CHAPTER 8 LEAST UPPER BOUNDS

This chapter reveals the most important property of the real numbers. Nevertheless, it is merely a sequel to Chapter 7; the path which must be followed has already been indicated, and further discussion would be useless delay.

DEFINITION

> A set A of real numbers is **bounded above** if there is a number x such that
> $$x \geq a \quad \text{for every } a \text{ in } A.$$
> Such a number x is called an **upper bound** for A.

Obviously A is bounded above if and only if there is a number x which is an upper bound for A (and in this case there will be lots of upper bounds for A); we often say, as a concession to idiomatic English, that "A has an upper bound" when we mean that there is a number which is an upper bound for A.

Notice that the term "bounded above" has now been used in two ways—first, in Chapter 7, in reference to functions, and now in reference to sets. This dual usage should cause no confusion, since it will always be clear whether we are talking about a set of numbers or a function. Moreover, the two definitions are closely connected: if A is the set $\{f(x) : a \leq x \leq b\}$, then the function f is bounded above on $[a, b]$ if and only if the set A is bounded above.

The entire collection $\mathbf{R}$ of real numbers, and the natural numbers $\mathbf{N}$, are both examples of sets which are *not* bounded above. An example of a set which *is* bounded above is
$$A = \{x : 0 \leq x < 1\}.$$

To show that A is bounded above we need only name some upper bound for A, which is easy enough; for example, 138 is an upper bound for A, and so are 2, $1\frac{1}{2}$, $1\frac{1}{4}$, and 1. Clearly, 1 is the least upper bound of A; although the phrase just introduced is self-explanatory, in order to avoid any possible confusion (in particular, to ensure that we all know what the superlative of "less" means), we define this explicitly.

DEFINITION

> A number x is a **least upper bound** of A if
> $$\begin{aligned} &(1) \quad x \text{ is an upper bound of } A, \\ \text{and} \quad &(2) \quad \text{if } y \text{ is an upper bound of } A, \text{ then } x \leq y. \end{aligned}$$

our assertion is true—and very important, definitely important enough to warrant consideration of details. We are finally ready to state the last property of the real numbers which we need.

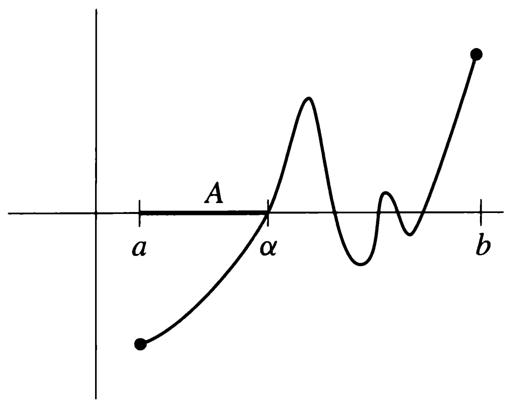


FIGURE 1

(P13) (The least upper bound property) If A is a set of real numbers, $A \neq \emptyset$, and A is bounded above, then A has a least upper bound.

Property P13 may strike you as anticlimactic, but that is actually one of its virtues. To complete our list of basic properties for the real numbers we require no particularly abstruse proposition, but only a property so simple that we might feel foolish for having overlooked it. Of course, the least upper bound property is not really so innocent as all that; after all, it does *not* hold for the rational numbers $\mathbf{Q}$. For example, if A is the set of all rational numbers x satisfying $x^2 < 2$, then there is no *rational* number y which is an upper bound for A and which is less than or equal to every other *rational* number which is an upper bound for A. It will become clear only gradually how significant P13 is, but we are already in a position to demonstrate its power, by supplying the proofs which were omitted in Chapter 7.

THEOREM 7-1 If f is continuous on $[a, b]$ and $f(a) < 0 < f(b)$, then there is some number x in $[a, b]$ such that $f(x) = 0$.

PROOF Our proof is merely a rigorous version of the outline developed at the end of Chapter 7—we will locate the smallest number x in $[a, b]$ with $f(x) = 0$.

Define the set A, shown in Figure 1, as follows:

$$A = \{x : a \leq x \leq b, \text{ and } f \text{ is negative on the interval } [a, x]\}.$$

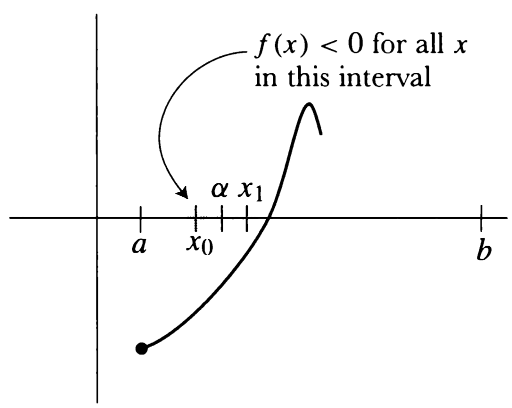


FIGURE 2

Clearly $A \neq \emptyset$, since a is in A; in fact, there is some $\delta > 0$ such that A contains all points x satisfying $a \leq x < a + \delta$; this follows from Problem 6-16, since f is continuous on $[a, b]$ and $f(a) < 0$. Similarly, b is an upper bound for A and, in fact, there is a $\delta > 0$ such that all points x satisfying $b - \delta < x \leq b$ are upper bounds for A; this also follows from Problem 6-16, since $f(b) > 0$.

From these remarks it follows that A has a least upper bound α and that $a < \alpha < b$. We now wish to show that $f(\alpha) = 0$, by eliminating the possibilities $f(\alpha) < 0$ and $f(\alpha) > 0$.

