Linear Algebra.

Chapter -I- Complex Numbers

I- Introduction:

- $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ set of natural integers

  $\mathbb{N}^* = \mathbb{N} - \{0\}$

- $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ set of integers.

  $\mathbb{Z}^* = \mathbb{Z} - \{0\}$.

example: solve $2x = 1$? $x = \frac{1}{2} \notin \mathbb{Z}$.

does not belong

Remark: * $0 \in \mathbb{N}$: zero belongs to the set $\mathbb{N}$. → symbol between element and set

* $\frac{1}{2} \notin \mathbb{Z}$: $\frac{1}{2}$ does not belong to the set $\mathbb{Z}$.
* $\mathbb{N} \subseteq \mathbb{Z}$: the set $\mathbb{N}$ is a subset of $\mathbb{Z}$, $\mathbb{N}$ is included in $\mathbb{Z}$
* $\not\subseteq$: is not a subset, is not included → between sets

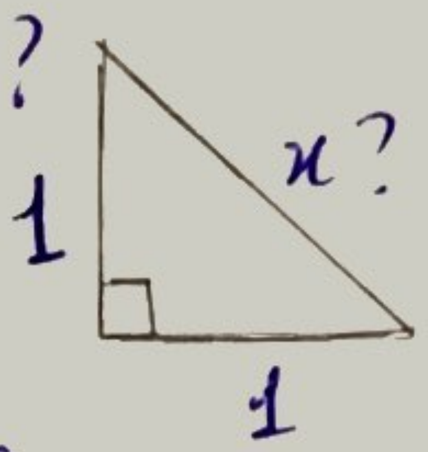
- $\mathbb{Q} = \{x = \frac{a}{b} ; a \in \mathbb{Z} ; b \in \mathbb{Z}^*\}$ set of rational numbers.

  the nb in $\mathbb{Q}$ have the form: (i) integer $\in \mathbb{Z}$ (ii) decimal as: $5.\underbrace{23}_{\text{finite nb of digits}}$

  $0.\underbrace{212121}_{\text{periodic}}\dots$

  $\mathbb{Q}^* = \mathbb{Q} - \{0\}$.

example: Find x ?



$x^2 = 1^2 + 1^2 = 2 \Rightarrow x = \sqrt{2} = 1.4142135\dots \notin \mathbb{Q}$.

- $\mathbb{R}$: set of real numbers

  $\mathbb{R} \setminus \mathbb{Q} = \mathbb{R} - \mathbb{Q}$: set of irrational numbers.

example: $\pi, \sqrt{2}, \sqrt{3}, e, \sqrt{5}, \dots$ are irrational numbers

Conclusion: $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq ??$

example: * $x^2 - 4 = 0 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2 \in \mathbb{R}$ (real solutions)

* $x^2 + 4 = 0 \Rightarrow x^2 = -4 \Rightarrow$ No real solutions

  $\Rightarrow$ we introduce number i such that $i^2 = -1$.

$\Rightarrow x^2 = -4 = (-1) \times 4 = 4i^2 \Rightarrow x = \pm 2i$ complex solutions

Definition: We suppose there exists a set $\mathbb{C}$ contains $\mathbb{R}$.

The elements of $\mathbb{C}$ are the complex numbers.

We assume that $\mathbb{C}$ contains a nb i such that $i^2 = -1$.

Definition: A complex number $z \in \mathbb{C}$ has the form:

$z = a + ib$ with $a \in \mathbb{R}$ and $b \in \mathbb{R}$.

example: * $z = \underbrace{2}_{a \in \mathbb{R}} + \underbrace{2}_{b \in \mathbb{R}} i \in \mathbb{C}$. * $8 = \underbrace{8}_{a} + i \times \underbrace{0}_{b} \in \mathbb{C} \Rightarrow$ every real number is a complex number ($\mathbb{R} \subseteq \mathbb{C}$).

* $\frac{1+2i}{3} = \frac{1}{3} + i\frac{2}{3} \in \mathbb{C}$.

* $2i = \underbrace{0}_{a} + \underbrace{2}_{b} i \in \mathbb{C}$ * $3 + i = 3 + i \times 1 \in \mathbb{C}$.

Definition: This writing of $z = a + ib$, with $a \in \mathbb{R}$ and $b \in \mathbb{R}$, is called the algebraic form of z.

$$\left.\begin{array}{l} a = R(z) : \text{real part of } z \\ b = I(z) : \text{imaginary part of } z \end{array}\right\} z = R(z) + i\, I(z).$$

Definition: • z is real if $I(z) = 0$.

• z is imaginary if $R(z) = 0$

• z is pure imaginary if $R(z) = 0$ and $z \neq 0$.

Proposition: let $z = a + ib \in \mathbb{C}$ with $a \in \mathbb{R}$, $b \in \mathbb{R}$ and $z' = a' + ib' \in \mathbb{C}$ with $a' \in \mathbb{R}$, $b' \in \mathbb{R}$.

1) Addition: $z + z' = (a + a') + i(b + b')$

example: * $z = 1 + 2i$, $z' = 3 - i$

$z + z' = 4 + i$.

* $z = 3 - i + 15$, $z' = i$.

$z + z' = 18$.

2) Product : $zz' = (aa' - bb') + i(ab' + a'b)$.

Proof: $zz' = (a + ib)(a' + ib')$

$= aa' + aib' + iba' + ibib'$

$= aa' + i\,ab' + i\,ba' + \underbrace{i^2}_{=-1} bb'$

$= aa' + i(ab' + ba') - bb' = (aa' - bb') + i(ab' + a'b)$

example: * $z = 1 + 2i$, $z' = 3 - i$

$zz' = (1 + 2i)(3 - i) = 3 - i + 6i - 2i^2 = 3 + 5i - 2(-1)$

$= 5 + 5i$

* $z^2 = (1 + 2i)^2 = 1^2 + 2(1)(2i) + (2i)^2$

$= 1 + 4i + 4i^2 = 1 + 4i - 4 = -3 + 4i$.

apply the rule:
$(a + b)^2 = a^2 + 2ab + b^2$

3) Inverse : $\frac{1}{z} = z^{-1} = \left(\frac{a}{a^2 + b^2}\right) + i\left(\frac{-b}{a^2 + b^2}\right)$

Proof: $\frac{1}{z} = \frac{1}{a + ib} = \frac{1}{a + ib} \times \frac{a - ib}{a - ib} = \frac{a - ib}{(a + ib)(a - ib)}$

$= \frac{a - ib}{a^2 - i^2 b^2} = \frac{a - ib}{a^2 + b^2} = \left(\frac{a}{a^2 + b^2}\right) + i\left(\frac{-b}{a^2 + b^2}\right)$

example: $\frac{1}{3 - 4i} = \frac{1}{3 - 4i} \times \frac{3 + 4i}{3 + 4i} = \frac{3 + 4i}{(3 - 4i)(3 + 4i)} = \frac{3 + 4i}{3^2 - (4i)^2}$

$= \frac{3 + 4i}{9 + 16} = \frac{3}{25} + i\frac{4}{25}$.

II- Conjugate of Complex Number:

Definition: The conjugate of $z = a + ib$, $a \in \mathbb{R}$ and $b \in \mathbb{R}$, is $\bar{z} = a - ib$

example: * $z = 3 + 4i \Rightarrow \bar{z} = 3 - 4i$

* $z = 2 - 2i \Rightarrow \bar{z} = 2 + 2i$

* $z = 2i \Rightarrow \bar{z} = -2i$

* $z = 2 \Rightarrow \bar{z} = 2$.

Theorem: (i) $z=0 \Leftrightarrow \bar{z}=0$

equivalent (if and only if)

($\Rightarrow$ and $\Leftarrow$)

(ii) $\overline{z+z'}=\bar{z}+\bar{z'}$

(iii) $\overline{zz'}=\bar{z}\,\bar{z'}$

(iv) $\overline{\left(\frac{z}{z'}\right)}=\frac{\bar{z}}{\bar{z'}}$, $z'\neq 0$.

(v) $\bar{\bar{z}}=z$

Proof: $z=a+ib \Rightarrow \bar{z}=a-ib=a+i(-b)$

$\Rightarrow \bar{\bar{z}}=a-i(-b)=a+ib=z$.

(vi). $R(\bar{z})=a=R(z)$

. $I(\bar{z})=-b=-I(z)$

(vii). $R(z)=\frac{z+\bar{z}}{2}$

. $I(z)=\frac{z-\bar{z}}{2i}$

Proof: . $\frac{z+\bar{z}}{2}=\frac{(a+ib)+(a-ib)}{2}=\frac{a+ib+a-ib}{2}=\frac{2a}{2}=a=R(z)$

. $\frac{z-\bar{z}}{2i}=\frac{(a+ib)-(a-ib)}{2i}=\frac{a+ib-a+ib}{2i}=\frac{2ib}{2i}=b=I(z)$

(viii). z is real $\Leftrightarrow z=\bar{z}$.

