Chapter-II - Polynomials

We denote by $\mathbb{K}$, the set $\mathbb{R}$ or $\mathbb{C}$.

I- Definitions

Definition: A polynomial over $\mathbb{K}$, with indeterminate x and coefficients $a_0, a_1, \dots, a_n$ in $\mathbb{K}$ such that $n \in \mathbb{N}$, every expression: $f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$.

example:
- $f(x) = x^2 - 1$; polynomial over $\mathbb{R}$ (real polynomial)
- $f(x) = i + 1 - 2x^2 + 5i x^3$: polynomial over $\mathbb{C}$ (complex polynomial)

Remark: $\left.\begin{array}{l} f(x) = a_0 + a_1 x + \dots + a_n x^n \\ g(x) = b_0 + b_1 x + \dots + b_m x^m \end{array}\right\}$ $f(x)$ and $g(x)$ are equal if $m = n$ and $a_i = b_i$, for all $1 \leq i \leq n$.

Notations:
- The set of all polynomials over $\mathbb{K}$ is $\mathbb{K}[x]$
- The zero polynomial is: $0 = 0 + 0.x + 0.x^2 + \dots$
- We can write : $a_0 1 + a_1 x + \dots + a_n x^n$

$$= a_0 \cdot x^0 + a_1 x^1 + \dots + a_n x^n$$
$$= \sum_{i=0}^{n} a_i x^i.$$

Definition: Let $u(x) \in \mathbb{K}[x]$ s.t. $u \neq 0$.
The degree of u is the greatest integer n such that $a_n \neq 0$.

example:
- $f(x) = x + 2x^5 + x^3 - x^7 + 3x^2$

$a_0 = 0$, $a_1 = 1$, $a_2 = 3$, $a_3 = 1$, $a_4 = 0$, $a_5 = 2$, $a_6 = 0$, $a_7 = -1$, $a_8 = a_9 = \dots = 0$.

so degree of $f = \deg(f) = 7$.

- $u = 5i + 1$; $\deg(u) = 0$. ($a_0 = 5i + 1$, $a_1 = a_2 = \dots = 0$).

Notations; Let $u = a_0 + a_1 x + \dots + a_n x^n$, $\deg(u) = n$
- a_n is the leading coefficient of u.
- a_0 is the constant term of u
- The constant polynomials are the numbers of $\mathbb{K}$ ($= a_0$).
- The degree of the non-zero constant polynomial is zero

II- Sum of Polynomials:

Definition. $\left.\begin{array}{l} u = a_0 + a_1 x + \dots + a_n x^n \\ v = b_0 + b_1 x + \dots + b_m x^m \end{array}\right\}$ $u + v = (a_0 + b_0) + (a_1 + b_1) x + (a_2 + b_2) x^2 + \dots$

example: $u = 1 + 2x^2 - 3x^3$

$v = -3 + 6x^2 + 2x^4 - x^5$

$u + v = 1 + 2x^2 - 3x^3 - 3 + 6x^2 + 2x^4 - x^5$

$= -2 + 8x^2 - 3x^3 + 2x^4 - x^5$.

Properties: 1) $f(x) + g(x) = g(x) + f(x)$.

2) $f(x) + \big(g(x) + h(x)\big) = \big(f(x) + g(x)\big) + h(x)$

3) $0 + f(x) = f(x)$

4) $f(x) = a_0 + a_1 x + \dots + a_n x^n$, $\underbrace{-f(x) = -a_0 - a_1 x - \dots - a_n x^n}_{\text{opposite of } f(x)}$,

$f(x) + (-f(x)) = 0$

example: $f(x) = -5 + 5x \Rightarrow -f(x) = 5 - 5x$

$f(x) + (-f(x)) = f(x) - f(x) = 0$.

III- Product with constants:

Definition: $u = a_0 + a_1 x + \dots + a_n x^n \in \mathbb{K}[x]$

$\alpha \in \mathbb{K}$

Then $\alpha u = \alpha a_0 + \alpha a_1 x + \dots + \alpha a_n x^n$.

example: $u(x) = x^2 - i$; $\alpha = 3i$

$\alpha u(x) = 3i x^2 - 3i^2 = 3i x^2 + 3$.

