Electromagnetic Fields

EPM 112

Course instructor

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Chapter (2)

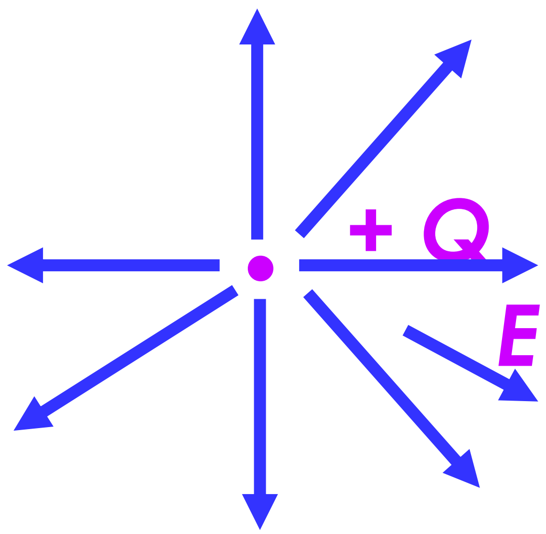
The Electric Flux Density and Gauss Law

2. 1. Introduction

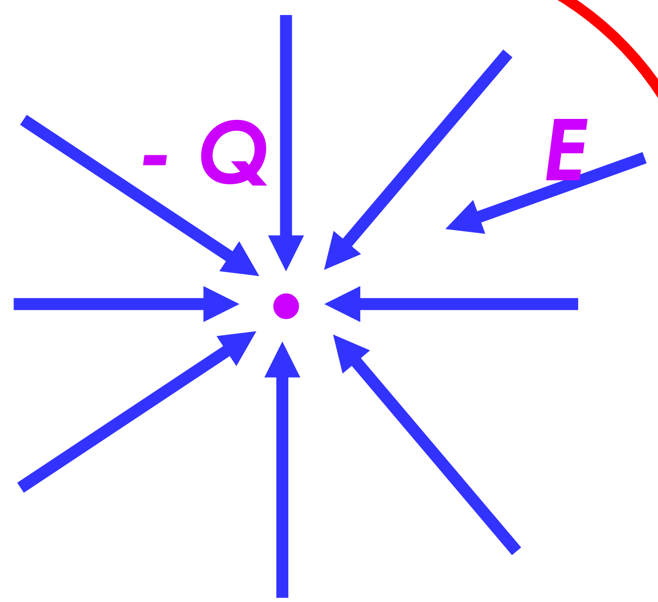
- **In previous chapter, the concept of vector force field acting in a point charge was used to define the electric field intensity $\bar{E}$.**
- **However, in this chapter, the concept of the electric flux lines and their density have been introduced.**
- **The concept of the electric flux density will lead us to *Gauss's law*, divergence, and divergence theorem.**
- **Through the use of Gauss's law, we will be able to readily solve many problems possessing charge symmetry.**

2. 2. Electric flux

- *Michael Faraday* performed several basic experiments in electrostatic that related to the concept of electric flux or electric flux lines.
- We shall use the electric flux concept to improve our visual picture of the vector force field about the charges.
- This concept is illustrated by *two - dimensional* plots in Fig. a, b, and c



(a)



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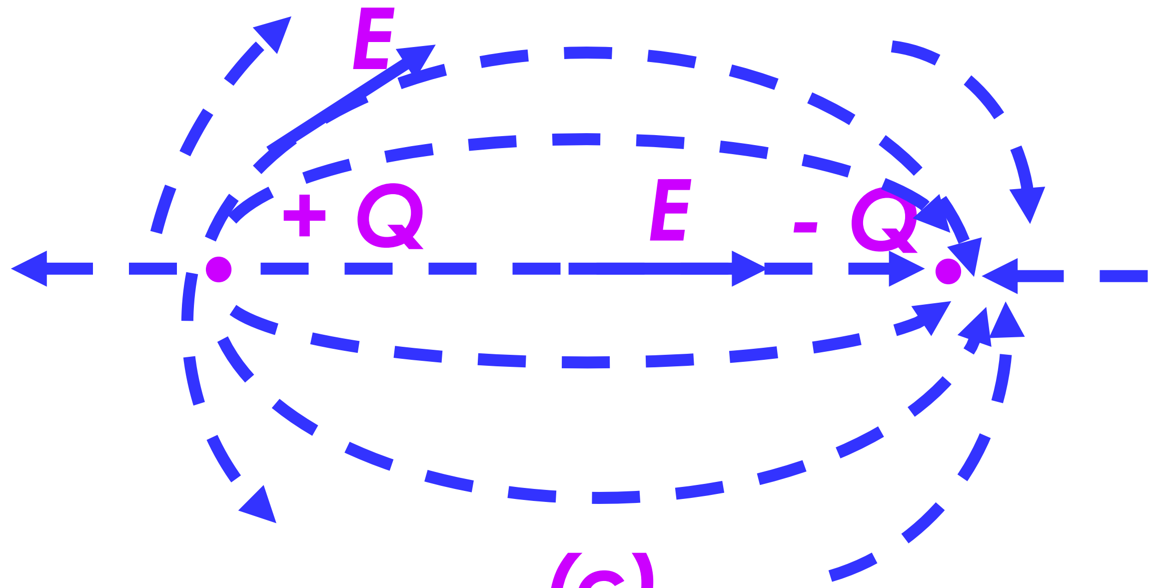
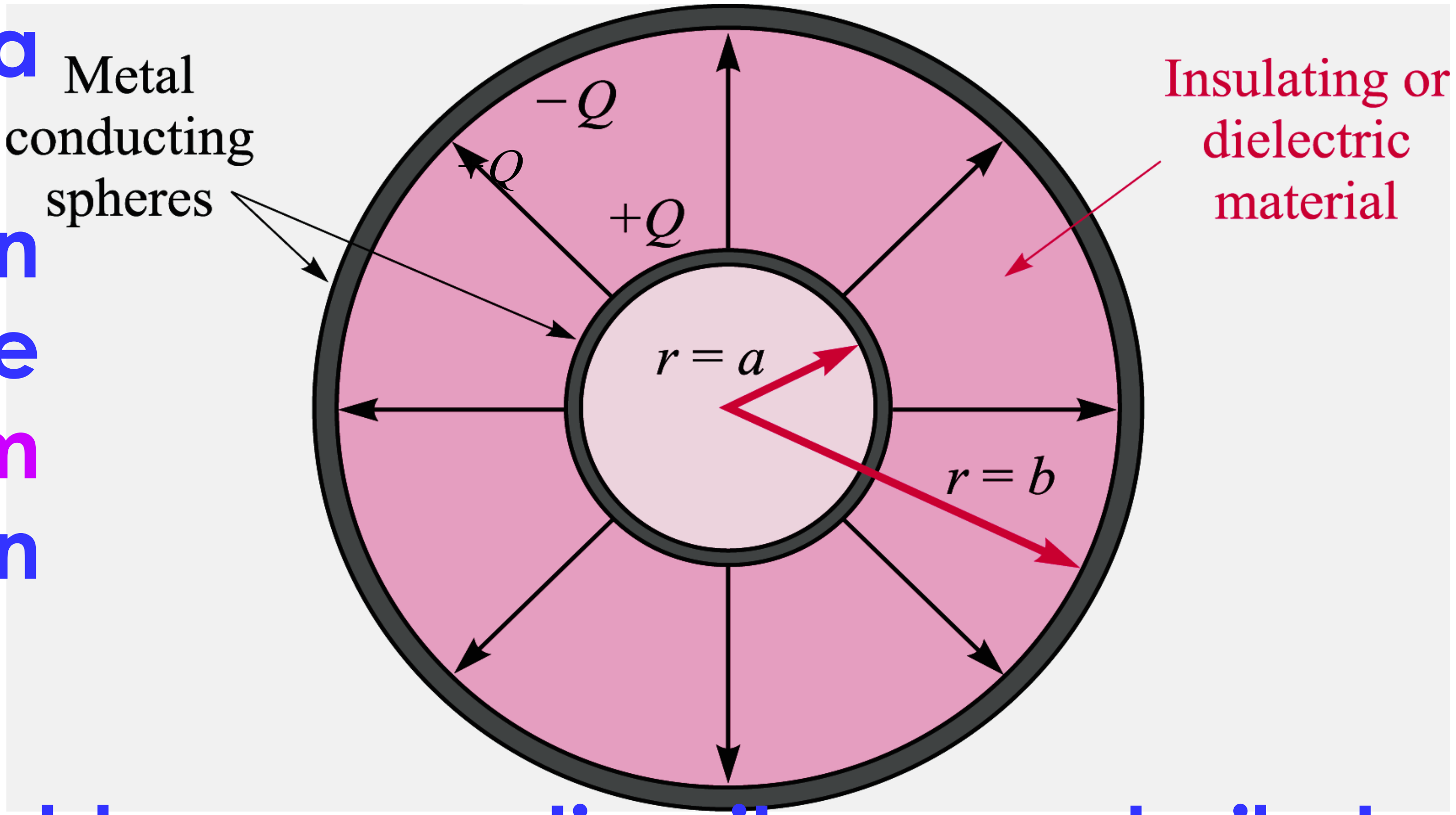


(c)

- Fig. (a) the electric flux generates from the positive charge + Q (c).
- Fig. (b) the electric flux for - Q (c).
- Fig. 2.1 (c) extends the flux concept to two point charges: one + Q (c) and the other - Q (c).