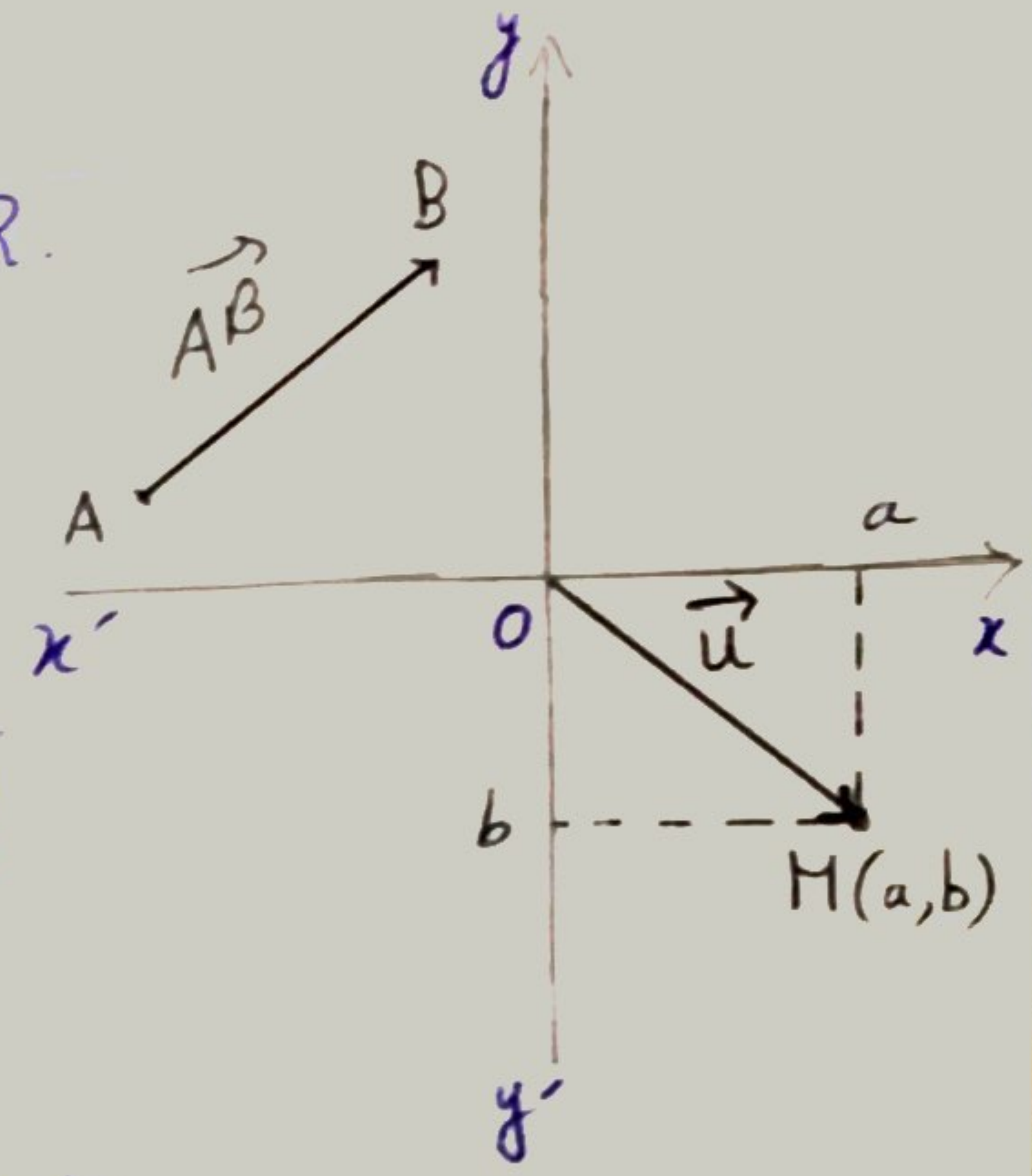
. z is imaginary $\Leftrightarrow z=-\bar{z}$.

Proof: z is real $\Leftrightarrow I(z)=0 \Leftrightarrow \frac{z-\bar{z}}{2i}=0 \Leftrightarrow z-\bar{z}=0$

$\Leftrightarrow z=\bar{z}$.

III: Trigonometric Form of a Complex Number:

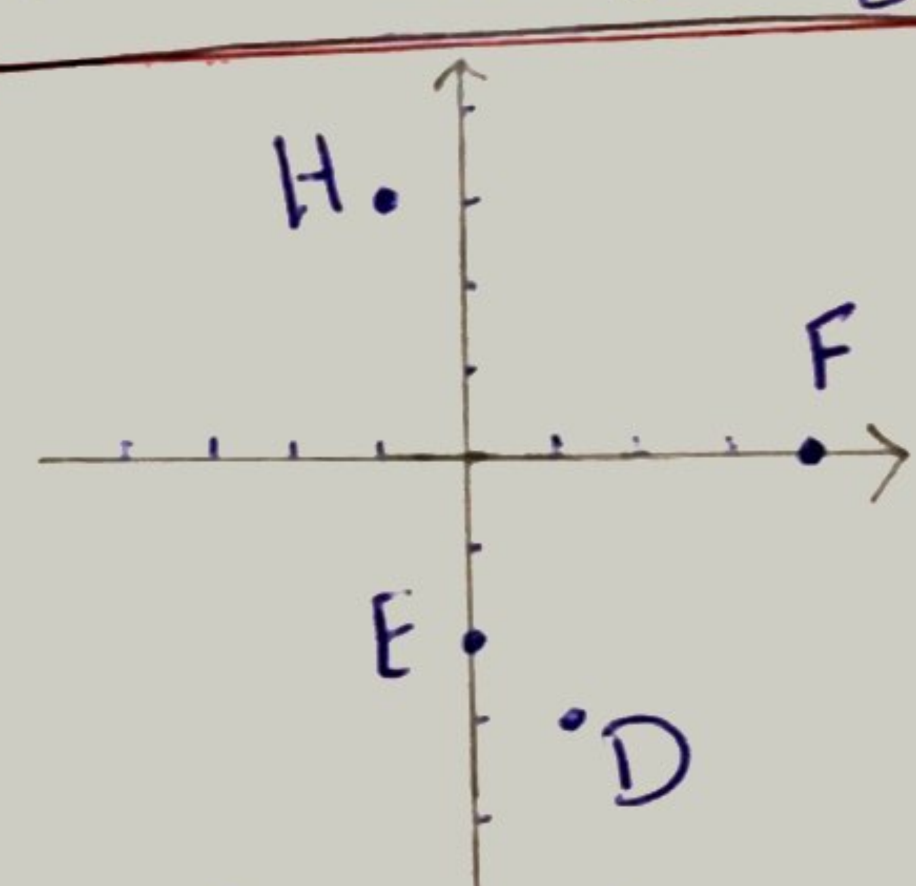
Let (Oxy) be an orthonormal system.



Definition: Let $z = a + ib \in \mathbb{C}$, with $a \in \mathbb{R}$ and $b \in \mathbb{R}$.

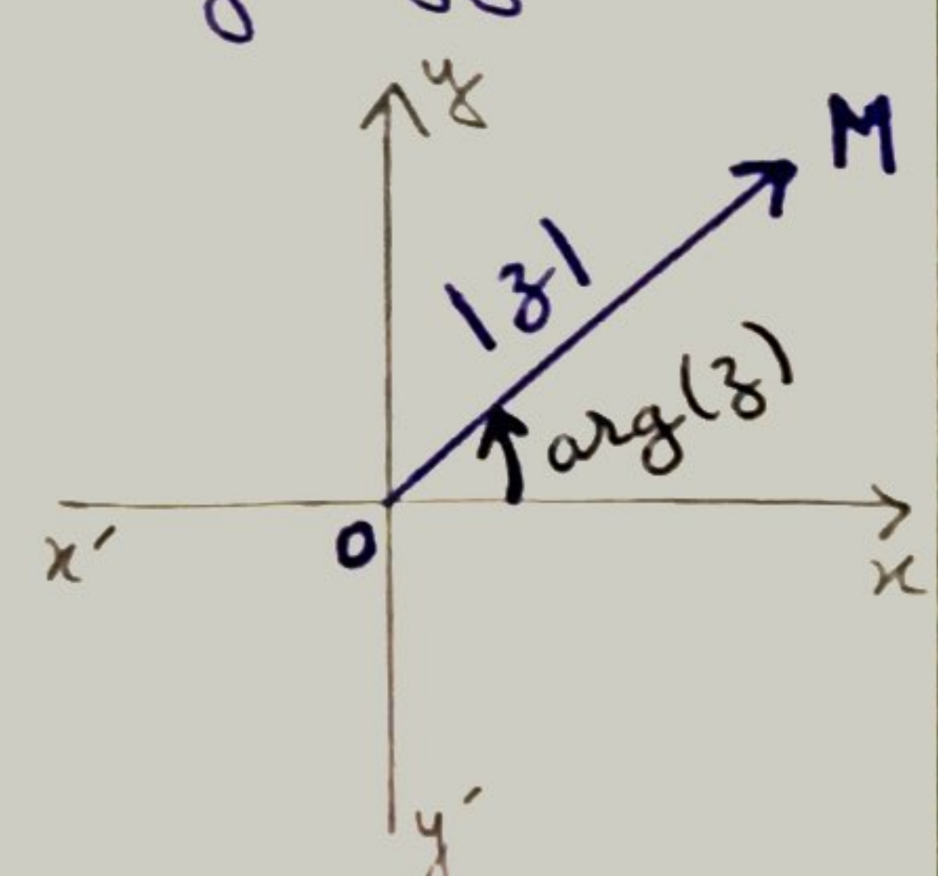
- Let M be a point of coordinates (a, b).
  M is the image of z in (Oxy)
  z is the affix of M, $z = \text{affix}(M) = z_M$
- Let $\vec{u}$ be a vector s.t. $\vec{u} = \vec{OM}$.
  so the coordinates of $\vec{u}$ are (a, b)
  $\vec{u}$ is the image of z and $z = \text{affix}(\vec{u})$
- $\vec{AB}$ is of coordinates $(x_B - x_A, y_B - y_A)$.
  $z_A = x_A + i y_A$ is the affix of $A(x_A, y_A)$
  $z_B = x_B + i y_B$ is the affix of $B(x_B, y_A)$.
  then the affix of $\vec{AB}$ is $z_{\vec{AB}} = (x_B - x_A) + i(y_B - y_A) = z_B - z_A$

