Cairo Uni., Engineering Faculty

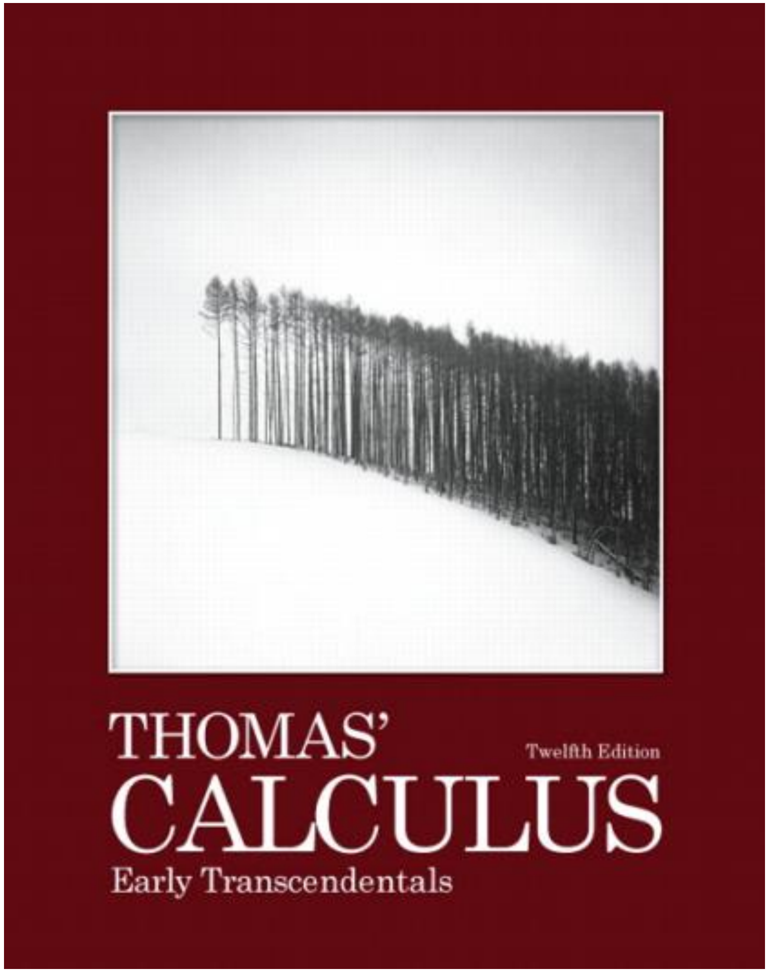
Multiple Integrals
Lecture Notes

FALL 2019

Contents

Main Reference Book

Thomas' Calculus 12th Edition



The Iterated Integral
(Nested Loops of Integration)

Example 1:- Solve the following double integration:-

$$\left(\int_2^4 \left(\int_0^1 4dx\right) dy\right)$$

Similar to the programming concepts, you should finalize the inner loop (Integral) first then discuss the outer loop (Integral) as follows:-

$$\left(\int_2^4 \left(\int_0^1 4dx\right) dy\right) = \left(\int_2^4 \left(4x\,\Big|_0^1\right) dy\right) = \left(\int_2^4 (4)dy\right)$$

$$= \int_2^4 4dy \; = 4y\,\Big|_2^4 = 4(4-2) = 8$$

Note:- The order of operation is important !!

The Double Integrals

Example 2:- Solve the following double integration:-

$$\left(\int_2^4 \left(\int_0^1 4e^x dx\right) dy\right)$$

Similar to the previous example:-

$$\left(\int_2^4 \left(\int_0^1 4e^x dx\right) dy\right) = \left(\int_2^4 \left(4e^x \Big|_0^1\right) dy\right)$$

$$= \left(\int_2^4 (4(e^1 - 1))dy\right)$$

$$= 4(e^1 - 1)\int_2^4 1dy = 4(e^1 - 1)y\Big|_2^4 = 8(e^1 - 1)$$

Exercise 1:- Solve the following double integration:-

$$\left(\int_2^4 \left(\int_0^1 (\sin^{-1} x)dx\right) dy\right)$$

Example 3:- Solve the following double integration:-

$$\left(\int_2^4 \left(\int_0^1 (2x+y^2)dx\right)dy\right)$$

The inner integral w.r.t. x, then y will be considered as constant

$$\left(\int_2^4 \left(\int_0^1 (2x+y^2)dx\right)dy\right) = \left(\int_2^4 \left((x^2+xy^2)\Big|_0^1\right)dy\right)$$

Note that the inner upper and lower limits w.r.t. x and not y as follows:

$$I = \left(\int_2^4 \left((1^2+1y^2)-(0^2+0y^2)\right)dy\right) = \int_2^4 (1+y^2)dy$$

Now the next integral w.r.t. y only ($x\ should\ be\ disappeared\ here$)

$$I = \int_2^4 (1+y^2)dy = \left(y+\frac{y^3}{3}\right)\Big|_2^4 = \left(4+\frac{64}{3}\right)-\left(2+\frac{8}{3}\right) = \cdots..$$

Exercise 2:- Solve the following double integration:-

$$\left(\int_2^4 \left(\int_0^1 (x^4+\cosh(y))dx\right)dy\right)$$

Example 4:- Solve the following double integration:-

$$\left(\int_2^4\left(\int_0^1 (e^y + ysec^2x)dx\right)dy\right)$$

The inner integral w.r.t. x, then y will be considered as constant

$$\left(\int_2^4\left(\int_0^1 (e^y + ysec^2x)dx\right)dy\right) = \left(\int_2^4\left((xe^y + ytanx)\Big|_0^1\right)dy\right)$$

Note that the inner upper and lower limits w.r.t. x and not y as follows:

$$I = \left(\int_2^4\big((1e^y + ytan1) - (0e^y + ytan0)\big)dy\right) = \int_2^4 (e^y + ytan1)dy$$

Now the next integral w.r.t. y only (x *should be disappeared here*)

$$I = \int_2^4 (e^y + ytan1)dy = \left(e^y + \frac{y^2}{2}\tan(1)\right)\Big|_2^4 = \cdots \ldots.\ldots..$$

Exercise 3:- Solve the following double integration:-

$$\left(\int_2^4\left(\int_0^1 (x^4)dx\right)dy\right)$$

Note that the upper and lower limits are, in general, variables. So, the generalized case can be written as follows:-

$$\int_{y=a}^{y=b}\left(\int_{x=g_1(y)}^{x=g_2(y)} f(x,y)dx\right)dy \quad \textbf{OR} \quad \int_{x=a}^{x=b}\left(\int_{y=g_1(x)}^{y=g_2(x)} f(x,y)dy\right)dx$$

Example 5:- Solve the following double integration:-

$$\int_0^1\left(\int_0^y (e^y + 2xy)dx\right)dy$$

$$\int_0^1\left(\int_0^y (e^y + 2xy)dx\right)dy = \int_0^1\left((xe^y + x^2y)\Big|_0^y\right)dy$$

$$= \int_0^1\big((ye^y + y^2y) - (0e^y + 0^2y)\big)dy = \int_0^1 (ye^y + y^3)dy = By\ parts \ \ldots\ldots$$

Example:- Solve the following double integration:-

$$\int_0^1 \left(\int_{\mathrm{x}}^{x^2} (2x^2y + 3x)dy \right) dx$$

$$\int_0^1 \left(\int_{\mathrm{x}}^{x^2} (2x^2y + 3x)dy \right) dx = \int_0^1 \left((x^2y^2 + 3xy) \Big|_{y=x}^{y=x^2} \right) dx$$

$$= \int_0^1 \left((x^2x^4 + 3xx^2) - \left((x^2x^2 + 3xx) \right) \right) dx$$

$$= \int_0^1 (x^6 - x^4 + 3x^3 - 3x^2) dx$$

$$= \left(\frac{x^7}{7} - \frac{x^5}{5} + \frac{3x^4}{4} - x^3 \right) \Big|_0^1$$

$$= \left(\frac{1}{7} - \frac{1}{5} + \frac{3}{4} - 1 \right) - (0) = \frac{20 - 28 + 105 - 140}{140} = \frac{-43}{140}$$

Example 1: Calculate:- $\iint_R (xye^x)\,dA$

Where R: $0 \le x \le 1, \quad 0 \le y \le 2$

Integrate with respect to y first (easier)

$$\int_0^1 \left(\int_0^2 (xye^x)\,dy \right) dx$$

$$= \int_0^1 \left(x\frac{y^2}{2}e^x \Big|_0^2 \right) dx$$

$$= \int_0^1 (2xe^x)\,dx$$

continue. by parts

Try the following :-

$$\iint_R xy\cos y\, dA, \qquad R: \quad -1 \le x \le 1, \quad 0 \le y \le \pi$$

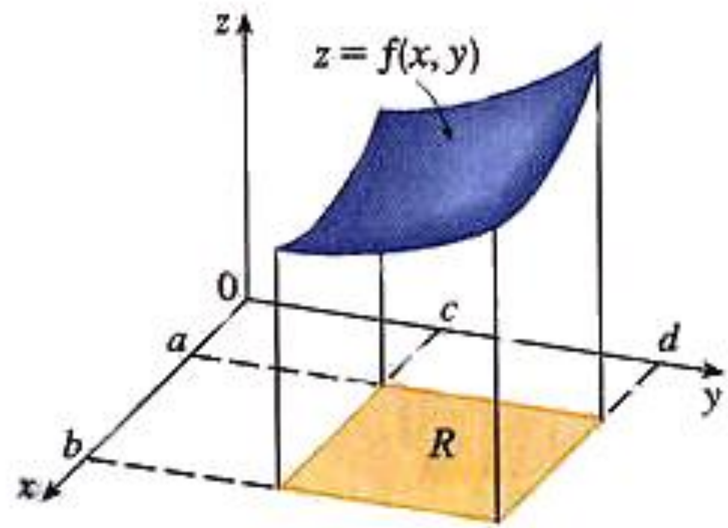
$$\iint_R y\sin(x+y)\, dA, \qquad R: \quad -\pi \le x \le 0, \quad 0 \le y \le \pi$$

Applications: Volume using Double Integrals

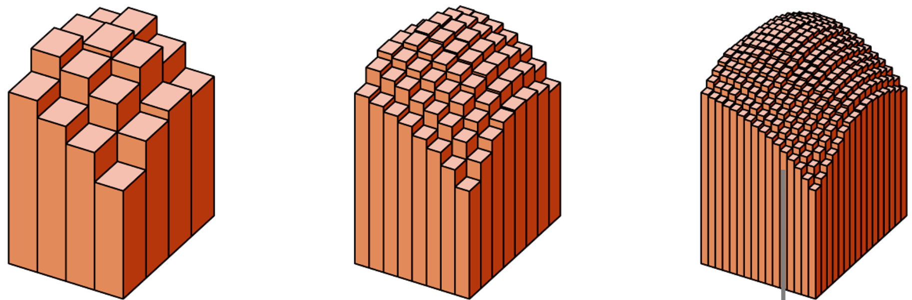
For the surface :

$$S = \{(x, y, z) in\ R^3 \mid 0 \le z \le f(x, y), \qquad (x, y)\ in\ R\}$$

$$z = f(x, y) \ge 0$$



The integral of ***f(x,y)*** over the region R is the Volume under the surface ***z=f(x,y)***



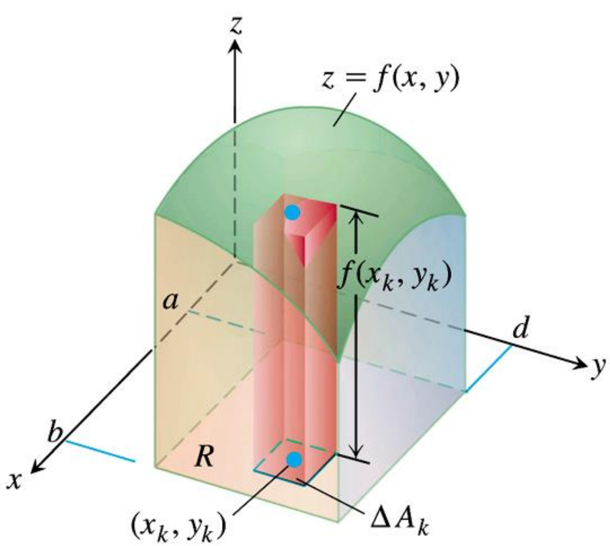
(a) $n = 16$ (b) $n = 64$ (c) $n = 256$

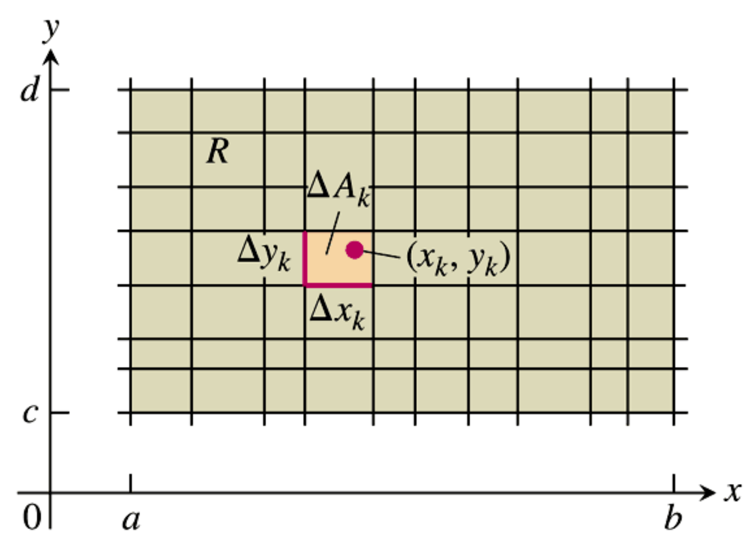
FIGURE 15.3 As n increases, the Riemann sum approximations approach the total

$$\text{Volume} = \lim_{n\to\infty} S_n = \iint_R f(x, y)\, dA,$$

where $\Delta A_k \to 0$ as $n \to \infty$.

