CHAPTER 25 COMPLEX NUMBERS

With the exception of the last few paragraphs of the previous chapter, this book has presented unremitting propaganda for the real numbers. Nevertheless, the real numbers do have a great deficiency—not every polynomial function has a root. The simplest and most notable example is the fact that no number x can satisfy $x^2 + 1 = 0$. This deficiency is so severe that long ago mathematicians felt the need to "invent" a number i with the property that $i^2 + 1 = 0$. For a long time the status of the "number" i was quite mysterious: since there is no number x satisfying $x^2 + 1 = 0$, it is nonsensical to say "let i be the number satisfying $i^2 + 1 = 0$." Nevertheless, admission of the "imaginary" number i to the family of numbers seemed to simplify greatly many algebraic computations, especially when "complex numbers" $a + bi$ (for a and b in $\mathbf{R}$) were allowed, and all the laws of arithmetical computation enumerated in Chapter 1 were assumed to be valid. For example, every quadratic equation

$$ax^2 + bx + c = 0 \qquad (a \neq 0)$$

can be solved formally to give

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

If $b^2 - 4ac \geq 0$, these formulas give correct solutions; when complex numbers are allowed the formulas seem to make sense in all cases. For example, the equation

$$x^2 + x + 1 = 0$$

has no real root, since

$$x^2 + x + 1 = (x + \tfrac{1}{2})^2 + \tfrac{3}{4} > 0, \quad \text{for all } x.$$

But the formula for the roots of a quadratic equation suggest the "solutions"

$$x = \frac{-1 + \sqrt{-3}}{2} \quad \text{and} \quad x = \frac{-1 - \sqrt{-3}}{2};$$

if we understand $\sqrt{-3}$ to mean $\sqrt{3 \cdot (-1)} = \sqrt{3} \cdot \sqrt{-1} = \sqrt{3}\, i$, then these numbers would be

$$-\frac{1}{2} + \frac{\sqrt{3}}{2} i \quad \text{and} \quad -\frac{1}{2} - \frac{\sqrt{3}}{2} i.$$

It is not hard to check that these, as yet purely formal, numbers do indeed satisfy the equation

$$x^2 + x + 1 = 0.$$

It is even possible to "solve" quadratic equations whose coefficients are themselves complex numbers. For example, the equation

$$x^2 + x + 1 + i = 0$$

ought to have the solutions

$$x = \frac{-1 \pm \sqrt{1 - 4(1 + i)}}{2} = \frac{-1 \pm \sqrt{-3 - 4i}}{2},$$

where the symbol $\sqrt{-3 - 4i}$ means a complex number $\alpha + \beta i$ whose square is $-3 - 4i$. In order to have

$$(\alpha + \beta i)^2 = \alpha^2 - \beta^2 + 2\alpha\beta i = -3 - 4i$$

we need

$$\begin{aligned} \alpha^2 - \beta^2 &= -3, \\ 2\alpha\beta &= -4. \end{aligned}$$

These two equations can easily be solved for real α and β; in fact, there are two possible solutions:

$$\begin{aligned} \alpha &= 1 \\ \beta &= -2 \end{aligned} \qquad \text{and} \qquad \begin{aligned} \alpha &= -1 \\ \beta &= 2. \end{aligned}$$

Thus the two "square roots" of $-3 - 4i$ are $1 - 2i$ and $-1 + 2i$. There is no reasonable way to decide which one of these should be called $\sqrt{-3 - 4i}$, and which $-\sqrt{-3 - 4i}$; the conventional usage of $\sqrt{x}$ makes sense only for real $x \geq 0$, in which case $\sqrt{x}$ denotes the (real) nonnegative root. For this reason, the solution

$$x = \frac{-1 \pm \sqrt{-3 - 4i}}{2}$$

must be understood as an abbreviation for:

$$x = \frac{-1 + r}{2}, \qquad \text{where } r \text{ is one of the square roots of } -3 - 4i.$$