example:
- $z_D = 1 - 3i \Rightarrow D(1, -3)$
- $z_E = -2i \Rightarrow E(0, -2)$
- $z_F = 4 \Rightarrow F(4, 0)$
- $z_H = -1 + 3i \Rightarrow H(-1, 3)$



Definition:
- $x'x$ is the real axis, the real complex nb belong to $x'x$.
- $y'y$ is the imaginary axis, the imaginary nb belong to $y'y$.

Definition: Let $z = a + ib \in \mathbb{C}$ with $a \in \mathbb{R}$, $b \in \mathbb{R}$.
Let $M(a, b)$ be the image of z in (Oxy)



- The module of z (or the modulus of z) is $|z| = OM \longrightarrow$ the geometric length of OM
- If $z \neq 0$, the argument of z is $\arg(z) = (\vec{Ox}, \vec{OM})$
  the angle of $\vec{Ox}$ and $\vec{OM}$.

Theorem: Let $z = a + ib \in \mathbb{C}$, $a \in \mathbb{R}$ and $b \in \mathbb{R}$

Then $|z|^2 = a^2 + b^2 = z \cdot \bar{z}$.

Proof: • $|z|^2 = \overline{OM}^2$

$= \overline{ON}^2 + \overline{NM}^2$ (Pythagorean in the right triangle OMN)

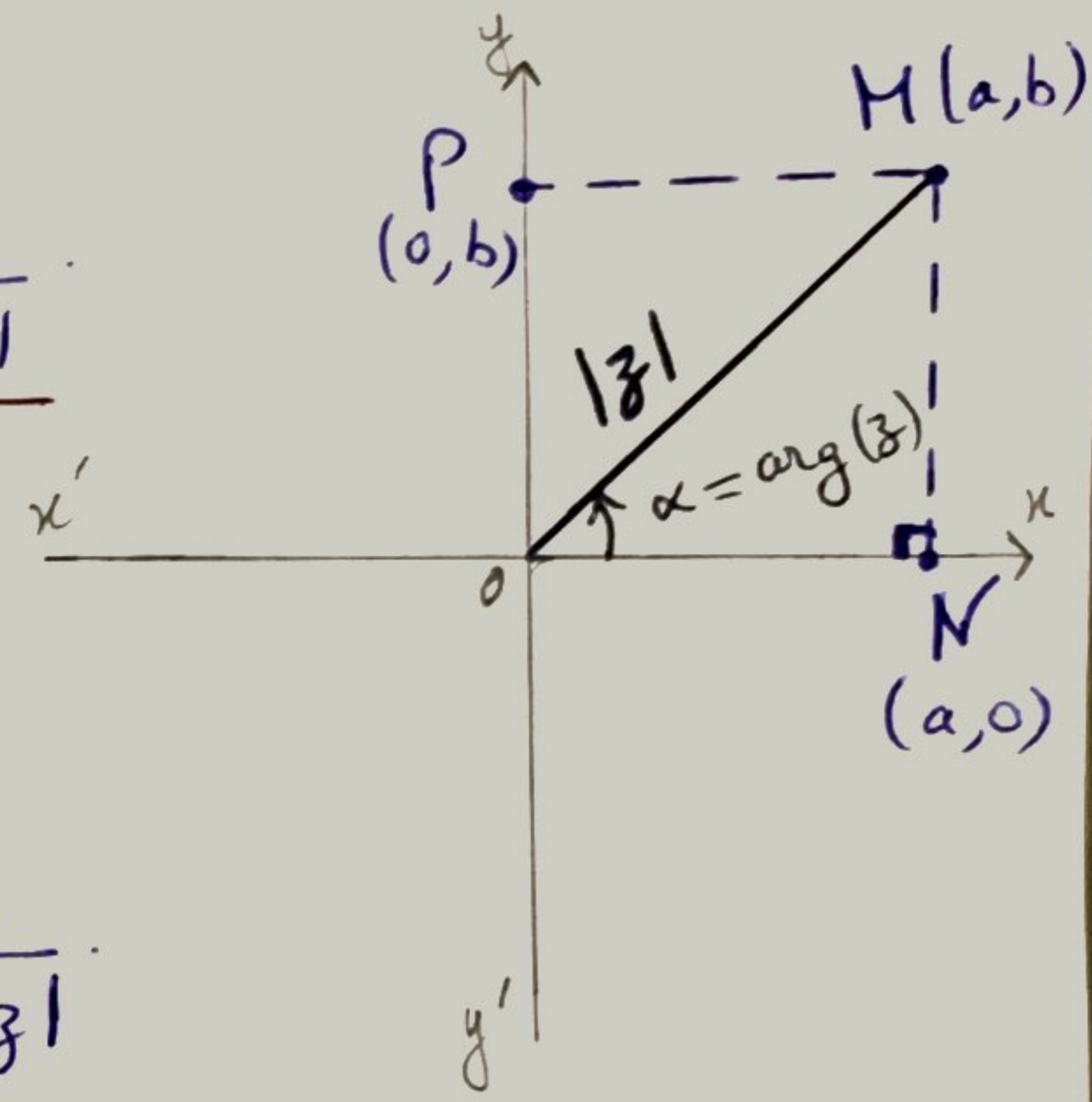
$= a^2 + \overline{OP}^2 = a^2 + b^2$.

• $z \cdot \bar{z} = (a + ib)(a - ib) = a^2 - (ib)^2 = a^2 - i^2 b^2 = a^2 + b^2 = |z|^2$.

Theorem: Let $z = a + ib \in \mathbb{C}$, with $a \in \mathbb{R}$ and $b \in \mathbb{R}$, such that $z \neq 0$

let $\alpha = \arg(z)$.

Then: $\cos \alpha = \dfrac{a}{|z|}$ and $\sin \alpha = \dfrac{b}{|z|}$.



Proof: In the right triangle OMN, we have:

$\cos \alpha = \dfrac{\text{adjacent side}}{\text{hypotenuse}} = \dfrac{ON}{OM} = \dfrac{a}{|z|}$

$\sin \alpha = \dfrac{\text{opposite side}}{\text{hypotenuse}} = \dfrac{NM}{OM} = \dfrac{OP}{OM} = \dfrac{b}{|z|}$.

Conclusion: $z = a + ib = (|z| \cos \alpha) + i(|z| \sin \alpha) = |z|(\cos \alpha + i \sin \alpha)$

because $\cos \alpha = \dfrac{a}{|z|}$ and $\sin \alpha = \dfrac{b}{|z|}$

Definition: The trigonometric form of $z \in \mathbb{C}$ is:

$z = r(\cos \alpha + i \sin \alpha)$ with $r \in \mathbb{R}, \alpha \in \mathbb{R}$ such that $r \geq 0$.

In this case: $r = |z| \geq 0$

and if $z \neq 0$ then $\arg(z) = \alpha + 2k\pi$ with $k \in \mathbb{Z}$.

Notation: $z = r(\cos \alpha + i \sin \alpha) = [r, \alpha]$.

Theorem: Let $z = r(\cos \alpha + i \sin \alpha)$, $z' = r'(\cos \alpha' + i \sin \alpha')$

with r, r', α, α' are in $\mathbb{R}$ such that $r \geq 0$ and $r' \geq 0$.

then (i) $zz' = rr'(\cos(\alpha+\alpha') + i\sin(\alpha+\alpha'))$

(ii) $\frac{z}{z'} = \frac{r}{r'}(\cos(\alpha-\alpha') + i\sin(\alpha-\alpha'))$

Conclusion:

- $|zz'| = rr' = |z|\,|z'|$
- $\arg(zz') = \alpha+\alpha' + 2k\pi = \arg(z) + \arg(z') + 2k\pi,\ k\in\mathbb{Z}$.
- $\left|\frac{z}{z'}\right| = \frac{r}{r'} = \frac{|z|}{|z'|}$
- $\arg\left(\frac{z}{z'}\right) = \alpha-\alpha' + 2k\pi = \arg(z) - \arg(z') + 2k\pi,\ k\in\mathbb{Z}$.

