CHAPTER 3 FUNCTIONS

Undoubtedly the most important concept in all of mathematics is that of a function—in almost every branch of modern mathematics functions turn out to be the central objects of investigation. It will therefore probably not surprise you to learn that the concept of a function is one of great generality. Perhaps it will be a relief to learn that, for the present, we will be able to restrict our attention to functions of a very special kind; even this small class of functions will exhibit sufficient variety to engage our attention for quite some time. We will not even begin with a proper definition. For the moment a provisional definition will enable us to discuss functions at length, and will illustrate the intuitive notion of functions, as understood by mathematicians. Later, we will consider and discuss the advantages of the modern mathematical definition. Let us therefore begin with the following:

PROVISIONAL DEFINITION A function is a rule which assigns, to each of certain real numbers, some other real number.

The following examples of functions are meant to illustrate and amplify this definition, which, admittedly, requires some such clarification.

Example 1 The rule which assigns to each number the square of that number.

Example 2 The rule which assigns to each number y the number

$$\frac{y^3 + 3y + 5}{y^2 + 1}.$$

Example 3 The rule which assigns to each number $c \neq 1, -1$ the number

$$\frac{c^3 + 3c + 5}{c^2 - 1}.$$

Example 4 The rule which assigns to each number x satisfying $-17 \leq x \leq \pi/3$ the number x^2.

Example 5 The rule which assigns to each number a the number 0 if a is irrational, and the number 1 if a is rational.

Example 6 The rule which assigns

to 2 the number 5,

to 17 the number $\dfrac{36}{\pi}$,

$$\text{to } \frac{\pi^2}{17} \text{ the number 28,}$$

$$\text{to } \frac{36}{\pi} \text{ the number 28,}$$

and to any $y \neq 2$, 17, $\pi^2/17$, or $36/\pi$, the number 16 if y is of the form $a + b\sqrt{2}$ for a, b in $\mathbf{Q}$.

Example 7 The rule which assigns to each number t the number $t^3 + x$. (This rule depends, of course, on what the number x is, so we are really describing infinitely many different functions, one for each number x.)

Example 8 The rule which assigns to each number z the number of 7's in the decimal expansion of z, if this number is finite, and $-\pi$ if there are infinitely many 7's in the decimal expansion of z.

One thing should be abundantly clear from these examples—a function is *any* rule that assigns numbers to certain other numbers, not just a rule which can be expressed by an algebraic formula, or even by one uniform condition which applies to every number; nor is it necessarily a rule which you, or anybody else, can actually apply in practice (no one knows, for example, what rule 8 associates to π). Moreover, the rule may neglect some numbers and it may not even be clear to which numbers the function applies (try to determine, for example, whether the function in Example 6 applies to π). The set of numbers to which a function *does* apply is called the *domain* of the function.

Before saying anything else about functions we badly need some notation. Since throughout this book we shall frequently be talking about functions (indeed we shall hardly ever talk about anything else) we need a convenient way of naming functions, and of referring to functions in general. The standard practice is to denote a function by a letter. For obvious reasons the letter "f" is a favorite, thereby making "g" and "h" other obvious candidates, but any letter (or any reasonable symbol, for that matter) will do, not excluding "x" and "y", although these letters are usually reserved for indicating numbers. If f is a function, then the number which f associates to a number x is denoted by $f(x)$—this symbol is read "f of x" and is often called the **value of f at x**. Naturally, if we denote a function by x, some other letter must be chosen to denote the number (a perfectly legitimate, though perverse, choice would be "f," leading to the symbol $x(f)$). Note that the symbol $f(x)$ makes sense only for x in the domain of f; for other x the symbol $f(x)$ is not defined.

If the functions defined in Examples 1–8 are denoted by f, g, h, r, s, θ, α_x, and y, then we can rewrite their definitions as follows:

(1) $f(x) = x^2$ for all x.

(2) $g(y) = \dfrac{y^3 + 3y + 5}{y^2 + 1}$ for all y.

(3) $h(c) = \dfrac{c^3 + 3c + 5}{c^2 - 1}$ for all $c \neq 1, -1$.

(4) $r(x) = x^2$ for all x such that $-17 \le x \le \pi/3$.

(5) $s(x) = \begin{cases} 0, & x \text{ irrational} \\ 1, & x \text{ rational.} \end{cases}$

(6) $\theta(x) = \begin{cases} 5, & x = 2 \\ \dfrac{36}{\pi}, & x = 17 \\ 28, & x = \dfrac{\pi^2}{17} \\ 28, & x = \dfrac{36}{\pi} \\ 16, & x \ne 2,\ 17,\ \dfrac{\pi^2}{17},\ \text{or } \dfrac{36}{\pi},\ \text{and } x = a + b\sqrt{2} \text{ for } a, b \text{ in } \mathbf{Q}. \end{cases}$

(7) $\alpha_x(t) = t^3 + x$ for all numbers t.

