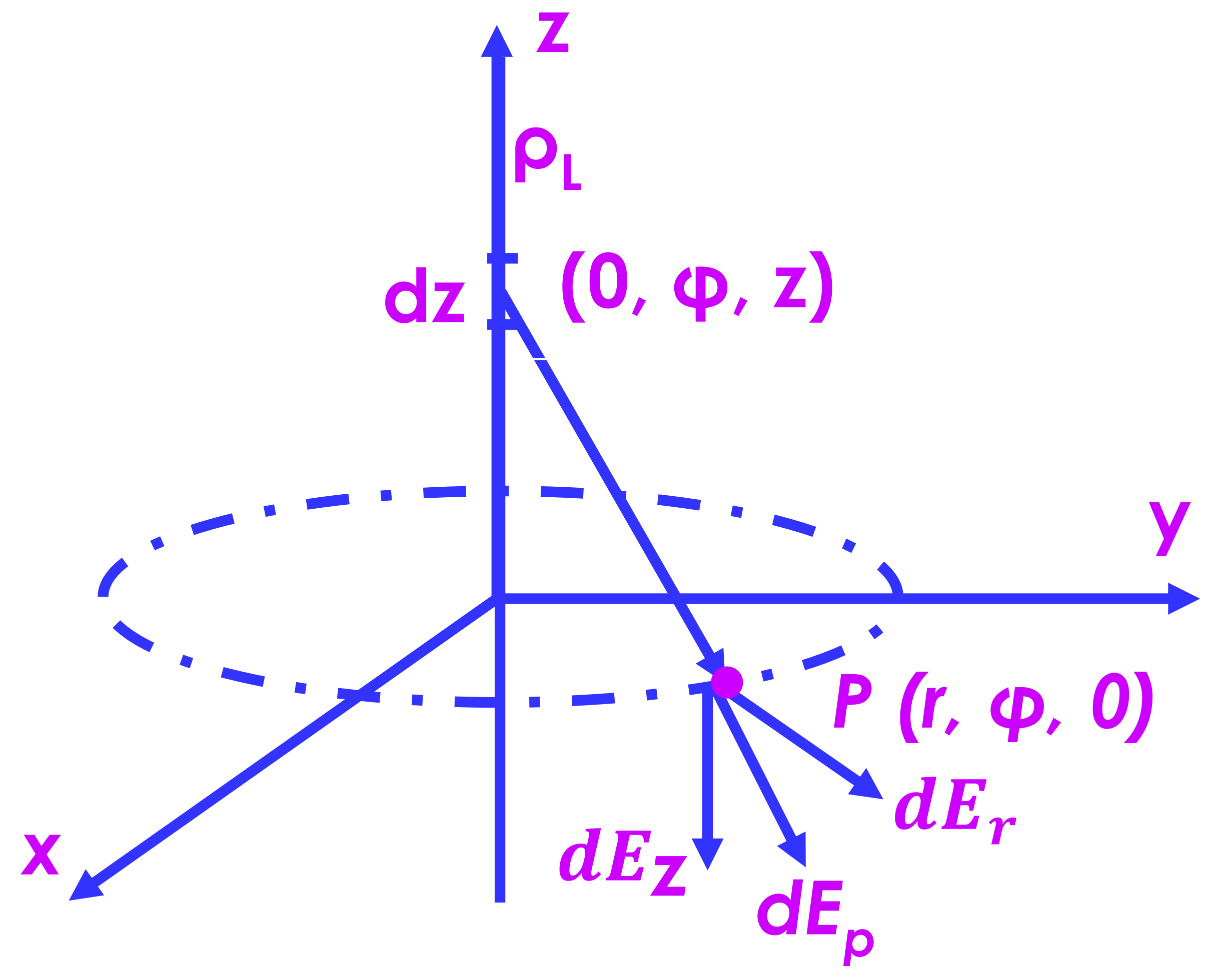
$$\overline{E} = \int_a^b d\overline{E} = \int_a^b \frac{\rho_L\,dL}{4\,\pi\,\varepsilon_o\,R^2}\,\overline{a}_R \qquad V/m$$

where the unit vector $\overline{a}_R$ is directed from the source point (dQ) to the field point $\overline{E}$ and the electric field intensity is measured in V/m.

Example (1-5)

Find the electric field intensity at any point in $z = 0$ plane due to an infinite line charge of uniform charge ρ_L distribution along z-axis and extended from $-\infty$ to ∞ (Hint: use cylindrical coordinates).

Solution



- The differential electric field at the point p is given by:

$$d\bar{E}_P = \frac{dQ}{4\,\pi\,\varepsilon_o\,R^2}\;\bar{a}_R$$

where $dQ = \rho_L\,dL = \rho_L\,dZ$

$$\bar{R} = r\,\bar{a}_r - z\,\bar{a}_Z \qquad |\bar{R}| = \sqrt{r^2 + z^2}$$

$$\bar{a}_R = \frac{\bar{R}}{|\bar{R}|} = \frac{r\,\bar{a}_r - z\,\bar{a}_z}{\sqrt{r^2 + z^2}}$$

- Substituting in the above equation to get

$$d\bar{E}_P = \frac{\rho_L\, dz}{4\,\pi\,\varepsilon_o}\,\frac{r\,\bar{a}_r - z\,\bar{a}_z}{(r^2 + z^2)^{3/2}}$$

- From symmetry, the component in z-direction will cancel each other, then the only component for $d\bar{E}$ will be in the r-direction and this component can be expressed as follows:

$$dE_r = \frac{\rho_L}{4\,\pi\,\varepsilon_o}\,\frac{r\,dz}{(r^2 + z^2)^{3/2}}$$

$$E_r = \frac{\rho_L}{4\,\pi\,\varepsilon_o}\int_{-\infty}^{+\infty}\frac{r\,dz}{(r^2 + z^2)^{3/2}} = \frac{\rho_L}{2\,\pi\,\varepsilon_o}\int_{0}^{+\infty}\frac{r\,dz}{(r^2 + z^2)^{3/2}}$$

$$= \frac{\rho_L}{2\,\pi\,\varepsilon_o}\,\frac{r\,z}{r^2\,(r^2 + z^2)^{1/2}}\Bigg|_0^{\infty} = \frac{\rho_L}{2\,\pi\,\varepsilon_o\,r} \qquad V/m$$