With this understanding we arrive at the solutions

$$\begin{aligned} x &= \frac{-1 + 1 - 2i}{2} = -i, \\ x &= \frac{-1 - 1 + 2i}{2} = -1 + i; \end{aligned}$$

as you can easily check, these numbers do provide formal solutions for the equation

$$x^2 + x + 1 + i = 0.$$

For cubic equations complex numbers are equally useful. Every cubic equation

$$ax^3 + bx^2 + cx + d = 0 \qquad (a \neq 0)$$

with real coefficients a, b, c, and d, has, as we know, a real root α, and if we divide $ax^3 + bx^2 + cx + d$ by $x - \alpha$ we obtain a second-degree polynomial whose roots are the other roots of $ax^3 + bx^2 + cx + d = 0$; the roots of this second-degree polynomial may be complex numbers. Thus a cubic equation will have either three real roots or one real root and 2 complex roots. The existence of the real root is guaranteed by our theorem that every odd degree equation has a real root, but it is not really necessary to appeal to this theorem (which is of no use at all if the coefficients are complex); in the case of a cubic equation we can, with sufficient cleverness, actually find a formula for all the roots. The following derivation is presented not only as an interesting illustration of the ingenuity of early mathematicians, but as further evidence for the importance of complex numbers (whatever they may be).

To solve the most general cubic equation, it obviously suffices to consider only equations of the form

$$x^3 + bx^2 + cx + d = 0.$$

It is even possible to eliminate the term involving x^2, by a fairly straight-forward manipulation. If we let

$$x = y - \frac{b}{3},$$

then

$$\begin{aligned} x^3 &= y^3 - by^2 + \frac{b^2y}{3} - \frac{b^3}{27}, \\ x^2 &= y^2 - \frac{2by}{3} + \frac{b^2}{9}, \end{aligned}$$

so

$$\begin{aligned} 0 &= x^3 + bx^2 + cx + d \\ &= \left(y^3 - by^2 + \frac{b^2y}{3} - \frac{b^3}{27}\right) + \left(by^2 - \frac{2b^2y}{3} + \frac{b^3}{9}\right) + \left(cy - \frac{bc}{3}\right) + d \\ &= y^3 + \left(\frac{b^2}{3} - \frac{2b^2}{3} + c\right)y + \left(\frac{b^3}{9} - \frac{b^3}{27} - \frac{bc}{3} + d\right). \end{aligned}$$

The right-hand side now contains no term with y^2. If we can solve the equation for y we can find x; this shows that it suffices to consider in the first place only equations of the form

$$x^3 + px + q = 0.$$

In the special case $p = 0$ we obtain the equation $x^3 = -q$. We shall see later on that every complex number does have a cube root, in fact it has three, so that this equation has three solutions. The case $p \neq 0$, on the other hand, requires quite an ingenious step. Let

$$(*) \qquad x = w - \frac{p}{3w}$$

Then

$$\begin{aligned}
0 = x^3 + px + q &= \left(w - \frac{p}{3w}\right)^3 + p\left(w - \frac{p}{3w}\right) + q \\
&= w^3 - \frac{3w^2 p}{3w} + \frac{3wp^2}{9w^2} - \frac{p^3}{27w^3} + pw - \frac{p^2}{3w} + q \\
&= w^3 - \frac{p^3}{27w^3} + q.
\end{aligned}$$

This equation can be written

$$27(w^3)^2 + 27q(w^3) - p^3 = 0,$$

which is a quadratic equation in w^3 (!!).

Thus

$$\begin{aligned}
w^3 &= \frac{-27q \pm \sqrt{(27)^2 q^2 + 4 \cdot 27 p^3}}{2 \cdot 27} \\
&= -\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}.
\end{aligned}$$

Remember that this really means:

$$w^3 = -\frac{q}{2} + r, \quad \text{where } r \text{ is a square root of } \frac{q^2}{4} + \frac{p^3}{27}.$$

We can therefore write

$$w = \sqrt[3]{\frac{-q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}};$$

this equation means that w is some cube root of $-q/2 + r$, where r is some square root of $q^2/4 + p^3/27$. This allows six possibilities for w, but when these are substituted into $(*)$, yielding

$$x = \sqrt[3]{\frac{-q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} - \frac{p}{3 \cdot \sqrt[3]{\frac{-q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}},$$

it turns out that only 3 different values for x will be obtained! An even more surprising feature of this solution arises when we consider a cubic equation all of whose roots are real; the formula derived above may still involve complex numbers in an essential way. For example, the roots of

$$x^3 - 15x - 4 = 0$$

are 4, $-2+\sqrt{3}$, and $-2-\sqrt{3}$. On the other hand, the formula derived above (with $p=-15$, $q=-4$) gives as one solution

$$\begin{aligned} x &= \sqrt[3]{2+\sqrt{4-125}} - \frac{-15}{3\cdot\sqrt[3]{2+\sqrt{4-125}}} \\ &= \sqrt[3]{2+11i} + \frac{15}{3\cdot\sqrt[3]{2+11i}}. \end{aligned}$$

