21-241 Matrices and Linear Transformations Notes

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Exam 3 Edition

Chapter 0

Disclaimer

This is not a textbook. Do not use this text to learn the concepts. This is a revision guide or notes used to refresh your memory of te theorems and algorithms. The best way to learn the content is to read the book, work through the guided exercises and do some practice problems.
Good luck,
SH

Chapter 1

Linear Equations in Linear Algebra

1.1 Systems of linear equations

Definition: A system of linear equations is a collection of one or more linear equations involving the same set of variables. A system has either:

- No solution
- Infinite solutions
- A unique solution

The info about a linear system can be recorded in a matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The one with just the coefficients is called **coefficient matrix**
The one with the last col with the product is called the **augmented matrix**

There are 3 elementary row operations to carry out on the matrix:

- Replacement: by a sum of itself and a multiple of another row
- Interchange: swap rows
- Scaling: scale a row by a scalar

1.2 Row Reduction and Echelon Forms

Definition: A matrix is in **echelon form** if

- All nonzero rows are above any rows of zeros
- Each leading entry of a row is in a column ro the right of the leading entry of the row above it
- All entries in a column below a leading entry are zeros

A matrix is in **reduced echelon form** if

- It is in echelon form
- The leading entry in each nonzero row is 1
- Each leading 1 is the only nonzero entry in its column

Theorem 1 (Uniqueness of the Reduced Echelon Form)**.** Each matrix is row equivalent to one and only one reduced echelon matrix

Definition 1 (Pivot Positon)**.** A **pivot position** in A is a location in A that corresponds to a leading 1 in the reduced echelon form of A

Row Reduction Algorithm

1. Start with left non-zero column. This is a pivot column. Pivot position is at the top
2. Choose a nonzero entry in the pivot column as a pivot. Interchange to move it into pivot position
3. Use row replacement to make zeros below pivot position
4. Ignore the row with the pivot and the ones above. Apply steps 1-3 to the remaining matrix until no more rows to modify
5. Starting with rightmost pivot and working up and left, make zeros above each pivot. Make every pivot a 1 by scaling

Variables corresponding to pivot columns are **basic**, the other ones are **free**

Theorem 2 (Existence and Uniqueness theorem)**.** A linear system is consistent iff the right column of the augmented matrix is NOT a pivot column i.e. iff echelon form has NO row of form:

$$\begin{bmatrix} 0 & \dots & 0 & b \end{bmatrix} \text{ with } b \text{ nonzero}$$

If a system is consistent then solution set contain either a unique solution if there are 0 free variables or infinite solutions if there is at least 1 free variable.

1.3 Vector Equations

Algebraic Properties of Vectors in $\mathbb{R}^n$

$\forall$ $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and scalars c and d:

1. **Commutativity of addition:** $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. **Associativity of addition:** $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
3. **Additive identity:** $\mathbf{u} + \mathbf{0} = \mathbf{u}$
4. **Additive inverse:** $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
5. **Distributivity of scalar multiplication over addition (vectors):** $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
6. **Distributivity of scalar multiplication over addition (scalars):** $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
7. **Associativity of scalar multiplication:** $(cd)\mathbf{u} = c(d\mathbf{u})$
8. **Scalar identity:** $1\mathbf{u} = \mathbf{u}$

A linear combination is a vector $\mathbf{y} = c_1\mathbf{v}_1 + \ldots + c_p\mathbf{v}_p$

Definition 2. Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is the set of all linear combinations of $\mathbf{v_1}, \ldots, \mathbf{v_p}$
It is the set of all vectors that can be written in the form $c_1\mathbf{v}_1 + \ldots + c_p\mathbf{v}_p$

1.4 The Matrix Equation Ax = b

Definition 3.