$$\bar{E}_P = \frac{\rho_L}{2\,\pi\,\varepsilon_o\,r}\;\bar{a}_r \qquad V/m$$

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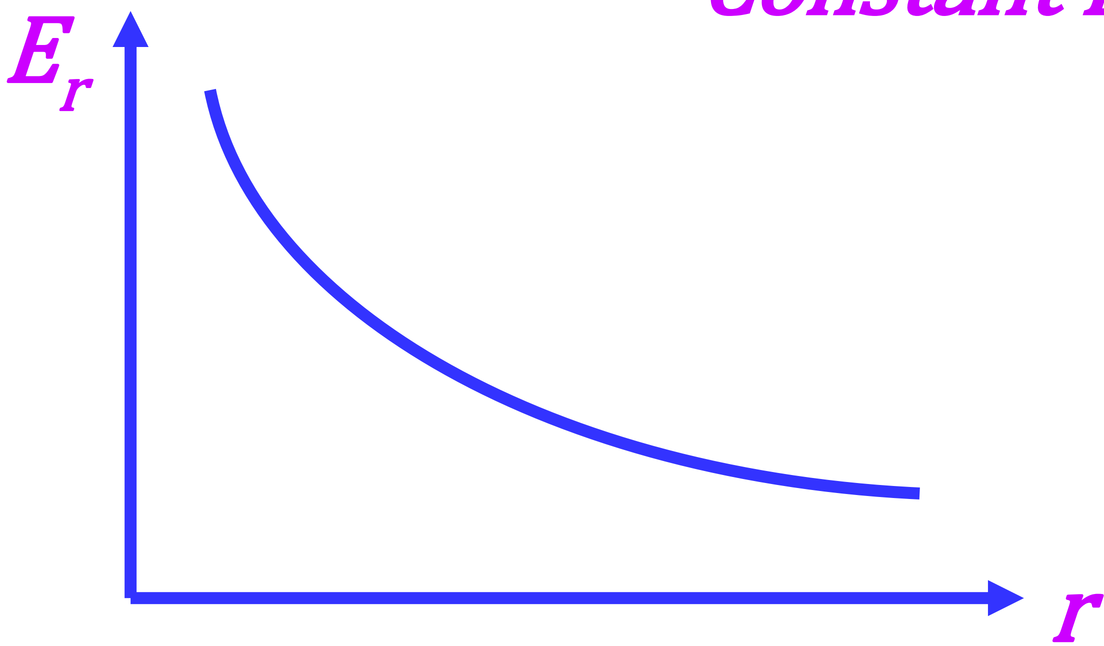
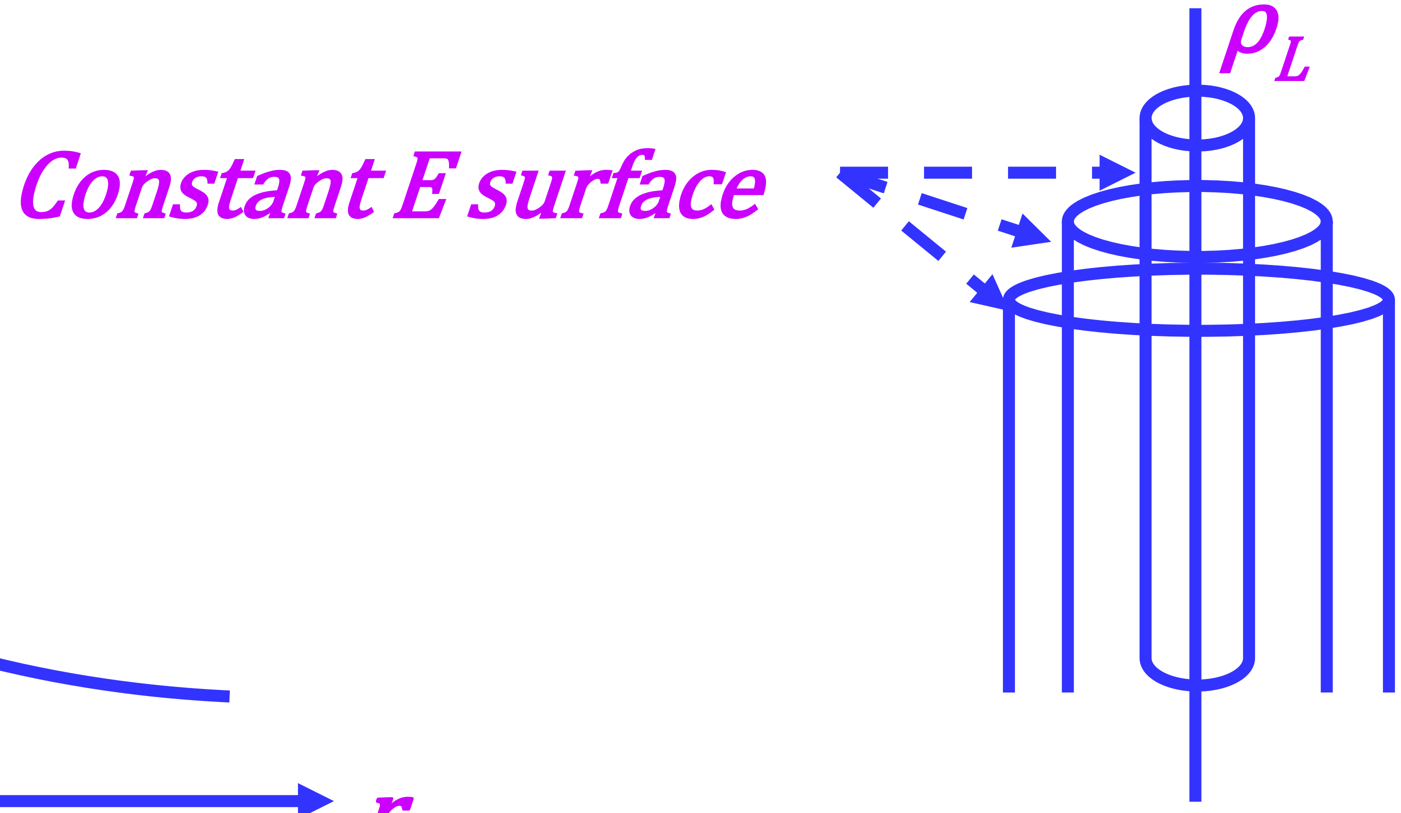
- The resultant electric field at the point P is given by

$$\bar{E} = \frac{\rho_L}{2\,\pi\,\varepsilon_o\,r}\,\bar{a}_r \qquad V/m$$

- The final result shows that the electric field intensity varies inversely with r.
- If r is kept constant, then $\bar{E}$ will be constant and the locus of constant $\bar{E}$ is cylindrical surface the line charge is its axis.



The variation of E with r.



The locus of constant E.

Example (1-6)

Consider a circular line charge of radius a in the $x-y$ plane in which $\rho_L = K\,sin\,\varphi$. Calculate the electric field intensity at any point $P\,(0,0,z)$ along $z - axis$.