Now,

$$\begin{aligned} (2+i)^3 &= 2^3+3\cdot 2^2 i+3\cdot 2\cdot i^2+i^3 \\ &= 8+12i-6-i \\ &= 2+11i, \end{aligned}$$

so one of the cube roots of $2+11i$ is $2+i$. Thus, for one solution of the equation we obtain

$$\begin{aligned} x &= 2+i+\frac{15}{6+3i} \\ &= 2+i+\frac{15}{6+3i}\cdot\frac{6-3i}{6-3i} \\ &= 2+i+\frac{90-45i}{36+9} \\ &= 4\ (!). \end{aligned}$$

The other roots can also be found if the other cube roots of $2+11i$ are known. The fact that even one of these real roots is obtained from an expression which depends on complex numbers is impressive enough to suggest that the use of complex numbers cannot be entirely nonsense. As a matter of fact, the formulas for the solutions of the quadratic and cubic equations can be interpreted entirely in terms of real numbers.

Suppose we agree, for the moment, to write all complex numbers as $a+bi$, writing the real number a as $a+0i$ and the number i as $0+1i$. The laws of ordinary arithmetic and the relation $i^2=-1$ show that

$$\begin{aligned} (a+bi)+(c+di) &= (a+c)+(b+d)i \\ (a+bi)\cdot(c+di) &= (ac-bd)+(ad+bc)i. \end{aligned}$$

Thus, an equation like

$$(1+2i)\cdot(3+1i) = 1+7i$$

may be regarded simply as an abbreviation for the *two* equations

$$\begin{aligned} 1\cdot 3-2\cdot 1 &= 1, \\ 1\cdot 1+2\cdot 3 &= 7. \end{aligned}$$

The solution of the quadratic equation $ax^2 + bx + c = 0$ with real coefficients could be paraphrased as follows:

$$\text{If} \begin{cases} u^2 - v^2 = b^2 - 4ac, \\ uv = 0, \end{cases}$$
$$\text{(i.e., if } (u + vi)^2 = b^2 - 4ac\text{)},$$

$$\text{then} \begin{cases} a\left[\left(\dfrac{-b+u}{2a}\right)^2 - \left(\dfrac{v}{2a}\right)^2\right] + b\left[\dfrac{-b+u}{2a}\right] + c = 0, \\ a\left[2\left(\dfrac{-b+u}{2a}\right)\left(\dfrac{v}{2a}\right)\right] + b\left[\dfrac{v}{2a}\right] = 0, \end{cases}$$

$$\left(\text{i.e., then } a\left(\frac{-b+u+vi}{2a}\right)^2 + b\left(\frac{-b+u+vi}{2a}\right) + c = 0\right).$$

It is not very hard to check this assertion about real numbers without writing down a single "i," but the complications of the statement itself should convince you that equations about complex numbers are worthwhile as abbreviations for pairs of equations about real numbers. (If you are still not convinced, try paraphrasing the solution of the cubic equation.) If we really intend to use complex numbers consistently, however, it is going to be necessary to present some reasonable definition.

One possibility has been implicit in this whole discussion. All mathematical properties of a complex number $a + bi$ are determined completely by the real numbers a and b; any mathematical object with this same property may reasonably be used to define a complex number. The obvious candidate is the ordered pair (a, b) of real numbers; we shall accordingly *define* a complex number to be a pair of real numbers, and likewise *define* what addition and multiplication of complex numbers is to mean.

DEFINITION

A **complex number** is an ordered pair of real numbers; if $z = (a, b)$ is a complex number, then a is called the **real part** of z, and b is called the **imaginary part** of z. The set of all complex numbers is denoted by $\mathbf{C}$. If (a, b) and (c, d) are two complex numbers we define

$$\begin{aligned} (a, b) + (c, d) &= (a + c, b + d) \\ (a, b) \cdot (c, d) &= (a \cdot c - b \cdot d, a \cdot d + b \cdot c). \end{aligned}$$

(The $+$ and $\cdot$ appearing on the left side are new symbols being defined, while the $+$ and $\cdot$ appearing on the right side are the familiar addition and multiplication for real numbers.)

