Electromagnetic Fields

EPM 112

Course instructor

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EPM 112

Chapter (3)

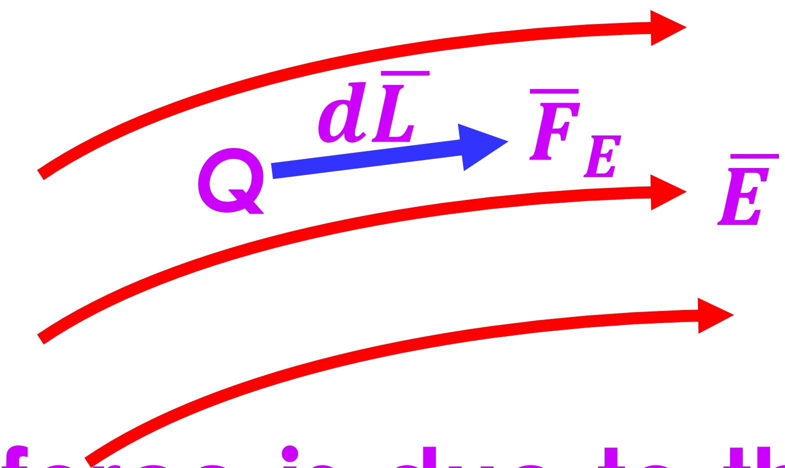
ENERGY AND ELECTRICAL POTENTIAL

3. 1. Introduction

- If an electric field is developed by moving a group of charges from infinity to their final position in space, then, owing to the mutual forces on the charges, work must be done in establishing the specified distribution of electric charges.
- Consequently, an electrostatic potential energy is associated with the charge distribution.
- If a dielectric medium is introduced into an electric field, work is done to establish the polarization, thereby changing the electrostatic potential energy of the system.

3. 2. Energy expanded in moving point charge in an electric field

- Suppose we wish to move a charge $Q\ (C)$ a distance $d\bar{L}$ in an electric field $\bar{E}$. The force on Q due to the electric field is: $\bar{F}_E = Q\,\bar{E}$ *Newton*

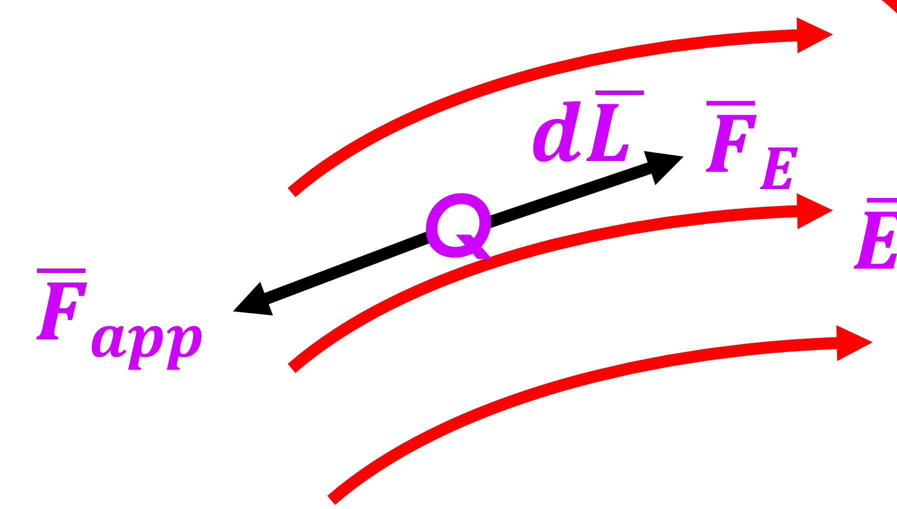


where the subscript reminds us that this force is due to the field, and the component of this force in the direction $d\bar{L}$.

- If the charge is free to move, the electric force $\bar{F}_E$ will accelerate the point charge and increase its kinetic energy.
- This means that energy is transferred from the field to the point charge
- The incremental work done on a charge by the electric field to move the charge distance $d\bar{L}$ is given by:

$$dW_E = \bar{F}_E \cdot d\bar{L} = Q\,\bar{E} \cdot d\bar{L} \quad Joul$$

- In order to move the charge in a direction opposite to the field, some external force should be applied.
- This force should be equal to and opposite in direction to the force exerted by the electric field.



$$\bar{F}_{app.} = -Q\,\bar{E} \quad Newton$$

- The differential work done by the external source for moving charge Q:

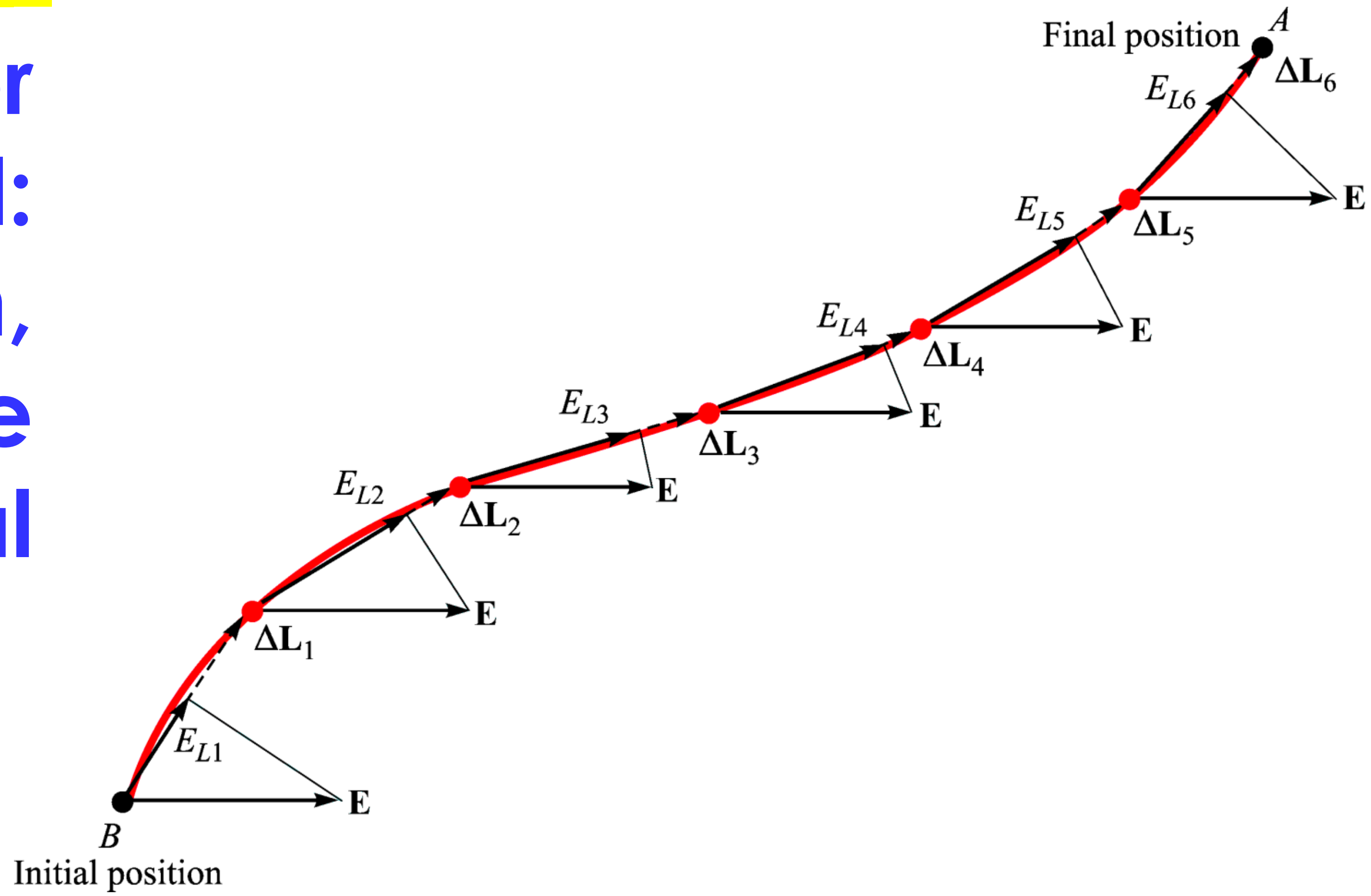
$$dW_{app} = \bar{F}_{app} \cdot d\bar{L} = -Q\,\bar{E} \cdot d\bar{L} \quad Joule$$

- The work required to move the charge a finite distance in an electric field must be determined by the integration:

$$W_{app} = -Q \int_{Intial}^{Final} \bar{E} \cdot d\bar{L} \quad Joule$$

3. 3. The Line Integral

- The integral expression for work is completely general: Any shape path may be taken, with the component of force evaluated on each differential path segment



$$W_{app} = -Q \int_{Intial}^{Final} E_L \, dL \quad Joule$$

where $E_L = Component\ \bar{E}\ of\ along\ d\bar{L}.$

- The integral expression involving the scalar product of the field with a differential path vector is called a line integral or a contour integral.

- For example, consider a uniform electric field applied along a path chosen from an initial position B to a final position A.
- The path is divided into six segments $\Delta L_1, \Delta L_2, \Delta L_3, \cdots, , \Delta L_6$ and the components of $\overline{E}$ along each segment denoted by $E_{L\,1}, E_{L\,2}, E_{L\,3}, \cdots, E_{L\,6}$.
- The work involved in moving a charge Q from B to A then approximately:

$$\Delta W_{app} = -Q\,(E_{L\,1}\Delta L_1 + E_{L\,2}\Delta L_2 + E_{L\,3}\Delta L_3 + \cdots + E_{L\,6}\Delta L_6)$$

- or using vector notation,

$$\Delta W_{app} = -Q\,(\overline{E}_1 \cdot \Delta\overline{L}_1 + \overline{E}_2 \cdot \Delta\overline{L}_2 + \overline{E}_3 \cdot \Delta\overline{L}_3 + \cdots + \overline{E}_6 \cdot \Delta\overline{L}_6)$$

- and since we have assumed a uniform field,

$$\overline{E}_1 = \overline{E}_2 = \overline{E}_3 = \cdots = \overline{E}_6 = \overline{E}$$

- Therefore:

$$\Delta W_{app} = -Q\,\bar{E} \cdot (\Delta\bar{L}_1 + \Delta\bar{L}_2 + \Delta\bar{L}_3 + \cdots + \Delta\bar{L}_6)$$

- The sum of the segments inside the parentheses is the vector directed from the initial point B to the final point A, $\bar{L}_{BA}$. Therefore: $\Delta W_{app} = -Q\,\bar{E} \cdot \Delta\bar{L}_{BA}$
- This result for the uniform can be obtained rapidly now from the integral expression:

$$W_{app} = -Q \int_{B}^{A} \bar{E} \cdot \Delta\bar{L}_{BA}$$

- As applied to the to a uniform field, yields to

$$W_{app} = -Q\,\bar{E} \cdot \int_{B}^{A} \Delta\bar{L}_{BA}$$

- where the last integral becomes L_{AB} and

$$W_{app} = -Q\,\bar{E} \cdot \bar{L}_{BA}$$

- The work is independent on the path on which the point charge can be carried in the field, as the electric field due to a static charge is conservative field,

$$\oint_L \overline{E} \cdot d\overline{L} = 0$$

- This equation is known as **Maxwell 2nd equation** and also known as **KVL** in electrostatic Field.

