15-382 Collective Intelligence

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First Edition

Disclaimer

This document aims to summarize the content of the slides for 15-382, including what the author considers important. As always, the definition of important is highly subjective so the author might have omitted something that was important to another person, or included something that is trivial to another.

Good luck,
SH

Dynamical Systems

Fingerprints of Complex Systems

- Multi-agent / multi-component
- Decentralized
- Local interactions
- Dynamic

Types of Abstract Models:

- **Agent-based:** mechanistic implemtation of the multi-compenent interactions
- **Mathematical (white-box)**: formally descibe the relations among the relevant components.
- **Black-box:**: Input-output pairs from the system are used to predict the output for a given input, or to tweak the internal parameters to get the desired output, no attempt made to understand how the system works on the inside
- **Statistical:** describing patterns and correlations betwen variables

Systems of ODEs

Here is a system of $n \geq 1$ Ordinary Differential Equations

$$\begin{cases} \frac{dx_1}{dt} = f_1(\mathbf{x}(t)) \\ \frac{dx_2}{dt} = f_2(\mathbf{x}(t)) \\ \vdots \\ \frac{dx_n}{dt} = f_n(\mathbf{x}(t)) \end{cases}$$

where $\mathbf{x}(t)$ is an n-dim vector.

A continuous-time Dynamical System is defined by a system of differential equations:

$$\frac{\mathbf{x}(t)}{t} = \mathbf{f}(\mathbf{x}, t; \theta)$$

where $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ specifies how each component of the state evolves as t changes. It can depend on a set of given parameters θ
Some definitions:

- Initial conditions: where the system is at the beginning of the evolution: $\mathbf{x}(t_0)$
- Phase space: space of all possible states
- Trajectory: the curve traces by $\mathbf{x}(t)$ in the phase space starting from $\mathbf{x}(t_0)$
- Solution: is in the form $\mathbf{x}(t; t_0)$ that defines a family of time trajectories in the phase space. Once we fix t_0, we fix a unique trajectory

Vector fields and flows

How are solutions built? At any point, $\mathbf{f}$ assigns a vector that shows where the point is heading (direction of motion). If we plot these arrows (vectors) in the phase space, we get can get an idea of how the system evolves.
Flow: $\Phi : \mathtt{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is the collection of all trajectories generated by all possible starting conditions.
$\Phi(t, \mathbf{x}_0) = \mathbf{x}(t; \mathbf{x}_0)$

A fundamental theorem guarantees that **two orbits corresponding to two different initial solutions never intersect with each other** , except at equilibrium

Basic Properites

An ODE is linear if

- $\mathbf{f}(\mathbf{x}) = A\mathbf{x}$ (Homogenous)
- $\mathbf{f}(\mathbf{x}) = A\mathbf{x} + b$ (Affine)

Linear ODE enjoys closed form solutions, non-linear ODEs usually not

A system is autonomous if time doesn't appear in expression of $\mathbf{f}$.
Facts:

- Any n-order ODE can be rewritten as a system of 1st order ODEs in $\mathbb{R}^n$
- Any Non-Autonomous ODE can be rewritten as an autonomous one

So we will focus on 1st order, autonomous and linear ODEs

Solving!

General form of linear ODE:

$$\dot{\mathbf{x}} = A\mathbf{x}, \quad x \in \mathbb{R}^n$$

A solution is a function $\mathbf{x}(t)$ that satisfies the vector field A.
(Lots of derivation out is skipped, here's how to solve)

1. Solve $\det(A - \lambda I) = 0$ for λ
2. The roots λ_i are eigenvalues of A
3. For each λ_i, there exists a non-null eigenvector $\mathbf{u}_i$
4. Together they yield one solution: $\mathbf{x}(t) = \mathbf{u}_i e^{\lambda_i t}$
5. Each distinct eigen-pair gives ONE independent vector solution
6. The general solution is then the combination of these terms: $\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{u}_1 + \cdots + c_n e^{\lambda_n t}\mathbf{u}_n$ (at most n terms)

Important: the above is strictly true only if all eigenvalues are distinct
Matrix Exponential representation: $\mathbf{x}(t) = e^{At}\mathbf{x}(0)$ where $\mathbf{x}(0)$ is a generic initial condition

Exponentials and Asymtotic Behavior

Since the solution is a sum of exponentials, stuff is being pulled in the direction of the eigenvectors, weighted by their corresponding signed eigenvalues.

If the real part of $\lambda_i > 0$, mode i is unstable/diverging.
If the real part of $\lambda_i < 0$, mode i is stable/contracting.
At each point, the solution **mixes** the modes.

