Fundamentals of Electromagnetic Fields_ EPM 112

CHAPTER (7)

Steady State Magnetic Fields

Lecture 1

Biot-Savart law
Ampere's Circuital Law
Magnetic Field Density
Magnetic Flux

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This chapter is divided into Three main parts as:

Part 1: Biot-Savart law

Part 2: Ampere's circuital law

Part 3: Magnetic field density & magnetic flux

N. B.: Steady state magnetic fields will be produced due to a constant current 'dc current' passes through a filament current conductor.

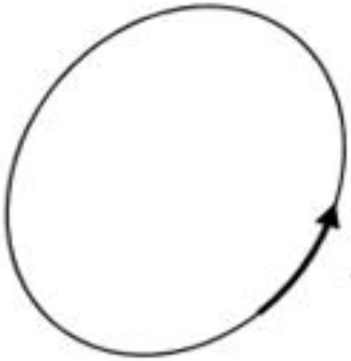
Part 1: Biot-Savart law

Sources of magnetic field:

1- Permanent magnet
2- Flow of current in conductors
3-Time varying of electric field inducing magnetic field

Current configurations:

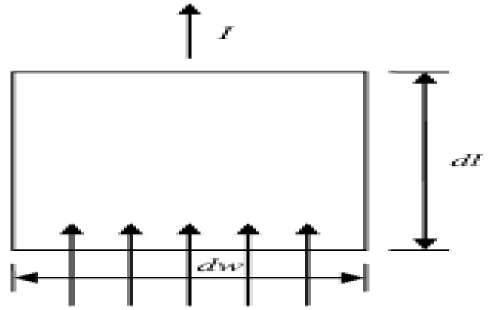
1- Filamentary current



2- Surface current: $\bar{J}_s \ (A/m)$

Current element: $\bar{J}_s \, ds \ (A.m)$,

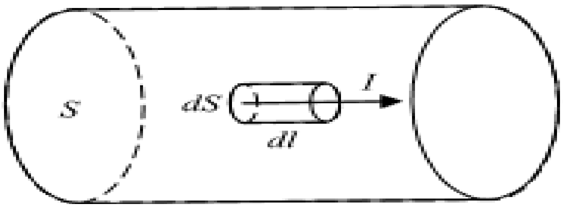
where: $J_s = \dfrac{I}{W} \quad A/m$



$\mathbf{J_s}$ is a surface current density.

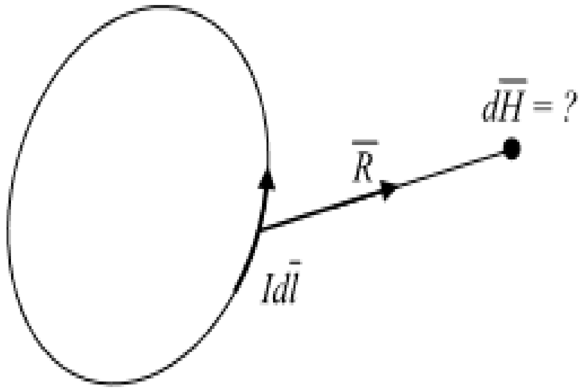
3- Volume current: $\bar{J} = \dfrac{I}{S} \, \hat{a}_n \ A/m^2$

Current element $\bar{J} \, dv \quad (A.m), \ dv = d\bar{S}.d\bar{l}$



Biot-Savart law:

$$d\overline{H} = \frac{Id\overline{l} \times \hat{a}_R}{4\pi R^2}$$



Where $\overline{H}$ = magnetic field intensity

$\hat{a}_R$ = unit vector from the current element to the point where we want to find $\overline{H}$ at it

$\overline{R}$ = distance between the current element and the point (p)

H = Magnetic field intensity $\overline{A/m}$

B = magnetic field or magnetic field density Wb/m^2 or Tesla

In Free space

$B = \mu_0 H$ where μ_0 is the permeability of free space

$\mu_0 = 4\pi * 10^{-7}$ H/m

- The magnetic field intensity circulates around its source, I.
- Using the right-hand rule, we can specify the direction of H.
- Right thumb in the direction of the current, fingers curl in the direction of H.

I = dc current

Example:

Find $\bar{H}$ in the x-y plane arising from a filament current I of infinite length is located on the z-axis.

