Four Fundamental Subspaces, use cases

Computational Intelligence, Lecture 4

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Fall 2020

Content

Finding fixed points

Given LTI system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$, where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{u} \in \mathbb{R}^m$, 1) find if there are states that can be made into fixed points, 2) find all states that can be made into fixed points with a constant control law.

Solution:

1. Yes, state $\mathbf{x} = \mathbf{0}$ becomes a fixed point under control law $\mathbf{u} = \mathbf{0}$ or $\mathbf{u} = -\mathbf{K}\mathbf{x}$
2. Let us find null space of the matrix $\begin{bmatrix} \mathbf{A} & \mathbf{B} \end{bmatrix}$ as $\mathbf{N} = \text{null}(\begin{bmatrix} \mathbf{A} & \mathbf{B} \end{bmatrix})$ We can find all pairs of $\mathbf{x}$ and $\mathbf{u}$ that produce fixed points as follows: $\begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} = \text{null}(\begin{bmatrix} \mathbf{A} & \mathbf{B} \end{bmatrix})\mathbf{z}, \forall \mathbf{z}$. Let $\mathbf{N}_x$ be the first n rows of $\mathbf{N}$. Then all states that can be made into fixed points are given as $\mathbf{x}^* = \mathbf{N}_x \mathbf{z}_x, \forall \mathbf{z}_x$

Checking fixed points

Given LTI system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$, where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{u} \in \mathbb{R}^m$, 1) check if $\mathbf{x}^*$ can be made into a fixed point, 2) find control constant $\mathbf{u}^*$ that does it, given control law $\mathbf{u} = -\mathbf{K}\mathbf{x} + \mathbf{u}^*$.

Solution:

1. We can check that $(\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}^* + \mathbf{B}\mathbf{u}^* = \mathbf{0}$ has a solution, in other words that $(\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}^* \in \text{col}(\mathbf{B})$. Resulting condition is: $(\mathbf{I} - \mathbf{B}\mathbf{B}^+)(\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}^* = \mathbf{0}$
2. This means finding such $\mathbf{u}^*$ that $(\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}^* + \mathbf{B}\mathbf{u}^* = \mathbf{0}$. This is done via pseudo-inverse, which provides exact solution, as long as it exists: $\mathbf{u}^* = -\mathbf{B}^+(\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}^*$.

Correcting fixed points

Given LTI system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$, where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{u} \in \mathbb{R}^m$, and a state $\mathbf{x}^d$ which can not be made into a fixed point under constant control law, find the closest to it state $\mathbf{x}^f$ which can be made into a fixed point.

As we know from the first example, for this LTI system all fixed points under constant control are given as $\mathbf{x}^f = \mathbf{N}_x \mathbf{z}_x$. We define shifting vector $\mathbf{x}_s = \mathbf{x}^f - \mathbf{x}^d$, and we can prove that $\mathbf{x}_s$ has to lie in the orthogonal compliment to the span of $\mathbf{N}_x$ (HW: prove it), or in other words, in its left null space.

The problem is solved by projecting $\mathbf{x}^d$ onto the left null space of $\mathbf{N}_x$: $\mathbf{x}_s^* = -(\mathbf{I} - \mathbf{N}_x \mathbf{N}_x^+)\mathbf{x}^d$ (note that $\mathbf{N}_x$ is not an orthonormal basis). We can check it by observing that $\mathbf{x}^f = \mathbf{x}_s^* + \mathbf{x}^d = \mathbf{x}^d - (\mathbf{I} - \mathbf{N}_x \mathbf{N}_x^+)\mathbf{x}^d = \mathbf{N}_x \mathbf{N}_x^+ \mathbf{x}^d \in \mathrm{col}(\mathbf{N}_x)$. Any other shift vector candidate $\mathbf{x}^s = \mathbf{x}_s^* + \mathbf{a}$, where $\mathbf{a} \in \mathrm{lnull}(\mathbf{N}_x)$ will produce $\mathbf{x}^f = \mathbf{N}_x \mathbf{N}_x^+ \mathbf{x}^d + \mathbf{a} \notin \mathrm{col}(\mathbf{N}_x)$.

