CHAPTER 26 COMPLEX FUNCTIONS

You will probably not be surprised to learn that a deeper investigation of complex numbers depends on the notion of functions. Until now a function was (intuitively) a rule which assigned real numbers to certain other real numbers. But there is no reason why this concept should not be extended; we might just as well consider a rule which assigns complex numbers to certain other complex numbers. A rigorous definition presents no problems (we will not even accord it the full honors of a formal definition): a function is a collection of pairs of complex numbers which does not contain two distinct pairs with the same first element. Since we consider real numbers to be certain complex numbers, the old definition is really a special case of the new one. Nevertheless, we will sometimes resort to special terminology in order to clarify the context in which a function is being considered. A function f is called **real-valued** if $f(z)$ is a real number for all z in the domain of f, and **complex-valued** to emphasize that it is not necessarily real-valued. Similarly, we will usually state explicitly that a function f is defined on [a subset of] $\mathbf{R}$ in those cases where the domain of f is [a subset of] $\mathbf{R}$; in other cases we sometimes mention that f is defined on [a subset of] $\mathbf{C}$ to emphasize that $f(z)$ is defined for complex z as well as real z.

Among the multitude of functions defined on $\mathbf{C}$, certain ones are particularly important. Foremost among these are the functions of the form

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0,$$

where $a_0, \ldots, a_n$ are complex numbers. These functions are called, as in the real case, polynomial functions; they include the function $f(z) = z$ (the "identity function") and functions of the form $f(z) = a$ for some complex number a ("constant functions"). Another important generalization of a familiar function is the "absolute value function" $f(z) = |z|$ for all z in $\mathbf{C}$.

Two functions of particular importance for complex numbers are Re (the "real part function") and Im (the "imaginary part function"), defined by

$$\begin{aligned} \operatorname{Re}(x+iy) &= x, \\ \operatorname{Im}(x+iy) &= y, \end{aligned} \qquad \text{for } x \text{ and } y \text{ real.}$$

The "conjugate function" is defined by

$$f(z) = \bar{z} = \operatorname{Re}(z) - i \operatorname{Im}(z).$$

Familiar real-valued functions defined on $\mathbf{R}$ may be combined in many ways to produce new complex-valued functions defined on $\mathbf{C}$—an example is the function

$$f(x+iy) = e^y \sin(x-y) + ix^3 \cos y.$$

The formula for this particular function illustrates a decomposition which is always possible. Any complex-valued function f can be written in the form

$$f = u + iv$$

for some real-valued functions u and v—simply define $u(z)$ as the real part of $f(z)$, and $v(z)$ as the imaginary part. This decomposition is often very useful, but not always; for example, it would be inconvenient to describe a polynomial function in this way.

One other function will play an important role in this chapter. Recall that an *argument* of a nonzero complex number z is a (real) number θ such that

$$z = |z|(\cos\theta + i\sin\theta).$$

There are infinitely many arguments for z, but just one which satisfies $0 \le \theta < 2\pi$. If we call this unique argument $\theta(z)$, then θ is a (real-valued) function (the "argument function") on $\{z \text{ in } \mathbf{C} : z \neq 0\}$.

"Graphs" of complex-valued functions defined on $\mathbf{C}$, since they lie in 4-dimensional space, are presumably not very useful for visualization. The alternative picture of a function mentioned in Chapter 4 can be used instead: we draw two copies of $\mathbf{C}$, and arrows from z in one copy, to $f(z)$ in the other (Figure 1).

