Electromagnetic Fields

EPM 112

Course instructor

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Electromagnetic Fields
EPM 112

Chapter (4)

Conductor, Dielectric and Capacitance
Part (1)

4. 1. Introduction

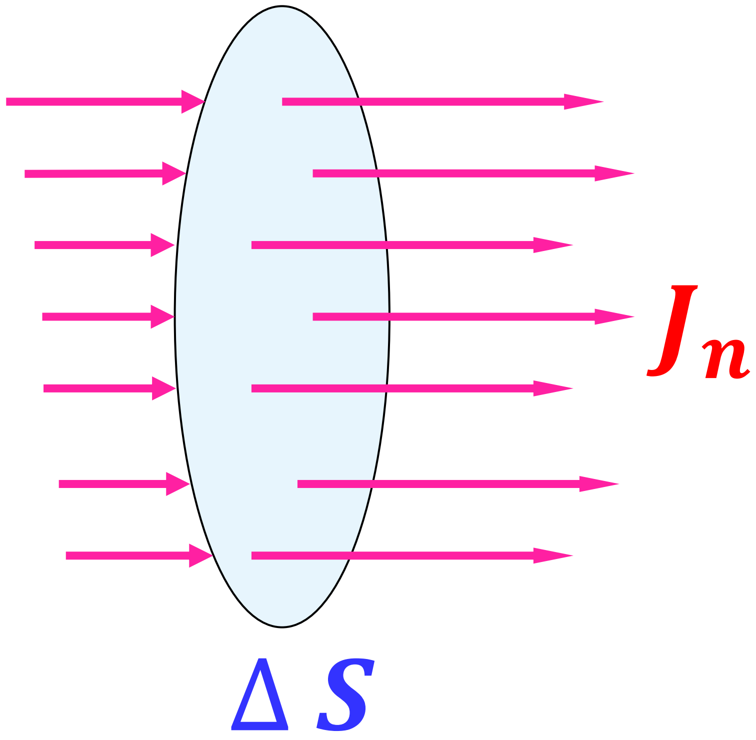
- In the previous chapters, the discussion was restricted to static (stationary) charges in free space.
- However, in this chapter we will allow the charge to move with a constant velocity and thus introduce the concept of current.
- The next discussion will be that of dielectric media whose widely known characteristic is that of electric polarization. The effect of polarization of the media on the electric field intensity and electric flux density fields will be studied.
- In addition, the concept of capacitance has been introduced.

4. 2. Current and Current Density

- Electric charges in motion constitute a *current (Amps)*. Current is a flux quantity and is defined as:

$$I = \frac{dQ}{dt}$$

- *Current density*, $\bar{J}$, measured in Amps/m², yields current in Amps when it is integrated over a cross-sectional area. The assumption would be that the direction of $\bar{J}$ is normal to the surface,



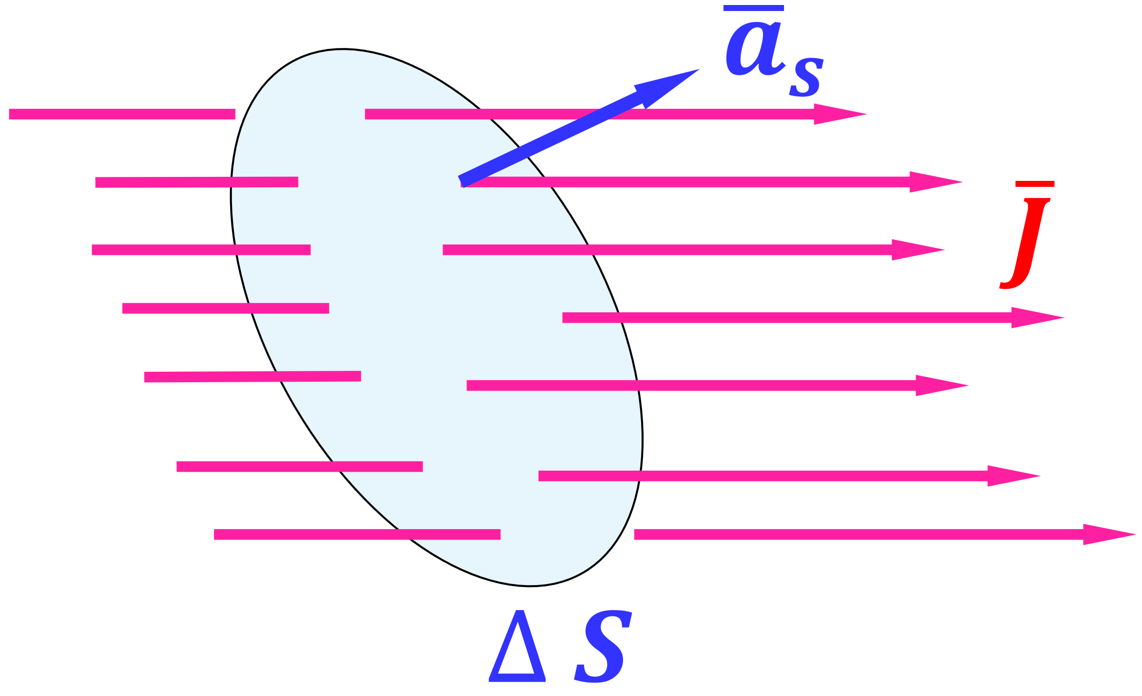
$$\Delta I = J_n \Delta s$$

- **In reality, the direction of current flow may not be normal to the surface, so we treat current density as a vector, and write the incremental flux through the small surface in the usual way:**

$$\Delta I = \bar{J} \cdot \Delta \bar{s}$$

- **Then, the current through a large surface is found through the flux integral:**

$$I = \int_{surf} \bar{J} \cdot d\bar{s}$$

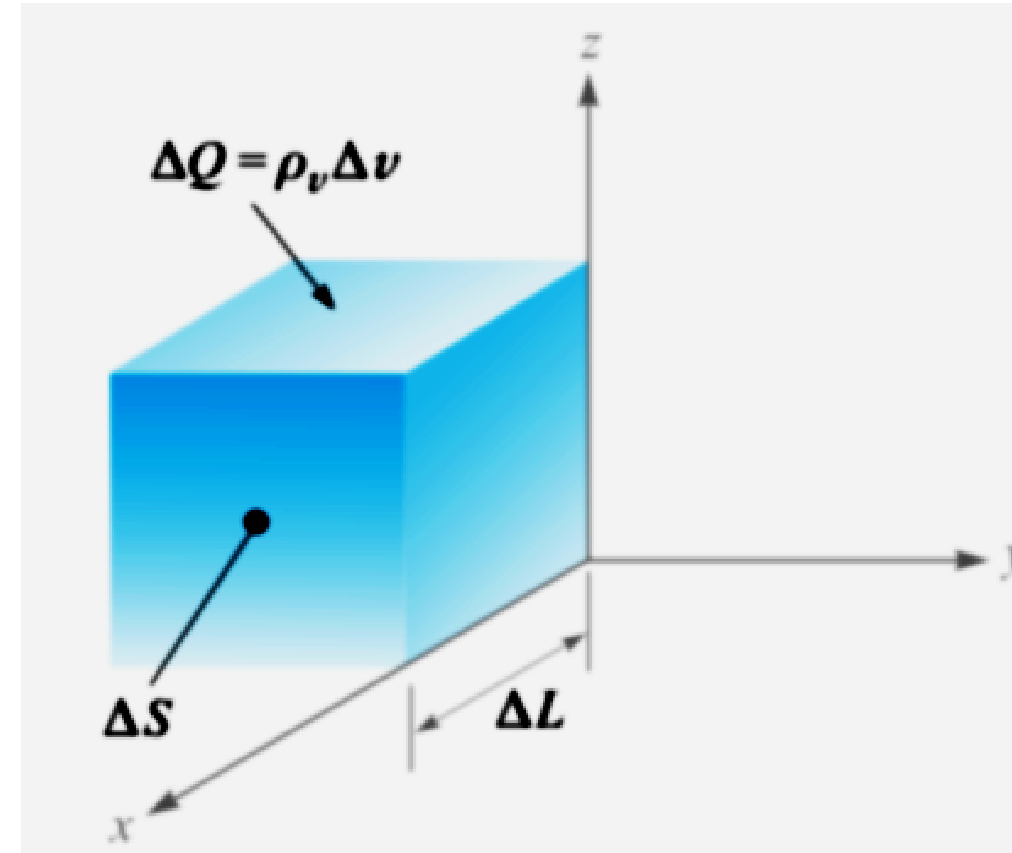


4. 2.1. Relation of Current to Charge Velocity

- Consider a charge ΔQ, occupying volume Δv, moving in the positive x direction at velocity v_x

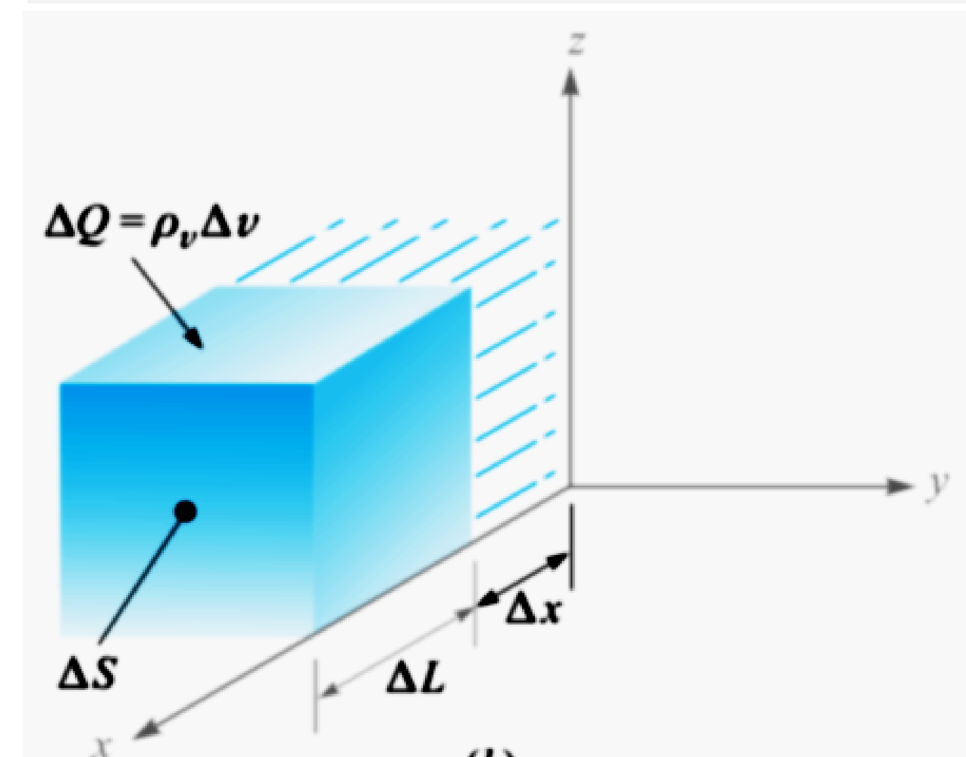
- In terms of the volume charge density, we may write:

$$\Delta Q = \rho_v \Delta v = \rho_v \Delta s \Delta L$$



- Suppose that in time Δt , the charge moves through a distance $\Delta x = \Delta L = v_x \Delta t$, then:

$$\Delta Q = \rho_v \Delta s \Delta x$$



- The motion of the charge represents a current given by:

$$\Delta I = \frac{\Delta Q}{\Delta t} = \rho_v \, \Delta s \, \frac{\Delta x}{\Delta t}$$

or:

$$\Delta I = \rho_v \, \Delta s \, v_x$$

where v_x represents the x component of the velocity v.

- In terms of the current density,

$$J_x = \frac{\Delta I}{\Delta s} = \rho_v \; v_x$$

- So that in general:

$$\bar{J} = \rho_v \; \bar{v}$$

This type of current is called a *convection current*, and $\rho_v \, \bar{v}$ is the *convection current density*.