Suppose first that $f(\alpha) < 0$. By Theorem 6-3, there is a $\delta > 0$ such that $f(x) < 0$ for $\alpha - \delta < x < \alpha + \delta$ (Figure 2). Now there is some number x_0 in A which satisfies $\alpha - \delta < x_0 < \alpha$ (because otherwise α would not be the *least* upper bound of A). This means that f is negative on the whole interval $[a, x_0]$. But if x_1 is a number between α and $\alpha + \delta$, then f is also negative on the whole interval $[x_0, x_1]$. Therefore f is negative on the interval $[a, x_1]$, so x_1 is in A. But this contradicts the fact that α is an upper bound for A; our original assumption that $f(\alpha) < 0$ must be false.

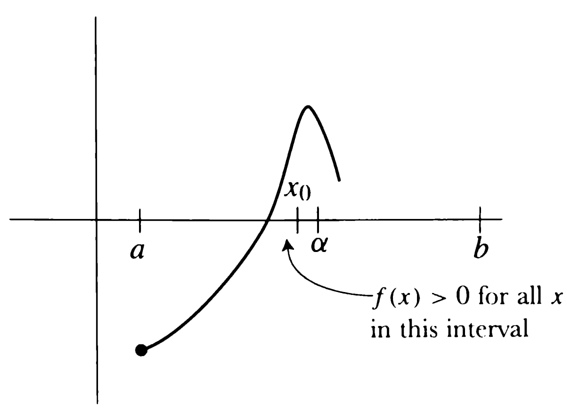


FIGURE 3

Suppose, on the other hand, that $f(\alpha) > 0$. Then there is a number $\delta > 0$ such that $f(x) > 0$ for $\alpha - \delta < x < \alpha + \delta$ (Figure 3). Once again we know that there is an x_0 in A satisfying $\alpha - \delta < x_0 < \alpha$; but this means that f is negative on $[a, x_0]$, which is impossible, since $f(x_0) > 0$. Thus the assumption $f(\alpha) > 0$ also leads to a contradiction, leaving $f(\alpha) = 0$ as the only possible alternative. ▮

The proofs of Theorems 2 and 3 of Chapter 7 require a simple preliminary result, which will play much the same role as Theorem 6-3 played in the previous proof.

THEOREM 1 If f is continuous at a, then there is a number $\delta > 0$ such that f is bounded above on the interval $(a - \delta, a + \delta)$ (see Figure 4).

PROOF Since $\lim\limits_{x \to a} f(x) = f(a)$, there is, for every $\varepsilon > 0$, a $\delta > 0$ such that, for all x,

$$\text{if } |x - a| < \delta, \text{ then } |f(x) - f(a)| < \varepsilon.$$

It is only necessary to apply this statement to some particular ε (any one will do), for example, $\varepsilon = 1$. We conclude that there is a $\delta > 0$ such that, for all x,

$$\text{if } |x - a| < \delta, \text{ then } |f(x) - f(a)| < 1.$$

It follows, in particular, that if $|x - a| < \delta$, then $f(x) - f(a) < 1$. This completes the proof: on the interval $(a - \delta, a + \delta)$ the function f is bounded above by $f(a) + 1$. ▮

It should hardly be necessary to add that we can now also prove that f is bounded below on some interval $(a - \delta, a + \delta)$, and, finally, that f is bounded on some open interval containing a.

A more significant point is the observation that if $\lim\limits_{x \to a^+} f(x) = f(a)$, then there

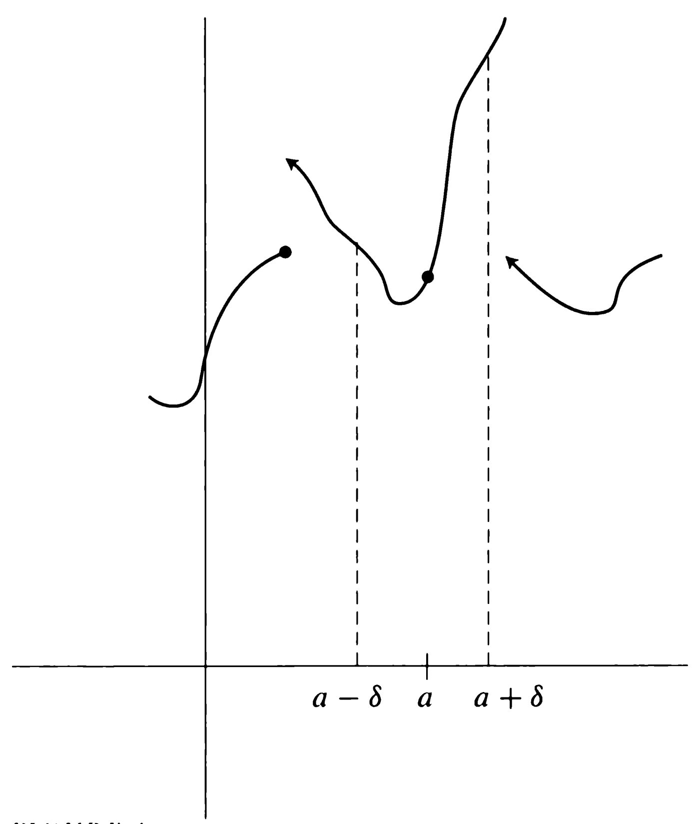


FIGURE 4

The use of the indefinite article "a" in this definition was merely a concession to temporary ignorance. Now that we have made a precise definition, it is easily seen that if x and y are both least upper bounds of A, then $x = y$. Indeed, in this case

$$\begin{aligned} x &\le y, && \text{since } y \text{ is an upper bound, and } x \text{ is a least upper bound,}\\ \text{and } y &\le x, && \text{since } x \text{ is an upper bound, and } y \text{ is a least upper bound;} \end{aligned}$$

it follows that $x = y$. For this reason we speak of *the* least upper bound of A. The term **supremum** of A is synonymous and has one advantage. It abbreviates quite nicely to

$$\sup A \qquad \text{(pronounced "soup } A\text{")}$$

and saves us from the abbreviation

$$\text{lub } A$$

(which is nevertheless used by some authors).

