CHAPTER 6 CONTINUOUS FUNCTIONS

If f is an arbitrary function, it is not necessarily true that

$$\lim_{x \to a} f(x) = f(a).$$

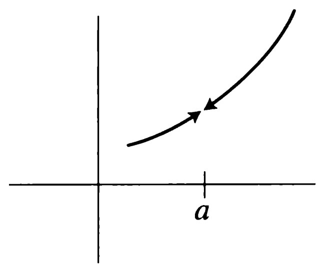


FIGURE 1

In fact, there are many ways this can fail to be true. For example, f might not even be defined at a, in which case the equation makes no sense (Figure 1).

Again, $\lim_{x \to a} f(x)$ might not exist (Figure 2). Finally, as illustrated in Figure 3, even if f is defined at a and $\lim_{x \to a} f(x)$ exists, the limit might not equal $f(a)$.

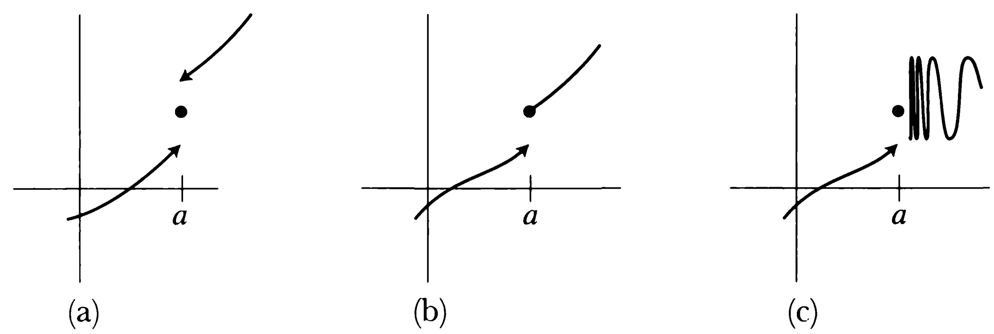


FIGURE 2

We would like to regard all behavior of this type as abnormal and honor, with some complimentary designation, functions which do not exhibit such peculiarities. The term which has been adopted is "continuous." Intuitively, a function f is continuous if the graph contains no breaks, jumps, or wild oscillations. Although this description will usually enable you to decide whether a function is continuous simply by looking at its graph (a skill well worth cultivating) it is easy to be fooled, and the precise definition is *very* important.

DEFINITION

The function f is **continuous at a** if

$$\lim_{x \to a} f(x) = f(a).$$

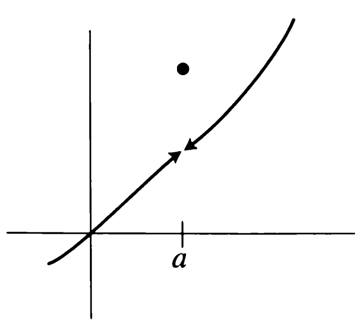


FIGURE 3

We will have no difficulty finding many examples of functions which are, or are not, continuous at some number a—every example involving limits provides an example about continuity, and Chapter 5 certainly provides enough of these.

The function $f(x) = \sin 1/x$ is not continuous at 0, because it is not even defined at 0, and the same is true of the function $g(x) = x \sin 1/x$. On the other hand, if we are willing to extend the second of these functions, that is, if we wish to define a new function G by

$$G(x) = \begin{cases} x \sin 1/x, & x \neq 0 \\ a, & x = 0, \end{cases}$$

then the choice of $a = G(0)$ can be made in such a way that G will be continuous at 0—to do this we can (if fact, we must) define $G(0) = 0$ (Figure 4). This sort of extension is not possible for f; if we define

$$F(x) = \begin{cases} \sin 1/x, & x \neq 0 \\ a, & x = 0, \end{cases}$$

then F will not be continuous at 0, no matter what a is, because $\lim_{x \to 0} f(x)$ does not exist.