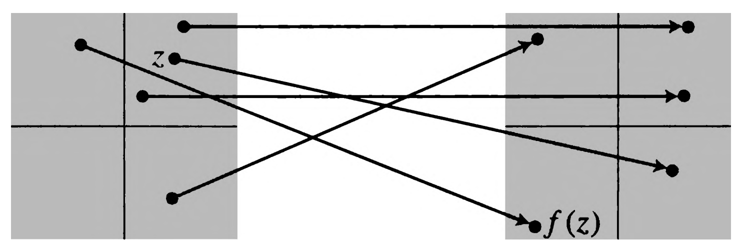
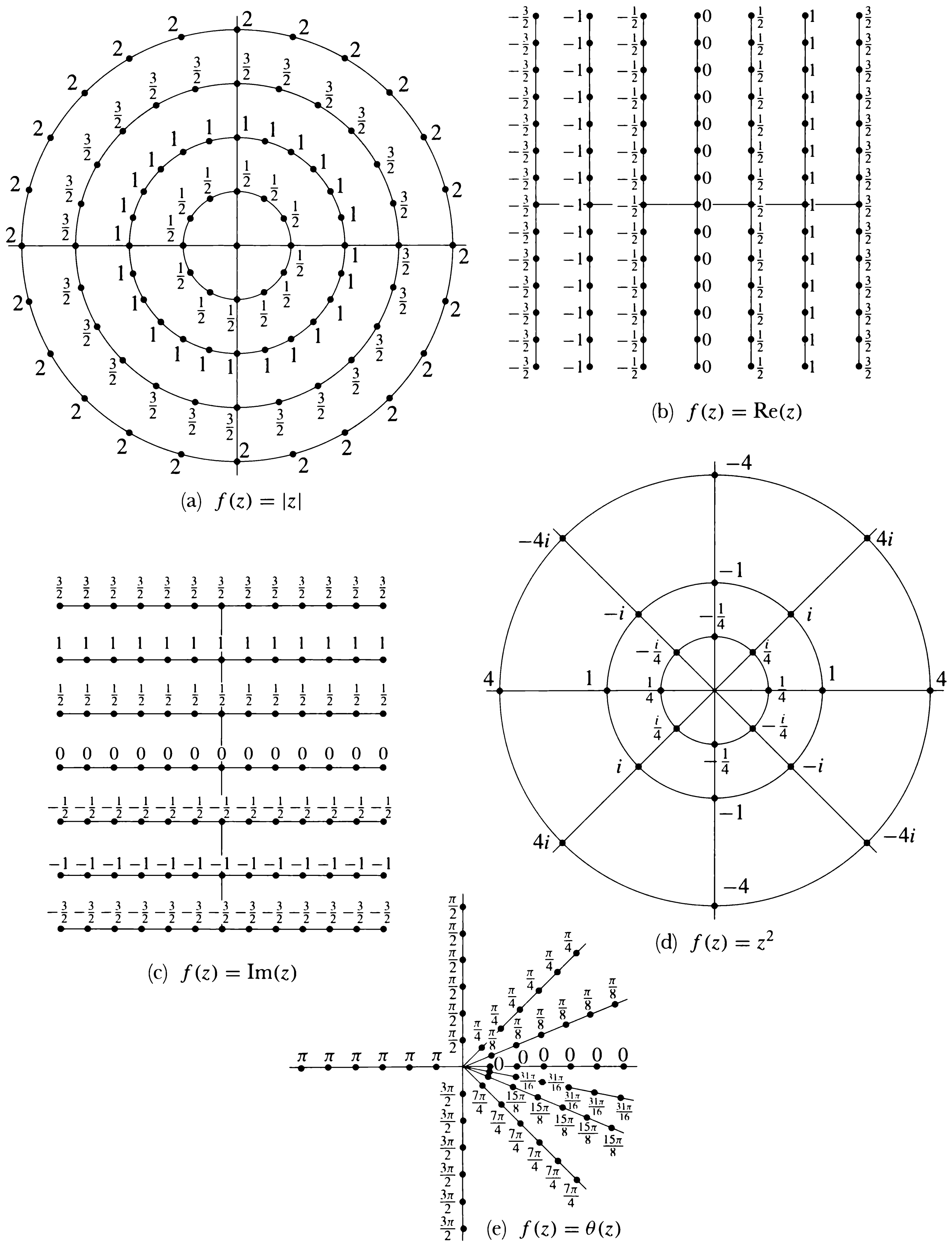


FIGURE 1

The most common pictorial representation of a complex-valued function is produced by labeling a point in the plane with the value $f(z)$, instead of with z (which can be estimated from the position of the point in the picture). Figure 2 shows this sort of picture for several different functions. Certain features of the function are illustrated very clearly by such a "graph." For example, the absolute value function is constant on concentric circles around 0, the functions Re and Im are constant on the vertical and horizontal lines, respectively, and the function $f(z) = z^2$ wraps the circle of radius r twice around the circle of radius r^2.

Despite the problems involved in visualizing complex-valued functions in general, it is still possible to define analogues of important properties previously defined for real-valued functions on $\mathbf{R}$, and in some cases these properties may be easier to visualize in the complex case. For example, the notion of limit can be defined as follows:

> $\lim_{z\to a} f(z) = l$ means that for every (real) number $\varepsilon > 0$ there is a (real) number $\delta > 0$ such that, for all z, if $0 < |z - a| < \delta$, then $|f(z) - l| < \varepsilon$.



(a) $f(z) = |z|$

(b) $f(z) = \mathrm{Re}(z)$

(c) $f(z) = \mathrm{Im}(z)$

(d) $f(z) = z^2$

(e) $f(z) = \theta(z)$

FIGURE 2

Although the definition reads precisely as before, the interpretation is slightly different. Since $|z - w|$ is the distance between the complex numbers z and w, the equation $\lim\limits_{z \to a} f(z) = l$ means that the values of $f(z)$ can be made to lie inside any given circle around l, provided that z is restricted to lie inside a sufficiently small circle around a. This assertion is particularly easy to visualize using the "two copy" picture of a function (Figure 3).

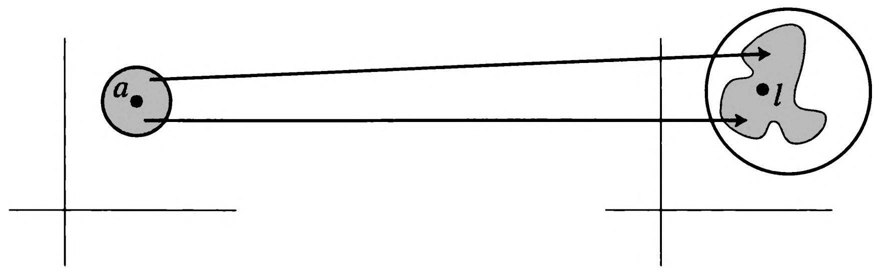


FIGURE 3

Certain facts about limits can be proved exactly as in the real case. In particular,

$$\lim_{z \to a} c = c,$$

$$\lim_{z \to a} z = a,$$

$$\lim_{z \to a} [f(z) + g(z)] = \lim_{z \to a} f(z) + \lim_{z \to a} g(z),$$

$$\lim_{z \to a} f(z) \cdot g(z) = \lim_{z \to a} f(z) \cdot \lim_{z \to a} g(z),$$

$$\lim_{z \to a} \frac{1}{g(z)} = \frac{1}{\lim\limits_{z \to a} g(z)}, \quad \text{if } \lim_{z \to a} g(z) \neq 0.$$

The essential property of absolute values upon which these results are based is the inequality $|z + w| \leq |z| + |w|$, and this inequality holds for complex numbers as well as for real numbers. These facts already provide quite a few limits, but many more can be obtained from the following theorem.