Theorem: De Moivre Formula.

$$(\cos\alpha + i\sin\alpha)^n = \cos(n\alpha) + i\sin(n\alpha), \text{ for any } n\in\mathbb{N}^*$$

example: $\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right)^2 = \cos^2\left(\frac{\pi}{3}\right) + 2i\cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{3}\right) + i^2\sin^2\left(\frac{\pi}{3}\right)$

$$= \left(\frac{1}{2}\right)^2 + 2i\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + (-1)\left(\frac{\sqrt{3}}{2}\right)^2$$

$$= \frac{1}{4} + i\frac{\sqrt{3}}{2} - \frac{3}{4}$$

$$= -\frac{1}{2} + i\frac{\sqrt{3}}{2}.$$

$$\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) = \cos\left(\pi - \frac{\pi}{3}\right) + i\sin\left(\pi - \frac{\pi}{3}\right)$$

$$= -\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)$$

$$= -\frac{1}{2} + i\frac{\sqrt{3}}{2}.$$

so $\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right)^2 = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)$

(De Moivre Formula: $\alpha = \frac{\pi}{3}$, $n = 2$).

This formula is important because it connects complex numbers and trigonometry.

example: $(\cos\alpha + i\sin\alpha)^2 = \cos^2\alpha + 2i\cos\alpha\sin\alpha + i^2\sin^2\alpha = (\cos^2\alpha - \sin^2\alpha) + i(2\cos\alpha\sin\alpha)$

$\|$

$\cos(2\alpha) + i\sin(2\alpha)$ (by De Moivre)

By comparison, we get $\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha$

and $\sin(2\alpha) = 2\cos\alpha\sin\alpha$.

Theorem: $|z^n| = |z|^n$ and $\arg(z^n) = n\arg(z) + 2k\pi$; for $k \in \mathbb{Z}$, for any $n \in \mathbb{N}^*$.

IV- Exponential Form of a Complex Number:

Definition: The Euler Equation is: $e^{i\theta} = \cos\theta + i\sin\theta$

Properties: (i) $e^{i\alpha} e^{i\beta} = e^{i(\alpha+\beta)}$

(ii) $\dfrac{e^{i\alpha}}{e^{i\beta}} = e^{i(\alpha-\beta)}$

Definition: Let $z = r(\cos\alpha + i\sin\alpha)$; with $r \in \mathbb{R}, \alpha \in \mathbb{R}$ s.t. $r \geq 0$.

then $z = re^{i\alpha}$ is the exponential form of z.

example: $z = 1 + i$

$|z| = \sqrt{a^2+b^2} = \sqrt{1^2+1^2} = \sqrt{2}$.

$z = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = \sqrt{2}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$= \sqrt{2}\, e^{i\frac{\pi}{4}}$

V- N^{th} Root of a Complex Number:

example: • The square roots or the 2^{th}-roots of $z = -4$ are:

$z_1 = 2i$; $z_2 = -2i$ because $(z_1)^2 = (z_2)^2 = 4i^2 = -4$

• The 3^{th}-roots or the cube roots of $z = 1$ are:

$z_1 = 1$, $z_2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$; $z_3 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$.

because $(z_1)^3 = (z_2)^3 = (z_3)^3 = 1$.

we say z_1, z_2, z_3 are the cube roots of unity $(z = 1)$

Definition: Let $z \in \mathbb{C}$ and $n \in \mathbb{N}^*$.

The n^{th} root of z is a complex number u s.t. $u^n = z$.

Definition: The n^{th} root of unity is u s.t. $u^n = 1$.

Theorem: Let $z = r(\cos\alpha + i\sin\alpha)$ with $r \in \mathbb{R}$, $\alpha \in \mathbb{R}$ such that $r \geq 0$.

Let $n \in \mathbb{N}^*$.

Then, the n^{th} roots of z are: $z_0, z_1, \dots, z_{n-1}$

where $z_t = \sqrt[n]{r}\left[\cos\left(\frac{\alpha + 2t\pi}{n}\right) + i\sin\left(\frac{\alpha + 2t\pi}{n}\right)\right]$; $0 \leq t \leq n-1$

Particular cases:

1) Calculation of square roots:

We may calculate the square roots of a complex number $z = a + ib$, with $a \in \mathbb{R}$ and $b \in \mathbb{R}$ as following:

let $u = x + iy$ s.t. $u^2 = z$ (u is square root of z).

How can we find x and y?

$u^2 = z \Rightarrow (x + iy)^2 = a + ib \Rightarrow (x^2 - y^2) + i(2xy) = a + ib$

on the other hand; $u^2 = z \Rightarrow |u|^2 = |z| \Rightarrow |x + iy|^2 = |a + ib|$

$\Rightarrow x^2 + y^2 = \sqrt{a^2 + b^2}$.

So, we get the system $\begin{cases} x^2 + y^2 = \sqrt{a^2 + b^2} & (1) \\ x^2 - y^2 = a & (2) \\ 2xy = b & (3) \end{cases}$

we add (1) and (2) to obtain two values of x.
Then we use (3) to obtain two values of y for the values of x.
Thus, we get the two values of the square roots u of z.

example: let $z = 3-4i = 3+i(-4)$

let $u = x+iy$ square root of z.

Then $\begin{cases} x^2+y^2 = \sqrt{3^2+(-4)^2} = \sqrt{9+16} = \sqrt{25} = 5. & (1) \\ x^2-y^2 = 3 & (2) \\ 2xy = -4 \Rightarrow xy = -2 & (3) \end{cases}$

(1) + (2) $\Rightarrow 2x^2 = 8 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2.$

(3) $\Rightarrow$ For $x = 2 \Rightarrow y = \frac{-2}{2} = -1$

For $x = -2 \Rightarrow y = \frac{-2}{-2} = 1.$

So, the roots of z are $u_1 = 2+i(-1) = 2-i$

$u_2 = -2+i(1) = -2+i.$

2) Calculation of the roots of a quadratic equation:

Let: $az^2+bz+c=0$ be a quadratic equation s.t. $a \in \mathbb{C}^*, b \in \mathbb{C}, c \in \mathbb{C}$.

The roots of this equation are: $z_1 = \frac{-b-w}{2a}$ and $z_2 = \frac{-b+w}{2a}$,

where w is a square root of the discriminant $\Delta = b^2-4ac$.

example: $iz^2+\sqrt{7}z+1-i=0.$

$\Delta = (\sqrt{7})^2 - 4(i)(1-i) = 7-4i+4i^2 = 3-4i.$

By the previous example: $w = 2-i$ is a square root of Δ.

thus the roots of this equation are:

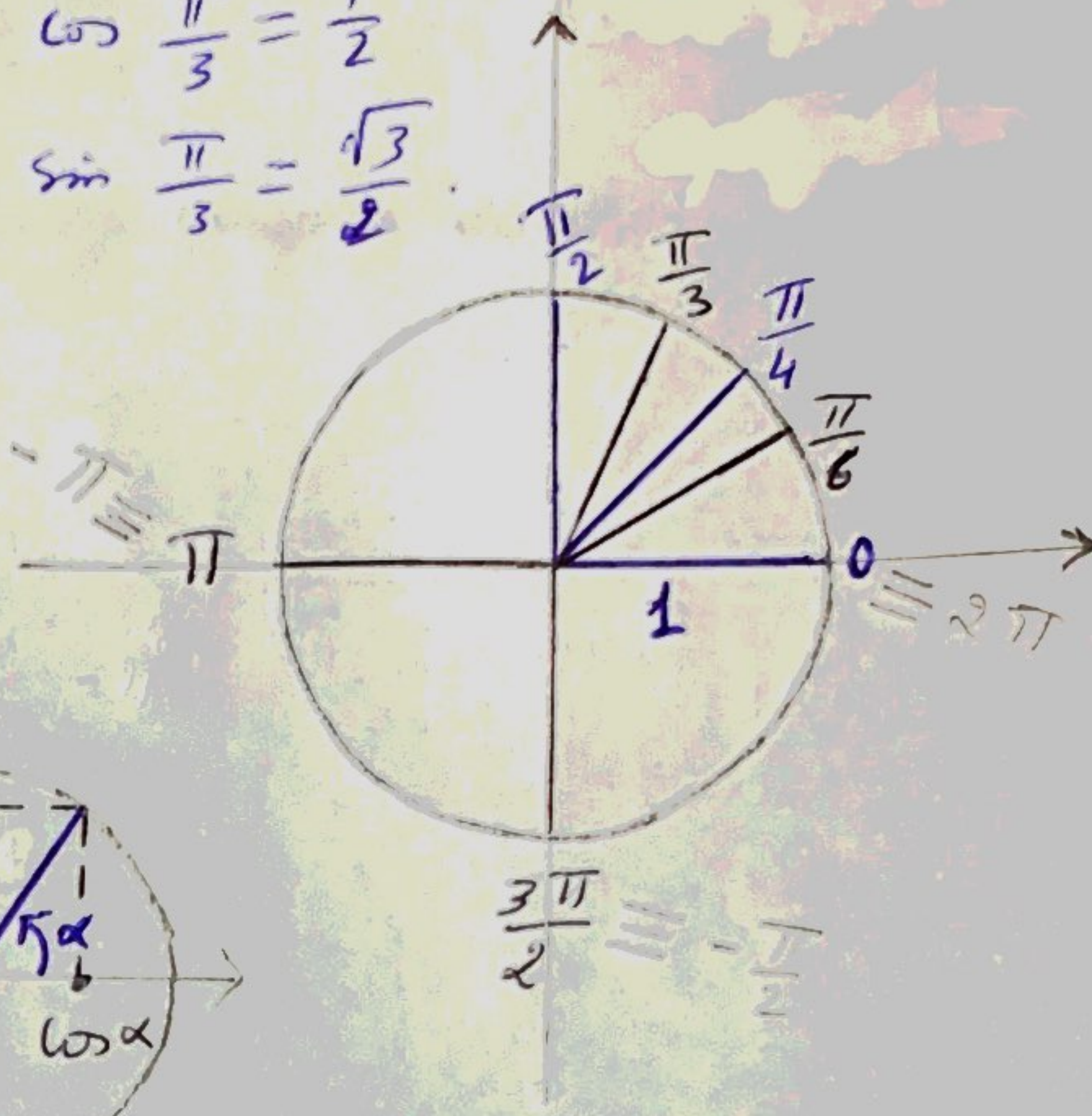
$$z_1 = \frac{-\sqrt{7}-2+i}{2i} \times \frac{-i}{-i} = \frac{\sqrt{7}i+2i-i^2}{-2i^2} = \frac{1}{2} + i\left(\frac{2+\sqrt{7}}{2}\right)$$