Properties: 1) $(\alpha + \beta) f(x) = \alpha f(x) + \beta f(x)$

example: $f(x) = 1 - x^2$

$\alpha = -3, \beta = 1$.

$(\alpha + \beta) f(x) = (-3 + 1) f(x) = -2 f(x) = -2 + 2x^2$

$\left.\begin{array}{l} \alpha \cdot f(x) = (-3) f(x) = -3 + 3x^2 \\ \beta \cdot f(x) = 1 \cdot f(x) = f(x) = 1 - x^2 \end{array}\right\}$ $\alpha f(x) + \beta f(x) = -2 + 2x^2$

$= (\alpha + \beta) f(x)$.

2) $\alpha(f(x)+g(x)) = \alpha f(x) + \alpha g(x)$

3) $\alpha(\beta f(x)) = (\alpha\beta).f(x)$

example: $f(x) = 1 - x^2$, $\alpha = -2$, $\beta = 3$.

$\beta . f(x) = 3(1-x^2) = 3 - 3x^2$.

$$\left.\begin{array}{l}\alpha.(\beta f(x)) = (-2)(3-3x^2) = -6+6x^2 \\ (\alpha\beta).f(x) = (-6)f(x) = -6+6x^2\end{array}\right\} \alpha.(\beta f(x)) = (\alpha\beta)f(x)$$

4) $1.f(x) = f(x)$, $0 \times f(x) = 0$; $(-1)\times f(x) = -f(x)$ opposite of $f(x)$.

IV- Product with Polynomials:

Definition: $u = a_0 + a_1 x + \dots + a_n x^n$

$v = b_0 + b_1 x + \dots + b_m x^m$

$u.v = c_0 + c_1 x + \dots + c_{n+m} x^{n+m}$

where $c_t = \sum_{t=i+j} a_i b_j$ $(c_0 = a_0 b_0,\ c_1 = a_0 b_1 + a_1 b_0,$
$c_2 = a_0 b_2 + a_1 b_1 + a_2 b_0, \dots, c_{n+m} = a_n b_m)$

example: $u = 1 - x + 3x^2$, $v = 2 + x - 3x^3$

$u.v = (1 - x + 3x^2)(2 + x - 3x^3)$

$= 1.(2+x-3x^3) - x(2+x-3x^3) + 3x^2(2+x-3x^3)$

$= 2 + x - 3x^3 - 2x - x^2 + 3x^4 + 6x^2 + 3x^3 - 9x^5$

$= 2 - x + 5x^2 + 3x^4 - 9x^5$

$(c_0 = 2 = 1\times 2 = a_0 b_0,\ c_1 = -1 = 1\times 1 + (-1)\times 2 = a_0 b_1 + a_1 b_0,$

$c_{n+m} = c_{2+3} = c_5 = -9 = 3\times(-3) = a_2 \times b_3)$.

Properties: 1) $f(x).g(x) = g(x)f(x)$

2) $f(x).(g(x).h(x)) = (f(x).g(x)).h(x)$

3) $f(x).(g(x)+h(x)) = f(x).g(x) + f(x).h(x)$

4) $\deg(uv) = \deg(u) + \deg(v)$

Remark : 1) If $uv = 0$ then $u = 0$ or $v = 0$.

2) If $\alpha u = 0$ then $\alpha = 0$ or $u = 0$

3) This is not true for addition: $u(x) + (-u(x)) = 0$ for any $u(x) \in \mathbb{K}[x]$

V- Euclidean Division:

Theorem: Let $u \in \mathbb{K}[x]$; $v \in \mathbb{K}[x] - \{0\}$ $(v \neq 0)$

Then there exist unique polynomials q and r such that:

$u = q \times v + r$ with $[r = 0]$ or $[r \neq 0$ and $\deg(r) < \deg(v)]$.

q: quotient of the division of u by v

r: remainder of the division of u by v

example: • $u = 5 + 2x + 2x^4$, $v = 3x^2 + 1$

quotient ? Remainder ? of the division of u by v.

