CHAPTER 22 INFINITE SEQUENCES

The idea of an infinite sequence is so natural a concept that it is tempting to dispense with a definition altogether. One frequently writes simply "an infinite sequence

$$a_1, a_2, a_3, a_4, a_5, \ldots ,"$$

the three dots indicating that the numbers a_i continue to the right "forever." A rigorous definition of an infinite sequence is not hard to formulate, however; the important point about an infinite sequence is that for each natural number, n, there is a real number a_n. This sort of correspondence is precisely what functions are meant to formalize.

DEFINITION An **infinite sequence** of real numbers is a function whose domain is **N**.

From the point of view of this definition, a sequence should be designated by a single letter like a, and particular values by

$$a(1), a(2), a(3), \ldots,$$

but the subscript notation

$$a_1, a_2, a_3, \ldots$$

is almost always used instead, and the sequence itself is usually denoted by a symbol like $\{a_n\}$. Thus $\{n\}$, $\{(-1)^n\}$, and $\{1/n\}$ denote the sequences α, β, and γ defined by

$$\alpha_n = n,$$
$$\beta_n = (-1)^n,$$
$$\gamma_n = \frac{1}{n}.$$

A sequence, like any function, can be graphed (Figure 1) but the graph is usually rather unrevealing, since most of the function cannot be fit on the page.

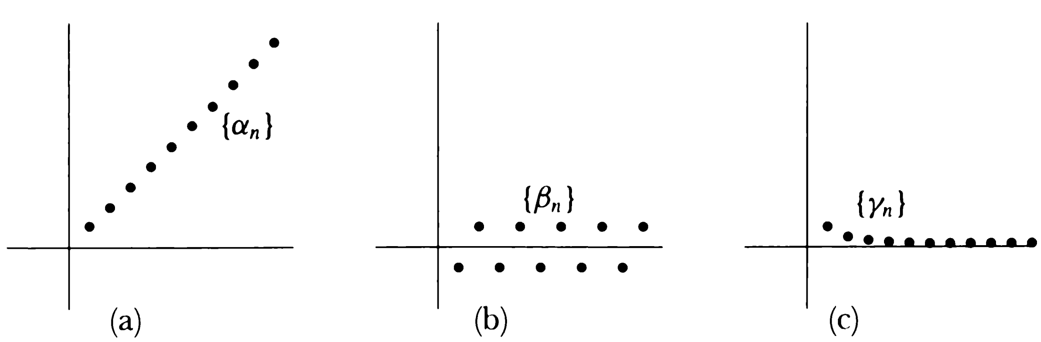

FIGURE 1

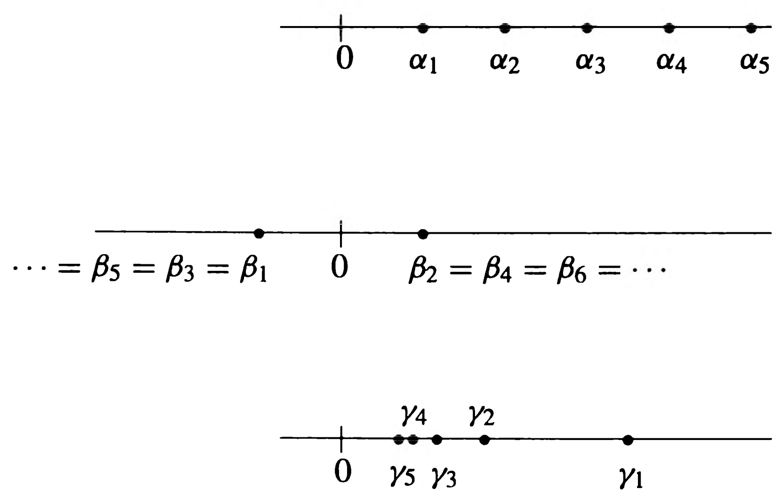

FIGURE 2

A more convenient representation of a sequence is obtained by simply labeling the points $a_1, a_2, a_3, \ldots$ on a line (Figure 2). This sort of picture shows where the sequence "is going." The sequence $\{\alpha_n\}$ "goes out to infinity," the sequence $\{\beta_n\}$ "jumps back and forth between -1 and 1," and the sequence $\{\gamma_n\}$ "converges to 0." Of the three phrases in quotation marks, the last is the crucial concept associated with sequences, and will be defined precisely (the definition is illustrated in Figure 3).

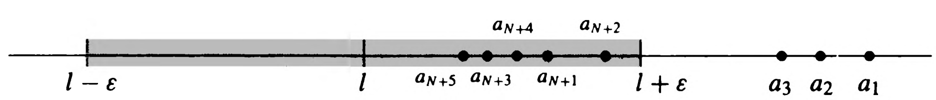

FIGURE 3

DEFINITION

A sequence $\{a_n\}$ **converges to l** (in symbols $\lim_{n\to\infty} a_n = l$) if for every $\varepsilon > 0$ there is a natural number N such that, for all natural numbers n,

$$\text{if } n > N, \text{ then } |a_n - l| < \varepsilon.$$

In addition to the terminology introduced in this definition, we sometimes say that the sequence $\{a_n\}$ **approaches l** or has the **limit l**. A sequence $\{a_n\}$ is said to **converge** if it converges to l for some l, and to **diverge** if it does not converge.

To show that the sequence $\{\gamma_n\}$ converges to 0, it suffices to observe the following. If $\varepsilon > 0$, there is a natural number N such that $1/N < \varepsilon$. Then, if $n > N$ we have

$$\gamma_n = \frac{1}{n} < \frac{1}{N} < \varepsilon, \qquad \text{so } |\gamma_n - 0| < \varepsilon.$$

The limit

$$\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n} = 0$$

will probably seem reasonable after a little reflection (it just says that $\sqrt{n+1}$ is practically the same as $\sqrt{n}$ for large n), but a mathematical proof might not be so obvious. To estimate $\sqrt{n+1}-\sqrt{n}$ we can use an algebraic trick:

$$\begin{aligned}\sqrt{n+1}-\sqrt{n} &= \frac{(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})}{\sqrt{n+1}+\sqrt{n}}\\ &= \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}}\\ &= \frac{1}{\sqrt{n+1}+\sqrt{n}}.\end{aligned}$$

It is also possible to estimate $\sqrt{n+1}-\sqrt{n}$ by applying the Mean Value Theorem to the function $f(x)=\sqrt{x}$ on the interval $[n, n+1]$. We obtain

$$\begin{aligned}\frac{\sqrt{n+1}-\sqrt{n}}{1} &= f'(x)\\ &= \frac{1}{2\sqrt{x}}, \qquad \text{for some } x \text{ in } (n, n+1)\\ &< \frac{1}{2\sqrt{n}}.\end{aligned}$$

Either of these estimates may be used to prove the above limit; the detailed proof is left to you, as a simple but valuable exercise.