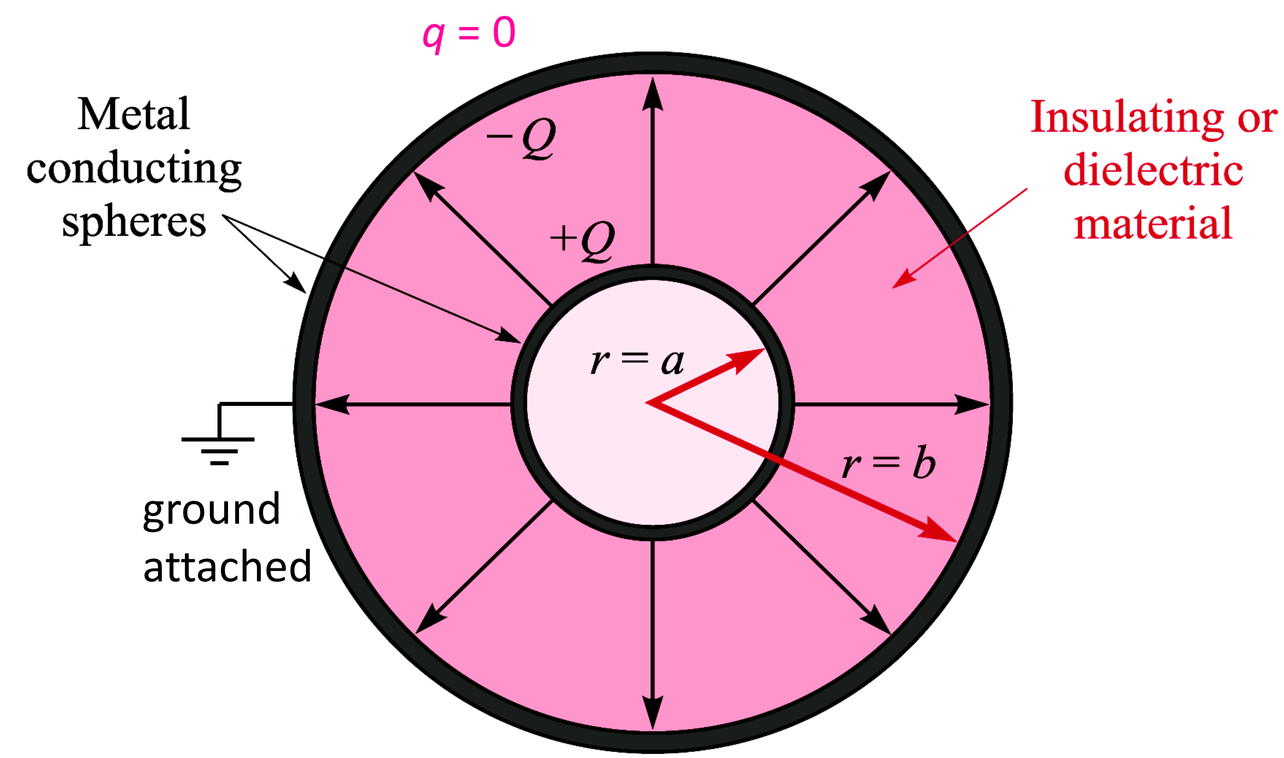
2. 3. Faraday Experiment

- Faraday started with a pair of metal spheres of different sizes; the larger one consisted of two hemispheres that could be assembled around the smaller sphere
- The inner sphere was given a known positive charge.
- The hemispheres were then clamped together around the charged sphere with about 2 cm of dielectric material between them.



- The outer sphere was discharged by connecting it momentarily to ground.
- The outer space was separated carefully, using tools made of insulating material in order not to disturb the induced charge on it, and the negative induced charge on each hemisphere was measured.

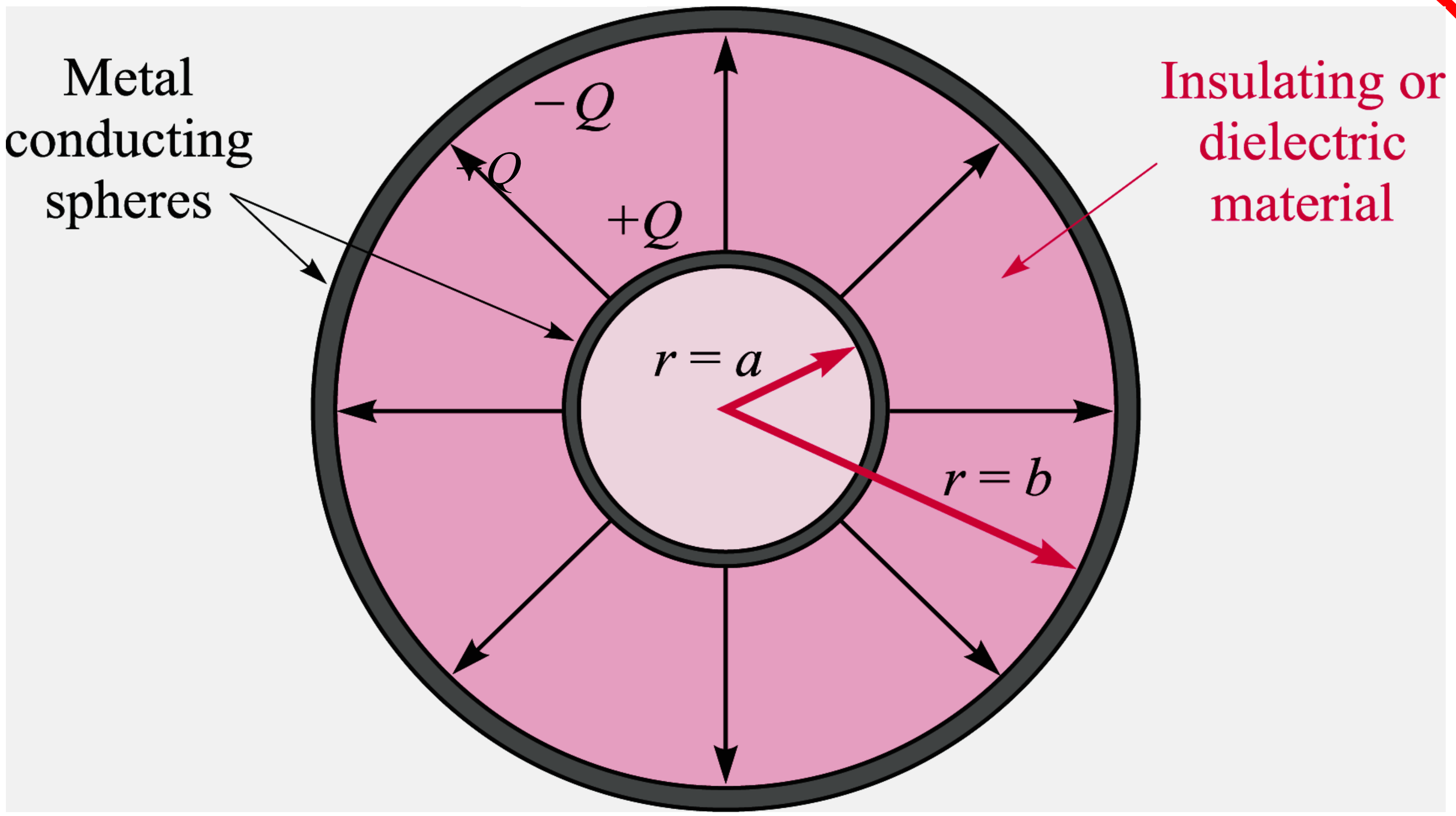
- The inner charge, **Q**, induces an equal and opposite charge, **- Q**, on the inside surface of the outer sphere, by attracting free electrons in the outer material toward the positive charge.



- This means that before the outer sphere is grounded, charge **+ Q** resides on the *outside* surface of the outer conductor.
- Attaching the ground connects the outer surface to an unlimited supply of free electrons, which then neutralize the positive charge layer. The net charge on the outer sphere is then the charge on the inner layer, or **- Q**.

- **Faraday** concluded that there occurred a charge "**displacement**" from the inner sphere to the outer sphere.
- **Displacement involves a *flow*** or ***flux***, Ψ, existing within the dielectric, and whose magnitude is equivalent to the amount of "**displaced**" charge.
- Specifically:

$$\Psi = Q \qquad C$$



- **From Faraday experiment, electric flux concept is**

1) ***The electric field intensity $\bar{E}$ is radial and outward from the positive charge (direction of the electric flux).***
2) ***The magnitude of the electric field intensity $\bar{E}$ is the same at a fixed radius (electric flux density is the same).***
3) ***The magnitude of the electric field intensity $\bar{E}$ the decreases with distance from the charge (electric flux density decreases with distance from the charge).***
4) ***The electric field intensity $\bar{E}$ is symmetrical about the $+Q\ (c)$ charge.***

- **The electric flux concept is based on the following rules:**

1) *Electric flux begins on the positive charge and terminates on negative charge.*

2) *Electric flux is in the same direction as electric field intensity $\bar{E}$.*

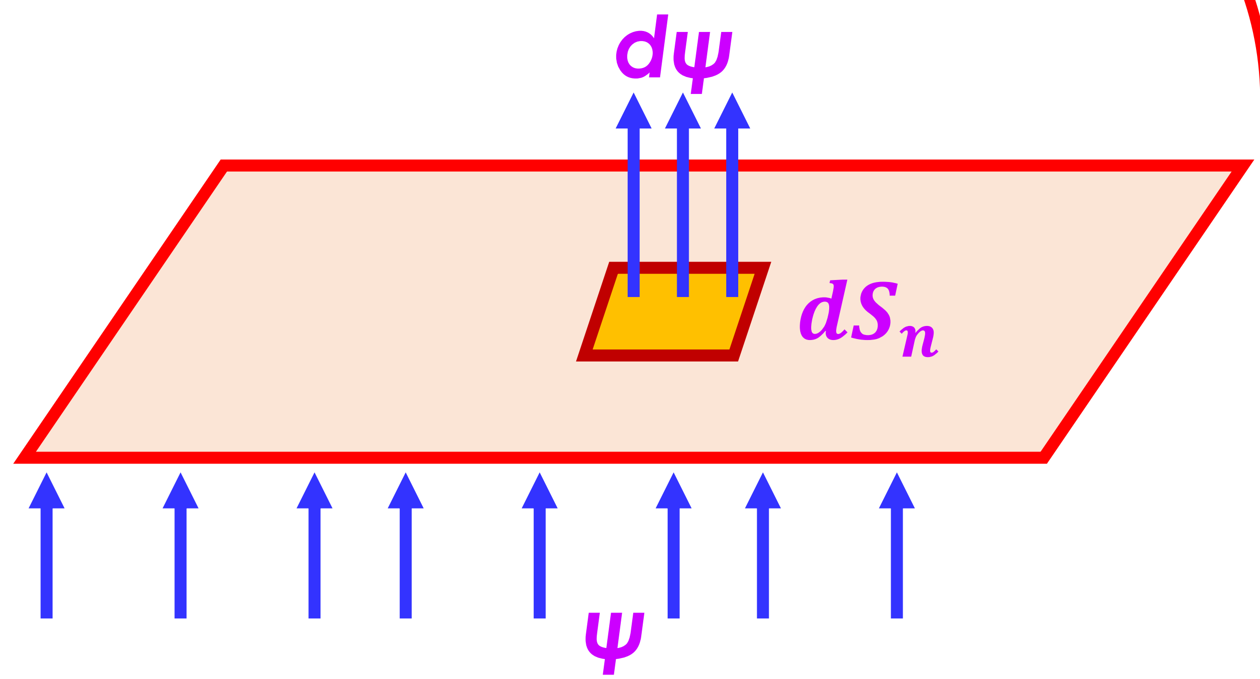
3) *Electric flux density is proportional to the magnitude of the electric field intensity $\bar{E}$.*

4) *In the SI system of units, the total flux emanating from a charge of Q (c) is $Q\,(c)$. A single line will emanate from $1\,C$ of charge.*

$$\Psi = Q \quad C$$

2. 4. Electric flux density ($\bar{D}$)

- In free space, the electric flux density vector $\bar{D}$ is defined as the number of electric flux lines per unit normal area and in the same direction of the electric flux lines (same as $\bar{E}$).