There is a series of important definitions, analogous to those just given, which can now be treated more briefly. A set A of real numbers is **bounded below** if there is a number x such that

$$x \le a \qquad \text{for every } a \text{ in } A.$$

Such a number x is called a **lower bound** for A. A number x is the **greatest lower bound** of A if

$$\begin{aligned} &(1) && x \text{ is a lower bound of } A,\\ \text{and } &(2) && \text{if } y \text{ is a lower bound of } A, \text{ then } x \ge y. \end{aligned}$$

The greatest lower bound of A is also called the **infimum** of A, abbreviated

$$\inf A;$$

some authors use the abbreviation

$$\text{glb } A.$$

One detail has been omitted from our discussion so far—the question of which sets have at least one, and hence exactly one, least upper bound or greatest lower bound. We will consider only least upper bounds, since the question for greatest lower bounds can then be answered easily (Problem 2).

If A is not bounded above, then A has no upper bound at all, so A certainly cannot be expected to have a least upper bound. It is tempting to say that A does have a least upper bound if it has *some* upper bound, but, like the principle of mathematical induction, this assertion can fail to be true in a rather special way. If $A = \emptyset$, then A is bounded above. Indeed, any number x is an upper bound for $\emptyset$:

$$x \ge y \quad \text{for every } y \text{ in } \emptyset$$

simply because there is no y in $\emptyset$. Since *every* number is an upper bound for $\emptyset$, there is surely no least upper bound for $\emptyset$. With this trivial exception however, is a $\delta > 0$ such that f is bounded on the set $\{x : a \le x < a + \delta\}$, and a similar observation holds if $\lim_{x \to b^-} f(x) = f(b)$. Having made these observations (and assuming that you will supply the proofs), we tackle our second major theorem.

THEOREM 7-2 If f is continuous on $[a, b]$, then f is bounded above on $[a, b]$.

PROOF Let

$$A = \{x : a \le x \le b \text{ and } f \text{ is bounded above on } [a, x]\}.$$

Clearly $A \ne \emptyset$ (since a is in A), and A is bounded above (by b), so A has a least upper bound α. Notice that we are here applying the term "bounded above" both to the set A, which can be visualized as lying on the horizontal axis, and to f, i.e., to the sets $\{f(y) : a \le y \le x\}$, which can be visualized as lying on the vertical axis (Figure 5).

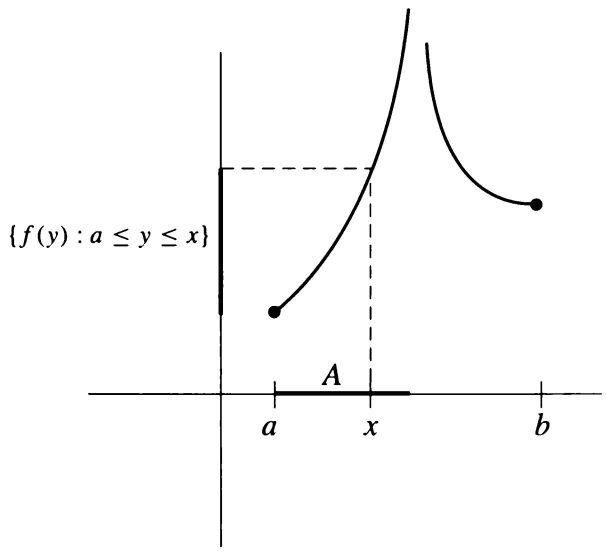


FIGURE 5

Our first step is to prove that we actually have $\alpha = b$. Suppose, instead, that $\alpha < b$. By Theorem 1 there is $\delta > 0$ such that f is bounded on $(\alpha - \delta, \alpha + \delta)$. Since α is the least upper bound of A there is some x_0 in A satisfying $\alpha - \delta < x_0 < \alpha$. This means that f is bounded on $[a, x_0]$. But if x_1 is any number with $\alpha < x_1 < \alpha + \delta$, then f is also bounded on $[x_0, x_1]$. Therefore f is bounded on $[a, x_1]$, so x_1 is in A, contradicting the fact that α is an upper bound for A. This contradiction shows that $\alpha = b$. One detail should be mentioned: this demonstration implicitly assumed that $a < \alpha$ [so that f would be defined on some interval $(\alpha - \delta, \alpha + \delta)$]; the possibility $a = \alpha$ can be ruled out similarly, using the existence of a $\delta > 0$ such that f is bounded on $\{x : a \le x < a + \delta\}$.

The proof is not quite complete—we only know that f is bounded on $[a, x]$ for every $x < b$, not necessarily that f is bounded on $[a, b]$. However, only one small argument needs to be added.

There is a $\delta > 0$ such that f is bounded on $\{x : b - \delta < x \le b\}$. There is x_0 in A such that $b - \delta < x_0 < b$. Thus f is bounded on $[a, x_0]$ and also on $[x_0, b]$, so f is bounded on $[a, b]$. ▌

To prove the third important theorem we resort to a trick.

THEOREM 7-3 If f is continuous on $[a, b]$, then there is a number y in $[a, b]$ such that $f(y) \ge f(x)$ for all x in $[a, b]$.

PROOF We already know that f is bounded on $[a, b]$, which means that the set

$$\{f(x) : x \text{ in } [a, b]\}$$

is bounded. This set is obviously not $\emptyset$, so it has a least upper bound α. Since $\alpha \ge f(x)$ for x in $[a, b]$ it suffices to show that $\alpha = f(y)$ for some y in $[a, b]$.

