

# 21-241 Matrices and Linear Transformations Notes

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Exam 2 Edition

# Chapter 1

## Linear Equations in Linear Algebra

### 1.1 Systems of linear equations

**Definition:** A system of linear equations is a collection of one or more linear equations involving the same set of variables. A system has either:

- No solution
- Infinite solutions
- A unique solution

The info about a linear system can be recorded in a matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The one with just the coefficients is called **coefficient matrix**

The one with the last col with the product is called the **augmented matrix**

There are 3 elementary row operations to carry out on the matrix:

- Replacement: by a sum of itself and a multiple of another row
- Interchange: swap rows
- Scaling: scale a row by a scalar

### 1.2 Row Reduction and Echelon Forms

**Definition:** A matrix is in **echelon form** if

- All nonzero rows are above any rows of zeros
- Each leading entry of a row is in a column to the right of the leading entry of the row above it
- All entries in a column below a leading entry are zeros

A matrix is in **reduced echelon form** if

- It is in echelon form
- The leading entry in each nonzero row is 1
- Each leading 1 is the only nonzero entry in its column

**Theorem 1** (Uniqueness of the Reduced Echelon Form). *Each matrix is row equivalent to one and only one reduced echelon matrix*

**Definition 1** (Pivot Position). A **pivot position** in  $A$  is a location in  $A$  that corresponds to a leading 1 in the reduced echelon form of  $A$

#### Row Reduction Algorithm

1. Start with left non-zero column. This is a pivot column. Pivot position is at the top
2. Choose a nonzero entry in the pivot column as a pivot. Interchange to move it into pivot position
3. Use row replacement to make zeros below pivot position
4. Ignore the row with the pivot and the ones above. Apply steps 1-3 to the remaining matrix until no more rows to modify
5. Starting with rightmost pivot and working up and left, make zeros above each pivot. Make every pivot a 1 by scaling

Variables corresponding to pivot columns are **basic**, the other ones are **free**

**Theorem 2** (Existence and Uniqueness theorem).

A linear system is consistent iff the right column of the augmented matrix is NOT a pivot column i.e. iff echelon form has NO row of form:

$$[0 \quad \dots \quad 0 \quad b] \text{ with } b \text{ nonzero}$$

If a system is consistent then solution set contain either a unique solution if there are 0 free variables or infinite solutions if there is at least 1 free variable.

## 1.3 Vector Equations

#### Algebraic Properties of Vectors in $\mathbb{R}^n$

$\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$  and scalars  $c$  and  $d$ :

1. **Commutativity of addition:**  $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. **Associativity of addition:**  $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
3. **Additive identity:**  $\mathbf{u} + \mathbf{0} = \mathbf{u}$
4. **Additive inverse:**  $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
5. **Distributivity of scalar multiplication over addition (vectors):**  $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
6. **Distributivity of scalar multiplication over addition (scalars):**  $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
7. **Associativity of scalar multiplication:**  $(cd)\mathbf{u} = c(d\mathbf{u})$
8. **Scalar identity:**  $1\mathbf{u} = \mathbf{u}$

A linear combination is a vector  $\mathbf{y} = c_1\mathbf{v}_1 + \dots + c_p\mathbf{v}_p$

**Definition 2.**  $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  is the set of all linear combinations of  $\mathbf{v}_1, \dots, \mathbf{v}_p$   
It is the set of all vectors that can be written in the form  $c_1\mathbf{v}_1 + \dots + c_p\mathbf{v}_p$

## 1.4 The Matrix Equation $\mathbf{Ax} = \mathbf{b}$

**Definition 3.**

If  $A$  is an  $m \times n$  matrix, with columns  $\mathbf{a}_1, \dots, \mathbf{a}_n$  and if  $\mathbf{x} \in \mathbb{R}^n$ ,

$$A\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$$

This is defined only if the number of columns of  $A$  equals number of entries in  $\mathbf{x}$