If A is an $m \times n$ matrix, with columns $\mathbf{a_1}, \ldots, \mathbf{a_n}$ and if $\mathbf{x} \in \mathbb{R}^n$,

$$A\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$$

This is defined only if the number of columns of A equals number of entires in $\mathbf{x}$

Theorem 3. If A is an $m \times n$ matrix and $\mathbf{b} \in \mathbb{R}^n$, then $A\mathbf{x} = \mathbf{b}$ has the same solution set as

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = b$$

which has the same solution set as the system of linear equations with matrix

$$\begin{bmatrix} \mathbf{a}_1 & \ldots & \mathbf{a}_n & \mathbf{b} \end{bmatrix}$$

Theorem 4. Let A be a $m \times n$ matrix. The following statements are equivalent

a. $\forall$ $\mathbf{b} \in \mathbb{R}^m, A\mathbf{x} = \mathbf{b}$ has a solution

b. $\forall$ $\mathbf{b}$ is a linear combination of the columns of A

c. The columns of A span $\mathbb{R}^m$

d. A has a pivot in every row

Warning: A is a **coefficient matrix** in this theorem

Theorem 5. If A is an $m \times n$ matrix and $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, and c is a scalar

a. $A\ (\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$;

b. $A\ (c\mathbf{u}) = c(A\mathbf{u})$;

1.5 Solution sets of Linear Systems

Homogenous Linear Systems: if it can be written in form $A\mathbf{x} = \mathbf{0}$
This system always has at least one solution ($\mathbf{x} = \mathbf{0}$): **trivial solution**
The equation has a **non-trivial** solution iff it has at least one free variable

Non-homogenous Linear Systems: if it can be written in form $A\mathbf{x} = \mathbf{b}$, $\mathbf{b}$ non-zero

Theorem 6. Let $A\mathbf{x} = \mathbf{b}$ be consistent for some $\mathbf{b}$, let $\mathbf{p}$ be a solution. Then the solution set is the set of all vectors in form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$, where $\mathbf{v}_h$ is any solution of $A\mathbf{x} = \mathbf{0}$

Writing a solution set in parametric vector form

1. Row reduce to reduced echelon form
2. Express each basic variable in terms of any free variables
3. Write a typical solution as a vector whose entries depend on free variables
4. Decompose it into a linear combination of vectors (with numbers only inside) using free variables as parameters

1.7 Linear Independence

Definition 4. $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is **linearly independent** if

$$x_1\mathbf{v}_1 + \ldots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution. It is said to be **linearly dependent** if $\exists c_1, \ldots, c_p$, not all zero, such that

$$c_1\mathbf{v}_1 + \ldots + c_p\mathbf{v}_p = \mathbf{0}$$

A set of two vectors is linearly dependent if at least one is a multiple of the other

Theorem 7 (Characterization of Linearly independent Sets)**.** A set of two or more vectors is linearly dependent iff at least one of them is a linear combination the others.

Theorem 8. If a set contains more vectors than entries in each vector, then it is linearly dependent

Theorem 9. If a set contains the zero vector, then it is linearly dependent

1.8 Introduction to Linear Transformations

A **transformation** T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns each vector $\mathbf{x}$ in $\mathbb{R}^n$ to a vector $T(\mathbf{x})$ in $\mathbb{R}^m$

- Domain: $\mathbb{R}^n$
- Co-domain: $\mathbb{R}^m$
- Range: set of all images $T(\mathbf{x})$

Definition 5. T is linear iff

- $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
- $T(c\mathbf{u}) = cT(\mathbf{u})$

If T is linear then:

- $T(\mathbf{0}) = \mathbf{0}$
- $T(c\mathbf{u}+d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$
- $T(c_1\mathbf{v}_1 + \ldots + c_p\mathbf{v}_p) = T(c_1\mathbf{v}_1) + \ldots + T(c_p\mathbf{v}_p)$

1.9 The Matrix of a Linear Transformation

Theorem 10. For a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$, there exists a unique matrix A such that

$$T(\mathbf{x}) = A\mathbf{x} \quad \forall\, \mathbf{x} \in \mathbb{R}^n.$$

This matrix A is given by

$$A = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \ldots & T(\mathbf{e}_n) \end{bmatrix},$$

where $\mathbf{e}_j$ denotes the j-th column of the identity matrix in $\mathbb{R}^n$.