- Also, the drift velocity $\bar{v}$ is directly proportional to the electric field intensity $\bar{E}$ as,

$$\bar{v} = \mu \bar{E}$$

where μ is *the mobility*, has the units $m^2/V.s$. The conduction current can be given as:

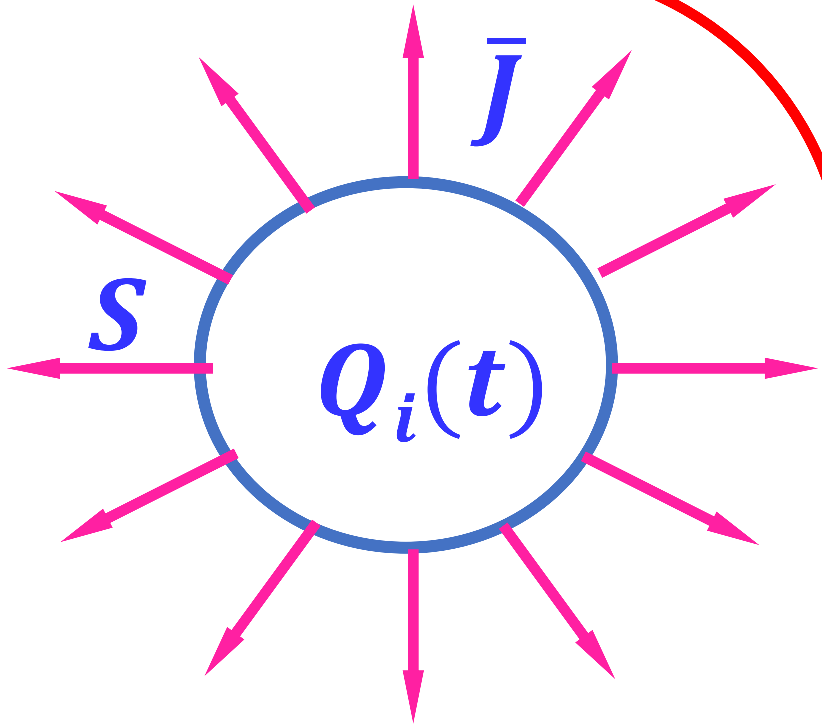
$$\bar{J} = \rho_v \bar{v} = \rho_v \mu \bar{E}$$

or

$$\bar{J} = \sigma \bar{E}$$

Where $\sigma = \rho_v \mu$ is the conductivity of the material, in *siemens* per meter (S/m or moh/m). This equation is called the point form of *Ohm's law*.

4. 2. 2. Continuity of Current



- Suppose that charge Q_i is escaping from a volume through closed surface S, to form current density J. Then the total current is:

$$I = \oint_{surf} \bar{J} \cdot d\bar{s} = -\frac{dQ_i}{dt} = -\frac{d}{dt}\int_{vol} \rho_v \, dv$$

where the minus sign is needed to produce *positive* outward flux, while the interior charge is *decreasing* with time.

- Apply the divergence theorem:

$$\oint_{surf} \bar{J} \cdot d\bar{s} = \int_{vol} (\bar{\nabla} \cdot \bar{J}) \, dv$$

- So that:

$$\int_{vol} (\bar{\nabla} \cdot \bar{J})\, dv = -\frac{d}{dt} \int_{vol} \rho_v \, dv$$

- Or

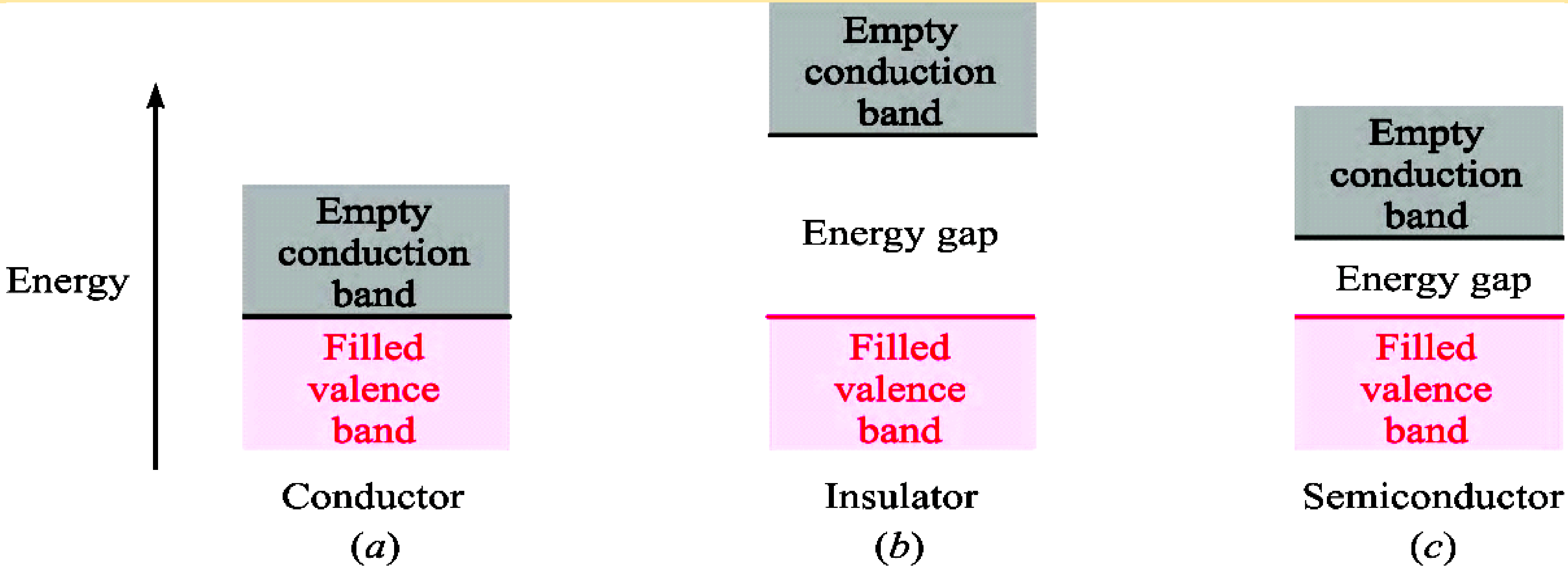
$$\int_{vol} (\bar{\nabla} \cdot \bar{J})\, dv = \int_{vol} \left(-\frac{\partial \rho_v}{\partial t}\right) dv$$

- The integrands of the last expression must be equal, leading to the ***Equation of Continuity:***

$$\bar{\nabla} \cdot \bar{J} = -\frac{\partial \rho_v}{\partial t}$$

✓ This equation indicates that the current, or charge per second, diverging from a small volume per unit volume is equal to the time rate of decrease of charge per unit volume at every point.

Energy Band Structure in Three Material Types



a) Conductors exhibit no energy gap between valence and conduction bands so electrons move freely

b) Insulators show large energy gaps, requiring large amounts of energy to lift electrons into the conduction band. When this occurs, the dielectric breaks down.

c) Semiconductors have a relatively small energy gap, so modest amounts of energy (applied through heat, light, or an electric field) may lift electrons from valence to conduction bands.

Example (4.1)

Find **the current in the cylindrical wire located along z axis, if the current density is $\bar{J} = 15\left(1 - e^{-1000\,r}\right)\bar{a}_z\ (A/m^2)$. The radius of the wire is $2\ mm$.**

Solution

- ✓ **A cross sectional area of the wire is $d\bar{s} = r\,dr\,d\varphi\,\bar{a}_z$.**
- ✓ **Then:**

$$d\,I = \bar{J}\cdot d\bar{s} = 15\left(1 - e^{-1000\,r}\right)\bar{a}_z . r\,dr\,d\varphi\,\bar{a}_z$$

$$= 15\left(1 - e^{-1000\,r}\right) r\,dr\,d\varphi$$

- ✓ **and**

$$I = \int_0^{2\pi}\int_0^{0.002} 15\left(1 - e^{-1000\,r}\right) r\,dr\,d\varphi = 0.133\ mA$$

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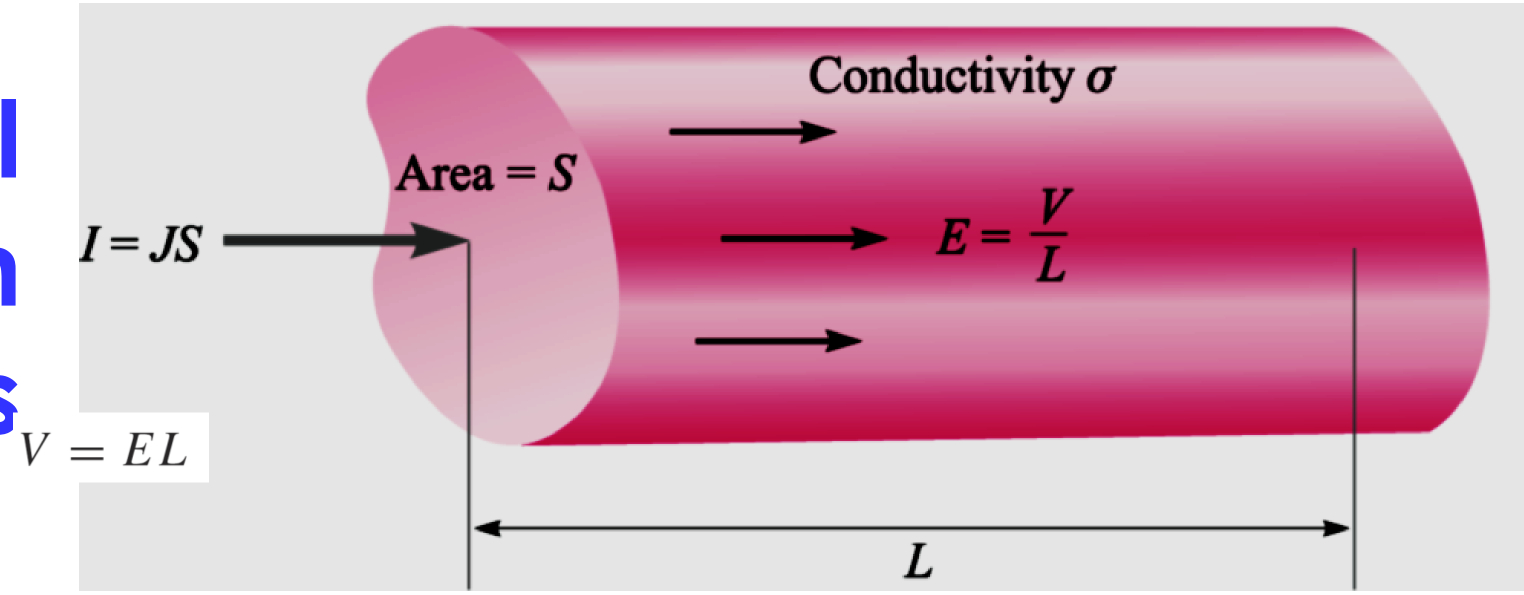
4. 2. 3. Resistance

- Consider the cylindrical conductor shown with voltage V applied across the ends.



- Current flows down the length, and is assumed to be uniformly distributed over the cross-section, S.

- First, we can write the voltage and current in the cylinder in terms of field quantities:

$$V_{ab} = -\int_b^a \overline{E}.d\overline{L} = -\overline{E}.\int_b^a d\overline{L} = -\overline{E}.\overline{L}_{ba} = \overline{E}.\overline{L}_{ab}$$

⇨⇨⇨ $V = E\,L$ **Volt**

$$I = \int_{surf} \bar{J} \cdot d\bar{s} = J\,S \quad \Rightarrow\Rightarrow \quad J = \frac{I}{S} = \sigma\,E = \sigma\,\frac{V}{L}$$

$$\Rightarrow\Rightarrow\Rightarrow \quad V = \frac{L}{\sigma\,S}\,I$$

- Using Ohm's Law:

$$V = I\,R$$

- We find the resistance of the cylinder:

$$R = \frac{L}{\sigma\,S}$$

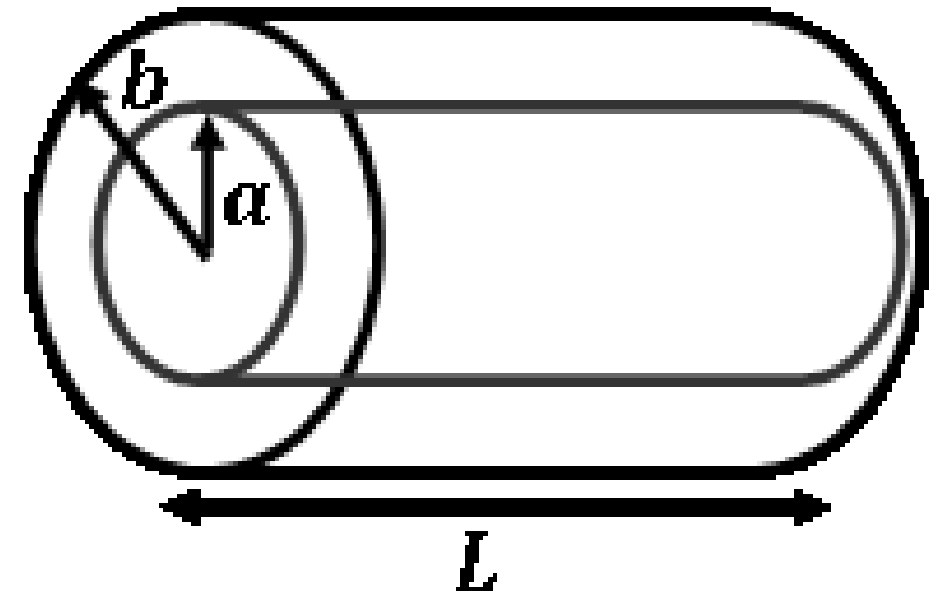
- General expression for resistance

$$R_{ab} = \frac{V_{ab}}{I} = \frac{-\int_b^a \bar{E} \cdot d\bar{L}}{\int_{surf} \bar{J} \cdot d\bar{s}} = \frac{-\int_b^a \frac{\bar{J}}{\sigma} \cdot d\bar{L}}{\int_{surf} \bar{J} \cdot d\bar{s}}$$

Example (4.2)

Determine the resistance of the insulation in a length L of coaxial cable of inner radius a and outer radius b.