Finding fixed points for affine systems

Given LTI system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{c}$, where $\mathbf{x} \in \mathbb{R}^n$, and control law $\mathbf{u} = -\mathbf{K}\mathbf{x} + \mathbf{u}^*$, find all states that can be made fixed points.

We are required to find all solutions to the equation $(\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}^* + \mathbf{B}\mathbf{u}^* + \mathbf{c} = \mathbf{0}$. Let us define state-control pairs $\mathbf{z} = \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix}$.

We can easily find particular solution to this linear system: $\mathbf{z}^p = -\begin{bmatrix} (\mathbf{A} - \mathbf{B}\mathbf{K}) & \mathbf{B} \end{bmatrix}^+ \mathbf{c}$.

Finding null space basis $\mathbf{N}$ for the matrix of this linear system: $\mathbf{N} = \text{null}(\begin{bmatrix} (\mathbf{A} - \mathbf{B}\mathbf{K}) & \mathbf{B} \end{bmatrix})$ we get the general solution as follows: $\mathbf{z}^* = \mathbf{z}^p + \mathbf{N}\mathbf{y}$. First n equations in this expression defining $\mathbf{z}^*$ give us $\mathbf{x}^*$, the rest - $\mathbf{u}^*$.

Given state $\mathbf{x}$, its estimation $\mathbf{x}_e$, and three correct measurements $\mathbf{y}_1 = \mathbf{C}_1\mathbf{x}$, $\mathbf{y}_2 = \mathbf{C}_2\mathbf{x}$ and $\mathbf{y}_3 = \mathbf{C}_3\mathbf{x}$, find a new better estimate of $\mathbf{x}$.

We combine all measurements into one: $\mathbf{y} = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \mathbf{y}_3 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} \mathbf{C}_1 \\ \mathbf{C}_2 \\ \mathbf{C}_3 \end{bmatrix}$.

Next we observe that since $\mathbf{y} = \mathbf{C}\mathbf{x}$, all correct state estimates lie on the affine plane $\mathbf{x} = \mathbf{x}_p + \mathbf{N}\mathbf{z}$, where $\mathbf{x}_p = \mathbf{C}^+\mathbf{y}$ is the particular solution and $\mathbf{N} = \text{null}(\mathbf{C})$ is the null space basis.

State estimation correction

part 2

We want to find closest to $\mathbf{x}_e$ state $\mathbf{x}_e^*$ with lies on the defined above affine plane. Defining a shift vector $\mathbf{x}_s = \mathbf{x}_e^* - \mathbf{x}_e$ we can write a condition: $\mathbf{x}_s + \mathbf{x}_e = \mathbf{x}_p + \mathbf{N}\mathbf{z}$. We can define a corrected shift vector $\mathbf{x}_c = \mathbf{x}_s - \mathbf{x}_p$, and then the problem becomes equivalent to the previously solved one: finding a state on a linear subspace, closest to the give one.

We remember that it is solved via projection onto the left null space: $\mathbf{x}_c = (\mathbf{I} - \mathbf{N}\mathbf{N}^\top)\mathbf{x}_e$, hence:

$$\mathbf{x}_e^* = \mathbf{N}\mathbf{N}^\top \mathbf{x}_e + \mathbf{x}_p \tag{1}$$

Homework

- Prove that in the third example $\mathbf{x}^s$ has to lie in the orthogonal space to $\mathbf{N}_x$.
- Implement all the examples in this lecture.
- Prove that affine and constant control laws provide exactly the same space of possible fixed points for an LTI system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$. Show how constant part of the control law changes for the same fixed point between an affine and constant control laws.

Lecture slides are available via Moodle.

You can help improve these slides at:
github.com/SergeiSa/Computational-Intelligence-Slides-Fall-2020

Check Moodle for additional links, videos, textbook suggestions.