Definition 6. T is **onto** (surjective) if

- Range of T = Co-domain
- $\forall\ \mathbf{v} \in \mathit{codomain}(B), \exists\ \mathbf{u} \in \mathit{domain}(A)$ such that $T(\mathbf{u}) = \mathbf{v}$
- Find standard matrix
$$A = \begin{bmatrix} T(c_1) & \ldots & T(c_n) \end{bmatrix}$$
Onto if $A\mathbf{u} = \mathbf{v}$ has a solution for each $\mathbf{v}$ in range

Definition 7. T is **one-to-one** (injective)if

- $T(\mathbf{u}_1) = T(\mathbf{u}_2) \implies \mathbf{u}_1 = \mathbf{u}_2$
- Find standard matrix
$$A = \begin{bmatrix} T(c_1) & \ldots & T(c_n) \end{bmatrix}$$
$A\mathbf{u} = \mathbf{b}$ has a unique solution for each $\mathbf{b} \in \mathit{Range}(T)$

Theorem 11. T is one-to-one iff $T(\mathbf{x}) = \mathbf{0}$ has a ONLY the trivial solution

Theorem 12. Let $T : \mathbb{R}^n \to \mathbb{R}^m$, let A be standard matrix for T. Then

- T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ iff the columns of A span $\mathbb{R}^m$
- T is one-to-one iff the columns of A are linearly independent

Chapter 2

Matrix Algebra

2.1 Matrix Operations

- **Square Matrix**: row and columns are the same
- **Diagonal entries**: $a_{11}, a_{22}, \ldots$
- **Diagonal matrix**: square $n \times n$ matrix who's non-diagonal entires are zero
- **Zero matrix**: $m \times n$ matrix where all entries are 0

$[a_{ij}]_{m\times n} \pm [b_{ij}]_{m\times n} = [a_{ij} \pm b_{ij}]_{m\times n}$

Two matrices are **equal** if they have the same size and their entries are equal (duh)

Theorem 1. Properties of Matrices

Let A, B and C be matrices of the same size, and let r and s be scalars

- $A + B = B + A$
- $(A + B) + C = A + (B + C)$
- $A + 0 = A$
- $r(A + B) = rA + rB$
- $(r + s)A = rA + sA$
- $r(sA) = (rs)A$

Definition 1. If A is a $m \times n$ matrix and B is an $n \times p$ matrix with columns $\mathbf{b}_1, \ldots, \mathbf{b}_p$, then

$$AB = A\begin{bmatrix}\mathbf{b}_1 \ldots \mathbf{b}_p\end{bmatrix} = \begin{bmatrix}A\mathbf{b}_1 \ldots A\mathbf{b}_p\end{bmatrix}$$

- Each column of AB is a linear combination of the columns of A using weights from columns of B
- **The number of columns of A must match the number of rows in B**
- Row column rule: $(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}$
- $row_i(AB) = row_i(A) \cdot B$

Theorem 2. More Properties of Matrices

Let A be an $m \times n$ matrix, and let B and C have sizes for which the indicated sums and products are defined.

$A(BC) = (AB)C$ (associative law of multiplication)

$A(B + C) = AB + AC$ (left distributive law)

$(B + C)A = BA + CA$ (right distributive law)

$r(AB) = (rA)B = A(rB)$ for any scalar r

$I_m A = A = AI_n$ (identity for matrix multiplication)

WARNING

Let A be an $m \times n$ matrix, and let B and C have sizes for which the indicated sums and products are defined.

- In general, $AB \neq BA$
- In general, $AB = AC$ **does not imply** $B = C$
- In general, $AB = 0$ **does not imply** $A = 0 \vee B = 0$

Power of a matrix: $A^k = \underbrace{AA \dots A}_{k}$

The transpose of a matrix: A^T is obtained by swapping its rows and columns of A.

Theorem 3. Transpose Properties

- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$
- $(rA)^T = rA^T$
- $(AB)^T = B^T A^T$

2.2 The Inverse of a Matrix

An $n \times n$ matrix A is said to be **invertible** if there is an $n \times n$ matrix C such that

$$CA = I_n \quad \text{and} \quad AC = I_n,$$

In this case, C is an **inverse** of A

$$A^{-1}A = I \text{ and } AA^{-1} = I$$

Theorem 4. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ If $ad - bc \neq 0$, then A is invertEBIL and

$$A^-1 = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$\det A = ad - bc$

Theorem 5. If A is invertible, then $\forall$ $\mathbf{b} \in \mathbb{R}^n$, $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$

Theorem 6. 1. If A invertible, then A^{-1} is invertEbil and $(A^{-1})^{-1} = A$

2. If A and B invertible, $(AB)^{-1} = B^{-1}A^{-1}$

3. If A invertible, $(A^T)^{-1} = (A^{-1})^T$

- An **elementary matrix** is a matrix obtained by performing a single elementary row operation on I
- If a row op is performed on A, the result can be written as EA, where E is made by performing the same row op on I
- Each E is an invertible and its inverse is the elementary matrix that transforms E back into I.