When complex numbers were first introduced, it was understood that real numbers were, in particular, complex numbers; if our definition is taken seriously this is not true—a real number is not a pair of real numbers, after all. This difficulty and the right side becomes

$$(a + [c + e], b + [d + f]);$$

these two are equal because P1 is true for real numbers. It is a good idea to check P2–P6 and P8 and P9. Notice that the complex numbers playing the role of 0 and 1 in P2 and P6 are $(0, 0)$ and $(1, 0)$, respectively. It is not hard to figure out what $-(a, b)$ is, but the multiplicative inverse for (a, b) required in P7 is a little trickier: if $(a, b) \neq (0, 0)$, then $a^2 + b^2 \neq 0$ and

$$(a, b) \cdot \left(\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2}\right) = (1, 0).$$

This fact could have been guessed in two ways. To find (x, y) with

$$(a, b) \cdot (x, y) = (1, 0)$$

it is only necessary to solve the equations

$$\begin{aligned} ax - by &= 1, \\ bx + ay &= 0. \end{aligned}$$

The solutions are $x = a/(a^2 + b^2)$, $y = -b/(a^2 + b^2)$. It is also possible to reason that if $1/(a + bi)$ means anything, then it should be true that

$$\frac{1}{a + bi} = \frac{1}{a + bi} \cdot \frac{a - bi}{a - bi} = \frac{a - bi}{a^2 + b^2}.$$

Once the existence of inverses has actually been proved (after guessing the inverse by some method), it follows that this manipulation is really valid; it is the easiest one to remember when the inverse of a complex number is actually being sought—it was precisely this trick which we used to evaluate

$$\begin{aligned} \frac{15}{6 + 3i} &= \frac{15}{6 + 3i} \cdot \frac{6 - 3i}{6 - 3i} \\ &= \frac{90 - 45i}{36 + 9}. \end{aligned}$$

Unlike P1–P9, the rules P10–P12 do not have analogues: it is easy to prove that there is *no* set P of *complex* numbers such that P10–P12 are satisfied for all *complex* numbers. In fact, if there were, then P would have to contain 1 (since $1 = 1^2$) and also -1 (since $-1 = i^2$), and this would contradict P10. The absence of P10–P12 will not have disastrous consequences, but it does mean that we cannot define $z < w$ for complex z and w. Also, you may remember that for the real numbers, P10–P12 were used to prove that $1 + 1 \neq 0$. Fortunately, the corresponding fact for complex numbers can be reduced to this one: clearly $(1, 0) + (1, 0) \neq (0, 0)$.

Although we will usually write complex numbers in the form $a + bi$, it is worth remembering that the set of all complex numbers $\mathbf{C}$ is just the collection of all pairs of real numbers. Long ago this collection was identified with the plane, and for this reason the plane is often called the "complex plane." The horizontal axis, which consists of all points $(a, 0)$ for a in $\mathbf{R}$, is often called the *real axis*, and the is only a minor annoyance, however. Notice that

$$(a,0)+(b,0)=(a+b,0+0)=(a+b,0),$$
$$(a,0)\cdot(b,0)=(a\cdot b-0\cdot 0,a\cdot 0+0\cdot b)=(a\cdot b,0);$$

this shows that the complex numbers of the form $(a,0)$ behave precisely the same with respect to addition and multiplication of complex numbers as real numbers do with their own addition and multiplication. For this reason we will adopt the convention that $(a,0)$ will be denoted simply by a. The familiar $a+bi$ notation for complex numbers can now be recovered if one more definition is made.

DEFINITION

$$\boxed{i=(0,1).}$$

Notice that $i^2=(0,1)\cdot(0,1)=(-1,0)=-1$ (the last equality sign depends on our convention). Moreover

$$\begin{aligned}(a,b)&=(a,0)+(0,b)\\&=(a,0)+(b,0)\cdot(0,1)\\&=a+bi.\end{aligned}$$

