$i^4 = 1$

Real polynomial

Exercise: 7.

Let $f(x) = x^5 + x^4 + 8x^3 + 8x^2 + 16x + 16$

1) Prove that $2i$ is a root of $f(x)$

$f(2i) = (2i)^5 + (2i)^4 + 8(2i)^3 + 8(2i)^2 + 16(2i) + 16$.

$= 32i + 16 + 64(-i) + 32(-1) + 32i + 16$.

$= 64i - 64i + 32 - 32 = 0$.

so $2i$ root of $f(x)$.

2) Without using the Euclidean Division Deduce that $f(x)$ is divisible by (x^2+4)

$f(x)$ Real polynomial

$2i$ complex root of $f(x)$.

so $\overline{2i} = -2i$ root of $f(x)$

$(x-2i)(x-(-2i))$ divids $f(x)$.

$2i \neq -2i$; distinct roots.

$(x-2i)(x+2i) = x^2 - (2i)^2 = x^2 - (-4) = x^2 + 4$.

so, $x^2 + 4$ divids $f(x)$.

3) Calculate $(x^2+4)^2$.

$(x^2+4)^2 = x^4 + 8x^2 + 16$.

4) By using Eulidean division of $f(x)$ by $(x^2+4)^2$, deduce that $(x+1)$ divids $f(x)$.

$$\begin{array}{l|l} x^5+x^4+8x^3+8x^2+16x+16 & x^4+8x^2+16 \\ \hline -\;(x^5+8x^3+16x) & x+1 \\ \hline x^4+8x^2+16 & \\ -\;(x^4+8x^2+16) & \\ \hline 0 & \end{array}$$

$f(x)=(x+1)(x^4+8x^2+16)+r^{=0}$

$=(x+1)(x^4+8x^2+16)$

so $(x+1)$ is a factor of $f(x)$

so $f(x)$ is divisible by $(x+1)$

5) Factorize $f(x)$ over $\mathbb{C}$

$f(x)=a(x-\alpha_1)\ldots(x-\alpha_n)$.

$f(x)=(x+1)(x^4+8x^2+16)$

$=(x+1)(x^2+4)^2$

Factorization over $\mathbb{C}$.

$=(x+1)\left((x-2i)(x+2i)\right)^2$

$=(x+1)(x-2i)^2(x+2i)^2$

Exercise: 8. Let $f(x)=2x^4-8x^3-10x^2$.

1) Prove that -1 and 5 are roots of $f(x)$.

$f(-1)=2(-1)^4-8(-1)^3-10(-1)^2=2+8-10=0$

so (-1) root of $f(x)$.

2nd Method: $(x+1)$ devids $f(x)$ $(r=0)$.

Horner Method

$f(5) = 2(5)^4 - 8(5)^3 - 10(5)^2$
$= 1250 - 1000 - 250 = 0$
so 5 root of $f(x)$.

2) Is $f(x)$ divisible by $(x+1)(x-5)$??
-1 and 5 are distinct roots.
and $(x+1)$ divids $f(x)$.
$(x-5)$ divids $f(x)$.
so $(x+1)(x-5)$ divids $f(x)$.

3) Calculate $(x+1)(x-5)$??
$(x+1)(x-5) = x^2 - 4x - 5$.

4) By using the Euclidean division of $f(x)$ by $(x+1)(x-5)$, factorize $f(x)$ over $\mathbb{C}$.

$$
\begin{array}{r|l}
2x^4 - 8x^3 - 10x^2 & x^2 - 4x - 5 \\
\hline
\ominus\ 2x^4 \oplus 8x^3 \oplus 10x^2 & 2x^2 \\
\hline
0 &
\end{array}
$$

so $f(x) = 2x^2(x^2 - 4x - 5) + 0$
$= 2x^2(x+1)(x-5)$
$= 2(x-0)^2(x+1)(x-5)$

Factorization of $f(x)$ over $\mathbb{R}$ and $\mathbb{C}$.

5) Deduce the roots of $f(x)$.

0: Double root.
-1: simple root
5: simple root.

Exercise: 9

Let $f(x) = x^3 - 9x^2 + 31x - 39$.

1) Prove that 3 is a root of $f(x)$.
Find it's multiplicity.

$f(3) = 3^3 - 9(3)^2 + 31(3) - 39 = 27 - 81 + 93 - 39 = 0$

so, 3 is a root of $f(x)$.

$(x-3)$ divids $f(x)$.

$f'(x) = 3x^2 - 18x + 31$.

$f'(3) = 3(3)^2 - 18(3) + 31 = 27 - 54 + 31 = 4 \neq 0$

so 3 is a root of multiplicity 1.

