

Linear Algebra, Four Fundamental Subspaces, part 2

Computational Intelligence, Lecture 3

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- Projection
 - How to find a projector
 - Use case
- Orthogonal compliment
- Column space
- Left null space
- Row space and left null space bases
- Example
- Homework

Projection

Previously we used the term *projection*. What does it mean?

Definition 1

For linear space $\mathcal{L} \subset \mathbb{R}^n$, an orthogonal projector \mathbf{P} onto it has properties:

- $\forall \mathbf{x} \in \mathbb{R}^n, \mathbf{P}\mathbf{x} \in \mathcal{L}$
- $\forall \mathbf{x} \in \mathcal{L}, \mathbf{P}\mathbf{x} = \mathbf{x}$
- $\forall \mathbf{y} \in \mathcal{L}, \mathbf{y}^\top (\mathbf{I} - \mathbf{P})\mathbf{x} = 0$

It follows that $\mathbf{P}\mathbf{P} = \mathbf{P}$. Also notice that a projection always maps the space onto itself: $\mathbf{P} : \mathbb{R}^n \rightarrow \mathbb{R}^n$.

In other words, an orthogonal projector takes the part of a vector that lies in the linear space, and cuts the rest off. We can refer to it as *projection*.

Projection

How to find a projector

As long as we know an orthonormal basis \mathbf{B} in the linear space \mathcal{L} , we can find a projector \mathbf{P} on that space as follows:

$$\mathbf{P} = \mathbf{B}\mathbf{B}^\top \quad (1)$$

You can think of it as follows: $\mathbf{B}\mathbf{B}^\top \mathbf{x}$ first finds a decomposition of \mathbf{x} on the orthonormal set of vectors, where all parts of \mathbf{x} not in the span of \mathbf{B} will be mapped to zero; next, it maps this decomposition back into the original coordinates, but the part not-in-the-span-of- \mathbf{B} is lost.

Projection

Column space projection and null space projection

In general, if linear space \mathcal{L} is given as a span of the columns of some matrix \mathbf{L} , the projector \mathbf{P} on that space can be generated as follows:

$$\mathbf{P} = \mathbf{L}\mathbf{L}^+ = \mathbf{L}(\mathbf{L}^\top \mathbf{L})^{-1} \mathbf{L}^\top \quad (2)$$

If $\mathbf{x} = \mathbf{x}_c + \mathbf{x}_{ln}$, where $\mathbf{x}_c \in \mathcal{C}(\mathbf{L})$ and $\mathbf{x}_{ln} \in \mathcal{C}(\mathbf{L})^\perp$, then $\mathbf{L}^+ \mathbf{x} = \mathbf{L}^+ (\mathbf{x}_c + \mathbf{x}_{ln}) = \mathbf{L}^+ \mathbf{x}_c \in \mathcal{N}(\mathbf{L})^\perp$. We observe that since \mathbf{x}_c is in the column space of \mathbf{L} , no residual solution to the least squares problem $\mathbf{L}\mathbf{z} = \mathbf{x}_c$ exists and as was pointed out before, it lies in the row space $\mathcal{N}(\mathbf{L})^\perp$. And therefore $\mathbf{P}\mathbf{x} = \mathbf{L}\mathbf{L}^+ \mathbf{x} = \mathbf{x}_c$

In the same way we can prove that the projection onto the row space of a matrix \mathcal{L} is given as:

$$\mathbf{P}_r = \mathbf{L}^+ \mathbf{L} \quad (3)$$

Projection

Use case

Remember the problem: find all solutions to the system of equations $\mathbf{Ax} = \mathbf{y}$. Assume we found a single solution \mathbf{x}^s . If we know an orthonormal basis \mathbf{N} in the null space of \mathbf{A} (which we get by calling `null()` in MATLAB, for example), then we can find a projector \mathbf{P} onto that space and find which part of \mathbf{x}^s lies outside it:

$$\mathbf{P} = \mathbf{N}\mathbf{N}^\top \quad (4)$$

$$\mathbf{x}^p = (\mathbf{I} - \mathbf{P})\mathbf{x}^s \quad (5)$$

Thus, all solutions are found as $(\mathbf{I} - \mathbf{N}\mathbf{N}^\top)\mathbf{x}^s + \mathbf{N}\mathbf{z}$, $\forall \mathbf{z}$

Orthogonal compliment

Remember how we defined row space: *Row space* of \mathbf{A} is the set of all inputs to \mathbf{A} that have a zero projection onto its null space.

Now we can observe that if \mathbf{P} is a projector onto the null space of \mathbf{A} , then the row space of \mathbf{A} is the set of all vectors \mathbf{x} , such that $\mathbf{Px} = \mathbf{0}$.