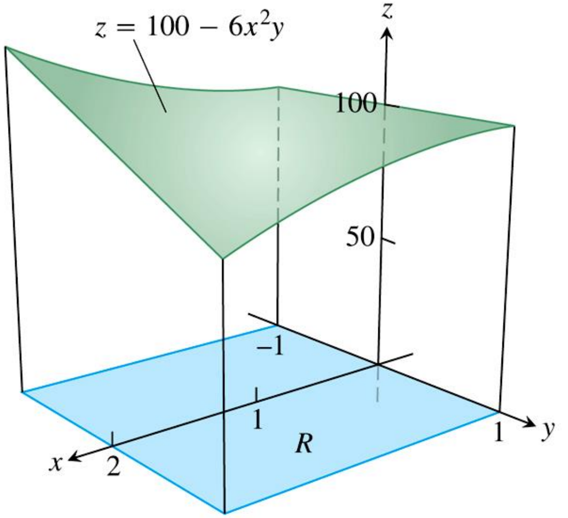
We can use this idea to implement the double integrals using any programming language. The value obtained will be approximate (not exact). The approximation depends on the number of divisions (n) and the method of summation.





Example 1:- find the volume under z $= (100 - 6x^2y)$ and over the region R,

Where R: $0 \le x \le 2, \quad -1 \le y \le 1$



V = $\int_{-1}^{1}\left(\int_{0}^{2}(100 - 6x^2y)dx\right)dy$

$$= \int_{-1}^{1}\left((100x - 2x^3y)\Big|_0^2\right)dy$$

$$= \int_{-1}^{1}(200 - 16y)dy$$

$$= (200y - 8y^2)\Big|_{-1}^{1} = 400$$

Note: :- Reversing the order of integration gives the same answer

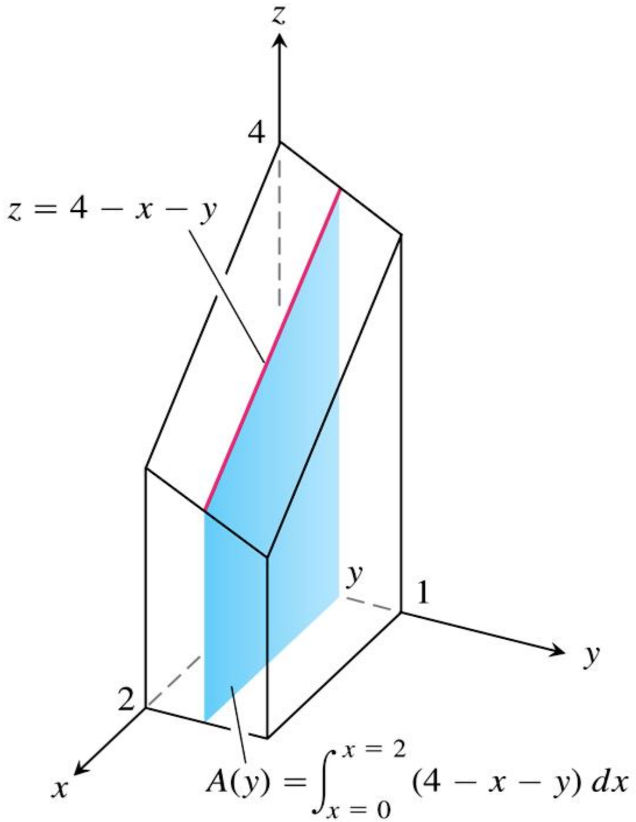
Example:- Find the volume under the plane $Z = 4 - x - y$ over the Rectangular region $R: 0 \le x \le 2, 0 \le y \le 1$ in the x-y plane

$$V = \int_0^1 \int_0^2 (4 - x - y)dxdy$$

$$V = \int_0^1 \left((4x - 0.5x^2 - yx) \Big|_0^2 \right) dy$$

$$= \int_0^1 (6 - 2y)dy$$

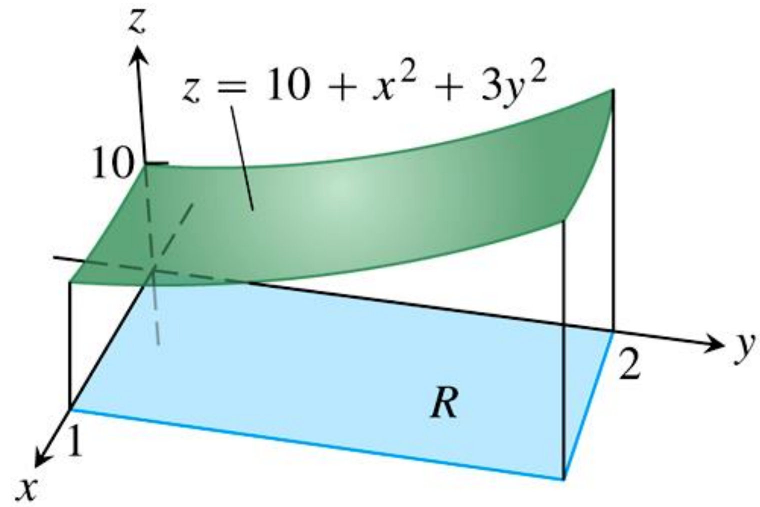
$$= (6y - y^2) \Big|_0^1 = 5$$



Note: :- Reversing the order of integration gives the same answer

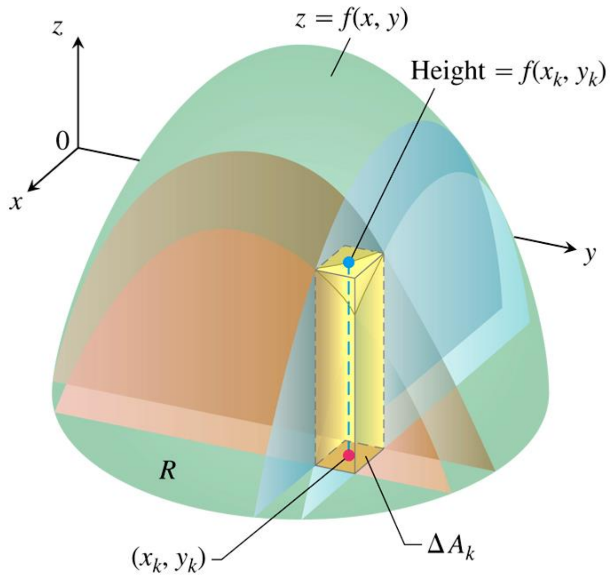
EXAMPLE 2:- Find the volume of the region bounded above by the elliptical paraboloid $z = 10 + x^2 + 3y^2$ and below by the rectangle $R = 0 \le x \le 1,\ 0 \le y \le 2$.

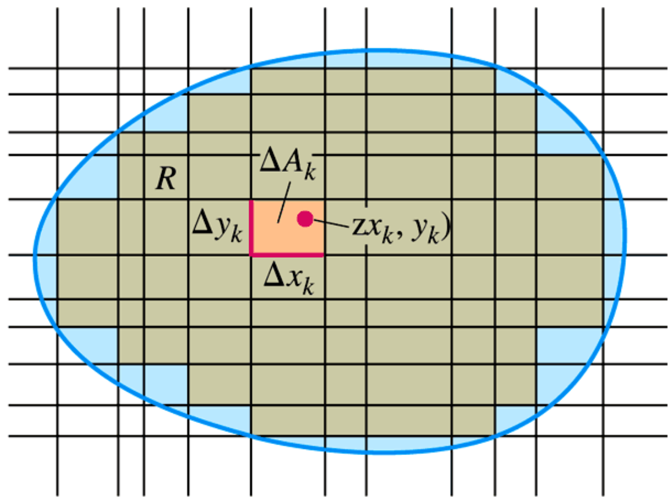
$$V = \int_0^1 \int_0^2 (10 + x^2 + 3y^2)\,dy\,dx$$



Continue..

Double Integrals over General Regions





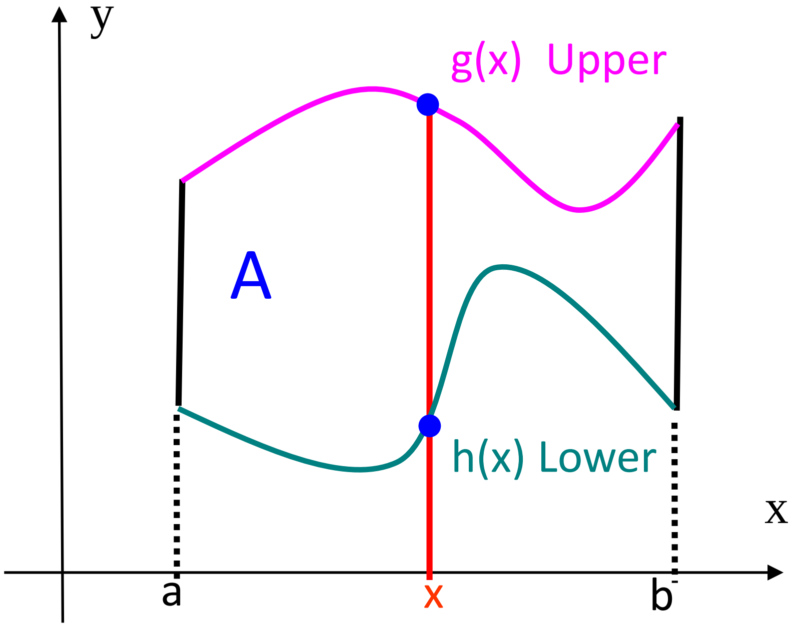
If R is defined by $a \leq x \leq b$, $g_1(x) \leq y \leq g_2(x)$, with g_1 and g_2 continuous on $[a, b]$, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

If R is defined by $c \leq y \leq d$, $h_1(y) \leq x \leq h_2(y)$, with h_1 and h_2 continuous on $[c, d]$, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

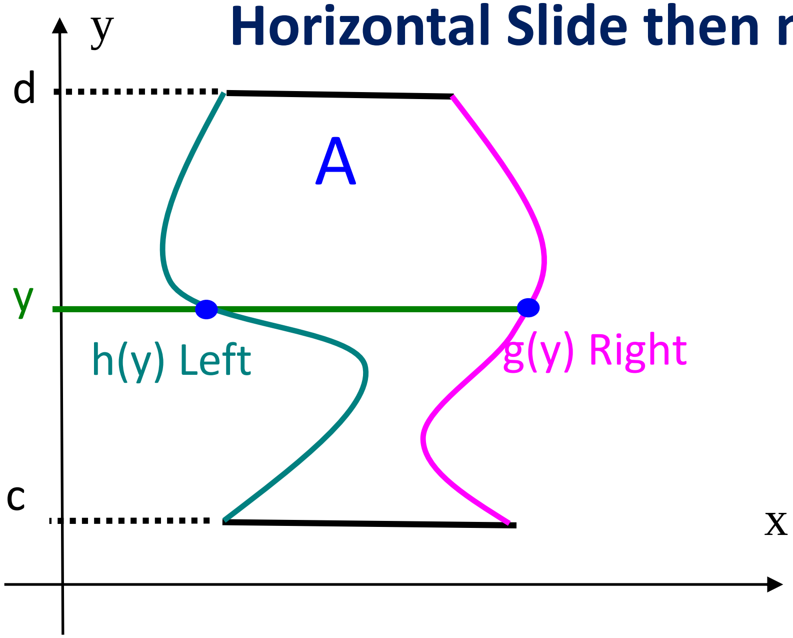
Vertical Slide then moving Horizontal



Type 1 Region

$$\iint_A f(x,y)\,dA = \int_a^b \int_{h(x)}^{g(x)} f(x,y)\,dy\,dx$$

Horizontal Slide then moving Vertical



Type 2 Region

$$\iint_R f(x,y)\,dA = \int_c^d \int_{h(y)}^{g(y)} f(x,y)\,dx\,dy$$

EXAMPLE Find the volume of the prism whose base is the triangle in the xy-plane bounded by the x-axis and the lines $y = x$ and $x = 1$ and whose top lies in the plane

$$z = f(x, y) = 3 - x - y.$$

Then the region of integration is as shown.