THEOREM 1 Let $f(z) = u(z) + iv(z)$ for real-valued functions u and v, and let $l = \alpha + i\beta$ for real numbers α and β. Then $\lim\limits_{z \to a} f(z) = l$ if and only if

$$\lim_{z \to a} u(z) = \alpha,$$

$$\lim_{z \to a} v(z) = \beta.$$

PROOF Suppose first that $\lim\limits_{z \to a} f(z) = l$. If $\varepsilon > 0$, there is $\delta > 0$ such that, for all z,

$$\text{if } 0 < |z - a| < \delta, \text{ then } |f(z) - l| < \varepsilon.$$

The second inequality can be written

$$\big|[u(z) - \alpha] + i[v(z) - \beta]\big| < \varepsilon,$$

or

$$[u(z) - \alpha]^2 + [v(z) - \beta]^2 < \varepsilon^2.$$

Since $u(z) - \alpha$ and $v(z) - \beta$ are both real numbers, their squares are positive; this inequality therefore implies that

$$[u(z) - \alpha]^2 < \varepsilon^2 \quad \text{and} \quad [v(z) - \beta]^2 < \varepsilon^2,$$

which implies that

$$|u(z) - \alpha| < \varepsilon \quad \text{and} \quad |v(z) - \beta| < \varepsilon.$$

Since this is true for all $\varepsilon > 0$, it follows that

$$\lim_{z \to a} u(z) = \alpha \quad \text{and} \quad \lim_{z \to a} v(z) = \beta.$$

Now suppose that these two equations hold. If $\varepsilon > 0$, there is a $\delta > 0$ such that, for all z, if $0 < |z - a| < \delta$, then

$$|u(z) - \alpha| < \frac{\varepsilon}{2} \quad \text{and} \quad |v(z) - \alpha| < \frac{\varepsilon}{2},$$

which implies that

$$\begin{aligned} |f(z) - l| &= \big|[u(z) - \alpha] + i[v(z) - \beta]\big| \\ &\le |u(z) - \alpha| + |i| \cdot |v(z) - \beta| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This proves that $\lim_{z \to a} f(z) = l$. ▮

In order to apply Theorem 1 fruitfully, notice that since we already know the limit $\lim_{z \to a} z = a$, we can conclude that

$$\begin{aligned} &\lim_{z \to a} \operatorname{Re}(z) = \operatorname{Re}(a), \\ &\lim_{z \to a} \operatorname{Im}(z) = \operatorname{Im}(a). \end{aligned}$$

A limit like

$$\lim_{z \to a} \sin(\operatorname{Re}(z)) = \sin(\operatorname{Re}(a))$$

follows easily, using continuity of sin. Many applications of these principles prove such limits as the following:

$$\lim_{z \to a} \bar{z} = \bar{a},$$

$$\lim_{z \to a} |z| = |a|,$$

$$\lim_{(x+iy) \to a+bi} e^y \sin x + ix^3 \cos y = e^b \sin a + ia^3 \cos b.$$

Now that the notion of limit has been extended to complex functions, the notion of continuity can also be extended: f is **continuous at a** if $\lim_{z \to a} f(z) = f(a)$, and

f is **continuous** if f is continuous at a for all a in the domain of f. The previous work on limits shows that all the following functions are continuous:

$$\begin{aligned} f(z) &= a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0, \\ f(z) &= \bar{z}, \\ f(z) &= |z|, \\ f(x+iy) &= e^y \sin x + ix^3 \cos y. \end{aligned}$$

Examples of discontinuous functions are easy to produce, and certain ones come up very naturally. One particularly frustrating example is the "argument function" θ, which is discontinuous at all nonnegative real numbers (see the "graph" in Figure 2). By suitably redefining θ it is possible to change the discontinuities; for example (Figure 4), if $\theta'(z)$ denotes the unique argument of z with $\pi/2 \le \theta'(z) < 5\pi/2$, then θ' is discontinuous at ai for every nonnegative real number a. But, no matter how θ is redefined, some discontinuities will always occur.