We can observe that $(\mathbf{I} - \mathbf{P})$ is a projector onto the row space. In a sense, row space contains everything left by the null space, and together then span all inputs on the matrix \mathbf{A} . They compliment each other, and they are orthogonal to each other. In other words, row space is an *orthogonal compliment* of null space, and vice versa.

Consider the problem. Current state of the system is given as $\mathbf{x}_i \in \mathbb{R}^n$, and we do not know it. At the next point in time, the state of the system is given as $\mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i$. What are all possible next states that we can expect?

To solve it, we need to find all possible outputs of the linear operator \mathbf{A} . This is called *column space* of \mathbf{A} .

In MATLAB it can be constructed by calling function `orth()`. Both `orth()` and `null()` (as well as `rank()` and `pinv()`) simply call `svd()` and perform minimal computations on the resulting decomposition. You can check it by typing `open orth` in MATLAB command window.

Left null space

Consider a dual problem. Current state of the system is given as $\mathbf{x}_i \in \mathbb{R}^n$, and we do not know it. At the next point in time, the state of the system is given as $\mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i$. Which values the next states will not assume?

Notice, that this is the same as asking what is the orthogonal complement of the column space of \mathbf{A} . It is called *left null space*.

Let \mathbf{P} be the projector onto the column space of \mathbf{A} : $\mathbf{P} = \mathbf{A}\mathbf{A}^+$. Then $\mathbf{Q} = (\mathbf{I} - \mathbf{P}) = (\mathbf{I} - \mathbf{A}\mathbf{A}^+)$ is a projector onto the left null space of \mathbf{A} .

Row space and left null space bases

While MATLAB does not provide tools to directly find orthonormal bases in the row space and left null space of a matrix, it can be done by tweaking the code for `orth()` and `null()`.

Your HW: write down formulas for `orth()`, `null()` and their orthogonal compliments, and implement those in MATLAB or your language of choice that provides an SVD decomposition implementation.

We will define operators `row()` and `lnull()` to refer to bases in those two spaces.

Example

Now that we have such powerful tools, we can solve difficult problems easily. Consider this one. Linear time-invariant (LTI) dynamical system is described as:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}_1\mathbf{u}_1 + \mathbf{B}_2\mathbf{u}_2 \quad (6)$$

Find such control inputs \mathbf{u}_1^* , \mathbf{u}_2^* that state \mathbf{x}^* becomes a fixed point. Additionally, assume that \mathbf{u}_1^* is free to use, while \mathbf{u}_2^* should be used as sparingly as possible.

This can be formulated in the language of optimization as follows:

$$\begin{aligned} & \underset{\mathbf{u}_1, \mathbf{u}_2}{\text{minimize}} && ||\mathbf{u}_2||, \\ & \text{subject to} && \mathbf{A}\mathbf{x}^* + \mathbf{B}_1\mathbf{u}_1 + \mathbf{B}_2\mathbf{u}_2 = \mathbf{0} \end{aligned} \quad (7)$$

Example

part 2

In order to check that the problem has at least one solution we need to make sure that there are such inputs \mathbf{u}_1^* , \mathbf{u}_2^* that state \mathbf{x}^* becomes a fixed point. In other words, vector $\mathbf{A}\mathbf{x}^*$ should lie in the span of the columns of the matrix $[\mathbf{B}_1 \ \mathbf{B}_2]$. Which is the same as saying that its projection on the complement on this column space (left null space of $[\mathbf{B}_1 \ \mathbf{B}_2]$) is zero:

$$(\mathbf{I} - [\mathbf{B}_1 \ \mathbf{B}_2][\mathbf{B}_1 \ \mathbf{B}_2]^+) \mathbf{A}\mathbf{x}^* = \mathbf{0} \quad (8)$$

All solutions to the problem $\mathbf{A}\mathbf{x}^* + \mathbf{B}_1\mathbf{u}_1 + \mathbf{B}_2\mathbf{u}_2 = \mathbf{0}$ can be found as:

$$\mathbf{u} = -[\mathbf{B}_1 \ \mathbf{B}_2]^+ \mathbf{A}\mathbf{x}^* + \text{null}([\mathbf{B}_1 \ \mathbf{B}_2])\mathbf{z}, \quad \forall \mathbf{z} \quad (9)$$

Example

part 3

Now we need to pick one solution out of all of them, based on the criteria that it minimizes \mathbf{u}_2 . We can solve it as an optimization (by thinking about the derivatives of the objective/cost function), but it can also be solved as a projection.