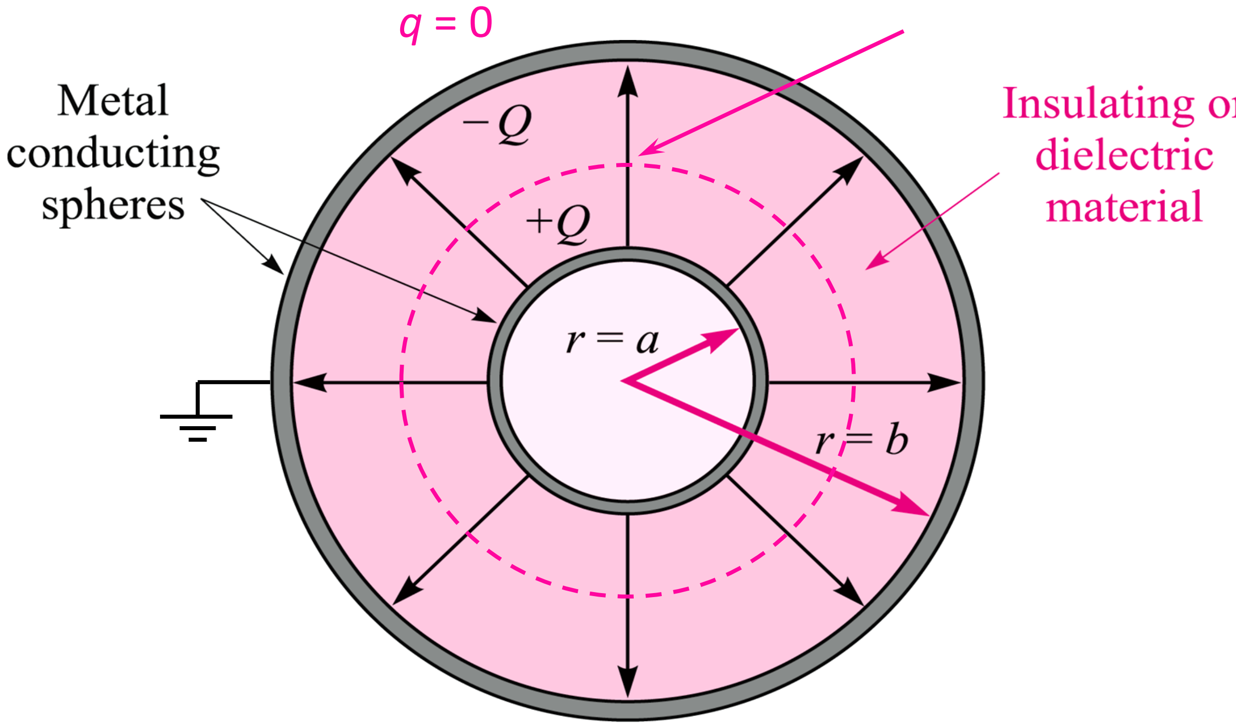


- The magnitude of the electric flux density is given by:

$$|\bar{D}| = \lim_{\Delta S_n \to 0} \left[\frac{\Delta \psi}{\Delta S_n} \right] \qquad (Lines/m^2 \;\; or \;\; C/m^2)$$

- Where $\Delta \psi$ equals the number of electric flux lines that are perpendicular to the surface ΔS_n as shown in Fig

- **Electric flux density, measured in coulombs per square meter (sometimes described as "*lines per square meter*" for each line is due to one coulomb), it is, also, called *displacement flux density* or *flux density*.**



- **Referring to Fig., the electric flux density is in the radial direction and has a value of:**

$$\bar{D}\Big|_a = \frac{Q}{4\,\pi\,a^2}\,\bar{a}_r\ (\,Inner\ Sphere) \qquad \bar{D}\Big|_b = \frac{Q}{4\,\pi\,b^2}\,\bar{a}_r\ (\,Outer\ Sphere)$$

- **and at a radial distance r, where $a \le r \le b$,**

$$\bar{D} = \frac{Q}{4\,\pi\,r^2}\,\bar{a}_r$$

Point Charge Fields

- If we now let the inner sphere become smaller and smaller, while still retaining a charge of Q, it becomes point charge in the limit, but the electric flux density $\overline{D}$ at a point r meters from the point charge is still given by:

$$\overline{D} = \frac{Q}{4\,\pi\,r^2}\,\overline{a}_r \qquad \frac{C}{m^2}$$

- The electric field intensity $\overline{E}$ will be

$$\overline{E} = \frac{Q}{4\,\pi\,\varepsilon_o\,r^2}\,\overline{a}_r \qquad \frac{V}{m}$$

- From the above equations, it can be concluded that; for free space we have:

$$\overline{D} = \varepsilon_o\,\overline{E} \qquad C/m^2$$

Finding E and D from Charge Distributions

- We learned in Chapter 1 that:

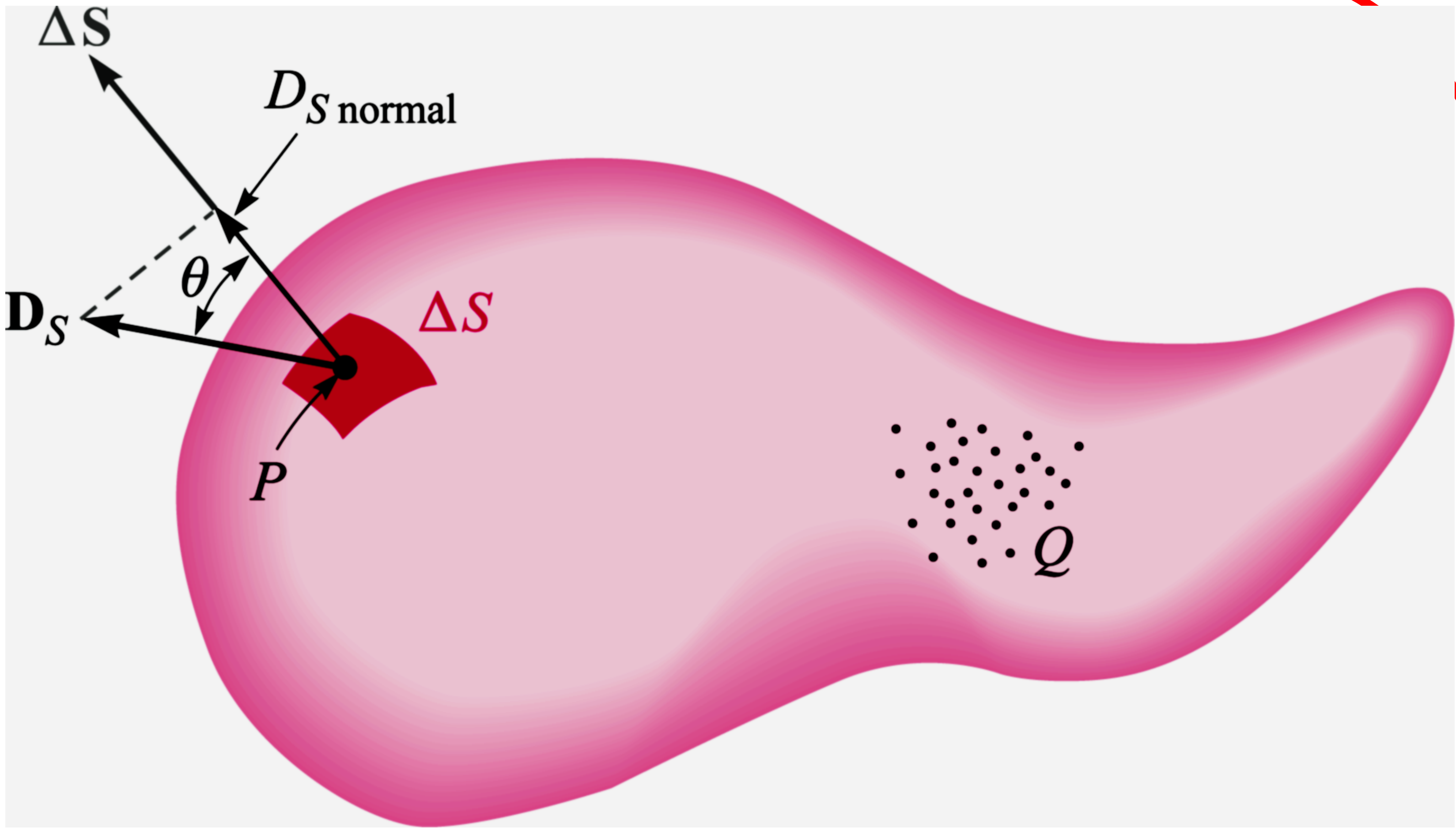
$$\overline{E} = \int_v \frac{\rho_v \, dv}{4\,\pi\,\varepsilon_o\,R^2}\,\overline{a}_R \qquad \textit{In Free Space}$$

- It now follows that:

$$\overline{D} = \int_v \frac{\rho_v \, dv}{4\,\pi\,R^2}\,\overline{a}_R \qquad \textit{In Free Space}$$

2. 5. Gauss's law

➤ Gauss's law states that: *'The electric flux passing through any closed surface is equal to the total charge enclosed by that surface'.*



➤ The flux crossing ΔS is then the product of the normal component of ΔS and $\bar{D}$

$$\Delta\psi = \text{Flux Crossing } \Delta S = D_{S\,norm}\Delta S = D_S \cos\theta\, \Delta S = \bar{D} \cdot d\bar{S}$$

➤ The total flux passing through the closed surface is obtained by adding the differential contributions crossing each surface element ΔS is given by:

$$\psi = \int d\,\psi = \oint_{\substack{\text{Closed}\\ \text{Surface}}} \bar{D} \cdot d\bar{S}$$

where $d\bar{S}$ is the differential surface element (it is called the surface vector).