Solution

- The electric field intensity at the point P on the z-axis is given as

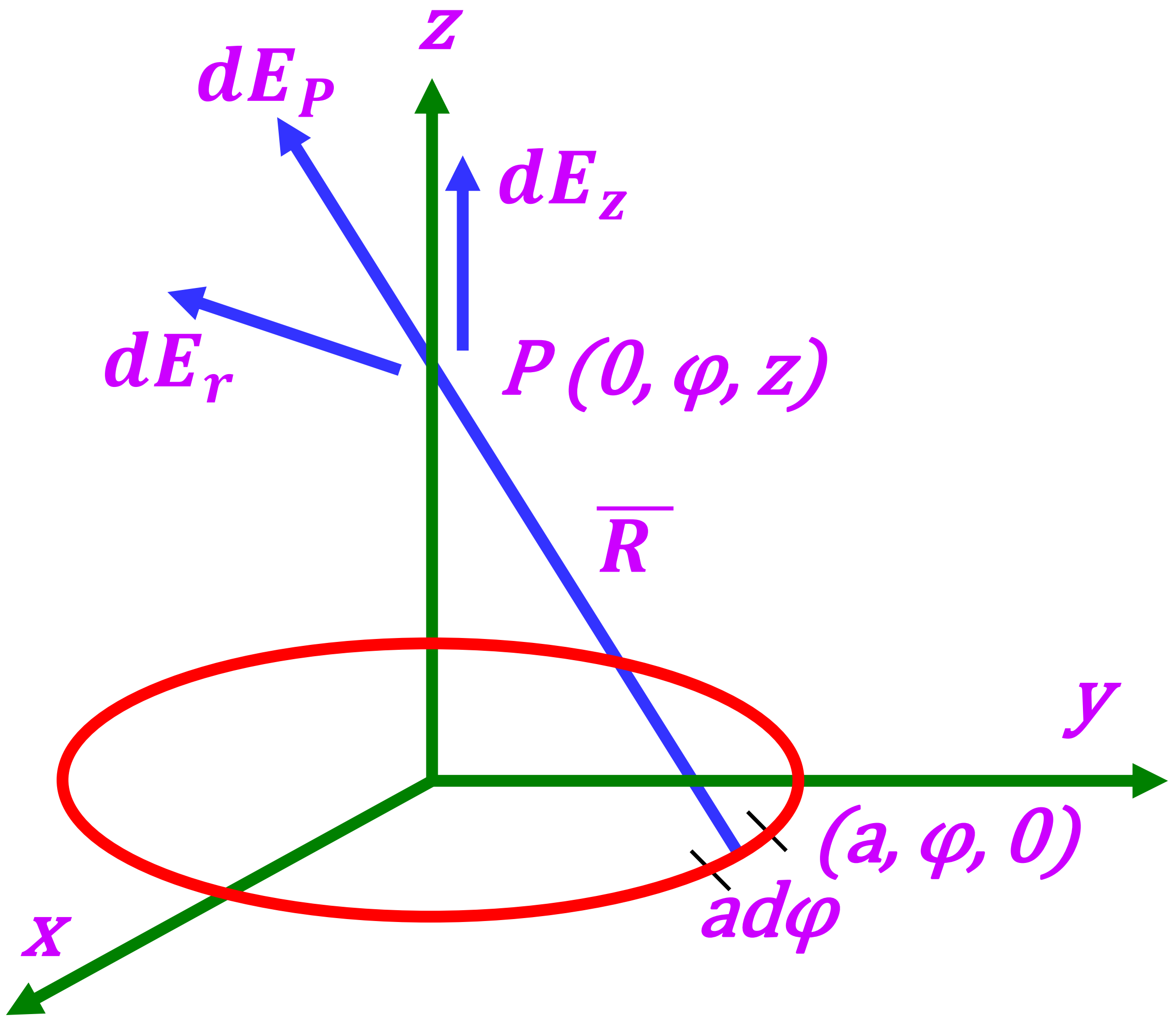
$$d\overline{E}_P = \frac{dQ}{4\,\pi\,\varepsilon_o\,R^2}\,\overline{a}_R$$

- where

$$dQ = \rho_L dL = \rho_L a\,d\varphi\,,$$

$$\overline{R} = -a\,\overline{a}_r + z\,\overline{a}_z\,, |R| = \sqrt{a^2 + z^2}$$

$$\overline{a}_R = \frac{\overline{R}}{|R|} = \frac{-a\,\overline{a}_r + \,z\,\overline{a}_z}{\sqrt{a^2 + \,z^2}}$$



- Substituting in the above equation to get:

$$d\bar{E}_P = \frac{\rho_L a\, d\varphi}{4\,\pi\,\varepsilon_o}\,\frac{(-\,a\,\bar{a}_r + \,z\,\bar{a}_z\,)}{(a^2 + \,z^2)^{3/2}}$$

- Given in this problem, $\rho_L = K\, sin\,\varphi$, then

$$d\bar{E}_P = \frac{K\,a\,sin\,\varphi\,d\varphi}{4\,\pi\,\varepsilon_o}\,\frac{(-\,a\,\bar{a}_r + \,z\,\bar{a}_z\,)}{(a^2 + \,z^2)^{3/2}}$$

- From the symmetry, the component in the *r* - direction:

$$d\bar{E}_P = -\,\frac{K\,a^2\,sin\,\varphi\,d\varphi}{4\,\pi\,\varepsilon_o(a^2 + \,z^2)^{3/2}}\bar{a}_r$$

- Since in cylindrical coordinate system, $\bar{a}_r = cos\,\varphi\,\bar{a}_x + sin\,\varphi\,\bar{a}_y$, therefore,

$$d\bar{E}_P = -\,\frac{K\,a^2\,sin\,\varphi\,d\varphi}{4\,\pi\,\varepsilon_o(a^2 + \,z^2)^{3/2}}(cos\,\varphi\,\bar{a}_x + sin\,\varphi\,\bar{a}_y)$$

$$= -\,\frac{K\,a^2\,d\varphi}{4\,\pi\,\varepsilon_o(a^2 + \,z^2)^{3/2}}\left[\frac{1}{2}sin\,2\varphi\,\bar{a}_x + \frac{1}{2}(1 - cos\,\varphi)\,\bar{a}_y\right]$$

- or

$$\overline{E}_P = -\int_0^{2\pi} \frac{K\,a^2\,d\varphi}{4\,\pi\,\varepsilon_o(a^2+\,z^2)^{3/2}}\left[\frac{1}{2}\sin 2\varphi\,\overline{a}_x + \frac{1}{2}(1-\cos\varphi)\,\overline{a}_y\right]$$

- $\sin 2\,\varphi$ and $\cos 2\,\varphi$ have zero integration over an entire period.
- Therefore, the electric field intensity will be:

$$\overline{E}_P = -\int_0^{2\pi} \frac{K\,a^2\,d\varphi}{8\,\pi\,\varepsilon_o(a^2+\,z^2)^{3/2}}\overline{a}_y = -\frac{K\,a^2\,d\varphi}{4\,\varepsilon_o(a^2+\,z^2)^{3/2}}\overline{a}_y \quad \frac{V}{m}$$

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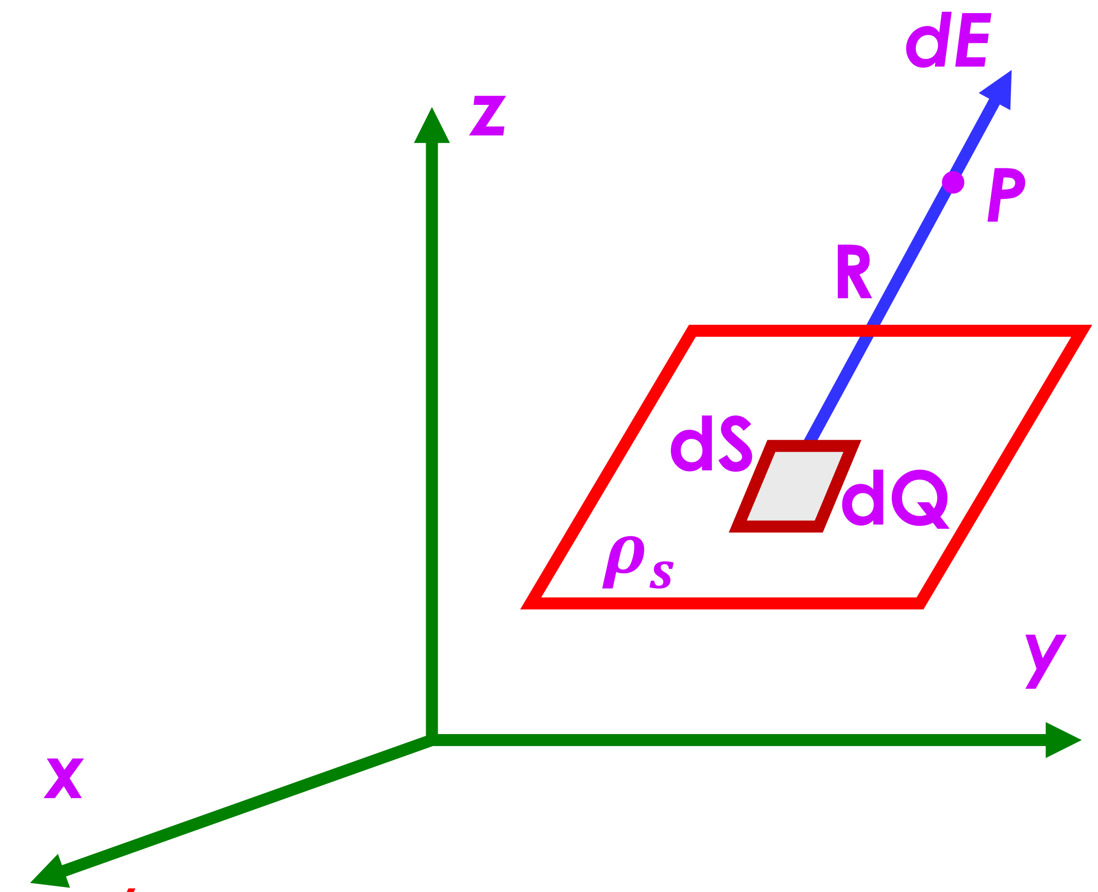
c) Electric Field due to a surface charge