**Theorem 3.**

If  $A$  is an  $m \times n$  matrix and  $\mathbf{b} \in \mathbb{R}^m$ , then  $A\mathbf{x} = \mathbf{b}$  has the same solution set as

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

which has the same solution set as the system of linear equations with matrix

$$\begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n & \mathbf{b} \end{bmatrix}$$

**Theorem 4.**

Let  $A$  be a  $m \times n$  matrix. The following statements are equivalent

- $\forall \mathbf{b} \in \mathbb{R}^m, A\mathbf{x} = \mathbf{b}$  has a solution
- $\forall \mathbf{b}$  is a linear combination of the columns of  $A$
- The columns of  $A$  span  $\mathbb{R}^m$
- $A$  has a pivot in every row

Warning:  $A$  is a **coefficient matrix** in this theorem

**Theorem 5.**

If  $A$  is an  $m \times n$  matrix and  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ , and  $c$  is a scalar

- $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$ ;
- $A(c\mathbf{u}) = c(A\mathbf{u})$ ;

## 1.5 Solution sets of Linear Systems

**Homogenous Linear Systems:** if it can be written in form  $A\mathbf{x} = \mathbf{0}$

This system always has at least one solution ( $\mathbf{x} = \mathbf{0}$ ): **trivial solution**

The equation has a **non-trivial** solution iff it has at least one free variable

**Nonhomogenous Linear Systems:** if it can be written in form  $A\mathbf{x} = \mathbf{b}$ ,  $\mathbf{b}$  non-zero

**Theorem 6.**

Let  $A\mathbf{x} = \mathbf{b}$  be consistent for some  $\mathbf{b}$ , let  $\mathbf{p}$  be a solution. Then the solution set is the set of all vectors in form  $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$ , where  $\mathbf{v}_h$  is any solution of  $A\mathbf{x} = \mathbf{0}$

Writing a solution set in parametric vector form

- Row reduce to reduced echelon form
- Express each basic variable in terms of any free variables
- Write a typical solution as a vector whose entries depend on free variables
- Decompose it into a linear combination of vectors (with numbers only inside) using free variables as parameters

## 1.6 Application of Linear Systems

idk how to make it skip it. So just ignore this

## 1.7 Linear Independence

**Definition 4.**

$\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  is **linearly independent** if

$$x_1\mathbf{v}_1 + \dots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution. It is said to be **linearly dependent** if  $\exists c_1, \dots, c_p$ , not all zero, such that

$$c_1\mathbf{v}_1 + \dots + c_p\mathbf{v}_p = \mathbf{0}$$

A set of two vectors is linearly dependent if at least one is a multiple of the other

**Theorem 7** (Characterization of Linearly independent Sets).

A set of two or more vectors is linearly dependent iff at least one of them is a linear combination the others.

**Theorem 8.**

If a set contains more vectors than entries in each vector, then it is linearly dependant

**Theorem 9.**

If a set contains the zero vector, then it is linearly dependant

## 1.8 Introduction to Linear Transformations

A **transformation**  $T$  from  $\mathbb{R}^n$  to  $\mathbb{R}^m$  is a rule that assigns each vector  $\mathbf{x}$  in  $\mathbb{R}^n$  to a vector  $T(\mathbf{x})$  in  $\mathbb{R}^m$

- Domain:  $\mathbb{R}^n$
- Codomain:  $\mathbb{R}^m$
- Range: set of all images  $T(\mathbf{x})$

**Definition 5.**  $T$  is linear iff

- $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
- $T(c\mathbf{u}) = cT(\mathbf{u})$

If  $T$  is linear then:

- $T(\mathbf{0}) = \mathbf{0}$
- $T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$
- $T(c_1\mathbf{v}_1 + \dots + c_p\mathbf{v}_p) = T(c_1\mathbf{v}_1) + \dots + T(c_p\mathbf{v}_p)$

## 1.9 The Matrix of a Linear Transformation

**Theorem 10.**

For  $T$ , there exists a unique  $A$  such that

$$T(\mathbf{x}) = A\mathbf{x} \quad \forall \mathbf{x} \in \mathbb{R}^n$$

Where  $A = [T(\mathbf{e}_1) \quad T(\mathbf{e}_2) \quad \dots \quad T(\mathbf{e}_n)]$