$$
\begin{array}{r|l}
\overset{u}{2x^{4} + 2x + 5} & \overset{v}{3x^{2} + 1} \\ \hline
-\quad 2x^{4} + \frac{2}{3}x^{2} \qquad & \underbrace{\frac{2}{3}x^{2} - \frac{2}{9}}_{\text{quotient}} \\ \cline{1-1}
-\frac{2}{3}x^{2} + 2x + 5 & \\
-\quad -\frac{2}{3}x^{2} - \frac{2}{9} \qquad & \\ \cline{1-1}
\underbrace{2x + \frac{47}{9}} & \\
\end{array}
$$

$\xleftarrow{\times}$ (The powers of x in u and v are in decreasing order).

remainder of degree $1 <$ degree $(v) = 2$.

$$2x^4 + 2x + 5 = \left(\frac{2}{3}x^2 - \frac{2}{9}\right) \times (3x^2 + 1) + \left(2x + \frac{47}{9}\right)$$
$$u \quad = \quad q \quad \times \quad v \quad + \quad r$$

• $f(x) = 4x^6 - 2x^5 + 4x^3 + 2x^4 - 2x + 6$

$g(x) = 3x^4 - 6x^2 + 8x - 5$

Division of $f(x)$ by $g(x)$.

$$
\begin{array}{l|l}
4x^6 - 2x^5 + 2x^4 + 4x^3 - 2x + 6 & 3x^4 - 6x^2 + 8x - 5 \\
\hline
-\left(4x^6 - 8x^4 + \frac{32}{3}x^3 - \frac{20}{3}x^2\right) & \frac{4}{3}x^2 - \frac{2}{3}x + \frac{10}{3} \\
\hline
-2x^5 + 10x^4 - \frac{20}{3}x^3 + \frac{20}{3}x^2 - 2x + 6 & \\
-\left(-2x^5 + 4x^3 - \frac{16}{3}x^2 + \frac{10}{3}x\right) & \\
\hline
10x^4 - \frac{32}{3}x^3 + 12x^2 - \frac{16}{3}x + 6 & \\
-\left(10x^4 - 20x^2 + \frac{80}{3}x - \frac{50}{3}\right) & \\
\hline
-\frac{32}{3}x^3 + 32x^2 - 32x + \frac{68}{3} &
\end{array}
$$

quotient is $q = \frac{4}{3}x^2 - \frac{2}{3}x + \frac{10}{3}$

remainder is $r = \frac{-32}{3}x^3 + 32x^2 - 32x + \frac{68}{3}$

Remark: If $\deg(u) < \deg(v)$ in the division of u by v then the quotient is zero ($q = 0$) and the remainder is u ($r = u$)

$u = q \times v + r = 0 \times v + r = r.$

example: $f(x) = 5x^2 + 2x - 4$

$g(x) = x^3 - x + 2.$

In the division of f by g, $\deg(f) = 2 < \deg(g) = 3$

then the quotient is $= 0$ and the remainder $= f(x)$.

Definition: In the division of u by v, if the remainder is equal to 0, then $u = q \cdot v$

In this case, we say: v is a factor of u, or, v divides u, or, u is divisible by v.

Definition: Let $\alpha \in \mathbb{K}$, α is a root of $f(x)$ if $f(\alpha) = 0$.

Theorem: α is a root of $f(x) \iff (x - \alpha)$ divides $f(x)$

$\iff f(x) = g(x) \times (x - \alpha)$

Theorem: $\alpha_1, \alpha_2, \ldots, \alpha_t$ roots of $f(x)$ pairwise distinct then $(x - \alpha_1) \times \cdots \times (x - \alpha_t)$ divides $f(x)$.

example: $f(x) = x^3 - 8x^2 + 17x - 10$

$f(1) = 0 \Rightarrow 1$ root of $f \Rightarrow (x-1)$ divides $f(x)$.

$f(5) = 0 \Rightarrow 5$ root of $f \Rightarrow (x-5)$ divides $f(x)$.

$1 \neq 5 \Rightarrow \underbrace{(x-1)\times(x-5)}_{= x^2 - 6x + 5}$ divides $f(x)$

Indeed,

$$\begin{array}{rl|l} & x^3 - 8x^2 + 17x - 10 & x^2 - 6x + 5 \\ \cline{3-3} - & x^3 - 6x^2 + 5x & x - 2 \\ \cline{1-2} & -2x^2 + 12x - 10 & \\ - & -2x^2 + 12x - 10 & \\ \cline{1-2} & 0 & \end{array}$$

remainder $= 0$ so $x^2 - 6x + 5$ divides $f(x)$.