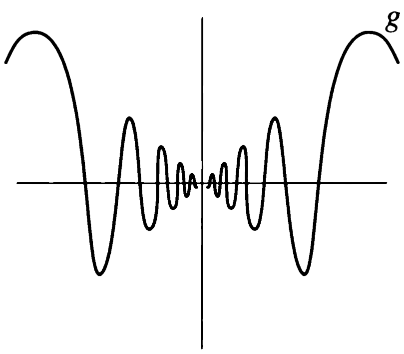
The function

$$f(x) = \begin{cases} x, & x \text{ rational} \\ 0, & x \text{ irrational} \end{cases}$$

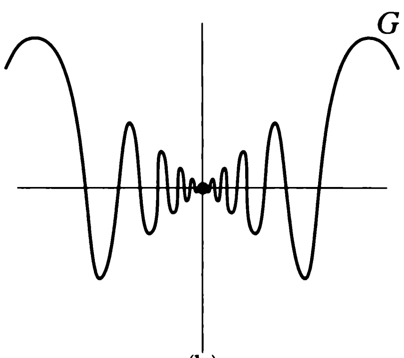
is not continuous at a, if $a \neq 0$, since $\lim_{x \to a} f(x)$ does not exist. However, $\lim_{x \to 0} f(x) = 0 = f(0)$, so f is continuous at precisely one point, 0.

The functions $f(x) = c$, $g(x) = x$, and $h(x) = x^2$ are continuous at all numbers a, since

$$\begin{aligned} \lim_{x \to a} f(x) &= \lim_{x \to a} c = c = f(a), \\ \lim_{x \to a} g(x) &= \lim_{x \to a} x = a = g(a), \\ \lim_{x \to a} h(x) &= \lim_{x \to a} x^2 = a^2 = h(a). \end{aligned}$$



(a)



(b)

FIGURE 4

Finally, consider the function

$$f(x) = \begin{cases} 0, & x \text{ irrational} \\ 1/q, & x = p/q \text{ in lowest terms.} \end{cases}$$

In Chapter 5 we showed that $\lim_{x \to a} f(x) = 0$ for all a (actually, only for $0 < a < 1$, but you can easily see that this is true for all a). Since $0 = f(a)$ only when a is irrational, this function is continuous at a if a is irrational, but not if a is rational.

It is even easier to give examples of continuity if we prove two simple theorems.

THEOREM 1 If f and g are continuous at a, then

(1) $f + g$ is continuous at a,
(2) $f \cdot g$ is continuous at a.

Moreover, if $g(a) \neq 0$, then

(3) $1/g$ is continuous at a.

PROOF Since f and g are continuous at a,

$$\lim_{x \to a} f(x) = f(a) \quad \text{and} \quad \lim_{x \to a} g(x) = g(a).$$

By Theorem 2(1) of Chapter 5 this implies that

$$\lim_{x \to a} (f + g)(x) = f(a) + g(a) = (f + g)(a),$$

which is just the assertion that $f + g$ is continuous at a. The proofs of parts (2) and (3) are left to you. ▮

Starting with the functions $f(x) = c$ and $f(x) = x$, which are continuous at a, for every a, we can use Theorem 1 to conclude that a function

$$f(x) = \frac{b_n x^n + b_{n-1} x^{n-1} + \cdots + b_0}{c_m x^m + c_{m-1} x^{m-1} + \cdots + c_0}$$

is continuous at every point in its domain. But it is harder to get much further than that. When we discuss the sine function in detail it will be easy to prove that sin is continuous at a for all a; let us assume this fact meanwhile. A function like

$$f(x) = \frac{\sin^2 x + x^2 + x^4 \sin x}{\sin^{27} x + 4x^2 \sin^2 x}$$

can now be proved continuous at every point in its domain. But we are still unable to prove the continuity of a function like $f(x) = \sin(x^2)$; we obviously need a theorem about the composition of continuous functions. Before stating this theorem, the following point about the definition of continuity is worth noting. If we translate the equation $\lim_{x \to a} f(x) = f(a)$ according to the definition of limits, we obtain

$$\begin{gathered}\text{for every } \varepsilon > 0 \text{ there is } \delta > 0 \text{ such that, for all } x,\\ \text{if } 0 < |x - a| < \delta, \text{ then } |f(x) - f(a)| < \varepsilon.\end{gathered}$$

But in this case, where the limit is $f(a)$, the phrase

$$0 < |x - a| < \delta$$

may be changed to the simpler condition

$$|x - a| < \delta,$$

since if $x = a$ it is certainly true that $|f(x) - f(a)| < \varepsilon$.