$$z_2 = \frac{-\sqrt{7}+2-i}{2i} \times \frac{-i}{-i} = \frac{\sqrt{7}i-2i+i^2}{-2i^2} = \frac{-1}{2} + i\left(\frac{\sqrt{7}-2}{2}\right).$$

VI- Trigonometric Circle:

1)

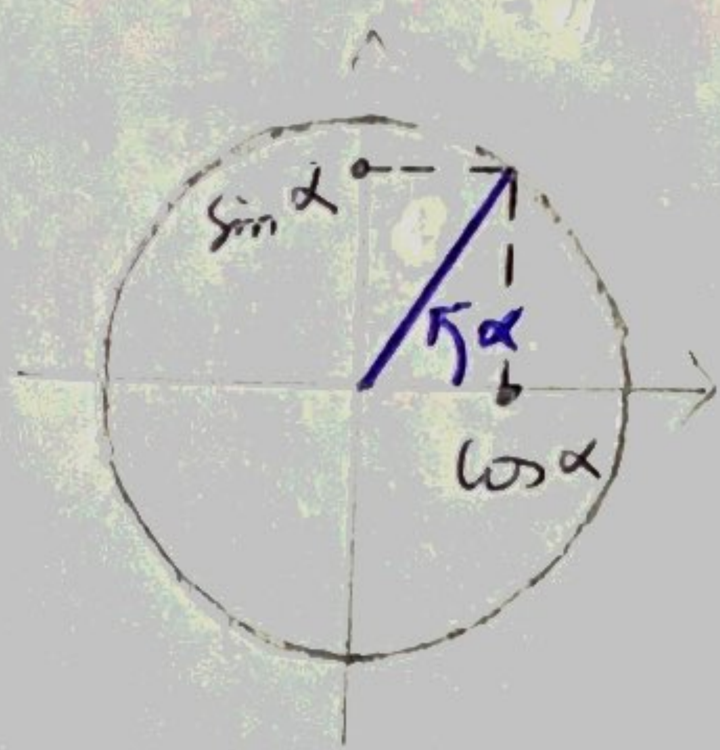
- $\cos 0 = 1$, $\sin 0 = 0$
- $\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$, $\sin \frac{\pi}{6} = \frac{1}{2}$.
- $\cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$, $\sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$
- $\cos \frac{\pi}{3} = \frac{1}{2}$, $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$.



- $\cos \frac{\pi}{2} = 0$, $\sin \frac{\pi}{2} = 1$
- $\cos \pi = -1$, $\sin \pi = 0$.
- $\cos \frac{3\pi}{2} = 0$, $\sin \frac{3\pi}{2} = -1$.

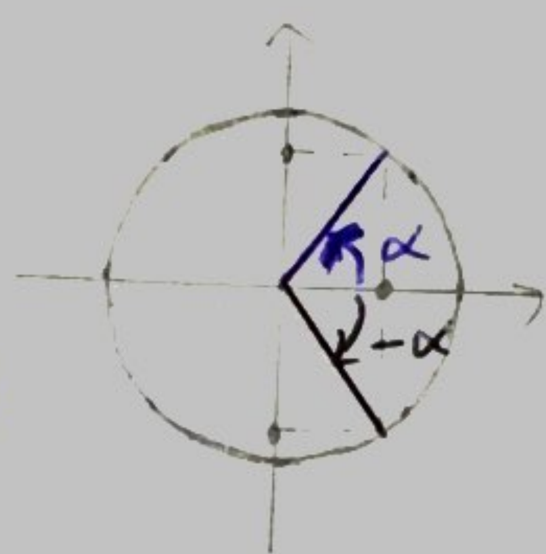
2)

- $\cos^2 \alpha + \sin^2 \alpha = 1$.
- $\cos(\alpha + 2K\pi) = \cos \alpha$; $K \in \mathbb{Z}$.
  $\sin(\alpha + 2K\pi) = \sin \alpha$; $K \in \mathbb{Z}$.

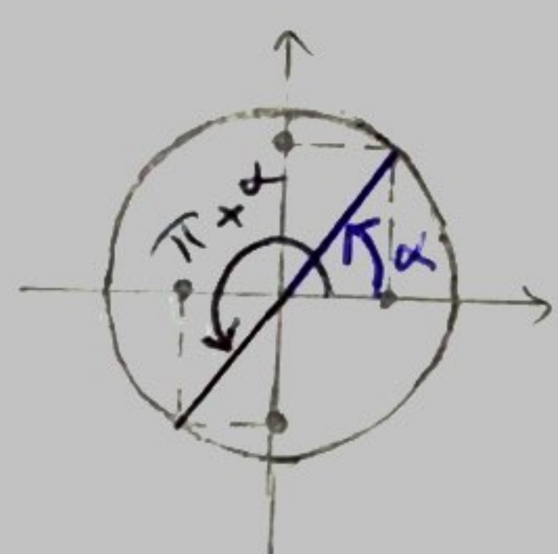


3)

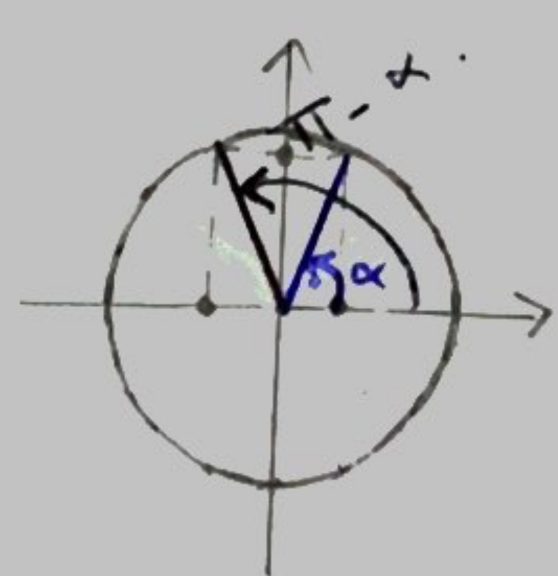
- $\cos(-\alpha) = \cos \alpha$
  $\sin(-\alpha) = -\sin \alpha$



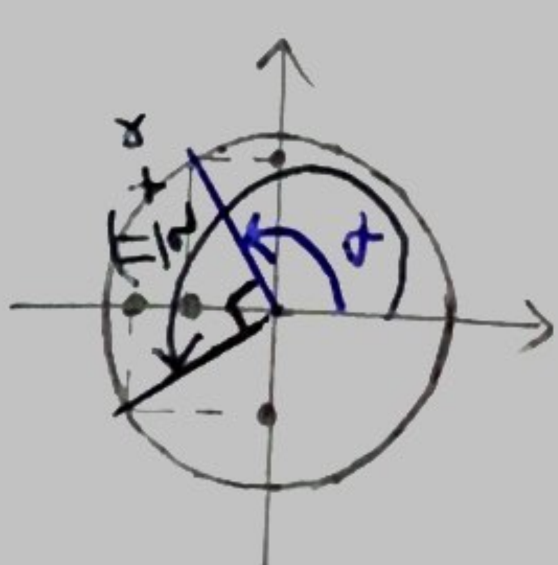
- $\cos(\pi + \alpha) = -\cos \alpha$
  $\sin(\pi + \alpha) = -\sin \alpha$



- $\cos(\pi - \alpha) = -\cos \alpha$
  $\sin(\pi - \alpha) = \sin \alpha$



- $\cos\left(\frac{\pi}{2} + \alpha\right) = -\sin \alpha$
  $\sin\left(\frac{\pi}{2} + \alpha\right) = \cos \alpha$.



example: $\cos\left(\frac{\pi}{2} - \frac{\pi}{6}\right) = \cos\underbrace{\left(\frac{\pi}{2} + \left(-\frac{\pi}{6}\right)\right)}_{\text{as } \frac{\pi}{2} + \alpha} = -\sin\left(-\frac{\pi}{6}\right) = -\left(-\sin\left(\frac{\pi}{6}\right)\right)$

$= +\sin \frac{\pi}{6} = \frac{1}{2}$.