Mathematical Statement of Gauss' Law

$$\psi = \oint_{S} \bar{D} \cdot d\bar{S} = Q = \text{Charge Enclosed}$$

- The charge enclosed might be several point charges, in which case:

$$Q = \sum Q_n$$

- For a line charge:

$$Q = \int_L \rho_L \, dL$$

- For a surface charge:

$$Q = \int_S \rho_S \, dS \quad (not\ necessarily\ a\ closed\ surface)$$

- For a volume charge distribution,

$$Q = \int_{Vol} \rho_V \, dV$$

- Gauss's law may be written in terms of the charge distribution as:

$$\oint_S \bar{D} \cdot d\bar{S} = \int_{Vol} \rho_V \, dV$$

Example (2-1)

Let $\bar{D} = \frac{r}{3}\,\bar{a}_r \quad nC/m^2$ in free space.

a) *Find* $\bar{E}$ at $r = 0.2\,m$.

b) *Find* the total charge within the sphere $r = 0.2\,m$.

c) *Find* the total flux leaving the sphere $r = 0.3\,m$.

Solution

a) Since the electric field intensity is

$$\bar{E} = \frac{\bar{D}}{\varepsilon_o} = \frac{r}{3\,\varepsilon_o}\,\bar{a}_r$$

Then at $r = 0.2\,m$, the electric field intensity $\bar{E}$ will be

$$\bar{E} = \frac{0.2 \times 10^{-9}}{3 \times 8.854 \times 10^{-12}}\,\bar{a}_r = 7.53\,\bar{a}_r \quad V/m$$

b) The total charge with the sphere $r = 0.2\ m$ is:

$$Q = \oint_S \bar{D} \cdot d\bar{S}$$

$$= \int_0^{2\pi} \int_0^{\pi} \frac{0.2 \times 10^{-9}}{3} \bar{a}_r \cdot (0.2)^2 \sin\theta \; d\theta \, d\varphi \, \bar{a}_r = 33.5 \;\; pC$$

c) The total flux leaving the sphere r = 0.3 m is

$$\psi = \oint_S \bar{D} \cdot d\bar{S} = \int_0^{2\pi} \int_0^{\pi} \frac{0.2 \times 10^{-9}}{3} \bar{a}_r \cdot (0.3)^2 \sin\theta \; d\theta \, d\varphi \, \bar{a}_r$$

$$= 113.1 \;\; pC$$

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Some symmetrical charge distribution

- Let us now consider how we may use Gauss's law:

$$Q = \oint_S \bar{D} \cdot d\bar{S}$$

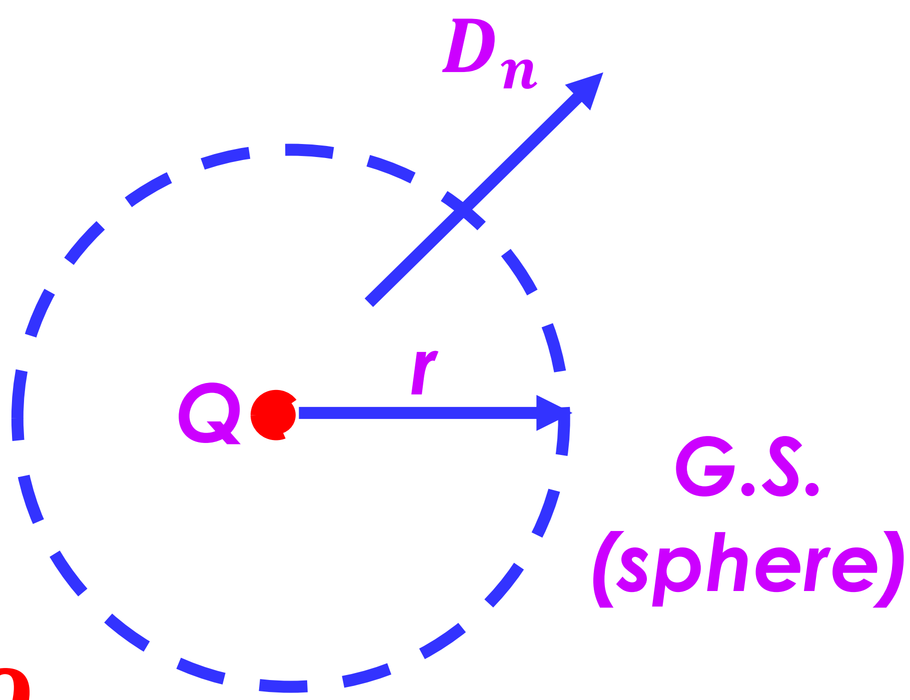
- Is used to determine $\bar{D}$ if the charge distribution is known.
- The solution is easy if we are able to choose a closed surface which satisfies two conditions *(special Gaussian surface)*:
  a) The surface is closed.
  b) Electric flux density $\bar{D}$ is everywhere either normal or tangential to the closed surface, so that $\bar{D} \cdot d\bar{S}$ becomes either $D_n \, dS_n$ or zero, respectively.
  c) On the portion of the closed surface for which $\bar{D} \cdot d\bar{S}$ is not zero, D_n is constant.

- If these three conditions are satisfied at the same time, then:

$$\oint_S \bar{D} \cdot d\bar{S} = D_n \times Area\ of\ Gaussian\ Surface.$$

Point charge

- For the point charge, the *Gaussian surface* is a sphere, the point charge is the center of the sphere. For this case:



$$\oint_S \bar{D} \cdot d\bar{S} = D_r \times 4\,\pi\, r^2 \qquad Q_{en} = Q$$

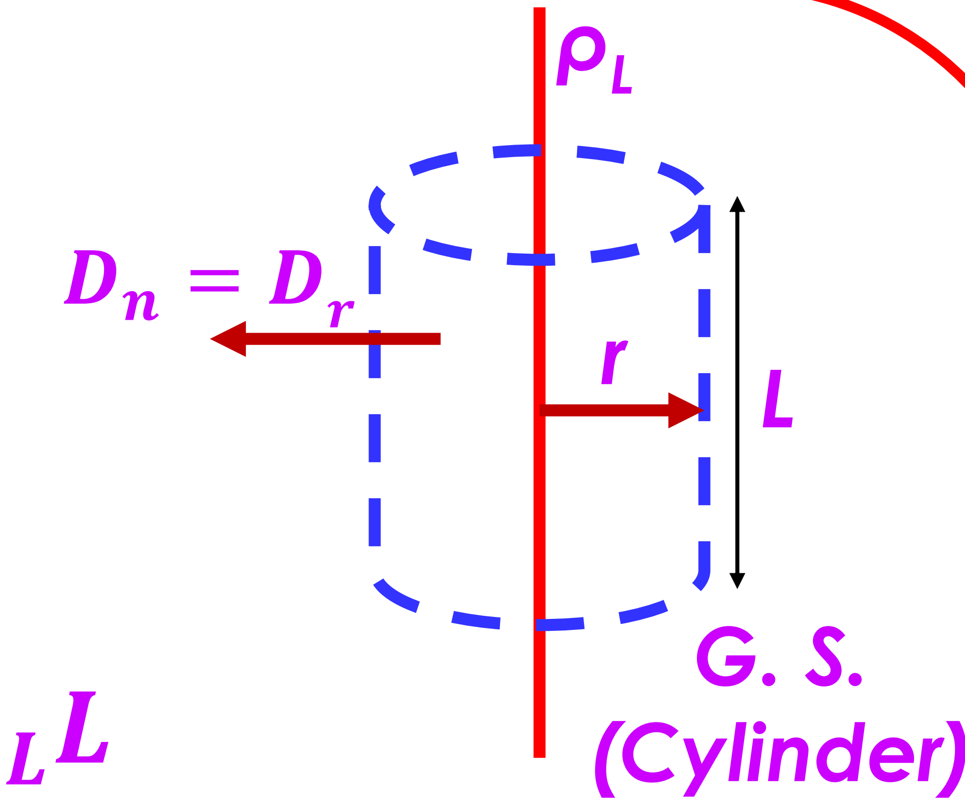
- Then:

$$D_r \times 4\,\pi\, r^2 = \mathrm{Q}$$

- Or:

$$D_r = \frac{\mathrm{Q}}{4\,\pi\, r^2} \quad or \quad \bar{D} = \frac{\mathrm{Q}}{4\,\pi\, r^2}\,\bar{a}_r \quad \frac{C}{m^2}$$

Line charge



- For the infinite line charge with uniform line charge density ρ_L, the *Gaussian surface* is a cylinder; the line charge is the axis of the cylinder.
- For this case: $D_n = D_r$, $and\ Q_{en} = \rho_L L$

$$Area\ of\ Gaussian\ Surface = 2\,\pi\,r\,L$$

$$\oint_S \bar{D} \cdot d\bar{S} = D_n \times 2\,\pi\,r\,L = D_r \times 2\,\pi\,r\,L$$

- Then:

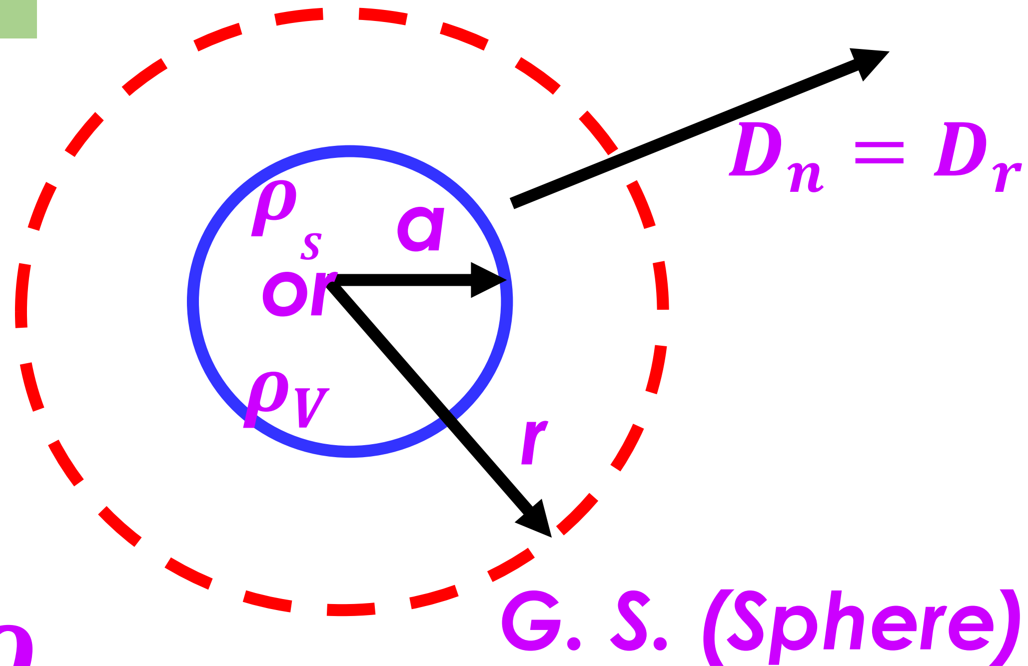
$$D_r \times 2\,\pi\,r\,L = \rho_L L$$

- Or:

$$D_r = \frac{\rho_L}{2\,\pi\,r} \qquad or \qquad \bar{D} = \frac{\rho_L}{2\,\pi\,r}\,\bar{a}_r \qquad \frac{C}{m^2}$$

Spherical surface of charge

- For the spherical surface or volume charge density (ρ_S or ρ_S either constant or function in r only), the *Gaussian surface* is a sphere.



