Quadratically constrained quadratic programming, Second-order cone programming

Computational Intelligence, Lecture 8

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Fall 2020

Content

- Quadratic programming: recap
- Quadratically constrained quadratic programming
  - General form
  - Domain
- Second-order cone programming
  - General form
  - Special cases
  - Friction cone
  - Friction cone, solution
- Homework

Quadratic programming

General form

Remember the general form of a quadratic program:

$$\begin{aligned}
&\underset{\mathbf{x}}{\text{minimize}} \quad && \mathbf{x}^\top \mathbf{H} \mathbf{x} + \mathbf{f}^\top \mathbf{x}, \\
&\text{subject to} && \begin{cases} \mathbf{A}\mathbf{x} \leq \mathbf{b}, \\ \mathbf{C}\mathbf{x} = \mathbf{d}. \end{cases}
\end{aligned} \tag{1}$$

where $\mathbf{H}$ is positive-definite and $\mathbf{A}\mathbf{x} \leq \mathbf{b}$ describe a convex region.

Quadratically constrained quadratic programming

General form

General form of a quadratically constrained quadratic program (QCQP) is given below:

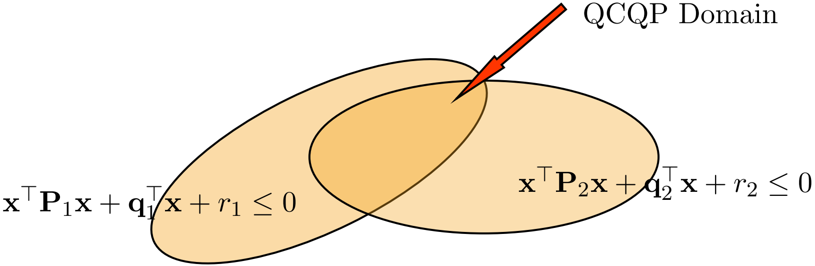
$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & \mathbf{x}^\top \mathbf{P}_0 \mathbf{x} + \mathbf{q}_0^\top \mathbf{x}, \\ \text{subject to} \quad & \begin{cases} \mathbf{x}^\top \mathbf{P}_i \mathbf{x} + \mathbf{q}_i^\top \mathbf{x} + r_i \leq 0, \\ \mathbf{A}\mathbf{x} = \mathbf{b}. \end{cases} \end{aligned} \tag{2}$$

where $\mathbf{P}_i$ are positive-definite.

Quadratically constrained quadratic programming

Domain

Domain of a QCQP without equality constraints and with no degenerate inequality constraints is an intersection of ellipses:



QCQP Domain

$\mathbf{x}^\top \mathbf{P}_1 \mathbf{x} + \mathbf{q}_1^\top \mathbf{x} + r_1 \leq 0$

$\mathbf{x}^\top \mathbf{P}_2 \mathbf{x} + \mathbf{q}_2^\top \mathbf{x} + r_2 \leq 0$

Second-order cone programming

General form

The general form of a Second-order cone program (SOCP) is:

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & \mathbf{f}^\top \mathbf{x}, \\ \text{subject to} \quad & \begin{cases} ||\mathbf{A}_i \mathbf{x} + \mathbf{b}_i||_2 \leq \mathbf{c}_i^\top \mathbf{x} + d_i, \\ \mathbf{F}\mathbf{x} = g. \end{cases} \end{aligned} \tag{3}$$

LP, QP and QCQP are subsets of SOCP.

Second-order cone programming

Special cases

We can write problem where our domain is a ball as SOCP:

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & \mathbf{f}^\top \mathbf{x}, \\ \text{subject to} \quad & ||\mathbf{x}||_2 \leq d_i \end{aligned} \tag{4}$$

Same for ellipsoidal constraints:

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & \mathbf{f}^\top \mathbf{x}, \\ \text{subject to} \quad & ||\mathbf{A}_i \mathbf{x}||_2 \leq d_i \end{aligned} \tag{5}$$

Second-order cone programming

Friction cone

Remember *friction cone*: the idea that for a reaction force should lie in a cone normal to the ground.

Let $\mathbf{f} = \mathbf{e}_n n + \mathbf{e}_{t,1} t_1 + \mathbf{e}_{t,2} t_2$ where $\mathbf{e}_n$ is a normal direction to the ground, and $\mathbf{e}_{t,i}$ are tangential directions (all unit vectors). Therefore $\mathbf{e}_n n$ is the normal reaction and $\mathbf{e}_{t,1} t_1 + \mathbf{e}_{t,2} t_2$ is the friction force.

The friction cone conditions could be written as:

$$\sqrt{t_1^2 + t_2^2} < \mu n \tag{6}$$

where μ is a friction coefficient.

Second-order cone programming

Friction cone, solution

We can represent condition (6) in the SOC form, taking $\mathbf{f}$ as our variable:

$$|| \begin{bmatrix} \mathbf{e}_{t,1} & \mathbf{e}_{t,2} \end{bmatrix}^\top \mathbf{f} || \leq \mu \mathbf{e}_n^\top \mathbf{f} \tag{7}$$

If we want to impose strict inequality, we should add a margin on μ, proposing to use $\mu^* < \mu$ instead.

Homework

Implement a program that finds right-most point of an intersection of two ellipsoids; visualise the problem and the solution.

Lecture slides are available via Moodle.

You can help improve these slides at:
github.com/SergeiSa/Computational-Intelligence-Slides-Fall-2020

Check Moodle for additional links, videos, textbook suggestions.