3. Compute the inverse of the following matrices:

i) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ ii) $\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

🕮 Let $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$$\therefore\ A^{-1} = \frac{1}{\cos^2\theta + \sin^2\theta}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

The Determinant equals unity & Rows/Columns are Orthonormal.

🕮 Let $A = \left[\begin{array}{cc:c} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ \hdashline 0 & 0 & 1 \end{array}\right]$

$$\therefore\ A^{-1} = \left[\begin{array}{c:c} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^{-1} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hdashline \begin{matrix} 0 & \quad 0 \end{matrix} & (1)^{-1} \end{array}\right] = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

4. Prove that the product of matrices

$A = \begin{bmatrix} \cos^2\theta & \cos\theta\,\sin\theta \\ \cos\theta\,\sin\theta & \sin^2\theta \end{bmatrix}$, $B = \begin{bmatrix} \cos^2\phi & \cos\phi\,\sin\phi \\ \cos\phi\,\sin\phi & \sin^2\phi \end{bmatrix}$

is zero when "θ" and "Φ" differ by an odd number of multiple of π/2.

$$AB = \begin{bmatrix} \cos^2\theta\cos^2\phi + \cos\theta\sin\theta\cos\phi\sin\phi & \cos^2\theta\cos\phi\sin\phi + \cos\theta\sin\theta\sin^2\phi \\ \cos\theta\sin\theta\cos^2\phi + \sin^2\theta\cos\phi\sin\phi & \cos\theta\sin\theta\cos\phi\sin\phi + \sin^2\phi\sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta\cos\phi(\cos\theta\cos\phi + \sin\theta\sin\phi) & \cos\theta\sin\phi(\cos\theta\cos\phi + \sin\theta\sin\phi) \\ \sin\theta\cos\phi(\cos\theta\cos\phi + \sin\theta\sin\phi) & \sin\theta\sin\phi(\cos\theta\cos\phi + \sin\theta\sin\phi) \end{bmatrix}$$

$$= (\cos\theta\ \cos\phi + \sin\theta\ \sin\phi) * \begin{bmatrix} \cos\theta\cos\phi & \cos\theta\ \sin\phi \\ \sin\theta\cos\phi & \sin\theta\ \sin\phi \end{bmatrix}$$

$$= \cos(\theta - \phi) * \begin{bmatrix} \cos\theta\cos\phi & \cos\theta \ \sin\phi \\ \sin\theta\cos\phi & \sin\theta \ \sin\phi \end{bmatrix}$$

$$\Rightarrow 0 \text{ if and only if: } \cos(\theta - \phi) = 0$$

$$(\theta - \phi) = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots \text{ i.e. odd multiples of } \frac{\pi}{2}$$

5. If $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, show that $A^k = \begin{bmatrix} 1+2k & -4k \\ k & 1-2k \end{bmatrix}$ where "k" is any positive integer.

$$A^2 = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 5 & -8 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} 1+2*2 & -4*2 \\ 2 & 1-2*2 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 5 & -8 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 7 & -12 \\ 3 & -5 \end{bmatrix} = \begin{bmatrix} 1+2*3 & -4*3 \\ 3 & 1-2*3 \end{bmatrix}$$

Thus the theorem is true for indices 2 and 3

Now, assume $A^k = \begin{bmatrix} 1+2k & -4k \\ k & 1-2k \end{bmatrix}$

Then:

$$A^{k+1} = A^k * A = \begin{bmatrix} 1+2k & -4k \\ k & 1-2k \end{bmatrix} \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3+2k & -4(k+1) \\ k+1 & -1-2k \end{bmatrix} = \begin{bmatrix} 1+2(k+1) & -4(k+1) \\ k+1 & 1-2(k+1) \end{bmatrix}$$

Thus if the law is true for A^k, it's also true for A^{k+1}

6. Prove:

i) $(A \pm B)^2 \neq A^2 \pm 2AB \pm B^2$,

ii) $A^2 - B^2 \neq (A + B)(A - B)$

$$\begin{aligned} (A+B)^2 &= (A+B)(A+B) \\ &= A(A+B) + B(A+B) \\ &= AA + AB + BA + BB \\ &= A^2 + AB + BA + B^2 \neq A^2 + 2AB + B^2 \end{aligned}$$

Since in general "$AB \neq BA$"; hence: "$AB + BA \neq 2AB$".