The limit

$$\lim_{n\to\infty} \frac{3n^3+7n^2+1}{4n^3-8n+63} = \frac{3}{4}$$

should also seem reasonable, because the terms involving n^3 are the most important when n is large. If you remember the proof of Theorem 7-9 you will be able to guess the trick that translates this idea into a proof—dividing top and bottom by n^3 yields

$$\frac{3n^3+7n^2+1}{4n^3-8n+63} = \frac{3+\dfrac{7}{n}+\dfrac{1}{n^3}}{4-\dfrac{8}{n^2}+\dfrac{63}{n^3}}.$$

Using this expression, the proof of the above limit is not difficult, especially if one uses the following facts:

If $\lim\limits_{n\to\infty} a_n$ and $\lim\limits_{n\to\infty} b_n$ both exist, then

$$\begin{aligned}\lim_{n\to\infty}(a_n+b_n) &= \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n,\\ \lim_{n\to\infty}(a_n\cdot b_n) &= \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n;\end{aligned}$$

moreover, if $\lim\limits_{n\to\infty} b_n \neq 0$, then $b_n \neq 0$ for all n greater than some N, and

$$\lim_{n\to\infty} a_n/b_n = \lim_{n\to\infty} a_n \Big/ \lim_{n\to\infty} b_n.$$

(If we wanted to be utterly precise, the third statement would have to be even more complicated. As it stands, we are considering the limit of the sequence $\{c_n\} = \{a_n/b_n\}$, where the numbers c_n might not even be defined for certain $n < N$. This doesn't really matter—we could define c_n any way we liked for such n—because the limit of a sequence is not changed if we change the sequence at a finite number of points.)

Although these facts are very useful, we will not bother stating them as a theorem—you should have no difficulty proving these results for yourself, because the definition of $\lim_{n\to\infty} a_n = l$ is so similar to previous definitions of limits, especially $\lim_{x\to\infty} f(x) = l$.

The similarity between the definitions of $\lim_{n\to\infty} a_n = l$ and $\lim_{x\to\infty} f(x) = l$ is actually closer than mere analogy; it is possible to define the first in terms of the second. If f is the function whose graph (Figure 4) consists of line segments joining

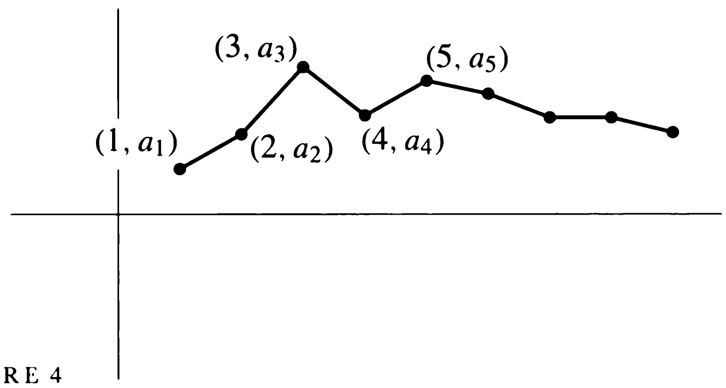

FIGURE 4

the points in the graph of the sequence $\{a_n\}$, so that

$$f(x) = (a_{n+1} - a_n)(x - n) + a_n \qquad n \le x \le n+1,$$

then

$$\lim_{n\to\infty} a_n = l \quad \text{if and only if} \quad \lim_{x\to\infty} f(x) = l.$$

Conversely, if f satisfies $\lim_{x\to\infty} f(x) = l$, and we set $a_n = f(n)$, then $\lim_{x\to\infty} a_n = l$.

This second observation is frequently very useful. For example, suppose that $0 < a < 1$. Then

$$\lim_{n\to\infty} a^n = 0.$$

To prove this we note that

$$\lim_{x\to\infty} a^x = \lim_{x\to\infty} e^{x\log a} = 0,$$

since $\log a < 0$, so that $x \log a$ is a negative and large in absolute value for large x. Notice that we actually have

$$\lim_{n\to\infty} a^n = 0 \quad \text{if } |a| < 1;$$

for if $a < 0$ we can write

$$\lim_{n\to\infty} a^n = \lim_{n\to\infty} (-1)^n |a|^n = 0.$$

The behavior of the logarithm function also shows that if $a > 1$, then a^n becomes arbitrarily large as n becomes large. This assertion is often written

$$\lim_{n\to\infty} a^n = \infty, \qquad a > 1,$$

and it is sometimes even said that $\{a^n\}$ approaches ∞. We also write equations like

$$\lim_{n\to\infty} -a^n = -\infty,$$

and say that $\{-a^n\}$ approaches $-\infty$. Notice, however, that if $a < -1$, then $\lim_{n\to\infty} a^n$ does not exist, even in this extended sense.

Despite this connection with a familiar concept, it is more important to visualize convergence in terms of the picture of a sequence as points on a line (Figure 3). There is another connection between limits of functions and limits of sequences which is related to *this* picture. This connection is somewhat less obvious, but considerably more interesting, than the one previously mentioned—instead of defining limits of sequences in terms of limits of functions, it is possible to reverse the procedure.

THEOREM 1 Let f be a function defined in an open interval containing c, except perhaps at c itself, with

$$\lim_{x\to c} f(x) = l.$$

Suppose that $\{a_n\}$ is a sequence such that

(1) each a_n is in the domain of f,
(2) each $a_n \neq c$,
(3) $\lim_{n\to\infty} a_n = c$.

Then the sequence $\{f(a_n)\}$ satisfies

$$\lim_{n\to\infty} f(a_n) = l.$$

Conversely, if this is true for every sequence $\{a_n\}$ satisfying the above conditions, then $\lim_{x\to c} f(x) = l$.