2) Find the quotient of division of $f(x)$ by $x-3$.

$$\begin{array}{r|l} x^3 - 9x^2 + 31x - 39 & x - 3 \\ \hline -\;\; x^3 - 3x^2 & x^2 - 6x + 13 \\ \hline -6x^2 + 31x - 39 & \text{Quotient.} \\ +\; -6x^2 + 18x & \\ \hline 13x - 39 & \\ -\;\; 13x - 39 & \\ \hline 0 & \end{array}$$

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3) Find the other roots of $f(x)$

$$F(x) = (x-3)(x^2 - 16x + 13)$$

roots of $f(x)$ are 3 and the roots of $x^2 - 6x + 13$

$$\Delta = 36 - 52 = -16 = (4i)^2$$

$$x_1 = \frac{6-4i}{2} = 3-2i.$$

$$x_2 = \frac{6+i4}{2} = 3+2i.$$

roots of $f(x)$ are 3, $x_1 = 3-2i$ and $x_2 = 3+2i$.

4) Give the factorization of $f(x)$ over $\mathbb{R}$ then over $\mathbb{C}$.

$$f(x) = (x-3)(x^2 - 6x + 13).$$

$(x-3)$: irreducible over $\mathbb{R}$ of degree 1.

$(x^2-6x+13)$: irreducible over $\mathbb{R}$: $\Delta < 0$.

$$f(x) = (x-3)(x-(3-2i))(x-(3+2i))$$

irreducible over $\mathbb{C}$ of degree 1

Factorization of $f(x)$ over $\mathbb{C}$.

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Exercise : 10.

$f(x) = x^6 + x^5 + 3x^4 + 2x^3 + 3x^2 + x + 1.$

1) Find the order of multiplicity of i in $f(x)$.

$i^4 = 1$

$i^6 = i^4 * i^2 = (1)*(-1) = -1$

$f(i) = i^6 + i^5 + 3i^4 + 2i^3 + 3i^2 + i + 1$

$= (-1) + i + 3 - 2i - 3 + i + 1 = 0$

$\Rightarrow i$ root of $f(x)$.

$f'(x) = 6x^5 + 5x^4 + 12x^3 + 6x^2 + 6x + 1.$

$f'(i) = 6i + 5 - 12i - 6 + 6i + 1 = 0$

$f''(x) = 30x^4 + 20x^3 + 36x^2 + 12x + 6.$

$f''(i) = 30 - 20i - 36 + 12i + 6$

$= -8i \neq 0$

so i is a root of multiplicity 2.

2) Find the quotient and the remainder of f by $(x^2+1)^2$.

$f(x) = (x-i)^2\, g(x)$, f real polynomial. i complex root double $\Rightarrow -i$ root double $\Rightarrow f(x) = (x-i)^2(x+i)^2 \times h(x) = (x^2+1)^2 \times h(x)$.

$(x^2+1)^2 = x^4 + 2x^2 + 1.$

$$
\begin{array}{l|l}
x^6 + x^5 + 3x^4 + 2x^3 + 3x^2 + x + 1 & x^4 + 2x^2 + 1 \\
\hline
x^6 + 2x^4 + x^2 & x^2 + x + 1 \\
\hline
x^5 + x^4 + 2x^3 + 2x^2 + x + 1 & \\
- \; x^5 + 2x^3 + x & \\
\hline
x^4 + 2x^2 + 1 & \\
- \; x^4 + 2x^2 + 1 & \\
\hline
0 &
\end{array}
$$

quotient : $x^2 + x + 1$.

Remainder : 0.

3) show x^2+x+1 is irreducible over $\mathbb{R}$
Deduce the factorization of $f(x)$ over $\mathbb{R}$.
and over $\mathbb{C}$.

x^2+x+1 : $\Delta = -3 < 0$.
so irreducible over $\mathbb{R}$.

$f(x) = (x^2+1)^2 (x^2+x+1)$

Factorization over $\mathbb{R}$.

$\Delta = -4 < 0$
irreducible

$\Delta = -3 < 0$
irreducible

$(x^2+1)^2 = (x-i)^2 (x+i)^2$

x^2+x+1 : $\Delta = -3 = (\sqrt{3}i)^2$.

$x_1 = \dfrac{-1-i\sqrt{3}}{2}$, $x_2 = \dfrac{-1+i\sqrt{3}}{2}$

so $f(x) = (x-i)^2 (x+i)^2 (x-x_1)(x-x_2)$
Factorize over $\mathbb{C}$.

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