THEOREM 2 If g is continuous at a, and f is continuous at $g(a)$, then $f \circ g$ is continuous at a. (Notice that f is required to be continuous at $g(a)$, not at a.)

PROOF Let $\varepsilon > 0$. We wish to find a $\delta > 0$ such that for all x,

$$\begin{aligned}\text{if } |x - a| < \delta, \text{ then } &|(f \circ g)(x) - (f \circ g)(a)| < \varepsilon,\\ \text{i.e., } &|f(g(x)) - f(g(a))| < \varepsilon.\end{aligned}$$

shows that this description is a little too optimistic, but it is nevertheless true that there are many important results involving functions which are continuous on an interval. There theorems are generally much harder than the ones in this chapter, but there is a simple theorem which forms a bridge between the two kinds of results. The hypothesis of this theorem requires continuity at only a single point, but the conclusion describes the behavior of the function on some interval containing the point. Although this theorem is really a lemma for later arguments, it is included here as a preview of things to come.

THEOREM 3 Suppose f is continuous at a, and $f(a) > 0$. Then $f(x) > 0$ for all x in some interval containing a; more precisely, there is a number $\delta > 0$ such that $f(x) > 0$ for all x satisfying $|x - a| < \delta$. Similarly, if $f(a) < 0$, then there is a number $\delta > 0$ such that $f(x) < 0$ for all x satisfying $|x - a| < \delta$.

PROOF Consider the case $f(a) > 0$. Since f is continuous at a, for every $\varepsilon > 0$ there is a $\delta > 0$ such that, for all x,

$$\text{if } |x - a| < \delta, \text{ then } |f(x) - f(a)| < \varepsilon,$$
$$\text{i.e., } -\varepsilon < f(x) - f(a) < \varepsilon.$$

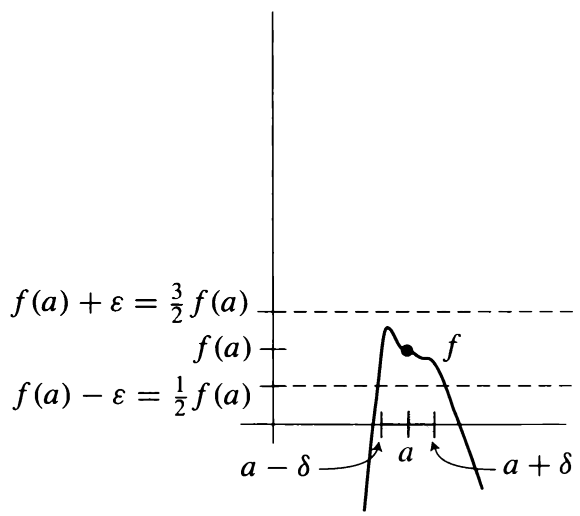


FIGURE 5

In particular, this must hold for $\varepsilon = \frac{1}{2}f(a)$, since $\frac{1}{2}f(a) > 0$ (Figure 5). Thus there is $\delta > 0$ so that for all x,

$$\text{if } |x - a| < \delta, \text{ then } -\tfrac{1}{2}f(a) < f(x) - f(a) < \tfrac{1}{2}f(a),$$

and this implies that $f(x) > \frac{1}{2}f(a) > 0$. (We could even have picked ε to be $f(a)$ itself, leading to a proof that is more elegant, but more confusing to picture.)

A similar proof can be given in the case $f(a) < 0$; take $\varepsilon = -\frac{1}{2}f(a)$. Or one can apply the first case to the function $-f$. ▮

PROBLEMS

1. For which of the following functions f is there a continuous function F with domain $\mathbf{R}$ such that $F(x) = f(x)$ for all x in the domain of f?

(i) $f(x) = \dfrac{x^2 - 4}{x - 2}$.

(ii) $f(x) = \dfrac{|x|}{x}$.

(iii) $f(x) = 0$, x irrational.

(iv) $f(x) = 1/q$, $x = p/q$ rational in lowest terms.