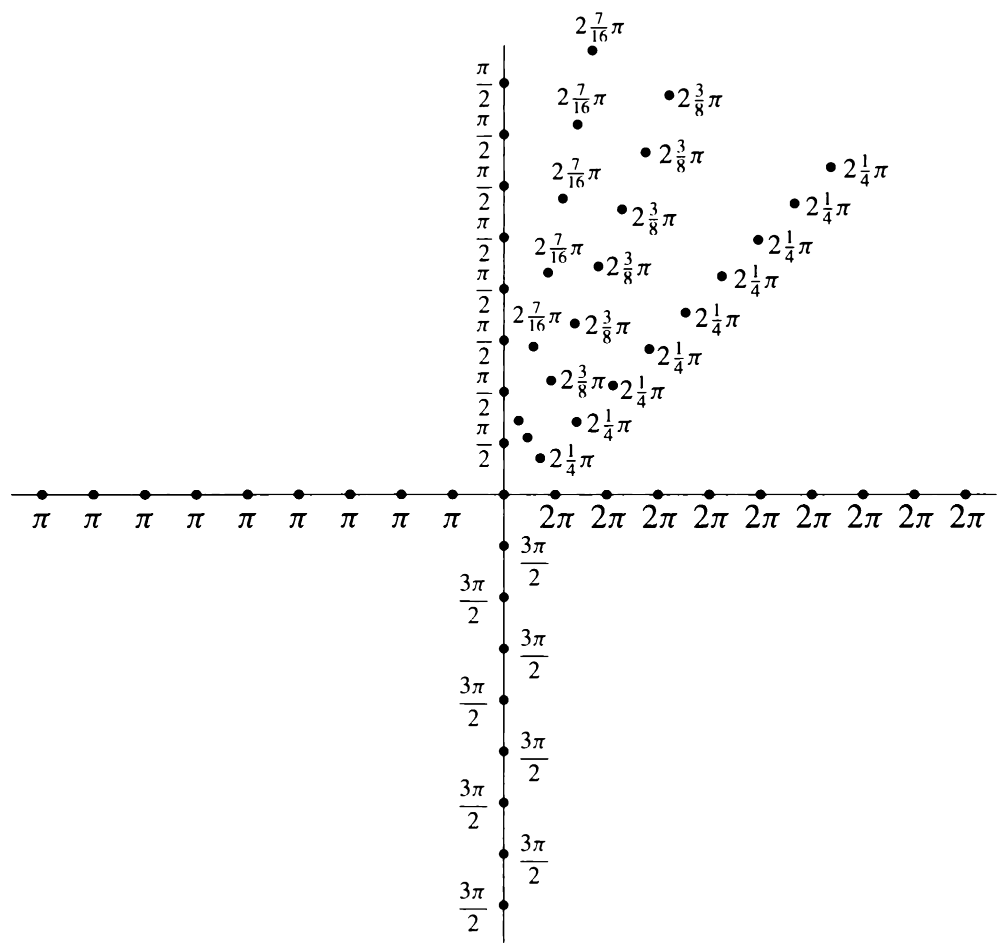


FIGURE 4

The discontinuity of θ has an important bearing on the problem of defining a "square-root function," that is, a function f such that $(f(z))^2 = z$ for all z. For real numbers the function $\sqrt{}$ had as domain only the nonnegative real numbers. If complex numbers are allowed, then every number has two square roots (except 0, which has only one). Although this situation may seem better, it is in some ways worse; since the square roots of z are complex numbers, there is no clear criterion for selecting one root to be $f(z)$, in preference to the other.

One way to define f is the following. We set $f(0) = 0$, and for $z \neq 0$ we set

$$f(z) = \sqrt{|z|}\left(\cos\frac{\theta(z)}{2} + i\sin\frac{\theta(z)}{2}\right).$$

Clearly $(f(z))^2 = z$, but the function f is discontinuous, since θ is discontinuous. As a matter of fact, it is impossible to find a continuous f such that $(f(z))^2 = z$ for all z. In fact, it is even impossible for $f(z)$ to be defined for all z with $|z| = 1$. To prove this by contradiction, we can assume that $f(1) = 1$ (since we could always replace f by $-f$). Then we claim that for all θ with $0 \leq \theta < 2\pi$ we have

$$(*) \qquad f(\cos\theta + i\sin\theta) = \cos\frac{\theta}{2} + i\sin\frac{\theta}{2}.$$

The argument for this is left to you (it is a standard type of least upper bound argument). But $(*)$ implies that

$$\begin{aligned}\lim_{\theta\to 2\pi} f(\cos\theta + i\sin\theta) &= \cos\pi + i\sin\pi \\ &= -1 \\ &\neq f(1),\end{aligned}$$

even though $\cos\theta + i\sin\theta \to 1$ as $\theta \to 2\pi$. Thus, we have our contradiction. A similar argument shows that it is impossible to define continuous "nth-root functions" for any $n \geq 2$.

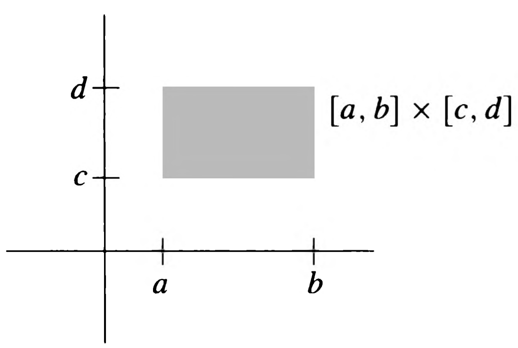


FIGURE 5

For continuous complex functions there are important analogues of certain theorems which describe the behavior of real-valued functions on closed intervals. A natural analogue of the interval $[a, b]$ is the set of all complex numbers $z = x + iy$ with $a \leq x \leq b$ and $c \leq y \leq d$ (Figure 5). This set is called a **closed rectangle**, and is denoted by $[a, b] \times [c, d]$.

If f is a continuous complex-valued function whose domain is $[a, b] \times [c, d]$, then it seems reasonable, and is indeed true, that f is bounded on $[a, b] \times [c, d]$. That is, there is some real number M such that

$$|f(z)| \leq M \quad \text{for all } z \text{ in } [a, b] \times [c, d].$$

It does not make sense to say that f has a maximum and a minimum value on $[a, b] \times [c, d]$, since there is no notion of order for complex numbers. If f is a real-valued function, however, then this assertion does make sense, and is true. In particular, if f is any complex-valued continuous function on $[a, b] \times [c, d]$, then $|f|$ is also continuous, so there is some z_0 in $[a, b] \times [c, d]$ such that

$$|f(z_0)| \leq |f(z)| \quad \text{for all } z \text{ in } [a, b] \times [c, d];$$

a similar statement is true with the inequality reversed. It is sometimes said that "f attains its maximum and minimum modulus on $[a, b] \times [c, d]$."

