Problem Sheet #3:

Problem 3.1:

a)

$$(x,y) \in ((A \cap B) \times (C \cap D))$$

$$\Leftrightarrow x \in (A \cap B) \text{ and } y \in (C \cap D)$$

$$\Leftrightarrow (x \in A \text{ and } x \in B) \text{ and } (y \in C \text{ and } y \in D)$$

$$\Leftrightarrow (x \in A \text{ and } y \in C) \text{ and } (x \in B \text{ and } y \in D)$$

$$\Leftrightarrow (x,y) \in (A \times C) \text{ and } (x,y) \in (B \times D)$$

$$\Leftrightarrow (x,y) \in (A \times C) \cap (B \times D)$$

$$(A \cap C)\times(B \cap D) = (A \times B) \cap (C \times D)$$

b)

let's give a *Counterexample to prove that*

$$\mathbf{(A\times B) \cup (C\times D) \neq (A \cup C) \times (B \cup D)}$$

$A=\{a\},B=\{b\},C=\{c\},D=\{d\}$
$A\cup C=\{a,c\}$
$B\cup D=\{b,d\}$

$$\mathbf{(A\cup C)\times(B\cup D)=\{(a,b),(a,d),(c,b),(c,d)\}}$$

$A\times B=\{(a,b)\}$
$C\times D=\{(c,d)\}$

$$\mathbf{(A\times B)\cup(C\times D)=\{(a,b),(c,d)\}}$$

Problem 3.2:

$R = \{(a, b) | a, b \in Z \land |a - b| \le 3\}$

a)

Reflexive:

For $\forall (a,b) \in Z \land |a - b| \le 3$ let's see if $\{(a,a),(b,b)\} \in R$:

$|a-a| = 0 \Leftrightarrow |a-a| \le 3$
$|b-b| = 0 \Leftrightarrow |b-b| \le 3$

Thus

$\{(a,a),(b,b)\} \in R$ and R is reflexive.

Symmetric:

$\forall (a,b) \in Z \land |a - b| \le 3$ we suppose that $(a,b) \in R$ let's see if $(b,a) \in R$:

$(a,b) \in R \Rightarrow |a - b| \le 3$
$\Rightarrow |-(-a + b)| \le 3$
$\Rightarrow |-(b-a)| \le 3$
$\Rightarrow |b-a| \le 3$

Thus,

$(a,b) \in R \Rightarrow (b,a) \in R$ and R is symmetric

Transitive:

$\forall a, b, c \in Z$ we suppose that $(a,b) \in R \land (b,c) \in R$ let's see if $(a,c) \in R$

for:
a=5
b=4
c=1
$|a - b| = |5 - 4| = 1 \le 3$
$|b-c| = |4 - 1| = 3 \le 3$

$|a - c| = |5-1| > 3$
$(a,c) \notin R$
We found a counterexample so R is not transitive.
b)
$R = \{(a, b) | a, b \in Z \land (a \bmod 10) = (b \bmod 10)\}$
Reflexive:
For $\forall (a,b) \in Z$ we suppose that $(a,b) \in R$ let's see if $\{(a,a),(b,b)\} \in R$

$(a \bmod 10) = (a \bmod 10)$
$(b \bmod 10) = (b \bmod 10)$

Thus

$\{(a,a),(b,b)\} \in R$ and R is reflexive.

Symmetric:
$\forall (a,b) \in Z$ we suppose that $(a,b) \in R$ let's see if $(b,a) \in R$:

$(a,b) \in R \Rightarrow (a \bmod 10) = (b \bmod 10)$
$\Rightarrow (b \bmod 10) = (a \bmod 10)$

Thus,
$(a,b) \in R \Rightarrow (b,a) \in R$ and R is symmetric

Transitive:
$\forall a, b, c \in Z$ we suppose that $(a,b) \in R \land (b,c) \in R$ let's see if $(a,c) \in R$

$(a,b) \in R \Rightarrow (a \bmod 10) = (b \bmod 10)$
$(b,c) \in R \Rightarrow (b \bmod 10) = (c \bmod 10)$
$\Rightarrow (a \bmod 10) = (c \bmod 10) \Rightarrow (a,c) \in R$

R is transitive.