15-382 Collective Intelligence

Salman Hajizada

First Edition

Disclaimer

This document aims to summarize the content of the slides for 15-382, including what the author considers important. As always, the definition of important is highly subjective so the author might have omitted something that was important to another person, or included something that is trivial to another.

Good luck,

SH

Dynamical Systems

Fingerprints of Complex Systems

- Multi-agent / multi-component
- Decentralized
- Local interactions
- Dynamic

Types of Abstract Models:

Agent-based Mechanistic implementation of the multi-component interactions.

Mathematical (white-box) Formally describe the relations among the relevant components.

Black-box Input-output pairs from the system are used to predict...

Statistical Describing patterns and correlations between variables.

Systems of ODEs

Here is a system of $n \geq 1$ Ordinary Differential Equations

$$\begin{cases} \frac{dx_1}{dt} = f_1(\mathbf{x}(t)) \\ \frac{dx_2}{dt} = f_2(\mathbf{x}(t)) \\ \vdots \\ \frac{dx_n}{dt} = f_n(\mathbf{x}(t)) \end{cases}$$

where $\mathbf{x}(t)$ is an n-dim vector.

A continuous-time Dynamical System is defined by a system of differential equations:

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{f}(\mathbf{x}, t; \theta) \quad \text{or} \quad \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t; \theta)$$

where $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ specifies how each component of the state evolves as t changes. It can depend on a set of given parameters θ

Some definitions:

- Initial conditions: where the system is at the beginning of the evolution: $\mathbf{x}(t_0)$
- Phase space: space of all possible states
- Trajectory: the curve traces by $\mathbf{x}(t)$ in the phase space starting from $\mathbf{x}(t_0)$
- Solution: is in the form $\mathbf{x}(t; t_0)$ that defines a family of time trajectories in the phase space. Once we fix t_0, we fix a unique trajectory

Vector fields and flows

How are solutions built? At any point, $\mathbf{f}$ assigns a vector that shows where the point is heading (direction of motion).

If we plot these arrows (vectors) in the phase space, we get can get an idea of how the system evolves.

Flow: $\Phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is the collection of all trajectories generated by all possible starting conditions.

$\Phi(t, \mathbf{x}_0) = \mathbf{x}(t; \mathbf{x}_0)$

A fundamental theorem guarantees that **two orbits corresponding to two different initial solutions never intersect with each other**, except at equilibrium

Basic Properties

An ODE is linear if

- $\mathbf{f}(\mathbf{x}) = A\mathbf{x}$ (Homogeneous)
- $\mathbf{f}(\mathbf{x}) = A\mathbf{x} + b$ (Affine)

Linear ODE enjoys closed form solutions, non-linear ODEs usually not

A system is autonomous if time doesn't appear in expression of $\mathbf{f}$.

Facts:

- Any n-order ODE can be rewritten as a system of 1st order ODEs in $\mathbb{R}^n$
- Any Non-Autonomous ODE can be rewritten as an autonomous one

So we will focus on 1st order, autonomous and linear ODEs

Solving!

General form of linear ODE:

$$\dot{\mathbf{x}} = A\mathbf{x}, \quad x \in \mathbb{R}^n$$

A solution is a function $\mathbf{x}(t)$ that satisfies the vector field A.

(Lots of derivation out is skipped, here's how to solve)

1. Solve $\det(A - \lambda I) = 0$ for λ
2. The roots λ_i are eigenvalues of A
3. For each λ_i, there exists a non-null eigenvector $\mathbf{u}_i$
4. Together they yield one solution: $\mathbf{x}(t) = \mathbf{u}_i e^{\lambda_i t}$
5. Each distinct eigen-pair gives ONE independent vector solution
6. The general solution is then the combination of these terms: $\mathbf{x}(t) = c_1 e^{\lambda_1 t} \mathbf{u}_1 + \cdots + c_n e^{\lambda_n t} \mathbf{u}_n$ (at most n terms)

Important: the above is strictly true only if all eigenvalues are distinct

Matrix Exponential representation: $\mathbf{x}(t) = e^{At}\mathbf{x}(0)$ where $\mathbf{x}(0)$ is a generic initial condition

Exponentials and Asymtotic Behavior

Since the solution is a sum of exponentials, stuff is being pulled in the direction of the eigenvectors, weighted by their corresponding signed eigenvalues.

