Four Fundamental Subspaces, use cases

Computational Intelligence, Lecture 4

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Finding fixed points

Given LTI system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$, where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{u} \in \mathbb{R}^m$, 1) find if there are states that can be made into fixed points, 2) find all states that can be made into fixed points with a constant control law.

Solution:

1. Yes, state $\mathbf{x} = \mathbf{0}$ becomes a fixed point under control law $\mathbf{u} = \mathbf{0}$ or $\mathbf{u} = -\mathbf{K}\mathbf{x}$
2. Let us find null space of the matrix $\begin{bmatrix} \mathbf{A} & \mathbf{B} \end{bmatrix}$ as $\mathbf{N} = \text{null}(\begin{bmatrix} \mathbf{A} & \mathbf{B} \end{bmatrix})$ We can find all pairs of $\mathbf{x}$ and $\mathbf{u}$ that produce fixed points as follows: $\begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} = \text{null}(\begin{bmatrix} \mathbf{A} & \mathbf{B} \end{bmatrix})\mathbf{z}, \forall \mathbf{z}$. Let $\mathbf{N}_x$ be the first n columns of $\mathbf{N}$. Then all states that can be made into fixed points are given as $\mathbf{x}^* = \mathbf{N}_x \mathbf{z}_x, \forall \mathbf{z}_x$

Checking fixed points

Given LTI system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$, where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{u} \in \mathbb{R}^m$, 1) check if $\mathbf{x}^*$ can be made into a fixed point, 2) find control constant $\mathbf{u}^*$ that does it, given control law $\mathbf{u} = -\mathbf{K}\mathbf{x} + \mathbf{u}^*$.

Solution:

1. We can check that $(\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}^* + \mathbf{B}\mathbf{u}^* = \mathbf{0}$ has a solution, in other words that $(\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}^* \in \mathrm{col}(\mathbf{B})$. Resulting condition is: $(\mathbf{I} - \mathbf{B}\mathbf{B}^+)(\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}^* = \mathbf{0}$
2. This means finding such $\mathbf{u}^*$ that $(\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}^* + \mathbf{B}\mathbf{u}^* = \mathbf{0}$. This is done via pseudo-inverse, which provides exact solution, as long as it exists: $\mathbf{u}^* = -\mathbf{B}^+(\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}^*$.

Correcting fixed points

Given LTI system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$, where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{u} \in \mathbb{R}^m$, and a state $\mathbf{x}^d$ which can not be made into a fixed point under constant control law, find the closest to it state $\mathbf{x}^f$ which can be made into a fixed point.

As we know from the first example, for this LTI system all fixed points under constant control are given as $\mathbf{x}^f = \mathbf{N}_x \mathbf{z}_x$. We define shifting vector $\mathbf{x}_s = \mathbf{x}^f - \mathbf{x}^d$, and we can prove that $\mathbf{x}_s$ has to lie in the orthogonal compliment to the span of $\mathbf{N}_x$ (HW: prove it), or in other words, in its left null space.

The problem is solved by projecting $\mathbf{x}^d$ onto the left null space of $\mathbf{N}_x$: $\mathbf{x}_s^* = -(\mathbf{I} - \mathbf{N}_x \mathbf{N}_x^\top)\mathbf{x}^d$. We can check it by observing that $\mathbf{x}^f = \mathbf{x}_s^* + \mathbf{x}^d = \mathbf{x}^d - (\mathbf{I} - \mathbf{N}_x \mathbf{N}_x^\top)\mathbf{x}^d = \mathbf{N}_x \mathbf{N}_x^\top \mathbf{x}^d \in \mathrm{col}(\mathbf{N}_x)$. Any other shirt vector candidate $\mathbf{x}^s = \mathbf{x}_s^* + \mathbf{a}$, where $\mathbf{a} \in \mathrm{lnull}(\mathbf{N}_x)$ will produce $\mathbf{x}^f = \mathbf{N}_x \mathbf{N}_x^\top \mathbf{x}^d + \mathbf{a} \notin \mathrm{col}(\mathbf{N}_x)$.

Finding fixed points for affine systems

Given LTI system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{c}$, where $\mathbf{x} \in \mathbb{R}^n$, and control law $\mathbf{u} = -\mathbf{K}\mathbf{x} + \mathbf{u}^*$, find all states that can be made fixed points.

We are required to find all solutions to the equation $(\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}^* + \mathbf{B}\mathbf{u}^* + \mathbf{c} = \mathbf{0}$. Let us define state-control pairs $\mathbf{z} = \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix}$.

We can easily find particular solution to this linear system: $\mathbf{z}^p = -\begin{bmatrix} (\mathbf{A} - \mathbf{B}\mathbf{K}) & \mathbf{B} \end{bmatrix}^+ \mathbf{c}$.

Finding null space basis $\mathbf{N}$ for the matrix of this linear system: $\mathbf{N} = \mathrm{null}(\begin{bmatrix} (\mathbf{A} - \mathbf{B}\mathbf{K}) & \mathbf{B} \end{bmatrix})$ we get the general solution as follows: $\mathbf{z}^* = \mathbf{z}^p + \mathbf{N}\mathbf{y}$. First n equations in this expression defining $\mathbf{z}^*$ give us $\mathbf{x}^*$, the rest - $\mathbf{u}^*$.

Given state $\mathbf{x}$, its estimation $\mathbf{x}_e$, and three correct measurements $\mathbf{y}_1 = \mathbf{C}_1\mathbf{x}$, $\mathbf{y}_2 = \mathbf{C}_2\mathbf{x}$ and $\mathbf{y}_3 = \mathbf{C}_3\mathbf{x}$, find a new better estimate of $\mathbf{x}$.

We combine all measurements into one: $\mathbf{y} = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \mathbf{y}_3 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} \mathbf{C}_1 \\ \mathbf{C}_2 \\ \mathbf{C}_3 \end{bmatrix}$.

Next we observe that since $\mathbf{y} = \mathbf{C}\mathbf{x}$, all correct state estimates lie on the affine plane $\mathbf{x} = \mathbf{x}_p + \mathbf{N}\mathbf{z}$, where $\mathbf{x}_p = \mathbf{C}^{+}\mathbf{y}$ is the particular solution and $\mathbf{N} = \text{null}(\mathbf{C})$ is the null space basis.

State estimation correction

part 2

We want to find closest to $\mathbf{x}_e$ state $\mathbf{x}_e^*$ with lies on the defined above affine plane. Defining a shift vector $\mathbf{x}_s = \mathbf{x}_e^* - \mathbf{x}_e$ we can write a condition: $\mathbf{x}_s + \mathbf{x}_e = \mathbf{x}_p + \mathbf{N}\mathbf{z}$. We can define a corrected shift vector $\mathbf{x}_c = \mathbf{x}_s - \mathbf{x}_p$, and then the problem becomes equivalent to the previously solved one: finding a state on a linear subspace, closest to the give one.

We remember that it is solved via projection onto the left null space: $\mathbf{x}_c = (\mathbf{I} - \mathbf{N}\mathbf{N}^\top)\mathbf{x}_e$, hence:

$$\mathbf{x}_e^* = \mathbf{N}\mathbf{N}^\top \mathbf{x}_e + \mathbf{x}_p \tag{1}$$

Homework

- Prove that in the third example $\mathbf{x}^s$ has to lie in the orthogonal space to $\mathbf{N}_x$.

Lecture slides are available via Moodle.

You can help improve these slides at:
github.com/SergeiSa/Computational-Intelligence-Slides-Fall-2020

Check Moodle for additional links, videos, textbook suggestions.