Solution:

since $d\bar{H} = \dfrac{I d\bar{l} \times \bar{a}_R}{4\pi R^2}$

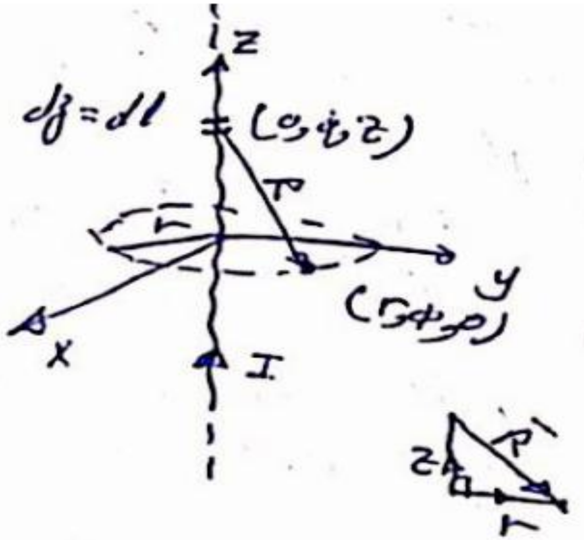
where $I d\bar{l} = I dz\, \bar{a}_z$

$\bar{R} = r\bar{a}_r - z\bar{a}_z$

$\bar{a}_R = \dfrac{\bar{R}}{R}$

$= \dfrac{r\bar{a}_r - z\bar{a}_z}{\sqrt{r^2 + z^2}}$

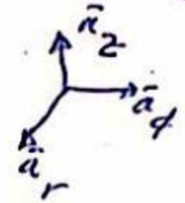
$\therefore d\bar{H} = \dfrac{I dz\,[\bar{a}_z \times (r\bar{a}_r - z\bar{a}_z)]}{4\pi (r^2 + z^2)^{3/2}}$



$$d\bar{H} = \frac{I r \, dz}{4\pi (r^2+z^2)^{3/2}} \overbrace{(\bar{a}_z \times \bar{a}_r)}^{\bar{a}_\phi} - \frac{I z \, dz}{4\pi (r^2+z^2)^{3/2}} \underbrace{(\bar{a}_z \times \bar{a}_z)}_{\text{zero}}$$

$$\therefore d\bar{H} = \frac{I r \, dz}{4\pi (r^2+z^2)^{3/2}} \bar{a}_\phi$$

$$\therefore \bar{H} = \frac{I}{4\pi} \int_{z=-\infty}^{z=\infty} \frac{r \, dz}{(r^2+z^2)^{3/2}} \bar{a}_\phi$$



N.B:

$$\int \frac{dx}{(x^2+a^2)^{3/2}} = \frac{x}{a^2\sqrt{x^2+a^2}}$$

or

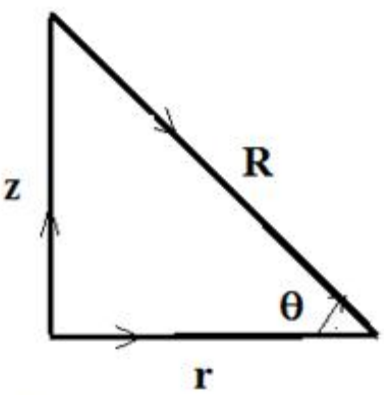
$$\int_{z=-\infty}^{\infty} \frac{r \, dz}{(r^2+z^2)^{3/2}} = \left[\frac{r z}{r^2\sqrt{r^2+z^2}}\right]_{-\infty}^{\infty} = \frac{1}{r}\left[\frac{1}{\sqrt{\left(\frac{r}{z}\right)^2+1}}\right]_{-\infty}^{\infty} = \frac{2}{r}$$