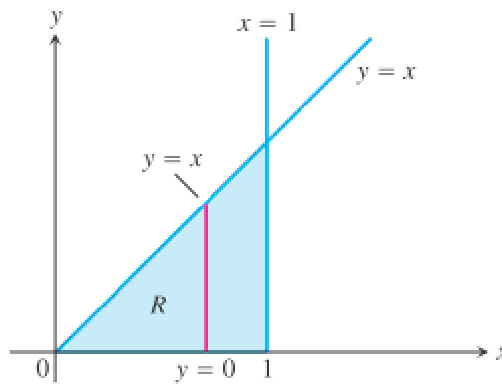
Type 1 Region (Upper and lower curves)

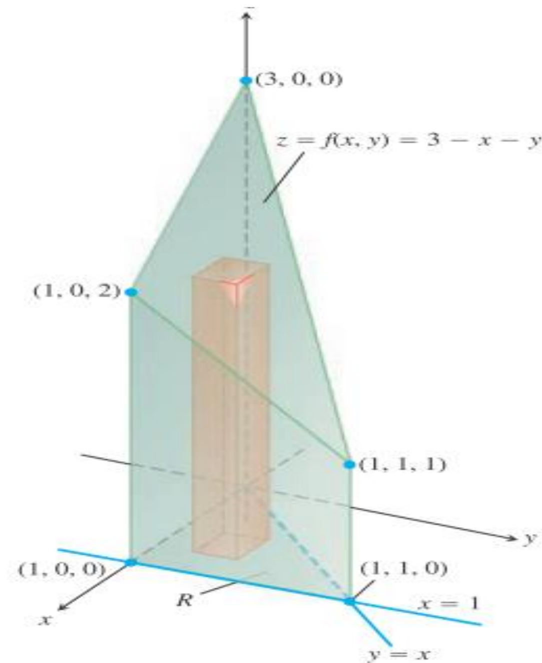
$$V = \int_0^1 \int_0^x (3 - x - y)\, dy\, dx$$

$$= \int_0^1 \left(3y - xy - \frac{y^2}{2} \Big|_0^x \right) dx$$

$$= \int_0^1 \left(3x - x^2 - \frac{x^2}{2} \right) dx$$

Continue.. (Answer: *V*=1)





Another solution with

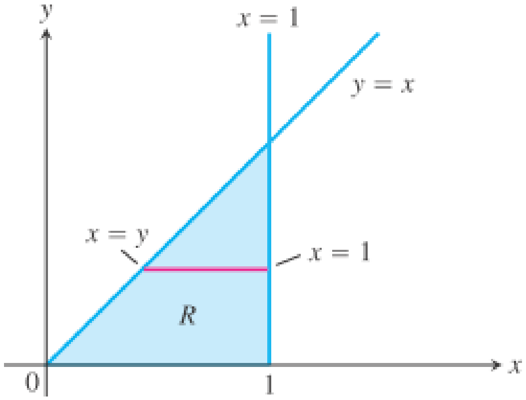
Type 2 Region (Left and Right curves)

$$V = \int_0^1 \int_y^1 (3 - x - y)\, dx\, dy$$

$$= \int_0^1 \left(3x - \frac{x^2}{2} - yx \Big|_y^1 \right) dy$$

$$= \int_0^1 \left(3(1) - \frac{(1)^2}{2} - y(1) \right) - \left(3(y) - \frac{(y)^2}{2} - y^2 \right) dy$$

$$= \int_0^1 \left(\frac{5}{2} - 4y - \frac{3(y)^2}{2} \right) dy$$



Continue.. (Answer: *V*=1)

Double Integrals : Change of order

EXAMPLE Sketch the region of integration for the integral

$$\int_0^2 \int_{x^2}^{2x} (4x + 2)\, dy\, dx$$

and write an equivalent integral with the order of integration reversed.

Solution:- $\int_{x=0}^{x=2} \int_{y=x^2}^{y=2x} (4x+2)\, dy\, dx$

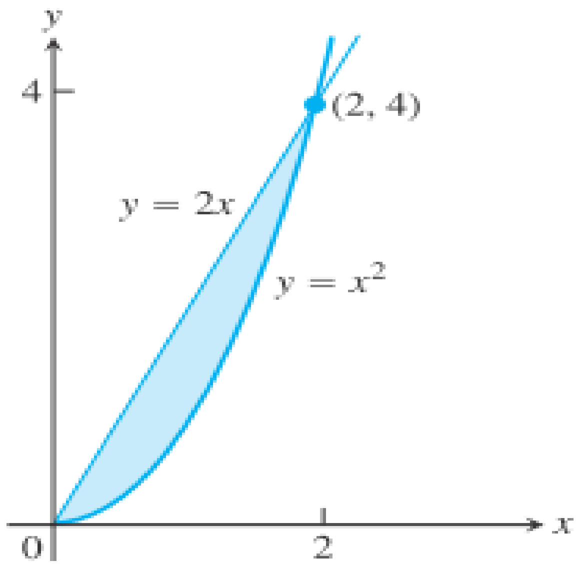
First, the order $dy\, dx$ means (Type 1Region)

$y = x^2\ (lower), y = 2x(upper)$

Then the region of integration is as shown.

By reverse the order $dx\, dy$

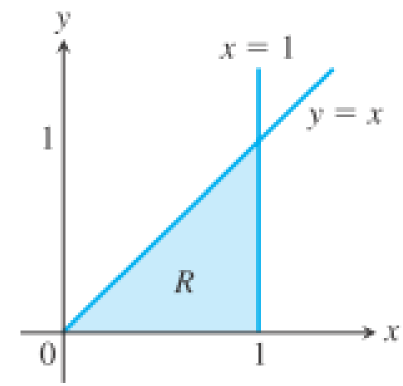
$$\int_{y=0}^{y=4} \int_{x=y/2}^{x=\sqrt{y}} (4x+2)\, dx\, dy$$



EXAMPLE Calculate $\displaystyle\iint_R \frac{\sin x}{x}\, dA,$

where R is the triangle in the xy-plane bounded by the x-axis, the line $y = x$, and the line $x = 1$.

Note: $\frac{sinx}{x}$ is not easy to be integrated w.r.t. x
and it is easy w.r.t. y, then the order should be $dy\, dx$
The intersection points are (0,0) and (1,1)

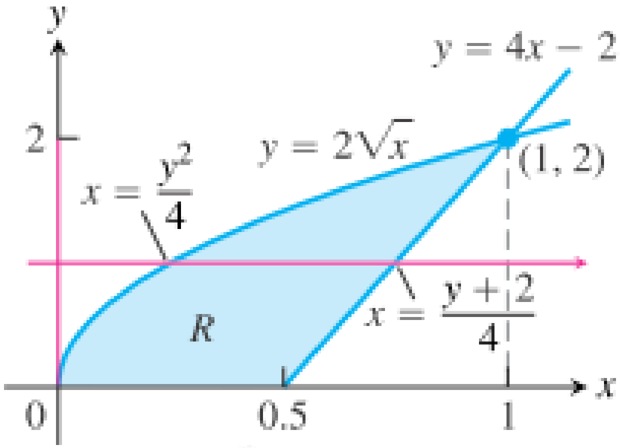


$$\int_{x=0}^{x=1}\int_{y=0}^{y=x} \frac{sinx}{x}\, dy\, dx$$

$$= \int_{x=0}^{x=1} \left(\frac{sinx}{x} y\right)\Big|_0^x dx = \int_{x=0}^{x=1} \left(\frac{sinx}{x} x\right) dx$$

$$= \int_{x=0}^{x=1} sinx\, dx = -\cos(x)\Big|_0^1 = 1 - \cos(1)$$

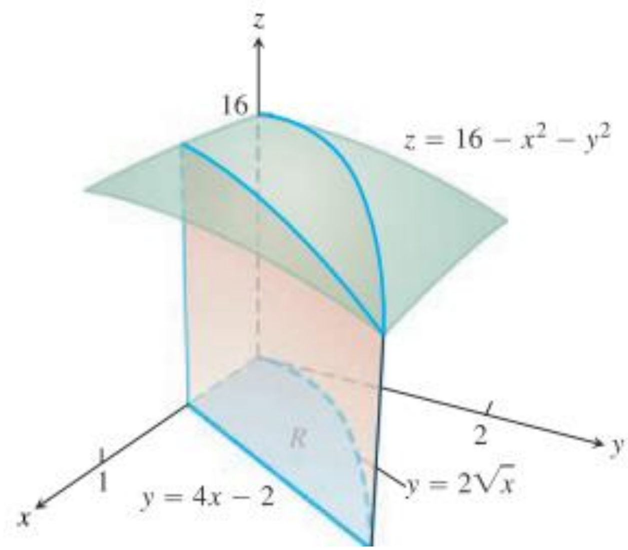
EXAMPLE Find the volume of the wedgelike solid that lies beneath the surface $z = 16 - x^2 - y^2$ and above the region R bounded by the curve $y = 2\sqrt{x}$, the line $y = 4x - 2$, and the x-axis.



Type 2 Region (easier)

Right curve $y = 4x - 2 \rightarrow \mathrm{x} = \frac{y+2}{4}$

Left curve $y = 2\sqrt{x} \rightarrow \mathrm{x} = \frac{y^2}{4}$



$$Volume = \int_0^2 \int_{x=\frac{y^2}{4}}^{x=\frac{y+2}{4}} (16 - x^2 - y^2)dx\,dy$$

Continue..

Finding Regions of Integration and Double Integrals

In Exercises 19–24, sketch the region of integration and evaluate the integral.

19. $\int_0^{\pi}\int_0^{x} x \sin y \, dy \, dx$

20. $\int_0^{\pi}\int_0^{\sin x} y \, dy \, dx$

21. $\int_1^{\ln 8}\int_0^{\ln y} e^{x+y} \, dx \, dy$

22. $\int_1^{2}\int_y^{y^2} dx \, dy$

23. $\int_0^{1}\int_0^{y^2} 3y^3 e^{xy} \, dx \, dy$

24. $\int_1^{4}\int_0^{\sqrt{x}} \frac{3}{2} e^{y/\sqrt{x}} \, dy \, dx$

In Exercises 47–56, sketch the region of integration, reverse the order of integration, and evaluate the integral.

47. $\int_0^{\pi}\int_x^{\pi} \frac{\sin y}{y} \, dy \, dx$

48. $\int_0^{2}\int_x^{2} 2y^2 \sin xy \, dy \, dx$

49. $\int_0^{1}\int_y^{1} x^2 e^{xy} \, dx \, dy$

50. $\int_0^{2}\int_0^{4-x^2} \frac{xe^{2y}}{4-y} \, dy \, dx$

51. $\int_0^{2\sqrt{\ln 3}}\int_{y/2}^{\sqrt{\ln 3}} e^{x^2} \, dx \, dy$

52. $\int_0^{3}\int_{\sqrt{x/3}}^{1} e^{y^3} \, dy \, dx$

53. $\int_0^{1/16}\int_{y^{1/4}}^{1/2} \cos(16\pi x^5) \, dx \, dy$

54. $\int_0^{8}\int_{\sqrt[3]{x}}^{2} \frac{dy \, dx}{y^4+1}$

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Applications : Area using Double Integrals

DEFINITION The **area** of a closed, bounded plane region R is

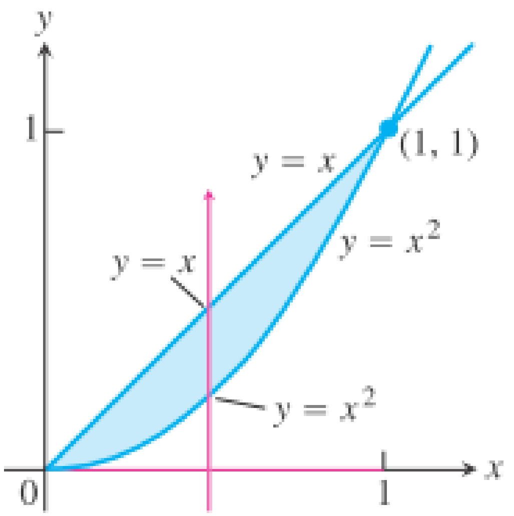
$$A = \iint_R dA.$$

EXAMPLE : Find the area of the region R bounded by $y = x$ and $y = x^2$ in the first quadrant.

$$Area = \int_0^1 \int_{x^2}^{x} (1)\,dy\,dx$$

$$Area = \int_0^1 \left(y \Big|_{x^2}^{x} \right) dx$$

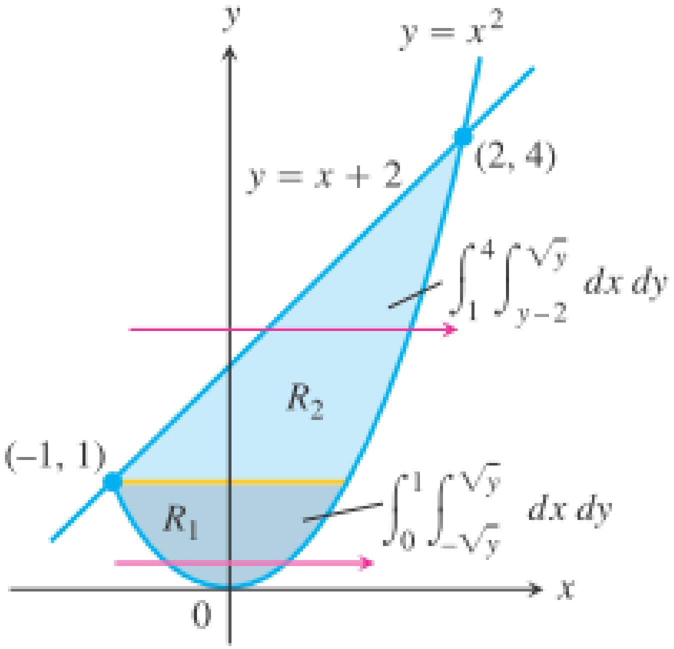
$$Area = \int_0^1 (x - x^2)\,dx$$



Continue..