PROOF Suppose first that $\lim_{x\to c} f(x) = l$. Then for every $\varepsilon > 0$ there is a $\delta > 0$ such that, for all x,

$$\text{if } 0 < |x - c| < \delta, \text{ then } |f(x) - l| < \varepsilon.$$

If the sequence $\{a_n\}$ satisfies $\lim_{n\to\infty} a_n = c$, then (Figure 3) there is a natural number N such that,

$$\text{if } n > N, \text{ then } |a_n - c| < \delta.$$

By our choice of δ, this means that

$$|f(a_n) - l| < \varepsilon,$$

showing that

$$\lim_{n\to\infty} f(a_n) = l.$$

Suppose, conversely, that $\lim_{n\to\infty} f(a_n) = l$ for every sequence $\{a_n\}$ with $\lim_{n\to\infty} a_n = c$. If $\lim_{x\to c} f(x) = l$ were *not* true, there would be some $\varepsilon > 0$ such that for *every* $\delta > 0$ there is an x with

$$0 < |x - c| < \delta \quad \text{but} \quad |f(x) - l| > \varepsilon.$$

In particular, for each n there would be a number x_n such that

$$0 < |x_n - c| < \frac{1}{n} \quad \text{but} \quad |f(x_n) - l| > \varepsilon.$$

Now the sequence $\{x_n\}$ clearly converges to c but, since $|f(x_n) - l| > \varepsilon$ for all n, the sequence $\{f(x_n)\}$ does not converge to l. This contradicts the hypothesis, so $\lim_{x\to c} f(x) = l$ must be true. ▮

Theorem 1 provides many examples of convergent sequences. For example, the sequences $\{a_n\}$ and $\{b_n\}$ defined by

$$a_n = \sin\left(13 + \frac{1}{n^2}\right)$$
$$b_n = \cos\left(\sin\left(1 + (-1)^n \cdot \frac{1}{n}\right)\right),$$

clearly converge to $\sin(13)$ and $\cos(\sin(1))$, respectively. It is important, however, to have some criteria guaranteeing convergence of sequences which are not obviously of this sort. There is one important criterion which is very easy to prove, but which is the basis for all other results. This criterion is stated in terms of concepts defined for functions, which therefore apply also to sequences: a sequence $\{a_n\}$ is **increasing** if $a_{n+1} > a_n$ for all n, **nondecreasing** if $a_{n+1} \geq a_n$ for all n, and **bounded above** if there is a number M such that $a_n \leq M$ for all n; there are similar definitions for sequences which are decreasing, nonincreasing, and bounded below.

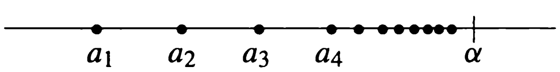

FIGURE 5

THEOREM 2 If $\{a_n\}$ is nondecreasing and bounded above, then $\{a_n\}$ converges (a similar statement is true if $\{a_n\}$ is nonincreasing and bounded below).

PROOF The set A consisting of all numbers a_n is, by assumption, bounded above, so A has a least upper bound α. We claim that $\lim_{n\to\infty} a_n = \alpha$ (Figure 5). In fact, if $\varepsilon > 0$, there is some a_N satisfying $\alpha - a_N < \varepsilon$, since α is the least upper bound of A. Then if $n > N$ we have

$$a_n \geq a_N, \quad \text{so} \quad \alpha - a_n \leq \alpha - a_N < \varepsilon.$$

This proves that $\lim_{n\to\infty} a_n = \alpha$. ▮

The hypothesis that $\{a_n\}$ is bounded above is clearly essential in Theorem 2: if $\{a_n\}$ is not bounded above, then (whether or not $\{a_n\}$ is nondecreasing) $\{a_n\}$ clearly diverges. Upon first consideration, it might appear that there should be little trouble deciding whether or not a given nondecreasing sequence $\{a_n\}$ is bounded above, and consequently whether or not $\{a_n\}$ converges. In the next chapter such sequences will arise very naturally and, as we shall see, deciding whether or not they converge is hardly a trivial matter. For the present, you might try to decide whether or not the following (obviously increasing) sequence is bounded above:

$$1,\ 1+\tfrac{1}{2},\ 1+\tfrac{1}{2}+\tfrac{1}{3},\ 1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4},\ \ldots.$$

Although Theorem 2 treats only a very special class of sequences, it is more useful than might appear at first, because it is always possible to extract from an arbitrary sequence $\{a_n\}$ another sequence which is either nonincreasing or else nondecreasing. To be precise, let us define a **subsequence** of the sequence $\{a_n\}$ to be a sequence of the form

$$a_{n_1}, a_{n_2}, a_{n_3}, \ldots,$$

where the n_j are natural numbers with

$$n_1 < n_2 < n_3 \cdots.$$

Then every sequence contains a subsequence which is either nondecreasing or nonincreasing. It is possible to become quite befuddled trying to prove this assertion, although the proof is very short if you think of the right idea; it is worth recording as a lemma.

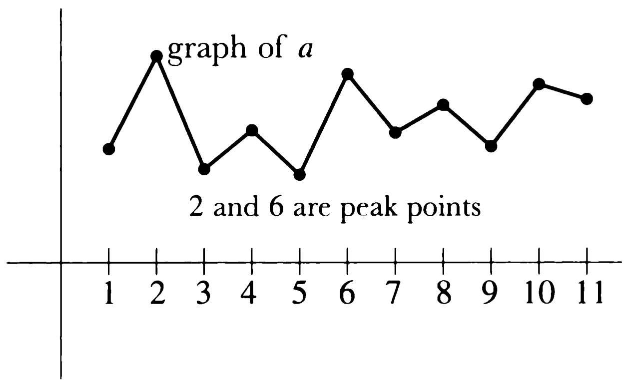

FIGURE 6

LEMMA Any sequence $\{a_n\}$ contains a subsequence which is either nondecreasing or nonincreasing.

PROOF Call a natural number n a "peak point" of the sequence $\{a_n\}$ if $a_m < a_n$ for all $m > n$ (Figure 6).

Case 1. The sequence has infinitely many peak points. In this case, if $n_1 < n_2 < n_3 < \cdots$ are the peak points, then $a_{n_1} > a_{n_2} > a_{n_3} > \cdots$, so $\{a_{n_k}\}$ is the desired (nonincreasing) subsequence.

Case 2. The sequence has only finitely many peak points. In this case, let n_1 be greater than all peak points. Since n_1 is not a peak point, there is some $n_2 > n_1$ such that $a_{n_2} \geq a_{n_1}$. Since n_2 is not a peak point (it is greater than n_1, and hence greater than all peak points) there is some $n_3 > n_2$ such that $a_{n_3} \geq a_{n_2}$. Continuing in this way we obtain the desired (nondecreasing) subsequence. ▮

If we assume that our original sequence $\{a_n\}$ is bounded, we can pick up an extra corollary along the way.

COROLLARY (THE BOLZANO-WEIERSTRASS THEOREM) Every bounded sequence has a convergent subsequence.