OR

let $\boxed{z = r\tan\theta} \Longrightarrow dz = r\sec^2\theta \, d\theta$

at $z = -\infty \Longrightarrow \theta = -\frac{\pi}{2}$

$z = \infty \Longrightarrow \theta = \frac{\pi}{2}$



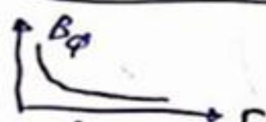
$$\therefore \bar{H} = \frac{I}{4\pi} \int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} \frac{r^2 \sec^2\theta \, d\theta}{r^3 \sec^3\theta} \bar{a}_\phi$$

$$= \frac{I}{4\pi r} \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta \, d\theta \; \bar{a}_\phi = \frac{I}{4\pi r}\left[\sin\frac{\pi}{2} - \sin\left(-\frac{\pi}{2}\right)\right] \bar{a}_\phi$$

$$\boxed{\therefore \bar{H} = \frac{I}{2\pi r} \bar{a}_\phi} = H_\phi \, \bar{a}_\phi \quad , \; H_\phi = \frac{I}{2\pi r}$$

Also, $\bar{B} = \mu_0 \bar{H} \Longrightarrow \boxed{\bar{B} = \frac{\mu_0 I}{2\pi r} \bar{a}_\phi}$

$H_\phi \propto \frac{1}{r}$



$B_\phi \propto \frac{1}{r}$

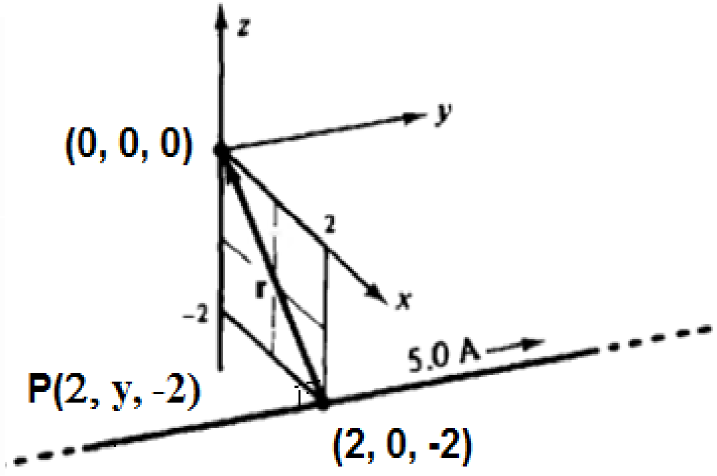
H and **B** are inversely proportional with r.

Example;

A current filament of 5.0 A in the $\mathbf{a}_y$ direction is parallel to the y axis at $x = 2$ m, $z = -2$ m. Find **H** at the origin.

Solution:

$$\mathbf{r} = -2\ \mathbf{a}_x + 2\ \mathbf{a}_z$$



The expression for **H** due to a straight current filament applies,

$$\mathbf{H} = \frac{I}{2\pi r}\mathbf{a}_\phi$$

where $r = 2\sqrt{2}$ and (use the right-hand rule)

$$\mathbf{a}_\phi = \frac{\mathbf{a}_x + \mathbf{a}_z}{\sqrt{2}}$$

Thus
$$\mathbf{H} = \frac{5.0}{2\pi(2\sqrt{2})}\left(\frac{\mathbf{a}_x + \mathbf{a}_z}{\sqrt{2}}\right) = (0.281)\left(\frac{\mathbf{a}_x + \mathbf{a}_z}{\sqrt{2}}\right) \text{ A/m}$$

Where

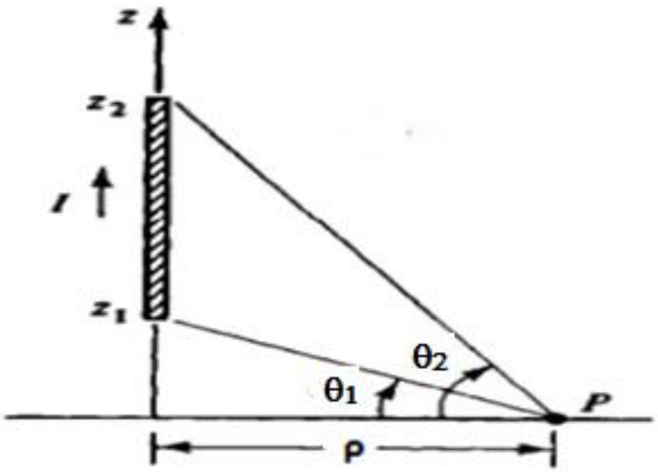
$$a_\emptyset = a_I x a_{r=} a_y x \frac{-2a_x + 2z}{\sqrt{2^2 + 2^2}}$$

$$= \frac{2a_z + 2a_x}{\sqrt{8}}$$

Example:

Show that the magnetic field intensity due to the finite current element shown in Fig.