If the real part of $\lambda_i > 0$, mode i is unstable/diverging.

If the real part of $\lambda_i < 0$, mode i is stable/contracting.

At each point, the solution **mixes** the modes.

Equilibrium points

A state $\mathbf{x}_e$ is an equilibrium state of a system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ if when at a time t_0 the sytem is at $\mathbf{x}_e$ then it stays there FOREVER

Why? Velocity of the field in $\mathbf{x}_e$ is null: $\mathbf{f}(\mathbf{x}_e) = 0$

For a linear ODE, the equilibrium points are the points of the **Null Space** (solutions to $A\mathbf{x} = 0$) Theres one trivial solution at $\mathbf{x} = \mathbf{0}$ if A is invertible, o.w. inifnitely many solutions

Taxonomy of equilibria

Equilibrium Type	System's Behavior	Trajectories
Equilibrium state	If at or arrives at $\mathbf{x_e}$, it stays at $\mathbf{x_e}$	Trajectory is constant: $\mathbf{x}(t) = \mathbf{x_e}$
Stable equilibrium (Lyapunov)	If started close to $\mathbf{x_e}$, stays close to $\mathbf{x_e}$ forever	Nearby trajectories remain in a neighborhood of $\mathbf{x_e}$
Asymptotically stable equilibrium	If started close to $\mathbf{x_e}$, stays close to $\mathbf{x_e}$ and moves toward $\mathbf{x_e}$ as $t \to \infty$	Nearby trajectories converge to $\mathbf{x_e}$
Unstable equilibrium	Even if started very close to $\mathbf{x_e}$, eventually diverges from $\mathbf{x_e}$	Nearby trajectories diverge from $\mathbf{x_e}$

TODO: go back and add dissipative/conservative flows or maybe not.

Attractors too. (But i think it will come up again soon)

Linear System Classification by Eigenvalues

(for the saddle case, keep in mind a product is negative only if exaclty one of the numbers is negative)

Eigenvalues	Critical Point	Stability
$r_1, r_2 > 0$	Node (real, distinct)	Unstable
$r_1, r_2 < 0$	Node (real, distinct)	Asymptotically stable
$r_1 r_2 < 0$	Saddle	Unstable
$r_1 = r_2 \neq 0$	Node / Improper node	Same as sign of r_1
$r_{1,2} = \lambda \pm i\mu$	Spiral (focus)	Same as sign of λ
$r_{1,2} = \pm i\mu$	Center	Neutrally stable

Perturbations

For Pure Imaginary Eigenvalue, small perturbations add a tiny real part to the eigenvalues:

- $\lambda > 0$ (positive real part) $\implies$ trajectories spiral outward (unstable spiral).
- $\lambda < 0$ (negative real part) $\implies$ trajectories spiral inward (stable spiral).

For Repeated Real Eigenvalues

- If the eigenvectors are linearly independent, the system stays a node, but may change to a saddle if the signs differents.

- If the eigenvectors are linearly dependent (a degenerate node), a small perturbation will typically turn it into either a spiral (if eigenvalues become complex) or a node (if eigenvalues become distinct real numbers).

Linearization around a Critical Point

Consider a nonlinear system:
$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n$$

Let $\mathbf{x}_0$ be a critical point: $\mathbf{f}(\mathbf{x}_0) = 0$.

