

# Linear Algebra, Four Fundamental Subspaces

## Computational Intelligence, Lecture 2

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# Motivating questions

You have a linear operator  $\mathbf{A}$ . Try to answer the following questions:

- What are all vectors this operator can produce as outputs?  
How to find them?
- Are there two inputs that make it produce the same output?
- Are there inputs that produce zero as an output?
- Are there outputs that cannot be produced?
- How to find all inputs that produce a unique output?

These questions are directly related to the idea of fundamental subspaces of a linear operator.

# Four Fundamental Subspaces

One of the key ideas in the linear algebra is that every linear operator has four fundamental subspaces:

- Null space
- Row space
- Column space
- Left null space

Our goal is to understand them. The usefulness of this understating is enormous.

# Null space

Consider the following task: find all solutions to the system of equations  $\mathbf{A}\mathbf{x} = \mathbf{0}$ .

It can be re-formulated as follows: find all elements of the *null space* of  $\mathbf{A}$ .

## Definition 1

*Null space* of  $\mathbf{A}$  is the set of all vectors  $\mathbf{x}$  that  $\mathbf{A}$  maps to  $\mathbf{0}$

We will denote null space as  $\mathcal{N}(\mathbf{A})$ . In the literature, it is often denoted as  $\ker(\mathbf{A})$ .

# Null space

## Example

Now we can find all solutions to the system of equations  $\mathbf{Ax} = \mathbf{0}$  by using functions that generate an *orthonormal basis* in the null space of  $\mathbf{A}$ . In MATLAB it is function `null`:

`N = null(A).`

That is it! Space of solutions of  $\mathbf{Ax} = \mathbf{0}$  is the span of the columns of  $\mathbf{N}$ , and all solutions  $\mathbf{x}^*$  can be represented as  $\mathbf{x}^* = \mathbf{Nz}$ ; for any  $\mathbf{z}$  we get a unique solution, and for any solution - a unique  $\mathbf{z}$ .

# Row space

## Part 1

Consider another task: find all solutions to the system of equations  $\mathbf{Ax} = \mathbf{y}$ .

Assume we have two solutions to the system:  $\mathbf{x}_1$  and  $\mathbf{x}_2$ . We know that  $\mathbf{Ax}_1 = \mathbf{Ax}_2 = \mathbf{y}$ , hence  $\mathbf{A}(\mathbf{x}_1 - \mathbf{x}_2) = \mathbf{0}$ . In other words, the difference between any two solutions lies in the null space of  $\mathbf{A}$ .

On the other hand, let  $\mathbf{x}^*$  be a solution, and  $\mathbf{x}^N \in \mathcal{N}(\mathbf{A})$  be a vector in the null space of  $\mathbf{A}$ . Then  $\mathbf{x}^* + \mathbf{x}^N$  is also a solution, since  $\mathbf{A}(\mathbf{x}^* + \mathbf{x}^N) = \mathbf{Ax}^* + \mathbf{Ax}^N = \mathbf{Ax}^* = \mathbf{y}$ .

Therefore, the solution space is given by a single particular solution  $\mathbf{x}^p \notin \mathcal{N}(\mathbf{A})$  and the whole null space of  $\mathbf{A}$ .

# Row space

## Part 2

There are infinitely many ways to choose  $\mathbf{x}^p$ , since if  $\mathbf{x}^p \notin \mathcal{N}(\mathbf{A})$ , then  $(\mathbf{x}^p + \mathbf{x}^N) \notin \mathcal{N}(\mathbf{A})$ , if  $\mathbf{x}^N \in \mathcal{N}(\mathbf{A})$ . However:

### Statement 1

we can find only a single  $\mathbf{x}^p$ , such that  $\mathbf{x}^p \cdot \mathbf{x}^N = 0$ ,  $\forall \mathbf{x}^N \in \mathcal{N}(\mathbf{A})$ , while  $\mathbf{A}(\mathbf{x}^p + \mathbf{x}^N) = \mathbf{y}$ .

Such  $\mathbf{x}^p$  lies in the *row space* of  $\mathbf{A}$ .

### Definition 2

*Row space* of  $\mathbf{A}$  is the set of all inputs to  $\mathbf{A}$  that have a zero projection onto its null space.



# Row space

## Example

Assuming  $\mathbf{Ax} = \mathbf{y}$  has at least one solution, we can find it by solving *least squares* problem, or equivalently, using a pseudo-inverse:

$$\mathbf{x}_p = \text{pinv}(\mathbf{A}) * \mathbf{y}.$$

Now all solutions to the original problem are given as  $\mathbf{x}^* = \mathbf{x}^p + \mathbf{Nz}$ , using previous notation.

# Homework

- Prove Statement 1.
- Come up with a way to check if a vector is in the row space of  $\mathbf{A}$ .

Lecture slides are available via Moodle.

You can help improve these slides at:

[github.com/SergeiSa/Computational-Intelligence-Slides-Fall-2020](https://github.com/SergeiSa/Computational-Intelligence-Slides-Fall-2020)

Check Moodle for additional links, videos, textbook suggestions.