$\cos\left(\frac{2\pi}{3}\right) = \cos\underbrace{\left(\pi - \frac{\pi}{3}\right)}_{\text{as } \pi - \alpha} = -\cos\left(\frac{\pi}{3}\right) = -\frac{1}{2}$.

Linear Algebra

Chapter -I- Complex Numbers

Exercises

Ex1 - Write in trigonometric form the following complex numbers:

(a) $z = i$

$z = i = 0 + i(1)$

$r = |z| = \sqrt{0^2 + 1^2} = \sqrt{1} = 1.$

$\cos\alpha = 0 = \cos\frac{\pi}{2}$

$\sin\alpha = 1 = \sin\frac{\pi}{2}$

So, $z = 1.\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right).$

(b) $z = -i$

$z = -i = 0 + i(-1)$

$r = |z| = \sqrt{0^2 + (-1)^2} = \sqrt{1} = 1.$

$\cos\alpha = 0 = \cos\frac{\pi}{2} = \cos\left(-\frac{\pi}{2}\right)$

$\sin\alpha = -1 = -\sin\frac{\pi}{2} = \sin\left(-\frac{\pi}{2}\right).$

So, $z = 1.\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right).$

(c) $z = 1 + i.$

$r = |z| = \sqrt{1^2 + 1^2} = \sqrt{2}.$

$z = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = \sqrt{2}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$

$\cos\alpha = \frac{\sqrt{2}}{2} = \cos\frac{\pi}{4}$

$\sin\alpha = \frac{\sqrt{2}}{2} = \sin\frac{\pi}{4}.$

$z = \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right).$

(d) $z = 1 - i$

$r = |z| = \sqrt{(1)^2 + (-1)^2} = \sqrt{2}.$

$z = \sqrt{2}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = \sqrt{2}\left(\frac{\sqrt{2}}{2} + i\left(\frac{-\sqrt{2}}{2}\right)\right)$

$$\left.\begin{aligned}\cos\alpha &= \frac{\sqrt{2}}{2} = \cos\frac{\pi}{4} = \cos\left(-\frac{\pi}{4}\right)\\ \sin\alpha &= -\frac{\sqrt{2}}{2} = -\sin\frac{\pi}{4} = \sin\left(-\frac{\pi}{4}\right)\end{aligned}\right\} \text{So, } z = \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right).$$

(e) $z = -1 - i$.

$r = |z| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$.

$z = \sqrt{2}\left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = \sqrt{2}\left(-\frac{\sqrt{2}}{2} + i\left(-\frac{\sqrt{2}}{2}\right)\right)$

$\cos\alpha = \frac{-1}{\sqrt{2}} = -\frac{\sqrt{2}}{2} = -\cos\left(\frac{\pi}{4}\right) = \cos\left(\pi + \frac{\pi}{4}\right)$

$\sin\alpha = \frac{-1}{\sqrt{2}} = -\frac{\sqrt{2}}{2} = -\sin\left(\frac{\pi}{4}\right) = \sin\left(\pi + \frac{\pi}{4}\right)$

So, $z = \sqrt{2}\left(\cos\left(\pi + \frac{\pi}{4}\right) + i\sin\left(\pi + \frac{\pi}{4}\right)\right) = \sqrt{2}\left(\cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right)\right)$

(f) $z = -1 + i\sqrt{3}$.

$r = |z| = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{4} = 2$.

$z = 2\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$

$\cos\alpha = -\frac{1}{2} = -\cos\frac{\pi}{3} = \cos\left(\pi - \frac{\pi}{3}\right)$

$\sin\alpha = \frac{\sqrt{3}}{2} = \sin\frac{\pi}{3} = \sin\left(\pi - \frac{\pi}{3}\right)$.

So, $z = 2\left(\cos\left(\pi - \frac{\pi}{3}\right) + i\sin\left(\pi - \frac{\pi}{3}\right)\right) = 2\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)$.

(g) $z = -1 - i\sqrt{3}$.

$r = |z| = \sqrt{(-1)^2 + (-\sqrt{3})^2} = \sqrt{4} = 2$.

$z = 2\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = 2\left(-\frac{1}{2} + i\left(-\frac{\sqrt{3}}{2}\right)\right)$

$\cos\alpha = -\frac{1}{2} = -\cos\frac{\pi}{3} = \cos\left(\pi + \frac{\pi}{3}\right)$

$\sin\alpha = -\frac{\sqrt{3}}{2} = -\sin\frac{\pi}{3} = \sin\left(\pi + \frac{\pi}{3}\right)$

So, $z = 2\left(\cos\left(\pi + \frac{\pi}{3}\right) + i\sin\left(\pi + \frac{\pi}{3}\right)\right) = 2\left(\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)\right)$

(h) $z = -\sqrt{3} + i$

$r = |z| = \sqrt{(-\sqrt{3})^2 + 1^2} = \sqrt{4} = 2$.

$z = 2\left(-\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)$

$\cos\alpha = -\frac{\sqrt{3}}{2} = -\cos\frac{\pi}{6} = \cos\left(\pi - \frac{\pi}{6}\right)$

$\sin\alpha = \frac{1}{2} = \sin\frac{\pi}{6} = \sin\left(\pi - \frac{\pi}{6}\right)$.

So,

$$z = 2\left(\cos\left(\pi - \frac{\pi}{6}\right) + i\sin\left(\pi - \frac{\pi}{6}\right)\right) = 2\left(\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right)$$

Ex2- Find the real numbers x and y in the following cases:

(i) $(1+2i)x + (3-5i)y = 1-3i$

$$(1+2i)x + (3-5i)y = 1-3i \Leftrightarrow x + 2ix + 3y - 5iy = 1-3i$$

$$\Leftrightarrow (x+3y) + i(2x-5y) = 1 + i(-3)$$

$$\Leftrightarrow \begin{cases} x+3y = 1 & (1) \\ 2x - 5y = -3 & (2) \end{cases}$$

$(2) - 2\times(1) \Rightarrow -5y - 6y = -3 - 2 \Rightarrow -11y = -5 \Rightarrow y = \frac{5}{11}$

$(1) \Rightarrow x = 1 - 3y = 1 - 3\left(\frac{5}{11}\right) = \frac{11-15}{11} = \frac{-4}{11}$

(ii) $(1+i)x - (1-i)y = 3i - 2$.

$$(1+i)x - (1-i)y = 3i - 2 \Leftrightarrow x + ix - y + iy = 3i - 2$$

$$\Leftrightarrow (x-y) + i(x+y) = -2 + i3.$$

$$\Leftrightarrow \begin{cases} x - y = -2 & (1) \\ x + y = 3 & (2) \end{cases}$$

$(2) - (1) \Rightarrow y - (-y) = 3 - (-2) \Rightarrow 2y = 5 \Rightarrow y = \frac{5}{2}$

$(1) \Rightarrow x = -2 + y = -2 + \frac{5}{2} = \frac{-4+5}{2} = \frac{1}{2}$.

Ex 3- Compute the square roots of the following complex numbers:

(a) $z = 2i + \sqrt{3} = \sqrt{3} + i2$, so $a = \sqrt{3}$, $b = 2$

Let $u = (x+iy)$ be a square root of z.

Then $\begin{cases} x^2 + y^2 = \sqrt{(\sqrt{3})^2 + 2^2} = \sqrt{3+4} = \sqrt{7}. & (1) \\ x^2 - y^2 = \sqrt{3} & (2) \\ 2xy = 2 \Rightarrow xy = 1. & (3) \end{cases}$

$① + ② \Rightarrow 2x^2 = \sqrt{7} + \sqrt{3} \Rightarrow x = \pm\sqrt{\frac{\sqrt{7}+\sqrt{3}}{2}}$

$③ \Rightarrow$ if $x = \sqrt{\frac{\sqrt{7}+\sqrt{3}}{2}}$ then $y = \frac{1}{\sqrt{\frac{\sqrt{7}+\sqrt{3}}{2}}} = \sqrt{\frac{2}{\sqrt{7}+\sqrt{3}}} = \sqrt{\frac{2(\sqrt{7}-\sqrt{3})}{7-3}}$

$$y = \sqrt{\frac{\sqrt{7}-\sqrt{3}}{2}}.$$

if $x = -\sqrt{\frac{\sqrt{7}+\sqrt{3}}{2}}$ then $y = -\sqrt{\frac{\sqrt{7}-\sqrt{3}}{2}}$

So the square roots of z are:

$u_1 = \sqrt{\frac{\sqrt{7}+\sqrt{3}}{2}} + i\sqrt{\frac{\sqrt{7}-\sqrt{3}}{2}}$ and $u_2 = -\sqrt{\frac{\sqrt{7}+\sqrt{3}}{2}} - i\sqrt{\frac{\sqrt{7}-\sqrt{3}}{2}}$.