Theorem 7. A is invertible iff A is row equivalent to I, and any sequence of row ops that reduces A to I also transforms I to A^{-1}

Finding A^{-1}: Row reduce $\begin{bmatrix} A & I \end{bmatrix}$. If A row-eq to I then $\begin{bmatrix} A & I \end{bmatrix}$is row-eq to $\begin{bmatrix} I & A^{-1} \end{bmatrix}$

2.3 Characterizations of Matrices

Theorem 8. The Invertible Matrix Theorem

The following statements about a square n by n matrix A are equivalent

- A is an invertible matrix.
- A is row equivalent to the $n \times n$ identity matrix.
- A has n pivot positions.
- The equation $Ax = 0$ has only the trivial solution.
- The columns of A form a linearly independent set.
- The linear transformation $x \mapsto Ax$ is one-to-one.
- The equation $Ax = b$ has at least one solution for each $b \in \mathbb{R}^n$.
- The columns of A span $\mathbb{R}^n$.
- The linear transformation $x \mapsto Ax$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.
- There is an $n \times n$ matrix C such that $CA = I$.
- There is an $n \times n$ matrix D such that $AD = I$.
- A^T is an invertible matrix.

If $AB = I$, then A and B invertible with $B = A^{-1}$ and $A = B^{-1}$
A linear transformation is invertible if $\exists S : \mathbb{R}^n \to \mathbb{R}^n$ such that:

$$\begin{aligned} S(T(\mathbf{x})) = x \quad & \text{for all } \mathbf{x} \in \mathbb{R}^n \\ T(S(\mathbf{x})) = x \quad & \text{for all } \mathbf{x} \in \mathbb{R}^n \end{aligned}$$

Theorem 9. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ and A be the matrix. T is invertible iff A is invertible. Then $S(\mathbf{x}) = A^{-1}\mathbf{x}$

2.4 Partitioned Matrices

Trivial stuff honestly just partition and multiply.
Ok actually there is one good theorem here

Theorem 10 (Column-Row Expansion of AB)**.** If A is $m \times n$ and B is $n \times p$:

$$AB = \begin{bmatrix} col_1(A) & \dots & col_n(A) \end{bmatrix} \begin{bmatrix} row_1(B) \\ \vdots \\ row_n(B) \end{bmatrix} = col_1(A)row_1(B) + \dots + col_n(A)row_n(B)$$

The inverse of a block upper triangular matrix:

$$A^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}^{-1} = \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{22}^{-1} \\ 0 & A_{22}^{-1} \end{bmatrix}$$

2.5 Matrix Factorizations

Assume A is $m \times n$ and can be reduced to Echelon Form without row interchanges.
Then $A = LU$,

- $L : m \times m$ lower triangular with 1's in diagonal
- $U : m \times n$ echelon form of A

$A\mathbf{x} = \mathbf{b}$ can be written as $L(U\mathbf{x}) = \mathbf{b}$. Writing $\mathbf{y} = U\mathbf{x}$, we get:

$$L(U\mathbf{x}) = \mathbf{b}$$

$$\begin{aligned} L\mathbf{y} &= \mathbf{b} \\ U\mathbf{x} &= \mathbf{y} \end{aligned}$$

We can get $\mathbf{x}$ by solving the easy pair of equations:

1. Solve $L\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$
2. Solve $U\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$

LU Factorization Algorithm (boring)

1. Reduce A to echelon form U using a sequence of row replacement operations
2. Place entries in L such that the same sequence of row operations reduces L to I

LU Factorization Algorithm (fast)

For a matrix $A : m \times n$

1. Initialize L as the identity matrix $m \times m$ and U as A.
2. For each row $i = 1, 2, \ldots, m$:
   - Use row operations to create zeros below the diagonal in U by subtracting multiples of the current row from the rows below.
   - For each row operation, divide the corresponding entries below the pivot in column i by the pivot, and store these multipliers in the corresponding positions of L.
3. Continue this process until U is in echelon form and L has recorded the multipliers such that $A = LU$.