Without some additional assumptions this is as far as we can go: it is easy to construct sequences having many, evenly infinitely many, subsequences converging to different numbers (see Problem 3). There is a reasonable assumption to add, which yields a necessary and sufficient condition for convergence of any sequence. Although this condition will not be crucial for our work, it does simplify many proofs. Moreover, this condition plays a fundamental role in more advanced investigations, and for this reason alone it is worth stating now.

If a sequence converges, so that the individual terms are eventually all close to the same number, then the difference of any two such individual terms should be very small. To be precise, if $\lim_{n\to\infty} a_n = l$ for some l, then for any $\varepsilon > 0$ there is an N such that $|a_n - l| < \varepsilon/2$ for $n > N$; now if both $n > N$ and $m > N$, then

$$|a_n - a_m| \le |a_n - l| + |l - a_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This final inequality, $|a_n - a_m| < \varepsilon$, which eliminates mention of the limit l, can be used to formulate a condition (the Cauchy condition) which is clearly necessary for convergence of a sequence.

DEFINITION

A sequence $\{a_n\}$ is a **Cauchy sequence** if for every $\varepsilon > 0$ there is a natural number N such that, for all m and n,

$$\text{if } m, n > N, \text{ then } |a_n - a_m| < \varepsilon.$$

(This condition is usually written $\lim_{m,n\to\infty} |a_m - a_n| = 0$.)

The beauty of the Cauchy condition is that it is also sufficient to ensure convergence of a sequence. After all our preliminary work, there is very little left to do in order to prove this.

THEOREM 3 A sequence $\{a_n\}$ converges if and only if it is a Cauchy sequence.

PROOF We have already shown that $\{a_n\}$ is a Cauchy sequence if it converges. The proof of the converse assertion contains only one tricky feature: showing that every Cauchy sequence $\{a_n\}$ is bounded. If we take $\varepsilon = 1$ in the definition of a Cauchy sequence we find that there is some N such that

$$|a_m - a_n| < 1 \qquad \text{for } m, n > N.$$

In particular, this means that

$$|a_m - a_{N+1}| < 1 \qquad \text{for all } m > N.$$

Thus $\{a_m : m > N\}$ is bounded; since there are only finitely many other a_i's the whole sequence is bounded.

The corollary to the Lemma thus implies that some subsequence of $\{a_n\}$ converges.

Only one point remains, whose proof will be left to you: if a subsequence of a Cauchy sequence converges, then the Cauchy sequence itself converges. ▌

PROBLEMS

1. Verify each of the following limits.

(i) $\displaystyle\lim_{n\to\infty} \frac{n}{n+1} = 1.$

(ii) $\displaystyle\lim_{n\to\infty} \frac{n+3}{n^3+4} = 0.$

(iii) $\displaystyle\lim_{n\to\infty} \sqrt[8]{n^2+1} - \sqrt[4]{n+1} = 0.$ Hint: You should at least be able to prove that $\displaystyle\lim_{n\to\infty} \sqrt[8]{n^2+1} - \sqrt[8]{n^2} = 0.$

(iv) $\displaystyle\lim_{n\to\infty} \frac{n!}{n^n} = 0.$

(v) $\displaystyle\lim_{n\to\infty} \sqrt[n]{a} = 1, \qquad a > 0.$

(vi) $\displaystyle\lim_{n\to\infty} \sqrt[n]{n} = 1.$

(vii) $\displaystyle\lim_{n\to\infty} \sqrt[n]{n^2+n} = 1.$

(viii) $\displaystyle\lim_{n\to\infty} \sqrt[n]{a^n+b^n} = \max(a,b), \qquad a, b \geq 0.$

(ix) $\displaystyle\lim_{n\to\infty} \frac{\alpha(n)}{n} = 0$, where $\alpha(n)$ is the number of primes which divide n. Hint: The fact that each prime is ≥ 2 gives a very simple estimate of how small $\alpha(n)$ must be.

*(x) $\displaystyle\lim_{n\to\infty} \frac{\sum_{k=1}^{n} k^p}{n^{p+1}} = \frac{1}{p+1}.$

2. Find the following limits.

(i) $\displaystyle\lim_{n\to\infty} \frac{n}{n+1} - \frac{n+1}{n}.$

(ii) $\displaystyle\lim_{n\to\infty} n - \sqrt{n+a}\sqrt{n+b}.$

(iii) $\displaystyle\lim_{n\to\infty} \frac{2^n+(-1)^n}{2^{n+1}+(-1)^{n+1}}.$

(iv) $\displaystyle\lim_{n\to\infty} \frac{(-1)^n\sqrt{n}\sin(n^n)}{n+1}.$

(v) $\displaystyle\lim_{n\to\infty} \frac{a^n-b^n}{a^n+b^n}.$

(vi) $\displaystyle\lim_{n\to\infty} nc^n, \qquad |c| < 1.$

(vii) $\displaystyle\lim_{n\to\infty} \frac{2^{n^2}}{n!}$.

3. (a) What can be said about the sequence $\{a_n\}$ if it converges and each a_n is an integer?

(b) Find all convergent subsequences of the sequence $1, -1, 1, -1, 1, -1, \dots$. (There are infinitely many, although there are only two limits which such subsequences can have.)

(c) Find all convergent subsequences of the sequence 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, ... (There are infinitely many limits which such subsequences can have.)

(d) Consider the sequence

$$\tfrac{1}{2}, \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{1}{4}, \tfrac{2}{4}, \tfrac{3}{4}, \tfrac{1}{5}, \tfrac{2}{5}, \tfrac{3}{5}, \tfrac{4}{5}, \tfrac{1}{6}, \dots.$$

For which numbers α is there a subsequence converging to α?

4. (a) Prove that if a subsequence of a Cauchy sequence converges, then so does the original Cauchy sequence.

(b) Prove that any subsequence of a convergent sequence converges.

5. (a) Prove that if $0 < a < 2$, then $a < \sqrt{2a} < 2$.

(b) Prove that the sequence

$$\sqrt{2},\ \sqrt{2\sqrt{2}},\ \sqrt{2\sqrt{2\sqrt{2}}},\ \dots$$

converges.

(c) Find the limit. Hint: Notice that if $\lim_{n\to\infty} a_n = l$, then $\lim_{n\to\infty} \sqrt{2a_n} = \sqrt{2l}$, by Theorem 1.

6. Let $0 < a_1 < b_1$ and define

$$a_{n+1} = \sqrt{a_n b_n}, \qquad b_{n+1} = \frac{a_n + b_n}{2}.$$

(a) Prove that the sequences $\{a_n\}$ and $\{b_n\}$ each converge.