(b) $z = \sqrt{3} - i\sqrt{2} = \sqrt{3} + i(-\sqrt{2})$ so $a = \sqrt{3}$ and $b = -\sqrt{2}$.

$u = x + iy$ square root of z.

$$\begin{cases} x^2 + y^2 = \sqrt{(\sqrt{3})^2 + (-\sqrt{2})^2} = \sqrt{3+2} = \sqrt{5} & ① \\ x^2 - y^2 = \sqrt{3} & ② \\ 2xy = -\sqrt{2} & ③ \end{cases}$$

$① + ② \Rightarrow 2x^2 = \sqrt{5} + \sqrt{3} \Rightarrow x = \pm\sqrt{\frac{\sqrt{5}+\sqrt{3}}{2}}.$

$③ \Rightarrow$ if $x = \sqrt{\frac{\sqrt{5}+\sqrt{3}}{2}}$ then $y = -\frac{\sqrt{2}}{2} \frac{1}{\sqrt{\frac{\sqrt{5}+\sqrt{3}}{2}}} = -\frac{\sqrt{2}\times\sqrt{2}}{2\sqrt{\sqrt{5}+\sqrt{3}}}$

$$y = \frac{-1}{\sqrt{\sqrt{5}+\sqrt{3}}} = -\sqrt{\frac{\sqrt{5}-\sqrt{3}}{5-3}} = -\sqrt{\frac{\sqrt{5}-\sqrt{3}}{2}}$$

if $x = -\sqrt{\frac{\sqrt{5}+\sqrt{3}}{2}}$ then $y = +\sqrt{\frac{\sqrt{5}-\sqrt{3}}{2}}.$

So the square roots of z are:

$u_1 = \sqrt{\frac{\sqrt{5}+\sqrt{3}}{2}} + i\left(-\sqrt{\frac{\sqrt{5}-\sqrt{3}}{2}}\right)$ and $u_2 = -\sqrt{\frac{\sqrt{5}+\sqrt{3}}{2}} + i\sqrt{\frac{\sqrt{5}-\sqrt{3}}{2}}$

Ex4 - Solve in $\mathbb{C}$ the following equations:

(a) $z^2 + z + 1 = 0$.

$\Delta = b^2 - 4ac = 1 - 4 = -3 = 3i^2$

$w = \sqrt{3}\,i$ is a square root of Δ.

The solutions are: $z_1 = \dfrac{-1-\sqrt{3}\,i}{2}$, $z_2 = \dfrac{-1+\sqrt{3}\,i}{2}$.

$z_1 = \frac{-1}{2} + i\left(-\frac{\sqrt{3}}{2}\right)$, $z_2 = \frac{-1}{2} + i\,\frac{\sqrt{3}}{2}$.

(b) $-z^2 + (7-i)z + 2i = 0$.

$\Delta = (7-i)^2 - 4(-1)(2i) = 49 - 14i + i^2 + 8i = 48 - 6i$.

$w = x + iy$ is a square root of Δ

$$\begin{cases} x^2+y^2 = \sqrt{(48)^2+(-6)^2} = \sqrt{2340} = 6\sqrt{65} & (1) \\ x^2 - y^2 = 48 & (2) \\ 2xy = -6 \Rightarrow xy = -3. & (3) \end{cases}$$

(1) + (2) $\Rightarrow 2x^2 = 48 + 6\sqrt{65} \Rightarrow x = \pm\sqrt{24 + 3\sqrt{65}}$.

(3) $\Rightarrow$ if $x = \sqrt{24+3\sqrt{65}} \Rightarrow y = \dfrac{-3}{\sqrt{24+3\sqrt{65}}} = \dfrac{-3\sqrt{3\sqrt{65}-24}}{\sqrt{585-576}}$

$y = -\sqrt{3\sqrt{65}-24}$.

if $x = -\sqrt{3\sqrt{65}+24} \Rightarrow y = +\sqrt{3\sqrt{65}-24}$.

so $w = \sqrt{3\sqrt{65}+24} + i\left(-\sqrt{3\sqrt{65}-24}\right)$.

The solutions are:

$z_1 = \dfrac{-(7-i)-w}{2(-1)} = \dfrac{7-i+w}{2}$.

$z_2 = \dfrac{-(7-i)+w}{2(-1)} = \dfrac{7-i-w}{2}$.

(c) $x^n - 14 = 0 \Rightarrow x^n = 14$

$14 = 14\,(\cos(0) + i\sin(0))$

The n^{th} roots of 14 are: $z_0, z_1, \dots, z_{n-1}$

where $z_t = \sqrt[n]{14}\left[\cos\left(\frac{0+2t\pi}{n}\right) + i\sin\left(\frac{0+2t\pi}{n}\right)\right]$

$= \sqrt[n]{14}\left[\cos\left(\frac{2t\pi}{n}\right) + i\sin\left(\frac{2t\pi}{n}\right)\right]; \quad 0 \leq t \leq n-1.$

(d) $x^4 - 1 - i = 0. \Rightarrow x^4 = 1 + i.$

$1 + i = \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)$

The 4^{th} roots of $1+i$ are z_0, z_1, z_2, z_3

where $z_t = \sqrt[4]{\sqrt{2}}\left[\cos\left(\frac{\frac{\pi}{4}+2t\pi}{4}\right) + i\sin\left(\frac{\frac{\pi}{4}+2t\pi}{4}\right)\right]; \quad 0 \leq t \leq 3.$

(e) $x^6 + 1 = 0. \Rightarrow x^6 = -1$

$-1 = 1.\,(\cos(\pi) + i\sin(\pi))$

The 6^{th} roots of -1 are: $z_0, z_1, \dots, z_5$

where $z_t = \sqrt[6]{1}\left[\cos\left(\frac{\pi+2t\pi}{6}\right) + i\sin\left(\frac{\pi+2t\pi}{6}\right)\right]; \quad 0 \leq t \leq 5.$

(f) $x^{12} = 2 - 2i\sqrt{3}$

$z = 2 - 2i\sqrt{3}$

$|z| = \sqrt{(2)^2 + (-2\sqrt{3})^2} = \sqrt{4+12} = \sqrt{16} = 4$

$z = 4\left(\frac{1}{2} + i\left(-\frac{\sqrt{3}}{2}\right)\right)$

$\cos\alpha = \frac{1}{2} = \cos\frac{\pi}{3} = \cos\left(-\frac{\pi}{3}\right)$

$\sin\alpha = -\frac{\sqrt{3}}{2} = -\sin\frac{\pi}{3} = \sin\left(-\frac{\pi}{3}\right)$

$\Rightarrow z = 4\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$

The 12^{th} roots of z are: $z_0, z_1, \dots, z_{11}$

where $z_t = \sqrt[12]{4}\left[\cos\left(\frac{-\frac{\pi}{3}+2t\pi}{12}\right) + i\sin\left(\frac{-\frac{\pi}{3}+2t\pi}{12}\right)\right]; \quad 0 \leq t \leq 11$

Ex 5- Find the algebraic form of each of the following complex numbers.

(1) $(1+i)^{2n}$

$1+i=\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{4}\right)\right)$

$(1+i)^{2n}=\left(\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{4}\right)\right)\right)^{2n}$

$=\left(\sqrt{2}\right)^{2n}\left(\cos\left(\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{4}\right)\right)^{2n}$ De Moivre

$=\left(\sqrt{2}^{2}\right)^{n}\left(\cos\left(2n\frac{\pi}{4}\right)+i\sin\left(2n\frac{\pi}{4}\right)\right)$

$=2^{n}\left(\cos\left(n\frac{\pi}{2}\right)+i\sin\left(\frac{n\pi}{2}\right)\right)$ De Moivre

$=2^{n}\left(\cos\left(\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{2}\right)\right)^{n}$

$=2^{n}\left(0+i(1)\right)^{n}=2^{n}i^{n}=(2i)^{n}$

2nd Method: $(1+i)^{2n}=\left((1+i)^{2}\right)^{n}=\left(1^{2}+2i+i^{2}\right)^{n}=(1+2i-1)^{n}=(2i)^{n}$.

(2) $\dfrac{(1+i)^{13}}{(1-i)^{11}}$ $\quad(1+i)^{2}=1+2i+i^{2}=2i$, $(1-i)^{2}=1-2i+i^{2}=-2i$.

$$\frac{(1+i)^{13}}{(1-i)^{11}}=\frac{(1+i)(1+i)^{12}}{(1-i)(1-i)^{10}}=\frac{(1+i)\,(2i)^{6}}{(1-i)\,(-2i)^{5}}=\frac{(1+i)\,2^{6}\,i^{6}}{-(1-i)\,2^{5}\,i^{5}}$$

$$=\frac{-(1+i)\,2i}{(1-i)}\times\frac{1+i}{1+i}=\frac{-(1+i)^{2}\,2i}{1^{2}-i^{2}}=\frac{-2i\times 2i}{2}=-2i^{2}=2.$$

(3) $\dfrac{(1-i)^{12}\,(1+i\sqrt{3})^{6}}{(1-i\sqrt{3})^{8}}$

$z=r(\cos\alpha+i\sin\alpha)=[r,\alpha]$

$z^{n}=r^{n}(\cos\alpha+i\sin\alpha)^{n}=r^{n}\left(\cos(n\alpha)+i\sin(n\alpha)\right)=[r^{n},n\alpha]$

So, $[r,\alpha]^{n}=[r^{n},n.\alpha]$.