You may feel that our definition was merely an elaborate device for defining complex numbers as "expressions of the form $a+bi$." That is essentially correct; it is a firmly established prejudice of modern mathematics that new objects must be defined as something specific, not as "expressions." Nevertheless, it is interesting to note that mathematicians were sincerely worried about using complex numbers until the modern definition was proposed. Moreover, the precise definition emphasizes one important point. Our aim in introducing complex numbers was to avoid the necessity of paraphrasing statements about complex numbers in terms of their real and imaginary parts. This means that we wish to work with complex numbers in the same way that we worked with rational or real numbers. For example, the solution of the cubic equation required writing $x=w-p/3w$, so we want to know that $1/w$ makes sense. Moreover, w^3 was found by solving a quadratic equation, which requires numerous other algebraic manipulations. In short, we are likely to use, at some time or other, any manipulations performed on real numbers. We certainly do not want to stop each time and justify every step. Fortunately this is not necessary. Since all algebraic manipulations performed on real numbers can be justified by the properties listed in Chapter 1, it is only necessary to check that these properties are also true for complex numbers. In most cases this is quite easy, and these facts will not be listed as formal theorems. For example, the proof of P1,

$$[(a,b)+(c,d)]+(e,f)=(a,b)+[(c,d)+(e,f)]$$

requires only the application of the definition of addition for complex numbers. The left side becomes

$$([a+c]+e,[b+d]+f),$$

vertical axis is called the *imaginary axis.* Two important definitions are also related to this geometric picture.

DEFINITION

If $z = x + iy$ is a complex number (with x and y real), then the **conjugate** $\bar{z}$ of z is defined as

$$\bar{z} = x - iy,$$

and the **absolute value** or **modulus** $|z|$ of z is defined as

$$|z| = \sqrt{x^2 + y^2}.$$

(Notice that $x^2 + y^2 \geq 0$, so that $\sqrt{x^2 + y^2}$ is defined unambiguously; it denotes the nonnegative real square root of $x^2 + y^2$.)

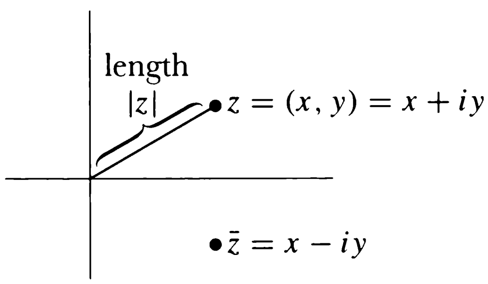

FIGURE 1

Geometrically, $\bar{z}$ is simply the reflection of z in the real axis, while $|z|$ is the distance from z to $(0, 0)$ (Figure 1). Notice that the absolute value notation for complex numbers is consistent with that for real numbers. The **distance** between two complex numbers z and w can be defined quite easily as $|z - w|$. The following theorem lists all the important properties of conjugates and absolute values.

THEOREM 1 Let z and w be complex numbers. Then

(1) $\bar{\bar{z}} = z$.
(2) $\bar{z} = z$ if and only if z is real (i.e., is of the form $a + 0i$, for some real number a).
(3) $\overline{z + w} = \bar{z} + \bar{w}$.
(4) $\overline{-z} = -\bar{z}$.
(5) $\overline{z \cdot w} = \bar{z} \cdot \bar{w}$.
(6) $\overline{z^{-1}} = (\bar{z})^{-1}$, if $z \neq 0$.
(7) $|z|^2 = z \cdot \bar{z}$.
(8) $|z \cdot w| = |z| \cdot |w|$.
(9) $|z + w| \leq |z| + |w|$.

PROOF Assertions (1) and (2) are obvious. Equations (3) and (5) may be checked by straightforward calculations and (4) and (6) may then be proved by a trick:

$$0 = \bar{0} = \overline{z + (-z)} = \bar{z} + \overline{-z}, \quad \text{so } \overline{-z} = -\bar{z},$$
$$1 = \bar{1} = \overline{z \cdot (z^{-1})} = \bar{z} \cdot \overline{z^{-1}}, \quad \text{so } \overline{z^{-1}} = (\bar{z})^{-1}.$$

Equations (7) and (8) may also be proved by a straightforward calculation. The only difficult part of the theorem is (9). This inequality has, in fact, already occurred (Problem 4-9), but the proof will be repeated here, using slightly different terminology.

