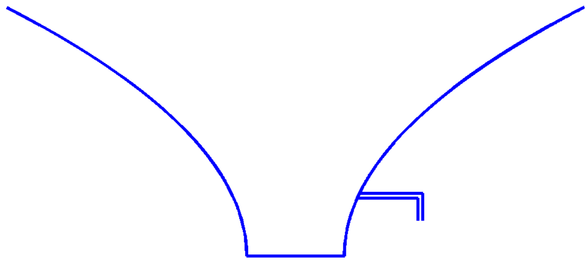
Tank System:

System Modeling:

For the following tank,



- q_{in} : Inlet volume flow rate
- q_{out} : Outlet volume flow rate
- a : Cross sectional area of tap
- h : Height of the tank
- $A(h)$: Cross sectional area of tank at height h
- A_o : Cross sectional area of tank at height h

$$
\begin{aligned}
q_{in} - q_{out} &= \frac{dV}{dt} = \frac{dV}{dh}\frac{dh}{dt} \\
dV &= A(h)\, dh \;\Rightarrow\; \frac{dV}{dh} = A(h) \\
q_{in} - q_{out} &= A(h)\,\frac{dh}{dt} \\
q_{out} &= \dot{V}_{out} = a v_{out} \;;\; v_{out}^2 = v_{in}^2 + 2gh \;,\;\; v_{in} \approx 0 \;\Rightarrow\; v_{out} = \sqrt{2gh} \\
q_{out} &= a\sqrt{2gh} \\
q_{out} - a\sqrt{2gh} &= A(h)\,\frac{dh}{dt}
\end{aligned}
$$

Linearization :

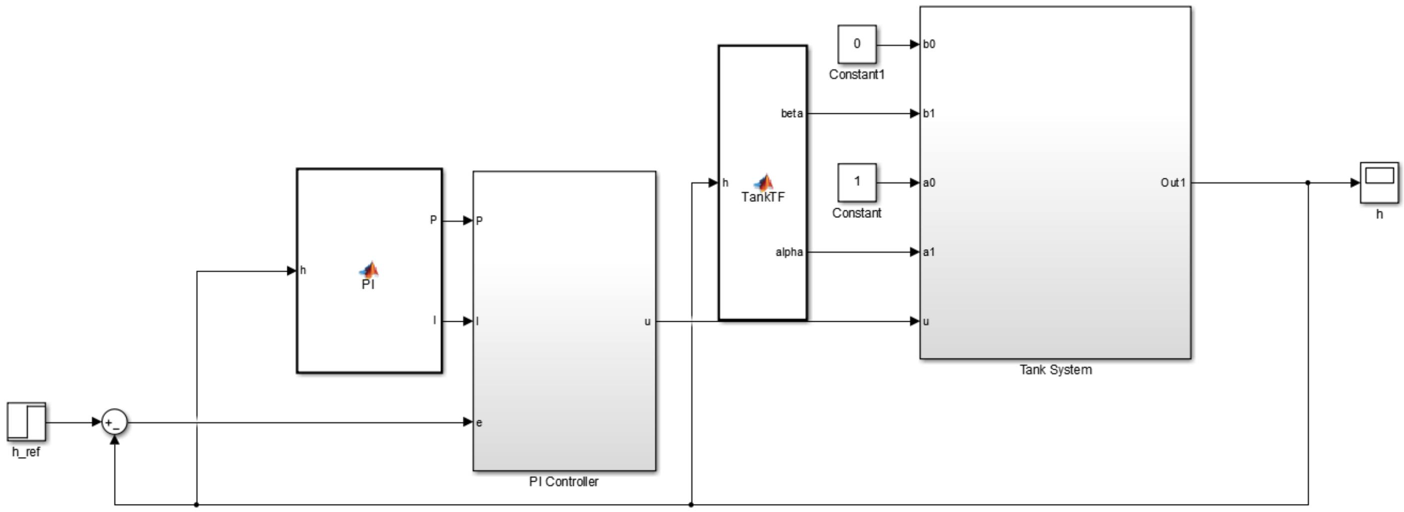
$$
\begin{aligned}
q_{in} &= q_{o,in} + \Delta q_{in} \;, \qquad h = h_o + \Delta h \\
q_{o,out} + \Delta q_{in} - a\sqrt{2g(h_o + \Delta h)} &= A(h_o + \Delta h)\,\frac{d}{dt}(h_o + \Delta h) \\
\frac{dh_o}{dt} &= 0 \;, \qquad A(h_o + \Delta h) \approx A(h_o) \\
\sqrt{2g(h_o + \Delta h)} &= \sqrt{2gh_o\left(1 + \frac{\Delta h}{h_o}\right)} \approx \sqrt{2gh_o}\left(1 + \frac{1}{2}\frac{\Delta h}{h_o}\right) \\
@\; steady\; state\; q_{o,in} - q_{o,out} &= 0 \;\Rightarrow\; q_{o,in} - a\sqrt{2gh_o} = 0 \\
\Delta q_{in} \;\rightarrow\; q_{in} \quad &, \qquad \Delta h \;\rightarrow\; h \\
q_{o,in} + q_{in} - a\sqrt{2gh_o} - a\sqrt{2gh_o}\left(\frac{h}{2h_o}\right) &= A(h_o)\frac{dh}{dt} \\
Q_{in} - \frac{a}{2}\sqrt{\frac{2g}{h_o}}\; H &= A_o H s \;; \qquad A_o = A(h_o) \\
let\; \beta &= \frac{1}{A_o} \;, \qquad \alpha = \frac{a}{2A_o}\sqrt{\frac{2g}{h_o}} \\
G(s) &= \frac{H}{Q_in}(s) = \frac{\beta}{s + \alpha} \\
\textbf{Use PI controller :}\; G_c(s) &= k\left(1 + \frac{1}{T_i s}\right)
\end{aligned}
$$