- For a surface charge distribution, as shown, the point charge is formulated by the charge on a differential surface ds.
- The differential point charge on dS is $dQ = \rho_S \, ds \quad (C)$. The differential electric field intensity $d\bar{E}$ becomes :



$$d\bar{E} = \frac{dQ}{4\,\pi\,\varepsilon_o\,R^2}\,\bar{a}_R = \frac{\rho_S\,dS}{4\,\pi\,\varepsilon_o\,R^2}\,\bar{a}_R \qquad V/m$$

- The total electric field intensity $\bar{E}$ is:

$$\bar{E} = \int_S d\,\bar{E} = \int_S \frac{\rho_S\,dS}{4\,\pi\,\varepsilon_o\,R^2}\,\bar{a}_R \qquad \frac{V}{m}$$

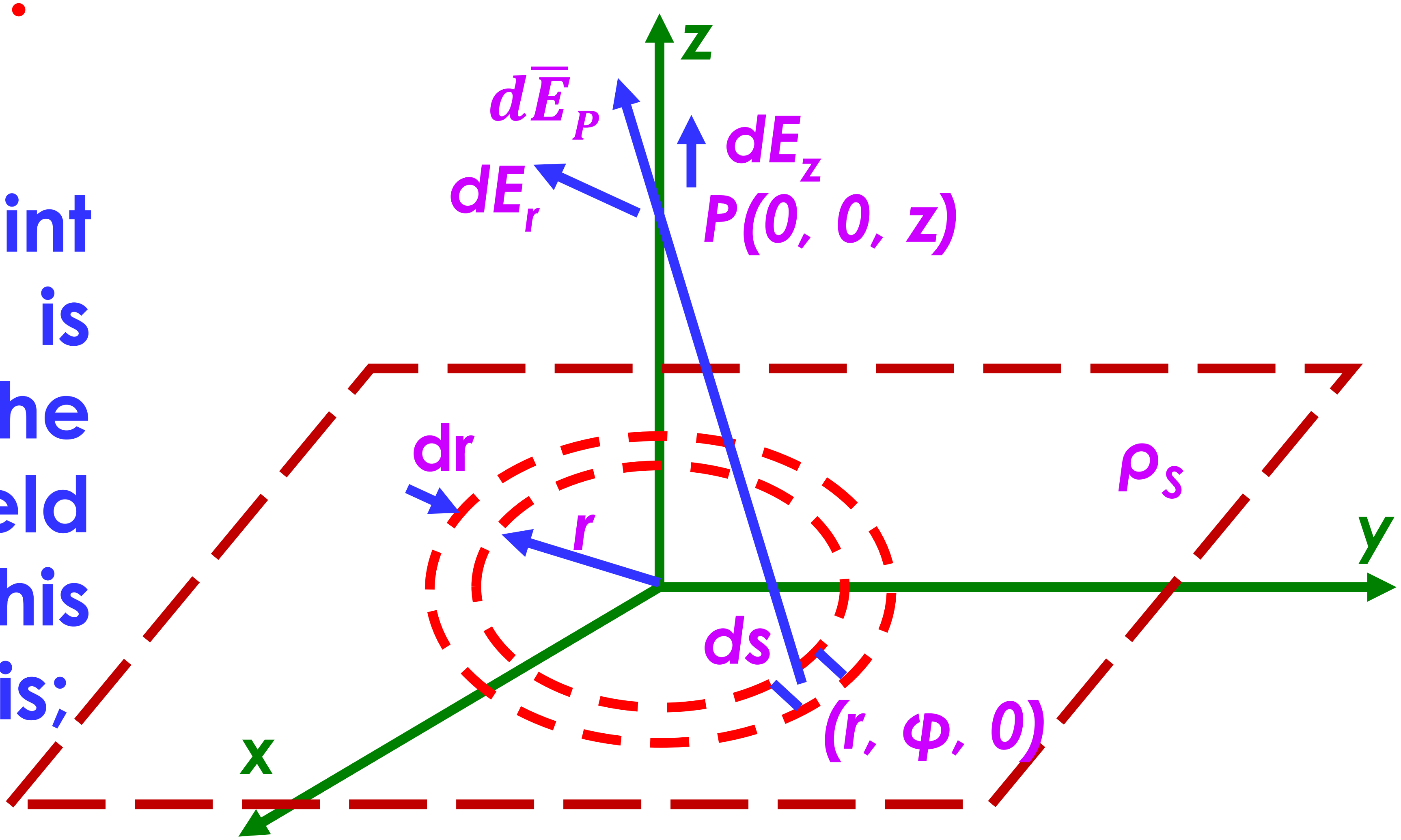
Example (1-7)

Through the use of cylindrical coordinates, find the electric field intensity at any point along the $z-axis$ due to infinite surface of uniform charge distribution ρ_S located on $z = 0$ plane.

