

# Linear inequality representation of convex domains

Computational Intelligence, Lecture 7

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# Convex polytopes

Before defining what a convex polytope is, let us look at examples:



# Half-spaces

## Definition

We can define half-space as a set of all points  $\mathbf{x}$ , such that  $\mathbf{a}^\top \mathbf{x} \leq b$ . It has a very clear geometric interpretation. In the following image, the filled space is **not** in the half space.



# Half-spaces

## Construction. Simple case

Consider half-space that passes through the origin, and defined by its normal vector  $\mathbf{n}$ :

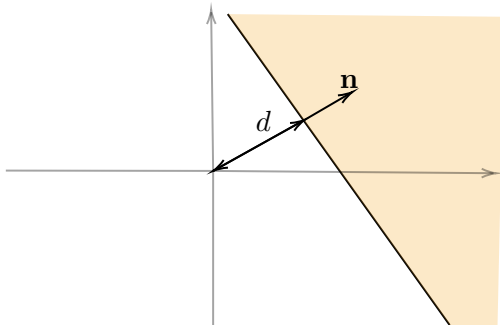


It is easy to see that this half-space can be defined as "all vectors  $\mathbf{x}$ , such that  $\mathbf{n} \cdot \mathbf{x} \leq 0$ ", which is the same as using  $\mathbf{n}$  instead of  $\mathbf{a}$  in our original definition, setting  $b = 0$ .

# Half-spaces

## Construction. General case

In the general case there is some distance between the boundary of the half-space and the origin, let's say  $d$ .



The same way we see, that the half space can be defined as "all vectors  $\mathbf{x}$ , such that  $\mathbf{n} \cdot \mathbf{x} \leq d$ ". This is the same as making  $\mathbf{a} = \mathbf{n}$  and  $b = d$  in our original definition. However, if  $\mathbf{a}$  is not a unit vector,  $b = d\|\mathbf{a}\|$ .

# Half-spaces

## Combination

We can define a region of space as an *intersection* of half-spaces  $\mathbf{a}_i^\top \mathbf{x} \leq b_i$ :



Resulting region will be easily described as 
$$\begin{bmatrix} \mathbf{a}_1^\top \\ \dots \\ \mathbf{a}_k^\top \end{bmatrix} \mathbf{x} \leq \begin{bmatrix} b_1 \\ \dots \\ b_k \end{bmatrix}$$

# Half-spaces

## Formal description via inequalities

The last result allows us to write any convex polytope as a matrix inequality:

$$\mathbf{Ax} \leq \mathbf{b} \tag{1}$$

And conversely, any matrix inequality (1) represents either an empty set or a convex polytope.



# Linear approximation of convex regions

Some convex regions can be easily approximated using polytopes.



Which allows to represent constraints on  $\mathbf{x}$  to belong in such a region as a matrix inequality

Represent in matrix inequality form the following figures:

- Equilateral triangle
- A square
- Parallelepiped
- Trapezoid

Lecture slides are available via Moodle.

You can help improve these slides at:

[github.com/SergeiSa/Computational-Intelligence-Slides-Fall-2020](https://github.com/SergeiSa/Computational-Intelligence-Slides-Fall-2020)

Check Moodle for additional links, videos, textbook suggestions.