Recall that: $[r,\alpha]\times[r',\alpha']=[rr',\alpha+\alpha']$, $\quad\dfrac{[r,\alpha]}{[r',\alpha']}=\left[\dfrac{r}{r'},\alpha-\alpha'\right]$.

and $[r,\alpha+2\pi]=[r,\alpha]$.

$$\frac{(1-i)^{12}\times(1+i\sqrt{3})^{6}}{(1-i\sqrt{3})^{8}}=\frac{\left[\sqrt{2},-\frac{\pi}{4}\right]^{12}\times\left[2,\frac{\pi}{3}\right]^{6}}{\left[\sqrt{2},-\frac{\pi}{3}\right]^{8}}$$

$$= \frac{[(\sqrt{2})^{12}, 12(-\frac{\pi}{4})] \times [2^6, 6(\frac{\pi}{3})]}{[2^8, 8(-\frac{\pi}{3})]}$$

$$= \frac{[2^6, -3\pi] \times [2^6, 2\pi]}{[2^8, -\frac{8\pi}{3}]}$$

$$= \frac{[2^6 \times 2^6, -3\pi + 2\pi]}{[2^8, -\frac{8\pi}{3}]}$$

$$= \frac{[2^{12}, -\pi]}{[2^8, -\frac{8\pi}{3}]} = [\frac{2^{12}}{2^8}, -\pi - (-\frac{8\pi}{3})] = [2^4, \frac{5\pi}{3}] = [16, \frac{6\pi - \pi}{3}]$$

$$= [16, 2\pi - \frac{\pi}{3}] = [16, -\frac{\pi}{3}] = 16(\cos(-\frac{\pi}{3}) + i\sin(-\frac{\pi}{3}))$$

$$= 16(\frac{1}{2} + i(-\frac{\sqrt{3}}{2})) = \underbrace{8 + i(-8\sqrt{3})}_{\text{algebraic form}}.$$

Ex 6 - Show that if $a, b, x, y \in \mathbb{R}$ and $n \in \mathbb{N}^*$
such that $x + iy = (a + ib)^n$
then $x^2 + y^2 = (a^2 + b^2)^n$.

we want to prove that $x^2 + y^2 = (a^2 + b^2)^n$?

$x + iy = (a + ib)^n$

$$\Rightarrow \underset{\underset{\sqrt{x^2+y^2}}{\parallel}}{|x + iy|} = |(a + ib)^n| = |a + ib|^n = (\sqrt{a^2 + b^2})^n$$

so $\sqrt{x^2 + y^2} = (\sqrt{a^2 + b^2})^n$

$\Rightarrow (\sqrt{x^2 + y^2})^2 = (\sqrt{a^2 + b^2})^{2n}$

$\Rightarrow x^2 + y^2 = (a^2 + b^2)^n$. ✓

Ex 7 - Let z and z' be two complex numbers

1) Show that $|z+z'| \leq |z| + |z'|$.

we want to prove that $|z+z'|^2 \leq (|z|+|z'|)^2$?

$z = r(\cos\alpha + i\sin\alpha)$

$z' = r'(\cos\alpha' + i\sin\alpha')$.

$z+z' = (r\cos\alpha + r'\cos\alpha') + i(r\sin\alpha + r'\sin\alpha')$

$|z+z'|^2 = (r\cos\alpha + r'\cos\alpha')^2 + (r\sin\alpha + r'\sin\alpha')^2$

$= r^2\cos^2\alpha + 2rr'\cos\alpha\cos\alpha' + r'^2\cos^2\alpha'$

$+ r^2\sin^2\alpha + 2rr'\sin\alpha\sin\alpha' + r'^2\sin^2\alpha'$

$= r^2\underbrace{(\cos^2\alpha + \sin^2\alpha)}_{=1} + 2rr'\underbrace{(\cos\alpha\cos\alpha' + \sin\alpha\sin\alpha')}_{=\cos(\alpha-\alpha')} + r'^2\underbrace{(\cos^2\alpha' + \sin^2\alpha')}_{=1}$

$= r^2 + r'^2 + 2rr'\cos(\alpha-\alpha')$

But $\cos(\alpha-\alpha') \leq 1 \xrightarrow[(2rr' \geq 0)]{\times 2rr'} 2rr'\cos(\alpha-\alpha') \leq 2rr'$

$\xrightarrow{+r^2+r'^2} r^2+r'^2+2rr'\cos(\alpha-\alpha') \leq r^2+r'^2+2rr'$

So $|z+z'|^2 = r^2+r'^2+2rr'\cos(\alpha-\alpha')$
$\leq r^2+r'^2+2rr' = (r+r')^2 = (|z|+|z'|)^2$

so $|z+z'| \leq |z|+|z'|$.

2) show that $|z+z'|^2 + |z-z'|^2 = 2(|z|^2+|z'|^2)$

Recall that, for any complex nb u, we have: $u.\bar{u} = |u|^2$

So, $|z+z'|^2 + |z-z'|^2 = (z+z').\overline{(z+z')} + (z-z')\overline{(z-z')}$

$= (z+z')(\bar{z}+\bar{z}') + (z-z')(\bar{z}-\bar{z}')$

$= z\bar{z} + \cancel{z\bar{z}'} + \cancel{z'\bar{z}} + z'\bar{z}' + z\bar{z} - \cancel{z\bar{z}'} - \cancel{z'\bar{z}} + z'\bar{z}' = 2z\bar{z} + 2z'\bar{z}'$

$= 2|z|^2 + 2|z'|^2$

$= 2(|z|^2+|z'|^2)$.

Ex8 - Find the complex nb z such that $\overline{z^n} = z$, with $n \in \mathbb{N}^*$.

Remark that, if $z = r(\cos\alpha + i\sin\alpha)$

then $\bar{z} = r(\cos\alpha - i\sin\alpha) = r(\cos(-\alpha) + i\sin(-\alpha))$

So, if $z = [r, \alpha]$ then $\bar{z} = [r, -\alpha]$.

We have $\overline{z^n} = z$

$\Rightarrow \overline{[r^n, n\alpha]} = [r, \alpha]$

$\Rightarrow [r^n, -n\alpha] = [r, \alpha]$

$\Rightarrow (r^n = r)$ and $(-n\alpha = \alpha + 2K\pi$, with $K \in \mathbb{Z})$

$\Rightarrow \left(r = 0 \text{ or } \frac{r^n}{r} = 1\right)$ and $((n+1)\alpha = -2K\pi,\ K \in \mathbb{Z})$

$\Rightarrow (r = 0 \text{ or } r^{n-1} = 1)$ and $((n+1)\alpha = 2K\pi,\ K \in \mathbb{Z})$

$\overset{r \in \mathbb{R}}{\Rightarrow} (r = 0 \text{ or } r = 1)$ and $\left(\alpha = \frac{2K\pi}{n+1},\ K \in \mathbb{Z}\right)$

So, $z = 0$ or $z = 1 \cdot \left(\cos\left(\frac{2K\pi}{n+1}\right) + i\sin\left(\frac{2K\pi}{n+1}\right)\right)$ with $K \in \mathbb{Z}$

Ex 9 - Express $\cos 3x$ and $\sin 3x$ in terms of $\cos x$ and $\sin x$.

Recall that $(a+b)^3 = (a+b)^2(a+b) = a^3 + 3a^2b + 3ab^2 + b^3$.

So, $(\cos x + i\sin x)^3 = \underline{\cos(3x)} + i\,\underline{\underline{\sin(3x)}}$ (De Moivre)

$//$

$(\cos x)^3 + 3(\cos x)^2(i\sin x) + 3(\cos x)(i\sin x)^2 + (i\sin x)^3$

$= \cos^3 x + i\,3\cos^2 x\sin x + i^2\,3\cos x\sin^2 x + i^3\sin^3 x$. $(i^3 = i^2 \cdot i = -i)$

$= \underline{(\cos^3 x - 3\cos x\sin^2 x)} + i\,\underline{\underline{(3\cos^2 x\sin x - \sin^3 x)}}$

So $\cos(3x) = \cos^3 x - 3\cos x\sin^2 x$

and $\sin(3x) = \cos^2 x\sin x - \sin^3 x$.

- $\cos(a+b) = \cos a \cos b - \sin a \sin b$

  $\cos(a-b) = \cos a \cos b + \sin a \sin b$

- $\sin(a+b) = \sin a \cos b + \cos a \sin b$

  $\sin(a-b) = \sin a \cos b - \cos a \sin b.$