(b) Prove that they have the same limit.

7. In Problem 2-16 we saw that any rational approximation k/l to $\sqrt{2}$ can be replaced by a better approximation $(k + 2l)/(k + l)$. In particular, starting with $k = l = 1$, we obtain

$$1, \frac{3}{2}, \frac{7}{5}, \dots.$$

(a) Prove that this sequence is given recursively by

$$a_1 = 1, \qquad a_{n+1} = 1 + \frac{1}{1 + a_n}.$$

(b) Prove that $\lim_{n\to\infty} a_n = \sqrt{2}$. This gives the so-called *continued fraction expansion*

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}.$$

Hint: Consider separately the subsequences $\{a_{2n}\}$ and $\{a_{2n+1}\}$.

(c) Prove that for any natural numbers a and b,

$$\sqrt{a^2+b} = a + \cfrac{b}{2a + \cfrac{b}{2a + \cdots}}.$$

8. Identify the function $f(x) = \lim_{n\to\infty}(\lim_{k\to\infty}(\cos n!\pi x)^{2k})$. (It has been mentioned many times in this book.)

9. Many impressive looking limits can be evaluated easily (especially by the person who makes them up), because they are really lower or upper sums in disguise. With this remark as hint, evaluate each of the following. (Warning: the list contains one red herring which can be evaluated by elementary considerations.)

(i) $\displaystyle \lim_{n\to\infty} \frac{\sqrt[n]{e} + \sqrt[n]{e^2} + \cdots + \sqrt[n]{e^n}}{n}.$

(ii) $\displaystyle \lim_{n\to\infty} \frac{\sqrt[n]{e} + \sqrt[n]{e^2} + \cdots + \sqrt[n]{e^{2n}}}{n}.$

(iii) $\displaystyle \lim_{n\to\infty} \left(\frac{1}{n+1} + \cdots + \frac{1}{2n}\right).$

(iv) $\displaystyle \lim_{n\to\infty} \left(\frac{1}{n^2} + \frac{1}{(n+1)^2} + \cdots + \frac{1}{(2n)^2}\right).$

(v) $\displaystyle \lim_{n\to\infty} \left(\frac{n}{(n+1)^2} + \frac{n}{(n+2)^2} + \cdots + \frac{n}{(n+n)^2}\right).$

(vi) $\displaystyle \lim_{n\to\infty} \left(\frac{n}{n^2+1} + \frac{n}{n^2+2^2} + \cdots + \frac{n}{n^2+n^2}\right).$

10. Although limits like $\lim_{n\to\infty} \sqrt[n]{n}$ and $\lim_{n\to\infty} a^n$ can be evaluated using facts about the behavior of the logarithm and exponential functions, this approach is vaguely dissatisfying, because integral roots and powers can be defined without using the exponential function. Some of the standard "elementary" arguments for such limits are outlined here; the basic tools are inequalities derived from the binomial theorem, notably

$$(1+h)^n \geq 1 + nh, \qquad \text{for } h > 0;$$

and, for part (c),

$$(1+h)^n \geq 1 + nh + \frac{n(n-1)}{2}h^2 \geq \frac{n(n-1)}{2}h^2, \qquad \text{for } h > 0.$$

(a) Prove that $\lim_{n\to\infty} a^n = \infty$ if $a > 1$, by setting $a = 1 + h$, where $h > 0$.
(b) Prove that $\lim_{n\to\infty} a^n = 0$ if $0 < a < 1$.
(c) Prove that $\lim_{n\to\infty} \sqrt[n]{a} = 1$ if $a > 1$, by setting $\sqrt[n]{a} = 1+h$ and estimating h.
(d) Prove that $\lim_{n\to\infty} \sqrt[n]{a} = 1$ if $0 < a < 1$.
(e) Prove that $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

11. (a) Prove that a convergent sequence is always bounded.
(b) Suppose that $\lim_{n\to\infty} a_n = 0$, and that some $a_n > 0$. Prove that the set of all numbers a_n actually has a maximum member.

12. (a) Prove that

$$\frac{1}{n+1} < \log(n+1) - \log n < \frac{1}{n}.$$

(b) If

$$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n,$$

show that the sequence $\{a_n\}$ is decreasing, and that each $a_n \geq 0$. It follows that there is a number

$$\gamma = \lim_{n\to\infty} \left(1 + \cdots + \frac{1}{n} - \log n\right).$$

This number, known as Euler's number, has proved to be quite refractory; it is not even known whether γ is rational.

13. (a) Suppose that f is increasing on $[1, \infty)$. Show that

$$f(1) + \cdots + f(n-1) < \int_1^n f(x)\,dx < f(2) + \cdots + f(n).$$

(b) Now choose $f = \log$ and show that

$$\frac{n^n}{e^{n-1}} < n! < \frac{(n+1)^{n+1}}{e^n};$$

it follows that

$$\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}.$$

This result shows that $\sqrt[n]{n!}$ is approximately n/e, in the sense that the ratio of these two quantities is close to 1 for large n. But we cannot conclude that $n!$ is close to $(n/e)^n$ in this sense; in fact, this is false. An estimate for $n!$ is very desirable, even for concrete computations, because $n!$ cannot be calculated easily even with logarithm tables. The standard (and difficult) theorem which provides the right information will be found in Problem 27-19.