(8) $y(x) = \begin{cases} n, & \text{exactly } n \text{ 7's appear in the decimal expansion of } x \\ -\pi, & \text{infinitely many 7's appear in the decimal expansion of } x. \end{cases}$

These definitions illustrate the common procedure adopted for defining a function f—indicating what $f(x)$ is for every number x in the domain of f. (Notice that this is exactly the same as indicating $f(a)$ for every number a, or $f(b)$ for every number b, etc.) In practice, certain abbreviations are tolerated. Definition (1) could be written simple

$$(1) \quad f(x) = x^2$$

the qualifying phrase "for all x" being understood. Of course, for definition (4) the only possible abbreviation is

$$(4) \quad r(x) = x^2, \qquad -17 \le x \le \pi/3.$$

It is usually understood that a definition such as

$$k(x) = \frac{1}{x} + \frac{1}{x-1}, \qquad x \ne 0, 1$$

can be shortened to

$$k(x) = \frac{1}{x} + \frac{1}{x-1};$$

in other words, *unless the domain is explicitly restricted further, it is understood to consist of all numbers for which the definition makes any sense at all.*

You should have little difficulty checking the following assertions about the functions defined above:

$$f(x+1) = f(x) + 2x + 1;$$
$$g(x) = h(x) \text{ if } x^3 + 3x + 5 = 0;$$
$$r(x+1) = r(x) + 2x + 1 \text{ if } -17 \le x \le \frac{\pi}{3} - 1;$$

$$s(x+y) = s(x) \text{ if } y \text{ is rational;}$$
$$\theta\left(\frac{\pi^2}{17}\right) = \theta\left(\frac{36}{\pi}\right);$$
$$\alpha_x(x) = x \cdot [f(x)+1];$$
$$y\left(\frac{1}{3}\right) = 0, \quad y\left(\frac{7}{9}\right) = -\pi.$$

If the expression $f(s(a))$ looks unreasonable to you, then you are forgetting that $s(a)$ is a number like any other number, so that $f(s(a))$ makes sense. As a matter of fact, $f(s(a)) = s(a)$ for all a. Why? Even more complicated expressions than $f(s(a))$ are, after a first exposure, no more difficult to unravel. The expression

$$f(r(s(\theta(\alpha_3(y(\tfrac{1}{3})))))),$$

formidable as it appears, may be evaluated quite easily with a little patience:

$$\begin{aligned} &f(r(s(\theta(\alpha_3(y(\tfrac{1}{3})))))) \\ &= f(r(s(\theta(\alpha_3(0))))) \\ &= f(r(s(\theta(3)))) \\ &= f(r(s(16))) \\ &= f(r(1)) \\ &= f(1) \\ &= 1. \end{aligned}$$

The first few problems at the end of this chapter give further practice manipulating this symbolism.

The function defined in (1) is a rather special example of an extremely important class of functions, the polynomial functions. A function f is a **polynomial function** if there are real numbers $a_0, \dots, a_n$ such that

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0, \qquad \text{for all } x$$

(when $f(x)$ is written in this form it is usually tacitly assumed that $a_n \neq 0$). The highest power of x with a nonzero coefficient is called the **degree** of f; for example, the polynomial function f defined by $f(x) = 5x^6 + 137x^4 - \pi$ has degree 6.

The functions defined in (2) and (3) belong to a somewhat larger class of functions, the **rational functions**; these are the functions of the form p/q where p and q are polynomial functions (and q is not the function which is always 0). The rational functions are themselves quite special examples of an even larger class of functions, very thoroughly studied in calculus, which are simpler than many of the functions first mentioned in this chapter. The following are examples of this kind of function:

(9) $f(x) = \dfrac{x + x^2 + x\sin^2 x}{x\sin x + x\sin^2 x}$

(10) $f(x) = \sin(x^2)$.

(11) $f(x) = \sin(\sin(x^2))$.

(12) $f(x) = \sin^2(\sin(\sin^2(x\sin^2 x^2)))\cdot\sin\left(\dfrac{x+\sin(x\sin x)}{x+\sin x}\right).$

By what criterion, you may feel impelled to ask, can such functions, especially a monstrosity like (12), be considered simple? The answer is that they can be built up from a few simple functions using a few simple means of combining functions. In order to construct the functions (9)–(12) we need to start with the "identity function" I, for which $I(x) = x$, and the "sine function" sin, whose value $\sin(x)$ at x is often written simple $\sin x$. The following are some of the important ways in which functions may be combined to produce new functions.

If f and g are any two functions, we can define a new function $f + g$, called the **sum** of f and g, by the equation

$$(f+g)(x) = f(x) + g(x).$$

Note that according to the conventions we have adopted, the domain of $f + g$ consists of all x for which "$f(x) + g(x)$" makes sense, i.e., the set of all x in both domain f and domain g. If A and B are any two sets, then $A \cap B$ (read "A intersect B" or "the intersection of A and B") denotes the set of x in both A and B; this notation allows us to write domain$(f + g)$ = domain $f \cap$ domain g.