Solution



- ✓ Assume a total current I flow from the inner conductor to the outer conductor. Then, at a radius distance r,

$$J = \frac{I}{S} = \frac{I}{2\,\pi\,r\,L}$$

- ✓ and so

$$E = \frac{J}{\sigma} = \frac{I}{2\,\pi\,\sigma\,r\,l}$$

- ✓ The potential difference between the conductors is then:

$$V_{ab} = -\int_b^a \bar{E}\,.\,d\bar{L} = -\int_b^a \frac{I}{2\,\pi\,\sigma\,r\,l}\,dr = \frac{I}{2\,\pi\,\sigma\,l}\ln\left(\frac{b}{a}\right)$$

- ✓ and the resistance is:

$$R_{ab} = \frac{V_{ab}}{I} = \frac{I}{2\,\pi\,\sigma\,l}\ln\left(\frac{b}{a}\right) \quad \Omega$$

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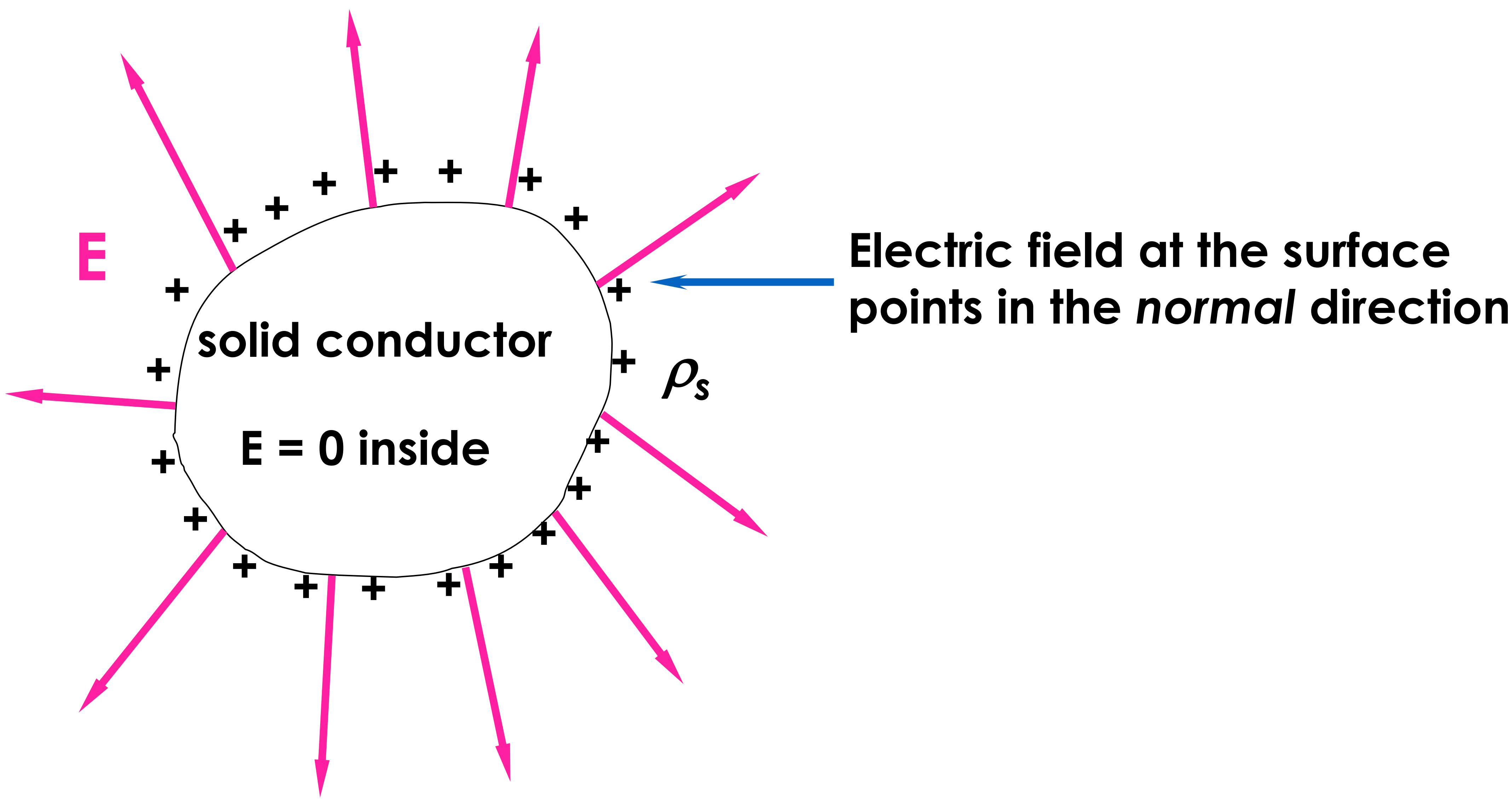
4. 2. 4. Electrostatic Properties of Conductors

- Consider a conductor, on which excess charge has been placed:



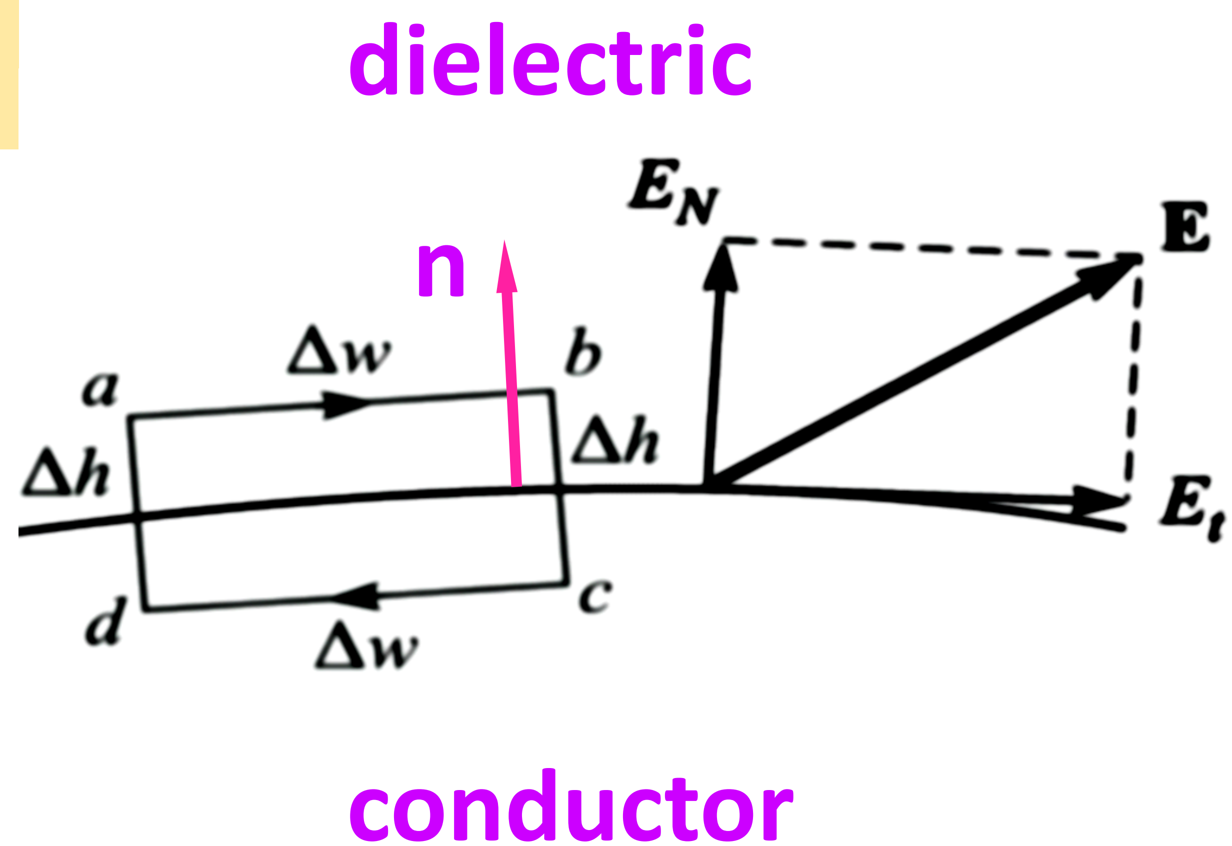
a) Charge can exist only on the surface as a surface charge density, ρ_s -- *not* in the interior.

b) Electric field *cannot* exist in the interior, nor can it possess a tangential component at the surface (as will be shown next slide).

c) It follows from condition (b) that the surface of a conductor is an *equipotential*.

4. 2. 5. Conductors Free Space Boundary Conditions

Tangential Electric Field



- Over the rectangular integration path, we use:

$$\oint_L \overline{E} \cdot d\overline{L} = 0$$

- or

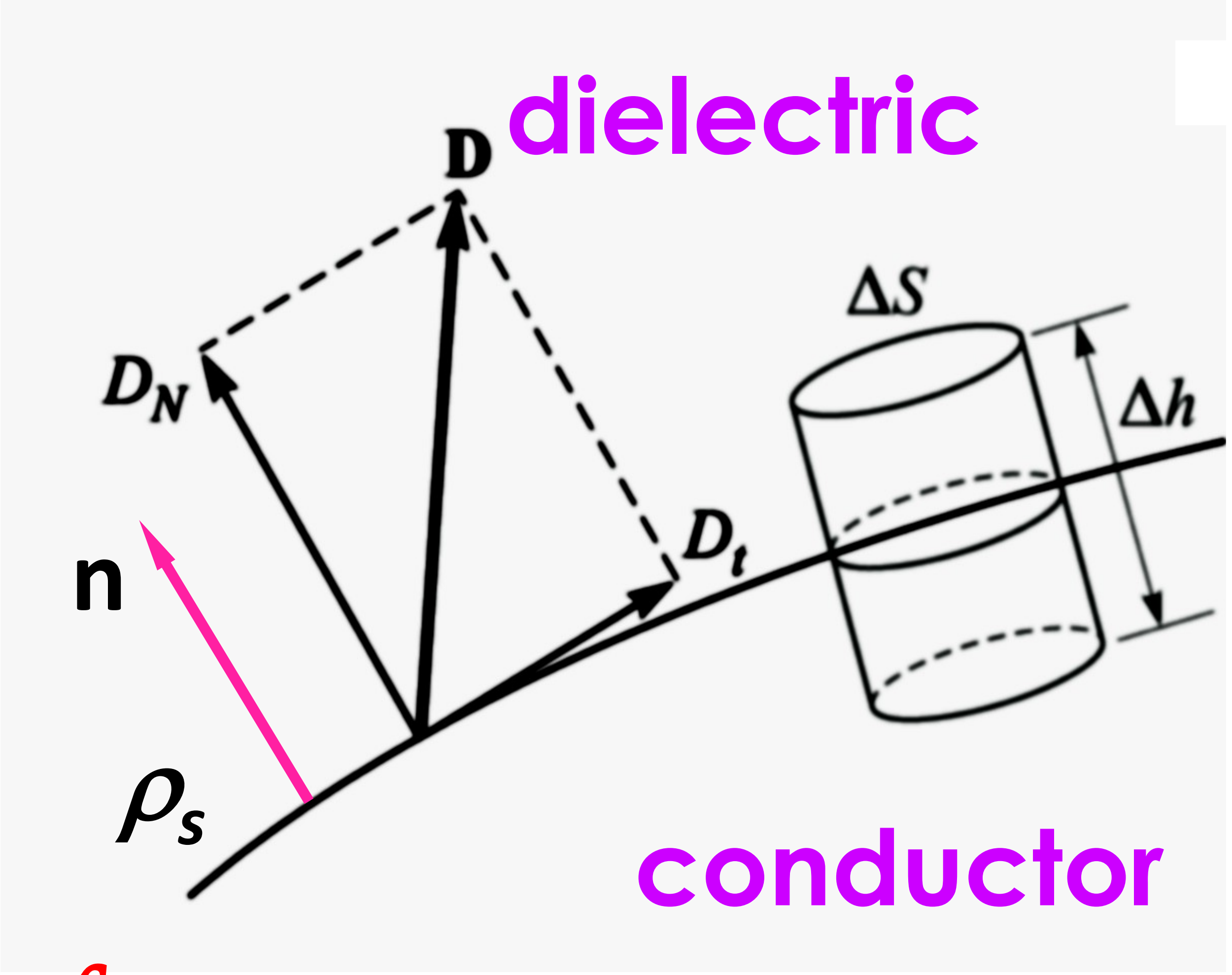
$$\int_a^b \overline{E}_1 . d\overline{L} + \int_b^c \overline{E} . d\overline{L} + \int_c^d \overline{E}_2 . d\overline{L} + \int_d^a \overline{E} . d\overline{L} = 0$$

$$E_t \, \Delta w - \underbrace{E_{N,\, at\, b} \times \frac{1}{2} \Delta h + E_{N,\, at\, a} \times \frac{1}{2} \Delta h}_{} = 0$$

These become negligible as Δh approaches zero.