EXAMPLE Find the area of the region R enclosed by the parabola $y = x^2$ and the line $y = x + 2$.

$$Area = \int_{-1}^{2}\int_{x^2}^{x+2}(1)dy\,dx$$



Try Type 1 and Type 2;
Compare and Comment

Note:

If $f(x,y) = g(x)h(y)$ and we are integrating over the rectangle $R = [a,b] \times [c,d]$ then,

$$\iint\limits_R f(x,y)\,dA = \iint\limits_R g(x)h(y)\,dA = \left(\int_a^b g(x)\,dx\right)\left(\int_c^d h(y)\,dy\right)$$

Example 2 Evaluate $\iint\limits_R x\cos^2(y)\,dA$, $R = [-2,3] \times \left[0, \frac{\pi}{2}\right]$.

Solution

Since the integrand is a function of *x* times a function of *y* we can use the fact.

$$\begin{aligned}
\iint\limits_R x\cos^2(y)\,dA &= \left(\int_{-2}^{3} x\,dx\right)\left(\int_0^{\frac{\pi}{2}} \cos^2(y)\,dy\right) \\
&= \left(\frac{1}{2}x^2\right)\Bigg|_{-2}^{3} \left(\frac{1}{2}\int_0^{\frac{\pi}{2}} 1 + \cos(2y)\,dy\right) \\
&= \left(\frac{5}{2}\right)\left(\frac{1}{2}\left(y + \frac{1}{2}\sin(2y)\right)\Bigg|_0^{\frac{\pi}{2}}\right) \\
&= \frac{5\pi}{8}
\end{aligned}$$

Properties of Double Integrals:

1. $\iint\limits_D f(x,y)+g(x,y)\,dA=\iint\limits_D f(x,y)\,dA+\iint\limits_D g(x,y)\,dA$

2. $\iint\limits_D cf(x,y)\,dA=c\iint\limits_D f(x,y)\,dA$, where c is any constant.

3. If the region D can be split into two separate regions D_1 and D_2 then the integral can be written as

$$\iint\limits_D f(x,y)\,dA=\iint\limits_{D_1} f(x,y)\,dA+\iint\limits_{D_2} f(x,y)\,dA$$

Example Evaluate the following integral:

$$\iint\limits_D \mathbf{e}^{\frac{x}{y}}\,dA,\quad D=\{(x,y)\,|\,1\le y\le 2,\ y\le x\le y^3\}$$

Solution

$$\iint\limits_D \mathbf{e}^{\frac{x}{y}}\,dA=\int_1^2\int_y^{y^3}\mathbf{e}^{\frac{x}{y}}\,dx\,dy=\int_1^2 y\,\mathbf{e}^{\frac{x}{y}}\Big|_y^{y^3}\,dy$$

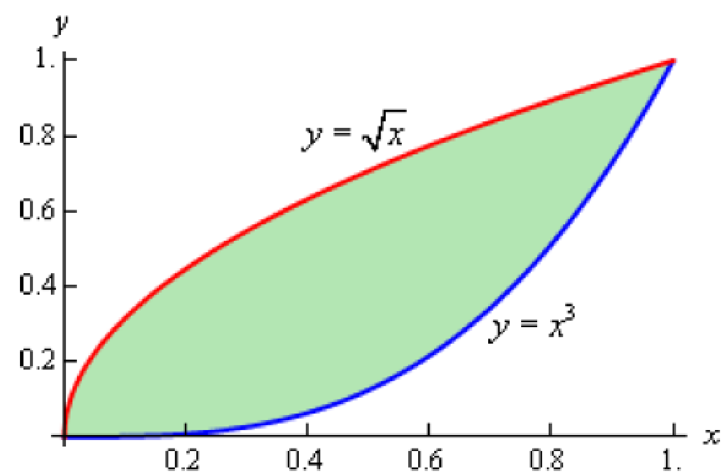
$$=\int_1^2 y\mathbf{e}^{y^2}-y\mathbf{e}^1\,dy=\left(\frac{1}{2}\mathbf{e}^{y^2}-\frac{1}{2}y^2\mathbf{e}^1\right)\Big|_1^2=\frac{1}{2}\mathbf{e}^4-2\mathbf{e}^1$$

Example Evaluate integral:

$$\iint_D 4xy - y^3\, dA,\ D \text{ is the region bounded by } y = \sqrt{x} \text{ and } y = x^3.$$

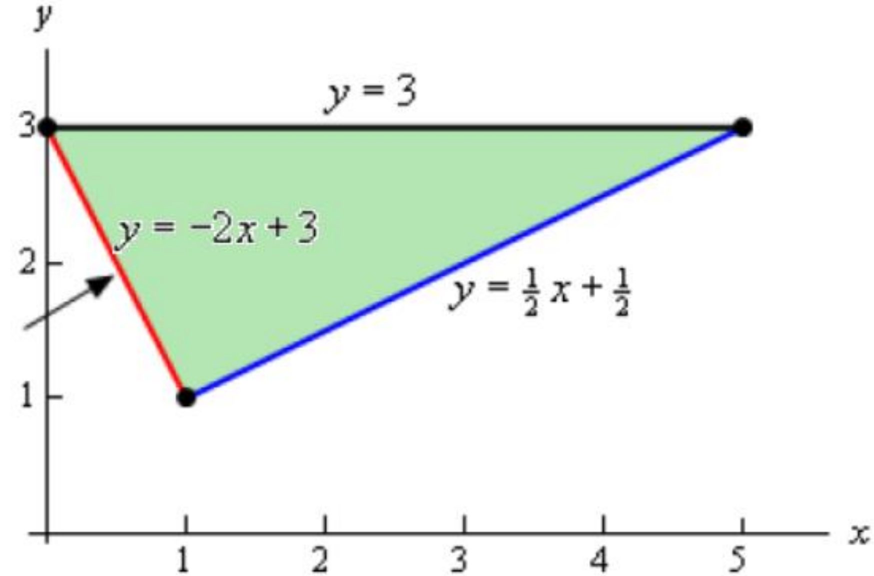
Solution

$$\begin{aligned}\iint_D 4xy - y^3\, dA &= \int_0^1 \int_{x^3}^{\sqrt{x}} 4xy - y^3\, dy\, dx \\ &= \int_0^1 \left(2xy^2 - \frac{1}{4}y^4\right)\Bigg|_{x^3}^{\sqrt{x}} dx \\ &= \int_0^1 \frac{7}{4}x^2 - 2x^7 + \frac{1}{4}x^{12}\, dx \\ &= \left(\frac{7}{12}x^3 - \frac{1}{4}x^8 + \frac{1}{52}x^{13}\right)\Bigg|_0^1 = \frac{55}{156}\end{aligned}$$



Example Evaluate $\iint\limits_D 6x^2 - 40y\,dA$, D is the triangle with vertices $(0,3)$, $(1,1)$, and $(5,3)$.

Solution



In this case the region would be given by

$D = D_1 \cup D_2$ where,

$$D_1 = \left\{(x,y) \mid 0 \le x \le 1,\ -2x+3 \le y \le 3\right\}$$

$$D_2 = \left\{(x,y) \mid 1 \le x \le 5,\ \frac{1}{2}x+\frac{1}{2} \le y \le 3\right\}$$

$$\iint\limits_D 6x^2 - 40y\,dA = \iint\limits_{D_1} 6x^2 - 40y\,dA + \iint\limits_{D_2} 6x^2 - 40y\,dA$$

$$= \int_0^1 \int_{-2x+3}^{3} 6x^2 - 40y\,dy\,dx + \int_1^5 \int_{\frac{1}{2}x+\frac{1}{2}}^{3} 6x^2 - 40y\,dy\,dx$$

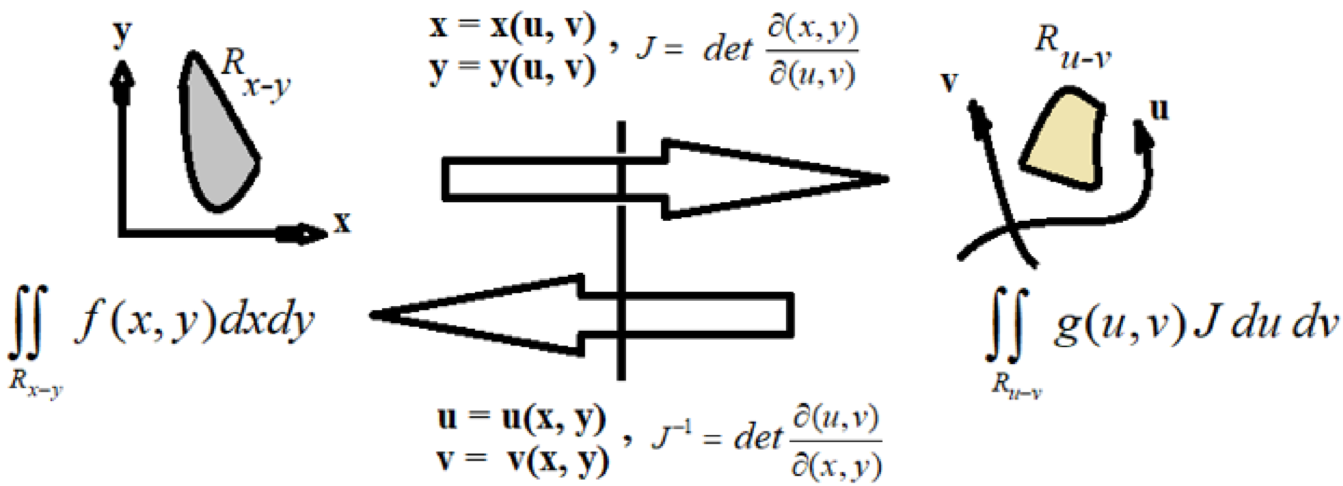
$$= \int_0^1 \left(6x^2y - 20y^2\right)\Big|_{-2x+3}^{3} dx + \int_1^5 \left(6x^2y - 20y^2\right)\Big|_{\frac{1}{2}x+\frac{1}{2}}^{3} dx$$

$$= \int_0^1 12x^3 - 180 + 20(3-2x)^2\,dx + \int_1^5 -3x^3 + 15x^2 - 180 + 20\left(\tfrac{1}{2}x+\tfrac{1}{2}\right)^2 dx$$

$$= \left(3x^4 - 180x - \tfrac{10}{3}(3-2x)^3\right)\Big|_0^1 + \left(-\tfrac{3}{4}x^4 + 5x^3 - 180x + \tfrac{40}{3}\left(\tfrac{1}{2}x+\tfrac{1}{2}\right)^3\right)\Big|_1^5 = -\frac{935}{3}$$

Change variables/axes of integration:

If there are difficulties to evaluate the integral, we may need to change the variables/axes of integration i.e. changing variables *x* & *y* to new variables *u* and *v*. This can be done by writing *x* and *y* in terms of the new variables *u* and *v*.



The integral is transformed as:

(1) Substituting $x=x(u, v)$ and $y=y(u, v)$ in $f(x, y)$ to get $g(u, v)=f(x(u,v), y(u,v))$
(2) Re-writing limits with respect to the new variables *u* and v
(3) Transforming the infinitesimal area element $dA = dx\,dy = J\,du\,dv$, where J is called the Jacobian of transformation.

The Jacobian of transformation J plays the role of "*scaling factor*" from *x-y* plane to *u-v* plane and it is evaluated as :

$$J = \det\left[\frac{\partial(x,y)}{\partial(u,v)}\right] = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

i.e. $$\iint_{R_{x-y}} f(x,y)\,dx\,dy = \iint_{R_{u-v}} g(u,v)\,J\,du\,dv$$

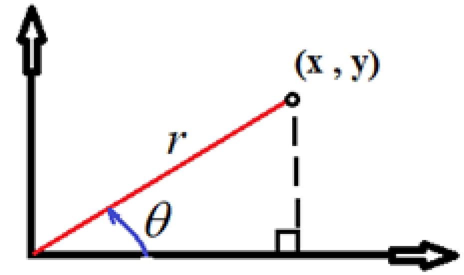
Note that: the Jacobian *J* should **not** be zero. If *J=zero*, this means the transformation maps the 2D x-y plane to a single 1D curve.

One common transformation is the use of the **Polar Coordinates** r, θ; where r is the distance from origin to point (x, y) and θ is the angle with the horizontal axis and is considered positive in the CCW direction (see the below figure).

