ICS 2020 Problem Sheet #6:

Problem 6.1:

We know that { AND, NOT } is a complete set of connectives (universal) so (nor) ↓ and (nand) ↑ are universal too. We can simply prove that → (implication) and ¬ (negation) are universal by expressing ↓ or ↑ with → and ¬.

p	q	p→q	¬p	¬q	p↓q	p↑q
F	F	T	T	T	T	T
F	T	T	T	F	F	T
T	F	F	F	T	F	T
T	T	T	F	F	F	F

From the table we deduce that:

$$(p\uparrow q) \equiv (p\rightarrow(\neg q))$$

$$(p\downarrow q) \equiv (\neg((\neg p)\rightarrow q))$$

Thus, implication and negation are universal.

Problem 6.2:

a) $\phi(A, B) = (\neg A \lor \neg B) \land (\neg A \lor B) \land (A \lor \neg B)$

$$\phi(A, B) = (\neg A \lor \neg B) \land (\neg A \lor B) \land (A \lor \neg B)$$
$$\phi(A, B) = \neg A \lor (\neg B \land B) \land (A \lor \neg B) \text{ (distributivity)}$$
$$\phi(A, B) = \neg A \lor (0) \land (A \lor \neg B) \text{ (identity)}$$
$$\phi(A, B) = \neg A \land (A \lor \neg B) \text{ (distributivity)}$$

$$\phi(A, B) = (\neg A \land A) \lor (\neg A \land \neg B)$$
$$\phi(A, B) = 0 \lor (\neg A \land \neg B) \text{ (identity)}$$
$$\phi(A, B) = (\neg A \land \neg B) \text{(de Morgan's laws)}$$
$$\phi(A, B) = \neg(A \land B)$$

b) $\phi(A, B, C) = (A \land \neg B) \lor (A \land \neg B \land C)$

$$\phi(A, B, C) = (A \land \neg B) \lor (A \land \neg B \land C)$$
$$\phi(A, B, C) = (A \land \neg B) \lor ((A \land \neg B) \land C) \text{ (associativity)}$$
$$\phi(A, B, C) = (A \land \neg B) \text{ (absorption laws)}$$

c) $\phi(A, B, C, D) = (A \lor \neg(B \land A)) \land (C \lor (D \lor C))$

$$\phi(A, B, C, D) = (A \lor \neg(B \land A)) \land (C \lor (D \lor C))$$
$$\phi(A, B, C, D) = (A \lor (\neg B \land \neg A)) \land (C \lor (D \lor C)) \text{(de Morgan's laws)}$$
$$\phi(A, B, C, D) = ((A \lor \neg B) \land (A \lor \neg A)) \land (C \lor (D \lor C)) \text{(distributivity)}$$
$$\phi(A, B, C, D) = ((A \lor \neg B) \land 1) \land (C \lor (D \lor C))$$
$$\phi(A, B, C, D) = (A \lor \neg B) \land (C \lor (D \lor C)) \text{(identity)}$$
$$\phi(A, B, C, D) = (A \lor \neg B) \land (D \lor (C \lor C)) \text{(associativity)}$$
$$\phi(A, B, C, D) = (A \lor \neg B) \land (D \lor C) \text{ (idempotency)}$$

d) $\phi(A, B, C) = (\neg(A \land B) \lor \neg C) \land (\neg A \lor B \lor \neg C)$

$$\phi(A, B, C) = (\neg(A \land B) \lor \neg C) \land (\neg A \lor B \lor \neg C)$$
$$\phi(A, B, C) = \neg((A \land B) \lor C) \land (\neg A \lor B \lor \neg C) \text{ (de Morgan's laws)}$$
$$\phi(A, B, C) = (\neg((A \land B) \lor C) \land \neg(A \lor \neg B \lor C)) \text{(double negation)}$$
$$\phi(A, B, C) = \neg(((A \land B) \lor C) \land (A \lor B \lor C)) \text{(de Morgan's laws)}$$
$$\phi(A, B, C) = \neg(((A \land B) \lor C) \land (A \lor B)) \lor ((A \land B) \lor C) \land C))$$
(distributivity)
$$\phi(A, B, C) = \neg(((A \land B) \lor C) \lor ((A \land B) \lor C)) \text{(idempotency)}$$
$$\phi(A, B, C) = \neg((A \land B) \lor C) \text{ (idempotency)}$$

e) $\phi(A, B) = (A \lor B) \land (\neg A \lor B) \land (A \lor \neg B) \land (\neg A \lor \neg B)$

$$\phi(A, B) = (A \lor B) \land (\neg A \lor B) \land (A \lor \neg B) \land (\neg A \lor \neg B)$$

$\phi(A, B) = (A \lor B) \land (\neg A \lor \neg B) \land (\neg A \lor B) \land (A \lor \neg B)$ (commutativity)

$\phi(A, B) = (A \lor B) \land \neg(A \lor B) \land (\neg A \lor B) \land (A \lor \neg B)$(de Morgan's laws)

$\phi(A, B) = (A \lor B) \land \neg(A \lor B) \land (\neg A \lor B) \land \neg(\neg A \lor B)$(double negation)

$\phi(A, B) = 0 \land 0$

$\phi(A, B) = 0$

Problem 6.3:

a) $\phi(P, Q, R, S) = (\neg P \lor Q) \land (\neg Q \lor R) \land (\neg R \lor S) \land (\neg S \lor P)$

P	Q	R	S	$(\neg P \lor Q)$	$(\neg Q \lor R)$	$(\neg R \lor S)$	$(\neg S \lor P)$	ϕ
F	F	F	F	T	T	T	T	T
F	F	F	T	T	T	T	F	F
F	F	T	F	T	T	F	T	F
F	F	T	T	T	T	T	F	F
F	T	F	F	T	F	T	F	F
F	T	F	T	T	F	T	F	F
F	T	T	T	T	T	T	F	F
F	T	T	T	T	T	T	F	F
T	F	F	F	F	T	T	T	F
T	F	F	T	F	T	T	T	F
T	F	T	F	F	T	F	T	F
T	F	T	T	F	T	T	T	F
T	T	F	F	T	F	T	T	F

T	T	F	T	T	F	T	T	F
T	T	T	F	T	T	F	T	F
T	T	T	T	T	T	T	T	T

From the truth table we see that 2 interpretation satisfy ϕ

b)

The two interpretations that satisfies ϕ require that P,Q,R,S are all TRUE or ALL false

DNF = $(P \land Q \land R \land S) \lor (\neg P \land \neg Q \neg R \land \neg S)$