- Therefore

$$\boxed{E_t = 0}$$



Normal Component

- Gauss' Law is applied to the cylindrical surface shown below:

$$\oint_{surf} \bar{D} \cdot d\bar{s} = \int_{Top} + \cancel{\int_{Bottom}} + \cancel{\int_{Side}} = Q_{en}$$

- This reduces to:

$$D_N \Delta S = Q = \rho_S \Delta S \quad \text{as } \Delta h \text{ approaches zero}$$

- Therefore:

$$D_N = \rho_S$$

Summary

1) The static electric field intensity inside a conductor is zero.

2) The static electric field intensity at the surface of a conductor is everywhere directed normal to that surface.

3) The conductor surface is an equipotential surface.

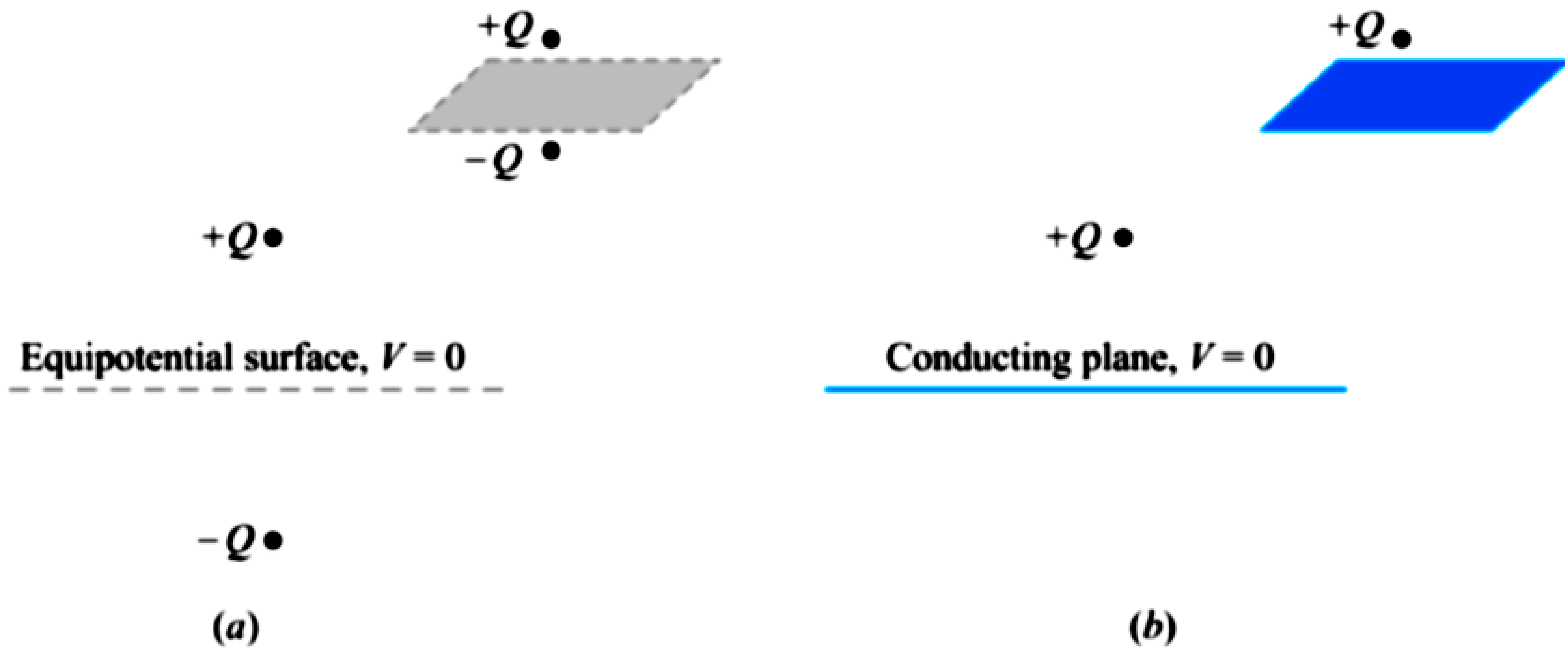
4. 3. Method of Images

- The *Theorem of Uniqueness* states that if we are given a configuration of charges and boundary conditions, there will exist *only one* potential and electric field solution.

- In the electric dipole, the surface along the plane of symmetry is an equipotential with $V = 0$.

- The same is true if a grounded conducting plane is located there.

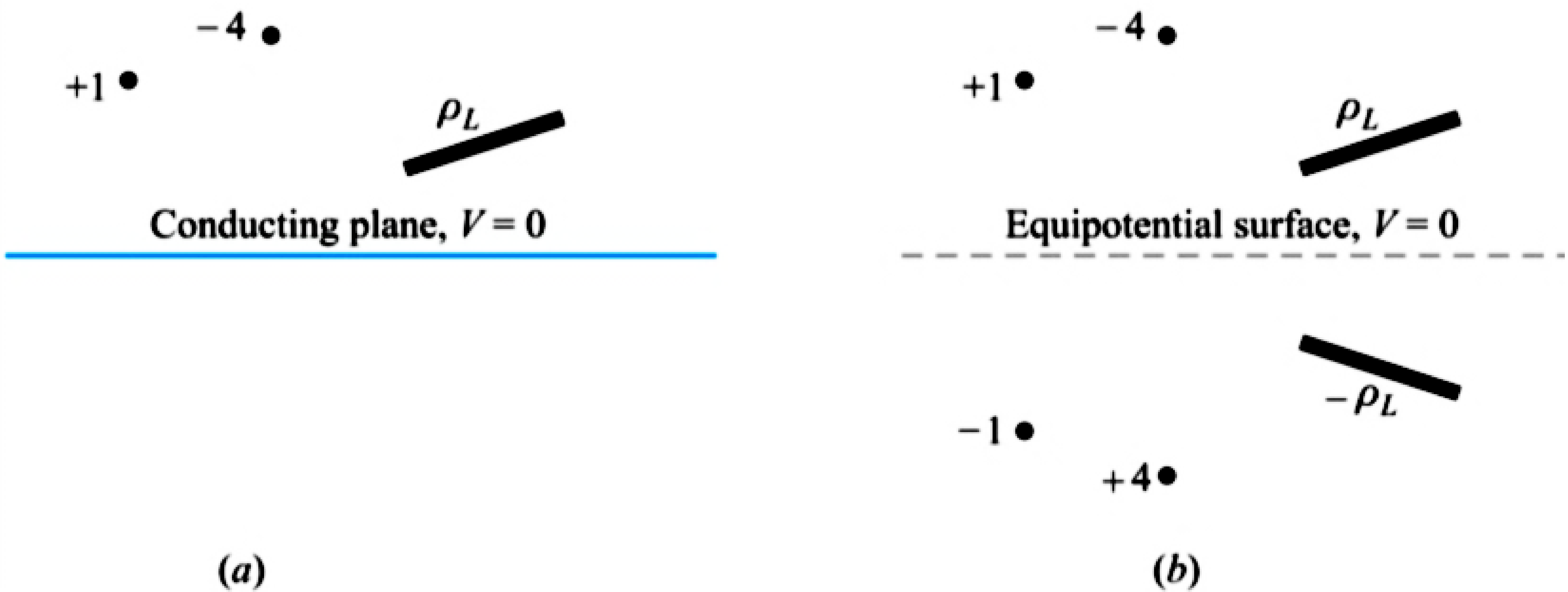
- So the boundary conditions and charges are *identical* in the *upper half* spaces of both configurations (not in the lower half).



a) Two equal but opposite charges may be replaced by

b) a single charge and a conducting plane without affecting the fields above the V = 0 surface.

Forms of Image Charges



a) A given charge configuration above an infinite conducting plane may be replaced by

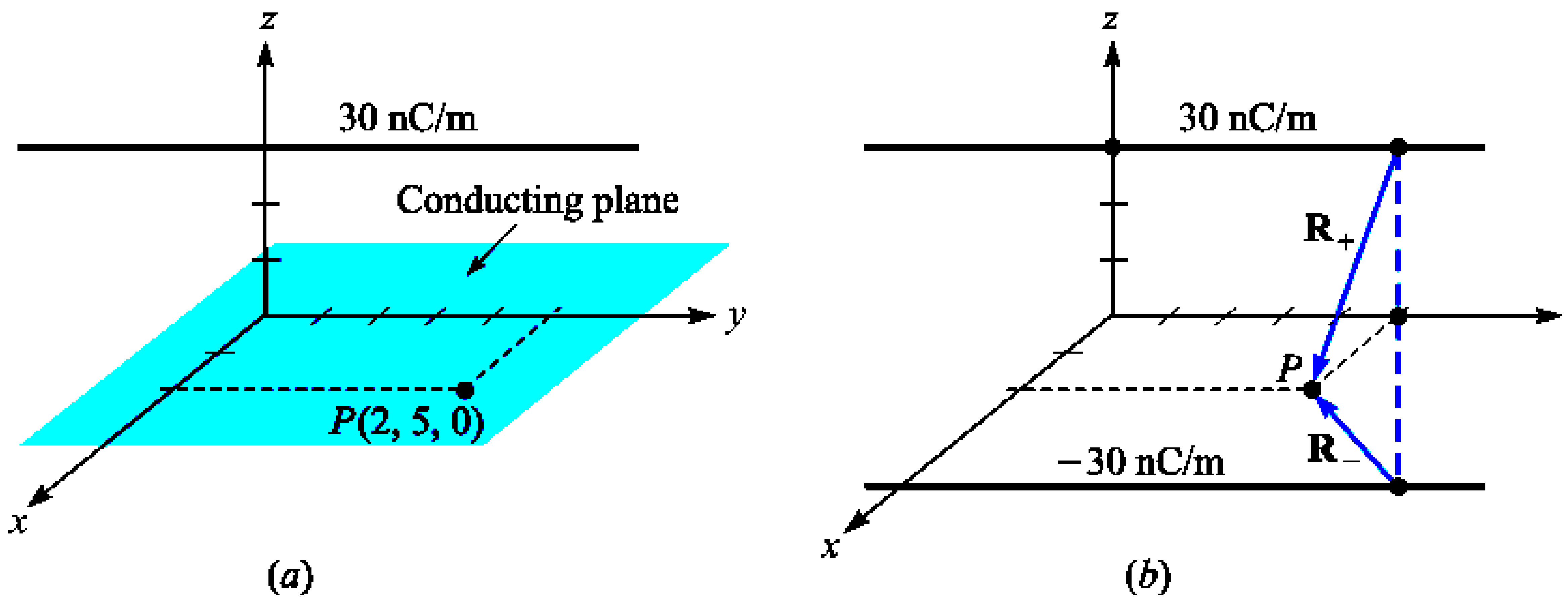
b) the given charge configuration plus the image configuration, without the conducting plane.

Example (4.3)

In this case, we are to find the surface charge density on the conducting plane at the point (2,5,0). A 30-nC line charge lies parallel to the y axis at z = 3.