$\Rightarrow f(x) = (x-2)\times(x^2 - 6x + 5) \Rightarrow (x-2)$ divides $f(x)$ so 2 is a root.

$= (x-2)\cdot(x-1)\cdot(x-5)$

$\Rightarrow 1$, 2 and 5 are simple roots of $f(x)$.

Definition: Let α be a root of $f(x)$.

α is of multiplicity $m \Leftrightarrow (x-\alpha)^m$ divides $f(x)$ and $(x-\alpha)^{m+1}$ does not divide $f(x)$.

example: $f(x) = (x-1)^{3}(x-4)^{2}(x+2)^{3}\underbrace{(2x+4)^5}_{= 2^5(x+2)^{5}}$

$\Rightarrow f(x) = 2^5(x-1)^3(x-4)^2(x+2)^8$

$(x-1)$, $(x-4)$ and $(x+2)$ divide $f(x)$ so 1, 4 and -2 are roots of $f(x)$.

$\Rightarrow 1$ root of multiplicity $3 \Rightarrow 1$ is triple root.

4 root of multiplicity $2 \Rightarrow 4$ is double root.

-2 root of multiplicity 8

VI- Derivatives of Polynomials:

Definition:
- If f is a constant polynomial then $f'(x) = 0$.
- If $\deg(f) \neq 0$, $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$

  then $f'(x) = a_1 + 2a_2 x + \cdots + n a_n x^{n-1}$

example: $f(x) = x^5 + 7x^4 + 16x^3 + 8x^2 - 16x - 16$

$$f'(x) = 5x^4 + 28x^3 + 48x^2 + 16x - 16$$

Definition: The derivative of order "s" or the s^{th} derivative is:

$$f^{(0)}(x) = f(x),\ f^{(1)}(x) = f'(x),\ f^{(2)}(x) = f''(x), \ldots, f^{(s)}(x) = \left(f^{(s-1)}(x)\right)'$$

example: In the previous example, we have:

$$f''(x) = 20x^3 + 84x^2 + 96x + 16$$
$$f'''(x) = 60x^2 + 168x + 96$$
$$f^{(4)}(x) = 120x + 168$$

Properties:
1) $(f(x) + g(x))' = f'(x) + g'(x)$.
2) $(\alpha f(x))' = \alpha f'(x)$
3) $(f(x) \cdot g(x))' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$

Theorem: $(x-\alpha)^m$ divides $f(x) \iff f(\alpha) = f'(\alpha) = \cdots = f^{(m-1)}(\alpha) = 0$.

example: $(x-1)^3$ divides $f(x) = x^4 - 2x^3 + 2x - 1$??

$$f(1) = 1 - 2 + 2 - 1 = 0$$
$$f'(x) = 4x^3 - 6x^2 + 2 \Rightarrow f'(1) = 4 - 6 + 2 = 0$$
$$f''(x) = 12x^2 - 12x \Rightarrow f''(1) = 12 - 12 = 0.$$

Thus, $f(1) = f'(1) = f''(1) = 0$

By the theorem, $(x-1)^3$ divides $f(x)$

(In this example, $m = 3 \Rightarrow m - 1 = 2$).

Theorem: α root of multiplicity m of $f(x)$

$$\Leftrightarrow \underbrace{f(\alpha) = f'(\alpha) = \cdots = f^{(m-1)}(\alpha) = 0}_{\Leftrightarrow (x-\alpha)^m \text{ divides } f(x)} \text{ and } \underbrace{f^{(m)}(\alpha) \neq 0}_{(x-\alpha)^{m+1} \text{ does not divide } f(x).}$$

example: multiplicity of $\alpha = -2$ in $f(x) = x^5 + 7x^4 + 16x^3 + 8x^2 - 16x - 16$?