2. At which points are the functions of Problems 4-17 and 4-19 continuous?

We first use continuity of f to estimate how close $g(x)$ must be to $g(a)$ in order for this inequality to hold. Since f is continuous at $g(a)$, there is a $\delta' > 0$ such that for all y,

$$(1) \qquad \text{if } |y - g(a)| < \delta', \text{ then } |f(y) - f(g(a))| < \varepsilon.$$

In particular, this means that

$$(2) \qquad \text{if } |g(x) - g(a)| < \delta', \text{ then } |f(g(x)) - f(g(a))| < \varepsilon.$$

We now use continuity of g to estimate how close x must be to a in order for the inequality $|g(x) - g(a)| < \delta'$ to hold. The number δ' is a positive number just like any other positive number; we can therefore take δ' as the ε (!) in the definition of continuity of g at a. We conclude that there is a $\delta > 0$ such that, for all x,

$$(3) \qquad \text{if } |x - a| < \delta, \text{ then } |g(x) - g(a)| < \delta'.$$

Combining (2) and (3) we see that for all x,

$$\text{if } |x - a| < \delta, \text{ then } |f(g(x)) - f(g(a))| < \varepsilon. \blacksquare$$

We can now reconsider the function

$$f(x) = \begin{cases} x \sin 1/x, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

We have already noted that f is continuous at 0. A few applications of Theorems 1 and 2, together with the continuity of sin, show that f is also continuous at a, for $a \neq 0$. Functions like $f(x) = \sin(x^2 + \sin(x + \sin^2(x^3)))$ should be equally easy for you to analyze.

The few theorems of this chapter have all been related to continuity of functions at a single point, but the concept of continuity doesn't begin to be really interesting until we focus our attention on functions which are continuous at all points of some interval. If f is continuous at x for all x in (a, b), then f is called **continuous on** (a, b); as a "special case", f is **continuous on** $\mathbf{R} = (-\infty, \infty)$ [see page 57] if it is continuous at x for all x in $\mathbf{R}$. Continuity on a closed interval must be defined a little differently; a function f is called **continuous on** $[a, b]$ if

(1) f is continuous at x for all x in (a, b),
(2) $\lim_{x \to a^+} f(x) = f(a)$ and $\lim_{x \to b^-} f(x) = f(b)$.

(We also often simply say that a function is continuous if it is continuous at x for all x in its domain.)

Functions which are continuous on an interval are usually regarded as especially well behaved; indeed continuity might be specified as the first condition which a "reasonable" function ought to satisfy. A continuous function is sometimes described, intuitively, as one whose graph can be drawn without lifting your pencil from the paper. Consideration of the function

$$f(x) = \begin{cases} x \sin 1/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

3. (a) Suppose that f is a function satisfying $|f(x)| \le |x|$ for all x. Show that f is continuous at 0. (Notice that $f(0)$ must equal 0.)
(b) Give an example of such a function f which is not continuous at any $a \neq 0$.
(c) Suppose that g is continuous at 0 and $g(0) = 0$, and $|f(x)| \le |g(x)|$. Prove that f is continuous at 0.

4. Give an example of a function f such that f is continuous nowhere, but $|f|$ is continuous everywhere.

5. For each number a, find a function which is continuous at a, but not at any other points.

6. (a) Find a function f which is discontinuous at $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$ but continuous at all other points.
(b) Find a function f which is discontinuous at $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$, and at 0, but continuous at all other points.

7. Suppose that f satisfies $f(x + y) = f(x) + f(y)$, and that f is continuous at 0. Prove that f is continuous at a for all a.

8. Suppose that f is continuous at a and $f(a) = 0$. Prove that if $\alpha \neq 0$, then $f + \alpha$ is nonzero in some open interval containing a.

9. (a) Suppose f is defined at a but is not continuous at a. Prove that for some number $\varepsilon > 0$ there are numbers x arbitrarily close to a with $|f(x) - f(a)| > \varepsilon$. Illustrate graphically.
(b) Conclude that for some number $\varepsilon > 0$ *either* there are numbers x arbitrarily close to a with $f(x) < f(a) - \varepsilon$ *or* there are numbers x arbitrarily close to a with $f(x) > f(a) + \varepsilon$.