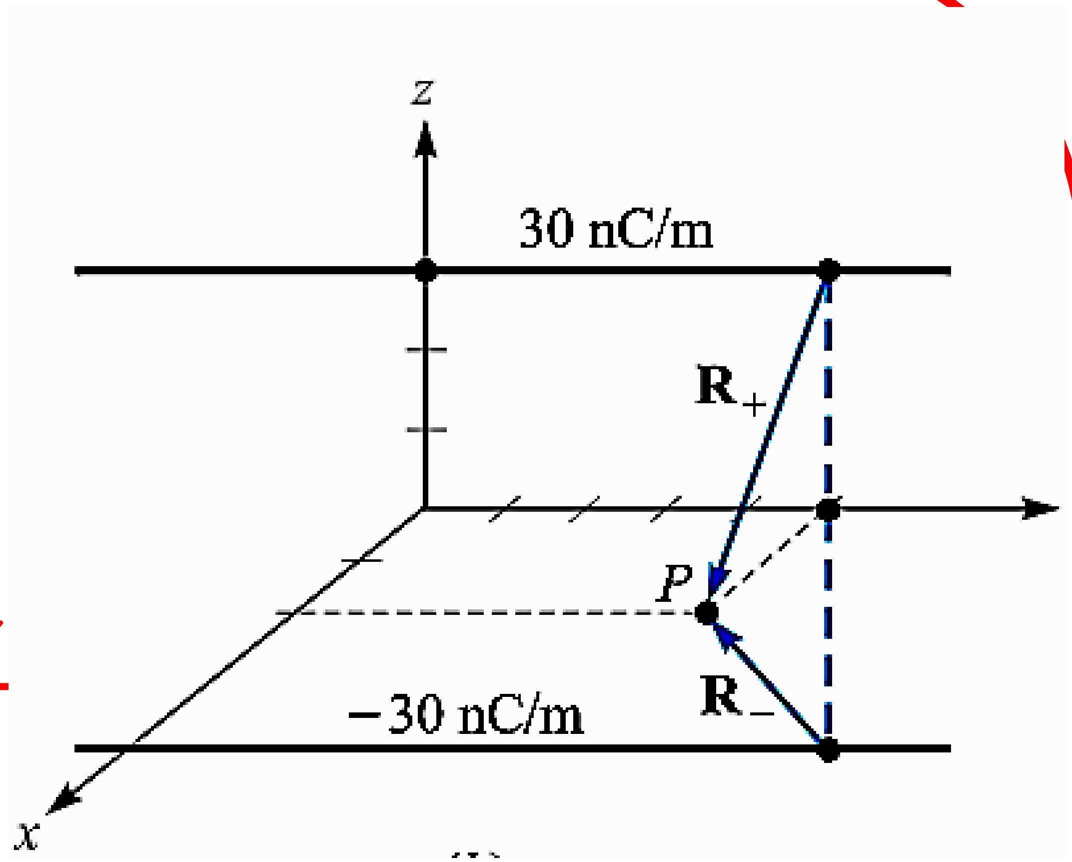
Solution

✓The first step is to replace the conducting plane by a line charge of -30 nC at z = -3.



✓ Referring to the figure, we find:

$$R_+ = 2\,\bar{a}_x - 3\,\bar{a}_z \qquad R_- = 2\,\bar{a}_x + 3\,\bar{a}_z$$



✓ Then

$$E_+ = \frac{\rho_L}{2\,\pi\,\varepsilon_0 R_+} = \frac{30\times 10^{-9}}{2\,\pi\,\varepsilon_0\sqrt{13}}\,\frac{2\,\bar{a}_x - 3\,\bar{a}_z}{\sqrt{13}}$$

$$E_- = \frac{-\rho_L}{2\,\pi\,\varepsilon_0 R_-} = -\frac{30\times 10^{-9}}{2\,\pi\,\varepsilon_0\sqrt{13}}\,\frac{2\,\bar{a}_x + 3\,\bar{a}_z}{\sqrt{13}}$$

✓ We now add the two fields to get electric field at point *P* :

$$E = -\frac{180\times 10^{-9}}{2\,\pi\,\varepsilon_0\,(13)}\,\bar{a}_z = -249\;\bar{a}_z \quad V/m$$

- ✓ Now

$$\overline{D} = \varepsilon_0 \overline{E} = -2.20\ \overline{a}_z \quad nC/m^2$$

- ✓ To find the charge density, use:

$$\rho_S = D_N = D_z = -2.20\ \ nC/m^2$$

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4. 4. Dielectric Materials

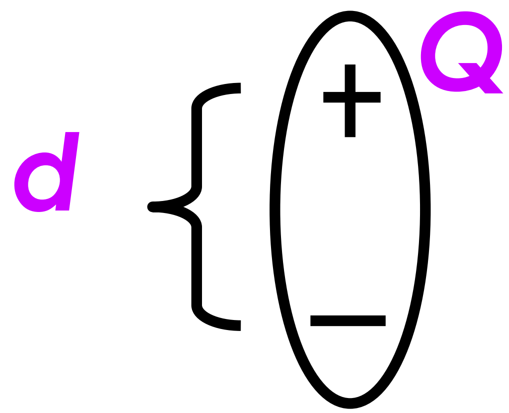
4. 4. 1. Electric Dipole and Dipole Moment

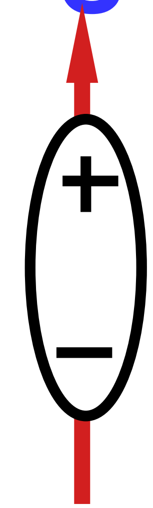
- In dielectric, charges are held in position (bound), and ideally there are no free charges that can move and form a current.

- Atoms and molecules may be polar (having separated positive and negative charges), or may be polarized by the application of an electric field.

- Consider such a polarized atom or molecule, which possesses a dipole moment, p, defined as the charge magnitude present, Q , times the positive and negative charge separation, d.

- **Dipole moment** is a vector that points from the negative to the positive charge.

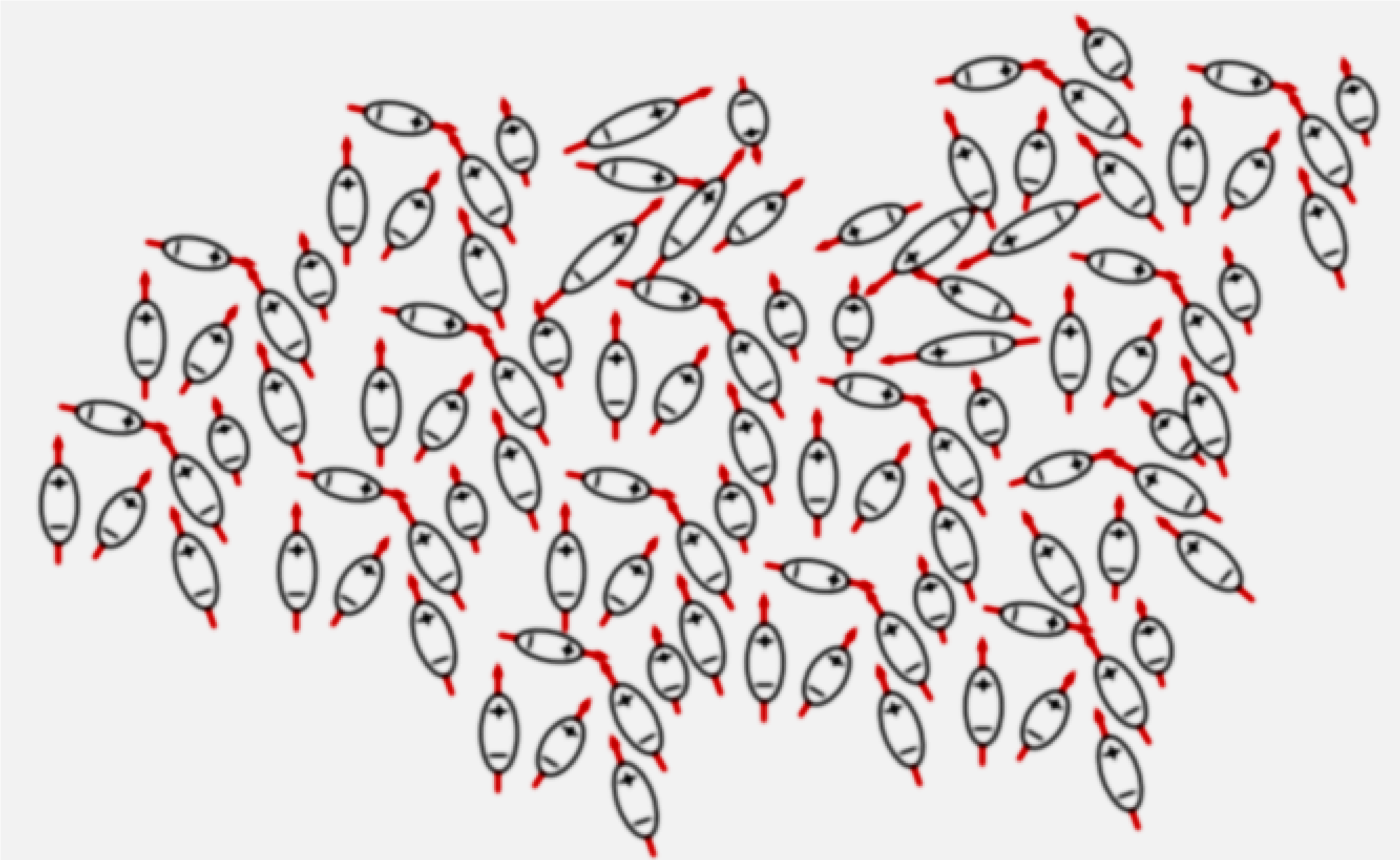




$$\overline{p} = Q\,d\ \overline{a}_x \quad \text{C m}$$

4. 4. 2. Electric Dipole and Dipole Moment

- A dielectric can be modeled as an ensemble of bound charges in free space, associated with the atoms and molecules that make up the material.
- Some of these may have intrinsic dipole moments, others not.
- In some materials (such as liquids), dipole moments are in random directions.

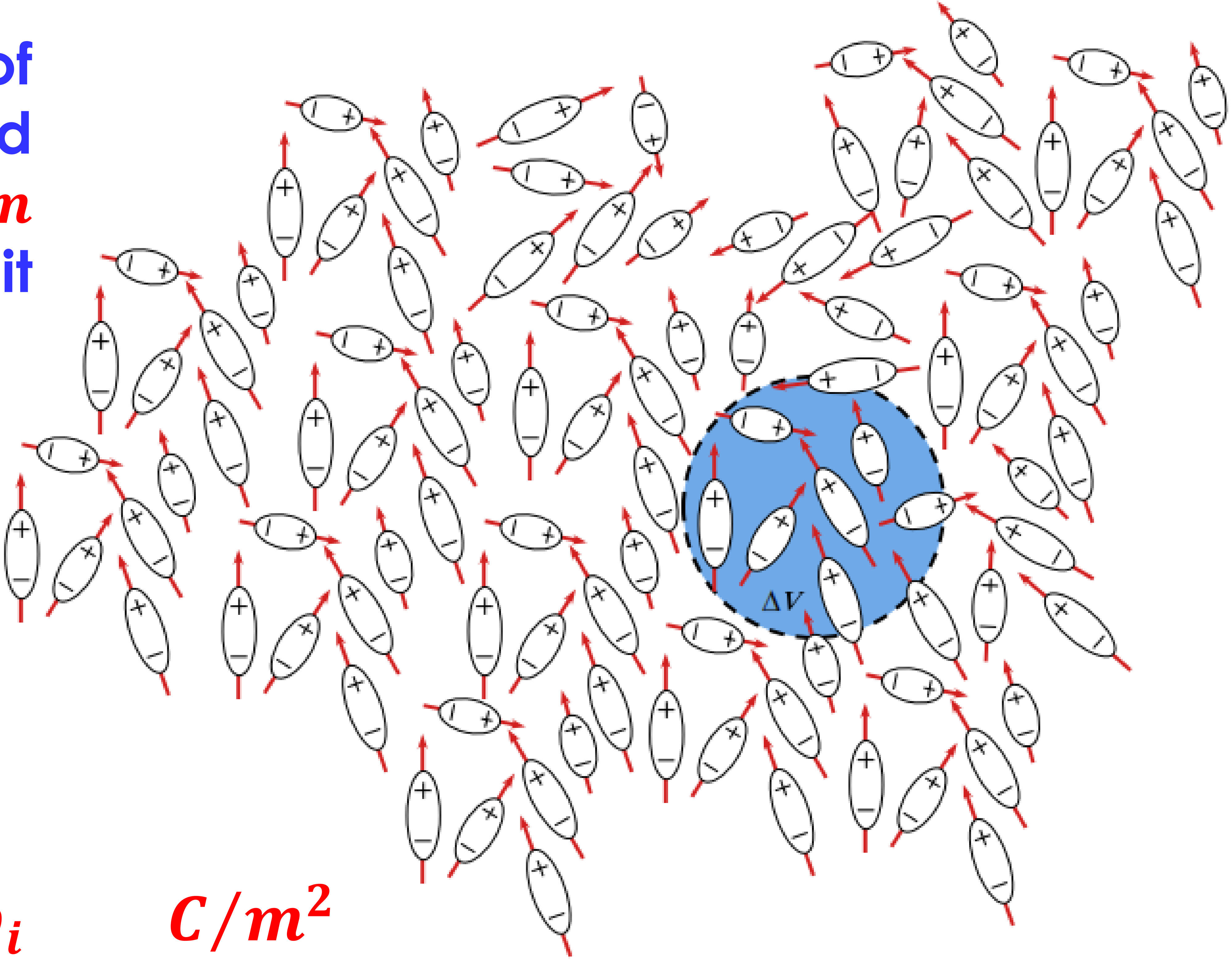


4. 4. 3. Polarization Field

- The number of dipoles is expressed as a density, n dipoles per unit volume.