Suppose instead that $\alpha \ne f(y)$ for all y in $[a, b]$. Then the function g defined by

$$g(x) = \frac{1}{\alpha - f(x)}, \quad x \text{ in } [a, b]$$

is continuous on $[a, b]$, since the denominator of the right side is never 0. On the other hand, α is the least upper bound of $\{f(x) : x \text{ in } [a, b]\}$; this means that

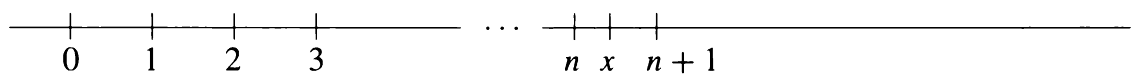
$$\text{for every } \varepsilon > 0 \text{ there is } x \text{ in } [a, b] \text{ with } \alpha - f(x) < \varepsilon.$$

This, in turn, means that

$$\text{for every } \varepsilon > 0 \text{ there is } x \text{ in } [a, b] \text{ with } g(x) > 1/\varepsilon.$$

But *this* means that g is not bounded on $[a, b]$, contradicting the previous theorem. ▌

At the beginning of this chapter the set of natural numbers **N** was given as an example of an unbounded set. We are now going to *prove* that **N** is unbounded. After the difficult theorems proved in this chapter you may be startled to find such an "obvious" theorem winding up our proceedings. If so, you are, perhaps, allowing the geometrical picture of **R** to influence you too strongly. "Look," you may say, "the real numbers look like



so every number x is between two integers n, $n + 1$ (unless x is itself an integer)." Basing the argument on a geometric picture is not a proof, however, and even the geometric picture contains an assumption: that if you place unit segments end-to-end you will eventually get a segment larger than any given segment. This axiom, often omitted from a first introduction to geometry, is usually attributed (not quite justly) to Archimedes, and the corresponding property for numbers, that **N** is not bounded, is called the *Archimedean property* of the real numbers. This property is *not* a consequence of P1–P12 (see reference [14] of the Suggested Reading), although it does hold for **Q**, of course. Once we have P13 however, there are no longer any problems.

THEOREM 2 **N** is not bounded above.

PROOF Suppose **N** were bounded above. Since $\mathbf{N} \neq \emptyset$, there would be a least upper bound α for **N**. Then

$$\alpha \geq n \qquad \text{for all } n \text{ in } \mathbf{N}.$$

Consequently,

$$\alpha \geq n + 1 \quad \text{for all } n \text{ in } \mathbf{N},$$

since $n + 1$ is in **N** if n is in **N**. But this means that

$$\alpha - 1 \geq n \quad \text{for all } n \text{ in } \mathbf{N},$$

and *this* means that $\alpha - 1$ is also an upper bound for **N**, contradicting the fact that α is the least upper bound. ▌

There is a consequence of Theorem 2 (actually an equivalent formulation) which we have very often assumed implicitly.

THEOREM 3 For any $\varepsilon > 0$ there is a natural number n with $1/n < \varepsilon$.

PROOF Suppose not; then $1/n \geq \varepsilon$ for all n in $\mathbf{N}$. Thus $n \leq 1/\varepsilon$ for all n in $\mathbf{N}$. But this means that $1/\varepsilon$ is an upper bound for $\mathbf{N}$, contradicting Theorem 2. ▮

A brief glance through Chapter 6 will show you that the result of Theorem 3 was used in the discussion of many examples. Of course, Theorem 3 was not available at the time, but the examples were so important that in order to give them some cheating was tolerated. As partial justification for this dishonesty we can claim that this result was never used in the proof of a *theorem*, but if your faith has been shaken, a review of all the proofs given so far is in order. Fortunately, such deception will not be necessary again. We have now stated every property of the real numbers that we will ever need. Henceforth, no more lies.

PROBLEMS

1. Find the least upper bound and the greatest lower bound (if they exist) of the following sets. Also decide which sets have greatest and least elements (i.e., decide when the least upper bound and greatest lower bound happens to belong to the set).

(i) $\left\{\dfrac{1}{n} : n \text{ in } \mathbf{N}\right\}$.

(ii) $\left\{\dfrac{1}{n} : n \text{ in } \mathbf{Z} \text{ and } n \neq 0\right\}$.

(iii) $\{x : x = 0 \text{ or } x = 1/n \text{ for some } n \text{ in } \mathbf{N}\}$.

(iv) $\{x : 0 \leq x \leq \sqrt{2} \text{ and } x \text{ is rational}\}$.

(v) $\{x : x^2 + x + 1 \geq 0\}$.

(vi) $\{x : x^2 + x - 1 < 0\}$.

(vii) $\{x : x < 0 \text{ and } x^2 + x - 1 < 0\}$.

(viii) $\left\{\dfrac{1}{n} + (-1)^n : n \text{ in } \mathbf{N}\right\}$.

2. (a) Suppose $A \neq \emptyset$ is bounded below. Let $-A$ denote the set of all $-x$ for x in A. Prove that $-A \neq \emptyset$, that $-A$ is bounded above, and that $-\sup(-A)$ is the greatest lower bound of A.

(b) If $A \neq \emptyset$ is bounded below, let B be the set of all lower bounds of A. Show that $B \neq \emptyset$, that B is bounded above, and that $\sup B$ is the greatest lower bound of A.

3. Let f be a continuous function on $[a, b]$ with $f(a) < 0 < f(b)$.

(a) The proof of Theorem 7-1 showed that there is a smallest x in $[a, b]$ with $f(x) = 0$. If there is more than one x in $[a, b]$ with $f(x) = 0$, is there necessarily a second smallest? Show that there is a largest x in

$[a, b]$ with $f(x) = 0$. (Try to give an easy proof by considering a new function closely related to f.)

(b) The proof of Theorem 7-1 depended upon considering $A = \{x : a \le x \le b$ and f is negative on $[a, x]\}$. Give another proof of Theorem 7-1, which depends upon consideration of $B = \{x : a \le x \le b$ and $f(x) < 0\}$. Which point x in $[a, b]$ with $f(x) = 0$ will this proof locate? Give an example where the sets A and B are not the same.

***4.** (a) Suppose that f is continuous on $[a, b]$ and that $f(a) = f(b) = 0$. Suppose also that $f(x_0) > 0$ for some x_0 in $[a, b]$. Prove that there are numbers c and d with $a \le c < x_0 < d \le b$ such that $f(c) = f(d) = 0$, but $f(x) > 0$ for all x in (c, d). Hint: The previous problem can be used to good advantage.