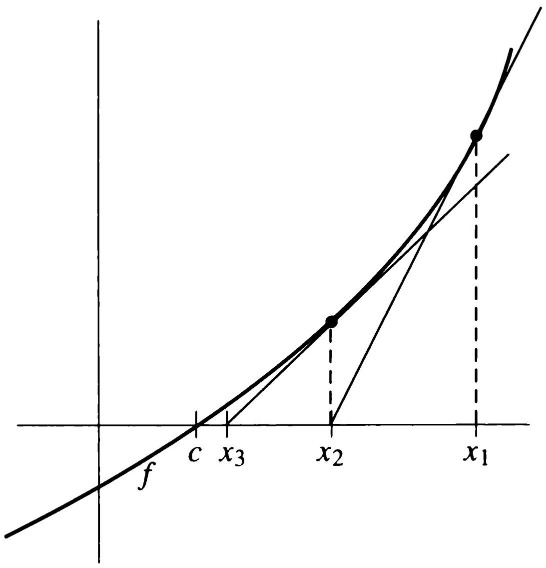

FIGURE 7

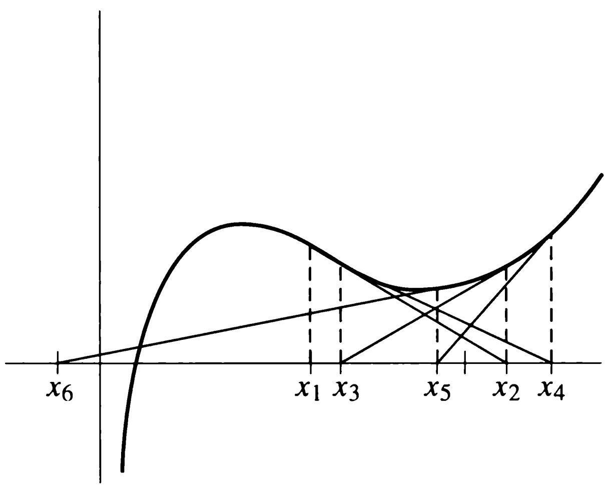

FIGURE 8

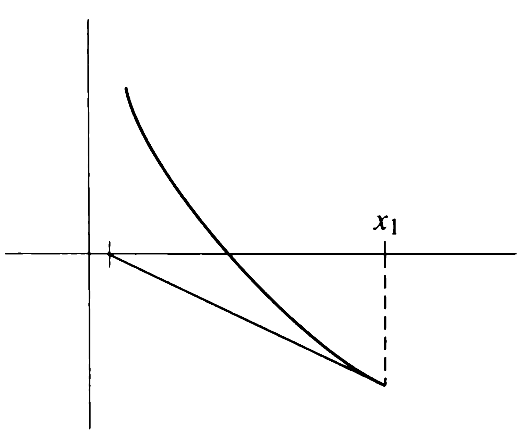

FIGURE 9

14. (a) Show that the tangent line to the graph of f at $(x_1, f(x_1))$ intersects the horizontal axis at $(x_2, 0)$, where

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

This intersection point may be regarded as a rough approximation to the point where the graph of f intersects the horizontal axis. If we now start at x_2 and repeat the process to get x_3, then use x_3 to get x_4, etc., we have a sequence $\{x_n\}$ defined inductively by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Figure 7 suggests that $\{x_n\}$ will converge to a number c with $f(c) = 0$; this is called *Newton's method* for finding a zero of f. In the remainder of this problem we will establish some conditions under which Newton's method works (Figures 8 and 9 show two cases where it doesn't). A few facts about convexity may be found useful; see Chapter 11, Appendix.

(b) Suppose that $f,' f'' > 0$, and that we choose x_1 with $f(x_1) > 0$. Show that $x_1 > x_2 > x_3 > \cdots > c$.

(c) Let $\delta_k = x_k - c$. Then

$$\delta_k = \frac{f(x_k)}{f'(\xi_k)}$$

for some ξ_k in (c, x_k). Show that

$$\delta_{k+1} = \frac{f(x_k)}{f'(\xi_k)} - \frac{f(x_k)}{f'(x_k)}.$$

Conclude that

$$\delta_{k+1} = \frac{f(x_k)}{f'(\xi_k) f'(x_k)} \cdot f''(\eta_k)(x_k - \xi_k)$$

for some η_k in (c, x_k), and then that

$$(*) \qquad \delta_{k+1} \le \frac{f''(\eta_k)}{f'(x_k)} \delta_k{}^2.$$

(d) Let $m = f'(c) = \inf f'$ on $[c, x_1]$ and let $M = \sup f''$ on $[c, x_1]$. Show that Newton's method works if $x_1 - c < m/M$.

(e) What is the formula for x_{n+1} when $f(x) = x^2 - A$?

If we take $A = 2$ and $x_1 = 1.4$ we get

$$\begin{aligned} x_1 &= 1.4 \\ x_2 &= 1.4142857 \\ x_3 &= 1.4142136 \\ x_4 &= 1.4142136, \end{aligned}$$

which is already correct to 7 decimals! Notice that the number of correct decimals at least doubled each time. This is essentially guaranteed by the inequality $(*)$ when $M/m < 1$.

15. Use Newton's method to estimate the zeros of the following functions.

(i)	$f(x) = \tan x - \cos^2 x$	near 0.
(ii)	$f(x) = \cos x - x^2$	near 0.
(iii)	$f(x) = x^3 + x - 1$	on $[0, 1]$.
(iv)	$f(x) = x^3 - 3x^2 + 1$	on $[0, 1]$.

***16.** Prove that if $\lim_{n\to\infty} a_n = l$, then

$$\lim_{n\to\infty} \frac{a_1 + \cdots + a_n}{n} = l.$$

Hint: This problem is very similar to (in fact, it is actually a special case of) Problem 13-40.

17. (a) Prove that if $\lim_{n\to\infty} a_{n+1} - a_n = l$, then $\lim_{n\to\infty} a_n/n = l$. Hint: See the previous problem.

(b) Suppose that f is continuous and $\lim_{x\to\infty} f(x+1) - f(x) = l$. Prove that $\lim_{x\to\infty} f(x)/x = l$. Hint: Let a_n and b_n be the inf and sup of f on $[n, n+1]$.

***18.** Suppose that $a_n > 0$ for each n and that $\lim_{n\to\infty} a_{n+1}/a_n = l$. Prove that $\lim_{n\to\infty} \sqrt[n]{a_n} = l$. Hint: This requires the same sort of argument that works in Problem 16, except using multiplication instead of addition, together with the fact that $\lim_{n\to\infty} \sqrt[n]{a} = 1$, for $a > 0$.

19. (a) Suppose that $\{a_n\}$ is a convergent sequence of points all in $[0, 1]$. Prove that $\lim_{n\to\infty} a_n$ is also in $[0, 1]$.

(b) Find a convergent sequence $\{a_n\}$ of points all in $(0, 1)$ such that $\lim_{n\to\infty} a_n$ is not in $(0, 1)$.

20. Suppose that f is continuous and that the sequence

$$x,\ f(x),\ f(f(x)),\ f(f(f(x))),\ \ldots$$

converges to l. Prove that l is a "fixed point" for f, i.e., $f(l) = l$. Hint: Two special cases have occurred already.

21. (a) Suppose that f is continuous on $[0, 1]$ and that $0 \le f(x) \le 1$ for all x in $[0, 1]$. Problem 7-11 shows that f has a fixed point (in the terminology of Problem 20). If f is *increasing*, a much stronger statement can be made: For any x in $[0, 1]$, the sequence

$$x,\ f(x),\ f(f(x)), \ldots$$

has a limit (which is necessarily a fixed point, by Problem 20). Prove this assertion, by examining the behavior of the sequence for $f(x) > x$ and $f(x) < x$, or by looking at Figure 10. A diagram of this sort is used in Littlewood's *Mathematician's Miscellany* to preach the value of drawing pictures: "For the professional the only proof needed is [this Figure]."