Let us define projector $\mathbf{P} = \mathbf{B}_1 \mathbf{B}_1^+$. We can prove that $\mathbf{P} \mathbf{B}_2 \mathbf{u}_2^* = \mathbf{0}$. Assume $\mathbf{P} \mathbf{B}_2 \mathbf{u}_2^* = \mathbf{a}$, then $\mathbf{u}_2^* = \mathbf{u}_2^0 + \mathbf{u}_2^a$, where \mathbf{u}_2^0 is in the null space of $\mathbf{P} \mathbf{B}_2$ and \mathbf{u}_2^a is in the row space of $\mathbf{P} \mathbf{B}_2$, or equivalently $\mathbf{P} \mathbf{B}_2 \mathbf{u}_2^0 = \mathbf{0}$, $(\mathbf{I} - \mathbf{P}) \mathbf{B}_2 \mathbf{u}_2^0 = \mathbf{b}$, and $\mathbf{P} \mathbf{B}_2 \mathbf{u}_2^a = \mathbf{a}$, $(\mathbf{I} - \mathbf{P}) \mathbf{B}_2 \mathbf{u}_2^a = \mathbf{0}$. This gives us solution:

$$\mathbf{A} \mathbf{x}^* + \mathbf{B}_1 \mathbf{u}_1^* + \mathbf{B}_2 \mathbf{u}_2^0 + \mathbf{B}_2 \mathbf{u}_2^a = \mathbf{0} \quad (10)$$

Example

part 4

But since $\mathbf{a} \in \text{span}(\mathbf{B}_1)$ we can also provide an alternative solution:

$$\mathbf{A}\mathbf{x}^* + \mathbf{B}_1\mathbf{u}_1^a + \mathbf{B}_2\mathbf{u}_2^0 = \mathbf{0} \quad (11)$$

where $\mathbf{u}_1^a = \mathbf{u}_1^* + \mathbf{B}_1^+ \mathbf{a}$.

Since \mathbf{u}_2^0 and \mathbf{u}_2^a belong to orthogonal subspaces, $\|\mathbf{u}_2^0 + \mathbf{u}_2^a\| \geq \|\mathbf{u}_2^0\|$. Therefore the alternative solution provides smaller norm \mathbf{u}_2 , hence $\mathbf{a} = \mathbf{0}$ and $\mathbf{u}_2^a = \mathbf{0}$ and $\mathbf{P}\mathbf{B}_2\mathbf{u}_2^* = \mathbf{0}$.

Since $\mathbf{P}\mathbf{B}_2\mathbf{u}_2^* = \mathbf{0}$, $(\mathbf{I} - \mathbf{P})\mathbf{B}_2\mathbf{u}_2^* = \mathbf{B}_2\mathbf{u}_2^*$, and while $(\mathbf{I} - \mathbf{P})\mathbf{B}_1 = \mathbf{0}$.

Example

part 5

Multiplying our constraint by $(\mathbf{I} - \mathbf{P})$ we attain:

$$(\mathbf{I} - \mathbf{P})\mathbf{A}\mathbf{x}^* + (\mathbf{I} - \mathbf{P})\mathbf{B}_2\mathbf{u}_2^* = \mathbf{0} \quad (12)$$

Which has an exact solution:

$$\mathbf{u}_2^* = -((\mathbf{I} - \mathbf{P})\mathbf{B}_2)^+(\mathbf{I} - \mathbf{P})\mathbf{A}\mathbf{x}^* \quad (13)$$

Which is the solution to the original problem; \mathbf{u}_1^* can be obtained as follows:

$$\mathbf{u}_1^* = -\mathbf{B}_1^+(\mathbf{A}\mathbf{x}^* + \mathbf{B}_2\mathbf{u}_2^*) \quad (14)$$

Homework

- Prove the following for an orthogonal projector \mathbf{P} :
 $(\mathbf{I} - \mathbf{P})(\mathbf{I} - \mathbf{P}) = (\mathbf{I} - \mathbf{P})$
- Write down formulas for `orth()`, `null()` and their orthogonal compliments, and implement those in MATLAB or your language of choice that provides an SVD decomposition implementation.
- Prove that (13) will always have a solution that turns $((\mathbf{I} - \mathbf{P})\mathbf{B}_2)\mathbf{u}_2^* = -(\mathbf{I} - \mathbf{P})\mathbf{A}\mathbf{x}^*$ into equality.
- Prove that (14) will always have a solution that turns $\mathbf{B}_1\mathbf{u}_1^* = -\mathbf{A}\mathbf{x}^* - \mathbf{B}_2\mathbf{u}_2^*$ into equality.

Suggested readings and videos

- Coursera, Projections, one-dimensional case
- Coursera, Projections, higher-dimensional case

Lecture slides are available via Moodle.

You can help improve these slides at:

github.com/SergeiSa/Computational-Intelligence-Slides-Fall-2020

Check Moodle for additional links, videos, textbook suggestions.