The various facts listed in the previous paragraph will not be proved here, although proofs are outlined in Problem 5. Assuming these facts, however, we can now give a proof of the Fundamental Theorem of Algebra, which is really quite surprising, since we have not yet said much to distinguish polynomial functions from other continuous functions.

THEOREM 2 (THE FUNDAMENTAL THEOREM OF ALGEBRA) Let $a_0, \dots, a_{n-1}$ be any complex numbers. Then there is a complex number z such that

$$z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_0 = 0.$$

PROOF Let

$$f(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0.$$

Then f is continuous, and so is the function $|f|$ defined by

$$|f|(z) = |f(z)| = |z^n + a_{n-1}z^{n-1} + \cdots + a_0|.$$

Our proof is based on the observation that a point z_0 with $f(z_0) = 0$ would clearly be a minimum point for $|f|$. To prove the theorem we will first show that $|f|$ does indeed have a smallest value on the *whole complex plane*. The proof will be almost identical to the proof, in Chapter 7, that a polynomial function of even degree (with real coefficients) has a smallest value on all of $\mathbf{R}$; both proofs depend on the fact that if $|z|$ is large, then $|f(z)|$ is large.

We begin by writing, for $z \neq 0$,

$$f(z) = z^n\left(1 + \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n}\right),$$

so that

$$|f(z)| = |z|^n \cdot \left|1 + \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n}\right|.$$

Let

$$M = \max(1, 2n|a_{n-1}|, \dots, 2n|a_0|).$$

Then for all z with $|z| \geq M$, we have $|z^k| \geq |z|$ and

$$\frac{|a_{n-k}|}{|z^k|} \leq \frac{|a_{n-k}|}{|z|} \leq \frac{|a_{n-k}|}{2n|a_{n-k}|} = \frac{1}{2n},$$

so

$$\left|\frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n}\right| \leq \left|\frac{a_{n-1}}{z}\right| + \cdots + \left|\frac{a_0}{z^n}\right| \leq \frac{1}{2},$$

which implies that

$$\left|1 + \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n}\right| \geq 1 - \left|\frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n}\right| \geq \frac{1}{2}.$$

This means that

$$|f(z)| \geq \frac{|z|^n}{2} \quad \text{for } |z| \geq M.$$

In particular, if $|z| \geq M$ and also $|z| \geq \sqrt[n]{2|f(0)|}$, then

$$|f(z)| \geq |f(0)|.$$

Now let $[a, b] \times [c, d]$ be a closed rectangle (Figure 6) which contains $\{z : |z| \le \max(M, \sqrt[n]{2|f(0)|})\}$, and suppose that the minimum of $|f|$ on $[a, b] \times [c, d]$ is attained at z_0, so that

$$(1) \quad |f(z_0)| \le |f(z)| \quad \text{for } z \text{ in } [a, b] \times [c, d].$$

It follows, in particular, that $|f(z_0)| \le |f(0)|$. Thus

$$(2) \quad \text{if } |z| \ge \max(M, \sqrt[n]{2|f(0)|}), \text{ then } |f(z)| \ge |f(0)| \ge |f(z_0)|.$$

Combining (1) and (2) we see that $|f(z_0)| \le |f(z)|$ for all z, so that $|f|$ attains its minimum value on the whole complex plane at z_0.

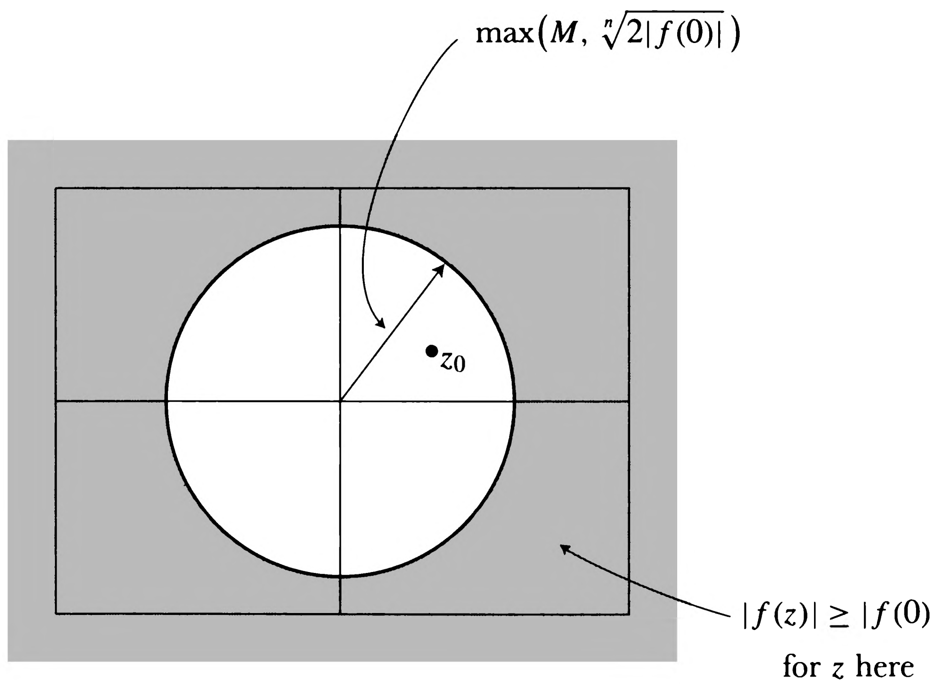


FIGURE 6

To complete the proof of the theorem we now show that $f(z_0) = 0$. It is convenient to introduce the function g defined by

$$g(z) = f(z + z_0).$$

Then g is a polynomial function of degree n, whose minimum absolute value occurs at 0. We want to show that $g(0) = 0$.