Polar Coordinates $r\,,\,\theta$:

From the figure, we can deduce the relation between *x, y* and $r\,,\,\theta$ as:

$$x = r\cos(\theta) \quad and \quad y = r\,sin(\theta)$$



$$\therefore \quad J = \begin{vmatrix} \partial x/\partial r & \partial x/\partial\theta \\ \partial y/\partial r & \partial y/\partial\theta \end{vmatrix} = \begin{vmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{vmatrix} = r$$

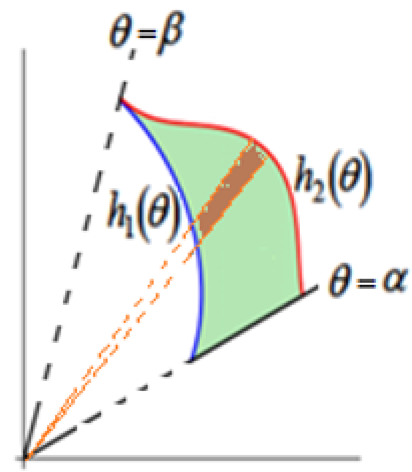
$$\therefore \quad dA = dx\,dy = r\,dr\,d\theta$$

Regions R_{x-y} and R_{u-v} are (graphically) the same, but we read the limits in a different way, as described below.

our general region will be defined by inequalities,

$$\alpha \le \theta \le \beta$$
$$h_1(\theta) \le r \le h_2(\theta)$$

$$\boxed{\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta}$$



Example Evaluate the following integral by converting into polar coordinates.
$\iint\limits_D 2x\,y\,dA$, D is the portion of the region between the circles of radius 2 and radius 5 centered at the origin that lies in the first quadrant.

Solution

$$2 \le r \le 5 \qquad \text{range of } \theta\text{'s.} \quad 0 \le \theta \le \frac{\pi}{2}$$

then

$$\iint\limits_D 2x\,y\,dA = \int_0^{\frac{\pi}{2}}\int_2^5 2(r\cos\theta)(r\sin\theta)\,r\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}}\int_2^5 r^3\sin(2\theta)\,dr\,d\theta \quad = \int_0^{\frac{\pi}{2}} \frac{1}{4}r^4\sin(2\theta)\Big|_2^5\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{609}{4}\sin(2\theta)\,d\theta \;= -\frac{609}{8}\cos(2\theta)\Big|_0^{\frac{\pi}{2}} \quad = \frac{609}{4}$$

Example Evaluate the following integral by converting into polar coordinates.

$$\iint_D \mathbf{e}^{x^2+y^2}\,dA$$, D is the unit disk centered at the origin.

Solution

the region D is defined by,

$$0 \le \theta \le 2\pi \qquad\qquad 0 \le r \le 1$$

In terms of polar coordinates the integral is then,

$$\iint_D \mathbf{e}^{x^2+y^2}\,dA = \int_0^{2\pi}\int_0^1 r\,\mathbf{e}^{r^2}\,dr\,d\theta$$

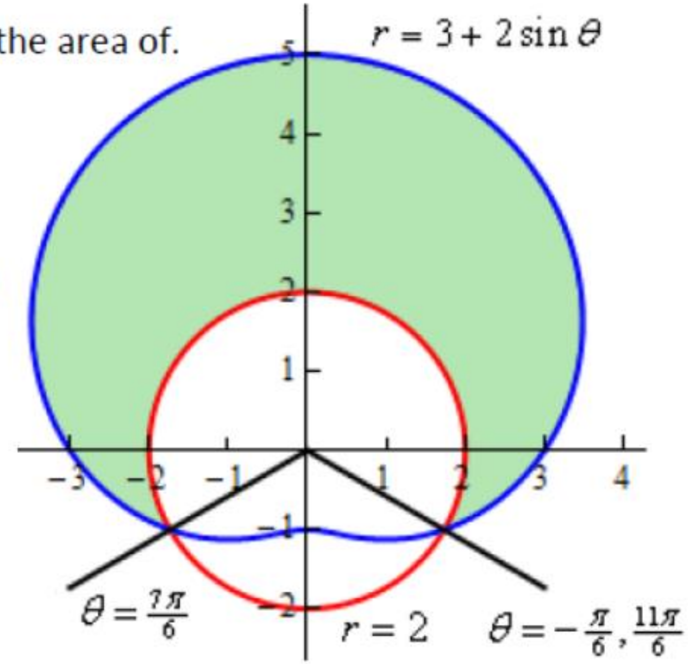
$$= \int_0^{2\pi} \frac{1}{2}\mathbf{e}^{r^2}\Big|_0^1\,d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}(\mathbf{e}-1)\,d\theta$$

$$= \pi(\mathbf{e}-1)$$

Example Determine the area of the region that lies inside $r = 3 + 2\sin\theta$ and outside $r = 2$.

Solution Here is a sketch of the region, *D*, that we want to determine the area of.



$$A = \iint_D dA = \int_{-\pi/6}^{7\pi/6} \int_2^{3+2\sin\theta} r\, dr\, d\theta$$

$$= \int_{-\pi/6}^{7\pi/6} \frac{1}{2} r^2 \Big|_2^{3+2\sin\theta} d\theta = \int_{-\pi/6}^{7\pi/6} \frac{5}{2} + 6\sin\theta + 2\sin^2\theta\, d\theta$$

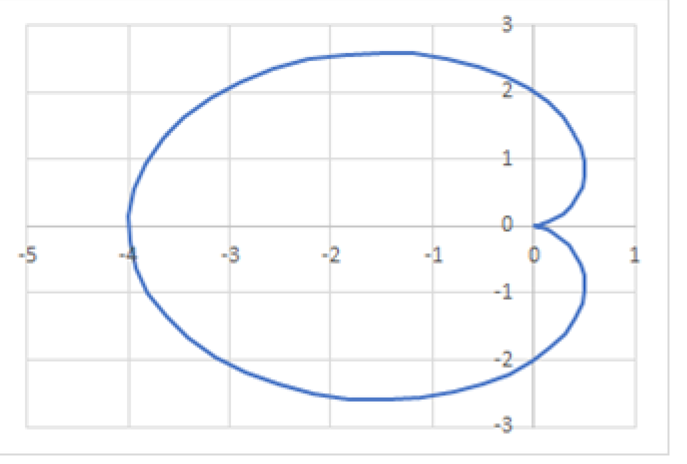
$$= \int_{-\pi/6}^{7\pi/6} \frac{7}{2} + 6\sin\theta - \cos(2\theta)\, d\theta$$

$$= \left(\frac{7}{2}\theta - 6\cos\theta - \frac{1}{2}\sin(2\theta) \right) \Big|_{-\frac{\pi}{6}}^{\frac{7\pi}{6}}$$

$$= \frac{11\sqrt{3}}{2} + \frac{14\pi}{3} = 24.187$$

Example: Find the area enclosed by the Cardoid $r = a(1-\cos\theta)$; $a > 0$.

We can select values for $0 \le \theta \le 2\pi$ and compute r from the Cardoid equation. The graph is as shown.



$$A = \int_R dA = \iint_R r\,dr\,d\theta$$

$$\therefore A = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=a(1-\cos\theta)} r\,dr\,d\theta$$

$$= \frac{1}{2} \int_{\theta=0}^{\theta=2\pi} \left(a^2 (1-\cos\theta)^2 \right) d\theta$$

$$= \frac{a^2}{2} (3\pi)$$

Example Determine the volume of the region that lies under the sphere $x^2 + y^2 + z^2 = 9$, above the plane $z = 0$ and inside the cylinder $x^2 + y^2 = 5$.

Solution

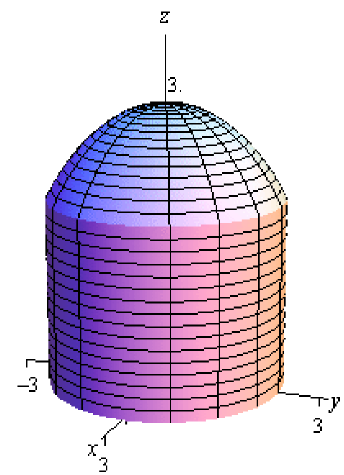
the volume of a region is, $V = \iint_D f(x, y)\, dA$

the region that we want the volume for is a cylinder with a cap that comes from the sphere.
in terms of polar coordinates the limits (in polar coordinates) for the region,

$$0 \le \theta \le 2\pi \qquad 0 \le r \le \sqrt{5}$$

and we'll need to convert the function to polar coordinates as well.

$$z = \sqrt{9 - \left(x^2 + y^2\right)} = \sqrt{9 - r^2}$$

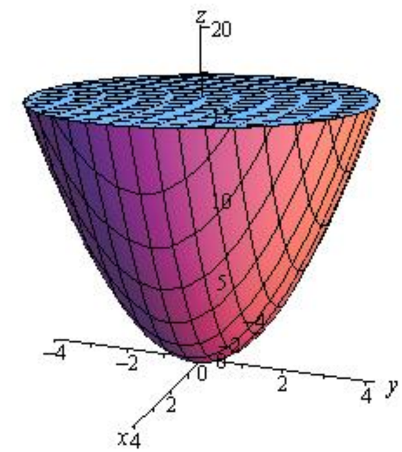


The volume is then, $V = \iint_D \sqrt{9 - x^2 - y^2}\, dA = \int_0^{2\pi} \int_0^{\sqrt{5}} r\sqrt{9 - r^2}\, dr\, d\theta$

$$= \int_0^{2\pi} -\frac{1}{3}\left(9 - r^2\right)^{\frac{3}{2}} \Big|_0^{\sqrt{5}} d\theta = \int_0^{2\pi} \frac{19}{3} d\theta = \frac{38\pi}{3}$$

Example Find the volume of the region that lies inside $z = x^2 + y^2$ and below the plane $z = 16$.

Solution $V = \iint_D f(x,y)\,dA$



$$V = \iint_D 16 - \left(x^2 + y^2\right) dA$$

integrating in terms of polar coordinates.

$$0 \le \theta \le 2\pi \qquad 0 \le r \le 4 \qquad z = 16 - r^2$$

The volume is then,

$$V = \iint_D 16 - \left(x^2 + y^2\right) dA$$

$$= \int_0^{2\pi} \int_0^4 r\left(16 - r^2\right) dr\, d\theta$$

$$= \int_0^{2\pi} \left(8r^2 - \frac{1}{4}r^4\right)\Bigg|_0^4 d\theta$$

$$= \int_0^{2\pi} 64\, d\theta \quad = 128\pi$$

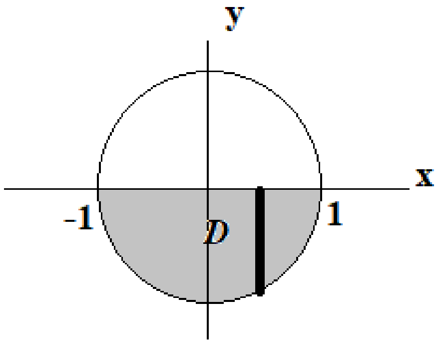
Example Evaluate the following integral by first converting to polar coordinates.

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{0}\cos\left(x^2+y^2\right)dy\,dx$$

Solution

notice that we cannot do this integral in Cartesian coordinates and so converting to polar

$$-1\le x\le 1 \qquad -\sqrt{1-x^2}\le y\le 0$$



in terms of polar coordinates are then, $\pi\le\theta\le 2\pi \qquad 0\le r\le 1$

$$dx\,dy=dA=r\,dr\,d\theta$$

and so the integral becomes,

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{0}\cos\left(x^2+y^2\right)dy\,dx=\int_{\pi}^{2\pi}\int_{0}^{1}r\cos\left(r^2\right)dr\,d\theta$$

Note that this is an integral that we can do. So, here is the rest of the work for this integral.

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{0}\cos\left(x^2+y^2\right)dy\,dx=\int_{\pi}^{2\pi}\frac{1}{2}\sin\left(r^2\right)\Big|_{0}^{1}\,d\theta$$

$$=\int_{\pi}^{2\pi}\frac{1}{2}\sin(1)\,d\theta \qquad =\frac{\pi}{2}\sin(1)$$

Applications of Double Integrals

1- Area in 2D of a region *R*

$$Area = \iint_R (1)\, dA$$

2- Volume in 3D, bounded above by function $f(x,y) \geq 0$ and below by region *R*

$$Volume = \iint_R f(x,y)\, dA$$

3- Average values of function $f(x,y)$ over a region *R* in 2D

$$Average = \frac{1}{Area} \iint_R f(x,y)\, dA$$, where $Area$ is the area of the region *R*

4- To find center of mass, moments of inertia in 2D

If the density is $\rho(x,y)$ over a region *R*, then the mass of a small area dA is given by:

$$dm = \rho(x,y)\, dA$$

Then the total mass of the region *R* is:

$$Mass = \iint_R dm = \iint_R \rho(x,y)\, dA$$

The center of mass/gravity $(\bar{x}, \bar{y})$ is then given as:

$$\bar{x} = \frac{1}{Mass} \iint_R x\,\rho(x,y)\,dA \qquad \text{and} \qquad \bar{y} = \frac{1}{Mass} \iint_R y\,\rho(x,y)\,dA$$

Moments of Inertia:

$$I_{xx} = \iint_R y^2\,\rho(x,y)\,dA \quad , \quad I_{yy} = \iint_R x^2\,\rho(x,y)\,dA$$

And the polar moment of inertia:

$$I_0 = I_{zz} = \iint_R \left(x^2 + y^2\right) \rho(x,y)\,dA$$

===

Example: Find the mass, center of mass, moments of inertia of a thin plate of uniform (constant) density ***k***. The plate covers a region *R* in *xy*-plane bounded by the curves *y=x* and $y=x^2$

Answers: Mass = $\frac{k}{6}$, center of gravity $(\bar{x}, \bar{y}) = \left(\frac{1}{2}, \frac{2}{5}\right)$, $I_x = \frac{k}{28}$, $I_y = \frac{k}{20}$, $I_0 = \frac{3k}{35}$

Triple Integrals

we'll use a **triple integral** to integrate over a three dimensional region.