$f(-2) = (-2)^5 + 7(-2)^4 + 16(-2)^3 + 8(-2)^2 - 16(-2) - 16$

$= -32 + 112 - 128 + 32 + 32 - 16 = 0$

$f'(x) = 5x^4 + 28x^3 + 48x^2 + 16x - 16$

$f'(-2) = 80 - 224 + 192 - 32 - 16 = 0$

$f''(x) = 20x^3 + 84x^2 + 96x + 16$

$f''(-2) = -160 + 336 - 192 + 16 = 0.$

$f'''(x) = 60x^2 + 168x + 96 \Rightarrow f'''(-2) = 240 - 336 + 96 = 0$

$f^{(4)}(x) = 120x + 168 \Rightarrow f^{(4)}(-2) = -240 + 168 = -72 \neq 0.$

Thus, $f(-2) = f'(-2) = f''(-2) = f'''(-2) = 0$ and $f^{(4)}(-2) \neq 0$

By the Theorem, -2 is a root of multiplicity $m = 4$ of $f(x)$.

VII- Real and Complex Polynomials:

Definition: • A real polynomial is a polynomial over $\mathbb{R}$ (the coefficients belong to $\mathbb{R}$), $\in \mathbb{R}[x]$

• A complex polynomial is a polynomial over $\mathbb{C}$ (the coefficients belong to $\mathbb{C}$), $\in \mathbb{C}[x]$

example: • $f(x) = 2x^7 - 2x^5 + 3x^2 + 2i - 5 \in \mathbb{C}[x], \notin \mathbb{R}[x]$

• $f(x) = x^2 + 2 \in \mathbb{R}[x]$ so $\in \mathbb{C}[x]$

Remark: Since $\mathbb{R} \subseteq \mathbb{C}$, then every real polynomial is complex polynomial
the contrary is not True.

Theorem: Every complex polynomial of degree $n \geq 1$ has n roots in $\mathbb{C}$.

Fundamental Theorem of Algebra: If $f(x)$ is a complex polynomial of degree $n \geq 1$
Then, there exist $u, \alpha_1, \dots, \alpha_n \in \mathbb{C}$ such
that $f(x) = u \times (x - \alpha_1) \times \dots \times (x - \alpha_n)$

Factorization of $f(x)$ in $\mathbb{C}[x]$.

example: $f(x) = 2x^3 + ix^2 + 4x + 2i$

The roots of $f(x)$ in $\mathbb{C}$ are: $-\frac{i}{\sqrt{2}}, \sqrt{2}\, i, -\sqrt{2}\, i$

Then $f(x) = 2\left(x - \left(-\frac{i}{\sqrt{2}}\right)\right)(x - \sqrt{2}\, i)(x - (-\sqrt{2}\, i))$

$= 2\left(x + \frac{i}{\sqrt{2}}\right)(x - \sqrt{2}\, i)(x + \sqrt{2}\, i)$ Factorization in $\mathbb{C}[x]$.

Theorem: Let $f(x)$ be a real polynomial
Let α be a complex root
then $\bar{\alpha}$ (the conjugate of α) is also a root of $f(x)$

example: • $f(x) = x^2 + 2$ real polynomial ✓

$\alpha = \sqrt{2}\, i$ complex root ✓ $\left(f(\sqrt{2}\, i) = (\sqrt{2}\, i)^2 + 2 = -2 + 2 = 0 \right)$

$\bar{\alpha} = -\sqrt{2}\, i$ is also a root ✓ $\left(f(-\sqrt{2}\, i) = (-\sqrt{2}\, i)^2 + 2 = -2 + 2 = 0 \right)$

$f(x) = (x - \sqrt{2}\, i)(x + \sqrt{2}\, i)$ factorization in $\mathbb{C}[x]$.

• $f(x) = 2x^3 - 4x^2 - 4x - 6$ real polynomial ✓

$= 2(x-3)(x^2 + x + 1)$ it is not a factorization in $\mathbb{C}[x]$

The roots of $x^2 + x + 1$ are $\frac{-1 - i\sqrt{3}}{2}$ and $\frac{-1 + i\sqrt{3}}{2}$

So, $f(x) = 2(x-3)\left(x - \underbrace{\frac{-1 - i\sqrt{3}}{2}}_{\text{complex root}}\right)\left(x - \underbrace{\frac{-1 + i\sqrt{3}}{2}}_{\text{conjugate of the complex root}}\right)$ factorization in $\mathbb{C}[x]$

Theorem: Every real polynomial of odd degree has at least one real root.