In a similar vein, we define the **product** $f \cdot g$ and the **quotient** $\dfrac{f}{g}$ (or f/g) of f and g by

$$(f \cdot g)(x) = f(x) \cdot g(x)$$

and

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}.$$

Moreover, if g is a function and c is a number, we define a new function $c \cdot g$ by

$$(c \cdot g)(x) = c \cdot g(x).$$

This becomes a special case of the notation $f \cdot g$ if we agree that the symbol c should also represent the function f defined by $f(x) = c$; such a function, which has the same value for all numbers x, is called a **constant function**.

The domain of $f \cdot g$ is domain $f \cap$ domain g, and the domain of $c \cdot g$ is simply the domain of g. On the other hand, the domain of f/g is rather complicated—it may be written domain $f \cap$ domain $g \cap \{x : g(x) \neq 0\}$, the symbol $\{x : g(x) \neq 0\}$ denoting the set of numbers x such that $g(x) \neq 0$. In general, $\{x : \dots\}$ denotes the set of all x such that " ... " is true. Thus $\{x : x^3 + 3 < 11\}$ denotes the set of all numbers x such that $x^3 < 8$, and consequently $\{x : x^3 + 3 < 11\} = \{x : x < 2\}$. Either of these symbols could just as well have been written using y everywhere instead of x. Variations of this notation are common, but hardly require any discussion. Any one can guess that $\{x > 0 : x^3 < 8\}$ denotes the set of positive numbers whose cube is less than 8; it could be expressed more formally as $\{x : x > 0 \text{ and } x^3 < 8\}$. Incidentally, this set is equal to the set $\{x : 0 < x < 2\}$. One variation is slightly less transparent, but very standard. The set $\{1, 3, 2, 4\}$, for example, contains just the four numbers 1, 2, 3, and 4; it can also be denoted by $\{x : x = 1 \text{ or } x = 3 \text{ or } x = 2 \text{ or } x = 4\}$.

Certain facts about the sum, product, and quotient of functions are obvious consequences of facts about sums, products, and quotients of numbers. For example, it is very easy to prove that

$$(f + g) + h = f + (g + h).$$

The proof is characteristic of almost every proof which demonstrates that two functions are equal—the two functions must be shown to have the same domain, and the same value at any number in the domain. For example, to prove that $(f+g)+h = f+(g+h)$, note that unraveling the definition of the two sides gives

$$\begin{aligned}[(f + g) + h](x) &= (f + g)(x) + h(x)\\ &= [f(x) + g(x)] + h(x)\end{aligned}$$

and

$$\begin{aligned}[f + (g + h)](x) &= f(x) + (g + h)(x)\\ &= f(x) + [g(x) + h(x)],\end{aligned}$$

and the equality of $[f(x)+g(x)]+h(x)$ and $f(x)+[g(x)+h(x)]$ is a fact about numbers. In this proof the equality of the two domains was not explicitly mentioned because this is obvious, as soon as we begin to write down these equations; the domain of $(f + g) + h$ and of $f + (g + h)$ is clearly domain $f \cap$ domain $g \cap$ domain h. We naturally write $f + g + h$ for $(f + g) + h = f + (g + h)$, precisely as we did for numbers.

It is just as easy to prove that $(f \cdot g) \cdot h = f \cdot (g \cdot h)$, and this function is denoted by $f \cdot g \cdot h$. The equations $f + g = g + f$ and $f \cdot g = g \cdot f$ should also present no difficulty.

Using the operations $+$, $\cdot$, $/$ we can now express the function f defined in (9) by

$$f = \frac{I + I \cdot I + I \cdot \sin \cdot \sin}{I \cdot \sin + I \cdot \sin \cdot \sin}.$$

It should be clear, however, that we cannot express function (10) this way. We require yet another way of combining functions. This combination, the composition of two functions, is by far the most important.

If f and g are any two functions, we define a new function $f \circ g$, the **composition** of f and g, by

$$(f \circ g)(x) = f(g(x));$$

the domain of $f \circ g$ is $\{x : x \text{ is in domain } g \text{ and } g(x) \text{ is in domain } f\}$. The symbol "$f \circ g$" is often read "$f$ circle g." Compared to the phrase "the composition of f and g" this has the advantage of brevity, of course, but there is another advantage of far greater import: there is much less chance of confusing $f \circ g$ with $g \circ f$, and these must *not* be confused, since they are not usually equal; in fact, almost any f and g chosen at random will illustrate this point (try $f = I \cdot I$ and $g = \sin$, for example). Lest you become too apprehensive about the operation of composition, let us hasten to point out that composition *is* associative:

$$(f \circ g) \circ h = f \circ (g \circ h)$$

(and the proof is a triviality); this function is denoted by $f \circ g \circ h$. We can now write the functions (10), (11), (12) as