With  $\mathbf{e}_j$  is the  $j$ th column of the identity matrix

**Definition 6.**

$T$  is **onto** (surjective) if

- Range of  $T = \text{Codomain}$
- $\forall \mathbf{v} \in \text{codomain}(B), \exists \mathbf{u} \in \text{domain}(A)$  such that  $T(\mathbf{u}) = \mathbf{v}$
- Find standard matrix

$$A = [T(c_1) \quad \dots \quad T(c_n)]$$

Onto if  $A\mathbf{u} = \mathbf{v}$  has a solution for each  $\mathbf{v}$  in range

**Definition 7.**

$T$  is **one-to-one** (injective) if

- $T(\mathbf{u}_1) = T(\mathbf{u}_2) \implies \mathbf{u}_1 = \mathbf{u}_2$
- Find standard matrix

$$A = [T(c_1) \quad \dots \quad T(c_n)]$$

$A\mathbf{u} = \mathbf{b}$  has a unique solution for each  $\mathbf{b} \in \text{Range}(T)$

**Theorem 11.**  $T$  is one-to-one iff  $T(\mathbf{x}) = \mathbf{0}$  has a *ONLY* the trivial solution

**Theorem 12.** Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ , let  $A$  be standard matrix for  $T$ . Then

- $T$  maps  $\mathbb{R}^n$  to  $\mathbb{R}^m$  iff the columns of  $A$  span  $\mathbb{R}^m$
- $T$  is one-to-one iff the columns of  $A$  are linearly independent

# Chapter 2

## Matrix Algebra

### 2.1 Matrix Operations

- **Square Matrix:** row and columns are the same
- **Diagonal entries:**  $a_{11}, a_{22}, \dots$
- **Diagonal matrix:** square  $n \times n$  matrix whose non-diagonal entries are zero
- **Zero matrix:**  $m \times n$  matrix where all entries are 0

$$[a_{ij}]_{m \times n} \pm [b_{ij}]_{m \times n} = [a_{ij} \pm b_{ij}]_{m \times n}$$

Two matrices are **equal** if they have the same size and their entries are equal (duh)

**Theorem 1.**

#### Properties of Matrices

Let  $A$ ,  $B$  and  $C$  be matrices of the same size, and let  $r$  and  $s$  be scalars

- $A + B = B + A$
- $(A + B) + C = A + (B + C)$
- $A + 0 = A$
- $r(A + B) = rA + rB$
- $(r + s)A = rA + sA$
- $r(sA) = (rs)A$

**Definition 1.** If  $A$  is a  $m \times n$  matrix and  $B$  is an  $n \times p$  matrix with columns  $\mathbf{b}_1, \dots, \mathbf{b}_p$ , then

$$AB = A [\mathbf{b}_1 \dots \mathbf{b}_p] = [A\mathbf{b}_1 \dots A\mathbf{b}_p]$$

- Each column of  $AB$  is a linear combination of the columns of  $A$  using weights from columns of  $B$
- **The number of columns of  $A$  must match the number of rows in  $B$**
- Row column rule:  $(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj}$
- $\text{row}_i(AB) = \text{row}_i(A) \cdot B$

**Theorem 2.**

## Properties of Matrices

Let  $A$  be an  $m \times n$  matrix, and let  $B$  and  $C$  have sizes for which the indicated sums and products are defined.

$$A(BC) = (AB)C \quad (\text{associative law of multiplication})$$

$$A(B + C) = AB + AC \quad (\text{left distributive law})$$

$$(B + C)A = BA + CA \quad (\text{right distributive law})$$

$$r(AB) = (rA)B = A(rB) \text{ for any scalar } r$$

$$I_m A = A = A I_n \quad (\text{identity for matrix multiplication})$$

## WARNING

Let  $A$  be an  $m \times n$  matrix, and let  $B$  and  $C$  have sizes for which the indicated sums and products are defined.