- The *Polarization Field* of the medium is defined as:

$$P = \lim_{\Delta v \to 0} \frac{1}{\Delta v} \sum_{i=1}^{i=n\,\Delta v} p_i \qquad C/m^2$$



Without Applied Electric field

With Applied Electric field

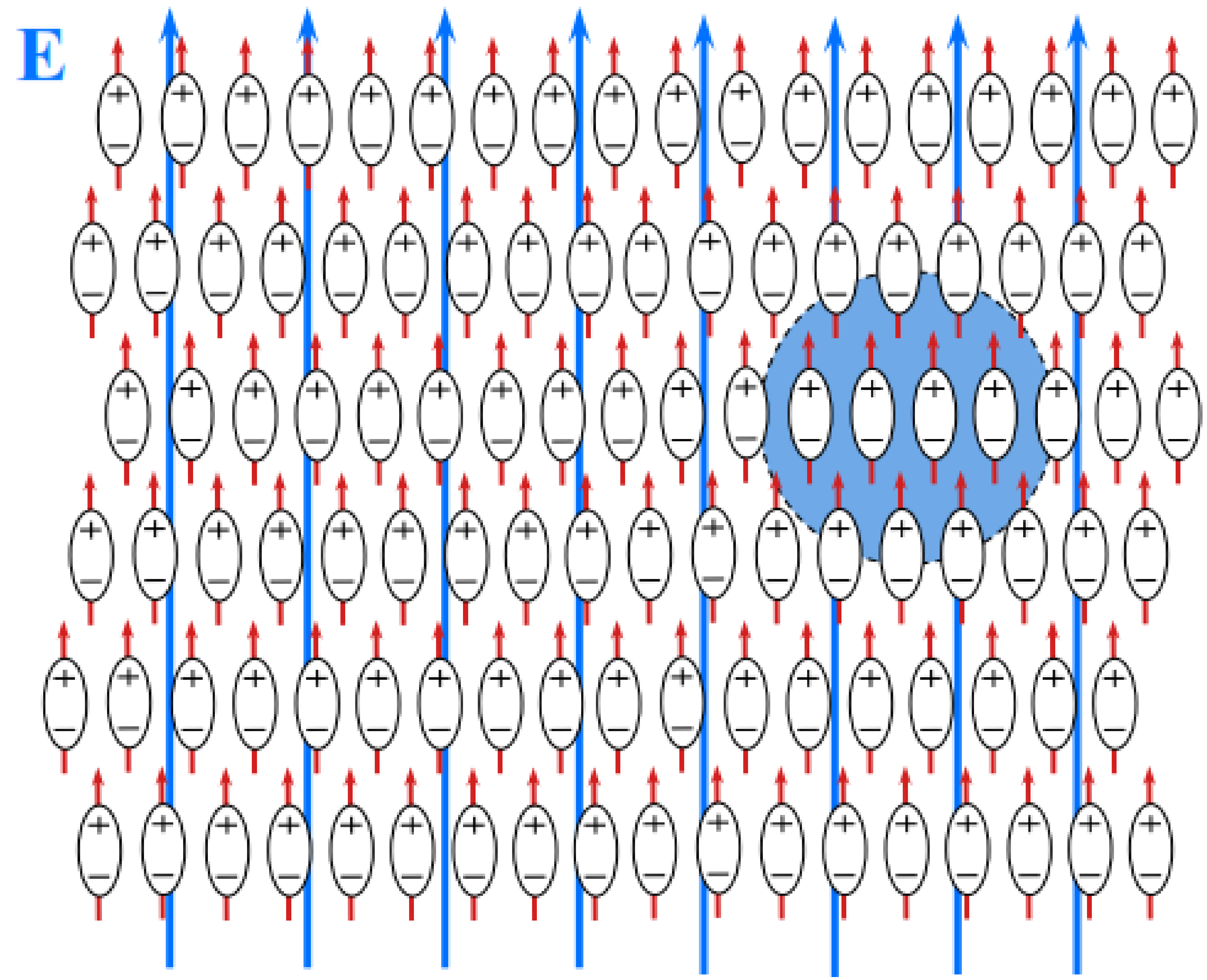
- Introducing an electric field may increase the charge separation in each dipole, and possibly re-orient dipoles so that there is some aggregate alignment, as shown.

- The effect is to increase P.

$$P = \lim_{\Delta v \to 0} \frac{1}{\Delta v} \sum_{i=1}^{i=n\,\Delta v} p_i \quad C/m^2$$

= np

if all dipoles are identical



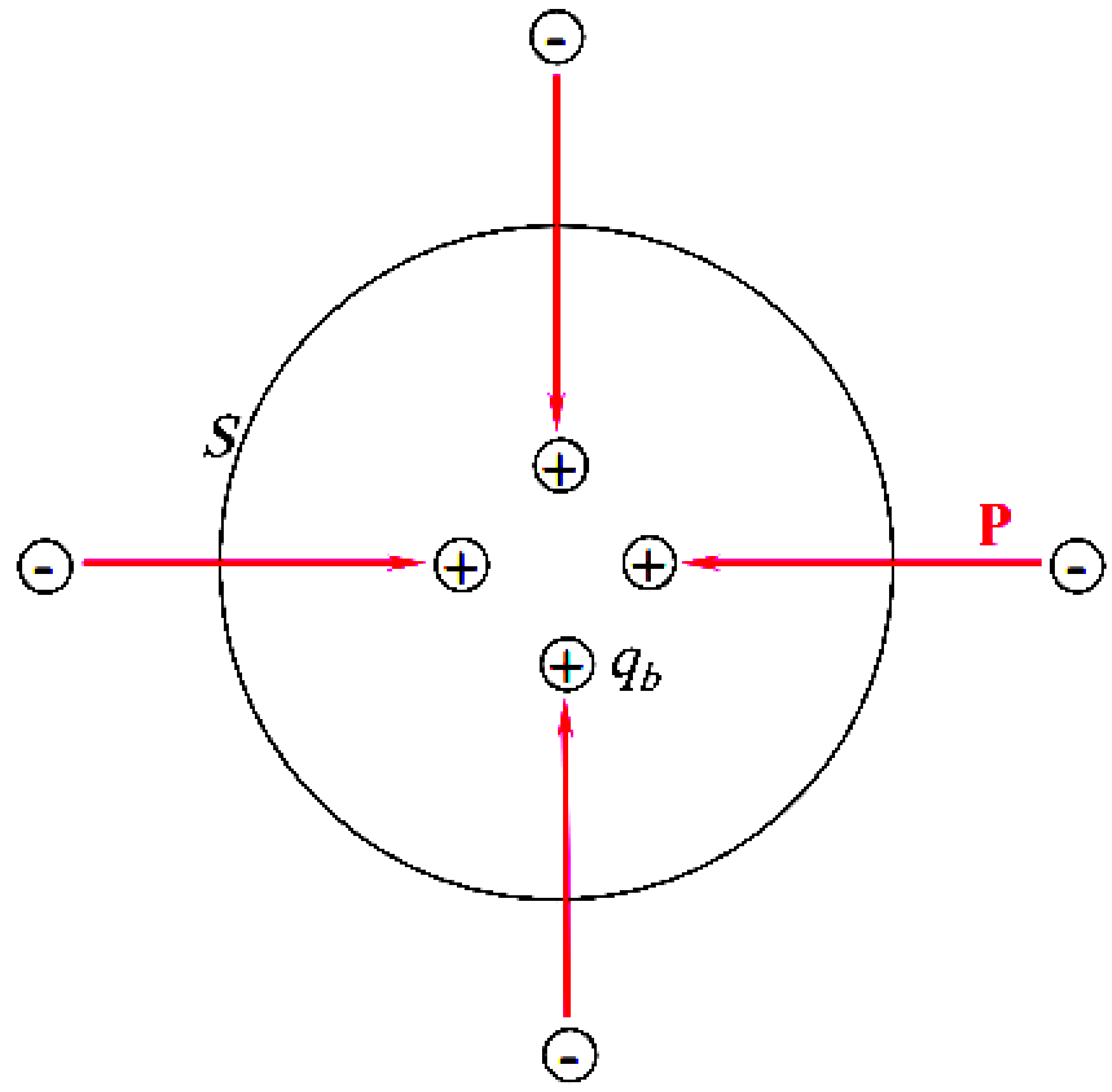
Polarization Flux Through a Closed Surface

- The accumulation of positive bound charge within a closed surface means that the polarization vector must be pointing *inward*. Therefore:

$$Q_b = -\oint_S \bar{P} \cdot d\bar{S}$$

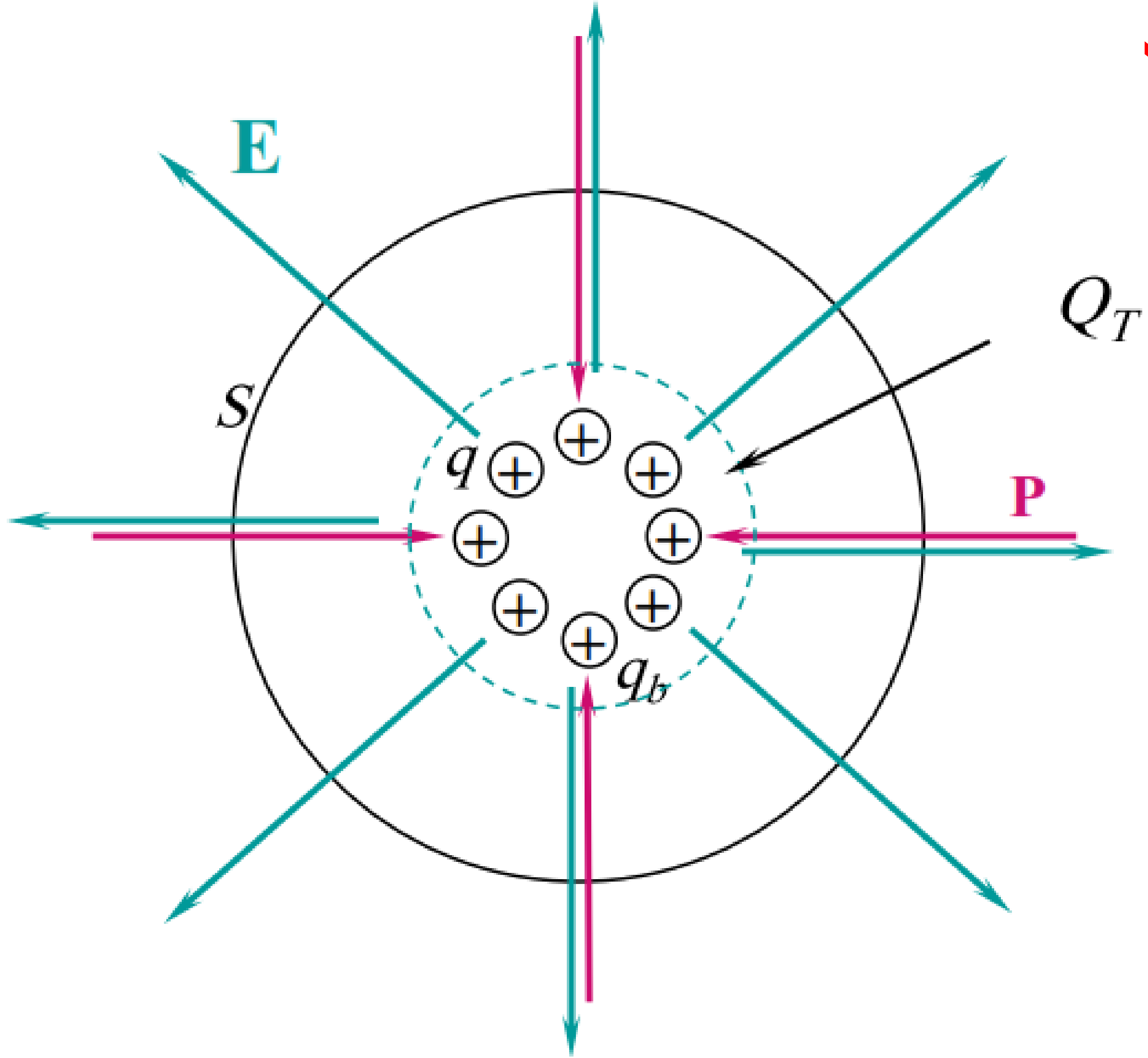
- and

$$Q_T = \oint_S \varepsilon_0 \bar{E} \cdot d\bar{S}$$



Bound and Free Charge

- Now consider the charge within the closed surface consisting of bound charges, Q_b, and free charges, Q.
- The total charge will be the sum of all bound and free charges.
- We write Gauss' Law in terms of the total charge, Q_T as:



$$Q_T = \oint_S \varepsilon_0 \overline{E} \cdot d\overline{S}$$

Where

$$Q_T = \underbrace{Q_b}_{\text{bound charge}} + \overbrace{Q}^{\text{free charge}}$$

Gauss Law for Free Charge

- We now have:

$$Q_b = -\oint_S \overline{P} \,.\, d\overline{S} \qquad \text{and} \qquad Q_T = \oint_S \varepsilon_0 \overline{E} \,.\, d\overline{S}$$

Where

$$Q_T = Q_b + Q$$

- Combining these, we write:

$$Q = Q_T - Q_b = \oint_S (\varepsilon_0 \overline{E} + \overline{P}) \,.\, d\overline{S}$$

- we thus identify

$$\overline{D} = \varepsilon_0 \overline{E} + \overline{P}$$

Electric Susceptibility and the Dielectric Constant

- A stronger electric field results in a larger polarization in the medium. In a *linear* medium, the relation between $\bar{P}$ and $\bar{E}$ is linear, and is given by:

$$\bar{P} = \chi_e \, \varepsilon_o \, \bar{E}$$

where χ_e is the electric *susceptibility* of the medium.