(10) $f = \sin \circ (I \cdot I)$,

(11) $f = \sin \circ \sin \circ (I \cdot I)$,

(12) $f = (\sin \cdot \sin) \circ \sin \circ (\sin \cdot \sin) \circ (I \cdot [(\sin \cdot \sin) \circ (I \cdot I)]) \cdot$

$$\sin \circ \left(\frac{I + \sin \circ (I \cdot \sin)}{I + \sin} \right).$$

One fact has probably already become clear. Although this method of writing functions reveals their "structure" very clearly, it is hardly short or convenient. The shortest name for the function f such that $f(x) = \sin(x^2)$ for all x unfortunately seems to be "the function f such that $f(x) = \sin(x^2)$ for all x." The need for abbreviating this clumsy description has been clear for two hundred years, but no reasonable abbreviation has received universal acclaim. At present the strongest contender for this honor is something like

$$x \to \sin(x^2)$$

(read "x goes to $\sin(x^2)$" or just "x arrow $\sin(x^2)$"), but it is hardly popular among writers of calculus textbooks. In this book we will tolerate a certain amount of ellipsis, and speak of "the function $f(x) = \sin(x^2)$." Even more popular is the quite drastic abbreviation: "the function $\sin(x^2)$." For the sake of precision we will never use this description, which, strictly speaking, confuses a number and a function, but it is so convenient that you will probably end up adopting it for personal use. As with any convention, utility is the motivating factor, and this criterion is reasonable so long as the slight logical deficiencies cause no confusion. On occasion, confusion *will* arise unless a more precise description is used. For example, "the function $x + t^3$" is an ambiguous phrase; it could mean either

$$x \to x + t^3, \text{ i.e., the function } f \text{ such that } f(x) = x + t^3 \text{ for all } x$$

or

$$t \to x + t^3, \text{ i.e., the function } f \text{ such that } f(t) = x + t^3 \text{ for all } t.$$

As we shall see, however, for many important concepts associated with functions, calculus has a notation which contains the "$x \to$" built in.

By now we have made a sufficiently extensive investigation of functions to warrant reconsidering our definition. We have defined a function as a "rule," but it is hardly clear what this means. If we ask "What happens if you break this rule?" it is not easy to say whether this question is merely facetious or actually profound.

A more substantial objection to the use of the word "rule" is that

$$f(x) = x^2$$

and

$$f(x) = x^2 + 3x + 3 - 3(x + 1)$$

are certainly *different* rules, if by a rule we mean the actual instructions given for determining $f(x)$; nevertheless, we want

$$f(x) = x^2$$

and

$$f(x) = x^2 + 3x + 3 - c(x + 1)$$

to define the same function. For this reason, a function is sometimes defined as an "association" between numbers; unfortunately the word "association" escapes the objections raised against "rule" only because it is even more vague.

There is, of course, a satisfactory way of defining functions, or we should never have gone to the trouble of criticizing our original definition. But a satisfactory definition can never be constructed by finding synonyms for English words which are troublesome. The definition which mathematicians have finally accepted for "function" is a beautiful example of the means by which intuitive ideas have been incorporated into rigorous mathematics. The correct question to ask about a function is not "What is a rule?" or "What is an association?" but "What does one have to know about a function in order to know all about it?" The answer to the last question is easy—for each number x one needs to know the number $f(x)$; we can imagine a table which would display all the information one could desire about the function $f(x) = x^2$:

x	$f(x)$
1	1
-1	1
2	4
-2	4
$\sqrt{2}$	2
$-\sqrt{2}$	2
π	π^2
$-\pi$	π^2

It is not even necessary to arrange the numbers in a table (which would actually be impossible if we wanted to list all of them). Instead of a two column array we can consider various pairs of numbers

$$(1, 1),\ (-1, 1),\ (2, 4),\ (-2, 4),\ (\pi, \pi^2),\ (\sqrt{2}, 2), \ldots$$

simply collected together into a set.* To find $f(1)$ we simply take the second number of the pair whose first member is 1; to find $f(\pi)$ we take the second number of the pair whose first member is π. We seem to be saying that a function might as well be defined as a collection of pairs of numbers. For example, if we were given the following collection (which contains just 5 pairs):

$$f = \{(1,7),\ (3,7),\ (5,3),\ (4,8),\ (8,4)\},$$

then $f(1) = 7$, $f(3) = 7$, $f(5) = 3$, $f(4) = 8$, $f(8) = 4$ and 1, 3, 4, 5, 8 are the only numbers in the domain of f. If we consider the collection

$$f = \{(1,7),\ (3,7),\ (2,5),\ (1,8),\ (8,4)\},$$

then $f(3) = 7$, $f(2) = 5$, $f(8) = 4$; but it is impossible to decide whether $f(1) = 7$ or $f(1) = 8$. In other words, a function cannot be defined to be any old collection of pairs of numbers; we must rule out the possibility which arose in this case. We are therefore led to the following definition.