Solution

- The differential point charge on dS is $dQ = \rho_S \, dS$, and the differential electric field intensity $d\bar{E}$ due to this differential point charge is;

$$d\bar{E} = \frac{dQ}{4\,\pi\,\varepsilon_o\,R^2}\,\bar{a}_R$$



- or

$$d\overline{E} = \frac{\rho_S \, dS}{4\,\pi\,\varepsilon_o\,R^2}\,\overline{a}_R$$

- From the figure, we have:

$$dQ = \rho_S dS = \rho_S r\,dr\,d\varphi$$

$$\overline{R} = -r\,\overline{a}_r + \,z\,\overline{a}_z\,, \qquad |R| = \sqrt{r^2 + \,z^2}$$

$$\overline{a}_R = \frac{\overline{R}}{|R|} = \frac{-r\,\overline{a}_r + \,z\,\overline{a}_z}{\sqrt{r^2 + \,z^2}}$$

- Thus:

$$\overline{E} = \frac{\rho_S\,r\,dr\,d\varphi}{4\,\pi\,\varepsilon_o\,(r^2 + \,z^2)^{3/2}}\,(-\,r\,\overline{a}_r + \,z\,\overline{a}_z) = d\,E_r\,\overline{a}_r + \,d\,E_z\,\overline{a}_z$$

- From the symmetry, the r-component in the is cancel:

$$dE_z = \frac{\rho_S\,z\,r\,dr\,d\varphi}{4\,\pi\,\varepsilon_o\,(r^2 + \,z^2)^{3/2}}$$

- or

$$E_z = \frac{\rho_S\, z}{4\,\pi\,\varepsilon_o} \int_0^{2\,\pi} \int_0^{\infty} \frac{r\, dr\, d\varphi}{(r^2 + z^2)^{3/2}} = \frac{\rho_S\, z}{2\,\varepsilon_o} \left[\frac{-1}{\sqrt{r^2 + z^2}} \right]_0^{\infty} = \frac{\rho_S}{2\,\varepsilon_o}$$

- Finally:

$$\overline{E} = \frac{\rho_S}{2\,\varepsilon_o}\, \overline{a}_z \qquad V/m$$

- From the above equation, we can deduce that there is only one component for the electric field intensity in case of an infinite sheet of charge having a uniform charge density ρ_S and this component is perpendicular to the sheet and outward from it.
- In general, the electric field is given by;

$$\overline{E} = \frac{\rho_S}{2\,\varepsilon_o}\, \overline{a}_n \quad V/m$$

- Where $\overline{a}_n$ is a unit vector perpendicular to the surface and outward from it.

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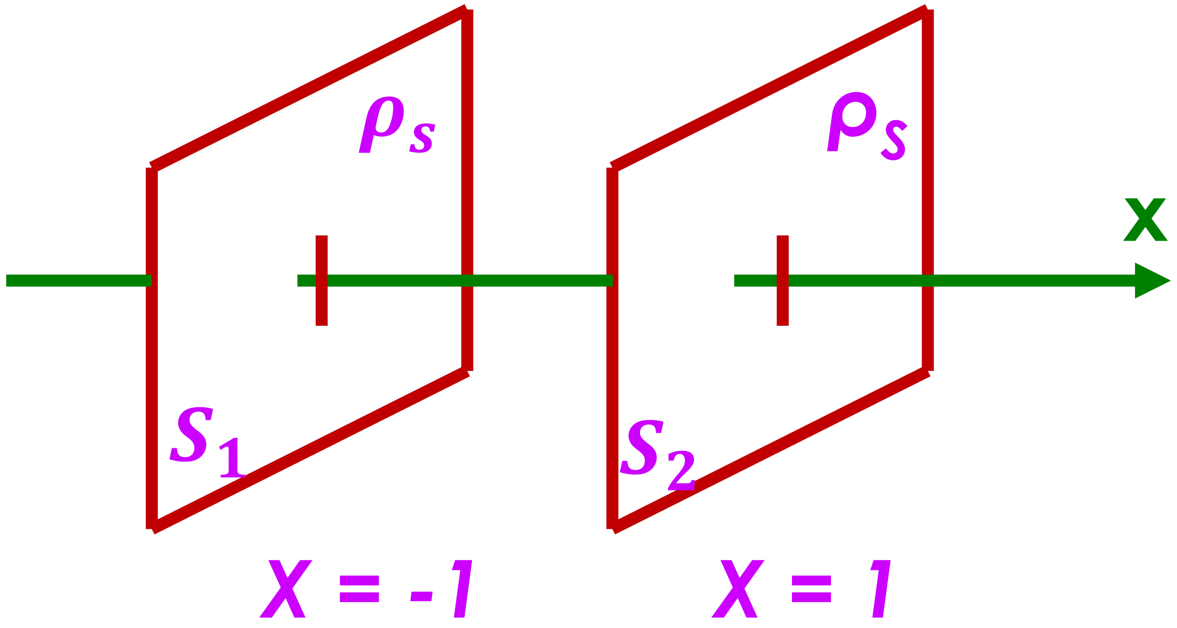
Example (1-8)

Two infinite uniform sheets of charge, each with density ρ_S, are located at $x = \pm 1$. Determine $\overline{E}$ in all regions.

Solution



- The electric field intensity due to infinite surface (sheet) is given by:

$$\overline{E} = \frac{\rho_S}{2\,\varepsilon_o}\,\overline{a}_n$$

- $Region\ I: -\infty \le x \le -1$

$$\overline{E}_I = \overline{E}_{S\,1} + \overline{E}_{S\,2}$$

$$\overline{E}_I = -\frac{\rho_S}{2\,\varepsilon_o}\,\overline{a}_x - \frac{\rho_S}{2\,\varepsilon_o}\,\overline{a}_x = -\frac{\rho_S}{\varepsilon_o}\,\overline{a}_x \qquad V/m$$

- $Region\ II: -1 \le x \le +1$

$$\bar{E}_{II} = \bar{E}_{S\,1} + \bar{E}_{S\,2}$$

$$\bar{E}_{II} = \frac{\rho_S}{2\,\varepsilon_o}\,\bar{a}_x - \frac{\rho_S}{2\,\varepsilon_o}\,\bar{a}_x = 0 \qquad V/m$$

- $Region\ III: +1 \le x \le +\infty$

$$\bar{E}_{III} = \bar{E}_{S\,1} + \bar{E}_{S\,2}$$

$$\bar{E}_{III} = \frac{\rho_S}{2\,\varepsilon_o}\,\bar{a}_x + \frac{\rho_S}{2\,\varepsilon_o}\,\bar{a}_x = \frac{\rho_S}{\varepsilon_o}\,\bar{a}_x \qquad V/m$$