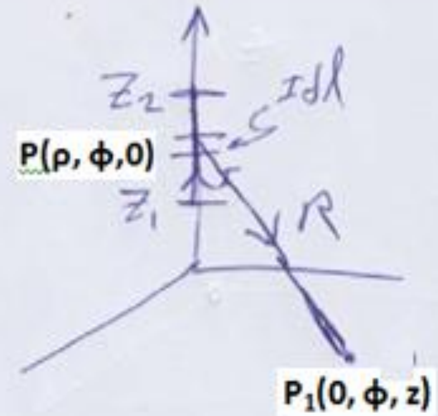
$$\mathbf{H} = \frac{I}{4\pi\rho}(\sin\alpha_2 - \sin\alpha_1)\mathbf{a}_\phi$$



$\theta_1 = \alpha_1$, $\theta_2 = \alpha_2$ & $r = \rho$

Solution:

$$\vec{H} = \int \frac{\vec{I\,dl} \times a_R}{4\pi R^2}$$

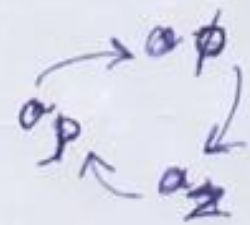


$$\vec{I\,dl} = I\,dz\,\hat{a}_z$$

$$\vec{R} = -z\,\hat{a}_z + \rho\,\hat{a}_\rho$$

$$\hat{a}_R = \frac{\rho\,\hat{a}_\rho - z\,\hat{a}_z}{\sqrt{z^2+\rho^2}}$$

$$\Rightarrow I\,dl \times \hat{a}_R = \frac{I\,dz}{\sqrt{z^2+\rho^2}}\left(\rho\,\hat{a}_\phi - 0\right) \quad (1)$$



$$\vec{H} = \int_{z_1}^{z_2} \frac{I \rho \, dz}{4\pi (z^2+\rho^2)^{3/2}} \hat{a}_\phi \quad (1)$$

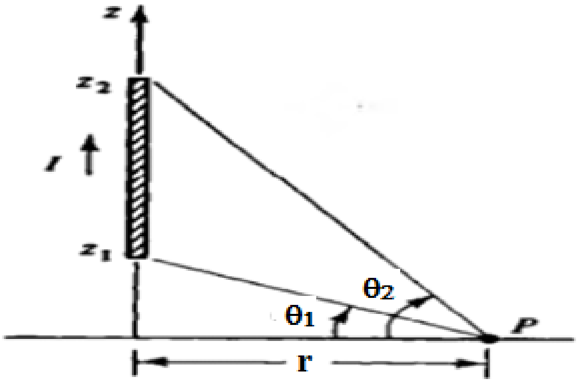
$$= \frac{I \rho}{4\pi \rho^2} \cdot \frac{z}{\sqrt{z^2+\rho^2}} \Bigg|_{z_1}^{z_2} \hat{a}_\phi$$

$$\vec{H} = \frac{I}{4\pi\rho} \cdot \left(\frac{z_2}{\sqrt{z_2^2+\rho^2}} - \frac{z_1}{\sqrt{z_1^2+\rho^2}} \right) \hat{a}_\phi$$

$$\Rightarrow \vec{H} = \frac{I}{4\pi\rho} (\sin\alpha_2 - \sin\alpha_1) \hat{a}_\phi \quad (1)$$

Example:

Find the magnetic field intensity H in the x-y plane at point P arising from a filament current I of finite length on the z-axis as shown in Fig.



Solution:

The solution is similarly as the previous example.
Comparing the figures, we can deduce that:

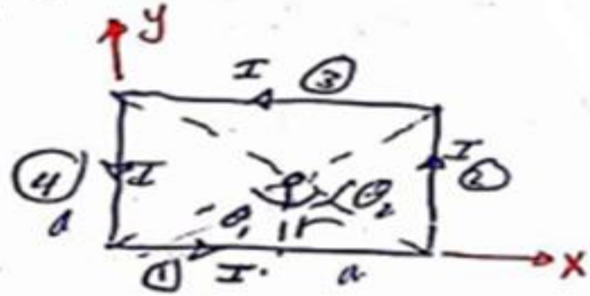
$\theta_1 = \alpha_1 \quad , \quad \theta_2 = \alpha_2 \;\&\; r = \rho$

Then, the equation of the magnetic field intensity can be written as:

$$\boxed{\overline{H} = \frac{I}{4\pi r}\left[\sin\theta_2 - \sin\theta_1\right]\bar{a}_\phi}$$

<u>Example</u>: Find **H** at the center of a square current loop of side a.