- In general,  $AB \neq BA$
- In general,  $AB = AC$  **does not imply**  $B = C$
- In general,  $AB = 0$  **does not imply**  $A = 0 \vee B = 0$

Power of a matrix:  $A^k = \underbrace{AA \dots A}_k$

The transpose of a matrix:  $A^T$  is obtained by swapping its rows and columns of  $A$ .

**Theorem 3.**

- $(A^T)^T = A$
- $(A + B)^T$
- $(rA)^T = rA^T$
- $(AB)^T = B^T A^T$

## 2.2 The Inverse of a Matrix

An  $n \times n$  matrix  $A$  is said to be **invertible** if there is an  $n \times n$  matrix  $C$  such that

$$CA = I_n \quad \text{and} \quad AC = I_n,$$

In this case,  $C$  is an **inverse** of  $A$

$$A^{-1}A = I \text{ and } AA^{-1} = I$$

**Theorem 4.** Let  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  If  $ad - bc \neq 0$ , then  $A$  is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\det A = ad - bc$$

**Theorem 5.** If  $A$  is invertible, then  $\forall \mathbf{b} \in \mathbb{R}^n$ ,  $A\mathbf{x} = \mathbf{b}$  has the unique solution  $\mathbf{x} = A^{-1}\mathbf{b}$



**Theorem 6.** 1. If  $A$  invertible, then  $A^{-1}$  too and  $(A^{-1})^{-1} = A$

2. If  $A$  and  $B$  invertible,  $(AB)^{-1} = B^{-1}A^{-1}$

3. If  $A$  invertible,  $(A^T)^{-1} = (A^{-1})^T$

- An **elementary matrix** is a matrix obtained by performing a single elementary row operation on  $I$
- If a row op is performed on  $A$ , the result can be written as  $EA$ , where  $E$  is made by performing the same row op on  $I$
- Each  $E$  is invertible and its inverse is the elementary matrix that transforms  $E$  back into  $I$ .

**Theorem 7.**  $A$  is invertible iff  $A$  is row equivalent to  $I$ , and any sequence of row ops that reduces  $A$  to  $I$  also transforms  $I$  to  $A^{-1}$

**Finding  $A^{-1}$ :** Row reduce  $[A \ I]$ . If  $A$  row-eq to  $I$  then  $[A \ I]$  is row-eq to  $[I \ A^{-1}]$

## 2.3 Characterizations of Matrices

**Theorem 8.**

The following statements about a square  $n$  by  $n$  matrix  $A$  are equivalent

- $A$  is an invertible matrix.
- $A$  is row equivalent to the  $n \times n$  identity matrix.
- $A$  has  $n$  pivot positions.
- The equation  $Ax = 0$  has only the trivial solution.
- The columns of  $A$  form a linearly independent set.
- The linear transformation  $x \mapsto Ax$  is one-to-one.
- The equation  $Ax = b$  has at least one solution for each  $b \in \mathbb{R}^n$ .
- The columns of  $A$  span  $\mathbb{R}^n$ .
- The linear transformation  $x \mapsto Ax$  maps  $\mathbb{R}^n$  onto  $\mathbb{R}^n$ .
- There is an  $n \times n$  matrix  $C$  such that  $CA = I$ .
- There is an  $n \times n$  matrix  $D$  such that  $AD = I$ .
- $A^T$  is an invertible matrix.

If  $AB = I$ , then  $A$  and  $B$  invertible with  $B = A^{-1}$  and  $A = B^{-1}$

A linear transformation is invertible if  $\exists S : \mathbb{R}^n \rightarrow \mathbb{R}^n$  such that:

$$\begin{aligned} S(T(\mathbf{x})) &= \mathbf{x} \quad \text{for all } \mathbf{x} \in \mathbb{R}^n \\ T(S(\mathbf{x})) &= \mathbf{x} \quad \text{for all } \mathbf{x} \in \mathbb{R}^n \end{aligned}$$

**Theorem 9.** Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$  and  $A$  be the matrix.  $T$  is invertible iff  $A$  is invertible. Then  $S(\mathbf{x}) = A^{-1}\mathbf{x}$

## 2.4 Partitioned Matrices

Trivial stuff honestly just partition and multiply.