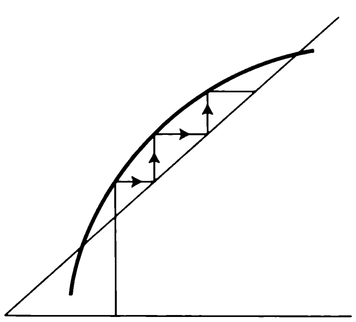

FIGURE 10

*(b) Suppose that f and g are two continuous functions on $[0, 1]$, with $0 \leq f(x) \leq 1$ and $0 \leq g(x) \leq 1$ for all x in $[0, 1]$, which satisfy $f \circ g = g \circ f$. Suppose, moreover, that f is increasing. Show that f and g have a common fixed point; in other words, there is a number l such that $f(l) = l = g(l)$. Hint: Begin by choosing a fixed point for g.

For a long time mathematicians amused themselves by asking whether the conclusion of part (b) holds without the assumption that f is increasing, but two independent announcements in the *Notices* of the American Mathematical Society, Volume 14, Number 2 give counterexamples, so it was probably a pretty silly problem all along.

The trick in Problem 20 is really much more valuable than Problem 20 might suggest, and some of the most important "fixed point theorems" depend upon looking at sequences of the form $x, f(x), f(f(x)), \dots$. A special, but representative, case of one such theorem is treated in Problem 23 (for which the next problem is preparation).

22. (a) Use Problem 2-5 to show that if $c \neq 1$, then

$$c^m + c^{m+1} + \cdots + c^n = \frac{c^m - c^{n+1}}{1 - c}.$$

(b) Suppose that $|c| < 1$. Prove that

$$\lim_{m,n \to \infty} c^m + \cdots + c^n = 0.$$

(c) Suppose that $\{x_n\}$ is a sequence with $|x_n - x_{n+1}| \leq c^n$, where $0 < c < 1$. Prove that $\{x_n\}$ is a Cauchy sequence.

23. Suppose that f is a function on $\mathbf{R}$ such that

$$(*) \qquad |f(x) - f(y)| \leq c|x - y|, \qquad \text{for all } x \text{ and } y,$$

where $c < 1$. (Such a function is called a *contraction*.)

(a) Prove that f is continuous.
(b) Prove that f has at most one fixed point.
(c) By considering the sequence

$$x,\ f(x),\ f(f(x)),\ \dots,$$

for any x, prove that f does have a fixed point. (This result, in a more general setting, is known as the "contraction lemma.")

24. (a) Prove that if f is differentiable and $|f'| < 1$, then f has at most one fixed point.
(b) Prove that if $|f'(x)| \leq c < 1$ for all x, then f has a fixed point.
(c) Give an example to show that the hypothesis $|f'(x)| \leq 1$ is not sufficient to ensure that f has a fixed point.

25. This problem is a sort of converse to the previous problem. Let b_n be a sequence defined by $b_1 = a$, $b_{n+1} = f(b_n)$. Prove that if $b = \lim_{n \to \infty} b_n$ exists and f' is continuous at b, then $|f'(b)| \le 1$ (provided that we don't already have $b_n = b$ for some n). Hint: If $|f'(b)| > 1$, then $|f'(x)| > 1$ for all x in an interval around b, and b_n will be in this interval for large enough n. Now consider f on the interval $[b, b_n]$.

26. This problem investigates for which $a > 0$ the symbol

$$a^{a^{a^{\cdot^{\cdot^{\cdot}}}}}$$

makes sense. In other words, if we define $b_1 = a$, $b_{n+1} = a^{b_n}$, when does $b = \lim_{n\to\infty} b_n$ exist? Note that if b exists, then $a^b = b$ by Problem 20.

(a) If b exists, then a can be written in the form $y^{1/y}$ for some y. Describe the graph of $g(y) = y^{1/y}$ and conclude that $0 < a \le e^{1/e}$.
(b) Suppose that $1 \le a \le e^{1/e}$. Show that $\{b_n\}$ is increasing and also $b_n \le e$. This proves that b exists (and also that $b \le e$).

The analysis for $a < 1$ is more difficult.

(c) Using Problem 25, show that if b exists, then $e^{-1} \le b \le e$. Then show that $e^{-e} \le a \le e^{1/e}$.

From now on we will suppose that $e^{-e} \le a < 1$.

(d) Show that the function

$$f(x) = \frac{a^x}{\log x}$$

is decreasing on the interval $(0, 1)$.
(e) Let b be the unique number such that $a^b = b$. Show that $a < b < 1$. Using part (e), show that if $0 < x < b$, then $x < a^{a^x} < b$. Conclude that $l = \lim_{n\to\infty} a_{2n+1}$ exists and that $a^{a^l} = l$.
(f) Using part (e) again, show that $l = b$.
(g) Finally, show that $\lim_{n\to\infty} a_{2n+2} = b$, so that $\lim_{n\to\infty} b_n = b$.

27. Let $\{x_n\}$ be a sequence which is bounded, and let

$$y_n = \sup\{x_n, x_{n+1}, x_{n+2}, \dots\}.$$

(a) Prove that the sequence $\{y_n\}$ converges. The limit $\lim_{n\to\infty} y_n$ is denoted by $\overline{\lim_{n\to\infty}}\, x_n$ or $\limsup_{n\to\infty} x_n$, and called the **limit superior**, or **upper limit**, of the sequence $\{x_n\}$.
(b) Find $\overline{\lim_{n\to\infty}}\, x_n$ for each of the following:

(i) $x_n = \dfrac{1}{n}$.

(ii) $x_n = (-1)^n \dfrac{1}{n}$.

(iii) $x_n = (-1)^n \left[1 + \dfrac{1}{n} \right]$.

(iv) $x_n = \sqrt[n]{n}$.

(c) Define $\varliminf_{n\to\infty} x_n$ (or $\liminf_{n\to\infty} x_n$) and prove that

$$\varliminf_{n\to\infty} x_n \le \varlimsup_{n\to\infty} x_n.$$

(d) Prove that $\lim_{n\to\infty} x_n$ exists if and only if $\varlimsup_{n\to\infty} x_n = \varliminf_{n\to\infty} x_n$ and that in this case $\lim_{n\to\infty} x_n = \varlimsup_{n\to\infty} x_n = \varliminf_{n\to\infty} x_n$.