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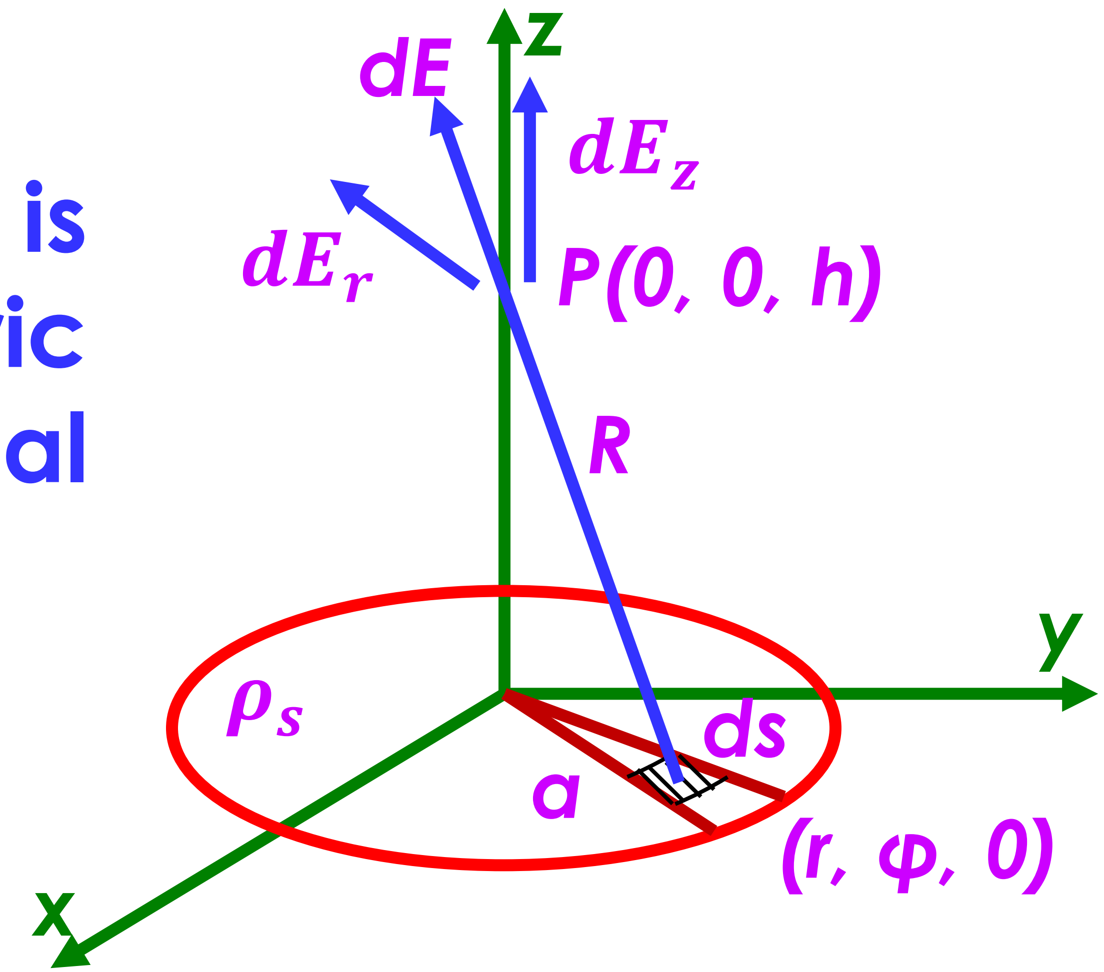
Example (1-9)

Find the electric field intensity $\overline{E}$ at $(0, 0, h)$ in cylindrical coordinates due to the uniformly charged disk $r \leq a,\ z = 0$.

Solution



- The differential point charge on ds is $dQ = \rho_S\, dS$, and the differential electric field intensity $d\overline{E}$ due to this differential point charge is;

$$d\overline{E} = \frac{dQ}{4\,\pi\,\varepsilon_o\,R^2}\,\overline{a}_R$$

- or

$$d\overline{E} = \frac{\rho_S\, dS}{4\,\pi\,\varepsilon_o\,R^2}\,\overline{a}_R$$

- From the shown figure, we have

$$dQ = \rho_S dS = \rho_S r\, dr\, d\varphi$$

$$\bar{R} = -r\,\bar{a}_r + h\,\bar{a}_z\,, \qquad |R| = \sqrt{r^2 + h^2}$$

$$\bar{a}_R = \frac{\bar{R}}{|R|} = \frac{-r\,\bar{a}_r + h\,\bar{a}_z}{\sqrt{r^2 + h^2}}$$

- Thus:

$$\bar{E} = \frac{\rho_S\, r\, dr\, d\varphi}{4\,\pi\,\varepsilon_o\,(r^2 + h^2)^{3/2}}\,(-\,r\,\bar{a}_r + h\,\bar{a}_z) = d\,E_r\,\bar{a}_r + d\,E_z\,\bar{a}_z$$

- From the symmetry, the r-component is canceled:

$$dE_z = \frac{\rho_S\, h\, r\, dr\, d\varphi}{4\,\pi\,\varepsilon_o\,(r^2 + h^2)^{3/2}}$$

- Therefore:

$$d\overline{E} = \frac{\rho_S\, h}{4\,\pi\,\varepsilon_o}\int_0^{2\,\pi}\int_0^{a}\frac{r\,dr\,d\varphi}{(r^2 + h^2)^{3/2}}\,\overline{a}_z$$

- or

$$\overline{E} = \frac{\rho_S\, h}{4\,\pi\,\varepsilon_o}\left[\frac{-1}{\sqrt{a^2 + h^2}} - \frac{1}{h}\right]\overline{a}_z$$

- Note that as $a \rightarrow \infty$,

$$\overline{E} \rightarrow \frac{\rho_S}{2\,\varepsilon_o}$$

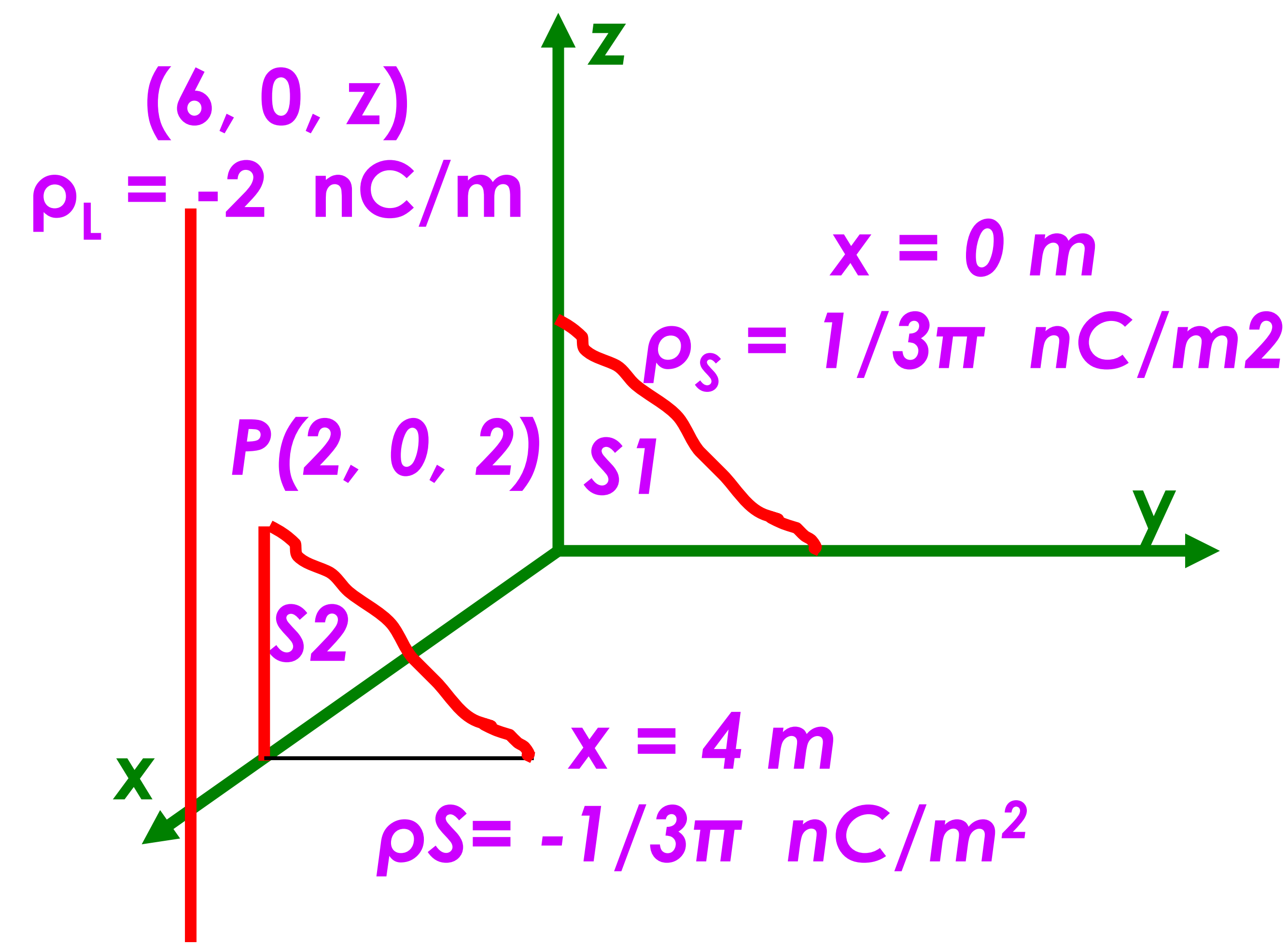
Which is the field due to a uniform plane sheet.