- For this case:

$$D_n = D_r, \, and \; Q_{en} = Q$$

$$Area\, of\, Gaussian\, Surface = 4\,\pi\, r^2$$

$$\oint_S \bar{D} \cdot d\bar{S} = D_n \times 4\,\pi\, r^2 = D_r \times 4\,\pi\, r^2$$

- Then:

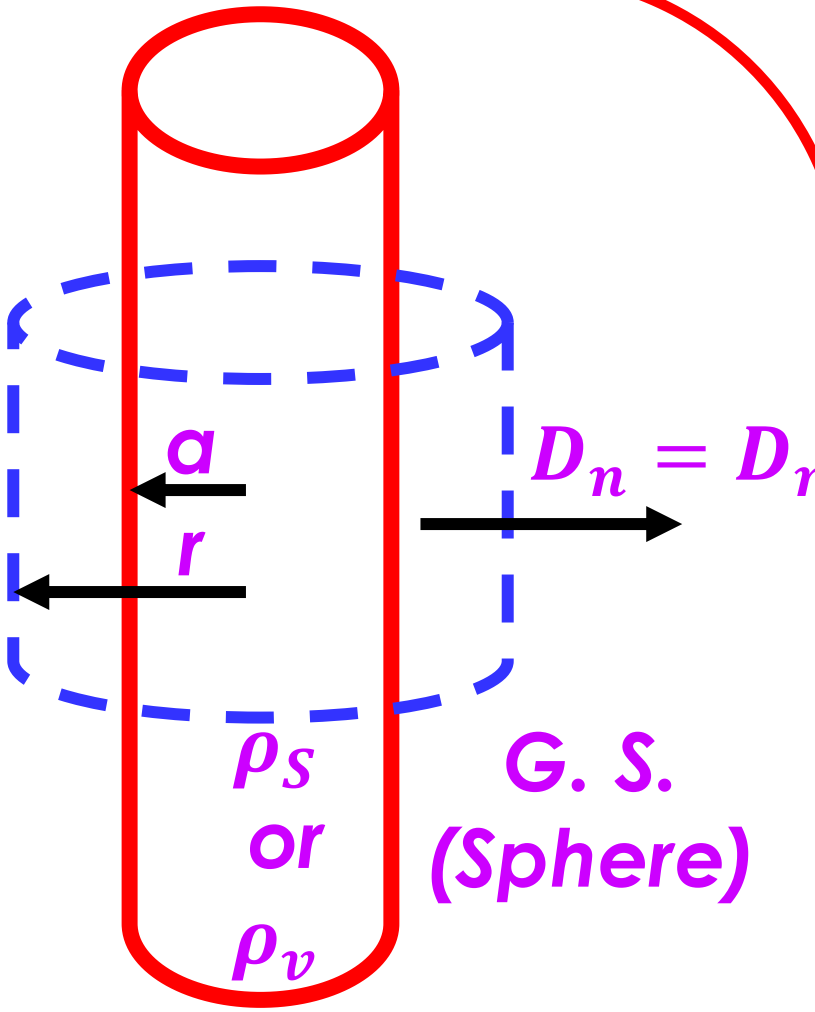
$$D_r \times 4\,\pi\, r^2 = Q$$

- Or:

$$D_r = \frac{Q}{4\,\pi\, r^2} \quad or \quad \bar{D} = \frac{Q}{4\,\pi\, r^2}\,\bar{a}_r \quad \frac{C}{m^2}$$

Cylindrical surface of charge

- For the infinite cylindrical surface or volume charge density (ρ_S or ρ_v either constant or function in r only), the *Gaussian surface* is a cylinder.



- For this case:

$$D_n = D_r, and\ Q_{en} = Q$$

$$Area\ of\ Gaussian\ Surface = 2\,\pi\,r\,L$$

- Then:

$$\oint_S \bar{D}\cdot d\bar{S} = D_n \times 2\,\pi\,r\,L = D_r \times 2\,\pi\,r\,L$$

$$D_r \times 2\,\pi\,r\,L = Q$$

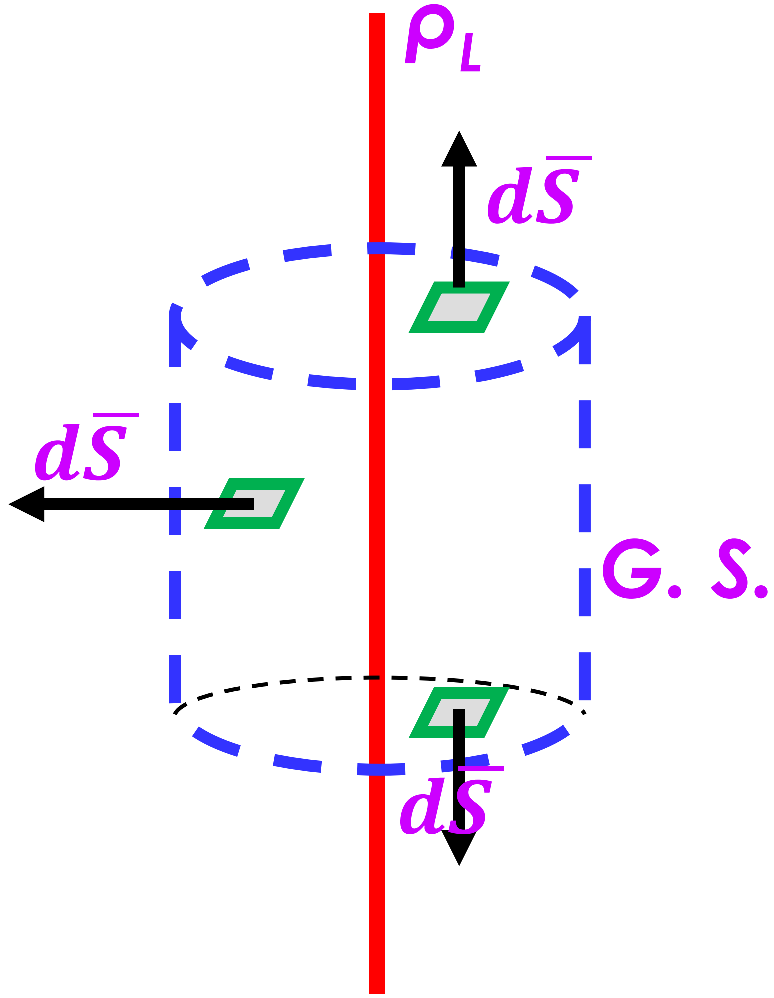
- Or:

$$D_r = \frac{Q}{2\,\pi\,r\,L} \quad or \quad \bar{D} = \frac{Q}{2\,\pi\,r\,L}\,\bar{a}_r \quad \frac{C}{m^2}$$

Example (2-2)

Calculate $\bar{D}$ and $\bar{E}$ at a distance r due to a uniformly line charge distribution of ρ_L and infinite length

Solution



- Since the uniform line charge has only a radial component of $\bar{D}$, or $\bar{D} = D_r\, a_r$
- The choice of a closed surface is simple, a cylindrical surface of radius *r* and length *L* is chosen in which D_r is everywhere normal to the sides of the cylinder.
- From *Gauss's law*:

$$Q = \oint_S \bar{D} \cdot d\bar{S} = \int_{Sides} \bar{D} \cdot d\bar{S} + \int_{Top} \bar{D} \cdot d\bar{S} + \int_{Bottoms} \bar{D} \cdot d\bar{S}$$

- The integral on the top or the bottom equal zero since $\bar{D}$ and $d\bar{S}$ are normal.

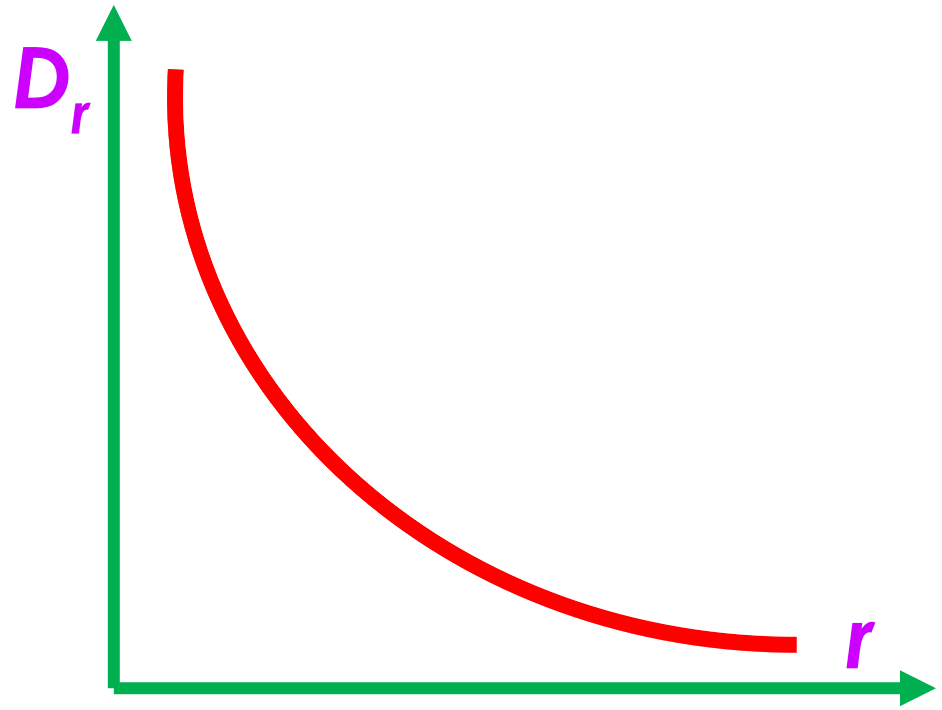
$$Q = \int_{Sides} \bar{D} \cdot d\bar{S} = D_r \int_0^L \int_0^{2\pi} r \, d\varphi \, dz = D_r \times 2\pi r L$$