$$
\begin{aligned}
\textbf{Closed loop TF : } \frac{H}{H_d} &= \frac{G_c(s)G(s)}{1+G_c(s)G(s)} = \frac{\beta k(T_i s+1)}{T_i s^2 + (\beta k T_i + \alpha T_i)s + \beta k} \\
&= \frac{\beta k(s+\frac{1}{T_i})}{s^2 + (\beta k + \alpha)s + \frac{\beta k}{T_i}} \\
2\zeta w_n &= \beta k + \alpha \;\Rightarrow\; k = \frac{2\zeta w_n - \alpha}{\beta} \\
w_n^2 &= \frac{\beta k}{T_i} \;\Rightarrow\; T_i = \frac{\beta k}{w_n^2} = \frac{2\zeta w_n - \alpha}{w_n^2} \\
\textbf{As : } \alpha &<< 2\zeta w_n \\
k &= \frac{2\zeta w_n}{\beta} = 2\zeta w_n A_o \;, \quad T_i = \frac{2\zeta}{w_n}
\end{aligned}
$$

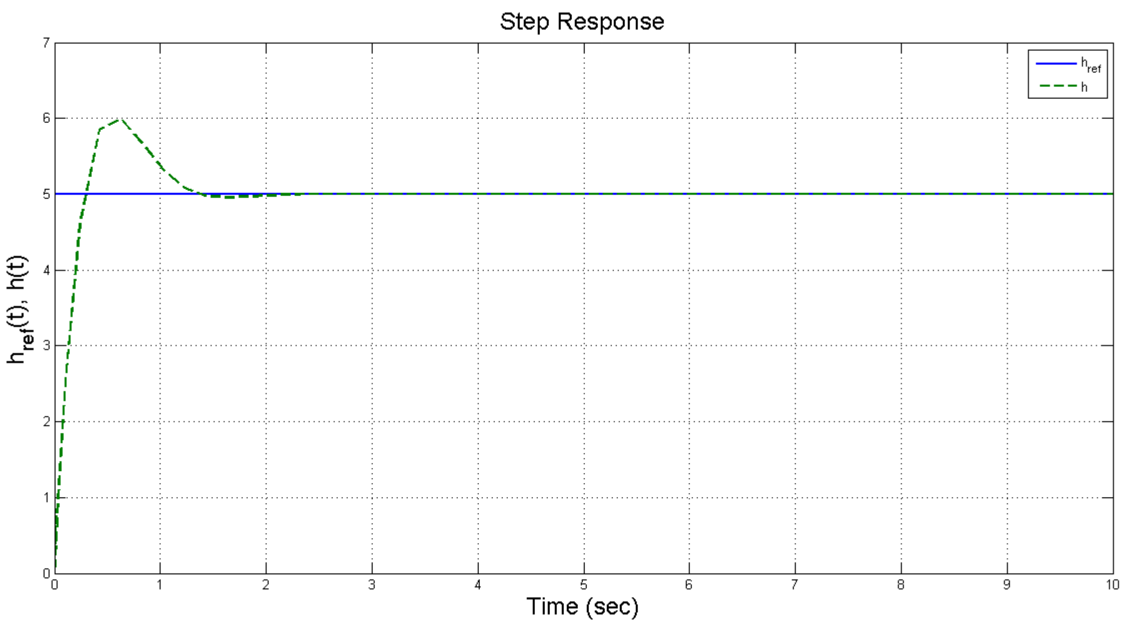
Let :

