Linear Algebra, Four Fundamental Subspaces

Computational Intelligence, Lecture 1, part 2

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Motivating questions

You have a linear operator $\mathbf{A}$. Try to answer the following questions:

- What are all vectors this operator can produce as outputs? How to find them?
- Are there two inputs that make it produce the same output?
- Are there inputs that produce zero as an output?
- Are there outputs that cannot be produced?
- How to find all inputs that produce a unique output?

These questions are directly related to the idea of fundamental subspaces of a linear operator.

Four Fundamental Subspaces

One of the key ideas in the linear algebra is that every linear operator has four fundamental subspaces:

- Null space
- Row space
- Column space
- Left null space

Our goal is to understand them. The usefulness of this understating is enormous.

Null space

Consider the following task: find all solutions to the system of equations $\mathbf{A}\mathbf{x} = \mathbf{0}$.

It can be re-formulated as follows: find all elements of the *null space* of $\mathbf{A}$.

Definition 1

Null space of $\mathbf{A}$ is the set of all vectors $\mathbf{x}$ that $\mathbf{A}$ maps to $\mathbf{0}$

We will denote null space as $\mathcal{N}(\mathbf{A})$. In the literature, it is often denoted as $\ker(\mathbf{A})$.

Now we can find all solutions to the system of equations $\mathbf{Ax} = \mathbf{0}$ by using functions that generate an orthonormal *basis* in the null space of $\mathbf{A}$. In MATLAB it is function `null`:

```
N = null(A).
```

That is it! Space of solutions of $\mathbf{Ax} = \mathbf{0}$ is the span of the columns of $\mathbf{N}$, and all solutions $\mathbf{x}^*$ can be represented as $\mathbf{x}^* = \mathbf{Nz}$; for any $\mathbf{z}$ we get a unique solution, and for any solution - a unique $\mathbf{z}$.

Consider another task: find all solutions to the system of equations $\mathbf{A}\mathbf{x} = \mathbf{y}$.

Assume we have two solutions to the system: $\mathbf{x}_1$ and $\mathbf{x}_2$. We know that $\mathbf{A}\mathbf{x}_1 = \mathbf{A}\mathbf{x}_2 = \mathbf{y}$, hence $\mathbf{A}(\mathbf{x}_1 - \mathbf{x}_2) = \mathbf{0}$. In other words, the difference between any two solutions lies in the null space of $\mathbf{A}$.

On the other hand, let $\mathbf{x}^*$ be a solution, and $\mathbf{x}^N \in \mathcal{N}(\mathbf{A})$ be a vector in the null space of $\mathbf{A}$. Then $\mathbf{x}^* + \mathbf{x}^N$ is also a solution, since $\mathbf{A}(\mathbf{x}^* + \mathbf{x}^N) = \mathbf{A}\mathbf{x}^* + \mathbf{A}\mathbf{x}^N = \mathbf{A}\mathbf{x}^* = \mathbf{y}$.

Therefore, the solution space is given by a single partial solution $\mathbf{x}^p \notin \mathcal{N}(\mathbf{A})$ and the whole null space of $\mathbf{A}$.

There are infinitely many ways to chose $\mathbf{x}^p$, since if $\mathbf{x}^p \notin \mathcal{N}(\mathbf{A})$, then $(\mathbf{x}^p + \mathbf{x}^N) \notin \mathcal{N}(\mathbf{A})$, if $\mathbf{x}^N \in \mathcal{N}(\mathbf{A})$. However:

Statement 1

we can find only a single $\mathbf{x}^p$, such that
$\mathbf{x}^p \cdot \mathbf{x}^N = 0, \ \forall \mathbf{x}^N \in \mathcal{N}(\mathbf{A})$, while $\mathbf{A}(\mathbf{x}^p + \mathbf{x}^N) = \mathbf{y}$.

Such $\mathbf{x}^p$ lies in the *row space* of $\mathbf{A}$.

Definition 2

Row space of $\mathbf{A}$ is the set of all inputs to $\mathbf{A}$ that have a zero projection onto its null space.

Assuming $\mathbf{Ax} = \mathbf{y}$ has at least one solution, we can find it by solving *least squares* problem, or equivalently, using a pseudo-inverse:

```
xp = pinv(A)*y.
```

Now all solutions to the original problem are given as $\mathbf{x}^* = \mathbf{x}^p + \mathbf{Nz}$, using previous notation.

Prove Statement 1.

Lecture slides are available via Moodle.

You can help improve these slides at:
github.com/SergeiSa/Computational-Intelligence-Slides-Fall-2020

Check Moodle for additional links, videos, textbook suggestions.