Step 1: Linear Approximation

$$\mathbf{f}(\mathbf{x}) \approx \mathbf{f}(\mathbf{x}_0) + J_{\mathbf{f}}(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)$$

Since $\mathbf{f}(\mathbf{x}_0) = 0$, the linearized system is

$$\dot{\mathbf{x}} \approx J_{\mathbf{f}}(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0), \quad \mathbf{y} = \mathbf{x} - \mathbf{x}_0$$

Step 2: Jacobian Matrix

$$J_{\mathbf{f}}(\mathbf{x}_0) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{\mathbf{x}=\mathbf{x}_0}$$

Step 3: Eigenvalues, Eigenvectors, and General Solution

Compute eigenvalues λ_i and eigenvectors $\mathbf{v}_i$ of $J_{\mathbf{f}}(\mathbf{x}_0)$. The general solution of the linearized system is

$$\mathbf{y}(t) = \sum_{i=1}^{n} c_i e^{\lambda_i t} \mathbf{v}_i, \quad \mathbf{x}(t) = \mathbf{x}_0 + \mathbf{y}(t)$$

Step 4: Stability Analysis

Stability is determined by the eigenvalues of the Jacobian $J_{\mathbf{f}}(\mathbf{x}_0)$, just like in the linear case.

Global Behavior and Nullclines

- **Basin of Attraction:** The set of all initial conditions that eventually lead a trajectory to the same stable equilibrium point.

- **Separatrix:** The boundary between different basins of attraction.

- **Isoline:** The set of points where a function takes the same value.

- **Isocline:** The set of points where the vector field has the same **slope**.

- **Nullcline:** A specific type of isocline. It's the set of points where the slope is either zero or infinite (i.e., where one component of the vector field is zero).

Nullcline fun facts:

1. The x-**nullcline** is the curve where $\dot{x} = f(x, y) = 0$.

2. The y-**nullcline** is the curve where $\dot{y} = g(x,y) = 0$.

3. On the x-nullcline, the vector field has no horizontal component, so vectors can only point vertically (up or down).

4. On the y-nullcline, the vector field has no vertical component, so vectors can only point horizontally (left or right).

5. **Equilibrium points** are exactly at the intersection of the x-nullclines and y-nullclines (since $\dot{x} = 0$ and $\dot{y} = 0$ simultaneously).

6. Nullclines divide the phase space into regions. In each region, the sign of $\dot{x}$ and $\dot{y}$ is constant.

Limit cycles

A periodic orbit is just any trajectory that forms a closed loop.

A limit cycle is an isolated closed trajectory. This means neighboring trajectories are not closed; they either spiral into the limit cycle (a stable limit cycle) or spiral away from it (an unstable limit cycle). There's also mixed scenarios (half-stable)

Example: Van der Pol Oscillator

2D is Boring

Theorem 1. Any closed trajectory muyst encolse at least one equilibrium

Theorem 2. Poincare-Bendixson Theorem:

IF a trajectory is trapped in a closed, bounded region R,

AND this region R contains no equilibrium points,

THEN the trajectory must eventually approach a limit cycle.

In other words, 2D systems cannot have chaos

Chaos and Strange Attractors

3 main types of attractors: Fixed Points, Limit Cycles and Strange Attractors (the last appears only in 3D+)

Properties of strange attractors:

- Sensitive dependence on initial conditions: two initial conditions very close to each other become very far apart as time goes on (but remain confined in the set that defines the attractor)
- Fractal dimensions (e.g. 2.06). Non-integer

Example: Lorenz Attractor

1. For a parameter r=21, trajectories spiral into one of two stable fixed points.

2. For r=28, the fixed points become unstable. Trajectories are still bounded, but they never settle down. They move from one "wing" of the attractor to the other in an aperiodic, unpredictable way.

Definition 1. Chaos: **aperiodic long-term behavior** in a **deterministic** system that exhibits **sensitive dependence on initial conditions**

Iterated Maps

Intro

Discrete-time dynamical systems have a state which is only defined at integer steps.