This description is very bad, please practice this on a non-square matrix

Chapter 3

Determinants

3.1 Intro to Determinants

First, we denote A_{ij} to be the sub-matrix of A formed by deleting the ith row and jth column of A

Definition 1. For $n \geq 2$, the **determinant** of a $n \times n$:

$$\begin{aligned}\det A &= a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + \ldots + (-1)^{1+n}a_{1n}\det(A_{1n}) = \\ &= \sum_{j=1}^{n}(-1)^{1+j}a_{1j}\det(A_{1j})\end{aligned}$$

The (i, j)-cofactor of A: $C_{ij} = (-1)^{i+j}\det(A_{ij})$

Theorem 1. The determinant of A can be computed by cofactor expansion across any row or column.
Across row i:

$$\det(A) = a_{i1}C_{i1} + \ldots + a_{in}C_{in}$$

Across column j:

$$\det(A) = a_{1j}C_{1j} + \ldots + a_{nj}C_{nj}$$

Signs of the cofactor:

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \\ + & - & + & \\ \vdots & & & \ddots \end{bmatrix}$$

Theorem 2. If A triangular, then $\det(A)$ is the product of the entires on the main diagonal of A

3.2 Properties of Determinants

Theorem 3. Row Operations

For a matrix $A : m \times n$

a. If a multiple of a row in A is added to another to produce B: $\det(B) = \det(A)$

b. If two rows of A are swapped to produce B: $\det(B) = -\det(A)$

c. If a row of A is multiplied by k to produce B: $\det(B) = k \cdot \det(A)$

Two main strats where this is useful:

- Row reduce a matrix to echelon form to simplify calculation (without row interchange!)
- Factor out a common factor from a row to simplify calculation

Theorem 4. A is invertible iff $\det(A) \neq 0$ (trivial)

Theorem 5. $\det(A) = \det(A^T)$

Theorem 6. If A and B $n \times n$, then $\det(AB) = \det(A) \cdot \det(B)$

Lemma 1. $\det(A^{-1}) = \frac{1}{\det(A)}$

3.3 Crammer's Rule

Let $A_i(\mathbf{b})$ be the matrix obtained from A by replacing column i by the vector $\mathbf{b}$

Theorem 7 (Cramer's Rule)**.** Let $A : n \times n$, invertible, the solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$ is given by:

$$x_i = \frac{\det(A_i(\mathbf{b}))}{\det(A)}, \text{ for } i = 1, 2, \ldots, n$$

The matrix of cofactors is called the **adjugate or classical adjoint** of A, denoted by adj A

Theorem 8 (an Inverse Forumla)**.**

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} C_{11} & C_{21} & \ldots & C_{n1} \\ C_{12} & C_{22} & \ldots & C_{n2} \\ \vdots & \vdots & & \\ C_{1n} & C_{2n} & \ldots & C_{nn} \end{bmatrix} = \frac{adj(A)}{\det(A)}$$

Theorem 9. If $A : 2 \times 2$, Area of parallelogram determined by columns of A is $\det(A)$
If $A : 3 \times 3$, Volume of parallelepiped determined by columns of A is $\det(A)$

Chapter 4

Vector Spaces and Subspaces

4.1 Vector Spaces and Subspaces

Definition 1. A **vector space** is a nonempty set V of objects called vectors, on which there are defined two operations: *addition* and *scalar multiplication*. The axioms below must hold for all $\mathbf{u}, \mathbf{w}, \mathbf{v} \in V$

Taweel Axioms

1. $\mathbf{u} + \mathbf{v} \in V$ (Closure under addition)
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
4. $\mathbf{0} \in V$, such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$
5. $\forall\ \mathbf{u} \in V$, $\exists\ -\mathbf{u} \in V$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
6. $c\mathbf{u} \in V$
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$
10. $1\mathbf{u} = \mathbf{u}$

Definition 2. A **subspace** of a space V is a subset H of V that:

- The zero vector of V is in H
- H is closed under vector addition: $\mathbf{u} + \mathbf{v} \in H$ for all $\mathbf{u}, \mathbf{v} \in H$
- H is closed under scalar multiplication: $c\mathbf{u} \in H$ for all scalars c and vectors $\mathbf{u} \in H$

Theorem 1. If $\mathbf{v}_1, \dots, \mathbf{v}_p$ are in a vector space V, then Span $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is a subspace of V

4.2 Null Spaces, Column Spaces, Row Spaces and Linear Transformations

Definition 3. The **null space** of an $m \times n$ matrix A, Nul A, is the set of all solutions to $A\mathbf{x} = \mathbf{0}$

$$\text{Nul } A = \{\mathbf{x} : \mathbf{x} \in \mathbb{R}^n \text{ and } A\mathbf{x} = \mathbf{0}\}$$

Theorem 2. For $A : m \times n$, Nul A is a subspace of $\mathbb{R}^n$

Definition 4. The **column space** of an $m \times n$ matrix A, Col A, is the set of all linear combinations of the columns of A. If $A = \begin{bmatrix} a_1 & \dots a_n \end{bmatrix}$ then

$$\text{Col } A = \text{Span } \{a_1 \dots a_n\}$$

Theorem 3. For $A : m \times n$, Col A is a subspace of $\mathbb{R}^n$

The column space of $A : m \times n$ is all of $\mathbb{R}^m$ iff $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b} \in \mathbb{R}^n$

Definition 5. The **row space** of an $m \times n$ matrix A, Row A, is the set of all linear combinations of the rows of A.

Definition 6. A **linear transformation** T from a vector space V into a vector space W is a mapping from $\mathbf{x} \in V$ to a unique vector $T(\mathbf{x}) \in W$ such that

(i) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in V, and

(ii) $T(c\mathbf{u}) = cT(\mathbf{u})$ for all $\mathbf{u}$ in V and all scalars c.

Definition 7. The **kernel** of T is the set of all $\mathbf{u} \in V$ such that $T(\mathbf{u}) = \mathbf{0}$

Definition 8. The **range** of T is the set of all $T(\mathbf{x}) \in W$ for some $\mathbf{x} \in V$

4.3 Linearly Independent Sets; Bases

Recap

The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is **linearly independent** if the vector equation $c_1\mathbf{v}_1 + \ldots + c_p\mathbf{v}_p = \mathbf{0}$ has **only** the trivial solution

Theorem 4. An indexed set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ of 2 or more vectors, $v_1 \neq \mathbf{0}$, is linearly dependent iff some $\mathbf{v}_j : j > 1$ is a linear combination of preceding vectors $\mathbf{v}_1, \ldots . \mathbf{v}_{j-1}$

Definition 9. Let H be a subspace of V. An set of vectors $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ is a **basis** for H iff

(i) $\mathcal{B}$ is a linearly independent set

(ii) $H = \text{Span}\ \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$

4.3.1 The Spanning Set Theorem

Theorem 5. Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$, $H = \text{Span}\ \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$

a. If one of the vectors is a linear combination of the remaining vectors, then the set formed by removing from S it still spans H.

b. If $H \neq \{\mathbf{0}\}$, some subset of S is a basis for H

4.3.2 Bases for Nul A and Col A

Bases for Nul A: the usual method (solve $A\mathbf{x} = \mathbf{0}$ and write the solution set) already provides the basis

Bases for Col A: columns of A have **exactly the same linear dependence relationships** as columns of B (reduced echelon form of A). For example, if $\mathbf{b}_2 = 4\mathbf{b}_1$ then $\mathbf{a}_2 = 4\mathbf{a}_1$

Theorem 6. The pivot columns of a matrix A forms a basis for Col A

WARNING: MAKE SURE TO USE THE PIVOT COLUMNS OF A ITSELF, NOT OF ITS ROW REDUCED COUNTERPART. ROW OPERATIONS CAN CHANGE THE COLUMNS SPACE OF A MATRIX.

Theorem 7 (Worthwillity of Row Reduction)**.** If two matrices A and B are row-equivalent, then **their row spaces are the same**. If B is echelon the nonzero rows of B form a basis for the row space of both A and B.