Suppose instead that $g(0) = \alpha \ne 0$. If m is the smallest positive power of z which occurs in the expression for g, we can write

$$g(z) = \alpha + \beta z^m + c_{m+1} z^{m+1} + \cdots + c_n z^n,$$

where $\beta \ne 0$. Now, according to Theorem 25-2 there is a complex number γ such that

$$\gamma^m = -\frac{\alpha}{\beta}.$$

Then, setting $d_k = c_k\gamma^k$, we have

$$\begin{aligned}|g(\gamma z)| &= |\alpha + \beta\gamma^m z^m + d_{m+1}z^{m+1} + \cdots + d_n z^n| \\ &= |\alpha - \alpha z^m + d_{m+1}z^{m+1} + \cdots| \\ &= \left|\alpha\left(1 - z^m + \frac{d_{m+1}}{\alpha}z^{m+1} + \cdots\right)\right| \\ &= \left|\alpha\left(1 - z^m + z^m\left[\frac{d_{m+1}}{\alpha}z + \cdots\right]\right)\right| \\ &= |\alpha| \cdot \left|1 - z^m + z^m\left[\frac{d_{m+1}}{\alpha}z + \cdots\right]\right|.\end{aligned}$$

This expression, so tortuously arrived at, will enable us to reach a quick contradiction. Notice first that if $|z|$ is chosen small enough, we will have

$$\left|\frac{d_{m+1}}{\alpha}z + \cdots\right| < 1.$$

If we choose, from among all z for which this inequality holds, some z which is *real and positive*, then

$$\left|z^m\left[\frac{d_{m+1}}{\alpha}z + \cdots\right]\right| < |z^m| = z^m.$$

Consequently, if $0 < z < 1$ we have

$$\begin{aligned}\left|1 - z^m + z^m\left[\frac{d_{m+1}}{\alpha}z + \cdots\right]\right| &\le |1 - z^m| + \left|z^m\left[\frac{d_{m+1}}{\alpha}z + \cdots\right]\right| \\ &= 1 - z^m + \left|z^m\left[\frac{d_{m+1}}{\alpha}z + \cdots\right]\right| \\ &< 1 - z^m + z^m \\ &= 1.\end{aligned}$$

This is the desired contradiction: for such a number z we have

$$|g(\gamma z)| < |\alpha|,$$

contradicting the fact that $|\alpha|$ is the minimum of $|g|$ on the whole plane. Hence, the original assumption must be incorrect, and $g(0) = 0$. This implies, finally, that $f(z_0) = 0$. ▮

Even taking into account our omission of the proofs for the basic facts about continuous complex functions, this proof verified a deep fact with surprisingly little work. It is only natural to hope that other interesting developments will arise if we pursue further the analogues of properties of real functions. The next obvious step is to define derivatives: a function f is **differentiable at a** if

$$\lim_{z\to 0} \frac{f(a+z) - f(a)}{z} \text{ exists,}$$

in which case the limit is denoted by $f'(a)$. It is easy to prove that

$$\begin{aligned} f'(a) &= 0 \qquad \text{if } f(z) = c, \\ f'(a) &= 1 \qquad \text{if } f(z) = z, \\ (f+g)'(a) &= f'(a) + g'(a), \\ (f \cdot g)'(a) &= f'(a)g(a) + f(a)g'(a), \\ \left(\frac{1}{g}\right)'(a) &= \frac{-g'(a)}{[g(a)]^2} \qquad \text{if } g(a) \neq 0, \\ (f \circ g)'(a) &= f'(g(a)) \cdot g'(a); \end{aligned}$$

the proofs of all these formulas are exactly the same as before. It follows, in particular, that if $f(z) = z^n$, then $f'(z) = nz^{n-1}$. These formulas only prove the differentiability of rational functions however. Many other obvious candidates are *not* differentiable. Suppose, for example, that

$$f(x+iy) = x - iy \quad (\text{i.e., } f(z) = \bar{z}).$$

If f is to be differentiable at 0, then the limit

$$\lim_{(x+iy)\to 0} \frac{f(x+iy) - f(0)}{x+iy} = \lim_{(x+iy)\to 0} \frac{x-iy}{x+iy}$$

must exist. Notice however, that

$$\text{if } y = 0, \text{ then } \frac{x-iy}{x+iy} = 1,$$

and

$$\text{if } x = 0, \text{ then } \frac{x-iy}{x+iy} = -1;$$

therefore this limit cannot possibly exist, since the quotient has both the values 1 and -1 for $x + iy$ arbitrarily close to 0.

In view of this example, it is not at all clear where other differentiable functions are to come from. If you recall the definitions of sin and exp, you will see that there is no hope at all of generalizing these definitions to complex numbers. At the moment the outlook is bleak, but all our problems will soon be solved.