- The expressions for $d\overline{L}$ in the three coordinate systems are:

$$d\overline{L} = dx\,\overline{a}_x + dy\,\overline{a}_y + dz\,\overline{a}_z \qquad (Cartesian)$$

$$d\,\overline{L} = dr\,\overline{a}_r + r\,d\varphi\,\overline{a}_\varphi + dz\,\overline{a}_z \qquad (Cylinderical)$$

$$d\,\overline{L} = dr\,\overline{a}_r + r\,d\theta\,\overline{a}_\theta + r\sin\theta\,d\varphi\,\overline{a}_\varphi \qquad (Spherical)$$

Note that, for

$$W_{app} = W_{AB} = -Q \int_{B}^{A} \overline{E} \cdot d\overline{L}$$

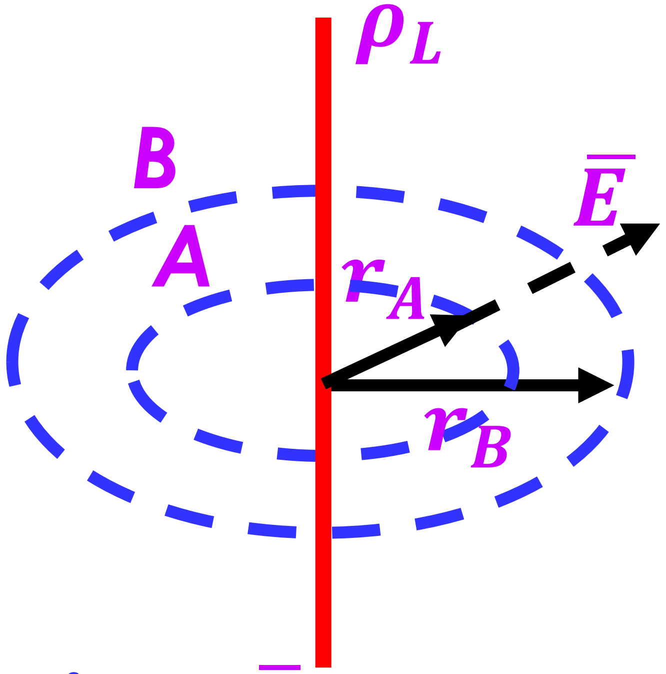
- If $W_{app} = +ve$, then,
  - The work is done by the external field.
  - The point charge is moving in the opposite direction of the field.
  - The potential difference (V_{AB}) is positive.

- If $W_{app} = -ve$, then,
  - The work is done by the field itself.
  - The point charge is moving in the same direction of the field.
  - The potential difference (V_{AB}) is nagtive.

Example (3-1)

Calculate the work done in carrying a charge of $Q\,(C)$ from r_A to r_B along a radial path due to field produced by infinite uniform line charge.

Solution



- The electric field intensity at a distance r produced by the infinite line charge with density of ρ_L is: $$\bar{E} = \frac{\rho_L}{2\,\pi\,\varepsilon_o\,r}\,\bar{a}_r$$
- The differential length in cylindrical coordinate is $d\bar{L} = dr\,\bar{a}_r$

$$W_{app} = -Q\int_{Initial}^{Final} \bar{E}\cdot d\bar{L} = -Q\int_{r_A}^{r_B} \frac{\rho_L}{2\,\pi\,\varepsilon_o\,r}\,\bar{a}_r\cdot dr\,\bar{a}_r$$

$$= -Q\int_{r_A}^{r_B} \frac{\rho_L}{2\,\pi\,\varepsilon_o\,r}\,dr = -\frac{Q\,\rho_L}{2\,\pi\,\varepsilon_o}\ln\left(\frac{r_B}{r_A}\right) \qquad Joule$$

The negative sign means that the work is done by the electric field.

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Example (3-2)

Consider the electric field given by $\overline{E} = x^2\,\overline{a}_y\ V/m$. Determine the work done by the field in carrying $3\,\mu C$ of charge from the point $A(0,0,0)$ to the point $B\,(1,1,0)$ along the straight line path.

Solution

- The work done will be:

$$W_{app} = W_{BA} = -Q\int_A^B \overline{E}\cdot d\overline{L}$$

- where $d\overline{L} = dx\,\overline{a}_x + dy\,\overline{a}_y + dz\,\overline{a}_z$

- Therefore,

$$W_{app} = W_{BA} = -\,Q\int_A^B x^2\,\overline{a}_y\cdot dy\,\overline{a}_y = -\,Q\int_A^B x^2\,dy$$

$$= -\,Q\int_A^B x^2\,dx = -3\times 10^{-6}\times\left[\frac{x^3}{3}\right]_{x=0}^{x=1} = -1\quad \mu\,Joule$$

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3. 4. The Potential Difference (V_{AB})

- "The potential difference V_{AB} between two points A and B is defined as the work in joule that must be expanded in moving a unit positive point charge from point B to point A in an electric field $\bar{E}$".

- We express this quantity in units of Joules/Coulomb, or *volts*:

$$Potential\ Difference = \frac{W_{AB}}{Q} = -\int_{B}^{A} \bar{E} \cdot d\bar{L}$$

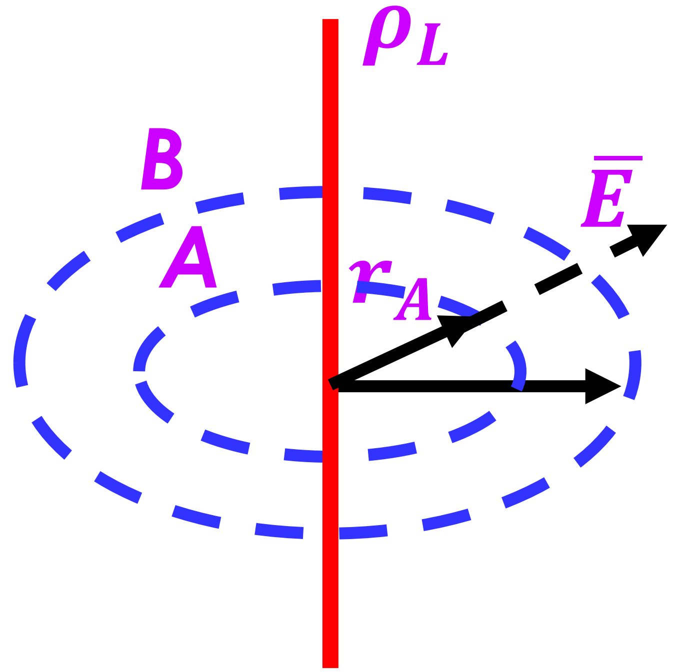
- The potential difference between *A* and *B* is:

$$V_{AB} = V_A - V_B = -\int_{B}^{A} \bar{E} \,.\, d\bar{L} \qquad Volt$$

- V_{AB} is positive if the work is done in carrying the positive charge from *B* to *A* against the direction of the field.

Example (3-3)

Calculate the potential difference between points r_B to r_A along a radial path due to the field of the infinite line charge.



Solution

- The electric field intensity at a distance r produced by the infinite line charge with density of ρ_L is: $\overline{E} = \dfrac{\rho_L}{2\,\pi\,\varepsilon_o\,r}\,\overline{a}_r$

- The differential length in cylindrical coordinate is $d\overline{L} = dr\,\overline{a}_r$

$$V_{AB} = -\int_{Initial}^{Final} \overline{E}\cdot d\overline{L} \quad = -\int_{r_B}^{r_A} \frac{\rho_L}{2\,\pi\,\varepsilon_o\,r}\,\overline{a}_r \cdot dr\,\overline{a}_r$$

$$= -\int_{r_B}^{r_A} \frac{\rho_L}{2\,\pi\,\varepsilon_o\,r}\,dr = \frac{\rho_L}{2\,\pi\,\varepsilon_o}\,ln\left(\frac{r_B}{r_A}\right) \qquad Volt$$

- If $r_B > r_A$, the potential difference V_{AB} is positive, indicating that energy is expanded by the external source in bringing the positive charge from r_B to r_A.

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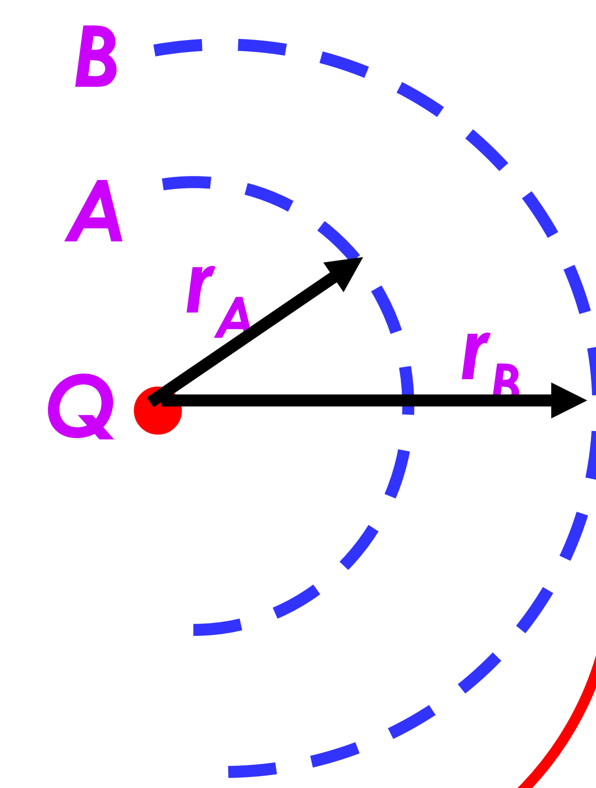
Example (3-3)

Calculate the potential difference between points r_B to r_A along a radial path due to the field of the point charge.