- We may now write:

$$\bar{D} = \varepsilon_o \, \bar{E} + \chi_e \, \varepsilon_o \, \bar{E} = \varepsilon_o \, (1 + \chi_e)\bar{E} = \varepsilon_o \, \varepsilon_r \, \bar{E}$$

where $\varepsilon_r = (1 + \chi_e) = \textit{Relative Permitittvity of the meduim}$
= Dielectric Constant

- The overall permittivity of the medium: $\varepsilon = \varepsilon_o \, \varepsilon_r$

and $\bar{D} = \varepsilon \, \bar{E}$

Isotropic vs. Anisotropic Media

- In an *isotropic medium*, the dielectric constant is invariant with *direction* of the applied electric field.
- This is not the case in an *anisotropic medium* (usually a crystal) in which the dielectric constant will vary as the electric field is rotated in certain directions.
- The electric flux density vector components must be evaluated separately through the dielectric *tensor*.

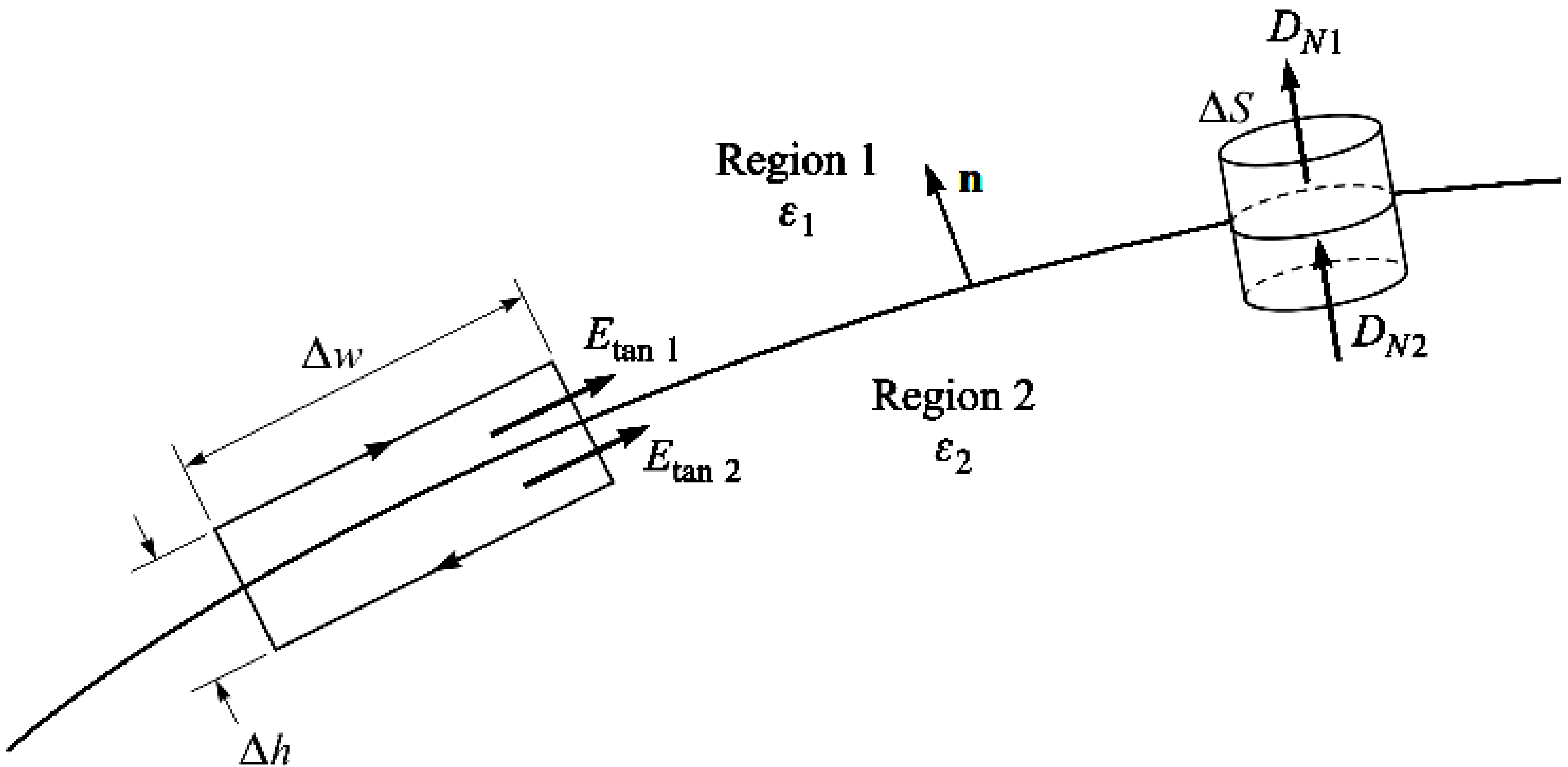
$$D_x = \varepsilon_{xx} E_x + \varepsilon_{xy} E_y + \varepsilon_{xz} E_z$$

$$D_y = \varepsilon_{yx} E_x + \varepsilon_{yy} E_y + \varepsilon_{yz} E_z$$

$$D_z = \varepsilon_{zx} E_x + \varepsilon_{zy} E_y + \varepsilon_{zz} E_z$$

Dielectric–Dielectric Boundary Cnditions

Consider the interface between two dielectrics having permittivities ε_1 and ε_2 and occupying regions 1 and 2, as shown.



The tangential components:

- We use the fact that E is conservative:

$$\oint_L \bar{E} \cdot d\bar{L} = 0$$

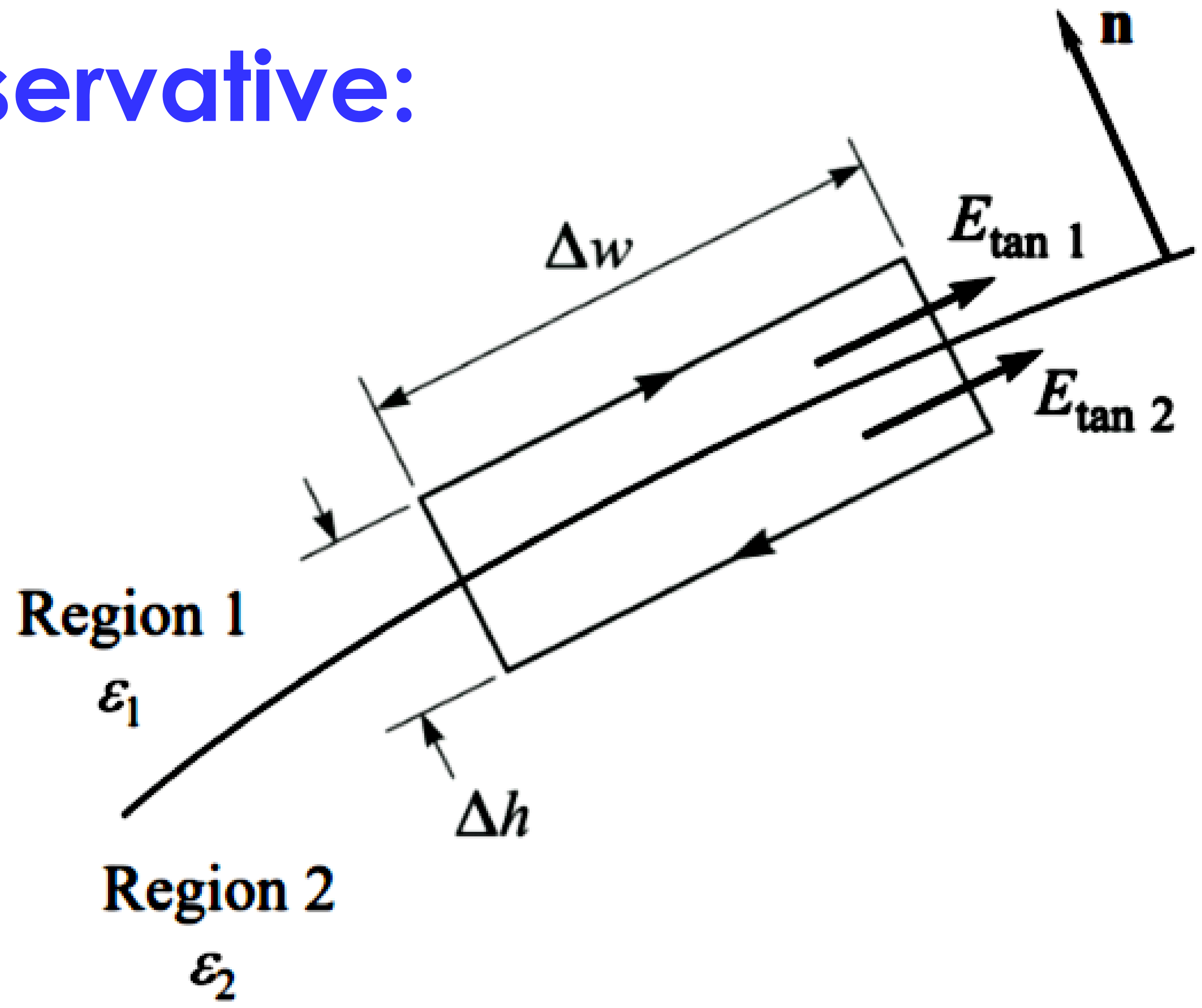
- Therefore:

$$E_{tan\,1}\,\Delta w - E_{tan\,2}\,\Delta w = 0$$

- Leading to:

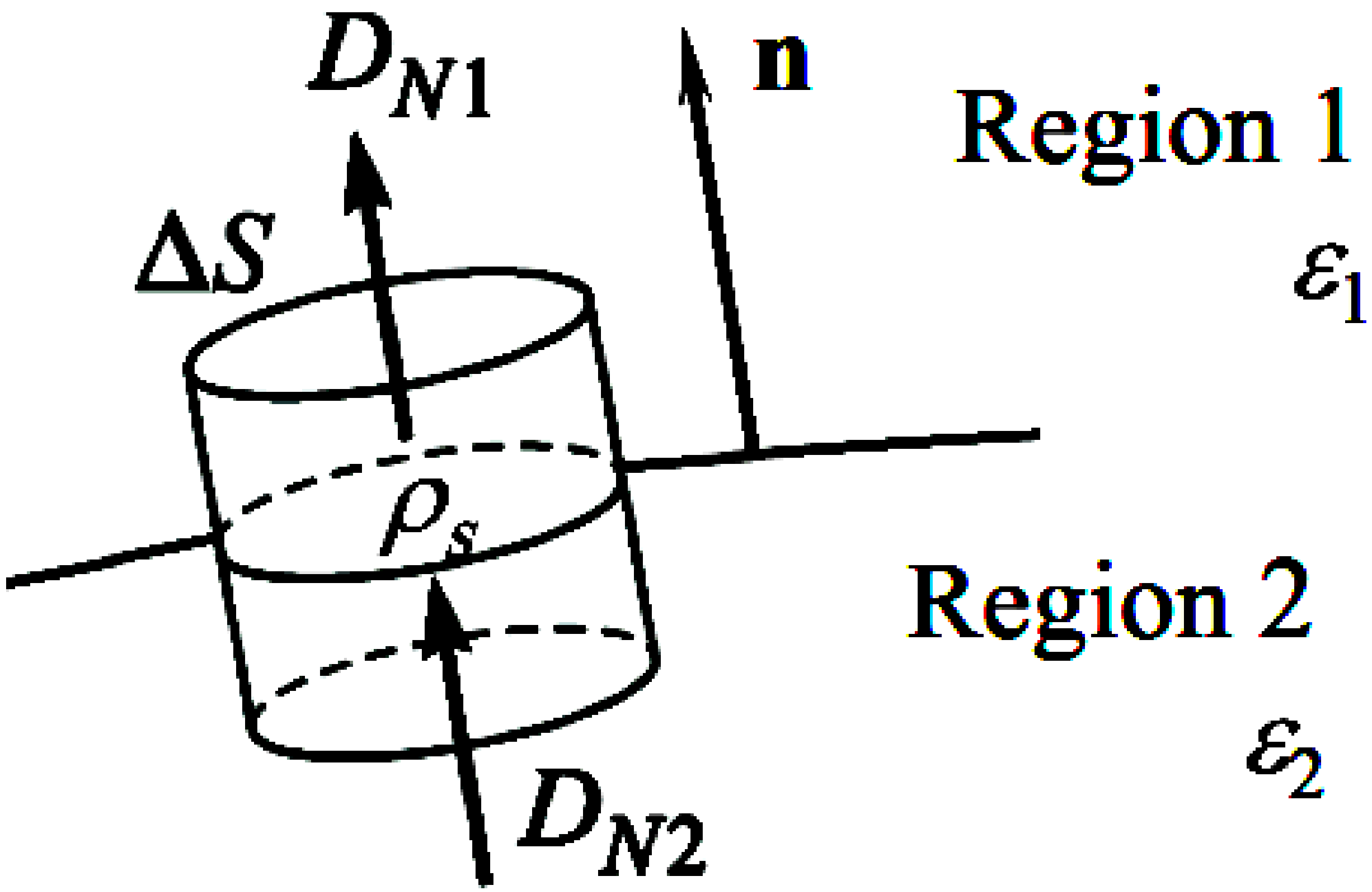
$$\boxed{E_{tan\,1} = E_{tan\,2}}$$

$$\boxed{\frac{D_{tan\,1}}{\varepsilon_1} = \frac{D_{tan\,2}}{\varepsilon_2}}$$



The normal components:

- Apply Gauss' Law to the cylindrical volume shown here, in which cylinder height is allowed to approach zero, and there is charge density ρ_s on the surface:



$$\oint_S \bar{D} \cdot d\bar{S} = Q$$

- The electric flux enters and exits only through the bottom and top surfaces, respectively.

$$D_{N\,1}\,\Delta s - D_{N\,2}\,\Delta s = \Delta\, Q = \rho_s\,\Delta s$$

- From which:

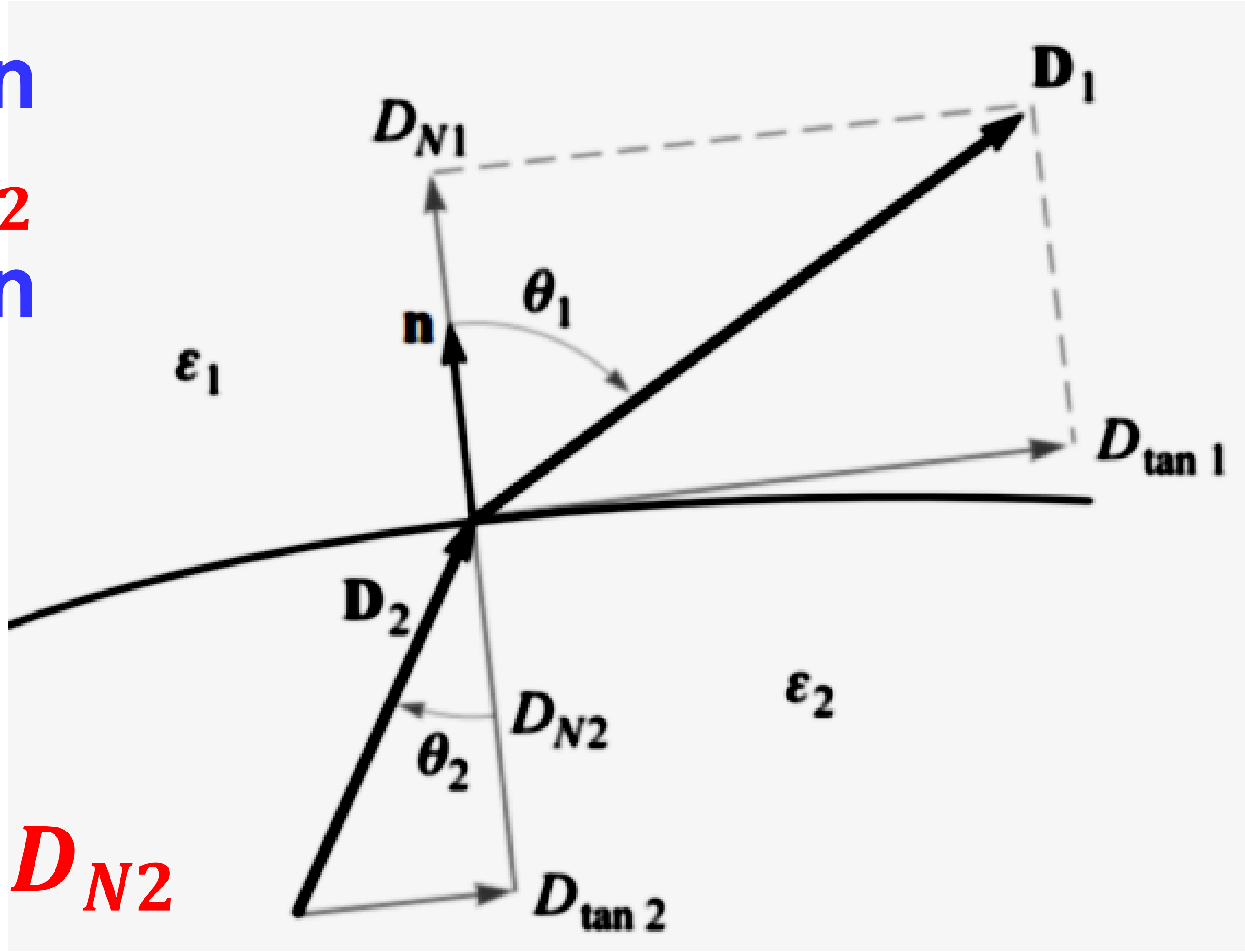
$$D_{N\,1} - D_{N\,2} = \rho_s$$

- and if the charge density is zero:

$$D_{N\,1} = D_{N\,2}$$

Example of the Use of Dielectric Boundary Conditions

- We wish to find the relation between the angles θ_1 and θ_2 assuming no charge density on the surface.
- The normal components of $\bar{D}$ will be continuous across the boundary, so that:



$$D_{N1} = D_1 \cos\theta_1 = D_2 \cos\theta_2 = D_{N2}$$

- Then, with tangential $\bar{E}$ continuous across the boundary, it follows that:

$$\frac{D_{tan\,1}}{D_{tan\,2}} = \frac{D_1 \sin\theta_1}{D_2 \sin\theta_2} = \frac{\varepsilon_1}{\varepsilon_2} \quad \Rightarrow\Rightarrow\Rightarrow \quad \frac{\tan\theta_1}{\tan\theta_2} = \frac{\varepsilon_1}{\varepsilon_2}$$

Example (4.4)

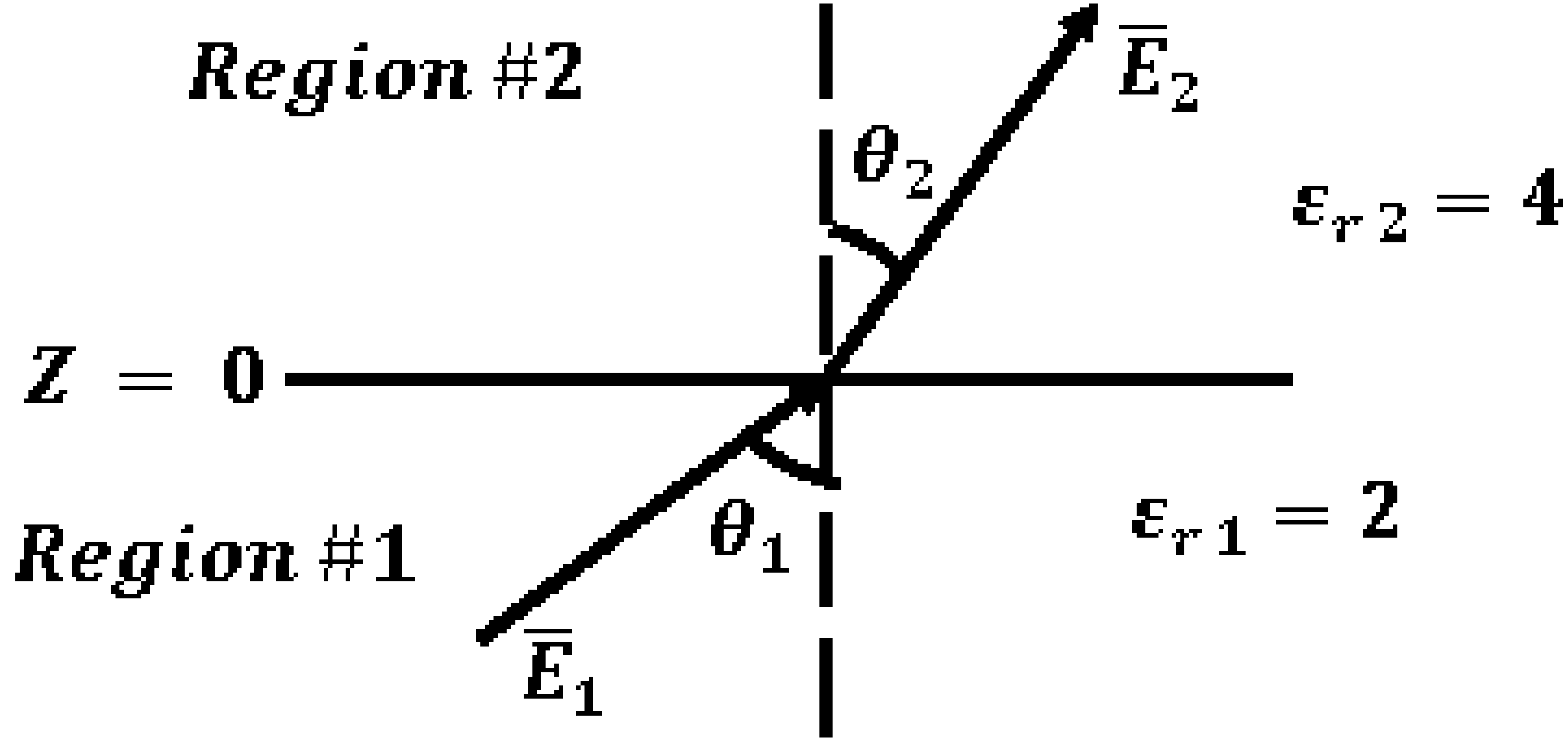
A $region\,\#1$, defined as $Z < 0$, contains a perfect dielectric of $\varepsilon_{r1} = 2.0$, while $region\,\#2$, $Z > 0$, contains a perfect dielectric material for which $\varepsilon_{r2} = 4.0$. If the electric field intensity in the $region\,\#1$ is $\bar{E}_1 = 3\,\bar{a}_x + 4\,\bar{a}_y - \bar{a}_z\ V/m$. Find:

a) $\bar{\mathrm{E}}_2$ and $\bar{\mathrm{D}}_2$.

b) The angles between $\bar{E}_2$ and a normal to the surface of interface

Solution

a) In the case of studying the boundary conditions between the different dielectric, we find that



$$D_{N\,1} = D_{N\,2}$$

or

$$D_{Z\,2} = D_{Z\,1} = -1 \times 2\,\varepsilon_o = -2\,\varepsilon_o \;\; C/m^2$$

also

$$E_{Z\,2} = \frac{D_{Z\,2}}{4\,\varepsilon_o} = -\,0.5 \quad V/m$$

and

$$E_{tan\,1} = E_{tan\,2}$$

or

$$\overline{E}_{tan\,2} = 3\,\overline{a}_x + \,3\,\overline{a}_y \qquad V/m$$

Therefore, the electric field intensity $\overline{E}_2$ in the $region$ #2 is:

$$\overline{E}_2 = 3\,\overline{a}_x + \,4\,\overline{a}_y - 0.5\,\overline{a}_z \qquad V/m$$

and the electric flux density $\overline{D}_2$ in the $region$ #2 is

$$\overline{D}_2 = \,\varepsilon_o\,\varepsilon_{r2}\,\overline{E}_2 = 4\,\varepsilon_o \times \left(3\,\overline{a}_x + \,4\,\overline{a}_y - \,0.5\,\overline{a}_z\right) \qquad C/m^2$$

b) The angle between $\bar{E}_2$ and a normal to the surface of interface is the angle θ_2.
Where:

$$\theta_2 = cos^{-1}\frac{|D_{N\,2}|}{|D_2|} = cos^{-1}\frac{0.5}{\sqrt{3^2 + 4^2 + 0.5^2}} = 84.3^0$$

The angle between $\bar{E}_1$ and a normal to the surface of interface is the angle θ_1. Where

$$\theta_1 = cos^{-1}\frac{|E_{N\,1}|}{|E_1|} = cos^{-1}\frac{1}{\sqrt{3^2 + 4^2 + 1^2}} = 78.7^0$$

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THE END OF PART (1) OF CHAPTER (4)