Ok actually there is one good theorem here

**Theorem 10** (Column-Row Expansion of  $AB$ ).

If  $A$  is  $m \times n$  and  $B$  is  $n \times p$ :

$$AB = \begin{bmatrix} \text{col}_1(A) & \dots & \text{col}_n(A) \end{bmatrix} \begin{bmatrix} \text{row}_1(B) \\ \vdots \\ \text{row}_n(B) \end{bmatrix} = \text{col}_1(A)\text{row}_1(B) + \dots + \text{col}_n(A)\text{row}_n(B)$$

The inverse of a block upper triangular matrix:

$$A^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{22}^{-1} \\ 0 & A_{22}^{-1} \end{bmatrix}$$

## 2.5 Matrix Factorizations

Assume  $A$  is  $m \times n$  and can be reduced to Echelon Form without row interchanges.

Then  $A = LU$ ,

- $L$  :  $m \times m$  lower triangular with 1's in diagonal
- $U$  :  $m \times n$  echelon form of  $A$

$A\mathbf{x} = \mathbf{b}$  can be written as  $L(U\mathbf{x}) = \mathbf{b}$ . Writing  $\mathbf{y} = U\mathbf{x}$ , we get:

$$L(U\mathbf{x}) = \mathbf{b}$$

$$L\mathbf{y} = \mathbf{b}$$

$$U\mathbf{x} = \mathbf{y}$$

We can get  $\mathbf{x}$  by solving the easy pair of equations:

1. Solve  $L\mathbf{y} = \mathbf{b}$  for  $\mathbf{y}$
2. Solve  $U\mathbf{x} = \mathbf{y}$  for  $\mathbf{x}$

### LU Factorization Algorithm (boring)

1. Reduce  $A$  to echelon form  $U$  using a sequence of row replacement operations
2. Place entries in  $L$  such that the same sequence of row operations reduces  $L$  to  $I$

### LU Factorization Algorithm (fast)

For a matrix  $A : m \times n$

1. Initialize  $L$  as the identity matrix  $m \times m$  and  $U$  as  $A$ .
2. For each row  $i = 1, 2, \dots, m$ :
  - Use row operations to create zeros below the diagonal in  $U$  by subtracting multiples of the current row from the rows below.
  - For each row operation, divide the corresponding entries below the pivot in column  $i$  by the pivot, and store these multipliers in the corresponding positions of  $L$ .
3. Continue this process until  $U$  is in echelon form and  $L$  has recorded the multipliers such that  $A = LU$ .

This description is very bad, please practice this on a non-square matrix

## Chapter 3

# Determinants

### 3.1 Intro to Determinants

First, we denote  $A_{ij}$  to be the sub-matrix of  $A$  formed by deleting the  $i$ th row and  $j$ th column of  $A$

**Definition 1.** For  $n \geq 2$ , the **determinant** of a  $n \times n$ :

$$\begin{aligned}\det A &= a_{11} \det(A_{11}) - a_{12} \det(A_{12}) + \dots + (-1)^{1+n} a_{1n} \det(A_{1n}) = \\ &= \sum_{j=1}^n (-1)^{1+j} a_{1j} \det(A_{1j})\end{aligned}$$

**The  $(i, j)$ -cofactor of  $A$ :**  $C_{ij} = (-1)^{i+j} \det(A_{ij})$

**Theorem 1.** The determinant of  $A$  can be computed by cofactor expansion across any row or column.  
Across row  $i$ :

$$\det(A) = a_{i1}C_{i1} + \dots + a_{in}C_{in}$$

Across column  $j$ :

$$\det(A) = a_{1j}C_{1j} + \dots + a_{nj}C_{nj}$$

Signs of the cofactor:

$$\begin{bmatrix} + & - & + & \dots \\ - & + & - & \\ + & - & + & \\ \vdots & & & \ddots \end{bmatrix}$$