Remark: If u is an n^{th} root of unity such that $u \neq 1$

then $1 + u + u^2 + \cdots + u^{n-1} = 0$.

example: If j is a cube root of unity with $j \neq 1$

then $1 + j + j^2 = 0$ (here $n = 3$ so $n - 1 = 2$)

Chapter-II- Polynomials

Exercises:

Ex 1 Let j be a cube root of unity ($j^3 = 1$)

Calculate $f+g$ and $f \cdot g$ where: $f = (5-i)x^2 + jx + i$

$g = 2jx - i$.

- $f+g = (5-i)x^2 + (j+2j)x + i - i = (5-i)x^2 + 3jx$.

- $fg = \left((5-i)x^2 + jx + i\right)(2jx - i)$

$= (5-i)2jx^3 - (5-i)ix^2 + 2j^2x^2 - ijx + 2jix - i^2$

$= 2(5-i)jx^3 + (2j^2 - 5i + i^2)x^2 + ijx + 1$

$= 2(5-i)j\, x^3 + (-1 - 5i + 2j^2)\, x^2 + ijx + 1$

Ex 2 Calculate the quotient and the remainder of the division of

$f = 2x^3 - 5x^2 + 2$ by $g = x^2 - 3x + 1$.

$$
\begin{array}{rr|l}
 & 2x^3 - 5x^2 + 2 & x^2 - 3x + 1 \\
\cline{3-3}
- & 2x^3 - 6x^2 + 2x & 2x + 1 \\
\cline{1-2}
 & x^2 - 2x + 2 & \\
- & x^2 - 3x + 1 & \\
\cline{2-2}
 & x + 1 &
\end{array}
$$

So, the quotient is $q = 2x + 1$

the remainder is $r = x + 1$

We have $f = q \times g + r$

$\Rightarrow 2x^3 - 5x^2 + 2 = (2x+1) \times (x^2 - 3x + 1) + (x + 1)$

Ex 3 Show that if j is a cube root of unity such that $j \neq 1$ then the polynomial $f(x) = x^4 + 2ix^3 + jx^2 - 2ij^2x + j^2$ is divisible by $(x-j)$, $(x+i)$ and $(x-j)(x+i)$.

Recall that α root of $f(x) \Leftrightarrow (x-\alpha)$ divides $f(x)$

- $f(j) = j^4 + 2ij^3 + j\,j^2 - 2ij^2 \cdot j + j^2$

$= j\,j^3 + 2i \times 1 + 1 - 2i \times 1 + j^2$

$= j \times 1 + 2i + 1 - 2i + j^2 = j + 1 + j^2 = 1 + j + j^2 = 0.$

So j is a root of $f(x)$ so $(x-j)$ divides $f(x)$.

- $f(-i) = (-i)^4 + 2i(-i)^3 + j(-i)^2 - 2ij^2(-i) + j^2$

$= i^4 - 2i \times i^3 + j \cdot i^2 + 2i^2j^2 + j^2 \quad (i^4 = i^2 \times i^2 = (-1)\times(-1) = 1)$

$= 1 - 2 \times 1 - j - 2j^2 + j^2$

$= -1 - j - j^2 = -(1 + j + j^2) = 0$

So $-i$ is a root of $f(x)$ so $(x-(-i)) = (x+i)$ divides $f(x)$.

- Recall that if $\alpha_1, \dots, \alpha_t$ are pairwise distinct roots of $f(x)$ then $(x-\alpha_1) \times \dots \times (x-\alpha_t)$ divides $f(x)$.

We have $-i \neq j$ because $\left.\begin{array}{l}(-i)^3 = -i^3 = -i\,i^2 = +i \neq 1 \\ \text{but } j^3 = 1.\end{array}\right\}$ so $-i \neq j$

Then $-i$ and j are distinct roots so $(x-j)\times(x+i)$ divides $f(x)$.

Ex 4 Find the reals a and b so that the real polynomial $f(x) = ax^{n+1} + bx^n + 1$ is divisible by $(x-1)^2$ where $n \geq 1$.