Solution



- The electric field intensity at a distance r produced by the point charge with density Q is:

$$\bar{E} = \frac{Q}{2\,\pi\,\varepsilon_o\,r^2}\,\bar{a}_r$$

- The differential length in spherical coordinate is:

$$d\bar{L} = dr\,\bar{a}_r$$

$$V_{AB} = -\int_{r_B}^{r_A} \bar{E} \cdot d\bar{L} = -\int_{r_B}^{r_A} \frac{Q}{4\,\pi\,\varepsilon_o\,r^2}\,\bar{a}_r \cdot dr\,\bar{a}_r$$

$$= -\int_{r_B}^{r_A} \frac{Q}{4\,\pi\,\varepsilon_o\,r^2}\,dr = \frac{Q}{4\,\pi\,\varepsilon_o}\left[\frac{1}{r_A} - \frac{1}{r_B}\right] \quad Volt$$

- If the point charge and line charge are present together, then the poetical difference (V_{AB}) is:

$$V_{AB} = V_{AB}\Big|_{Line\ Charge} + V_{AB}\Big|_{Point\ Charge}$$

$$V_{AB} = \frac{\rho_L}{2\,\pi\,\varepsilon_o}\,ln\left(\frac{r'_B}{r'_A}\right) + \frac{Q}{4\,\pi\,\varepsilon_o}\left[\frac{1}{r_A} - \frac{1}{r_B}\right] \quad Volt$$

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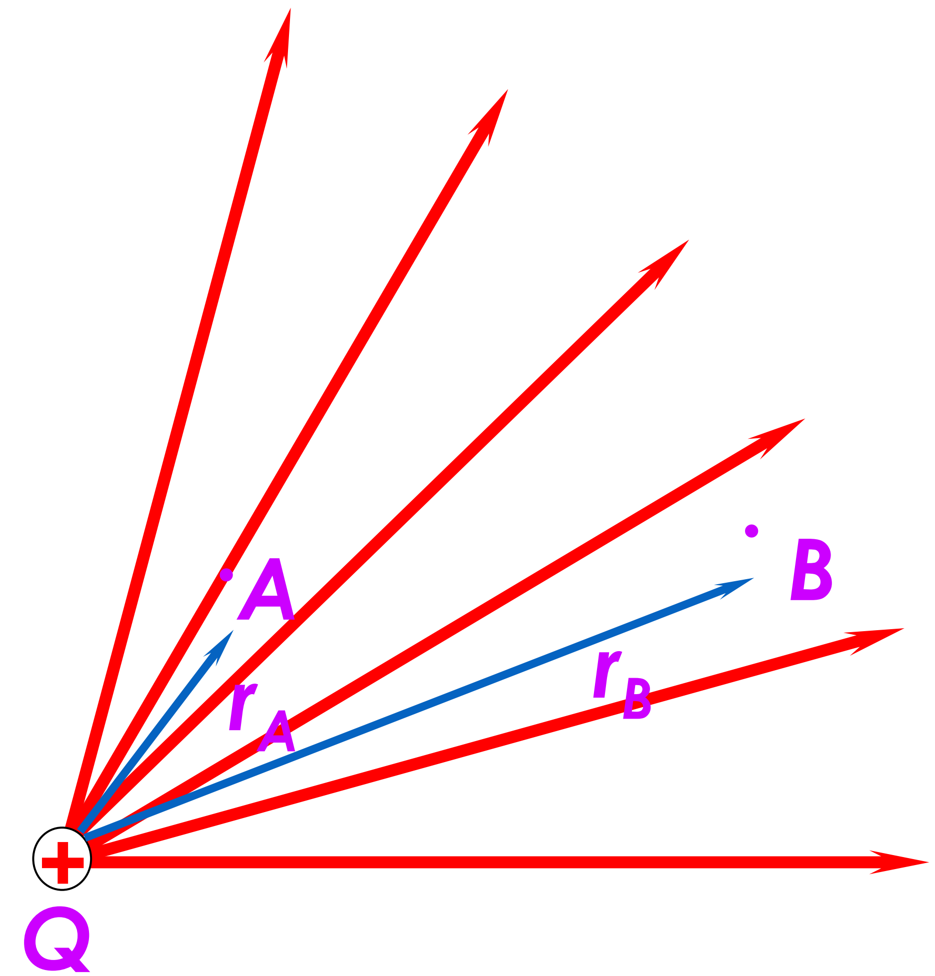
3. 5. The Absolute Potential (V_A)

- Assume a point charge $Q\ (C)$ located at the origin.
- The electric field intensity at a distance r from the origin is given by:

$$\bar{E} = \frac{Q}{4\,\pi\,\varepsilon_o\,r^2}\,\bar{a}_r \quad V/m$$

- The potential difference between two points r_A and r_B (the two points are on the radial distance) is V_{AB}. For general:



$$V_{AB} = -\int_{r_B}^{r_A} \bar{E}\cdot d\bar{L} = -\int_{r_B}^{r_A} \frac{Q}{4\,\pi\,\varepsilon_o\,r^2}\,\bar{a}_r\cdot dr\,\bar{a}_r$$

$$= -\int_{r_B}^{r_A} \frac{Q}{4\,\pi\,\varepsilon_o\,r^2}\,dr = \frac{Q}{4\,\pi\,\varepsilon_o}\left[\frac{1}{r_A} - \frac{1}{r_B}\right] \qquad Volt$$

- To define a zero reference for the potential, it is assumed $V = 0$ at infinity.
- If we let the point at $r_A = r_B$ recede to infinity, the potential at r_A becomes:

$$V_A = \frac{Q}{4\,\pi\,\varepsilon_o\,r_A}$$

- Generally, the potential at any point distance r from a charge of Q at the origin is given by:

$$\boxed{V = \frac{Q}{4\,\pi\,\varepsilon_o\,r}}$$

- The potential at infinity radius being taken at the zero reference.
- This means that the work that must be done in carrying a $1\,C$ charge from infinity to any point r from the charge Q is given by $\frac{Q}{4\,\pi\,\varepsilon_o\,r}$ Joules.

Note that

a) The *potential difference* V_{AB} *is independent of the reference point.*

b) Absolute *potential* V_A *depends on the chosen reference point, for example, let:*

$$V = \frac{Q}{4\,\pi\,\varepsilon_o\,r} + C$$

where $C = Constant$ $\quad C = V_o \; at \; r = 0$

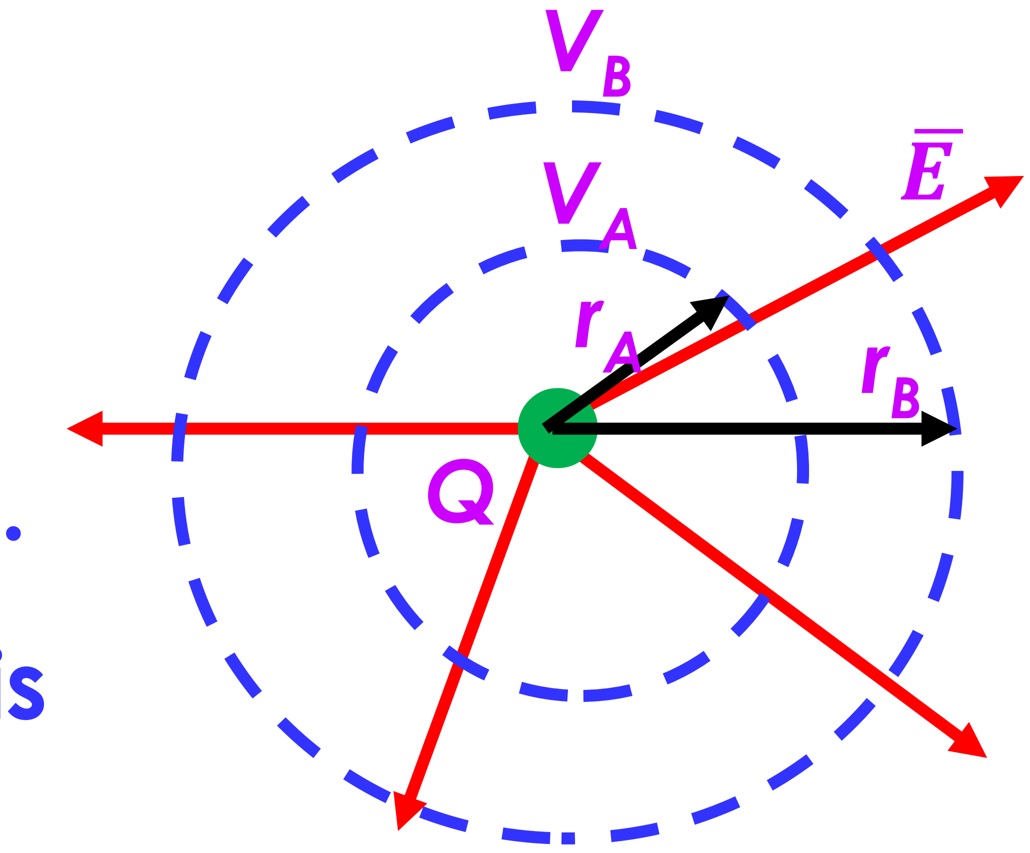
$C = 0 \quad at \; r = \infty$

c) The potential V_A or V_{AB} is a scalar field.

3. 6. The Equi-Potential Surface

- The potential field around the point charge is given by:

$$V = \frac{Q}{4\,\pi\,\varepsilon_o\,r}$$



- for constant r, the potential is constant.
- The surface of constant voltage is called *equi-potential surface*.
- The *equi-potential surface* is defined as a surface that composed of all those points having the same value of potential.

The properties of the equi-potential surface can be classified to:

a) The equi-potential surface is defined as the surface that composed of all those points having the same value of potential.

b) No work is done to in moving a unit positive point charge around any equi-potential surface because there is no potential difference between any two points on the surface.

c) Electric flux lines are everywhere perpendicular to the equi-potential surface.