The notation for the general triple integrals is,

$$\iiint_E f(x, y, z)\, dV$$

The triple integral over the box, $B = [a,b] \times [c,d] \times [r,s]$ is,

$$\iiint_B f(x, y, z)\, dV = \int_r^s \int_c^d \int_a^b f(x, y, z)\, dx\, dy\, dz$$

Example Evaluate the following integral.

$$\iiint_B 8xyz\, dV, \quad B = [2,3] \times [1,2] \times [0,1]$$

Solution

$$\iiint_B 8xyz\, dV = \int_1^2 \int_2^3 \int_0^1 8xyz\, dz\, dx\, dy \quad = \int_1^2 \int_2^3 4xyz^2 \Big|_0^1 dx\, dy$$

$$= \int_1^2 \int_2^3 4xy\, dx\, dy \quad = \int_1^2 2x^2 y \Big|_2^3 dy \quad = \int_1^2 10y\, dy = 15$$

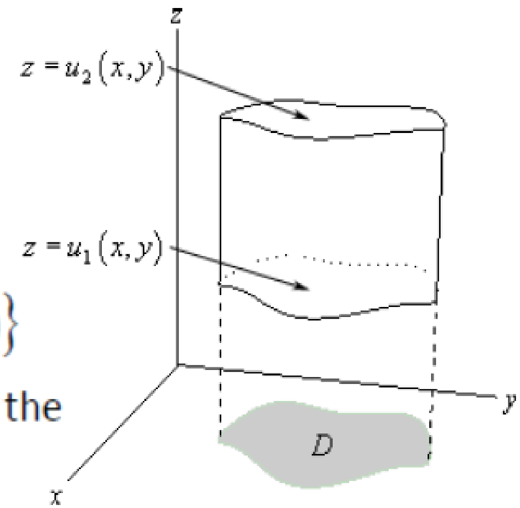
Volumes for general regions are computed easily with the triple integrals.

The volume of the three-dimensional region E is given by the integral,

$$V = \iiint_E dV$$

We have three different possibilities for a general region.

Here is a sketch of the first possibility.



In this case we define the region E as follows,

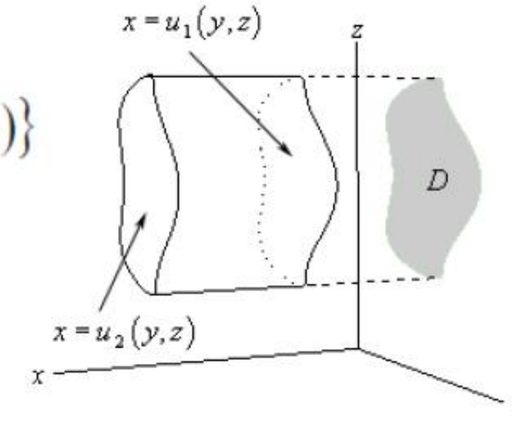
$$E = \{(x,y,z) \mid (x,y) \in D,\ u_1(x,y) \le z \le u_2(x,y)\}$$

where $(x,y) \in D$ is the notation that means that the point (x,y) lies in the region D from the xy- plane.

In this case we will evaluate the triple integral as follows,

$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\right] dA$$

the second possible three-dimensional region



For this possibility we define the region E as follows,

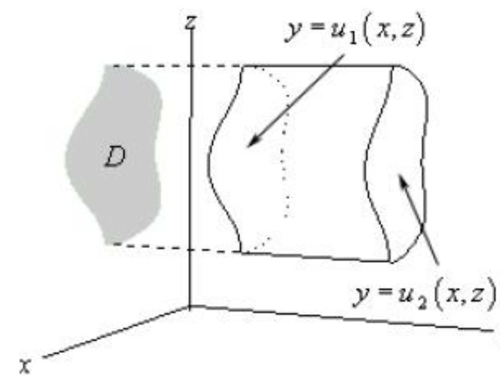
$$E = \{(x,y,z) \mid (y,z) \in D,\ u_1(y,z) \le x \le u_2(y,z)\}$$

So, the region D will be a region in the yz-plane.

Here is how we will evaluate these integrals.

$$\iiint\limits_E f(x,y,z)\,dV = \iint\limits_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\,dx \right] dA$$

In this final case E is defined as,



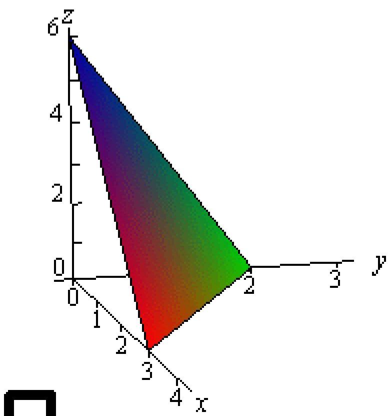
$$E = \{(x,y,z) \mid (x,z) \in D,\ u_1(x,z) \le y \le u_2(x,z)\}$$

and here the region D will be a region in the xz-plane.

Here is how we will evaluate these integrals.

$$\iiint\limits_E f(x,y,z)\,dV = \iint\limits_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\,dy \right] dA$$

Example Evaluate $\iiint\limits_E 2x\,dV$ where E is the region under the plane $2x+3y+z=6$ that lies in the first octant.



Solution we have the following limits for z.

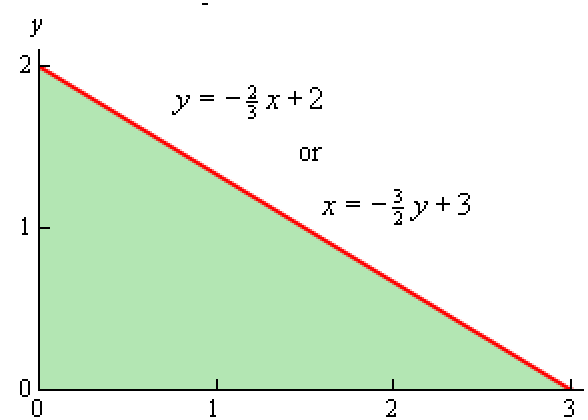
$$0 \le z \le 6-2x-3y$$

$$\iiint\limits_E 2x\,dV = \iint\limits_D \left[\int_0^{6-2x-3y} 2x\,dz\right] dA \quad = \iint\limits_D 2xz\Big|_0^{6-2x-3y}\,dA$$

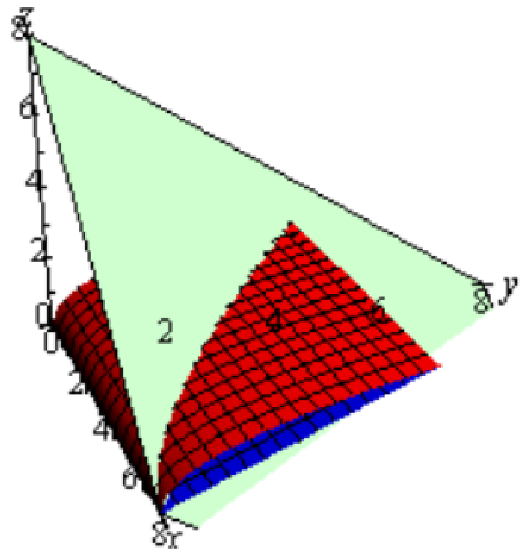
$$= \int_0^3 \int_0^{-\frac{2}{3}x+2} 2x(6-2x-3y)\,dy\,dx$$

$$= \int_0^3 \left(12xy-4x^2y-3xy^2\right)\Big|_0^{-\frac{2}{3}x+2} dx = \int_0^3 \frac{4}{3}x^3-8x^2+12x\,dx$$

$$= \left(\frac{1}{3}x^4-\frac{8}{3}x^3+6x^2\right)\Bigg|_0^3 \quad = 9$$



Example Determine the volume of the region that lies behind the plane $x+y+z=8$ and in front of the region in the *yz*-plane that is bounded by $z=\frac{3}{2}\sqrt{y}$ and $z=\frac{3}{4}y$.



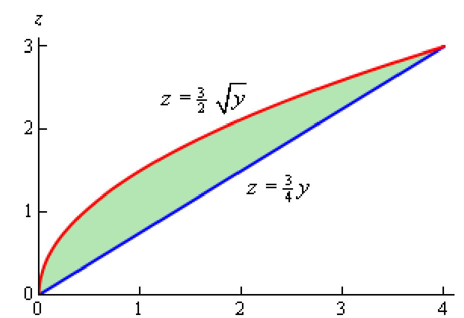
Solution

limits for each of the variables. $0 \le y \le 4$ $\quad \frac{3}{4}y \le z \le \frac{3}{2}\sqrt{y}$ $\quad 0 \le x \le 8-y-z$

$$V=\iiint_E dV=\iint_D\left[\int_0^{8-y-z} dx\right]dA = \int_0^4\int_{3y/4}^{3\sqrt{y}/2} 8-y-z\,dz\,dy$$

$$=\int_0^4\left(8z-yz-\frac{1}{2}z^2\right)\Bigg|_{\frac{3y}{4}}^{\frac{3\sqrt{y}}{2}} dy = \int_0^4 12y^{\frac{1}{2}}-\frac{57}{8}y-\frac{3}{2}y^{\frac{3}{2}}+\frac{33}{32}y^2\,dy$$

$$=\left(8y^{\frac{3}{2}}-\frac{57}{16}y^2-\frac{3}{5}y^{\frac{5}{2}}+\frac{11}{32}y^3\right)\Bigg|_0^4=\frac{49}{5}$$

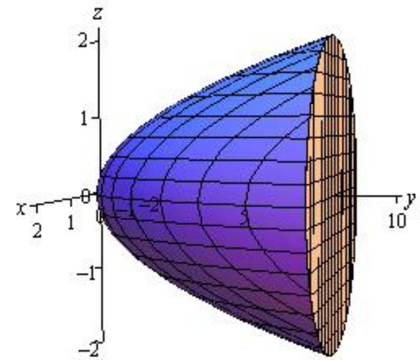


Example Evaluate $\iiint\limits_E \sqrt{3x^2+3z^2}\, dV$ where E is the solid bounded by $y=2x^2+2z^2$ and the plane $y=8$.

Solution

Intersecting two surfaces

$$2x^2+2z^2=8 \quad \Rightarrow \quad x^2+z^2=4$$



$$\iiint\limits_E \sqrt{3x^2+3z^2}\, dV$$

$$=\iint\limits_D \left[\int_{2x^2+2z^2}^{8} \sqrt{3x^2+3z^2}\, dy\right] dA$$

$$=\iint\limits_D \left(y\sqrt{3x^2+3z^2}\right)\Big|_{2x^2+2z^2}^{8} dA$$

$$=\iint\limits_D \sqrt{3\left(x^2+z^2\right)}\left(8-\left(2x^2+2z^2\right)\right) dA$$

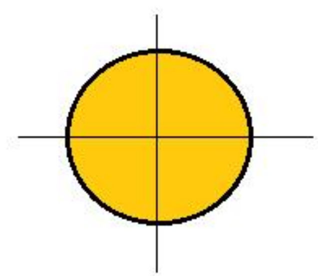
using polar coordinates :

$x=r\cos\theta \qquad z=r\sin\theta$

$0\le r\le 2 \qquad 0\le\theta\le 2\pi$

$$=\sqrt{3}\int_0^{2\pi}\int_0^{2}\left(8r-2r^3\right)r\,dr\,d\theta \;=\sqrt{3}\int_0^{2\pi}\left(\frac{8}{3}r^3-\frac{2}{5}r^5\right)\Big|_0^2 d\theta$$

$$=\sqrt{3}\int_0^{2\pi}\frac{128}{15}d\theta \;=\frac{256\sqrt{3}\,\pi}{15}$$



Triple Integrals

Example Evaluate $\iiint\limits_E \sqrt{3x^2+3z^2}\, dV$ where E is the solid bounded by $y = 2x^2+2z^2$ and the plane $y = 8$.