Recall that $(x-\alpha)^m$ divides $f(x) \Leftrightarrow f(\alpha) = f'(\alpha) = \dots = f^{(m-1)}(\alpha) = 0.$

So $(x-1)^2$ divides $f(x) \Leftrightarrow f(1) = f'(1) = 0.$

$f(1) = 0 \Rightarrow a \cdot 1^{n+1} + b \cdot 1^n + 1 = 0 \Rightarrow a + b + 1 = 0 \Rightarrow a + b = -1$

$f'(x) = a(n+1)x^n + bnx^{n-1}$

$f'(1) = 0 \Rightarrow a \cdot (n+1) 1^n + b \cdot n \cdot 1^{(n-1)} = 0 \Rightarrow (n+1)a + nb = 0$

So, $\begin{cases} a + b = -1 & (1) \\ (n+1)a + nb = 0 & (2) \end{cases}$

$n \times (1) - (2) \Rightarrow na + nb - (n+1)a - nb = -n - 0$

$\cancel{na} + \cancel{nb} - \cancel{na} - a - \cancel{nb} = -n$

$-a = -n \Rightarrow \underline{a = n}$

$(1) \Rightarrow \underline{b = -1 - a = -1 - n}$

Ex 5 Compute the multiplicity of α in $f(x)$.

i) $f(x) = x^5 - 5x^4 + 7x^3 - 2x^2 + 4x - 8$, $\alpha = 2$.

Recall that α is root of multiplicity m of $f(x)$

$\Leftrightarrow f(\alpha) = f'(\alpha) = \cdots = f^{(m-1)}(\alpha) = 0$ and $f^{(m)}(\alpha) \neq 0$.

$f(2) = 2^5 - 5 \times 2^4 + 7 \times 2^3 - 2 \times 2^2 + 4 \times 2 - 8$

$= 32 - 80 + 56 - 8 + 8 - 8 = 0$. (so, 2 is a root of $f(x)$)

$f'(x) = 5x^4 - 20x^3 + 21x^2 - 4x + 4$

$f'(2) = 80 - 160 + 84 - 8 + 4 = 0$

$f''(x) = 20x^3 - 60x^2 + 42x - 4$

$f''(2) = 160 - 240 + 84 - 4 = 0$

$f'''(x) = 60x^2 - 120x + 42$

$f'''(2) = 240 - 240 + 42 = 42 \neq 0$

So $f(2) = f'(2) = f''(2) = 0$ and $f'''(2) \neq 0$

- Hence, 2 is a root of multiplicity 3

(it means that we can write $f(x) = (x-2)^3 \times g(x)$ but $f(x) \neq (x-2)^4 g(x)$ where $g(x)$ is a polynomial)

ii) $f(x) = x^4 - 2x^3 - x + 2$, $\alpha = 1$.

$f(1) = 1 - 2 - 1 + 2 = 0$. so 1 is a root of $f(x)$.

$f'(x) = 4x^3 - 6x^2 - 1$

$f'(1) = 4 - 6 - 1 = -3 \neq 0$

So 1 is a root of multiplicity 1 ($\Rightarrow$ $(x-1)$ divides $f(x)$ and $(x-1)^2$ does not divide $f(x)$)

Ex 6 Find the remainder of the division of $(\cos\alpha + x\sin\alpha)^n$ by x^2+1 where $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}^*$.

Let $r(x)$ be the remainder and $q(x)$ the quotient of this division

We have $\deg(r(x)) < \deg(x^2+1) = 2$

So $\deg(r(x)) \leq 1$. so $r(x)$ has the form $r(x) = ax + b$ where $a \in \mathbb{R}$, $b \in \mathbb{R}$.

Also, by the Euclidean division:

$$(\cos\alpha + x\sin\alpha)^n = q(x) \times (x^2+1) + r(x)$$
$$= q(x) \times (x^2+1) + ax + b.$$

For $x = i \Rightarrow \underbrace{(\cos\alpha + i\sin\alpha)^n}_{\text{De Moivre}} = q(i) \cdot \underbrace{(i^2+1)}_{=-1+1=0} + ai + b.$

$$\Rightarrow \cos(n\alpha) + i\sin(n\alpha) = 0 + ai + b = ai + b$$

By comparison: $a = \sin(n\alpha)$ and $b = \cos(n\alpha)$

So, $r(x) = \sin(n\alpha)\, x + \cos(n\alpha)$.