(e) Recall the definition, in Problem 8-18, of $\overline{\lim}\, A$ for a bounded set A. Prove that if the numbers x_n are distinct, then $\varlimsup_{n\to\infty} x_n = \overline{\lim}\, A$, where $A = \{x_n : n \text{ in } \mathbf{N}\}$.

28. In the Appendix to Chapter 8 we defined uniform continuity of a function on an interval. If $f(x)$ is defined only for rational x, this concept still makes sense: we say that f is uniformly continuous on an interval if for every $\varepsilon > 0$ there is some $\delta > 0$ such that, if x and y are rational numbers in the interval and $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$.

(a) Let x be any (rational or irrational) point in the interval, and let $\{x_n\}$ be a sequence of *rational* points in the interval such that $\lim_{n\to\infty} x_n = x$. Show that the sequence $\{f(x_n)\}$ converges.

(b) Prove that the limit of the sequence $\{f(x_n)\}$ doesn't depend on the choice of the sequence $\{x_n\}$.

We will denote this limit by $\bar{f}(x)$, so that $\bar{f}$ is an extension of f to the whole interval.

(c) Prove that the extended function $\bar{f}$ is uniformly continuous on the interval.

29. Let $a > 0$, and for rational x let $f(x) = a^x$, as defined in the usual elementary algebraic way. This problem shows directly that f can be extended to a continuous function $\bar{f}$ on the whole line. Problem 28 provides the necessary machinery.

(a) For rational $x < y$, show that $a^x < a^y$ for $a > 1$ and $a^x > a^y$ for $a < 1$.

(b) Using Problem 10, show that for any $\varepsilon > 0$ we have $|a^x - 1| < \varepsilon$ for rational numbers x close enough to 0.

(c) Using the equation $a^x - a^y = a^y(a^{x-y} - 1)$, prove that on any closed interval f is uniformly continuous, in the sense of Problem 28.

(d) Show that the extended function $\bar{f}$ of Problem 28 is increasing for $a > 1$ and decreasing for $a < 1$ and satisfies $\bar{f}(x + y) = \bar{f}(x)\bar{f}(y)$.

***30.** The Bolzano-Weierstrass Theorem is usually stated, and also proved, quite differently than in the text—the classical statement uses the notion of limit points. A point x is a **limit point** of the set A if for every $\varepsilon > 0$ there is a point a in A with $|x - a| < \varepsilon$ but $x \neq a$.

(a) Find all limit points of the following sets.

(i) $\left\{\frac{1}{n} : n \text{ in } \mathbf{N}\right\}$.

(ii) $\left\{\frac{1}{n} + \frac{1}{m} : n \text{ and } m \text{ in } \mathbf{N}\right\}$.

(iii) $\left\{(-1)^n\left[1 + \frac{1}{n}\right] : n \text{ in } \mathbf{N}\right\}$.

(iv) $\mathbf{Z}$.

(v) $\mathbf{Q}$.

(b) Prove that x is a limit point of A if and only if for every $\varepsilon > 0$ there are infinitely many points a of A satisfying $|x - a| < \varepsilon$.

(c) Prove that $\overline{\lim}\, A$ is the largest limit point of A, and $\underline{\lim}\, A$ the smallest.

The usual form of the Bolzano-Weierstrass Theorem states that if A is an infinite set of numbers contained in a closed interval $[a, b]$, then some point of $[a, b]$ is a limit point of A. Prove this in two ways:

(d) Using the form already proved in the text. Hint: Since A is infinite, there are distinct numbers $x_1, x_2, x_3, \dots$ in A.

(e) Using the Nested Intervals Theorem. Hint: If $[a, b]$ is divided into two intervals, at least one must contain infinitely many points of A.

31. (a) Use the Bolzano-Weierstrass Theorem to prove that if f is continuous on $[a, b]$, then f is bounded above on $[a, b]$. Hint: If f is not bounded above, then there are points x_n in $[a, b]$ with $f(x_n) > n$.

(b) Also use the Bolzano-Weierstrass Theorem to prove that if f is continuous on $[a, b]$, then f is uniformly continuous on $[a, b]$ (see Chapter 8, Appendix).

****32.** (a) Let $\{a_n\}$ be the sequence

$$\tfrac{1}{2}, \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{1}{4}, \tfrac{2}{4}, \tfrac{3}{4}, \tfrac{1}{5}, \tfrac{2}{5}, \tfrac{3}{5}, \tfrac{4}{5}, \tfrac{1}{6}, \tfrac{2}{6}, \dots.$$

Suppose that $0 \le a < b \le 1$. Let $N(n; a, b)$ be the number of integers $j \le n$ such that a_j is in (a, b). (Thus $N(2; \frac{1}{3}, \frac{2}{3}) = 2$, and $N(4; \frac{1}{3}, \frac{2}{3}) = 3$.) Prove that

$$\lim_{n \to \infty} \frac{N(n; a, b)}{n} = b - a.$$

(b) A sequence $\{a_n\}$ of numbers in $[0, 1]$ is called **uniformly distributed** in $[0, 1]$ if

$$\lim_{n \to \infty} \frac{N(n; a, b)}{n} = b - a$$

for all a and b with $0 \le a < b \le 1$. Prove that if s is a step function defined on $[0, 1]$, and $\{a_n\}$ is uniformly distributed in $[0, 1]$, then

$$\int_0^1 s = \lim_{n\to\infty} \frac{s(a_1) + \cdots + s(a_n)}{n}.$$

(c) Prove that if $\{a_n\}$ is uniformly distributed in $[0, 1]$ and f is integrable on $[0, 1]$, then

$$\int_0^1 f = \lim_{n\to\infty} \frac{f(a_1) + \cdots + f(a_n)}{n}.$$

****33.** (a) Let f be a function defined on $[0, 1]$ such that $\lim_{y\to a} f(y)$ exists for all a in $[0, 1]$. For any $\varepsilon > 0$ prove that there are only finitely many points a in $[0, 1]$ with $|\lim_{y\to a} f(y) - f(a)| > \varepsilon$. Hint: Show that the set of such points cannot have a limit point x, by showing that $\lim_{y\to x} f(y)$ could not exist.

(b) Prove that, in the terminology of Problem 21-5, the set of points where f is discontinuous is countable. This finally answers the question of Problem 6-17: If f has only removable discontinuities, then f is continuous except at a countable set of points, and in particular, f cannot be discontinuous everywhere.