3. 7. The Potential Field of System of Charge

- The potential due to two charges Q_1 and Q_2 is a function only of r_1 and r_2, the distances from Q_1 and Q_2 to the field point, respectively.

$$V = \frac{Q_1}{4\,\pi\,\varepsilon_o\,r_1} + \frac{Q_2}{4\,\pi\,\varepsilon_o\,r_2}$$

- The potential due to n charges is:

$$V = \frac{Q_1}{4\,\pi\,\varepsilon_o\,r_1} + \frac{Q_2}{4\,\pi\,\varepsilon_o\,r_2} + \cdots + \frac{Q_m}{4\,\pi\,\varepsilon_o\,r_m} = \sum_{m=1}^{m=n} \frac{Q_m}{4\,\pi\,\varepsilon_o\,r_m}$$

- For small element of volume charge distribution $\rho_V\,\Delta v$

$$dV = \frac{dQ}{4\,\pi\,\varepsilon_o\,r}$$

- And

$$V = \int \frac{dQ}{4\,\pi\,\varepsilon_o\,r}$$

For line charge

- The differential charge $dQ = \rho_L \, dL$
- The absolute potential function will be:

$$V = \int_L \frac{\rho_L \, dL}{4 \pi \varepsilon_o r}$$

For surface charge

- The differential charge is $dQ = \rho_S \, dS$
- The absolute potential function will be:

$$V = \int_S \frac{\rho_S \, dS}{4 \pi \varepsilon_o r}$$

For surface charge

- The differential charge is $dQ = \rho_v \, dv$
- The absolute potential function will be:

$$V = \int_S \frac{\rho_V \, dV}{4 \pi \varepsilon_o r}$$

For zero reference at infinity, then:

a) The potential due to a single point charge is the work done in carrying a unit positive charge from infinity to the point at which we desire the potential, and the work is independent of the path chosen between those two points.

b) The potential field in the presence of a number of point charges is the sum of the individual potential fields arising from each charge.

c) The potential due to a number of point charges or any continues charge distribution may therefore found by carrying a unit positive charge from infinity to the required point along any chosen path.

- In other words, the expression for the potential (*zero reference at infinity*)

$$V_A = -\int_{\infty}^{A} \bar{E} \, . \, d\bar{L}$$

- or the potential difference,

$$V_{AB} = V_A - V_B = -\int_B^A \bar{E} \, . \, d\bar{L}$$

<u>Note that:</u> For the potential calculation

- If the electric field intensity $\bar{E}$ is given as a function or easy to be calculated (*Point charge*, *Infinite Line charge*, *Infinite Surface of charge*)

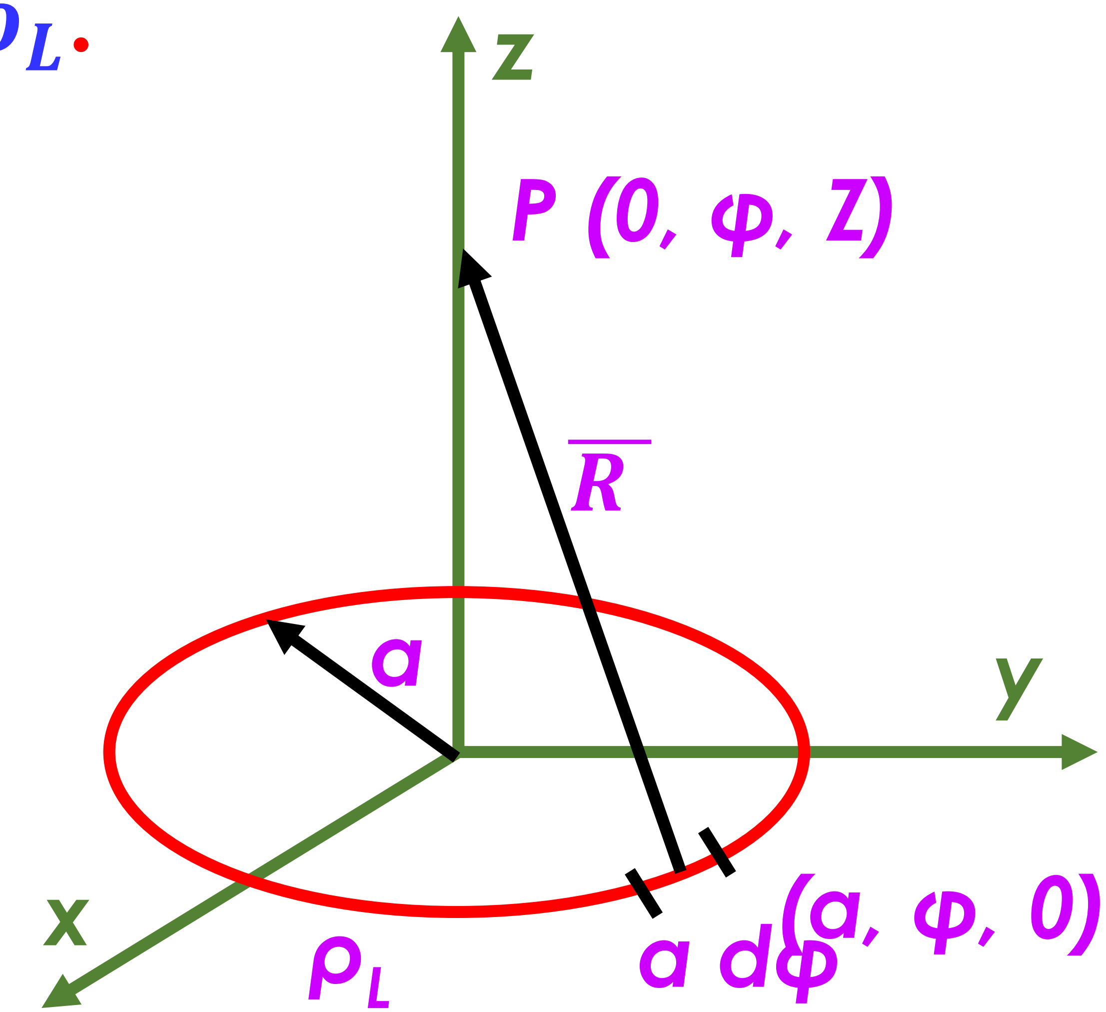
$$V_{AB} = -\int_B^A \bar{E} \, . \, d\bar{L}$$

- If the electric field intensity $\bar{E}$ is not easy to be calculated (*charge distribution*), then

$$dV = \frac{dQ}{4 \, \pi \, \varepsilon_o \, r}$$

Example (3-4)

Find the potential at a point P on the axis of a ring $r = a$ in the $z = 0$ plane having a uniform charge ρ_L.



Solution

- The potential in case of line charge is given by:

$$V = \int_L \frac{\rho_L \, dL}{4 \pi \varepsilon_o R}$$

where $dL = a \, d\varphi$ and $R = \sqrt{a^2 + Z^2}$

- Then:

$$V = \int_0^{2\pi} \frac{\rho_L \, a \, d\varphi}{4 \pi \varepsilon_o \sqrt{a^2 + z^2}} = \frac{\rho_L \, a}{2 \varepsilon_o \sqrt{a^2 + z^2}} \quad Volt$$

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Example (3-5)

In cylindrical coordinates, the following electric fields are $\bar{E} = \frac{10}{r}\,\bar{a}_r \;\; V/m$ for $0 < r \leq 2\,m$ and $\bar{E} = \frac{12.5}{r}\,\bar{a}_r \;\; V/m$ for $r > 2\,m$. **Find:**

a) The potential difference V_{AB} between the points $A(1,0,0)$ and $B(5,0,0)$.

b) The value of the volume charge density ρ_v at the points A and B.

Solution

a) The potential difference between A and B is:

$$V_{AB} = -\int_B^A \bar{E}\,.\,d\bar{L}$$

➢ Therefore:

$$V_{AB} = -\int_5^2 \frac{2.5}{r}\,dr - \int_2^1 \frac{10}{r}\,dr = \left[2.5\ \ln\left(\frac{2}{5}\right) + 10\ \ln\left(\frac{1}{2}\right)\right]$$

$$= 2.5\ \ln\left(\frac{5}{2}\right) + 10\ \ln(2) = 9.222\quad V$$

b) The volume charge density is:

$$\rho_v = \bar{\nabla}\cdot\bar{D} = \varepsilon_o\,\bar{\nabla}\cdot\bar{E} = \varepsilon_o \frac{1}{r}\frac{\partial}{\partial r}(r\,E_r) = \varepsilon_o \frac{1}{r}\frac{d}{dr}(12.5) = 0$$

Since $E \propto \frac{1}{r}$ in the two regions, then the value of ρ_v at the points A and B are zero.

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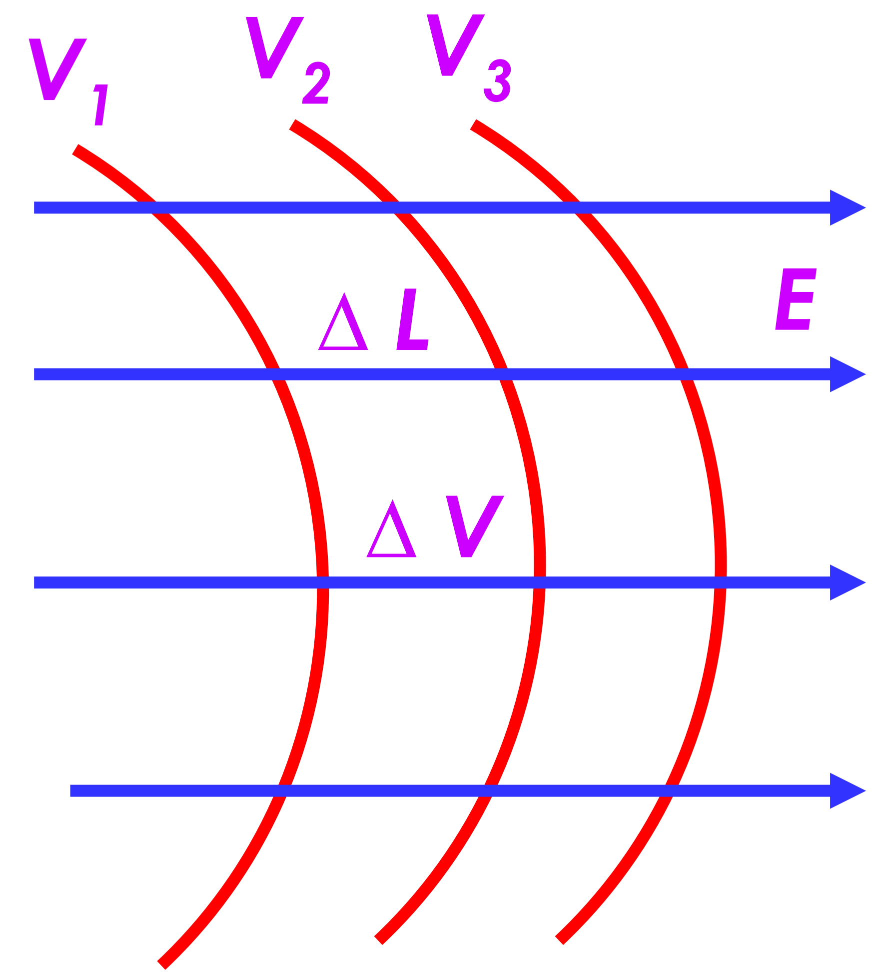
3. 8. The Potential Gradient



- An incremental potential difference ΔV is given by:

$$\Delta V = -\overline{E} \cdot \Delta \overline{L} \quad \text{or} \quad dV = -\overline{E} \cdot d\overline{L}$$

The negative sign means that the potential is decreasing in the direction of $\overline{E}$.