- and $Q = \rho_L L$

- Therefore,

$$\bar{D} = \frac{\rho_L}{2\pi r} \bar{a}_r \quad C/m^2 \qquad and \quad \bar{E} = \frac{\bar{D}}{\varepsilon_o} = \frac{\rho_L}{2\pi \varepsilon_o r} \bar{a}_r \quad V/m$$

❖ The variation of D_r versus r for infinite line charge with uniform ρ_L will be as shown

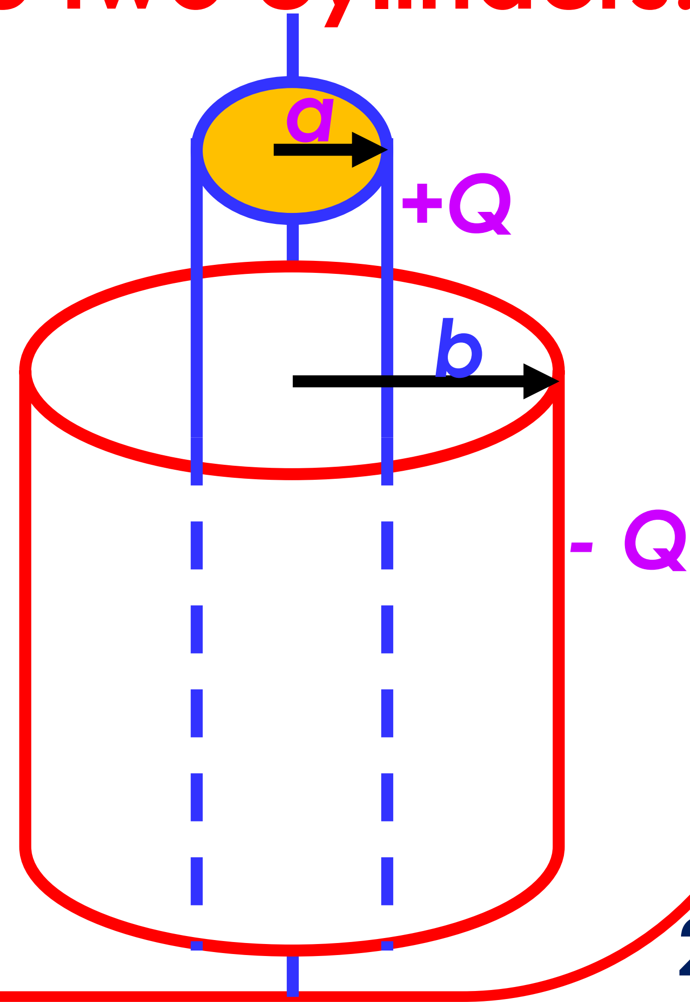


Example (2-3)

Two concentric cylindrical conductors of radius $a = 0.01\,m$ and $b = 0.08\,m$. The inner cylinder has a charge density $\rho_{S\,a} = 40\ pC/m^2$ while the outer cylinder has $\rho_{S\,b}$ such that $\bar{D}\ and\ \bar{E}$ field exists between the two cylinders but they are zero elsewhere.

a) Find $\rho_{S\,b}$.

b) Drive an expression for $\bar{D}\ and\ \bar{E}$ between the two cylinders.



Solution

a) The density $\rho_{S\,b}$ can be found as following: since $\bar{D}$ and $\bar{E}$ are equal zero in the region $r > b$, then:

$$Q_{en} = Q_a - Q_b = 0 \qquad \text{or} \qquad Q_a = -Q_b$$

$$\rho_{S\,a}\, 2\,\pi\, a\, L = -\rho_{S\,b}\, 2\,\pi\, b\, L$$

- Then,

$$\rho_{S\,b} = -\rho_{S\,b}\left(\frac{a}{b}\right) = -40 \times 10^{-12} \times \frac{0.01}{0.08} = -5\,pC/m^2$$

b) From symmetry, the field between the two cylinders must be radial and a function of r only. The application of *Gauss's law* results:

$$\oint_S \bar{D} \cdot d\bar{S} = Q_{en} \quad or \quad D_r \times 2\,\pi\,r\,L = Q_{en}$$

<u>*0 < r < a*</u>

$$Q_{en} = 0$$ (There is no charge inside the inner conductor)

Therefore, $\bar{D} = 0 \; and \; \bar{E} = 0$

<u>*a < r < b*</u>

$$Q = \int_0^L \int_0^{2\pi} \rho_{S\,a}\, a\, d\varphi\, dz = 2\,\pi\,a\,L\,\rho_{S\,a}$$

- Therefore

$$D_r = \frac{\rho_S\, a}{r} = \frac{40 \times 10^{-12} \times 0.01}{r} = \frac{4 \times 10^{-13}}{r} \; C/m^2$$

- and

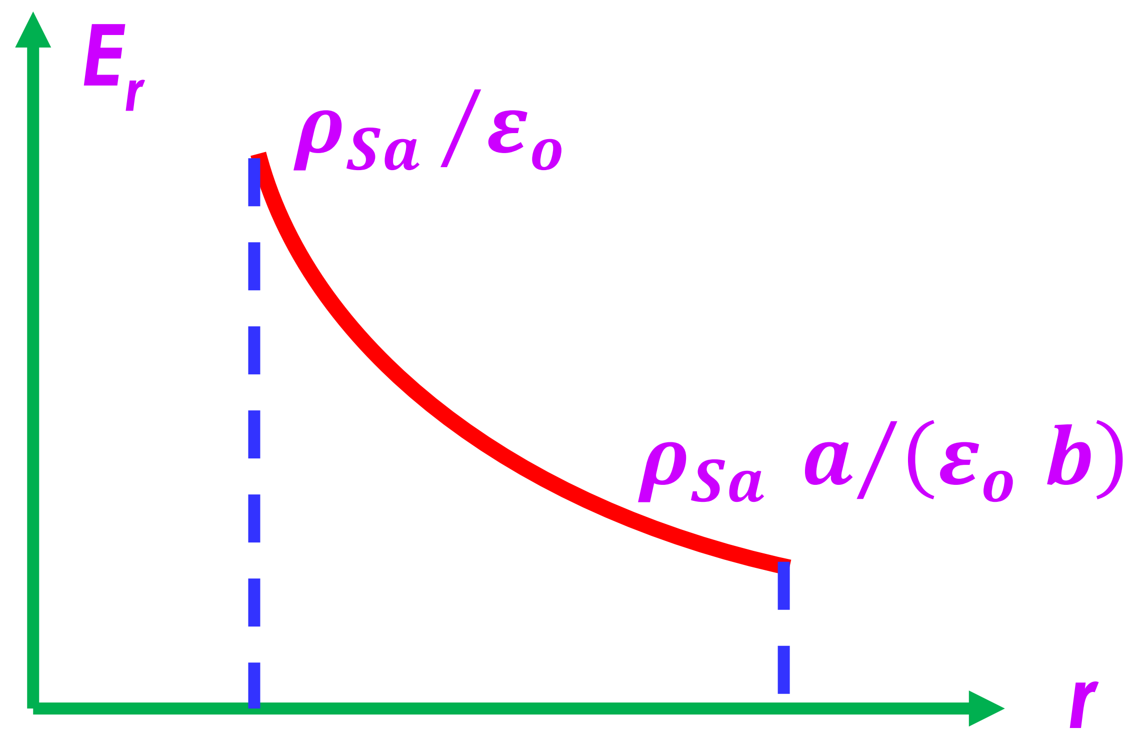
$$E_r = \frac{\rho_S\, a}{\varepsilon_o r} = \frac{4.52\ \times 10^{-2}}{r}\ V/m^2$$

<u>$b < r < \infty$</u>

$$Q_{en} = Q - Q = 0$$

Therefore, $\overline{D} = 0\ and\ \overline{E} = 0$

- The relationship between the field E_r as a function of the radial distance r is as shown in figure:



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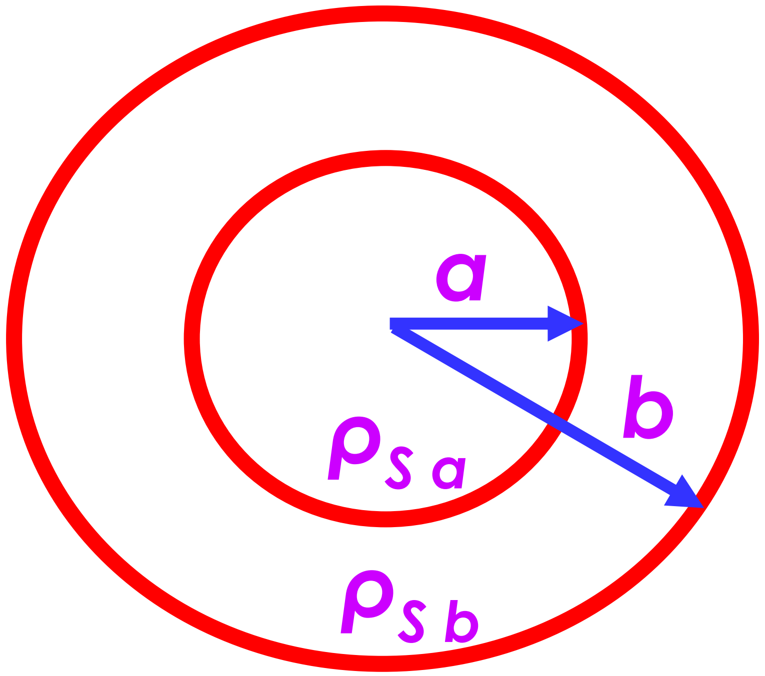
Example (2-4)

Two spherical conducting shells of radii a and b, $b > a$. Assume a surface charge density of $\rho_{S\,a}$ on the outer surface of the inner sphere. *Find*

(a) Q_a (b) $\overline{D}$ (c) Q_b (d) ρ_{sb}.