Equilibrium points

A state $\mathbf{x}_e$ is an equilibrium state of a system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ if when at a time t_0 the sytem is at $\mathbf{x}_e$ then it stays there FOREVER
Why? Velocity of the field in $\mathbf{x}_e$ is null: $\mathbf{f}(\mathbf{x}_e) = 0$

For a linear ODE, the equilibrium points are the points of the **Null Space** (solutions to $A\mathbf{x} = 0$) Theres one trivial solution at $\mathbf{x} = \mathbf{0}$ if A is invertible, o.w. inifnitely many solutions

Taxonomy of equilibria

Equilibrium Type	**System's Behavior**	**Trajectories**
Equilibrium state	If at or arrives at $\mathbf{x_e}$, it stays at $\mathbf{x_e}$	Trajectory is constant: $\mathbf{x}(t) = \mathbf{x_e}$
Stable equilibrium (Lyapunov)	If started close to $\mathbf{x_e}$, stays close to $\mathbf{x_e}$ forever	Nearby trajectories remain in a neighborhood of $\mathbf{x_e}$
Asymptotically stable equilibrium	If started close to $\mathbf{x_e}$, stays close to $\mathbf{x_e}$ and moves toward $\mathbf{x_e}$ as $t \to \infty$	Nearby trajectories converge to $\mathbf{x_e}$
Unstable equilibrium	Even if started very close to $\mathbf{x_e}$, eventually diverges from $\mathbf{x_e}$	Nearby trajectories diverge from $\mathbf{x_e}$

TODO: go back and add dissipative/conservative flows or maybe not.
Attractors too. (But i think it will come up again soon)

Linear System Classification by Eigenvalues

Consider the linear system:

$$\dot{\mathbf{x}} = A\mathbf{x}, \quad A \in \mathbb{R}^{2\times 2}$$

We classify equilibria based on the eigenvalues of A.

1. Two Distinct Real Eigenvalues of Opposite Sign
Behavior:

- Trajectories diverge along one eigenvector and converge along the other.
- Equilibrium is a **saddle point (unstable)**.

2. Two Distinct Real Eigenvalues with Same Sign
Behavior:

- **Stable node** if both eigenvalues < 0.
- **Unstable node** if both eigenvalues > 0.
- Trajectories are straight lines along eigenvectors.

3. One Repeated Real Eigenvalue, Two Independent Eigenvectors (Star / Proper Node)
Behavior:

- Repeated real eigenvalue (algebraic multiplicity = 2).
- Two linearly independent eigenvectors.
- Trajectories are straight lines toward/away from the origin.
- **Stable** if $\lambda < 0$, **unstable** if $\lambda > 0$.

4. One Repeated Real Eigenvalue, One Eigenvector (Improper / Degenerate Node)
Behavior:

- Repeated real eigenvalue with only one eigenvector.
- Requires a generalized eigenvector.
- Trajectories are curved, tangent to the direction of the eigenvector.
- **Stable** if $\lambda < 0$, **unstable** if $\lambda > 0$.

5. Complex Eigenvalues with Nonzero Real Part (Spiral / Focus)

$$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \quad \lambda_{1,2} = a \pm bi$$

$$\mathbf{x}(t) = e^{at} \begin{pmatrix} \cos(bt) & -\sin(bt) \\ \sin(bt) & \cos(bt) \end{pmatrix} \mathbf{x}(0)$$

Behavior:

- Complex conjugate eigenvalues.
- **Stable spiral** if $a < 0$, **unstable spiral** if $a > 0$.
- Trajectories rotate while exponentially approaching or diverging from origin.

6. Pure Imaginary Eigenvalues (Center)

$$A = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}, \quad \lambda_{1,2} = \pm i\omega$$

$$\mathbf{x}(t) = \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{pmatrix} \mathbf{x}(0)$$

Behavior:

- Purely imaginary eigenvalues, no real part.
- Trajectories are closed orbits (circles or ellipses).
- Neither converge nor diverge — **neutrally stable**.