DEFINITION

A **function** is a collection of pairs of numbers with the following property: if (a,b) and (a,c) are both in the collection, then $b = c$; in other words, the collection must not contain two different pairs with the same first element.

This is our first full-fledged definition, and illustrates the format we shall always use to define significant new concepts. These definitions are so important (at least as important as theorems) that it is essential to know when one is actually at hand, and to distinguish them from comments, motivating remarks, and casual explanations. They will be preceded by the word **DEFINITION**, contain the term being defined in boldface letters, and constitute a paragraph unto themselves.

There is one more definition (actually defining two things at once) which can now be made rigorously:

DEFINITION

If f is a function, the **domain** of f is the set of all a for which there is some b such that (a,b) is in f. If a is in the domain of f, it follows from the definition of a function that there is, in fact, a *unique* number b such that (a,b) is in f. This unique b is denoted by $\boldsymbol{f(a)}$.

With this definition we have reached our goal: the important thing about a function f is that a number $f(x)$ is determined for each number x in its domain. You may feel that we have also reached the point where an intuitive definition has been replaced by an abstraction with which the mind can hardly grapple. Two consolations may be offered. First, although a function has been defined as a

*The pairs occurring here are often called "ordered pairs," to emphasize that, for example, (2, 4) is not the same pair as (4, 2). It is only fair to warn that we are going to define functions in terms of ordered pairs, another undefined term. Ordered pairs *can* be defined, however, and an appendix to this chapter has been provided for skeptics.

collection of pairs, there is nothing to stop you from *thinking* of a function as a rule. Second, neither the intuitive nor the formal definition indicates the best way of thinking about functions. The best way is to draw pictures; but this requires a chapter all by itself.

PROBLEMS

1. Let $f(x) = 1/(1+x)$. What is

(i) $f(f(x))$ (for which x does this make sense?).
(ii) $f\left(\dfrac{1}{x}\right)$.
(iii) $f(cx)$.
(iv) $f(x+y)$.
(v) $f(x)+f(y)$.
(vi) For which numbers c is there a number x such that $f(cx) = f(x)$. Hint: There are a lot more than you might think at first glance.
(vii) For which numbers c is it true that $f(cx) = f(x)$ for two different numbers x?

2. Let $g(x) = x^2$, and let

$$h(x) = \begin{cases} 0, & x \text{ rational} \\ 1, & x \text{ irrational.} \end{cases}$$

(i) For which y is $h(y) \le y$?
(ii) For which y is $h(y) \le g(y)$?
(iii) What is $g(h(z)) - h(z)$?
(iv) For which w is $g(w) \le w$?
(v) For which ε is $g(g(\varepsilon)) = g(\varepsilon)$?

3. Find the domain of the functions defined by the following formulas.

(i) $f(x) = \sqrt{1-x^2}$.
(ii) $f(x) = \sqrt{1-\sqrt{1-x^2}}$.
(iii) $f(x) = \dfrac{1}{x-1} + \dfrac{1}{x-2}$.
(iv) $f(x) = \sqrt{1-x^2} + \sqrt{x^2-1}$.
(v) $f(x) = \sqrt{1-x} + \sqrt{x-2}$.

4. Let $S(x) = x^2$, let $P(x) = 2^x$, and let $s(x) = \sin x$. Find each of the following. In each case you answer should be a *number.*

(i) $(S \circ P)(y)$.
(ii) $(S \circ s)(y)$.
(iii) $(S \circ P \circ s)(t) + (s \circ P)(t)$.
(iv) $s(t^3)$.

5. Express each of the following functions in terms of S, P, s, using only $+$, $\cdot$, and $\circ$ (for example, the answer to (i) is $P \circ s$). In each case your answer should be a *function.*

(i) $f(x) = 2^{\sin x}$.
(ii) $f(x) = \sin 2^x$.
(iii) $f(x) = \sin x^2$.
(iv) $f(x) = \sin^2 x$ (remember that $\sin^2 x$ is an abbreviation for $(\sin x)^2$).
(v) $f(t) = 2^{2^t}$. (Note: a^{b^c} *always* means $a^{(b^c)}$; this convention is adopted because $(a^b)^c$ can be written more simply as a^{bc}.)
(vi) $f(u) = \sin(2^u + 2^{u^2})$.
(vii) $f(y) = \sin\big(\sin\big(\sin\big(2^{2^{2^{\sin y}}}\big)\big)\big)$.
(viii) $f(a) = 2^{\sin^2 a} + \sin(a^2) + 2^{\sin(a^2+\sin a)}$.

Polynomial functions, because they are simple, yet flexible, occupy a favored role in most investigations of functions. The following two problems illustrate their flexibility, and guide you through a derivation of their most important elementary properties.