PROBLEMS

1. (a) For any real number y, define $\alpha(x) = x + iy$ (so that α is a complex-valued function defined on $\mathbf{R}$). Show that α is continuous. (This follows immediately from a theorem in this chapter.) Show similarly that $\beta(y) = x + iy$ is continuous.

(b) Let f be a continuous function defined on $\mathbf{C}$. For fixed y, let $g(x) = f(x+iy)$. Show that g is a continuous function (defined on $\mathbf{R}$). Show similarly that $h(y) = f(x+iy)$ is continuous. Hint: Use part (a).

2. (a) Suppose that f is a continuous real-valued function defined on a closed rectangle $[a,b] \times [c,d]$. Prove that if f takes on the values $f(z)$ and $f(w)$

for z and w in $[a, b] \times [c, d]$, then f also takes all values between $f(z)$ and $f(w)$. Hint: Consider $g(t) = f(tz + (1-t)w)$ for t in $[0, 1]$.

*(b) If f is a continuous complex-valued function defined on $[a, b] \times [c, d]$, the assertion in part (a) no longer makes any sense, since we cannot talk of complex numbers between $f(z)$ and $f(w)$. We might conjecture that f takes on all values on the line segment between $f(z)$ and $f(w)$, but even this is false. Find an example which shows this.

3. (a) Prove that if $a_0, \dots, a_{n-1}$ are any complex numbers, then there are complex numbers $z_1, \dots, z_n$ (not necessarily distinct) such that

$$z^n + a_{n-1}z^{n-1} + \cdots + a_0 = \prod_{i=1}^{n}(z - z_i).$$

(b) Prove that if $a_0, \dots, a_{n-1}$ are *real*, then $z^n + a_{n-1}z^{n-1} + \cdots + a_0$ can be written as a product of linear factors $z+a$ and quadratic factors z^2+az+b all of whose coefficients are real. (Use Problem 25-7.)

4. In this problem we will consider only polynomials with real coefficients. Such a polynomial is called a **sum of squares** if it can be written as ${h_1}^2 + \cdots + {h_n}^2$ for polynomials h_i with real coefficients.

(a) Prove that if f is a sum of squares, then $f(x) \geq 0$ for all x.
(b) Prove that if f and g are sums of squares, then so is $f \cdot g$.
(c) Suppose that $f(x) \geq 0$ for all x. Show that f is a sum of squares. Hint: First write $f(x) = \prod_{i=1}^{k}(x - a_i)^2 g(x)$, where $g(x) > 0$ for all x. Then use Problem 3(b).

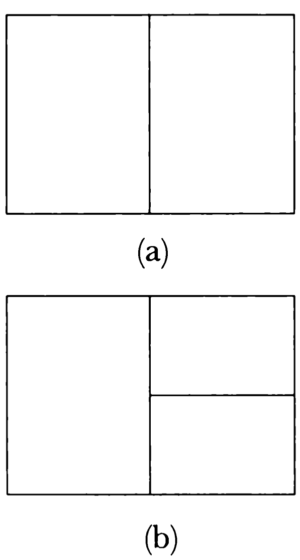


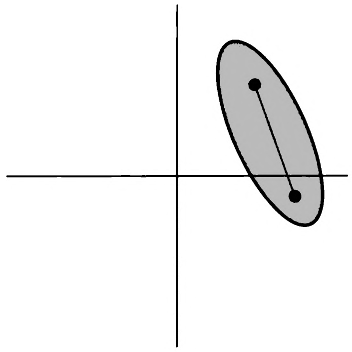
FIGURE 7

5. (a) Let A be a set of complex numbers. A number z is called, as in the real case, a **limit point** of the set A if for every (real) $\varepsilon > 0$, there is a point a in A with $|z - a| < \varepsilon$ but $z \neq a$. Prove the two-dimensional version of the Bolzano-Weierstrass Theorem: If A is an infinite subset of $[a, b] \times [c, d]$, then A has a limit point in $[a, b] \times [c, d]$. Hint: First divide $[a, b] \times [c, d]$ in half by a vertical line as in Figure 7(a). Since A is infinite, at least one half contains infinitely many points of A. Divide this in half by a horizontal line, as in Figure 7(b). Continue in this way, alternately dividing by vertical and horizontal lines.

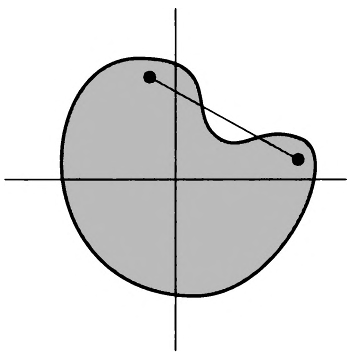
(The two-dimensional bisection argument outlined in this hint is so standard that the title "Bolzano-Weierstrass" often serves to describe the method of proof, in addition to the theorem itself. See, for example, H. Petard, "A Contribution to the Mathematical Theory of Big Game Hunting," *Amer. Math. Monthly*, **45** (1938), 446–447.)

(b) Prove that a continuous (complex-valued) function on $[a, b] \times [c, d]$ is bounded on $[a, b] \times [c, d]$. (Imitate Problem 22-31.)

(c) Prove that if f is a real-valued continuous function on $[a, b] \times [c, d]$, then f takes on a maximum and minimum value on $[a, b] \times [c, d]$. (You can use the same trick that works for Theorem 7-3.)