Summary of types of eq

(I might remove above stuff, this is better)
(for the saddle case, keep in mind a product is negative only if exaclty one of the numbers is negative)

Eigenvalues	Critical Point	Stability
$r_1, r_2 > 0$	Node (real, distinct)	Unstable
$r_1, r_2 < 0$	Node (real, distinct)	Asymptotically stable
$r_1 r_2 < 0$	Saddle	Unstable
$r_1 = r_2 \neq 0$	Node / Improper node	Same as sign of r_1
$r_{1,2} = \lambda \pm i\mu$	Spiral (focus)	Same as sign of λ
$r_{1,2} = \pm i\mu$	Center	Neutrally stable

Perturbations

For Pure Imaginary Eigenvalue, small perturbations add a tiny real part to the eigenvalues:

- $\lambda > 0$ (positive real part) $\implies$ trajectories spiral outward (unstable spiral).
- $\lambda < 0$ (negative real part) $\implies$ trajectories spiral inward (stable spiral).

For Repeated Real Eigenvalues

- If the eigenvectors are linearly independent, the system stays a node, but may change to a saddle if the signs differents.
- If the eigenvectors are linearly dependent, it will become a spiral.

Linearization around a Critical Point

Consider a nonlinear system:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n$$

Let $\mathbf{x}_0$ be a critical point: $\mathbf{f}(\mathbf{x}_0) = 0$.

Step 1: Linear Approximation

$$\mathbf{f}(\mathbf{x}) \approx \mathbf{f}(\mathbf{x}_0) + J_{\mathbf{f}}(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)$$

Since $\mathbf{f}(\mathbf{x}_0) = 0$, the linearized system is

$$\dot{\mathbf{x}} \approx J_{\mathbf{f}}(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0), \quad \mathbf{y} = \mathbf{x} - \mathbf{x}_0$$

Step 2: Jacobian Matrix

$$J_{\mathbf{f}}(\mathbf{x}_0) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{\mathbf{x}=\mathbf{x}_0}$$

Step 3: Eigenvalues, Eigenvectors, and General Solution

Compute eigenvalues λ_i and eigenvectors $\mathbf{v}_i$ of $J_{\mathbf{f}}(\mathbf{x}_0)$. The general solution of the linearized system is

$$\mathbf{y}(t) = \sum_{i=1}^{n} c_i e^{\lambda_i t}\mathbf{v}_i, \quad \mathbf{x}(t) = \mathbf{x}_0 + \mathbf{y}(t)$$

Step 4: Stability Analysis

Analyze how you would analyze a normal linear system.

Global Behavior and Nullclines

- **Basin of Attraction:** set of all initial conditions that
- **Separatrix:** boundary between different basins of attraction eventually lead a trajectory to the same stable equilibrium point
- **Isoline:** set of points where a function takes the same value
- **Isocline:** set of points where a function has the same **slope**
- **Nullcline:** set of point where a function has the same **null slope**

Nullcline fun facts:

1. The x-nullcline is curve where $\dot{\mathbf{x}} = 0$, the y-nullcline is curve where $\dot{\mathbf{y}} = 0$
2. On x-nullcline, the vector field can only point vertically (no horizontal motion).
3. On y-nullcline, the vector field can only point horizontally (no vertical motion).
4. Equilibrium points are exaclty at the intersection of the nullclines
5. Nullclines divide the phase space into regions. In each region, the sign of $\dot{\mathbf{x}}$ and $\dot{\mathbf{y}}$ is constant.

Limit cycles

A periodic orbit is just any trajectory that forms a closed loop.
A limit cycle is an isolated closed trajectory. This means neighboring trajectories are not closed; they either spiral into the limit cycle (a stable limit cycle) or spiral away from it (an unstable limit cycle). There's also mixed scenarios (half-stable)
Example: Van der Pol Oscillator

2D is Boring

Theorem 1. Any closed trajectory muyst encolse at least one equilibrium

Theorem 2. Poincare-Bendixson Theorem:
IF a trajectory is trapped in a closed, bounded region R,
AND this region R contains no equilibrium points,
THEN the trajectory must eventually approach a limit cycle.

In other words, 2D systems cannot have chaos

Chaos and Strange Attractors

3 main types of attractors: Fixed Points, Limit Cycles and Strange Attractors (the last appears only in 3D+)
Properties of strange attractors:

- Sensitive dependence on initial conditions: two initial conditions very close to each other become very far apart as time goes on (but remain confined in the set that defines the attractor)
- Fractal dimensions (e.g. 2.06). Non-integer

Example: Lorenz Attractor

1. For a parameter r=21, trajectories spiral into one of two stable fixed points.
2. For r=28, the fixed points become unstable. Trajectories are still bounded, but they never settle down. They move from one "wing" of the attractor to the other in an aperiodic, unpredictable way.

Definition 1. Chaos: **aperiodic long-term behavior** in a **deterministic** system that exhibits **sensitive dependence on initial conditions**