(b) Suppose that f is continuous on $[a, b]$ and that $f(a) < f(b)$. Prove that there are numbers c and d with $a \le c < d \le b$ such that $f(c) = f(a)$ and $f(d) = f(b)$ and $f(a) < f(x) < f(d)$ for all x in (c, d).

5. (a) Suppose that $y - x > 1$. Prove that there is an integer k such that $x < k < y$. Hint: Let l be the largest integer satisfying $l \le x$, and consider $l + 1$.

(b) Suppose $x < y$. Prove that there is a rational number r such that $x < r < y$. Hint: If $1/n < y - x$, then $ny - nx > 1$. (Query: Why have parts (a) and (b) been postponed until this problem set?)

(c) Suppose that $r < s$ are rational numbers. Prove that there is an irrational number between r and s. Hint: As a start, you know that there is an irrational number between 0 and 1.

(d) Suppose that $x < y$. Prove that there is an irrational number between x and y. Hint: It is unnecessary to do any more work; this follows from (b) and (c).

***6.** A set A of real numbers is said to be **dense** if every open interval contains a point of A. For example, Problem 5 shows that the set of rational numbers and the set of irrational numbers are each dense.

(a) Prove that if f is continuous and $f(x) = 0$ for all numbers x in a dense set A, then $f(x) = 0$ for all x.

(b) Prove that if f and g are continuous and $f(x) = g(x)$ for all x in a dense set A, then $f(x) = g(x)$ for all x.

(c) If we assume instead that $f(x) \ge g(x)$ for all x in A, show that $f(x) \ge g(x)$ for all x. Can $\ge$ be replaced by $>$ throughout?

7. Prove that if f is continuous and $f(x + y) = f(x) + f(y)$ for all x and y, then there is a number c such that $f(x) = cx$ for all x. (This conclusion can be demonstrated simply by combining the results of two previous problems.) Point of information: There *do* exist *noncontinuous* functions f satisfying $f(x + y) = f(x) + f(y)$ for all x and y, but we cannot prove this now; in fact, this simple question involves ideas that are usually never mentioned in undergraduate courses (see reference [7] in the Suggested Reading).

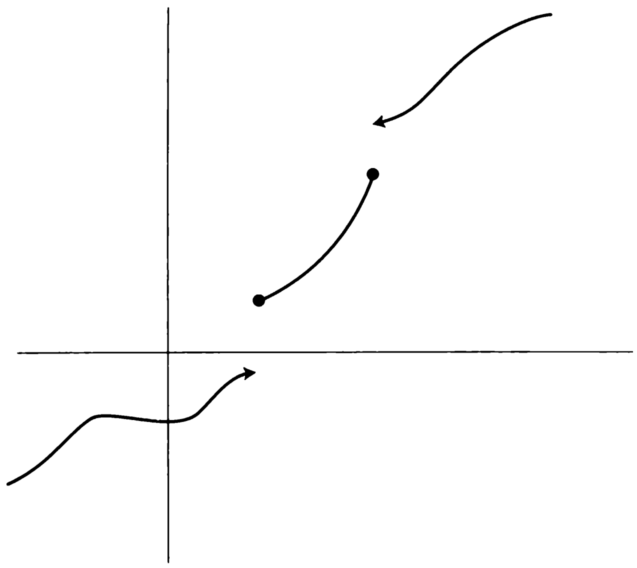


FIGURE 6

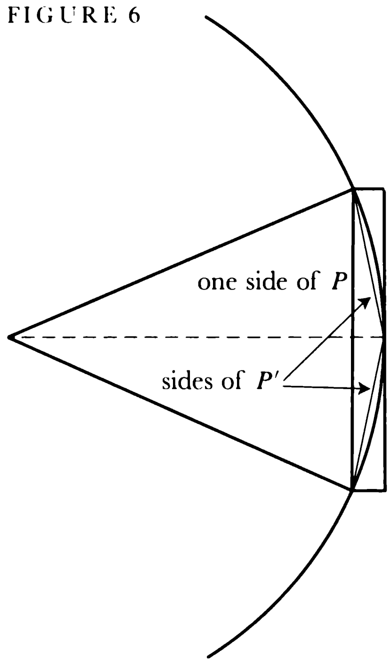


FIGURE 7

***8.** Suppose that f is a function such that $f(a) \leq f(b)$ whenever $a < b$ (Figure 6).

(a) Prove that $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ both exist. Hint: Why is this problem in this chapter?
(b) Prove that f never has a removable discontinuity (this terminology comes from Problem 6-17).
(c) Prove that if f satisfies the conclusions of the Intermediate Value Theorem, then f is continuous.

***9.** If f is a bounded function on $[0, 1]$, let $|||f||| = \sup\{|f(x)| : x \text{ in } [0, 1]\}$. Prove analogues of the properties of $\| \ \|$ in Problem 7-14.

10. Suppose $\alpha > 0$. Prove that every number x can be written uniquely in the form $x = k\alpha + x'$, where k is an integer, and $0 \leq x' < \alpha$.

11. (a) Suppose that $a_1, a_2, a_3, \ldots$ is a sequence of positive numbers with $a_{n+1} \leq a_n/2$. Prove that for any $\varepsilon > 0$ there is some n with $a_n < \varepsilon$.
(b) Suppose P is a regular polygon inscribed inside a circle. If P' is the inscribed regular polygon with twice as many sides, show that the difference between the area of the circle and the area of P' is less than half the difference between the area of the circle and the area of P (use Figure 7).
(c) Prove that there is a regular polygon P inscribed in a circle with area as close as desired to the area of the circle. In order to do part (c) you will need part (a). This was clear to the Greeks, who used part (a) as the basis for their entire treatment of proportion and area. By calculating the areas of polygons, this method ("the method of exhaustion") allows computations of π to any desired accuracy; Archimedes used it to show that $\frac{223}{71} < \pi < \frac{22}{7}$. But it has far greater theoretical importance:
*(d) Using the fact that the areas of two regular polygons with the same number of sides have the same ratio as the square of their sides, prove that the areas of two circles have the same ratios as the square of their radii. Hint: Deduce a contradiction from the assumption that the ratio of the areas is greater, or less, than the ratio of the square of the radii by inscribing appropriate polygons.