And $(A - B)^2 = (A - B)(A - B)$
$= A(A - B) + B(A - B)$
$= A A - A B - B A - B B$
$= A^2 - AB - BA + B^2 \neq A^2 - 2AB - B^2$

Since in general "$AB \neq BA$"; hence: "$-AB - BA \neq -2AB$".

$(A + B)(A - B) = A(A - B) + B(A - B)$
$= A A - A B - B A - B B$
$= A^2 - AB + BA + B^2 \neq A^2 - B^2$

Since in general "$AB \neq BA$"; hence: "$-AB + BA \neq 0$".

7. If $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, find the values of "α, and β" such that:

$(\alpha I + \beta A)^2 = A^2$.

$$A^2 = A * A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$$

$$(\alpha I + \beta A)^2 = \alpha^2 I^2 + \beta^2 A^2 + \alpha\beta I A + \alpha\beta A I$$
$$= \alpha^2 I + \beta^2(-I) + 2\alpha\beta A = (\alpha^2 - \beta^2) I + 2\alpha\beta A$$
$$= (\alpha^2 - \beta^2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 2\alpha\beta\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} \alpha^2 - \beta^2 & 2\alpha\beta \\ -2\alpha\beta & \alpha^2 - \beta^2 \end{bmatrix}$$

Given

$(\alpha I + \beta A)^2 = A$

Hence

$$\begin{bmatrix} \alpha^2 - \beta^2 & 2\alpha\beta \\ -2\alpha\beta & \alpha^2 - \beta^2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Thus

$(\alpha^2 - \beta^2) = -1$, And $2\alpha\beta = 0$.

i.e. $\alpha = 0$, $\rightarrow$ $-\beta^2 = -1$ $\rightarrow$ $\beta^2 = 1$ $\rightarrow$ $\beta = \pm 1$

Or:

$\beta = 0$, $\rightarrow$ $\alpha^2 = -1$ $\rightarrow$ $\alpha = \pm\sqrt{(-1)}$ $\rightarrow$ $\alpha = \pm i$

8. Show that the possible square roots of the tow-rowed unit matrix "I" are: "± I" and $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$, where "$1 - \alpha^2 = \beta\gamma$".

Let $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ be a possible square root of the two-rowed unit matrix,

Then:

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} \alpha^2+\beta\gamma & \alpha\beta+\beta\delta \\ \gamma\alpha+\delta\gamma & \gamma\beta+\delta^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Hence:

$\alpha^2 + \beta\gamma = 1$, "i"

$\gamma\beta + \delta^2 = 1$, "ii"

$\beta(\alpha + \delta) = 0$, "iii"

And $\gamma(\alpha + \delta) = 0$ "iv"

From 1st two equations, by subtraction:

$$\alpha^2 - \delta^2 = 0 \rightarrow \alpha^2 = \delta^2 \rightarrow \alpha = \pm\delta.$$

Case I

If $\alpha = -\delta$, last two equations are automatically true and first tow equations reduce to $1 - \alpha^2 = \beta\gamma$.

Therefore,

In this case the square root matrix becomes $\begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}$, where $1 - \alpha^2 = \beta\gamma$.

Case II

- When $\alpha = \delta$, the last two equations give:

$$\beta = 0 = \gamma.$$

- Then from first two equations

$$\alpha = \delta = \pm 1.$$

- Thus the other possible square roots of the two - rowed unit matrix are :

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \quad \& \quad \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I.$$