Solution

a) On the surface of the inner sphere, $r = a$ and



$$Q_a = \int_S \rho_{s\,a}\, dS$$

$$= \rho_{s\,a} \int_0^{2\pi} \int_0^{\pi} a^2 \sin\theta \, d\theta \, d\varphi = 4\pi a^2 \rho_{s\,a} \;\; C$$

b) Use *Gauss's theorem*, from spherical symmetry, only the $r-$ component of $\bar{D}$ exists, therefore,

$$\oint_S \bar{D}\cdot d\bar{S} = Q_{en} \quad or \quad D_r \times 4\,\pi\, r^2 = \rho_{S\,a}\, 4\,\pi\, a^2 \quad and \quad D_r = \left(\frac{a}{r}\right)^2 \rho_{S\,a}$$

c) For spherical capacitor, Q_b must be equal and opposite to Q_a,

$$Q_b = -Q_a = -\rho_{S\,a}\, 4\,\pi\, a^2 \qquad C$$

d) The charge density on the spherical shell b is then

$$Q_b = \rho_{S\,b}\, 4\,\pi\, b^2 = -\rho_{S\,a}\, 4\,\pi\, ab^2$$

- or

$$\rho_{S\,b} = -\left(\frac{a}{b}\right)^2 \rho_{S\,a}$$

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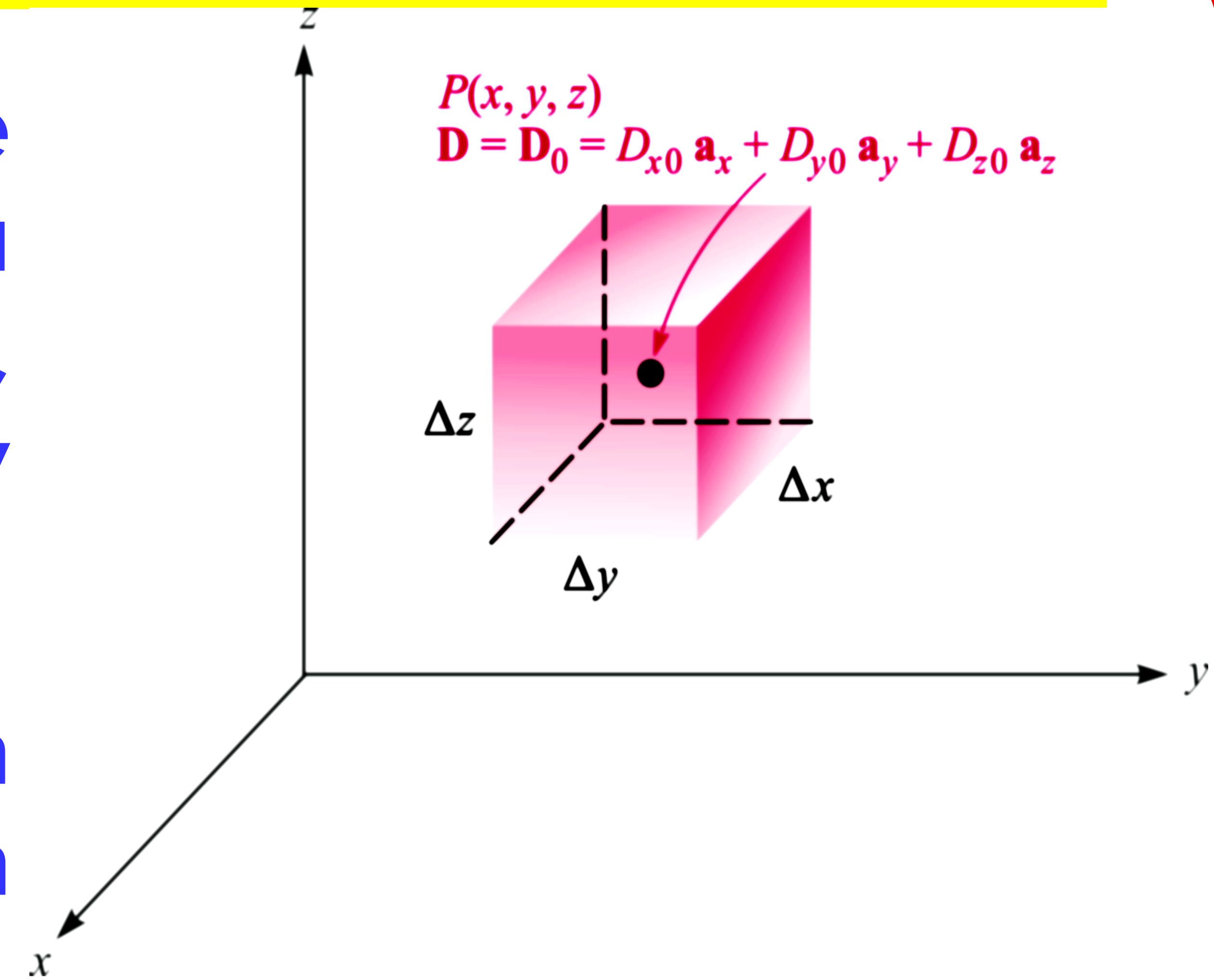
2. 5. Electric Flux Within a Differential Volume Element

- We are now going to apply the methods of *Gauss's law* to a slightly different type of problem, one that does not posses any symmetry at all.

- Let the point $P(x_o, y_o, z_o)$, shown in Fig., located by a Cartesian coordinate system.
- The value of $\bar{D}$ at any point P may be expressed in Cartesian components as follows:



$$\bar{D} = \bar{D}_o = D_{xo}\,\bar{a}_x + D_{yo}\,\bar{a}_{yx} + D_{zo}\,\bar{a}_z$$

- We choose the closed surface as a small rectangular box, centered at P, having sides of Δx, Δy and Δz as shown in Fig. Applying *Gauss's law*

$$\oint_S \bar{D} \cdot d\bar{S} = Q_{en}$$

- In order to evaluate the integral over the closed surface, the integral must be broken up into six integrals, one over each face,

$$\oint_S \bar{D} \cdot d\bar{S} = \int_{Front} + \int_{Back} + \int_{Left} + \int_{Right} + \int_{Top} + \int_{Bottom}$$

- Taking the front surface, for example, we have:

$$\int_{Front} = \bar{D}_{Front} \cdot \Delta\bar{S}_{Front} = \bar{D}_{Front} \cdot \Delta y\, \Delta z\, \bar{a}_x = D_{x,Front}\, \Delta y\, \Delta z$$

- The front face is at a distance of $\Delta x/2$ from P, and hence:

$$D_{x,Front} = D_{x\,o} + \frac{\Delta\, x}{2} \times Rate\ of\ change\ of\ D_x\ with\ x = D_{x\,o} + \frac{\Delta\, x}{2} \times \frac{\partial D_x}{\partial x}$$

- We have now
$$\int_{Front} = \left(D_{x\,o} + \frac{\Delta\, x}{2}\frac{\partial D_x}{\partial x}\right)\Delta\, y\ \Delta z$$

- Consider now the back surface,
$$\int_{Back} = \left(-D_{x\,o} + \frac{\Delta\, x}{2}\frac{\partial D_x}{\partial x}\right)\Delta\, y\ \Delta z$$

minus sign because D_{x0} is *inward* flux through the back surface

- If we combine these two integrals, we have
$$\int_{Front} + \int_{Back} = \frac{\partial D_x}{\partial x}\Delta\, x\,\Delta\, y\ \Delta z$$

- By exactly the same process we find that:

$$\int_{Right} + \int_{Left} = \frac{\partial D_y}{\partial y}\,\Delta x\,\Delta y\,\Delta z \quad and \quad \int_{Top} + \int_{Bottom} = \frac{\partial D_z}{\partial z}\,\Delta x\,\Delta y\,\Delta z$$

- **All results are assembled to yield:**

$$\oint_S \bar{D}\cdot d\bar{S} = \left(\frac{\partial\, D_x}{\partial x} + \frac{\partial\, D_y}{\partial y} + \frac{\partial\, D_z}{\partial z}\right)\Delta x\,\Delta y\,\Delta z$$

- **or**

$$\oint_S \bar{D}\cdot d\bar{S} = \left(\frac{\partial\, D_x}{\partial x} + \frac{\partial\, D_y}{\partial y} + \frac{\partial\, D_z}{\partial z}\right)\Delta v = Q$$

where Q is the charge enclosed within volume Δv.

2. 6. Divergence

- To obtain the exact expression, the differential volume element Δv has to go to zero,

$$\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \lim_{\Delta v \to 0} \frac{\oint_S \bar{D} \cdot d\bar{S}}{\Delta v} = \lim_{\Delta v \to 0} \frac{Q}{\Delta v} = \rho_v$$

- This equation contains too much information to discuss all at once, and we shall write it as two separate equations:

$$\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \lim_{\Delta v \to 0} \frac{\oint_S \bar{D} \cdot d\bar{S}}{\Delta v}$$

and

$$\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \rho_v$$

- any vector $\bar{A}$ to find $\oint_S \bar{A} \cdot d\bar{S}$ for a small closed surface, leading to

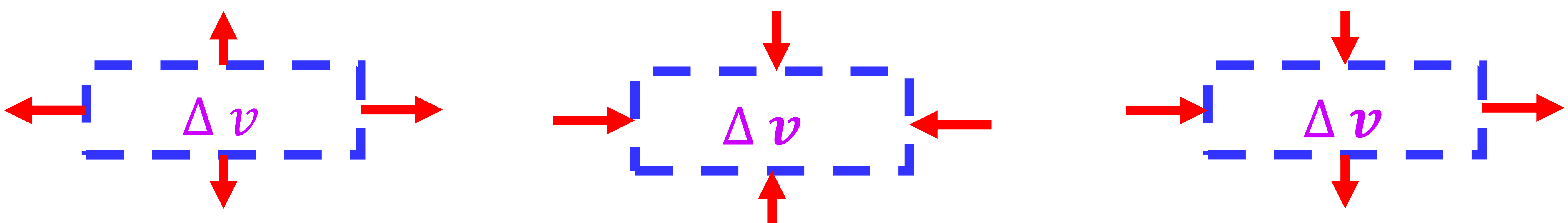
$$\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \lim_{\Delta v \to 0} \frac{\oint_S \bar{A} \cdot d\bar{S}}{\Delta v}$$

where $\bar{A}$ could represent velocity, temperature gradient, force, or any other vector field.