6. (a) If $x_1, \dots, x_n$ are distinct numbers, find a polynomial function f_i of degree $n-1$ which is 1 at x_i and 0 at x_j for $j \neq i$. Hint: the product of all $(x - x_j)$ for $j \neq i$, is 0 at x_j if $j \neq i$. (This product is usually denoted by

$$\prod_{\substack{j=1 \\ j\neq i}}^{n} (x - x_j),$$

the symbol Π (capital pi) playing the same role for products that Σ plays for sums.)

(b) Now find a polynomial function f of degree $n-1$ such that $f(x_i) = a_i$, where $a_1, \dots, a_n$ are given numbers. (You should use the functions f_i from part (a). The formula you will obtain is called the "Lagrange interpolation formula.")

7. (a) Prove that for any polynomial function f, and any number a, there is a polynomial function g, and a number b, such that $f(x) = (x-a)g(x)+b$ for all x. (The idea is simply to divide $(x-a)$ into $f(x)$ by long division, until a constant remainder is left. For example, the calculation

$$
\begin{array}{r|rrrr}
 & x^2 & +x & -2 & \\ \hline
x-1 & x^3 & & -3x & +1 \\
 & x^3 & -x^2 & & \\ \cline{3-4}
 & & x^2 & -3x & \\
 & & x^2 & -x & \\ \cline{3-4}
 & & & -2x & +1 \\
 & & & -2x & +2 \\ \cline{4-5}
 & & & & -1
\end{array}
$$

shows that $x^3 - 3x + 1 = (x - 1)(x^2 + x - 2) - 1$. A formal proof is possible by induction on the degree of f.)

(b) Prove that if $f(a) = 0$, then $f(x) = (x - a)g(x)$ for some polynomial function g. (The converse is obvious.)

(c) Prove that if f is a polynomial function of degree n, then f has at most n roots, i.e., there are at most n numbers a with $f(a) = 0$.

(d) Show that for each n there is a polynomial function of degree n with n roots. If n is even find a polynomial function of degree n with no roots, and if n is odd find one with only one root.

8. For which numbers a, b, c, and d will the function

$$f(x) = \frac{ax + b}{cx + d}$$

satisfy $f(f(x)) = x$ for all x (for which this equation makes sense)?

9. (a) If A is any set of real numbers, define a function C_A as follows:

$$C_A(x) = \begin{cases} 1, & x \text{ in } A \\ 0, & x \text{ not in } A. \end{cases}$$

Find expressions for $C_{A \cap B}$ and $C_{A \cup B}$ and $C_{\mathbf{R}-A}$, in terms of C_A and C_B. (The symbol $A \cap B$ was defined in this chapter, but the other two may be new to you. They can be defined as follows:

$$A \cup B = \{x : x \text{ is in } A \text{ or } x \text{ is in } B\},$$
$$\mathbf{R} - A = \{x : x \text{ is in } \mathbf{R} \text{ but } x \text{ is not in } A\}.)$$

(b) Suppose f is a function such that $f(x) = 0$ or 1 for each x. Prove that there is a set A such that $f = C_A$.

(c) Show that $f = f^2$ if and only if $f = C_A$ for some set A.

10. (a) For which functions f is there a function g such that $f = g^2$? Hint: You can certainly answer this question if "function" is replaced by "number."

(b) For which functions f is there a function g such that $f = 1/g$?

*(c) For which functions b and c can we find a function x such that

$$(x(t))^2 + b(t)x(t) + c(t) = 0$$

for all numbers t?

*(d) What conditions must the functions a and b satisfy if there is to be a function x such that

$$a(t)x(t) + b(t) = 0$$

for all numbers t? How many such functions x will there be?

11. (a) Suppose that H is a function and y is a number such that $H(H(y)) = y$. What is

$$\underbrace{H(H(H(\cdots(H(y)\cdots)}_{80 \text{ times}}?$$

(b) Same question if 80 is replaced by 81.
(c) Same question if $H(H(y)) = H(y)$.
*(d) Find a function H such that $H(H(x)) = H(x)$ for all numbers x, and such that $H(1) = 36$, $H(2) = \pi/3$, $H(13) = 47$, $H(36) = 36$, $H(\pi/3) = \pi/3$, $H(47) = 47$. (Don't try to "solve" for $H(x)$; there are many functions H with $H(H(x)) = H(x)$. The extra conditions on H are supposed to suggest a way of finding a suitable H.)
*(e) Find a function H such that $H(H(x)) = H(x)$ for all x, and such that $H(1) = 7$, $H(17) = 18$.

12. A function f is **even** if $f(x) = f(-x)$ and **odd** if $f(x) = -f(-x)$. For example, f is even if $f(x) = x^2$ or $f(x) = |x|$ or $f(x) = \cos x$, while f is odd if $f(x) = x$ or $f(x) = \sin x$.

(a) Determine whether $f + g$ is even, odd, or not necessarily either, in the four cases obtained by choosing f even or odd, and g even or odd. (Your answers can most conveniently be displayed in a 2×2 table.)
(b) Do the same for $f \cdot g$.
(c) Do the same for $f \circ g$.
(d) Prove that every even function f can be written $f(x) = g(|x|)$, for infinitely many functions g.