****

Example (1-10)

Determin $\overline{E}$ at $(2, 0, 2)\ m$ due to three standard charge distribution as follows: a uniform sheet at $x = 0\ m$ with $\rho_S = (1/3\,\pi)\ nC/m^2$, a uniform sheet at $x = 4\ m$ with $\rho_S = (-1/3\,\pi)\ nC/\,m^2$, and a uniform line at $x = 6\,m$, $y = 0\ m$ with $\rho_L = -2\ nC/m$

Solution



- Due to the sheet charge s_1:

$$\overline{E}_{S\,1} = \frac{\rho_{S\,1}}{2\,\varepsilon_o}\ \overline{a}_n$$

- At P, $\overline{a}_n = \overline{a}_x$ and then:

$$\overline{E}_{S\,1} = 6\,\overline{a}_x \qquad V/m$$

- Due to the sheet charge s_2:

$$\bar{E}_{S\,2} = \frac{\rho_{S\,2}}{2\,\varepsilon_o}\;\bar{a}_n$$

- At P, $\bar{a}_n = -\,\bar{a}_x$ and then:

$$E_{2\,S} = 6\,\bar{a}_x \qquad V/m$$

- Due to the line charge:

$$\bar{E}_L = \frac{\rho_L}{2\,\pi\,\varepsilon_o\,r}\,a_r$$

- At P, $\bar{r} = -\,4\,\bar{a}_x$, $|r| = 4$, $\bar{a}_r = \frac{\bar{r}}{|r|} = -\,a_x$, and then $\bar{E}_L = 9\,\bar{a}_x\;\; V/m$

- The total electric field is the sum of the tow fields:

$$\bar{E}_P = \bar{E}_{S\,1} + \bar{E}_{S\,2} + \bar{E}_L = 6\,\bar{a}_x + 6\,\bar{a}_x + 9\,\bar{a}_x = 21\,\bar{a}_x \qquad V/m$$

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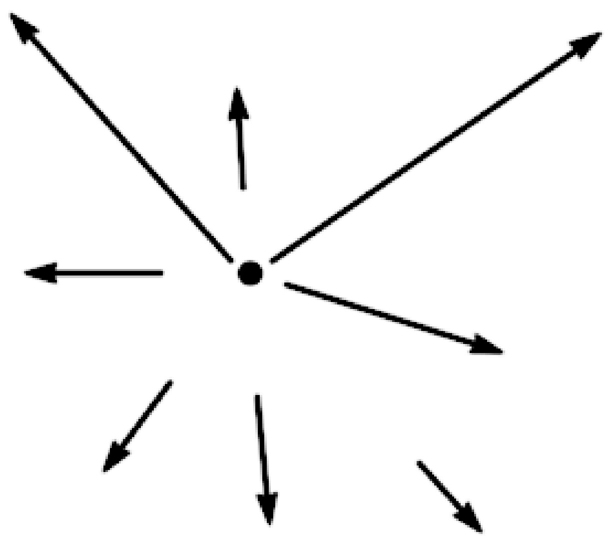
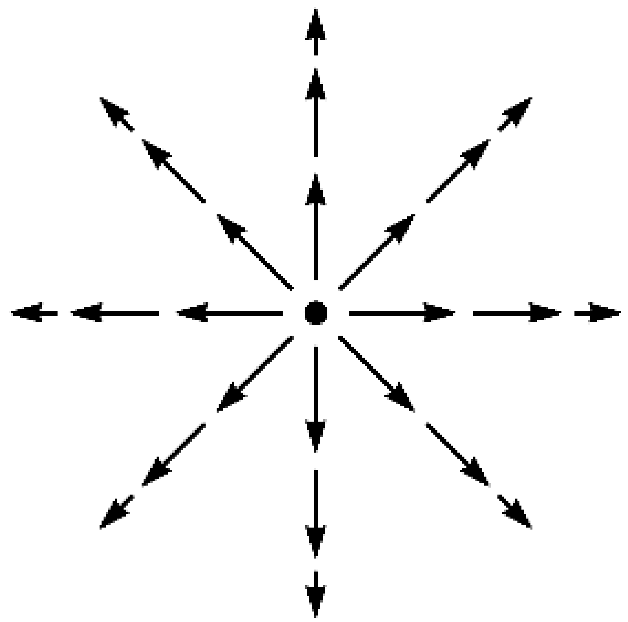
1. 5. Streamlines and Sketches of the Field

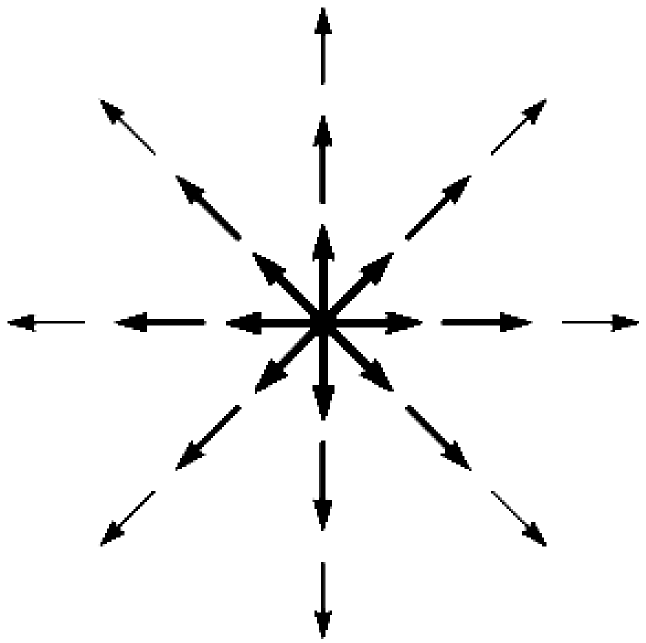
- These lines are usually called *streamlines*, although other terms such as *flux lines* and *direction lines* are also used.