Two views of a basis

- A basis is a spanning set that is as small as possible
- A basis is a linearly independent set that is as large as possible

4.4 Coordinate Systems (Counts as a systems elective)

Theorem 8 (The Unique Ebil Theorem). Let $\mathcal{B}$ be a basis for a vector space V. $\forall \mathbf{x} \in V$, there exists a unique set of scalars $c_1, \ldots, c_n$ such that

$$\mathbf{x} = c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n$$

Definition 10. Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V and $\mathbf{x} \in V$. The **coordinates of x relative to the basis** $\mathcal{B}$ are the weights $c_1, \ldots, c_n$ such that $\mathbf{x} = c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n$

If $c_1, \ldots, c_n$ are the $\mathcal{B}$-Coordinates of $\mathbf{x}$ then

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

is the **B-coordinate vector of x**
The mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is the **coordinate mapping**

When a basis for $\mathbb{R}^n$ is fixed, the $\mathcal{B}$-coordinate vector of an $\mathbf{x}$ becomes trivial.
Basically:

$$\begin{aligned} \mathcal{B} &= \{\mathbf{b}_1, \ldots, \mathbf{b}_n\} \\ [\mathbf{x}]_{\mathcal{B}} &=? \\ \\ \mathbf{x} &= c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n \\ \mathbf{x} &= \begin{bmatrix} \mathbf{b}_1 & \ldots & \mathbf{b}_n \end{bmatrix} \begin{bmatrix} c_1 \\ \ldots \\ c_n \end{bmatrix} \end{aligned}$$

This equation can be solved by row reduction or another method, which will give you the value of $c_1, \ldots, c_n$, which makes

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \ldots \\ c_n \end{bmatrix}$$

Let $P_{\mathcal{B}} = \begin{bmatrix} \mathbf{b}_1 & \ldots & \mathbf{b}_n \end{bmatrix}$ Then the vector equation for $\mathbf{x}$ becomes

$$\mathbf{x} = P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}$$

$P_{\mathcal{B}}$ is the **change-of-coordinate matrix** from $\mathcal{B}$ to the standard basis in $\mathbb{R}^n$. Left multiplication by it transforms $[\mathbf{x}]_{\mathcal{B}}$ into $\mathbf{x}$.
Likewise:

$$P_{\mathcal{B}}^{-1}\mathbf{x} = [\mathbf{x}]_{\mathcal{B}}$$

Theorem 9. Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for V. Then the coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is a 1-1 linear transformation from V to $\mathbb{R}^n$

Definition 11. Isomorphism from V onto W: a 1-1 linear transformation from a vector space V onto W. Every vector space calculation in V is accurately reproduced in W and vice versa.

Example of where its useful: $\mathbb{P}_n$ is isomorphic to $\mathbb{R}^{n+1}$. The coordinates will become the coefficients of the polynomial and you can convert from $\mathbb{P}_n$ to $\mathbb{R}^{n+1}$, do your business in a nice clean matrix and enjoy.

4.5 The Dimension of a Vector Space

Theorem 10. If a vector space has basis with n vectors, any set containing more than n vectors is linearly dependent.

Theorem 11. If a vector space has basis with n vectors, every basis of the space must consist of exactly n vectors

Definition 12. Dimension of V, $\dim V$, is the number of vectors in a basis for V. Dimension of $\{\mathbf{0}\}$ is defined to be 0. This only applies if V is spanned by a finite set, which makes it **finite-dimensional**. Otherwise, it is **infinite-dimensional**

For example: $\dim \mathbb{R}^n = n$, $\dim \mathbb{P}_n = n + 1$

4.5.1 Subspaces of a Finite-Dimensional Space

Theorem 12. Let H be a subspace of finite-dimensional V

- Any linearly independent set in H can be expanded to a basis of H
- H is finite-dimensional
- $\dim H \leq \dim V$

Next theorem in a nutshell: if the set has the right number of elements, its sufficient to show either that the set is linearly independent OR that it spans the space. (Linear Independence is sometimes easier to verify than spanning)

Theorem 13 (The Basis Theorem)**.** Let V be a p-dimensional vector space, $p \geq 1$.