Solution

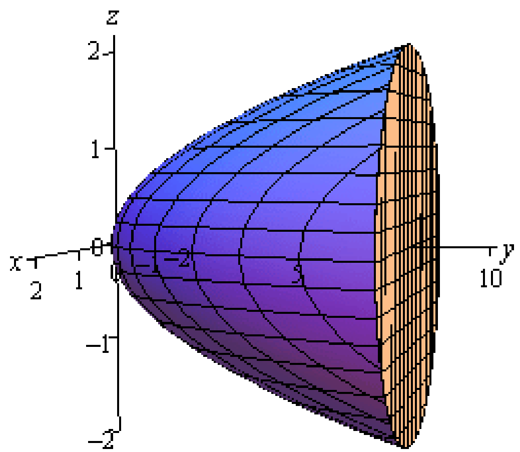
$$\iiint\limits_E \sqrt{3x^2+3z^2}\, dV = \iint\limits_D \left[\int_{2x^2+2z^2}^{8} \sqrt{3x^2+3z^2}\, dy\right] dA$$

$$= \iint\limits_D \left(y\sqrt{3x^2+3z^2}\right)\Big|_{2x^2+2z^2}^{8} dA$$

$$= \iint\limits_D \sqrt{3\left(x^2+z^2\right)}\left(8-\left(2x^2+2z^2\right)\right) dA$$

$$= \iint\limits_D \sqrt{3}\,\left(8r-2r^3\right) dA \quad = \sqrt{3}\int_0^{2\pi}\int_0^{2}\left(8r-2r^3\right) r\, dr\, d\theta$$

$$= \sqrt{3}\int_0^{2\pi}\left(\frac{8}{3}r^3-\frac{2}{5}r^5\right)\Big|_0^2 d\theta \quad = \sqrt{3}\int_0^{2\pi}\frac{128}{15}\, d\theta \quad = \frac{256\sqrt{3}\,\pi}{15}$$



Triple Integrals in Cylindrical Coordinates

The following are the conversion formulas for cylindrical coordinates.

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

$$\boxed{dV = r\,dz\,dr\,d\theta}$$

In terms of cylindrical coordinates a triple integral is,

$$\boxed{\iiint\limits_E f(x,y,z)\,dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta,r\sin\theta)}^{u_2(r\cos\theta,r\sin\theta)} r\,f(r\cos\theta, r\sin\theta, z)\,dz\,dr\,d\theta}$$

Note: The cylindrical coordinates can be considered/seen as polar coordinates with additional *z* variable. So, we can integrate first w.r.t. the variable *z* and then the resulting integral in *x* , *y* is transformed to polar coordinates.

Example Evaluate $\iiint_E y\,dV$ where *E* is the region that lies below the plane $z = x+2$ above the *xy*-plane and between the cylinders $x^2+y^2=1$ and $x^2+y^2=4$.

Solution

$$\iiint_E y\,dV = \int_0^{2\pi}\int_1^2\int_0^{r\cos\theta+2}(r\sin\theta)r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_1^2 r^2\sin\theta(r\cos\theta+2)\,dr\,d\theta$$

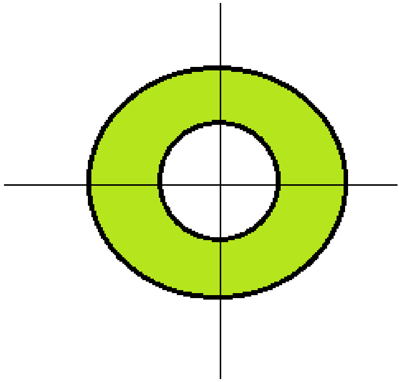
$$= \int_0^{2\pi}\int_1^2 \frac{1}{2}r^3\sin(2\theta)+2r^2\sin\theta\,dr\,d\theta$$

$$= \int_0^{2\pi}\left(\frac{1}{8}r^4\sin(2\theta)+\frac{2}{3}r^3\sin\theta\right)\Bigg|_1^2 d\theta$$

$$= \int_0^{2\pi}\frac{15}{8}\sin(2\theta)+\frac{14}{3}\sin\theta\,d\theta$$

$$= \left(-\frac{15}{16}\cos(2\theta)-\frac{14}{3}\cos\theta\right)\Bigg|_0^{2\pi}$$

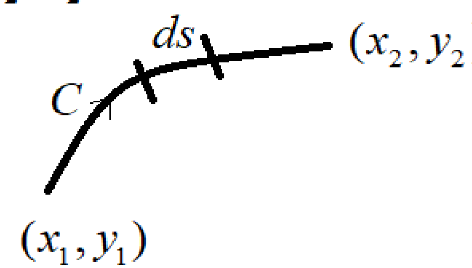
$$= 0$$



Line Integrals:

Line integral is an integral along curve C from point (x_1, y_1) to point (x_2, y_2).

It can appear in one of the following forms:



1- $I = \int_C P(x,y)\,dx + Q(x,y)\,dy$

2- $I = \int_C f(x,y)\,ds$, where $ds \equiv dl$ is a segment along curve C.

PROPERTIES OF THE LINE INTEGRALS

1) $\int_c P(x,y)dx + Q(x,y)dy = \int_c P(x,y)dx + \int_c Q(x,y)dy$

2) $\int_{A(a_1,b_1)}^{B(a_2,b_2)} P(x,y)dx + Q(x,y)dy = -\int_{B(a_2,b_2)}^{A(a_1,b_1)} P(x,y)dx + Q(x,y)dy$

3) $\int_{A(a_1,b_1)}^{B(a_2,b_2)} P(x,y)dx + Q(x,y)dy = \int_{A(a_1,b_1)}^{D(a_3,b_3)} P(x,y)dx + Q(x,y)dy + \int_{D(a_3,b_3)}^{B(a_2,b_2)} P(x,y)dx + Q(x,y)dy$

where $D(a_3, b_3)$ is a point on the curve C.

EVALUATION OF THE LINE INTEGRALS

1- If the curve C in the xy-plane is given by the relation $y = f(x)$, then the line integral on C $\int_{c} P(x, y)dx + Q(x, y)dy$

can be evaluated by putting $y = f(x)$, $dy = f'(x)dx$, in the line integral which results in the following definite integral

$$\int_{x=a_1}^{x=a_2} P(x, f(x))dx + Q(x, f(x))f'(x)dx$$

and this is can be evaluated by usual methods of integration.

2- If the curve C is given by the relation $x = g(y)$, then $dx=g'(y)dy$, and the line integral can take the form

$$\int_{y=b_1}^{y=b_2} P(g(y), y)g'(y)dy + Q(g(y), y)dy$$

3- If C is given by $x = x(t)$, $y = y(t)$, $t_A \le t \le t_B$, then the line integral will be

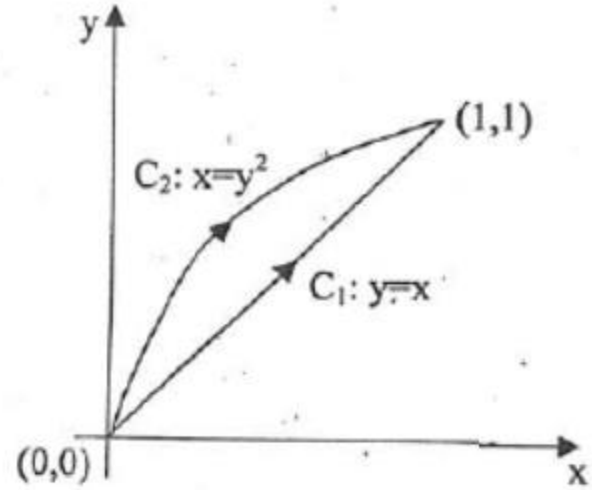
$$\int_{t_A}^{t_B} P(x(t), y(t))(\frac{dx}{dt})dt + Q(x(t), y(t))(\frac{dy}{dt})dt$$

If is to be noticed that the some of above methods or all of them can be used to evaluate line integrals.

Example

Evaluate $\int_{(0,0)}^{(1,1)} (2x^2 - 4xy)dx + (2x^2 - y^2)dy$, along

i) The curve $y = x$ ii) The curve $x = y^2$



Solution

i) $C_1: y = x,\ dy = dx$, then the line integral

$$I = \int_{x=0}^{x=1} \left[2x^2 - 4x(x)\right]dx + \left[2x^2 - (x)^2\right]dx$$

$$= \int_{x=0}^{x=1} \left(-2x^2 + x^2\right)dx = \int_{x=0}^{x=1} x^2 dx = \left[-\frac{x^3}{3}\right]_{x=0}^{x=1} = -\frac{1}{3}$$

ii) $C_2: x = y^2,\ dx = 2y\,dy$, then the line integral

$$I = \int_{y=0}^{y=1} \left[2\left(y^2\right)^2 - 4\left(y^2\right)y\right](2ydy) + \left[2\left(y^2\right)^2 - y^2\right]dy$$

$$= \int_{y=0}^{y=1} \left[2y^4 - 4y^3\right](2y)dy + \left[2y^4 - y^2\right]dy$$

$$= \int_{y=0}^{y=1} \left[2y^5 - 6y^4 - y^2\right]dy = \left[\frac{1}{3}y^6 - \frac{6}{5}y^5 - \frac{1}{3}y^3\right]_{y=0}^{y=1} = -\frac{6}{5}$$

Example

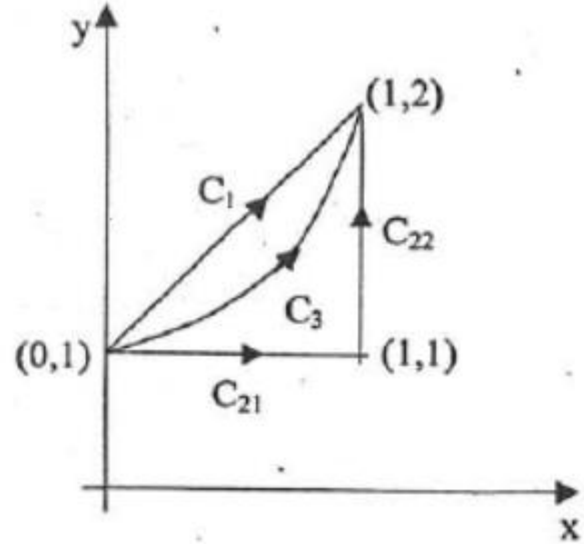
Evaluate $\int\limits_{(0,1)}^{(1,2)} \left(x^2 - y\right)dx + \left(y^2 + x\right)dy$, along

i) The straight line from (0,1) to (1,2)

ii) The straight lines from (0,1) to (1,1),

then from (1,1) to (1,2)

iii) The parabola $x = t$, $y = t^2 + 1$



Solution

i) The equation of line joining the two points (0,1) and (1,2) is $y = x+1$.

Then C_1: $y = x+1$, $dy = dx$ and the line integral

$$I = \int\limits_{x=0}^{x=1} \left[x^2 - (x+1)\right]dx + \left[(x+1)^2 + x\right]dx$$

$$= 2\int\limits_{x=0}^{1} \left[x^2 + x\right]dx = 2\left[\frac{x^3}{3} + \frac{x^2}{2}\right]_{x=0}^{x=1} = \frac{5}{3}$$

ii) Along the straight line from (0,1) to (1,1)

C_{21}: $y = 1$, $dy = 0$, then the line integral

$$I_1 = \int\limits_{x=0}^{x=1} \left[x^2 - (1)\right]dx + 0 = \left[\frac{x^3}{3} - x\right]_{x=0}^{x=1} = -\frac{2}{3}.$$

Along the straight line C_{22} from (1,1) to (1,2)

C_{22}: $x = 1$, $dx = 0$, then the line integral

$$I_2 = \int_{y=1}^{y=2} 0 + \left[y^2 + (1)\right]dy = \left[\frac{y^3}{3} + y\right]_{y=1}^{y=2} = \frac{10}{3}$$

The path $C_2 = C_{21} + C_{22}$, then the required line integral

$$I = I_1 + I_2 = -\frac{2}{3} + \frac{10}{3} = \frac{8}{3}$$

iii) Along C_3: $x = t$, $y = t^2 + 1$

$dx = dt$, $dy = 2t\,dt$

at $t = 0$, $x = 0$, $y = 1$

at $t = 1$, $x = 1$, $y = 2$

then the line integral

$$I = \int_{t=0}^{t=1} \left[t^2 - \left(t^2 + 1\right)\right]dt + \left[\left(t^2 + 1\right)^2 + t\right](2t)dt$$

$$I = \int_{t=0}^{t=1} \left(2t^5 + 4t^3 + 2t^2 + 2t - 1\right)dt = 2$$

PHYSICAL MEANING OF THE LINE INTEGRAL

It is suitable to express the line integral in terms of vectors to treat some physical and engineering concepts. In this respect, the line integral can be expressed as follows:

$$\int_c P(x,y)dx + Q(x,y)dy = \int_c \left[P(x,y)\underline{i} + Q(x,y)\underline{j}\right]\left[dx\,\underline{i} + dy\,\underline{j}\right]$$

$$= \int_c \underline{F}\cdot\underline{ds}$$

where $\underline{F} = P(x,y)\,\underline{i} + Q(x,y)\,\underline{j}$, $\quad \underline{ds} = dx\,\underline{i} + dy\,\underline{j}$