***13.** (a) Prove that any function f with domain $\mathbf{R}$ can be written $f = E + O$, where E is even and O is odd.
(b) Prove that this way of writing f is unique. (If you try to do part (b) first, by "solving" for E and O you will probably find the solution to part (a).)

14. If f is any function, define a new function $|f|$ by $|f|(x) = |f(x)|$. If f and g are functions, define two new functions, $\max(f, g)$ and $\min(f, g)$, by

$$\max(f, g)(x) = \max(f(x), g(x)),$$
$$\min(f, g)(x) = \min(f(x), g(x)).$$

Find an expression for $\max(f, g)$ and $\min(f, g)$ in terms of $|\ \ |$.

15. (a) Show that $f = \max(f, 0) + \min(f, 0)$. This particular way of writing f is fairly useful; the functions $\max(f, 0)$ and $\min(f, 0)$ are called the **positive** and **negative parts** of f.
(b) A function f is called **nonnegative** if $f(x) \geq 0$ for all x. Prove that any function f can be written $f = g - h$, where g and h are nonnegative, in infinitely many ways. (The "standard way" is $g = \max(f, 0)$ and $h = -\min(f, 0)$.) Hint: Any *number* can certainly be written as the difference of two nonnegative *numbers* in infinitely many ways.

***16.** Suppose f satisfies $f(x + y) = f(x) + f(y)$ for all x and y.

(a) Prove that $f(x_1 + \cdots + x_n) = f(x_1) + \cdots + f(x_n)$.
(b) Prove that there is some number c such that $f(x) = cx$ for all *rational* numbers x (at this point we're not trying to say anything about $f(x)$ for irrational x). Hint: First figure out what c must be. Now prove that

$f(x) = cx$, first when x is a natural number, then when x is an integer, then when x is the reciprocal of an integer and, finally, for all rational x.

***17.** If $f(x) = 0$ for all x, then f satisfies $f(x+y) = f(x) + f(y)$ for all x and y, and also $f(x \cdot y) = f(x) \cdot f(y)$ for all x and y. Now suppose that f satisfies these two properties, but that $f(x)$ is not always 0. Prove that $f(x) = x$ for all x, as follows:

(a) Prove that $f(1) = 1$.
(b) Prove that $f(x) = x$ if x is rational.
(c) Prove that $f(x) > 0$ if $x > 0$. (This part is tricky, but if you have been paying attention to the philosophical remarks accompanying the problems in the last two chapters, you will know what to do.)
(d) Prove that $f(x) > f(y)$ if $x > y$.
(e) Prove that $f(x) = x$ for all x. Hint: Use the fact that between any two numbers there is a rational number.

***18.** Precisely what conditions must f, g, h, and k satisfy in order that $f(x)g(y) = h(x)k(y)$ for all x and y?

***19.** (a) Prove that there do *not* exist functions f and g with either of the following properties:

(i) $f(x) + g(y) = xy$ for all x and y.
(ii) $f(x) \cdot g(y) = x + y$ for all x and y.

Hint: Try to get some information about f or g by choosing particular values of x and y.

(b) Find functions f and g such that $f(x+y) = g(xy)$ for all x and y.

***20.** (a) Find a function f, other than a constant function, such that $|f(y) - f(x)| \le |y - x|$.
(b) Suppose that $f(y) - f(x) \le (y-x)^2$ for all x and y. (Why does this imply that $|f(y) - f(x)| \le (y-x)^2$?) Prove that f is a constant function. Hint: Divide the interval from x to y into n equal pieces.

21. Prove or give a counterexample for each of the following assertions:

(a) $f \circ (g + h) = f \circ g + f \circ h$.
(b) $(g + h) \circ f = g \circ f + h \circ f$.
(c) $\dfrac{1}{f \circ g} = \dfrac{1}{f} \circ g$.
(d) $\dfrac{1}{f \circ g} = f \circ \left(\dfrac{1}{g}\right)$.

22. (a) Suppose $g = h \circ f$. Prove that if $f(x) = f(y)$, then $g(x) = g(y)$.
(b) Conversely, suppose that f and g are two functions such that $g(x) = g(y)$ whenever $f(x) = f(y)$. Prove that $g = h \circ f$ for some function h. Hint: Just try to define $h(z)$ when z is of the form $z = f(x)$ (these are the only z that matter) and use the hypotheses to show that your definition will not run into trouble.

23. Suppose that $f \circ g = I$, where $I(x) = x$. Prove that

(a) if $x \neq y$, then $g(x) \neq g(y)$;
(b) every number b can be written $b = f(a)$ for some number a.

***24.** (a) Suppose g is a function with the property that $g(x) \neq g(y)$ if $x \neq y$. Prove that there is a function f such that $f \circ g = I$.
(b) Suppose that f is a function such that every number b can be written $b = f(a)$ for some number a. Prove that there is a function g such that $f \circ g = I$.