12. Suppose that A and B are two nonempty sets of numbers such that $x \leq y$ for all x in A and all y in B.

(a) Prove that $\sup A \leq y$ for all y in B.
(b) Prove that $\sup A \leq \inf B$.

13. Let A and B be two nonempty sets of numbers which are bounded above, and let $A+B$ denote the set of all numbers $x+y$ with x in A and y in B. Prove that $\sup(A+B) = \sup A + \sup B$. Hint: The inequality $\sup(A+B) \leq \sup A + \sup B$ is easy. Why? To prove that $\sup A + \sup B \leq \sup(A+B)$ it suffices to prove that $\sup A + \sup B \leq \sup(A+B) + \varepsilon$ for all $\varepsilon > 0$; begin by choosing x in A

and y in B with $\sup A - x < \varepsilon/2$ and $\sup B - y < \varepsilon/2$.

FIGURE 8

14. (a) Consider a sequence of closed intervals $I_1 = [a_1, b_1]$, $I_2 = [a_2, b_2]$, ... Suppose that $a_n \le a_{n+1}$ and $b_{n+1} \le b_n$ for all n (Figure 8). Prove that there is a point x which is in every I_n.
(b) Show that this conclusion is false if we consider open intervals instead of closed intervals.

The simple result of Problem 14(a) is called the "Nested Interval Theorem." It may be used to give alternative proofs of Theorems 1 and 2. The appropriate reasoning, outlined in the next two problems, illustrates a general method, called a "bisection argument."

***15.** Suppose f is continuous on $[a, b]$ and $f(a) < 0 < f(b)$. Then either $f((a+b)/2) = 0$, or f has different signs at the end points of the interval $[a, (a+b)/2]$, or f has different signs at the end points of $[(a+b)/2, b]$. Why? If $f((a+b)/2) \neq 0$, let I_1 be the interval on which f changes sign. Now bisect I_1. Either f is 0 at the midpoint, or f changes sign on one of the two intervals. Let I_2 be that interval. Continue in this way, to define I_n for each n (unless f is 0 at some midpoint). Use the Nested Interval Theorem to find a point x where $f(x) = 0$.

***16.** Suppose f were continuous on $[a, b]$, but not bounded on $[a, b]$. Then f would be unbounded on either $[a, (a+b)/2]$ or $[(a+b)/2, b]$. Why? Let I_1 be one of these intervals on which f is unbounded. Proceed as in Problem 15 to obtain a contradiction.

17. (a) Let $A = \{x : x < \alpha\}$. Prove the following (they are all easy):

(i) If x is in A and $y < x$, then y is in A.
(ii) $A \neq \emptyset$.
(iii) $A \neq \mathbf{R}$.
(iv) If x is in A, then there is some number x' in A such that $x < x'$.

(b) Suppose, conversely, that A satisfies (i)–(iv). Prove that $A = \{x : x < \sup A\}$.

***18.** A number x is called an **almost upper bound** for A if there are only finitely many numbers y in A with $y \ge x$. An **almost lower bound** is defined similarly.

(a) Find all almost upper bounds and almost lower bounds of the sets in Problem 1.
(b) Suppose that A is a bounded infinite set. Prove that the set B of all almost upper bounds of A is nonempty, and bounded below.

(c) It follows from part (b) that inf B exists; this number is called the **limit superior** of A, and denoted by $\overline{\lim}\, A$ or lim sup A. Find $\overline{\lim}\, A$ for each set A in Problem 1.
(d) Define $\underline{\lim}\, A$, and find it for all A in Problem 1.

***19.** If A is a bounded infinite set prove

(a) $\underline{\lim}\, A \le \overline{\lim}\, A$.

(b) $\overline{\lim}\, A \le \sup A$.

(c) If $\overline{\lim}\, A < \sup A$, then A contains a largest element.
(d) The analogues of parts (b) and (c) for $\underline{\lim}$.

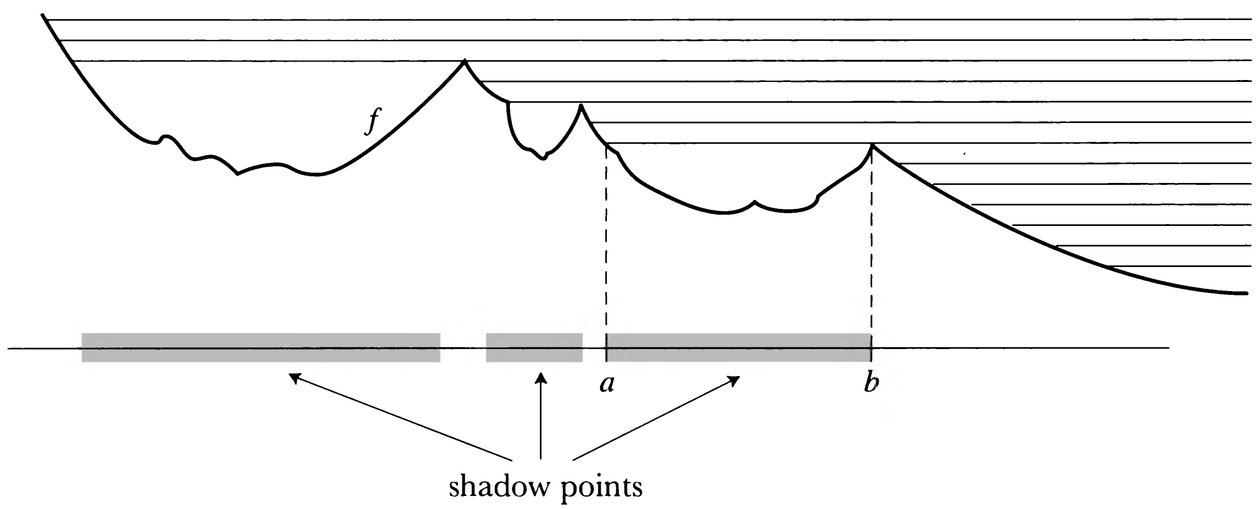


FIGURE 9

20. Let f be a continuous function on **R**. A point x is called a **shadow point** of f if there is a number $y > x$ with $f(y) > f(x)$. The rationale for this terminology is indicated in Figure 9; the parallel lines are the rays of the sun rising in the east (you are facing north). Suppose that all points of (a, b) are shadow points, but that a and b are not shadow points. Clearly, $f(a) \ge f(b)$.