<u>For a square current loop</u>



$$\theta_1 = -\frac{\pi}{4}, \quad \theta_2 = \frac{\pi}{4}$$

$$\bar{H} = \bar{H}_1 + \bar{H}_2 + \bar{H}_3 + \bar{H}_4$$

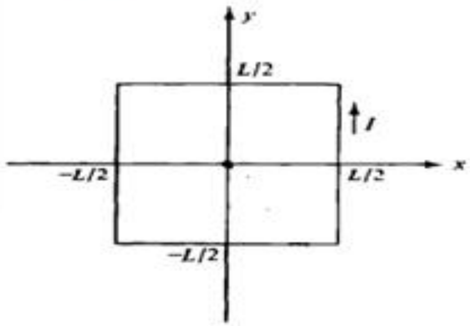
$$= 4 * \frac{I}{4\pi r}\left[\sin\frac{\pi}{4} - \sin\left(-\frac{\pi}{4}\right)\right]\bar{a}_\phi$$

$$= \frac{I}{\pi r}\, 2 * \frac{1}{\sqrt{2}} = \frac{2I}{\sqrt{2}\,\pi r}\bar{a}_\phi = \frac{2I}{\sqrt{2}\,\pi r}\bar{a}_z$$

where $\bar{a}_\phi = \bar{a}_I * \bar{a}_r = \vec{a}_x \times \vec{a}_y = \bar{a}_z$

$$r = a/2$$

$$\therefore \bar{H} = \frac{2}{\sqrt{2}}\,\frac{I}{\pi a/2}\,\bar{a}_\phi = \frac{4}{\sqrt{2}}\,\frac{I}{\pi a}$$



Example:

Find $\overline{H}$ at the point (0, 0, z) on z axis and also at the center of circular loop carrying current *I*.

Solution:

$$d\overline{H} = \frac{Id\overline{l} \times \hat{a}_R}{4\pi R^2}$$

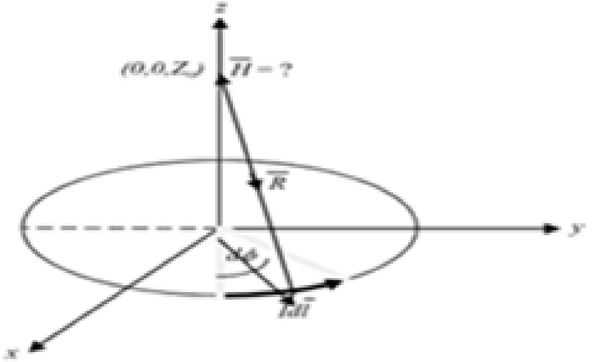
$$Id\overline{l} = Ia\,d\phi\,\hat{\phi}$$

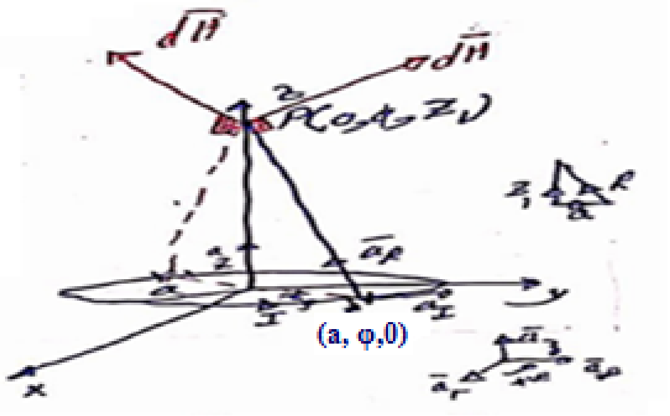
$$\overline{R} = -a\hat{r}_c + z_0\hat{z}$$

$$R = \sqrt{a^2 + z_o^2}$$

$$\hat{a}_R = \frac{\overline{R}}{R} = \frac{-a\hat{r}_c + z_0\hat{z}}{\sqrt{a^2 + z_0^2}}$$

