

Linear inequality representation of convex domains

Computational Intelligence, Lecture 7

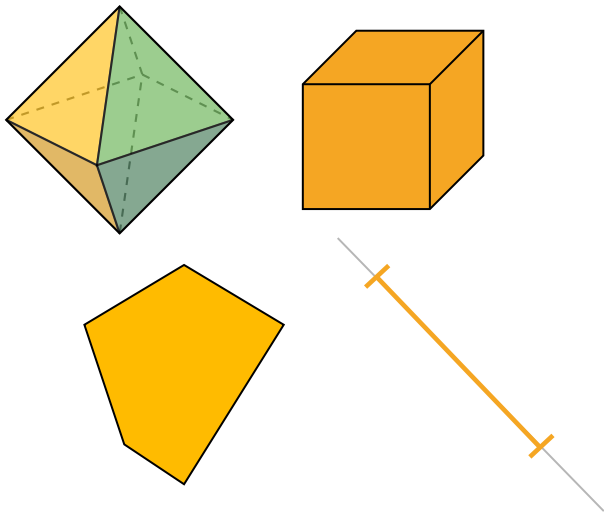
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Fall 2020

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Convex polytopes

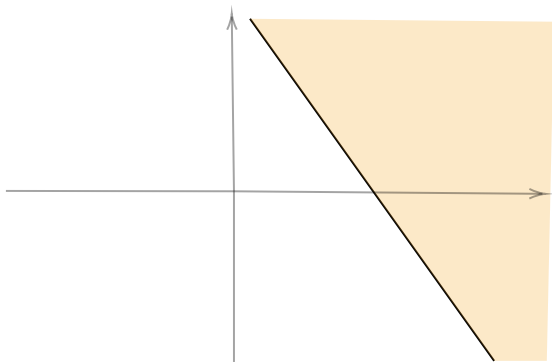
Before defining what a convex polytope is, let us look at examples:



Half-spaces

Definition

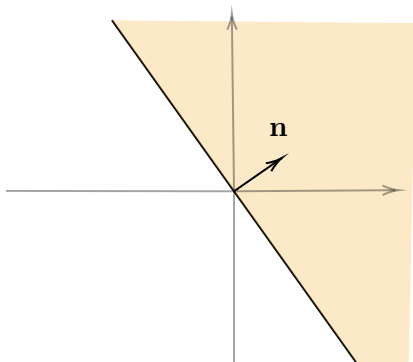
We can define half-space as a set of all points \mathbf{x} , such that $\mathbf{a}^\top \mathbf{x} \leq b$. It has a very clear geometric interpretation. In the following image, the filled space is **not** in the half space.



Half-spaces

Construction. Simple case

Consider half-space that passes through the origin, and defined by its normal vector \mathbf{n} :

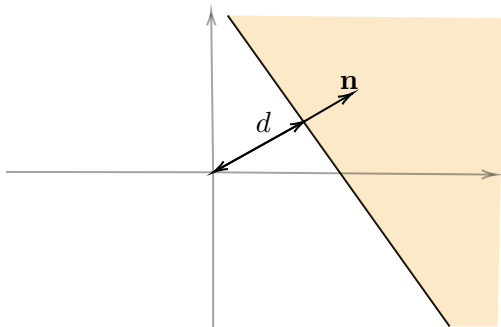


It is easy to see that this half-space can be defined as "all vectors \mathbf{x} , such that $\mathbf{n} \cdot \mathbf{x} \leq 0$ ", which is the same as using \mathbf{n} instead of \mathbf{a} in our original definition, setting $b = 0$.

Half-spaces

Construction. General case

In the general case there is some distance between the boundary of the half-space and the origin, let's say d .

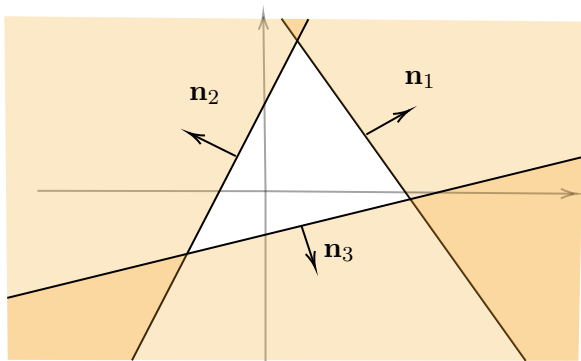


The same way we see, that the half space can be defined as "all vectors \mathbf{x} , such that $\mathbf{n} \cdot \mathbf{x} \leq d$ ". This is the same as making $\mathbf{a} = \mathbf{n}$ and $b = d$ in our original definition. However, if \mathbf{a} is not a unit vector, $b = d\|\mathbf{a}\|$.

Half-spaces

Combination

We can define a region of space as an *intersection* of half-spaces $\mathbf{a}_i^\top \mathbf{x} \leq b_i$:



Resulting region will be easily described as
$$\begin{bmatrix} \mathbf{a}_1^\top \\ \dots \\ \mathbf{a}_k^\top \end{bmatrix} \mathbf{x} \leq \begin{bmatrix} b_1 \\ \dots \\ b_k \end{bmatrix}$$

Half-spaces

Formal description via inequalities

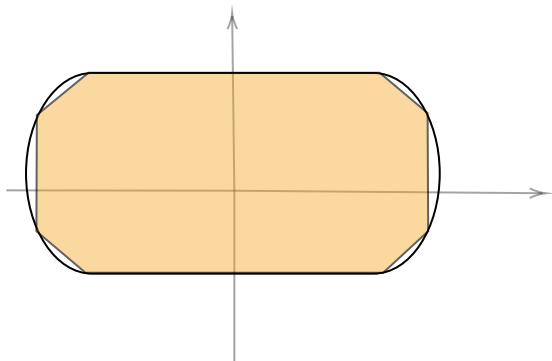
The last result allows us to write any convex polytope as a matrix inequality:

$$\mathbf{Ax} \leq \mathbf{b} \tag{1}$$

And conversely, any matrix inequality (1) represents either an empty set or a convex polytope.

Linear approximation of convex regions

Some convex regions can be easily approximated using polytopes.



Which allows to represent constraints on \mathbf{x} to belong in such a region as a matrix inequality

Homework

Represent in matrix inequality form the following figures:

- Equilateral triangle
- A square
- Parallelepiped
- Trapezoid

Lecture slides are available via Moodle.

You can help improve these slides at:

github.com/SergeiSa/Computational-Intelligence-Slides-Fall-2020

Check Moodle for additional links, videos, textbook suggestions.