(a) Suppose that $f(a) > f(b)$. Show that the point where f takes on its maximum value on $[a, b]$ must be a.
(b) Then show that this leads to a contradiction, so that in fact we must have $f(a) = f(b)$.

This little result, known as the Rising Sun Lemma, is instrumental in proving several beautiful theorems that do not appear in this book; see page 450.

APPENDIX. UNIFORM CONTINUITY

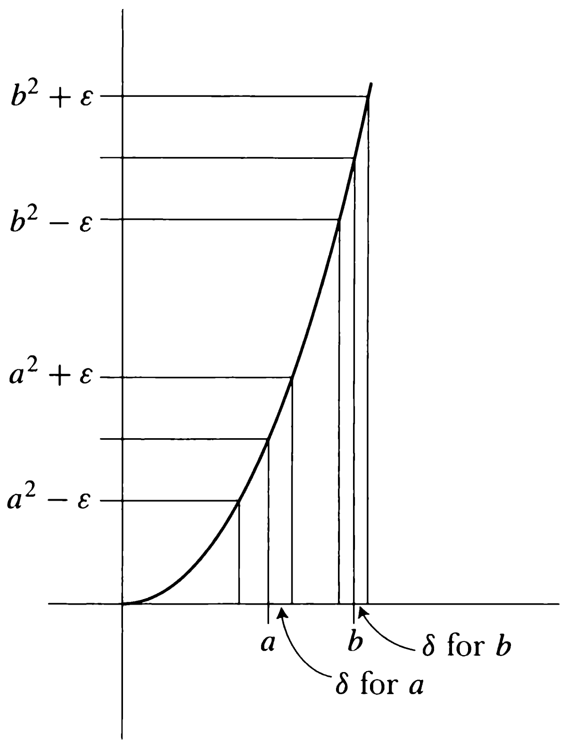


FIGURE 1

Now that we've come to the end of the "foundations," it might be appropriate to slip in one further fundamental concept. This notion is not used crucially in the rest of the book, but it can help clarify many points later on.

We know that the function $f(x) = x^2$ is continuous at a for all a. In other words,

> if a is any number, then for every $\varepsilon > 0$ there is some $\delta > 0$ such that, for all x, if $|x - a| < \delta$, then $|x^2 - a^2| < \varepsilon$.

Of course, δ depends on ε. But δ *also depends* on a—the δ that works at a might not work at b (Figure 1). Indeed, it's clear that given $\varepsilon > 0$ there is no one $\delta > 0$ that works for all a, or even for all positive a. In fact, the number $a + \delta/2$ will certainly satisfy $|x - a| < \delta$, but if $a > 0$, then

$$\left|\left(a + \frac{\delta}{2}\right)^2 - a^2\right| = \left|a\delta + \frac{\delta^2}{4}\right| \geq a\delta,$$

and this won't be $< \varepsilon$ once $a > \varepsilon/\delta$. (This is just an admittedly confusing computational way of saying that f is growing faster and faster!)

On the other hand, for any $\varepsilon > 0$ there *will* be one $\delta > 0$ that works for all a in any interval $[-N, N]$. In fact, the δ which works at N or $-N$ will also work everywhere else in the interval.

As a final example, consider the function $f(x) = \sin 1/x$, or the function whose graph appears in Figure 18 on page 62. It is easy to see that, so long as $\varepsilon < 1$, there will not be one $\delta > 0$ that works for these functions at all points a in the open interval $(0, 1)$.

These examples illustrate important distinctions between the behavior of various continuous functions on certain intervals, and there is a special term to signal this distinction.

DEFINITION

> The function f is **uniformly continuous on an interval** A if for every $\varepsilon > 0$ there is some $\delta > 0$ such that, for all x and y in A,
>
> $$\text{if } |x - y| < \delta, \text{ then } |f(x) - f(y)| < \varepsilon.$$

We've seen that a function can be continuous on the whole line, or on an open interval, without being uniformly continuous there. On the other hand, the function $f(x) = x^2$ did turn out to be uniformly continuous on any closed interval. This shouldn't be too surprising—it's the same sort of thing that occurs when we ask whether a function is bounded on an interval—and we would be led to suspect that any continuous function on a closed interval is also uniformly continuous on that interval. In order to prove this, we'll need to deal first with one subtle point.

Suppose that we have two intervals $[a, b]$ and $[b, c]$ with the common endpoint b, and a function f that is continuous on $[a, c]$. Let $\varepsilon > 0$ and suppose that the following two statements hold:

(i) if x and y are in $[a, b]$ and $|x - y| < \delta_1$, then $|f(x) - f(y)| < \varepsilon$,
(ii) if x and y are in $[b, c]$ and $|x - y| < \delta_2$, then $|f(x) - f(y)| < \varepsilon$.