***25.** Find a function f such that $g \circ f = I$ for some g, but such that there is no function h with $f \circ h = I$.

***26.** Suppose $f \circ g = I$ and $h \circ f = I$. Prove that $g = h$. Hint: Use the fact that composition is associative.

27. (a) Suppose $f(x) = x + 1$. Are there any functions g such that $f \circ g = g \circ f$?
(b) Suppose f is a constant function. For which functions g does $f \circ g = g \circ f$?
(c) Suppose that $f \circ g = g \circ f$ for *all* functions g. Show that f is the identity function, $f(x) = x$.

28. (a) Let F be the set of all functions whose domain is $\mathbf{R}$. Prove that, using $+$ and $\cdot$ as defined in this chapter, all of properties P1–P9 except P7 hold for F, provided 0 and 1 are interpreted as constant functions.
(b) Show that P7 does not hold.
*(c) Show that P10–P12 cannot hold. In other words, show that there is no collection P of functions in F, such that P10–P12 hold for P. (It is sufficient, and will simplify things, to consider only functions which are 0 except at two points x_0 and x_1.)
(d) Suppose we define $f < g$ to mean that $f(x) < g(x)$ for all x. Which of P′10–P′13 (in Problem 1-8) now hold?
(e) If $f < g$, is $h \circ f < h \circ g$? Is $f \circ h < g \circ h$?

APPENDIX. ORDERED PAIRS

Not only in the definition of a function, but in other parts of the book as well, it is necessary to use the notion of an ordered pair of objects. A definition has not yet been given, and we have never even stated explicitly what properties an ordered pair is supposed to have. The one property which we will require states formally that the ordered pair (a, b) should be determined by a and b, and the order in which they are given:

$$\text{if } (a, b) = (c, d), \text{ then } a = c \text{ and } b = d.$$

Ordered pairs may be treated most conveniently by simply introducing (a, b) as an undefined term and adopting the basic property as an axiom—since this property is the only significant fact about ordered pairs, there is not much point worrying about what an ordered pair "really" is. Those who find this treatment satisfactory need read no further.

The rest of this short appendix is for the benefit of those readers who will feel uncomfortable unless ordered pairs are somehow defined so that this basic property becomes a theorem. There is no point in restricting our attention to ordered pairs of numbers; it is just as reasonable, and just as important, to have available the notion of an ordered pair of any two mathematical objects. This means that our definition ought to involve only concepts common to all branches of mathematics. The one common concept which pervades all areas of mathematics is that of a set, and ordered pairs (like everything else in mathematics) can be defined in this context; an ordered pair will turn out to be a set of a rather special sort.

The set $\{a, b\}$, containing the two elements a and b, is an obvious first choice, but will not do as a definition for (a, b), because there is no way of determining from $\{a, b\}$ which of a or b is meant to be the first element. A more promising candidate is the rather startling set:

$$\{\{a\}, \{a, b\}\}.$$

This set has two members, both of which are *themselves* sets; one member is the set $\{a\}$, containing the single member a, the other is the set $\{a, b\}$. Shocking as it may seem, we are going to define (a, b) to be this set. The justification for this choice is given by the theorem immediately following the definition—the definition works, and there really isn't anything else worth saying.

DEFINITION

$$\boxed{\mathbf{(a, b)} = \{\{a\}, \{a, b\}\}.}$$

THEOREM 1 If $(a, b) = (c, d)$, then $a = c$ and $b = d$.

PROOF The hypothesis means that

$$\{\{a\}, \ \{a, b\}\} = \{\{c\}, \ \{c, d\}\}.$$

Now $\{\{a\}, \ \{a, b\}\}$ contains just two members, $\{a\}$ and $\{a, b\}$; and a is the only common element of these two members of $\{\{a\}, \ \{a, b\}\}$. Similarly, c is the unique common member of both members of $\{\{c\},\ \{c,d\}\}$. Therefore $a = c$. We therefore have

$$\{\{a\},\ \{a,b\}\} = \{\{a\},\ \{a,d\}\},$$

and only the proof that $b = d$ remains. It is convenient to distinguish 2 cases.

Case 1. $b = a$. In this case, $\{a,b\} = \{a\}$, so the set $\{\{a\},\ \{a,b\}\}$ really has only one member, namely, $\{a\}$. The same must be true of $\{\{a\},\ \{a,d\}\}$, so $\{a,d\} = \{a\}$, which implies that $d = a = b$.

Case 2. $b \neq a$. In this case, b is in one member of $\{\{a\},\ \{a,b\}\}$ but not in the other. It must therefore be true that b is in one member of $\{\{a\},\ \{a,d\}\}$ but not in the other. This can happen only if b is in $\{a,d\}$, but b is not in $\{a\}$; thus $b = a$ or $b = d$, but $b \neq a$; so $b = d$. ▮