- The divergence of $\bar{A}$ is defined as

$$Divergence\ of\ \bar{A} = Div\ \bar{A} = \lim_{\Delta v \to 0} \frac{\oint_S \bar{A} \cdot d\bar{S}}{\Delta v}$$

- The definition of the divergence is ***"The divergence of the vector flux density $\bar{A}$ is the outflow of flux from a small closed surface per unit volume as the volume goes to zero"***



+ve divergence (Source of flux) | ***- ve divergence (Sink of flux)*** | ***Zero divergence (Zero flux)***

Properties of the divergence:

a) The divergence tells us how flux is leaving small volume, but it does not describe its direction.

b) Divergence is a mathematical operation which performed on a vector and the result is a scalar.

c) If the divergence of $\bar{A}$ exists at a point in the space, then lines of $\bar{A}$ diverge or emanate from that point.

- **The mathematical expression of the divergence is given as:**

$$\bar{\nabla} \cdot \bar{A} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \quad (Cartesian)$$

$$\bar{\nabla} \cdot \bar{A} = \frac{1}{r}\frac{\partial (r A_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z} \quad (Cylindrical)$$

$$\bar{\nabla} \cdot \bar{A} = \frac{1}{r^2}\frac{\partial (r^2 A_r)}{\partial r} + \frac{1}{r \sin\theta}\frac{\partial (A_\theta \sin\theta)}{\partial \theta} + \frac{1}{r \sin\theta}\frac{\partial A_\varphi}{\partial \varphi} \quad (Spherical)$$

2. 7. Maxwell's First Equation

In Electrostatic Fields

- From *Gauss's law*, we have

$$\oint_S \bar{D} \cdot d\bar{S} = Q_{en}$$

- Per unit volume

$$\frac{\oint_S \bar{D} \cdot d\bar{S}}{\Delta v} = \frac{Q_{en}}{\Delta v}$$

- As the volume goes to zero

$$\lim_{\Delta v \to 0} \frac{\oint_S \bar{D} \cdot d\bar{S}}{\Delta v} = \lim_{\Delta v \to 0} \frac{Q_{en}}{\Delta v}$$

- we should see div $\bar{D}$ on the left and the volume charge density on the right,

$$Div\,\bar{D} = \rho_v$$

- **This is the first of *Maxwell's four equations* as they apply to electrostatics and steady magnetic fields.**
- **It *states that "the electric flux per unit volume leaving a small volume is exactly equal to the volume charge density there"*.**
- **This equation is called *the point form of Gauss's law*.**
- ***Gauss's law* relates the flux leaving any closed surface to the charge enclosed,**
- ***Maxwell's first equation* makes an identical statement on a per unit volume basis for a small volume, or at a point.**
- ***Maxwell's first equation* is also described as the differential equation form of the *Gauss's law*, and *Gauss's law* is recognized as the integral form of *Maxwell's first equation*.**

2. 8. The vector operator $\bar{\nabla}$

- The *del* (*nippla*) *operator* $\bar{\nabla}$ is defined as a *vector operator*,

$$\bar{\nabla} = \frac{\partial}{\partial x}\,\bar{a}_x + \frac{\partial}{\partial y}\,\bar{a}_y + \frac{\partial}{\partial z}\,\bar{a}_z$$

- Consider $\bar{\nabla} \cdot \bar{D}$, signifying

$$\bar{\nabla} \cdot \bar{D} = \left(\frac{\partial}{\partial x}\,\bar{a}_x + \frac{\partial}{\partial y}\,\bar{a}_y + \frac{\partial}{\partial z}\,\bar{a}_z\right) \cdot \left(D_x\,\bar{a}_x + D_y\,\bar{a}_y + D_z\,\bar{a}_z\right)$$

$$= \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

- This is recognized as the divergence of $\bar{D}$, so that we have:

$$Div\,\bar{D} = \bar{\nabla} \cdot \bar{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

2. 9. The Divergence Theorem

- From *Gauss's law*, we have

$$\oint_S \bar{D} \cdot d\bar{S} = Q_{en} = \int_{vol} \rho_v \, dv$$

- and then replacing $(\bar{\nabla} \cdot \bar{D}) = \rho_v$, we can get:

$$\oint_S \bar{D} \cdot d\bar{S} = Q_{en} = \int_{vol} \rho_v \, dv = \int_{vol} (\bar{\nabla} \cdot \bar{D}) \, dv$$

- The first and last expressions constitute the divergence theorem,

$$\oint_S \bar{D} \cdot d\bar{S} = \int_{vol} (\bar{\nabla} \cdot \bar{D}) \, dv$$

" The integral of the normal component of any vector field over a closed surface is equal to the integral of the divergence of this vector field throughout the volume enclosed by the closed surface ".

Example (2-5)

Given that $\bar{D} = \frac{10\, r^3}{4}\, \bar{a}_r \;\; C/m^2$ in the region $0 < r < 3\, m$ in cylindrical coordinates and $\bar{D} = \frac{810}{4r}\, \bar{a}_r \;\; C/m^2$ elsewhere. *Find* the charge density?

Solution

- For *0 < r < 3* Div $\bar{D}$ in cylindrical coordinate is:

$$Div\, \bar{D} = \frac{1}{r}\frac{\partial\, (r\, D_r\,)}{\partial\, r} + \frac{1}{r}\frac{\partial\;\, D_\varphi}{\partial\, \varphi} + \frac{\partial\;\, D_z}{\partial\, z}$$

- Since $\bar{D}$ is a function of r only (in radial direction), then

$$Div\, \bar{D} = \frac{1}{r}\frac{\partial\, (r\, D_r\,)}{\partial\, r}\frac{1}{r}\frac{\partial\, (10\, r^2/4\;\,)}{\partial\, r} = 10\, r^2 \quad C/m^3$$

- Then

$$\rho_v = 10\, r^2 \quad C/m^3$$

- For $r > 3$

$$Div\,\bar{D} = \frac{1}{r}\frac{\partial\,(810\,r\,/4\,r\ )}{\partial\,r} = 0$$

- Then

$$\rho_v = 0 \quad C/m^3$$

****

Example (2-6)

Given that $\bar{D} = 30\,e^{-r}\,\bar{a}_r - 2\,z\,\bar{a}_z \ \ C/m^2$ in cylindrical coordinates, *evaluate* both sides of the divergence theorem for the volume enclosed by $r = 2\,m$, $Z = 0$ and $z = 5\,m$.

Solution

- The *divergence theorem* is:

$$\oint_S \bar{D}\cdot d\bar{S} = \int_{vol} (\bar{\nabla}\cdot\bar{D})\,dv$$

- It is noted that $D_z = 0$ for $z = 0$ and hence $\bar{D}\cdot d\bar{S}$ is $zero$ over that part of the surface.

$$\oint_S \bar{D} \cdot d\bar{S} = \int_{side} \bar{D} \cdot d\bar{S} + \int_{top} \bar{D} \cdot d\bar{S}$$

$$= \int_0^5 \int_0^{2\pi} D_r\, \bar{a}_r \cdot 2\, d\varphi\, dz\, \bar{a}_r + \int_0^5 \int_0^{2\pi} D_z\, \bar{a}_z \cdot r\, dr\, d\varphi\, \bar{a}_z = 129.4\ C$$

- For the right hand side of the divergence theorem:

$$Div\, \bar{D} = \frac{1}{r} \frac{\partial\, (r\, D_r)}{\partial\, r} + \frac{1}{r} \frac{\partial\, D_\varphi}{\partial\, \varphi} + \frac{\partial\, D_z}{\partial\, z} = \frac{1}{r} \frac{\partial\, (30\, r\, e^{-r})}{\partial\, r} + \frac{\partial\, (-2\, z)}{\partial\, z}$$

$$= \frac{30\, e^{-r}}{r} - 30\, e^{-r} - 2$$

- and

$$\int_{vol} (\bar{\nabla} \cdot \bar{D})\, dv = \int_0^5 \int_0^{2\pi} \int_0^2 \left(\frac{30\, e^{-r}}{r} - 30\, e^{-r} - 2 \right) r\, dr\, d\varphi\, dz$$

$$= 129.4\ C$$

****

Example (2-5)

Given that $\bar{D} = \frac{5\,r^2}{4}\,\bar{a}_r\;\;C/m^2$ in spherical coordinates, *evaluate* both sides of the *divergence theorem* for the volume enclosed by $r = 1$, and $r = 2m$.

Solution

- The *divergence theorem* is:

$$\oint_S \bar{D}\cdot d\bar{S} = \int_{vol} (\bar{\nabla}\cdot\bar{D})\, dv$$

- For the right hand side of the *divergence theorem*:

$$Div\,\bar{D} = \frac{1}{r^2}\frac{\partial\left(r^2\,D_r\right)}{\partial\,r} = \frac{1}{r^2}\frac{\partial\left(5\,r^4/4\right)}{\partial\,r} = 5\,r$$

- and

$$\int_{vol} (\bar{\nabla}\cdot\bar{D})\, dv = \int_0^2\int_0^{\pi}\int_0^{2\,\pi} (5\,r)\; r^2\,sin\,\theta\,dr\,d\theta\,d\varphi$$

$$= 2\,\pi\int_0^2 \left(5\,r^3\right) dr \int_0^{\pi} sin\,\theta\;d\theta\; = 75\,\pi \quad C$$

➢ For the left hand side of the divergence theorem:

$$\oint_S \bar{D} \cdot d\bar{S} = \int_{outer} \bar{D} \cdot d\bar{S} + \int_{inner} \bar{D} \cdot d\bar{S}$$

$$= \int_0^{\pi} \int_0^{2\pi} D_r \bar{a}_r \cdot r^2 \sin\theta \, d\theta \, d\varphi \;\bar{a}_r \Big|_{r=2}$$

$$+ \int_0^{\pi} \int_0^{2\pi} D_r \bar{a}_r \cdot \left(- r^2 \sin\theta \, d\theta \, d\varphi \;\bar{a}_r\right) \Big|_{r=1}$$

$$= \int_0^{\pi} \int_0^{2\pi} \left(\frac{5}{4}\right) (2)^2 \sin\theta \, d\theta \, d\varphi - \int_0^{\pi} \int_0^{2\pi} \left(\frac{5}{4}\right) (1)^2 \sin\theta \, d\theta \, d\varphi$$

$$= 20\,(2)\,(2\pi) - \left(\frac{5}{4}\right)(2)\,(2\pi) = 75\,\pi \;\; C$$

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THE END

OF CHAPTER (2)