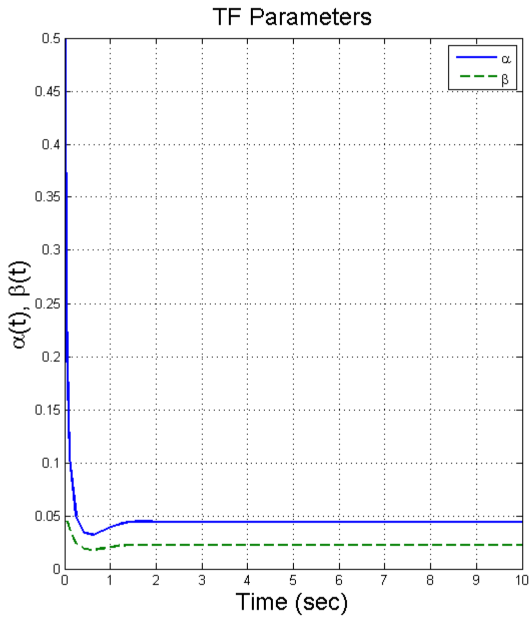
$$
A(h) = A(0) + h^2 \;, \quad A(0) = 20 \;, \quad h = 7 \;, \quad a = 0.1A(0) = 2 \;, \quad \zeta = 0.7 \;, \quad w_n = 4
$$

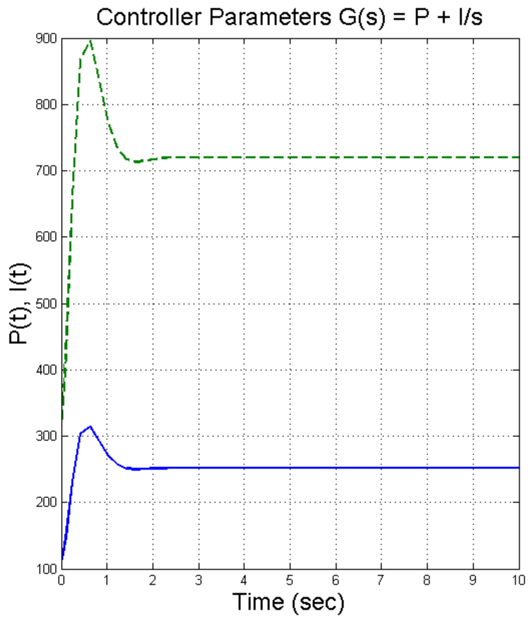
Block Diagram:

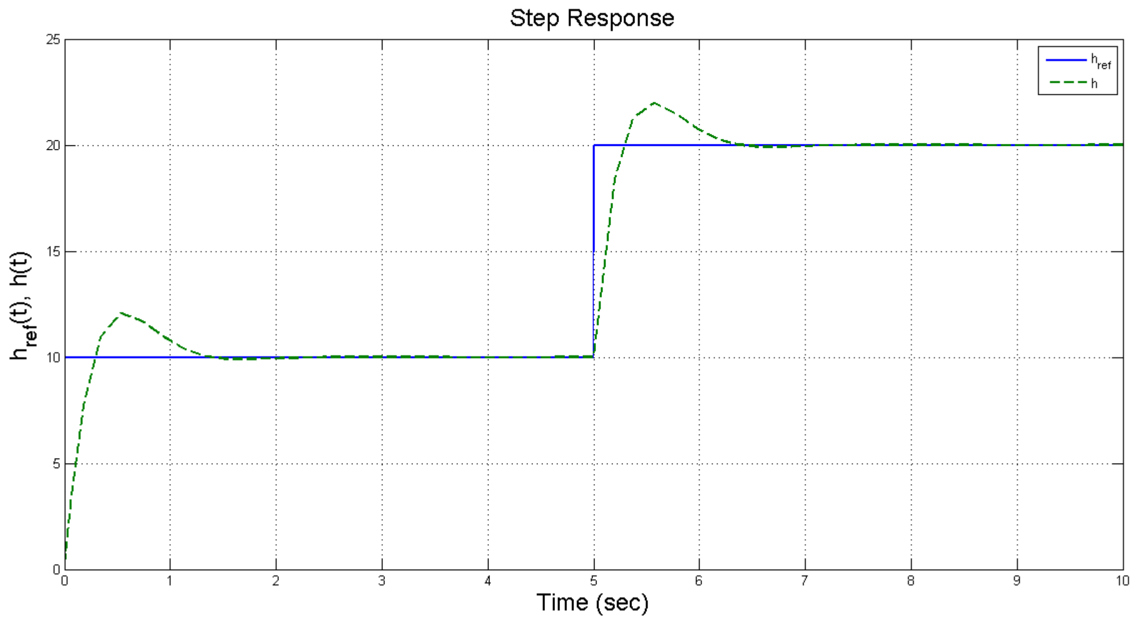


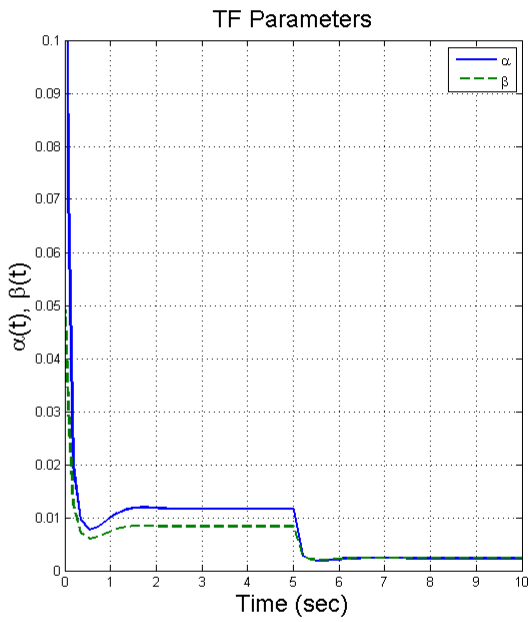
Some Simulation Results

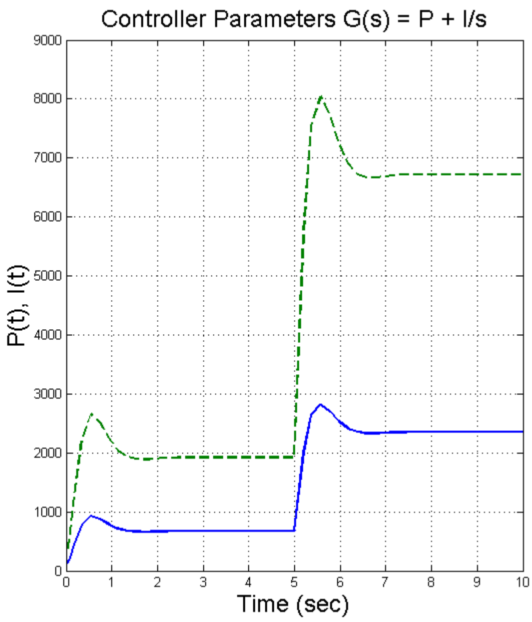



TF Parameters
α
β
α(t), β(t)
Time (sec)


Controller Parameters G(s) = P + I/s
P(t), I(t)
Time (sec)


Step Response
h_ref
h
h_ref(t), h(t)
Time (sec)


TF Parameters
α
β
α(t), β(t)
Time (sec)


Controller Parameters G(s) = P + I/s
P(t), I(t)
Time (sec)