**Theorem 2.** If  $A$  triangular, then  $\det(A)$  is the product of the entries on the main diagonal of  $A$

### 3.2 Properties of Determinants

**Theorem 3.**

#### Row Operations

For a matrix  $A : m \times n$

- If a multiple of a row in  $A$  is added to another to produce  $B$ :  $\det(B) = \det(A)$
- If two rows of  $A$  are swapped to produce  $B$ :  $\det(B) = -\det(A)$
- If a row of  $A$  is multiplied by  $k$  to produce  $B$ :  $\det(B) = k \cdot \det(A)$

Two main strats where this is useful:

- Row reduce a matrix to echelon form to simplify calculation (without row interchange!)
- Factor out a common factor from a row to simplify calculation

**Theorem 4.**  $A$  is invertible iff  $\det(A) \neq 0$  (trivial)

**Theorem 5.**  $\det(A) = \det(A^T)$

**Theorem 6.** If  $A$  and  $B$   $n \times n$ , then  $\det(AB) = \det(A) \cdot \det(B)$

**Lemma 1.**  $\det(A^{-1}) = \frac{1}{\det(A)}$

### 3.3 Crammers Rule

Let  $A_i(\mathbf{b})$  be the matrix obtained from  $A$  by replacing column  $i$  by the vector  $\mathbf{b}$

**Theorem 7** (Cramer's Rule). Let  $A : n \times n$ , invertible, the solution  $\mathbf{x}$  of  $A\mathbf{x} = \mathbf{b}$  is given by:

$$x_i = \frac{\det(A_i(\mathbf{b}))}{\det(A)}, \text{ for } i = 1, 2, \dots, n$$

The matrix of cofactors is called the **adjugate or classical adjoint** of  $A$ , denoted by  $\text{adj } A$

**Theorem 8** (an Inverse Formula).

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \vdots & \vdots & & \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{bmatrix} = \frac{\text{adj}(A)}{\det(A)}$$

**Theorem 9.** If  $A : 2 \times 2$ , Area of parallelogram determined by columns of  $A$  is  $|\det(A)|$

If  $A : 3 \times 3$ , Volume of parallelepiped determined by columns of  $A$  is  $|\det(A)|$

## Chapter 4

# Vector Spaces and Subspaces

### 4.1 Vector Spaces and Subspaces

**Definition 2.** A *vector space* is a nonempty set  $V$  of objects called vectors, on which there are defined two operations: addition and scalar multiplication. The axioms below must hold for all  $\mathbf{u}, \mathbf{w}, \mathbf{v} \in V$

1.  $\mathbf{u} + \mathbf{v} \in V$  (Closure under addition)
2.  $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3.  $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
4.  $\mathbf{0} \in V$ , such that  $\mathbf{u} + \mathbf{0} = \mathbf{u}$
5.  $\forall \mathbf{u} \in V, \exists -\mathbf{u} \in V$  such that  $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
6.  $c\mathbf{u} \in V$
7.  $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
8.  $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
9.  $c(d\mathbf{u}) = (cd)\mathbf{u}$
10.  $1\mathbf{u} = \mathbf{u}$

**Definition 3.** A *subspace* of a space  $V$  is a subset  $H$  of  $V$  that:

- The zero vector of  $V$  is in  $H$
- $H$  is closed under vector addition:  $\mathbf{u} + \mathbf{v} \in H$  for all  $\mathbf{u}, \mathbf{v} \in H$
- $H$  is closed under scalar multiplication:  $c\mathbf{u} \in H$  for all scalars  $c$  and vectors  $\mathbf{u} \in H$

**Theorem 10.** If  $\mathbf{v}_1, \dots, \mathbf{v}_p$  are in a vector space  $V$ , then  $\text{Span}\mathbf{v}_1, \dots, \mathbf{v}_p$  is a subspace of  $V$