# Domain, Convex Domains Computational Intelligence, Lecture 6

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#### Content

- Domain
- Bounded and unbounded domains
- Convex domains
- Examples of convex domains
- Examples of non-convex domains
- Convex functions
- Convex functions examples
- Convex programming
- Homework

#### Domain

Problem 1. Find minimum of the function  $f = x^2 + 2y^2$  if  $x \in \mathbb{R}$  and  $y \in \mathbb{R}$ . Solution is x = 0, y = 0.

Problem 2. Find minimum of the function  $f = x^2 + 2y^2$  if  $x \in [1\ 2]$  and  $y \in [2\ 5]$ . Solution is x = 1, y = 2.

Note that solutions of problems 1 and 2 are different, and this is only due to the difference of the allowed values that the decision variables x and y can assume.

#### Definition 1

Space of all allowed values that decision variables can assume is called *domain* of the optimization problem.

# Bounded and unbounded domains

Part 1

Problem 3. Find minimum of the function  $f = -x^2$  if  $x \in [-3\ 2]$ . Solution is x = -3.

Problem 4. Find minimum of the function  $f = -x^2$  if  $x \in \mathbb{R}$ . The problem has no solution.

Problem 5. Find minimum of the function  $f = -x^2$  if  $x \in [-\infty 2]$ . The problem has no solution.

The major difference between domains of the problems 2, 3 vs problems 1, 4 and 5 is that the later are *not bound* (i.e., you can construct a sequence of the values in the domain that would approach infinity).

We can see that in the case of problems 3-5, bounding the domain allows the problem to obtain a solution.

# Bounded and unbounded domains

Part 2

Problem 6. Find maximum of the function  $f = x^2$  if  $1 \le x \le 2$ . It has no solution.

Problem 7. Find minimum of the function  $f = x^2$  if  $1 \le x \le 2$ . Solution is x = 1.

This time, the fact that the one of the boundaries of the domain were not included lead problem 6 to have no solution, while problem 7 had one. For problem 6 we can pick a value arbitrary close to x=2, approaching it from the left, but for any pick of the decision variable we can make, there always will be other values of the decision variable closer to x=2 and hence producing larger values of f.

### Convex domains

#### Definition 2

Domain is *convex* iff for any two points in the domain, the line segment connecting them is also in the domain.

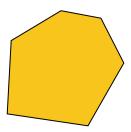


Figure 1: Convex domain

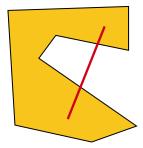


Figure 2: Non-convex domain

## Examples of convex domains

$$\mathbf{x} \in \mathcal{X}, \, \mathcal{X} = \mathbb{R}^n$$
 is convex. Prove it.

$$\mathbf{x} \in \mathcal{X}, \ \mathcal{X} = \{\mathbf{x} \in \mathbb{R}^n : \ \mathbf{x} \leq \mathbf{h}\} \text{ is convex. Prove it.}$$

$$\mathbf{x} \in \mathcal{X}, \ \mathcal{X} = \{\mathbf{x} \in \mathbb{R}^2: \ x_1^2 + x_2^2 \le h^2\}$$
 is convex. Prove it.

 $\mathbf{x} \in \mathcal{X}, \ \mathcal{X} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}^\top \mathbf{H} \mathbf{x} \leq h\}, \text{ where } \mathbf{H} \succ 0 \text{ is positive-definite symmetric matrix is convex. Prove it.}$ 

In the proofs it is convenient to remember that for any two points  $\mathbf{x}_1$  and  $\mathbf{x}_2$ , all points in the line segment connecting them are given as  $\mathbf{x}_l = \alpha \mathbf{x}_1 + (1 - \alpha)\mathbf{x}_2$ , where  $\alpha \in [0 \ 1]$ . This is called *convex combination*.

## Examples of non-convex domains

$$\mathbf{x} \in \mathcal{X}, \ \mathcal{X} = \{\mathbf{x} \in \mathbb{R}^n : \ \mathbf{x} \ge \mathbf{h}\}$$
 is not convex. Prove it.

$$x \in \mathcal{X}, \, \mathcal{X} = [-1 \,\, 2] \cup [3 \,\, 7]$$
 is not convex. Prove it.

$$\mathbf{x} \in \mathcal{X}, \ \mathcal{X} = \{\mathbf{x} \in \mathbb{R}^2 : \ x_1^2 + x_2^2 \ge h^2\}$$
 is not convex. Prove it.

 $\mathbf{x} \in \mathcal{X}, \ \mathcal{X} = \{\mathbf{x} \in \mathbb{R}^n : \ \mathbf{x}^\top \mathbf{H} \mathbf{x} \ge h\}, \text{ where } \mathbf{H} \text{ is positive-definite symmetric matrix is not convex. Prove it.}$ 

These proves simply require one counter-example to show that the defining property of convex domains does not hold.

#### Convex functions

#### Definition 3

Function  $f(\mathbf{x})$  defined on a domain  $\mathcal{D}$ , for which it holds that  $\forall \mathbf{x}_1, \mathbf{x}_2 \in \mathcal{D}$ ,  $f(\alpha \mathbf{x}_1 + (1 - \alpha)\mathbf{x}_2) \leq \alpha f(\mathbf{x}_1) + (1 - \alpha)f(\mathbf{x}_2)$  is called a convex function.

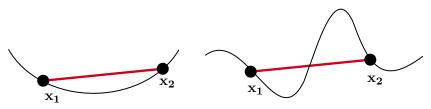


Figure 3: Convex function

Figure 4: Non-convex function

## Convex functions - examples

Here are some single-variable convex functions:

• 
$$f(x) = 1$$

• 
$$f(x) = x$$
,  $f(x) = x + 1$ ,  $f(x) = 6x + 3$ 

• 
$$f(x) = x^2$$
,  $f(x) = (x-5)^2$ ,  $f(x) = (x+1)^2 - 10$ 

• 
$$f(x) = x^3$$
, if  $x > 0$ 

Here are some multi-variable convex functions:

$$f(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$$

$$f(\mathbf{x}) = \mathbf{x}^{\top} \mathbf{H} \mathbf{x}, \, \mathbf{H} \succ 0$$

# Convex programming

#### Definition 3

If the domain of the optimization problem is convex and the cost function is convex, it is called a *convex optimization* problem.

Additionally, we will always assume that the domain of the convex optimization problem contains its boundary. Also, without the loss of generality, we will consider only minimization problems.

There are a few important properties of convex optimization problems (with our additional assumption):

- If the domain is non-empty, there is a solution.
- The problem has no local minima. We can find a path from any point to the solution, along which the cost function will not increase.

## Homework

 $\bullet$  Make formal proofs asked for in this lecture.

# Self-study

• Convex Optimization, lecture 3, S. Boyd. Stanford. Convex functions.

Lecture slides are available via Moodle.

You can help improve these slides at: github.com/SergeiSa/Computational-Intelligence-Slides-Fall-2020

Check Moodle for additional links, videos, textbook suggestions.