- But the electric field intensity is given by:

$$\overline{E} = E_x\,\overline{a}_x + E_y\,\overline{a}_y + E_z\,\overline{a}_z$$

- and

$$d\overline{L} = dx\,\overline{a}_x + dy\,\overline{a}_y + dz\,\overline{a}_z$$

- Therefore:

$$dV = -E_x\,dx - E_y\,dy - E_z\,dz \qquad (I)$$

- Since potential V is a scalar field is function in x, y, and z. Then,

$$dV = \frac{\partial V}{\partial x}\,dx + \frac{\partial V}{\partial y}\,dy + \frac{\partial V}{\partial z}\,dz \qquad (II)$$

- Comparing equation (I) with equation (II), we can get:

$$E_x = -\frac{\partial V}{\partial x}\,, \qquad E_y = -\frac{\partial V}{\partial y}\,, \qquad and \quad E_z = -\frac{\partial V}{\partial z}$$

- Therefore:

$$\bar{E} = -\frac{\partial V}{\partial x}\,\bar{a}_x - \frac{\partial V}{\partial y}\,\bar{a}_y - \frac{\partial V}{\partial z}\,\bar{a}_z = -Grade\,V$$

- In Cartesian coordinates, the gradient of V is given by

$$Grade\,V = \bar{\nabla}\,V = \frac{\partial V}{\partial x}\,\bar{a}_x + \frac{\partial V}{\partial y}\,\bar{a}_y + \frac{\partial V}{\partial z}\,\bar{a}_z$$

- or

$$\bar{E} = -\,Grade\,V = -\bar{\nabla}\,V$$

"The gradient of the potential at a point is defined as the potential rise ΔV across an element ΔL along the electric field line divided by ΔL".

- **The gradient of V in the three coordinate systems is given by:**

$$\bar{\nabla}\,V = \frac{\partial V}{\partial x}\,\bar{a}_x + \frac{\partial V}{\partial y}\,\bar{a}_y + \frac{\partial V}{\partial z}\,\bar{a}_z \qquad (Cartesian)$$

$$\bar{\nabla}\,V = \frac{\partial V}{\partial r}\,\bar{a}_r + \frac{1}{r}\frac{\partial V}{\partial \varphi}\,\bar{a}_\varphi + \frac{\partial V}{\partial z}\,\bar{a}_z \qquad (Cylindrical)$$

$$\bar{\nabla}\,V = \frac{\partial V}{\partial r}\,\bar{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\,\bar{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \varphi}\,\bar{a}_\varphi \qquad (Spherical)$$

Example (3-6)

The potential distribution $V = \dfrac{12}{x^2 + y^2}$ refers to a non-uniform field. Here V is in volts and x and y in centimeters. With respect to z there is no variation, therefore the distribution is in two-dimensional. ***Evaluate:***

a) Expression for the gradient of the potential.
b) The value of the gradient at the point $(2, 1)\ Cm$.
c) The strength of electric field at this point.

Solution

a) Since the potential distribution is independent of z, the gradient of V will be

$$\bar{\nabla} V = \frac{\partial V}{\partial x}\,\bar{a}_x + \frac{\partial V}{\partial y}\,\bar{a}_y = -\frac{20}{(x^2 + y^2)^2}\left(x\,\bar{a}_x + y\,\bar{a}_y\right) \quad V/m$$

b) At point $(2, 1)$ cm:

$$\bar{\nabla} V = -\frac{20}{25}\left(2\,\bar{a}_x + \bar{a}_y\right) = -1.6\,\bar{a}_x - 0.4\,\bar{a}_y = 1.79\,\angle 206^o \quad V/m$$

c) The electric field has the opposite direction of the gradient:

$$\bar{\mathrm{E}} = -\bar{\nabla} V = 1.6\,\bar{a}_x + 0.4\,\bar{a}_y = -\,1.79\,\angle 206^o \; = 1.79\,\angle 46^o \quad V/m$$

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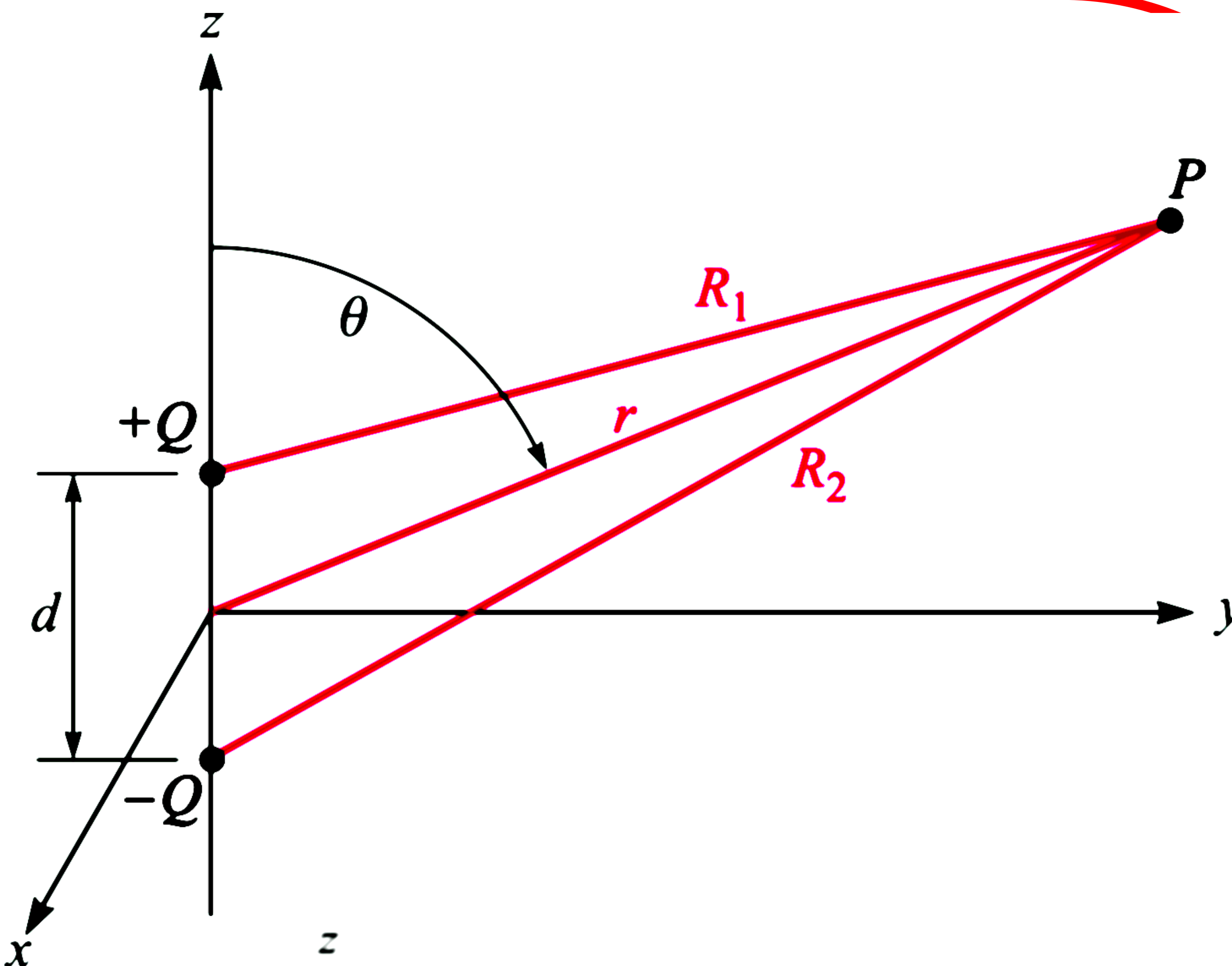
3. 9. Electric Dipole

- The dipole fields are quite important because they form the basis for the behavior of dielectric materials in electric fields.
- Moreover, this development will serve to illustrate the importance of the potential concept.
- An electric dipole, or simply a dipole, is the name given to two point charges of equal magnitude and opposite sign, separated by a distance that is small compared to the distance to the point **P** at which we want to know the electric and potential fields.
- The objective of this section is to find the potential due to both charges at point ***P***, and then from the potential function, determine the electric field.

- The potential will be just the sum of the two potential functions associated with each point charge:

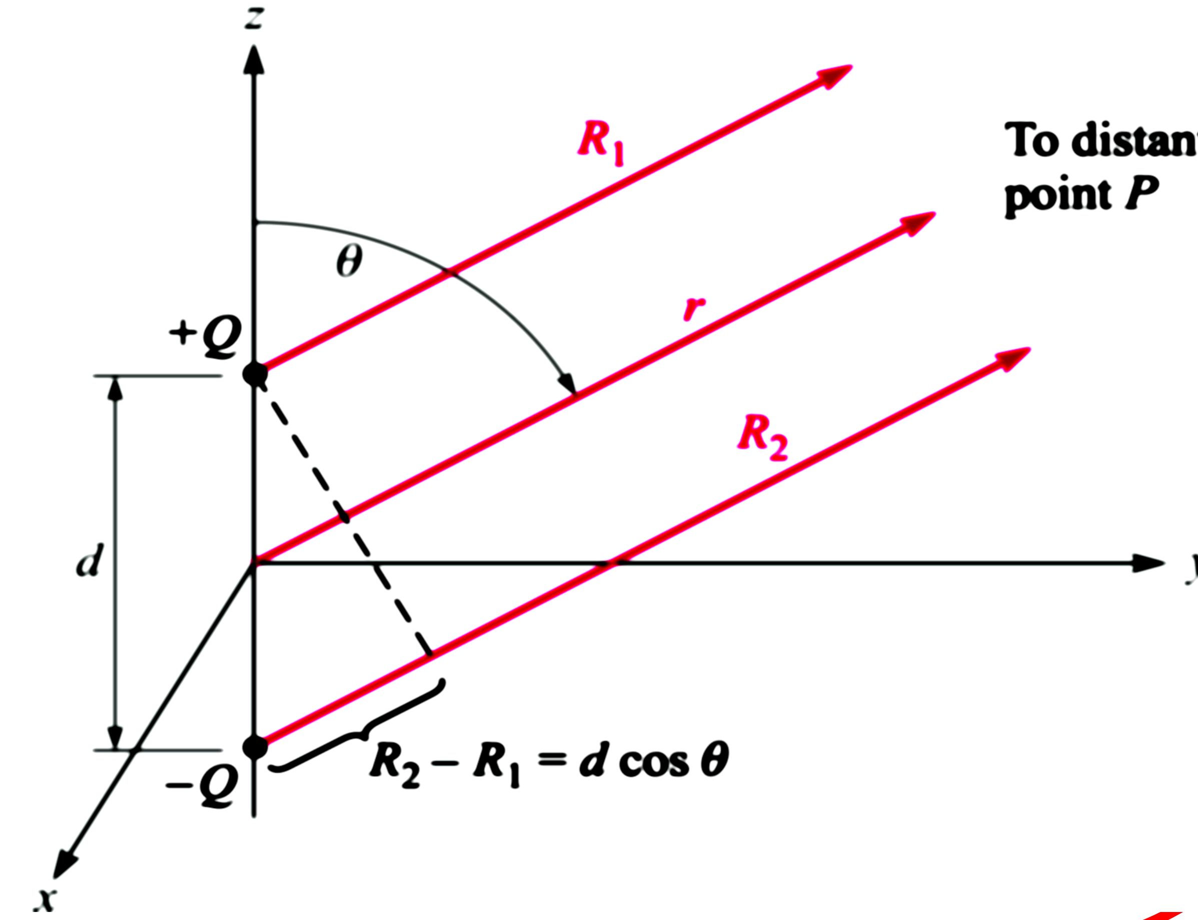
$$V = \frac{Q}{4\,\pi\,\varepsilon_0}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = \frac{Q}{4\,\pi\,\varepsilon_0}\,\frac{R_2 - R_1}{R_1 R_2}$$



Far-Field Approximation

- Under the condition $r >> d$, the three position vectors are approximately parallel.
- This means that we may use the approximations:

$$R_1 R_2 = r^2 \; and \; R_2 - R_1 = \mathrm{d}\cos\theta$$



- We can get finally:

$$V = \frac{Q\, d\, \cos\theta}{4\,\pi\,\varepsilon_0\, r^2}$$

Electric Field of the Dipole

- Electric field is found by taking the negative gradient:

$$\bar{E} = -\bar{\nabla}\, V = -\left(\frac{\partial V}{\partial r}\,\bar{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\,\bar{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \varphi}\,\bar{a}_\varphi\right)$$

- or:

$$\bar{E} = -\left(-\frac{Q\, d\, \cos\theta}{2\,\pi\,\varepsilon_0\, r^3}\bar{a}_r - \frac{Q\, d\, \sin\theta}{4\,\pi\,\varepsilon_0\, r^3}\bar{a}_\theta\right)$$

- From which we can get finally:

$$\bar{E} = \frac{Q\, d}{4\,\pi\,\varepsilon_0\, r^3}\left(2\,\cos\theta\,\bar{a}_r + \sin\theta\,\bar{a}_\theta\right)$$

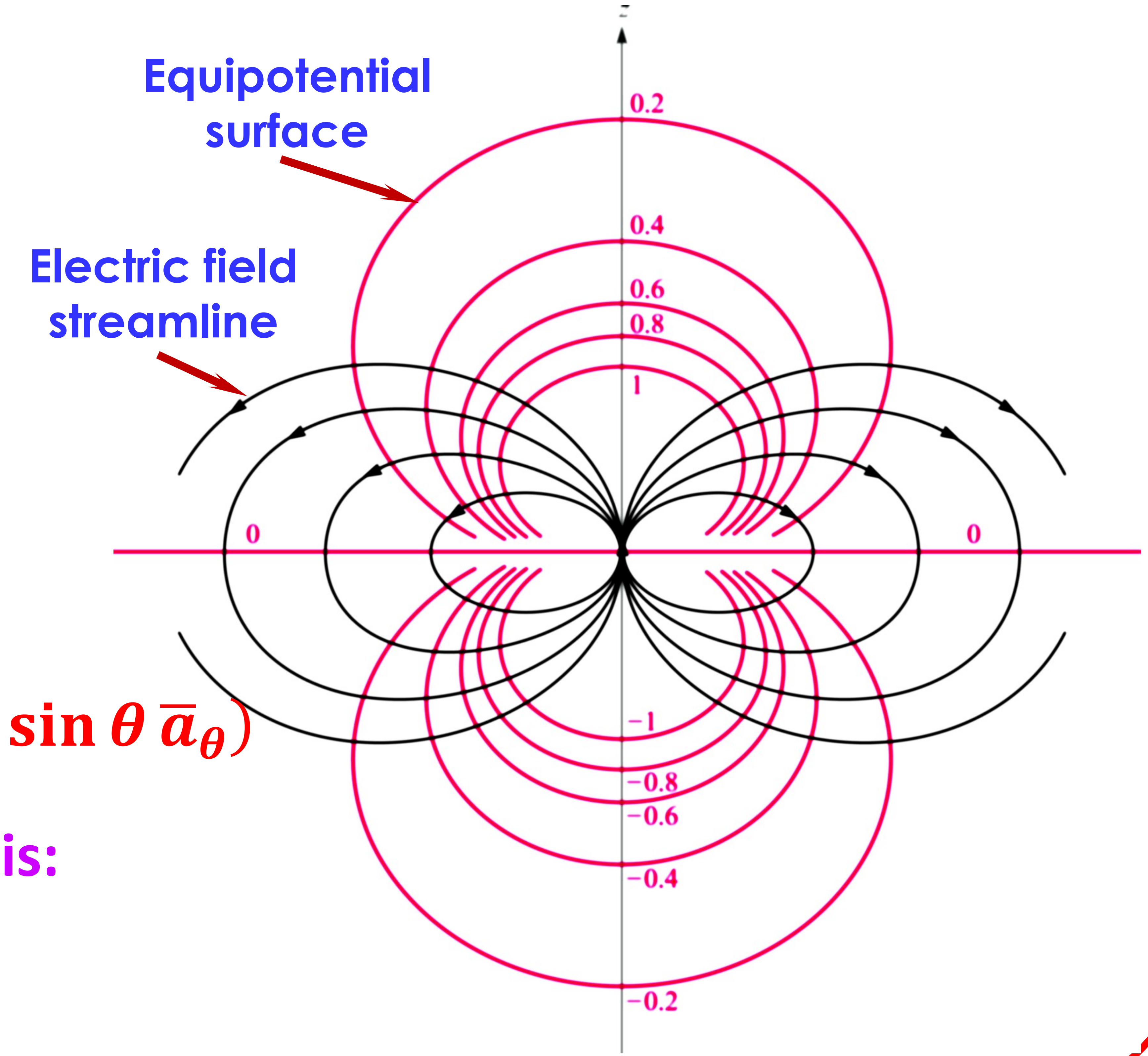
Electric Dipole Field and Equipotentials

$$V = \frac{Q\,d\,\cos\theta}{4\,\pi\,\varepsilon_0\,r^2}$$

- $let\ \ \frac{Q\,d}{4\,\pi\,\varepsilon_0} = 1,$

$$\cos\theta = V\,r^2$$

- Then $V = 0, +0.2, +0.4,$
  $+0.6, +0.8, +1$



$$\bar{E} = \frac{Q\,d}{4\,\pi\,\varepsilon_0\,r^3}\left(2\ \cos\theta\,\bar{a}_r + \sin\theta\,\bar{a}_\theta\right)$$

- The streamlines equation is:

$$r = \ C_1\ \sin^2\theta$$

- $C_1 = \ 1, 1.5, 2, and\ 2.5.$

Electric Dipole Moment

- The dipole moment vector is directed from the negative charge to the positive charge, and is defined as:

$$P = Q\,d \qquad C.m.$$

- In the charge configuration we have used, the direction of p is $\bar{a}_z$, and therefore:

$$\bar{d} \cdot \bar{a}_r = d\,\cos\theta \quad \text{and} \quad \bar{p} \cdot \bar{a}_r = Q\,d\,\cos\theta$$

- So we may write:

$$V = \frac{\bar{p} \cdot \bar{a}_r}{4\,\pi\,\varepsilon_0\,r^2}$$ which would account for any orientation of p.

- In general, for a dipole at any orientation, positioned off-origin:

$$V = \frac{1}{4\,\pi\,\varepsilon_0\,|r - r'|^2}\bar{p} \cdot \frac{\bar{r} - \bar{r}'}{|r - r'|} = \frac{\bar{p} \cdot (\bar{r} - \bar{r}')}{4\,\pi\,\varepsilon_0\,|r - r'|^3}$$

where r locates the field point P, and r' determines the dipole center.

Example (3-7)

An electric dipole located at the origin in free space has a moment $\bar{p} = 3\,\bar{a}_x - 2\,\bar{a}_y + \bar{a}_z\ nC.m.$ Find V at $P_A\,(2, 3, 4)$.

Solution

- Using the general expression for the potential in the far field:

$$V = \frac{\bar{p} \cdot (\bar{r} - \bar{r}')}{4\,\pi\,\varepsilon_0\,|r - r'|^3}$$

$$\bar{r} - \bar{r}' = 2\,\bar{a}_x + 3\,\bar{a}_y + 4\,\bar{a}_z$$

$$V = \frac{\left(3\,\bar{a}_x - 2\,\bar{a}_y + \bar{a}_z\right) \cdot \left(2\,\bar{a}_x + 3\,\bar{a}_y + 4\,\bar{a}_z\right) \times 10^{-9}}{4\,\pi\,\varepsilon_0\,[4 + 9 + 16]^{1.5}} = 0.2302\ \ V$$

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Example (3-8)

Two point charges, $1\ nC$ at (0, 0, 0.1) and $-1\ nC$ at (0, 0, −0.1), are in free space.

a) Calculate V at P (0.3, 0, 0.4).
b) Calculate $|E|$ at P.
c) Now treat the two charges as a dipole at the origin and find V at P.