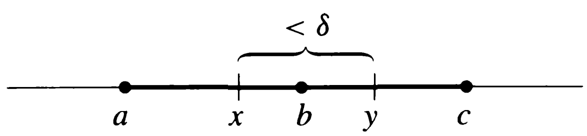


FIGURE 2

We'd like to know if there is some $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever x and y are points in $[a, c]$ with $|x - y| < \delta$. Our first inclination might be to choose δ as the minimum of δ_1 and δ_2. But it is easy to see what goes wrong (Figure 2): we might have x in $[a, b]$ and y in $[b, c]$, and then neither (i) nor (ii) tells us anything about $|f(x) - f(y)|$. So we have to be a little more cagey, and also use continuity of f at b.

LEMMA Let $a < b < c$ and let f be continuous on the interval $[a, c]$. Let $\varepsilon > 0$, and suppose that statements (i) and (ii) hold. Then there is a $\delta > 0$ such that,

$$\text{if } x \text{ and } y \text{ are in } [a, c] \text{ and } |x - y| < \delta, \text{ then } |f(x) - f(y)| < \varepsilon.$$

PROOF Since f is continuous at b, there is a $\delta_3 > 0$ such that,

$$\text{if } |x - b| < \delta_3, \text{ then } |f(x) - f(b)| < \frac{\varepsilon}{2}.$$

It follows that

(iii) if $|x - b| < \delta_3$ and $|y - b| < \delta_3$, then $|f(x) - f(y)| < \varepsilon$.

Choose δ to be the minimum of δ_1, δ_2, and δ_3. We claim that this δ works. In fact, suppose that x and y are any two points in $[a, c]$ with $|x - y| < \delta$. If x and y are both in $[a, b]$, then $|f(x) - f(y)| < \varepsilon$ by (i); and if x and y are both in $[b, c]$, then $|f(x) - f(y)| < \varepsilon$ by (ii). The only other possibility is that

$$x < b < y \qquad \text{or} \qquad y < b < x.$$

In either case, since $|x - y| < \delta$, we also have $|x - b| < \delta$ and $|y - b| < \delta$. So $|f(x) - f(y)| < \varepsilon$ by (iii). ▮

THEOREM 1 If f is continuous on $[a, b]$, then f is uniformly continuous on $[a, b]$.

PROOF It's the usual trick, but we've got to be a little bit careful about the mechanism of the proof. For $\varepsilon > 0$ let's say that f is *ε-good* on $[a, b]$ if there is some $\delta > 0$ such that, for all y and z in $[a, b]$,

$$\text{if } |y - z| < \delta, \text{ then } |f(y) - f(z)| < \varepsilon.$$

Then we're trying to prove that f is ε-good on $[a, b]$ for all $\varepsilon > 0$.

Consider any particular $\varepsilon > 0$. Let

$$A = \{x : a \leq x \leq b \text{ and } f \text{ is } \varepsilon\text{-good on } [a, x]\}.$$

Then $A \neq \emptyset$ (since a is in A), and A is bounded above (by b), so A has a least upper bound α. We really should write α_ε, since A and α might depend on ε. But we won't since we intend to prove that $\alpha = b$, no matter what ε is.

Suppose that we had $\alpha < b$. Since f is continuous at α, there is some $\delta_0 > 0$ such that, if $|y-\alpha| < \delta_0$, then $|f(y) - f(\alpha)| < \varepsilon/2$. Consequently, if $|y-\alpha| < \delta_0$ and $|z-\alpha| < \delta_0$, then $|f(y) - f(z)| < \varepsilon$. So f is surely ε-good on the interval $[\alpha - \delta_0, \alpha + \delta_0]$. On the other hand, since α is the least upper bound of A, it is also clear that f is ε-good on $[a, \alpha - \delta_0]$. Then the Lemma implies that f is ε-good on $[a, \alpha + \delta_0]$, so $\alpha + \delta_0$ is in A, contradicting the fact that α is an upper bound.

To complete the proof we just have to show that $\alpha = b$ is actually in A. The argument for this is practically the same: Since f is continuous at b, there is some $\delta_0 > 0$ such that, if $b - \delta_0 < y < b$, then $|f(y) - f(b)| < \varepsilon/2$. So f is ε-good on $[b - \delta_0, b]$. But f is also ε-good on $[a, b - \delta_0]$, so the Lemma implies that f is ε-good on $[a, b]$. ▮

PROBLEMS

1. (a) For which of the following values of α is the function $f(x) = x^\alpha$ uniformly continuous on $[0, \infty)$: $\alpha = 1/3, 1/2, 2, 3$?
(b) Find a function f that is continuous and bounded on $(0, 1]$, but not uniformly continuous on $(0, 1]$.
(c) Find a function f that is continuous and bounded on $[0, \infty)$ but which is not uniformly continuous on $[0, \infty)$.

2. (a) Prove that if f and g are uniformly continuous on A, then so is $f + g$.
(b) Prove that if f and g are uniformly continuous and bounded on A, then fg is uniformly continuous on A.
(c) Show that this conclusion does not hold if one of them isn't bounded.
(d) Suppose that f is uniformly continuous on A, that g is uniformly continuous on B, and that $f(x)$ is in B for all x in A. Prove that $g \circ f$ is uniformly continuous on A.

3. Use a "bisection argument" (page 142) to give another proof of Theorem 1.

4. Derive Theorem 7-2 as a consequence of Theorem 1.

PART 3
DERIVATIVES AND INTEGRALS

In 1604, at the height of
his scientific career, Galileo argued
that for a rectilinear motion
in which speed increases proportionally
to distance covered,
the law of motion should be
just that ($x = ct^2$)
which he had discovered
in the investigation of falling bodies.
Between 1695 and 1700
not a single one of the monthly issues
of Leipzig's Acta Eruditorum was published
without articles of Leibniz,
the Bernoulli brothers
or the Marquis de l'Hôpital treating,
with notation only slightly different from
that which we use today,
the most varied problems of
differential calculus, integral calculus
and the calculus of variations.
Thus in the space of almost precisely
one century
infinitesimal calculus or,
as we now call it in English,
The Calculus,
the calculating tool par excellence,
had been forged;
and nearly three centuries of
constant use have not completely dulled
this incomparable instrument.

NICHOLAS BOURBAKI