$$\therefore d\overline{H} = \frac{Ia\,d\phi\hat{\phi}\times(-a\hat{r}_c + z_0\hat{z})}{4\pi(a^2 + z_0^2)^{3/2}}$$





$$\therefore d\overline{H} = \frac{Ia^2 d\phi\,\hat{z} + Ia \;\; z_0 d\phi\,\hat{r}_c}{4\pi(a^2 + z_0^2)^{\frac{3}{2}}}$$

$$\overline{H} = \int d\overline{H} = \frac{Ia^2}{4\pi(a^2 + z_0^2)^{\frac{3}{2}}} \int_0^{2\Pi} d\phi.\hat{z} + \frac{Ia\, z_0}{4\pi(a^2 + z_0^2)^{\frac{3}{2}}} \int_0^{2\Pi} d\phi\,\hat{r}_c$$

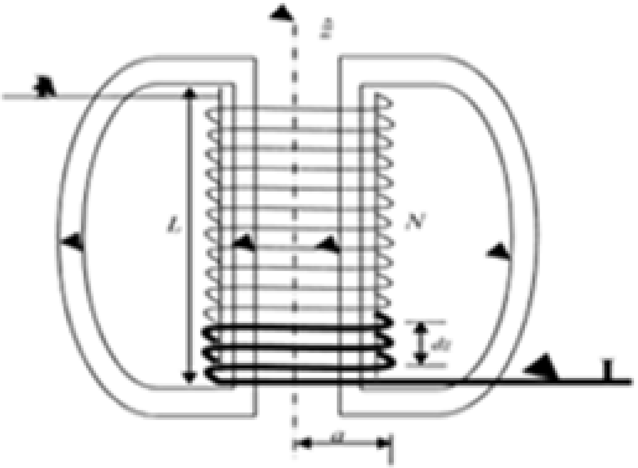
$$\overline{H} = \frac{2\pi I a^2}{4\pi\left(a^2 + z_0^2\right)^{\frac{3}{2}}}\hat{z} = \frac{Ia^2}{2\left(a^2 + z_0^2\right)^{\frac{3}{2}}}\hat{z}$$

To determine H at the center of the circular current loop, in this case, substitute by $Z_o = 0$.

Example:

Find the magnetic field intensity at the center of a solenoid (coil) of radius a and length L and the number of N turns carrying current I.

Solution:



Current in length dz is

$$I = \frac{(NI)}{l} dz$$

Since

L ➡ NI

dz ➡ mmf?

Since $\overline{H}$ due to circular loop of current I and radius a is given by

$$\overline{H} = \frac{Ia^2}{2\left(a^2 + z_0^2\right)^{\frac{3}{2}}} \hat{z}$$

So

$$d\overline{H} = \frac{\frac{(NI)}{l}\, dz.a^2}{2\left(a^2 + z^2\right)^{\frac{3}{2}}}\,\hat{z}$$

$\overline{H}$ **at the center of the solenoid is given by**

$$\overline{H} = \frac{NIa^2}{2l}\int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{dz}{(a^2+z^2)^{\frac{3}{2}}}\,\hat{z} = \frac{NIa^2}{2l}\left(\frac{z}{a^2(a^2+z^2)^{\frac{1}{2}}}\right)_{-\frac{L}{2}}^{\frac{L}{2}}\hat{z}$$

$$= \frac{2\left(\frac{L}{2}\right)NI}{2l\left[a^2 + \left(\frac{L}{2}\right)^2\right]^{\frac{1}{2}}}\,\hat{z} = \frac{NI}{2\left(a^2 + \frac{L^2}{4}\right)^{\frac{1}{2}}}\,\hat{z} = \frac{NI}{2\left(\frac{4a^2+L^2}{4}\right)^{\frac{1}{2}}}\,\hat{z}$$

$$\therefore \overline{H} = \frac{NI}{\left(4a^2 + L^2\right)^{\frac{1}{2}}} \hat{z}$$

If $L >> a$: $$\boxed{\therefore \overline{H} = \frac{NI}{L} \hat{z}}$$

Notes:

For the magnetic field at the end of the solenoid we must integrate from

$0 \rightarrow L$, so $\overline{H}$ at the end:

Thanks