Solution

a) V at P (0.3, 0, 0.4) is:

$$V_p = \frac{Q_+}{4\,\pi\,\varepsilon_0 R_+} + \frac{Q_-}{4\,\pi\,\varepsilon_0 R_-} = \frac{10^{-9}}{4\,\pi\,\varepsilon_0}\left[\frac{1}{0.4243} - \frac{1}{0.5831}\right] = 5.769\ \ V$$

b) $\bar{E}$ at P (0.3, 0, 0.4) is:

$$\bar{E}_p = \frac{Q_+}{4\,\pi\,\varepsilon_0 R_+^3}\bar{R}_+ + \frac{Q_-}{4\,\pi\,\varepsilon_0 R_-^3}\bar{R}_-$$

$$\bar{R}_{+} = 0.3\,\bar{a}_x + 0.3\,\bar{a}_z \qquad \bar{R}_{-} = 0.3\,\bar{a}_x + 0.5\,\bar{a}_z$$

$$\bar{E}_p = \frac{10^{-9}}{4\,\pi\,\varepsilon_0 (0.424)^{1.5}}(0.3\,\bar{a}_x + 0.3\,\bar{a}_z) + \frac{-\,10^{-9}}{4\,\pi\,\varepsilon_0 (0.583)^{1.5}}(0.3\,\bar{a}_x + 0.5\,\bar{a}_z)$$

$$\bar{E}_p = \frac{10^{-9}}{4\,\pi\,\varepsilon_0}(2.4218\bar{a}_x + 1.4124\,\bar{a}_z)$$

Taking the magnitude of the above, we find

$$|\bar{E}_p| = 25.2 \quad V/m$$

c) Now treat the two charges as a dipole at the origin and find V at P: In spherical coordinates, P is located at

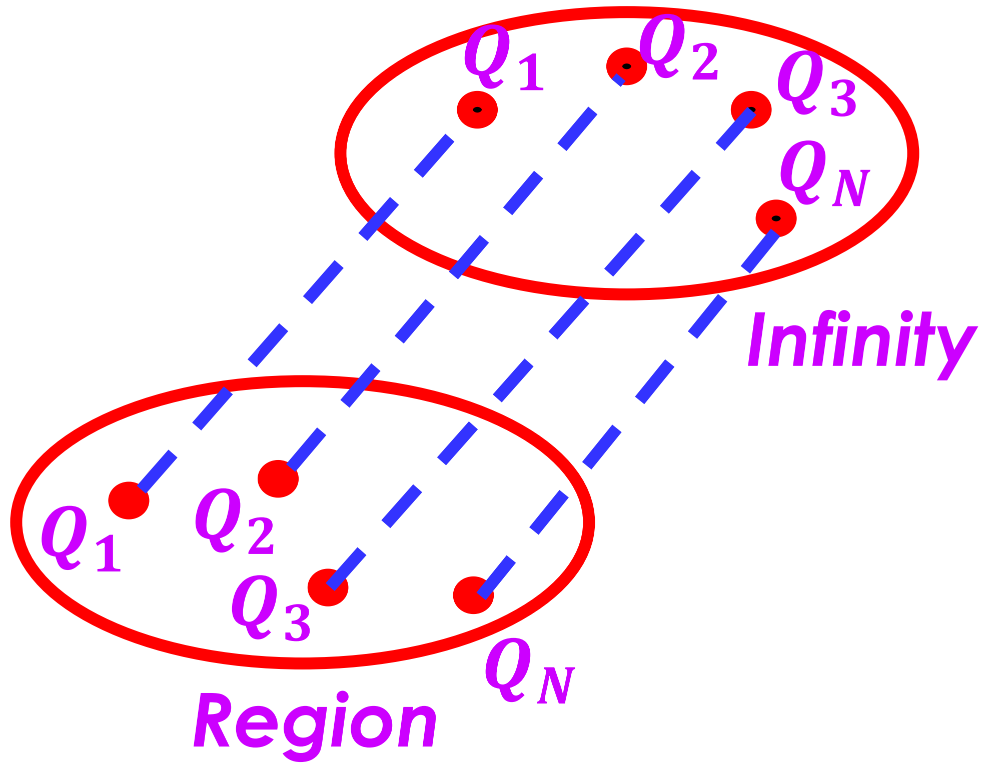
$$r = \sqrt{0.3^2 + 0.4^2} = 0.5\ m \quad and \quad \theta = \sin^{-1}(0.3/0.5) = 36.9^0$$

$$V_p = \frac{Q\,d\,\cos\theta}{4\,\pi\,\varepsilon_0\,r^2} = \frac{10^{-9} \times 0.2 \times \cos 36.9}{4\,\pi\,\varepsilon_0 \times 0.5^2} = 5.75 \quad V$$

****

3. 10. The Energy in Electrostatic Fields

- In order to find the potential energy present in a system of charges, we must find the work done by an external source in positioning the charges.
- Bringing a charge of Q_1 from infinity to any position requires no work, for there no field present.



- The position of Q_2 at a point in the field of Q_1 requires an amount of work given by the point of the charge Q_2 and the potential at that point due to Q_1.
- If we represent this potential as V_{21}, where the first subscript indicates the location and the second subscript the source. Also, V_{12} is the potential at the location of Q_2 due to Q_1.

- The work require to position Q_2 is equal to $Q_2 V_{21}$.
- Similarly, we may express the work required to position each charge in the field of all those already present.

$$\text{Work required to position } Q_2 = Q_2 V_{21}$$

$$\text{Work required to position } Q_3 = Q_3 V_{31} + Q_3 V_{32}$$

$$\text{Work required to position } Q_4 = Q_4 V_{41} + Q_4 V_{42} + Q_4 V_{43}$$

- The total work required to position the charge is equal to the potential energy of the field of charge or:

$$\text{Total Positioning Work} = \text{Potential Energy of the Field}$$

- Therefore,

$$W_E = Q_2 V_{21} + Q_3 V_{31} + Q_3 V_{32} + Q_4 V_{41} + Q_4 V_{42} + \cdots \quad \text{(I)}$$

- Using $$Q_3 V_{31} = Q_3 \frac{Q_1}{4\pi\varepsilon_o R_{13}} = Q_1 \frac{Q_3}{4\pi\varepsilon_o R_{31}} = Q_1 V_{13}$$

- If each term of the total energy expression is replaced by its equal, we have:

$$W_E = Q_1 V_{12} + Q_1 V_{13} + Q_2 V_{23} + Q_1 V_{14} + Q_2 V_{24} + \cdots (II)$$

- Adding equation (I) and equation (II) gives:

$$2 W_E = Q_1 (V_{12} + V_{13} + V_{14} + \cdots) + Q_2 (V_{21} + V_{23} + V_{24} + \cdots) + Q_3 (V_{31} + V_{32} + V_{34} + \cdots) + Q_4 (V_{41} + V_{42} + V_{43} + \cdots) + \cdots$$

$$But \quad V_1 = V_{12} + V_{13} + V_{14} + V_{15} + \cdots$$

- The potential at the location of Q_1 due to the presence of Q_2, Q_3, ... we therefore have:

$$W_E = \frac{1}{2}(Q_1 V_1 + Q_2 V_2 + Q_3 V_3 + \cdots) = \frac{1}{2}\sum_{m=1}^{m=N} Q_m V_m$$

- For continues charge distribution, each charge is replaced by $\rho_v \, dv$ and the summation becomes an integral:

$$W_E = \frac{1}{2} \int_{vol} \rho_v \, V \, dv$$

- Using *Maxwell's first equation*, replace ρ_v by its equal $\overline{\nabla} \cdot \overline{D}$ and make use of a vector identity:

$$\overline{\nabla} \cdot (V \, \overline{D}) = V \, (\overline{\nabla} \cdot \overline{D}) + \overline{D} \cdot \overline{\nabla} \, V$$

- Then

$$W_E = \frac{1}{2} \int_{vol} \rho_v \, V \, dv = \frac{1}{2} \int_{vol} (\overline{\nabla} \cdot \overline{D}) \, V \, dv$$

$$W_E = \frac{1}{2} \int_{vol} [\overline{\nabla} \cdot (V \, \overline{D}) - \overline{D} \cdot \overline{\nabla} \, V \,] dv$$

- Using the divergence theorem

$$W_E = \frac{1}{2}\oint_{Surf} (V\,\bar{D}) \cdot d\bar{s} - \frac{1}{2}\int_{vol} (\bar{D} \cdot \bar{\nabla}\, V)\, dv$$

- The surface integral is equal to zero, for over this closed surface.
- Substituting $\bar{E} = -\bar{\nabla}\, V$ in the remaining volume integral, yields to:

$$\boxed{W_E = \frac{1}{2}\int_{vol} (\bar{D} \cdot \bar{\mathrm{E}})\, dv = \frac{1}{2}\int_{vol} \varepsilon_o\, |E|^2\, dv}$$

- The differential form become (*Energy density J/m³*)

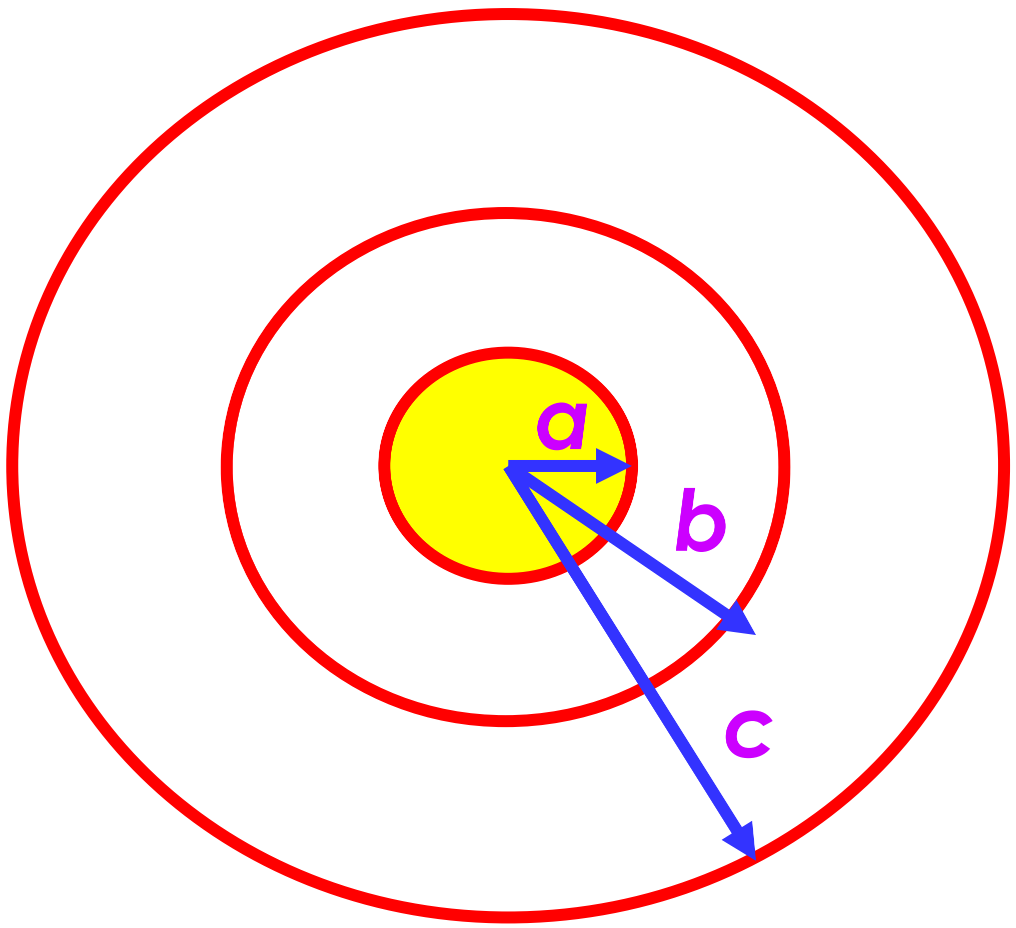
$$W_E = \frac{1}{2}(\bar{D} \cdot \bar{\mathrm{E}})\, dv$$