(a) a convex subset of the plane



(b) a nonconvex subset of the plane

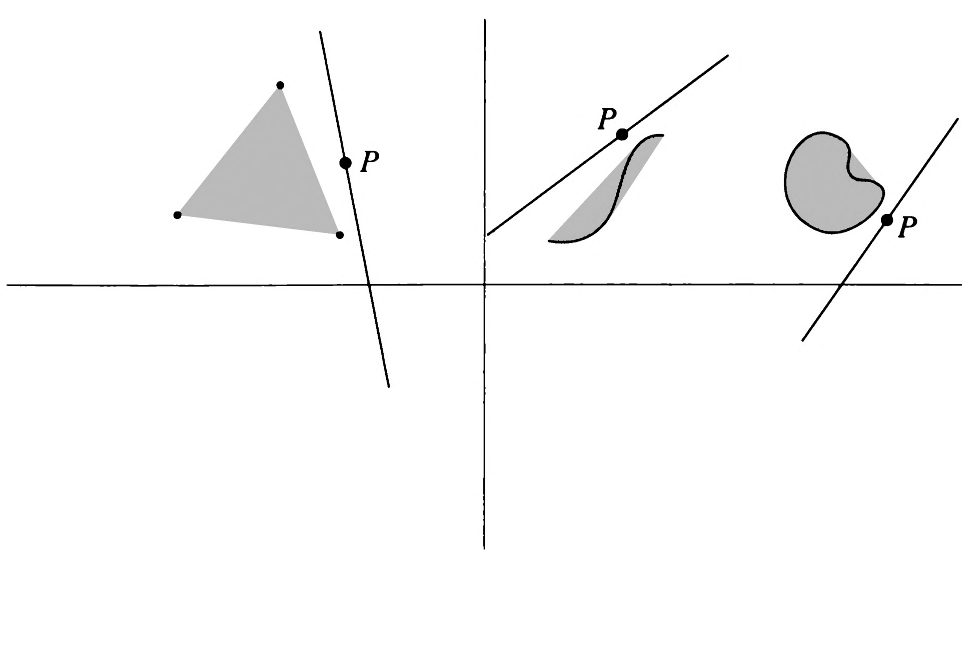
FIGURE 8

***6.** The proof of Theorem 2 cannot be considered to be completely elementary because the possibility of choosing γ with $\gamma^m = -\alpha/\beta$ depends on Theorem 25-2, and thus on the trigonometric functions. It is therefore of some interest to provide an elementary proof that there is a solution for the equation $z^n - c = 0$.

(a) Make an explicit computation to show that solutions of $z^2 - c = 0$ can be found for any complex number c.
(b) Explain why the solution of $z^n - c = 0$ can be reduced to the case where n is odd.
(c) Let z_0 be the point where the function $f(z) = z^n - c$ has its minimum absolute value. If $z_0 \neq 0$, show that the integer m in the proof of Theorem 2 is equal to 1; since we can certainly find γ with $\gamma^1 = -\alpha/\beta$, the remainder of the proof works for f. It therefore suffices to show that the minimum absolute value of f does not occur at 0.
(d) Suppose instead that f has its minimum absolute value at 0. Since n is odd, the points $\pm\delta, \pm\delta i$ go under f into $-c\pm\delta^n, -c\pm\delta^n i$. Show that for small δ at least one of these points has smaller absolute value than $-c$, thereby obtaining a contradiction.

7. Let $f(z) = (z - z_1)^{m_1} \cdot \ldots \cdot (z - z_k)^{m_k}$ for $m_1, \ldots, m_k > 0$.

(a) Show that $f'(z) = (z - z_1)^{m_1} \cdot \ldots \cdot (z - z_k)^{m_k} \cdot \displaystyle\sum_{\alpha=1}^{k} m_\alpha (z - z_\alpha)^{-1}$.
(b) Let $g(z) = \displaystyle\sum_{\alpha=1}^{k} m_\alpha (z - z_\alpha)^{-1}$. Show that if $g(z) = 0$, then $z_1, \ldots, z_k$ cannot all lie on the same side of a straight line through z. Hint: Use Problem 25-11.
(c) A subset K of the plane is **convex** if K contains the line segment joining any two points in it (Figure 8). For any set A, there is a smallest convex set containing it, which is called the **convex hull** of A (Figure 9); if a



point P is not in the convex hull of A, then all of A is contained on one side of some straight line through P. Using this information, prove that the roots of $f'(z) = 0$ lie within the convex hull of the set $\{z_1, \ldots, z_k\}$. Further information on convex sets will be found in reference [18] of the Suggested Reading.

8. Prove that if f is differentiable at z, then f is continuous at z.

***9.** Suppose that $f = u + iv$ where u and v are real-valued functions.

(a) For fixed y_0 let $g(x) = u(x + iy_0)$ and $h(x) = v(x + iy_0)$. Show that if $f'(x_0 + iy_0) = \alpha + i\beta$ for real α and β, then $g'(x_0) = \alpha$ and $h'(x_0) = \beta$.
(b) On the other hand, suppose that $k(y) = u(x_0+iy)$ and $l(y) = v(x_0+iy)$. Show that $l'(y_0) = \alpha$ and $k'(y_0) = -\beta$.
(c) Suppose that $f'(z) = 0$ for all z. Show that f is a constant function.

10. (a) Using the expression

$$f(x) = \frac{1}{1+x^2} = \frac{1}{2i}\left(\frac{1}{x-i} - \frac{1}{x+i}\right),$$

find $f^{(k)}(x)$ for all k.
(b) Use this result to find $\arctan^{(k)}(0)$ for all k.