10. (a) Prove that if f is continuous at a, then so is $|f|$.
(b) Prove that every function f continuous on $\mathbf{R}$ can be written $f = E + O$, where E is even and continuous and O is odd and continuous.
(c) Prove that if f and g are continuous, then so are $\max(f, g)$ and $\min(f, g)$.
(d) Prove that every continuous f can be written $f = g - h$, where g and h are nonnegative and continuous.

11. Prove Theorem 1(3) by using Theorem 2 and continuity of the function $f(x) = 1/x$.

***12.** (a) Prove that if f is continuous at l and $\lim_{x \to a} g(x) = l$, then $\lim_{x \to a} f(g(x)) = f(l)$. (You can go right back to the definitions, but it is easier to consider the function G with $G(x) = g(x)$ for $x \neq a$, and $G(a) = l$.)
(b) Show that if continuity of f at l is not assumed, then it is not generally true that $\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x))$. Hint: Try $f(x) = 0$ for $x \neq l$, and $f(l) = 1$.

13. (a) Prove that if f is continuous on $[a, b]$, then there is a function g which is continuous on $\mathbf{R}$, and which satisfies $g(x) = f(x)$ for all x in $[a, b]$. Hint: Since you obviously have a great deal of choice, try making g constant on $(-\infty, a]$ and $[b, \infty)$.

(b) Give an example to show that this assertion is false if $[a, b]$ is replaced by (a, b).

14. (a) Suppose that g and h are continuous at a, and that $g(a) = h(a)$. Define $f(x)$ to be $g(x)$ if $x \geq a$ and $h(x)$ if $x \leq a$. Prove that f is continuous at a.

(b) Suppose g is continuous on $[a, b]$ and h is continuous on $[b, c]$ and $g(b) = h(b)$. Let $f(x)$ be $g(x)$ for x in $[a, b]$ and $h(x)$ for x in $[b, c]$. Show that f is continuous on $[a, c]$. (Thus, continuous functions can be "pasted together".)

15. Prove that if f is continuous at a, then for any $\varepsilon > 0$ there is a $\delta > 0$ so that whenever $|x - a| < \delta$ and $|y - a| < \delta$, we have $|f(x) - f(y)| < \varepsilon$.

16. (a) Prove the following version of Theorem 3 for "right-hand continuity": Suppose that $\lim_{x \to a^+} f(x) = f(a)$, and $f(a) > 0$. Then there is a number $\delta > 0$ such that $f(x) > 0$ for all x satisfying $0 \leq x - a < \delta$. Similarly, if $f(a) < 0$, then there is a number $\delta > 0$ such that $f(x) < 0$ for all x satisfying $0 \leq x - a < \delta$.

(b) Prove a version of Theorem 3 when $\lim_{x \to b^-} f(x) = f(b)$.

17. If $\lim_{x \to a} f(x)$ exists, but is $\neq f(a)$, then f is said to have a **removable discontinuity** at a.

(a) If $f(x) = \sin 1/x$ for $x \neq 0$ and $f(0) = 1$, does f have a removable discontinuity at 0? What if $f(x) = x \sin 1/x$ for $x \neq 0$, and $f(0) = 1$?

(b) Suppose f has a removable discontinuity at a. Let $g(x) = f(x)$ for $x \neq a$, and let $g(a) = \lim_{x \to a} f(x)$. Prove that g is continuous at a. (Don't work very hard; this is quite easy.)

(c) Let $f(x) = 0$ if x is irrational, and let $f(p/q) = 1/q$ if p/q is in lowest terms. What is the function g defined by $g(x) = \lim_{y \to x} f(y)$?

*(d) Let f be a function with the property that every point of discontinuity is a removable discontinuity. This means that $\lim_{y \to x} f(y)$ exists for all x, but f may be discontinuous at some (even infinitely many) numbers x. Define $g(x) = \lim_{y \to x} f(y)$. Prove that g is continuous. (This is not quite so easy as part (b).)

**(e) Is there a function f which is discontinuous at every point, and which has only removable discontinuities? (It is worth thinking about this problem now, but mainly as a test of intuition; even if you suspect the correct answer, you will almost certainly be unable to prove it at the present time. See Problem 22-33.)