or

$$\frac{dW_E}{dv} = \frac{1}{2}(\bar{D} \cdot \bar{\mathrm{E}}) \;\; J/m^3$$

Example (3-9)

A small isolated conducting sphere of radius a is charged with $+Q\ C$. Surrounding this sphere and concentric with it is a conducting spherical shell, which possesses no net charge. The inner radius of the shell is b, and the outer radius is c. All non-conducting spaces is air. It is required to *find* the absolute potential of each sphere.



Solution

- Assume that zero potential at infinity

$$0 \le r \le a$$

$$\overline{E} = 0 \qquad (Inside\ the\ conductor)$$

$$a \le r \le b$$

$$\overline{E} = \frac{Q}{4\pi\varepsilon_o r^2}\,\overline{a}_r$$

$$b \leq r \leq c$$

$$\bar{E} = 0 \qquad (\textit{Inside the conductor})$$

$$r > c$$

$$\bar{E} = \frac{Q}{4\,\pi\,\varepsilon_o\,r^2}\,\bar{a}_r$$

$$V_{c\,\infty} = V_c - V_\infty = -\int_\infty^c \bar{E}\cdot d\bar{L}$$

- Since $V_\infty = 0$ (Reference at infinity), then

$$V_c = -\int_\infty^c \bar{E}\cdot d\bar{L} = -\int_\infty^c \frac{Q}{4\,\pi\,\varepsilon_o\,r^2}\,dr = \frac{Q}{4\,\pi\,\varepsilon_o\,c} \qquad V$$

- The conductor is an *equipotential surface*; the absolute potential of the outer sphere is given by

$$V_c = V_b = \frac{Q}{4\,\pi\,\varepsilon_o\,c} \qquad V$$

➢ also,

$$V_{ab} = V_a - V_b = -\int_b^a \frac{Q}{4\,\pi\,\varepsilon_o\,r^2}\,dr = \frac{Q}{4\,\pi\,\varepsilon_o}\left[\frac{1}{a} - \frac{1}{b}\right] \qquad V$$

➢ Then, the absolute oh the inner sphere is given by

$$V_a = \frac{Q}{4\,\pi\,\varepsilon_o}\left[\frac{1}{a} + \frac{1}{c} - \frac{1}{b}\right] \qquad V$$

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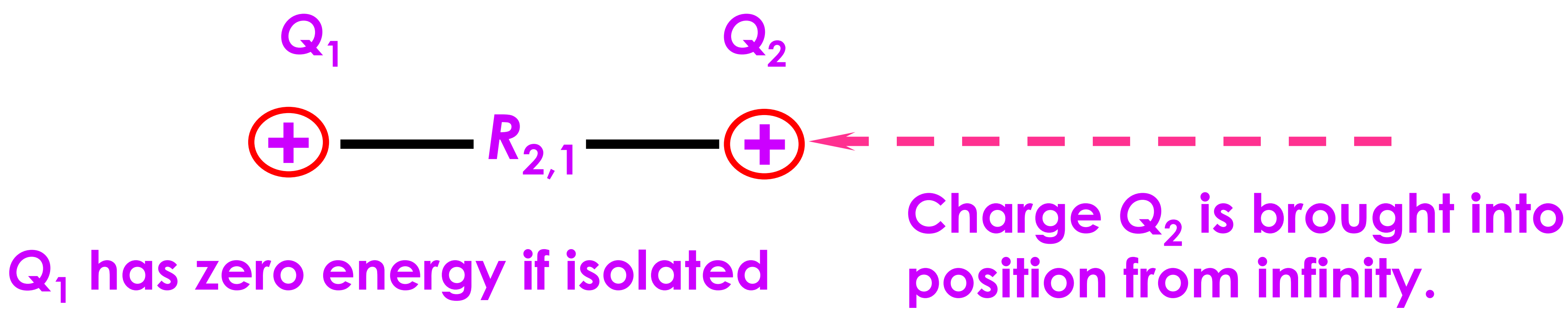
3. 11. Potential Energy in a System of Point Charges

Two Point Charges

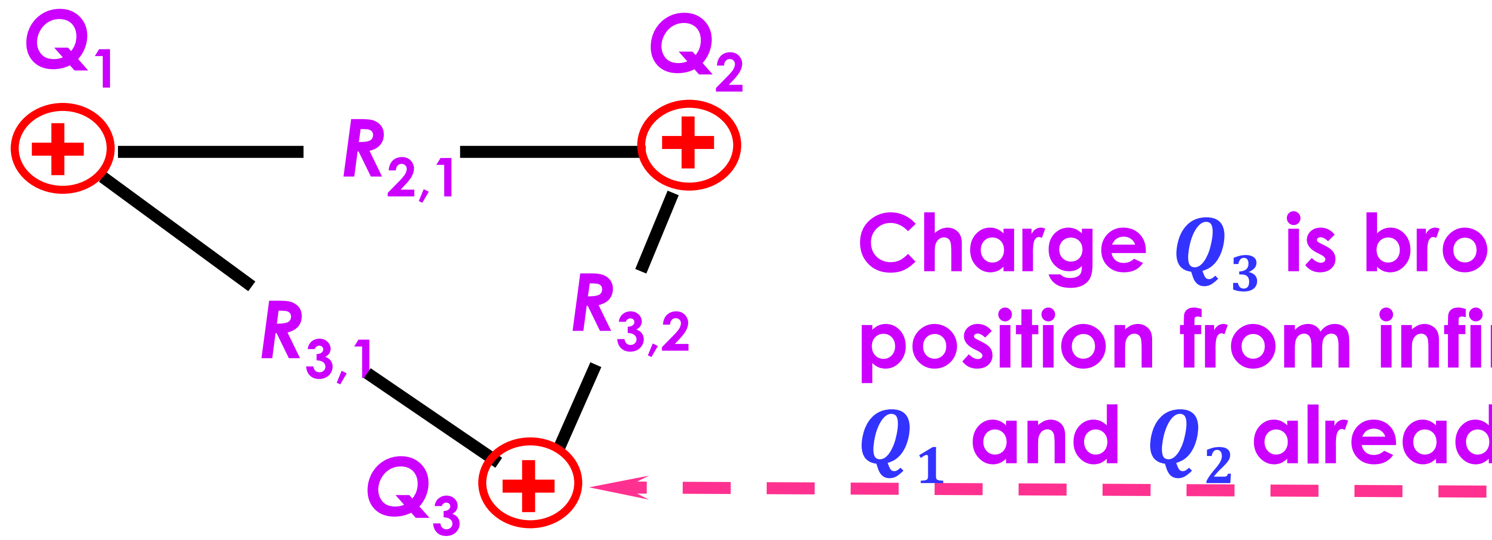


- The work done in bringing Q_2 into position is:

$$W_E = Q_2 V_{21} = \frac{Q_2\, Q_1}{4\,\pi\,\varepsilon_o\, R_{21}}$$

This is the stored energy in the "system", consisting of the two assembled charges.

Three Point Charges



Charge Q_3 is brought into position from infinity, with Q_1 and Q_2 already situated.

- The system energy is now the previous 2-charge energy plus the work done in bringing Q_3 into position

$$W_E = \frac{1}{2}[Q_1 V_1 + Q_2 V_2 + Q_3 V_3] \quad \text{J}$$

$$V_1 = V_{12} + V_{13} = \frac{Q_2}{4\,\pi\,\varepsilon_o\,R_{12}} + \frac{Q_3}{4\,\pi\,\varepsilon_o\,R_{13}}$$

$$V_2 = V_{21} + V_{23} = \frac{Q_1}{4\,\pi\,\varepsilon_o\,R_{21}} + \frac{Q_3}{4\,\pi\,\varepsilon_o\,R_{23}}$$

$$V_3 = V_{31} + V_{32} = \frac{Q_1}{4\,\pi\,\varepsilon_o\,R_{31}} + \frac{Q_2}{4\,\pi\,\varepsilon_o\,R_{32}}$$

Example (3-10)

Four $0.8\,nC$ point charges are located in free space at the corners of a square $4\,cm$ on aside. Find the total potential energy stored.

Solution

- The total potential energy stored is given by:

$$W_E = \frac{1}{2}\sum_{m=1}^{m=4} Q_m V_m$$

- where V_m in this case is the potential at the location of any one of the point charges that arises from the other three. This will be (for charge 1)

$$V_1 = V_{12} + V_{13} + V_{14} = \frac{0.8\times 10^{-9}}{4\,\pi\,\varepsilon_o}\left[\frac{1}{0.04} + \frac{1}{0.04} + \frac{1}{0.04\sqrt{2}}\right] = 486.61\ V$$

- Taking the summation produces a factor of **4**, since the situation is the same at all four points. Consequently,

$$W_E = \frac{1}{2}\sum_{m=1}^{m=4} Q_m V_m = \frac{1}{2} \times 4 \times \left(0.8 \times 10^{-9}\right) \times 486.61$$

$$W_E = 7.79 \times 10^{-7} \quad J$$

****

THE END

OF CHAPTER (3)