- Any linearly independent set of p elements in V is a basis
- Any set of p elements in V that spans V is a basis

4.5.2 The Dimensions of Nul A, Col A and Row A

Definition 13. The **rank** of A is the dimension of the column space of A

Theorem 14 (The Real Theorem)**.** For $A : m \times n$,

- $\dim(\text{Col } A) = \dim(\text{Row } A) = \text{rank } A$
- $\text{rank } A + \dim(\text{Nul } A) = n$

Now we are able to get the Nul, Col and Row space with one sequence of row operations!

Getting the trio

Let A be a matrix.

1. Row reduce A to an echelon form B
2. The non-zero rows of B form a basis for the row space of B and also A
3. The pivot columns, identified from B, can be taken from A and they will form a basis for the column space
4. Now get the reduced echelon form, C
5. $A\mathbf{x} = \mathbf{0}$ is equivalent to $C\mathbf{x} = \mathbf{0}$. Do the usual stuff to find the solution set, which will be the null space.

Unlike Col space, Null and Row have no simple connection with the entries in A itself.

4.5.3 Rank and the Invertible Matrix Theorem

As if it wasn't enough, there's now more statements for it. Namely, the following statements are equivalent to the statement that A is invertible:

- The columns of A form a basis of $\mathbb{R}^n$
- $\text{Col } A = \mathbb{R}^n$
- $\dim(\text{Col } A) = n$
- $\text{rank } A = n$
- $\text{Nul } A = \{\mathbf{0}\}$
- $\dim(\text{Nul } A) = 0$

4.6 Change of Basis

Motivation:
Given a B-coordinate vector of an $\mathbf{x}$ we wanna get a C-coordinate vector of the same $\mathbf{x}$. How?

This may look scary but its not - there's simply a matrix, and if you left multiply a B-coordinate by it, the result will be the C-coordinate. Hence, **change-of-coordinate matrix from $\mathcal{B}$ to $\mathcal{C}$**

Theorem 15. Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}, \mathcal{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ be bases of V. Then there is a unique matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$ such that

$$[\mathbf{x}]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}$$

The columns of the $P_{\mathcal{C}\leftarrow\mathcal{B}}$ are the $\mathcal{C}$-coordinate vectors in the basis $\mathcal{B}$. Or:

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix}[\mathbf{b}_1]_{\mathcal{C}} & \dots & [\mathbf{b}_n]_{\mathcal{C}}\end{bmatrix}$$

Again, this looks scary but you can just use the equation.

The columns of this matrix are linearly independent. It must be invertible then. So:

$$\begin{aligned}[][\mathbf{x}]_{\mathcal{C}} &= P_{\mathcal{C}\leftarrow\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} \\ (P_{\mathcal{C}\leftarrow\mathcal{B}})^{-1}[\mathbf{x}]_{\mathcal{C}} &= [\mathbf{x}]_{\mathcal{B}}\end{aligned}$$

As a result, $(P_{\mathcal{C}\leftarrow\mathcal{B}})^{-1}$ changes a $\mathcal{C}$-coordinate to a $\mathcal{B}$-coordinate, hence $(P_{\mathcal{C}\leftarrow\mathcal{B}})^{-1} = P_{\mathcal{B}\leftarrow\mathcal{C}}$. Follow the arrows and you won't be confused.

Chapter 5

Eigenvalues and Eigenvectors

5.1 Eigenvalues and Eigenvectors

Definition 1. An **eigenvector** of $A : n \times n$ is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$. The scalar λ is called an **eigenvalue** of A if there is a nontrivial solution of $A\mathbf{x} = \lambda\mathbf{x}$.

λ is an eigenvalue of A iff $(A - \lambda I)\mathbf{x} = \mathbf{0}$ has a nontrivial solution.
The set of all the solutions of the above equation is just Nul $(A-\lambda I)$, which is called the **eigenspace** of A corresponding to λ.
It consists of the zero vector and the all eigenvectors.

Theorem 1. The eigenvalues of a triangular matrix are the entries on its main diagonal

Theorem 2. If $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are eigenvectors corresponding to $\lambda_1, \ldots, \lambda_r$, then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is linearly independent.

5.2 The Characteristic Equation

The problem of finding eigenvalues of A is equivalent to finding all λ such that the matrix $A - \lambda I$ is NOT invertible.
This happens precisely when the determinant is zero.
Hence, this equation gives the eigenvalues of A:

$$\det(A - \lambda I) = 0$$

Now there's more stuff to add to... ill finish this later