The state is:
$$\mathbf{x}(n) = (x_1(n), \ldots, x_k(n))$$
where k is the dimension of the system and n is an integer step parameter

and the rule is:

$$\mathbf{x}(n) = (f)((\mathbf{x}(n-1), \ldots, \mathbf{x}(n-m)))$$

(so the system here depends on m previous states)

Can also be written nicely as $\mathbf{x}_n = \mathbf{f}(\mathbf{x}_{n-1}, \ldots, \mathbf{x}_{n-m})$

Next state is obtained by directly apply the map $\mathbf{f}$

1D Iterated Maps and Cobweb Plots

We focus on the simplest case: a 1D map $x_n = f(x_{n-1})$

- **Orbit:** sequence of points generated starting at x_0 and keep applying the map

- **Fixed point:** Point that maps to itself: $f(x^*) = x^*$

Sawtooth diagram: Plotting the $(x, f(x))$ diagram. No time shown, all it tells you is if the system is at x now, the next state will be at $f(x)$

Cobweb Plot construction algorithm:

1. Draw $y = f(x)$ and $y = x$

2. Start at initial point x_0 on horizontal axis

3. Move **vertically** to $y = f(x)$ (this is point x_0, x_1)

4. Move **horizontally** to $y = x$ (this is point x_1, x_1)

5. Move **vertically** to $y = f(x)$ again (this is point x_1, x_2)

6. Keep going lil bro

The intersections of $y = f(x)$ and $y = x$ are fixed points. We can analyze stability of a fixed point by looking at the plot.

Stability of fixed points

Some simple derivation because its interesting:

Consider a point near a fixed point, $x_n = x^* + \epsilon_n$. Next point $x_{n+1} = f(x^* + \epsilon_n) = f(x^*) + f'(x^*)\epsilon_n + O(\epsilon_n^2)$

So a linear approximation shows that $\epsilon_{n+1} = f'(x^*)\epsilon_n$. Let $\lambda = f'(x^*)$

Solving: $\epsilon_n = \lambda^n \epsilon_0$. If you know some basics you can infer stability from this alone but I will write a table.

$\lambda = f'(x^*)$	Stability Type	Behavior Near x^*
Stable (Attracting)		
$0 < \lambda < 1$	Monotonic	Converges from one side
$-1 < \lambda < 0$	Oscillatory	Zig-zag convergence (alternating sides)
$\lambda = 0$	Superstable	Very fast convergence
Unstable (Repelling)		
$\lambda > 1$	Monotonic	Diverges from one side
$\lambda < -1$	Oscillatory	Alternating divergence
Marginal		
$|\lambda| = 1$	Neutral	Linearization inconclusive

Logistic map and chaos

Logistic Map: $x_{n+1} = rx_n(1 - x_n)$ where $x_n \in [0, 1]$ is population, $r \in [0, 4]$ is growth rate

- $1 < r < 3$: The population converges to a single, stable fixed point $x^* = 1 - 1/r$.

- $r = 3$: The fixed point becomes unstable.

- $3 < r < 3.449\ldots$: The system no longer settles to one point. It settles into a stable period-2 cycle, oscillating between two values. This is a bifurcation.

- $r > 3.449\ldots$: The 2-cycle becomes unstable and splits into a stable period-4 cycle. This continues, creating an 8-cycle, 16-cycle, etc., in a period-doubling cascade.

- $r > r^{\inf} \approx 3.5699\ldots$: The cascade finishes, and the system enters the chaotic regime. The orbit becomes aperiodic, never settling down and seemingly random.

This behavior can be summarized in an orbit diagram .

- The x-axis is the parameter r.

- The y-axis plots the long-term attractor points for that r.

A **bifurcation** is a qualitative change in the long-term behavior of a system with a smooth variation of a parameter

A bunch of math deriving Lyapunov Exponents, but here is the final result:

$$\lambda = \lim_{n \to \inf} \frac{1}{n} \sum_{k=0}^{n-1} \ln |f'(x_k)|$$

The approximation for λ can be constructed numerically, by iterating the map!

- $\lambda < 0$: For stable fixed points and cycles

- $\lambda > 0$: For chaotic attractors

- $\lambda = 0$: This is the marginal case, which occurs at bifurcation points.

- λ is the same for all points in the basin of attraction of an attractor