Types of Streamline Sketches of Fields:

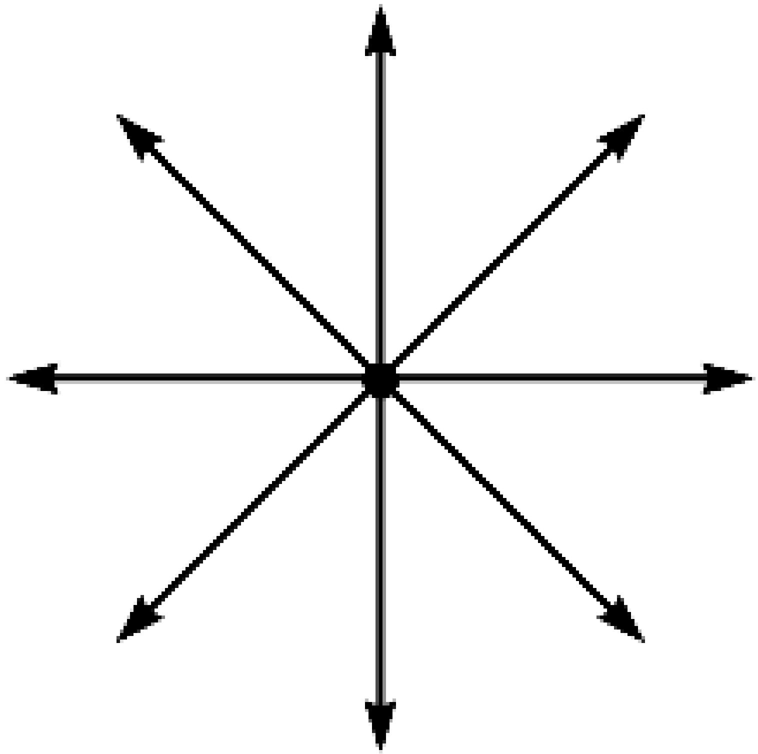


One very poor sketch





Two fair sketches

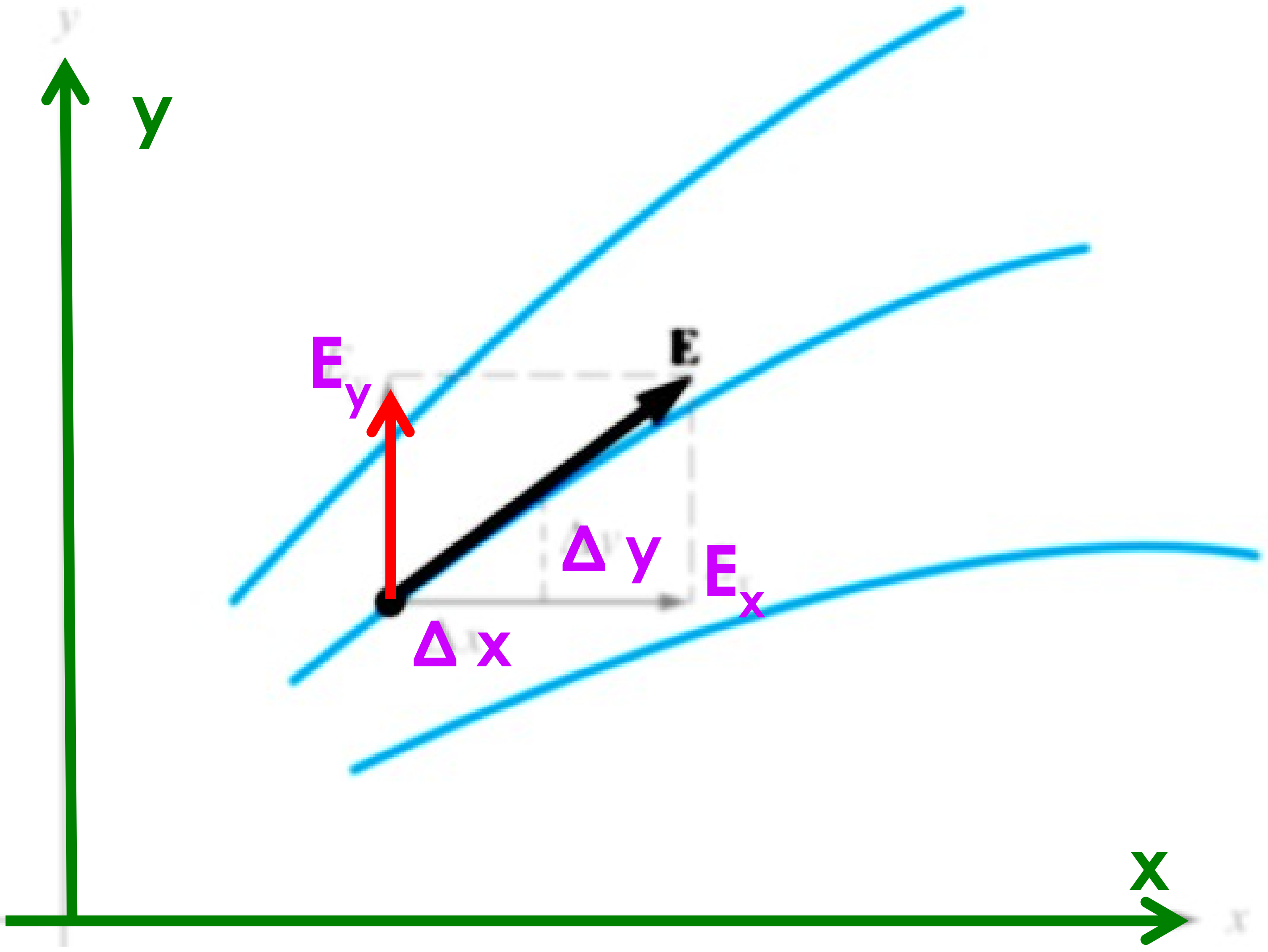


The usual form of a streamline sketch.

the arrows show the direction of the field at every point along the line, and the spacing of the lines is inversely proportional to the strength of the field.

- The ratio of the *y* and *x* field components gives the slope of the field plot in the *x*-*y* plane.

$$\frac{E_y}{E_x} = \frac{dy}{dx}$$



- In cylindrical Coordinate, the stream line equation is given by:

$$\frac{dr}{r d\phi} = \frac{E_r}{E_\phi}$$

Example (1-11)

Given the electric field $\bar{E} = 15\,x^2\,y\,\bar{a}_x + 5\,x^3\,\bar{a}_y$, find

a) The equation of the streamline that passes through point $P(2,3,-4)$,

b) A unit vector $\bar{a}_E$ specifying the direction of $\bar{E}$ at P,

c) A unit vector $\bar{a}_\perp = (l, m, 0)$ that is perpendicular to $\bar{a}_E$ at P and has $l > 0$.

Solution

a) The differential equation is given by

$$\frac{dy}{dx} = \frac{E_y}{E_x} = \frac{x}{3\,y} \quad or \quad 3\,y\,dy = x\,dx. \; Therefore:$$

$$\frac{3\,y^2}{2} = \frac{x^2}{2} + C$$

➢ At P(2, 3, -4)

$$C = -\frac{3\,(3)^2}{2} + \frac{2^2}{2} = 11.5$$

➢ From which the equation of the streamlines are obtained:

$$3\,y^2 - x^2 = 23$$

b) At P(2, 3, -4)

$$\overline{E} = 180\,\overline{a}_x + 40\,\overline{a}_y \quad and \quad |\overline{E}| = \sqrt{180^2 + 40^2} = 184.4 \quad V/m$$

Therefore,

$$\overline{a}_E = \frac{\overline{E}}{|E|} = 0.976\,\overline{a}_x + 0.217\,\overline{a}_y$$

c) Let $\overline{a}_\perp = l\,\overline{a}_x + m\,\overline{a}_y$, then

$\overline{a}_E \cdot \overline{a}_\perp = 0$ and, $0.976\,l + 0.217\,m = 0$

but $l > 0$, Then $l = 0.217$ and $m = -0.976$. The unit vector perpendicular to $\overline{a}_E$ will be

$$\overline{a}_\perp = 0.217\,\overline{a}_x - 0.976\,\overline{a}_y.$$

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THE END
OF CHAPTER (1)