In this case the line integral can be interpreted as the work done by a force $\underline{F}$ along the path C connecting the two points A and B. i.e.

$$w = \int_c \underline{F}\cdot\underline{ds}$$

In this context, the line integral can be defined along a curve in the three dimensional space as follows

$$\text{Line Integral} = \int_{A(a_1,b_1,c_1)}^{B(a_2,b_2,c_2)} P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz$$

$$= \int_A^B \underline{F}\cdot\underline{ds}\ ,$$

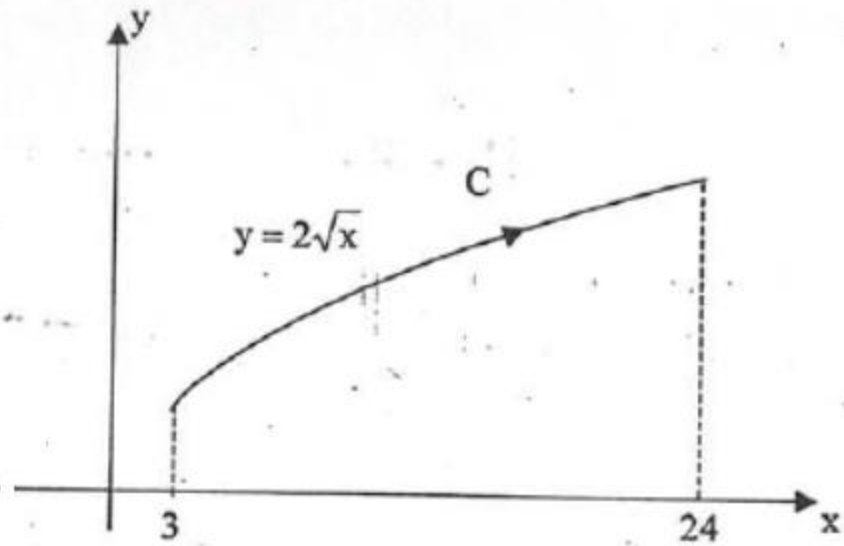
where $\underline{F} = P(x,y,z)\underline{i} + Q(x,y,z)\underline{j} + R(x,y,z)\underline{k}$ $\qquad \underline{ds} = dx\,\underline{i} + dy\,\underline{j} + dz\,\underline{k}$

Example

Evaluate $\int_C y\,ds$, where C is the curve in the xy-plane given by

$y = 2\sqrt{x}$ from $x = 3$ to $x = 24$

Solution



as $ds = \sqrt{\overline{dx}^2 + \overline{dy}^2} = \sqrt{1 + (\frac{dy}{dx})^2}\,dx$

along the curve $y = 2\sqrt{x}$, $\frac{dy}{dx} = \frac{1}{\sqrt{x}}$,

hence $ds = \sqrt{1 + (\frac{1}{\sqrt{x}})^2}\,dx = \sqrt{1 + \frac{1}{x}}\,dx$.

The line integral will be

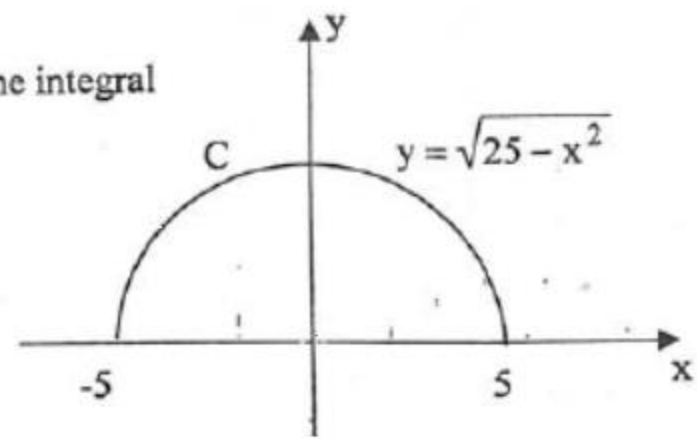
$$\int_C y\,ds = \int_{x=3}^{x=24} 2\sqrt{x}\,(\sqrt{1 + \frac{1}{x}})\,dx = 2\int_{x=3}^{x=24} \sqrt{x+1}\,dx$$

$$= \frac{4}{3}\Big[(x+1)^{3/2}\Big]_{x=3}^{x=24} = 156$$

Example

Suppose that the semicircular wire has the equation $y = \sqrt{25 - x^2}$ and that its mass density is given by $\rho(x,y)=15-y$. Find the mass of the wire

Solution



The mass M of the wire can be expressed as the line integral

$$M = \int_C \rho(x, y)ds = \int_C (15 - y)ds$$

Along the semicircle C, to evaluate this integral

we will express C parametrically as

$$x = 5 \cos t \quad , \qquad y = 5 \sin t \quad , \qquad 0 \le t \le \pi$$

Then, $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \sqrt{25 \sin^2 t + 25 \cos^2 t}\, dt \;= 5\, dt$

Thus,

$$M = \int_{t=0}^{t=\pi} (15 - 5\sin t)5\, dt = 5 \int_{t=0}^{t=\pi} (15 - 5 \sin t)\, dt$$

$$= 5[15\, t + 5 \cos t]_{t=0}^{\pi} = 75\pi - 50$$

CONDITION FOR THE LINE INTEGRAL TO BE INDPENDENT OF PATH

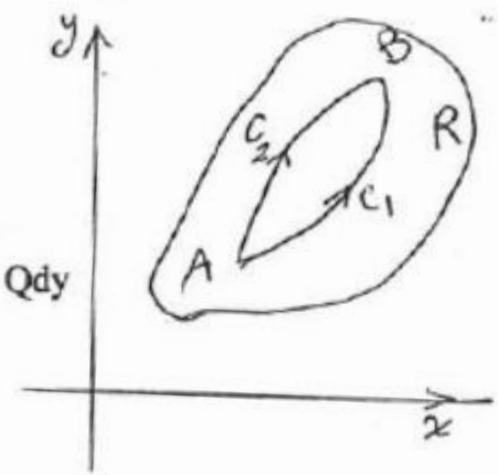
Theorem: A necessary and sufficient condition for the line integral

$$\int_C P(x,y)dx + Q(x,y)dy,$$

to be independent of the path C joining any two given points in a region R is that in R: $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$

Where it is supposed that these partial derivatives are continous in R.

To prove sufficiency: let $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$ and let the path $C = C_1 + (-C_2)$ as shown in Fig, then



$$\oint_C Pdx + Qdy = \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$= 0$$

$$\oint_C Pdx + Qdy = 0 = \int_C Pdx + Qdy + \int_{(-C_2)} Pdx + Qdy$$

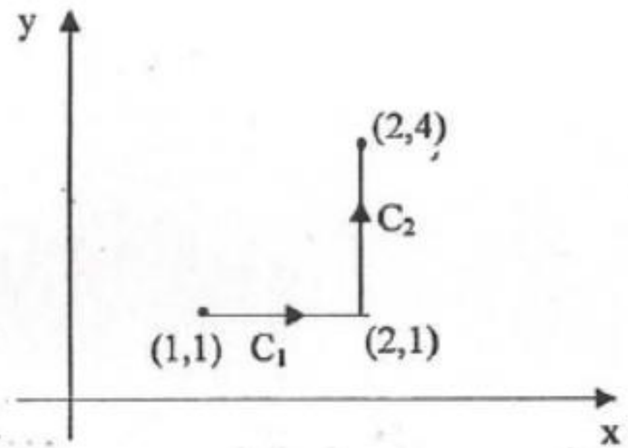
$$= \int_{C_1} Pdx + Qdy - \int_{C_2} Pdx + Qdy$$

$$\therefore \int_{C_1} Pdx + Qdy = \int_{C_2} Pdx + Qdy$$

$\therefore$ the line integral is independent of path

Example

Prove that the line integral $\int_{(1,1)}^{(2,4)} (2xy^3)dx + (1+3x^2y^2)dy$ is independent of path, then evaluate it.



$P = 2xy^3$, $Q = (1+3x^2y^2)$

$\frac{\partial P}{\partial y} = 6xy^2$ $\frac{\partial Q}{\partial x} = 6xy^2$

$\therefore \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, $\therefore$ the integral is independent of path.

To evaluate the line integral we choose any path joining the two points (1,1) and (2,4). For Simplicity, we choose the path parallel to the coordinate axes such that $C = C_1 + C_2$ as shown in Fig.

Along C_1: $y = 1, dy = 0$

$$I_1 = \int_{x=1}^{x=2} 2x(1)dx + 0 = [x^2]_{x=1}^{x=2} = 3$$

Along C_2: $x = 2, dx = 0$

$$I_2 = \int_{y=1}^{y=4} 0 + (1+12y^2)dy = [y + 4y^3]_{y=1}^{y=4} = 255$$

$$\therefore I = I_1 + I_2 = 255 + 3 = 258$$

Green's Theorem

Let P, Q, $\frac{\partial P}{\partial y}$ and $\frac{\partial Q}{\partial x}$ be single valued and continuous functions of (x,y) in a simply connected region R bounded by a simple closed curve C such that a line parallel to either axis cuts C in at most two points, then

$$\oint_C P(x,y)dx + Q(x,y)dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy,$$

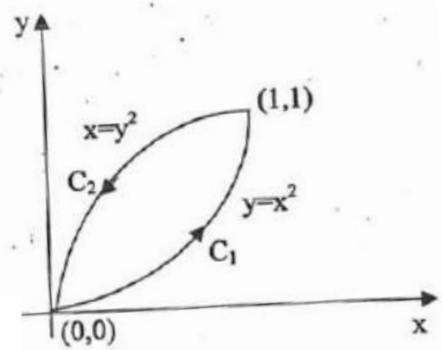
where the symbol $\oint$ is used to indicate that C is closed and that it is described in the positive direction.

Example

Verify Green's theorem for the line integral

$$\oint_C \left(2xy - x^2\right)dx + \left(x + y^2\right)dy$$

Where C is the closed curve of the region bounded by the curves $y = x^2$ and $y^2 = x$



Solution

a) we divide the line integral into two line integrals along the curves C_1: $y = x^2$ and C_2: $x = y^2$ such that $C = C_1 + C_2$

Along C_1: $y = x^2$, $dy = 2x\,dx$

$$I_1 = \int_{x=0}^{x=1} \left(2x^3 - x^2\right)dx + \left(x + x^4\right)(2x)dx = \int_{x=0}^{x=1} \left[x^2 + 2x^3 + 2x^5\right]dx = \frac{7}{6}$$

Along C_2: $x = y_2$, $dx = 2y\,dy$

$$I_1 = \int_{y=1}^{y=0} \left(2y^3 - y^4\right)(2y\,dy) + \left(y^2 + y^2\right)dy$$

$$= \int_{y=1}^{y=0} \left[2y^2 + 4y^4 - 2y^5\right]dy = -\frac{17}{15}$$

$$\therefore \oint_C \left(2xy - x^2\right)dx + \left(x + y^2\right)dy = I_1 + I_2 = \frac{7}{6} - \frac{17}{15} = \frac{1}{30}$$

b) Applying Green's theorem

$P = 2xy - x^2$, $Q = x + y^2$

$\dfrac{\partial P}{\partial y} = 2x$ $\qquad \dfrac{\partial Q}{\partial x} = 1$

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} (1 - 2x)\,dy\,dx$$

$$= \int_{x=0}^{1} \left[(1 - 2x)y\right]_{y=x^2}^{y=\sqrt{x}} dx$$

$$= \int_{x=0}^{1} (1 - 2x)\left(\sqrt{x} - x^2\right)dx = \int_{x=0}^{1} \left[\sqrt{x} - 2x^{3/2} - x^2 + 2x^3\right]dx$$

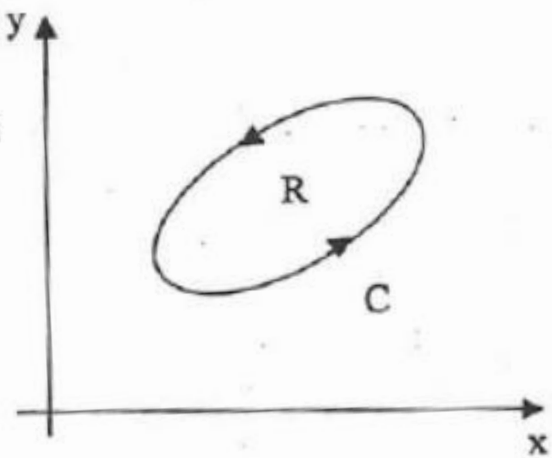
$$= \frac{1}{30}$$

Hence, Green's theorem is verified

Example

Show that the area enclosed by the closed curve C can be given by the line integral $\frac{1}{2}\oint_C x\,dy - y\,dx$

Solution



By applying Green's theorem

$P = -y \quad , \quad Q = x,$ then

$\frac{\partial P}{\partial y} = -1 \quad , \quad \frac{\partial Q}{\partial x} = 1$

$$\oint x\,dy - y\,dx = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$= \iint_R 2dx\,dy = 2A$$

Hence, the required area is

$$A = \frac{1}{2}\oint_C x\,dy - y\,dx$$

End of the Lectures