It is clear that equality holds in (9) if $z = 0$ or $w = 0$. It is also easy to see that (9) is true if $z = \lambda w$ for any real number λ (consider separately the cases $\lambda > 0$ and $\lambda < 0$). Suppose, on the other hand, that $z \neq \lambda w$ for any real number λ, and that

$w \neq 0$. Then, for all real numbers λ,

$$(*) \qquad \begin{aligned} 0 < |z - \lambda w|^2 &= (z - \lambda w) \cdot \overline{(z - \lambda w)} \\ &= (z - \lambda w) \cdot (\bar{z} - \lambda \bar{w}) \\ &= z\bar{z} + \lambda^2 w\bar{w} - \lambda(w\bar{z} + z\bar{w}) \\ &= \lambda^2 |w|^2 + |z|^2 - \lambda(w\bar{z} + z\bar{w}). \end{aligned}$$

Notice that $w\bar{z} + z\bar{w}$ is real, since

$$\overline{w\bar{z} + z\bar{w}} = \bar{w}\bar{\bar{z}} + \bar{z}\bar{\bar{w}} = \bar{w}z + \bar{z}w = w\bar{z} + z\bar{w}.$$

Thus the right side of $(*)$ is a quadratic equation in λ with real coefficients and no real solutions; its discriminant must therefore be negative. Thus

$$(w\bar{z} + z\bar{w})^2 - 4|w|^2 \cdot |z|^2 < 0;$$

it follows, since $w\bar{z} + \bar{z}w$ and $|w| \cdot |z|$ are real numbers, and $|w| \cdot |z| \geq 0$, that

$$(w\bar{z} + z\bar{w}) < 2|w| \cdot |z|.$$

From this inequality it follows that

$$\begin{aligned} |z + w|^2 &= (z + w) \cdot (\bar{z} + \bar{w}) \\ &= |z|^2 + |w|^2 + (w\bar{z} + z\bar{w}) \\ &< |z|^2 + |w|^2 + 2|w| \cdot |z| \\ &= (|z| + |w|)^2, \end{aligned}$$

which implies that

$$|z + w| < |z| + |w|. \blacksquare$$

The operations of addition and multiplication of complex numbers both have important geometric interpretations. The picture for addition is very simple (Figure 2). Two complex numbers $z = (a, b)$ and $w = (c, d)$ determine a parallelogram having for two of its sides the line segment from $(0, 0)$ to z, and the line segment from $(0, 0)$ to w; the vertex opposite $(0, 0)$ is $z + w$ (a proof of this geometric fact is left to you [compare Appendix 1 to Chapter 4]).

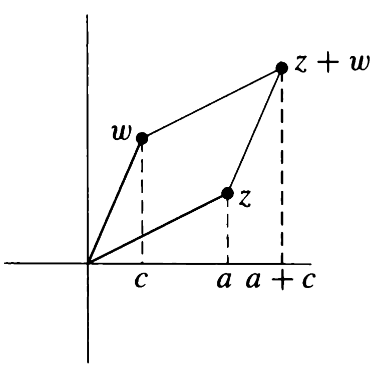

FIGURE 2

The interpretation of multiplication is more involved. If $z = 0$ or $w = 0$, then $z \cdot w = 0$ (a one-line computational proof can be given, but even this is unnecessary—the assertion has already been shown to follow from P1–P9), so we may restrict our attention to nonzero complex numbers. We begin by putting every nonzero complex number into a special form (compare Appendix 3 to Chapter 4).

For any complex number $z \neq 0$ we can write

$$z = |z| \frac{z}{|z|};$$

in this expression, $|z|$ is a positive real number, while

$$\left| \frac{z}{|z|} \right| = \frac{|z|}{|z|} = 1,$$

so that $z/|z|$ is a complex number of absolute value 1. Now any complex number $a = x + iy$ with $1 = |a| = x^2 + y^2$ can be written in the form

$$a = (\cos\theta, \sin\theta) = \cos\theta + i\sin\theta$$

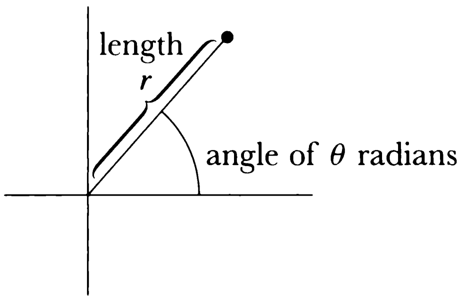

FIGURE 3

for some number θ. Thus every nonzero complex number z can be written

$$z = r(\cos\theta + i\sin\theta)$$

for some $r > 0$ and some number θ. The number r is unique (it equals $|z|$), but θ is not unique; if θ_0 is one possibility, then the others are $\theta_0 + 2k\pi$ for k in $\mathbf{Z}$—any one of these numbers is called an **argument** of z. Figure 3 shows z in terms of r and θ. (To find an argument θ for $z = x + iy$ we may note that the equation

$$x + iy = z = |z|(\cos\theta + i\sin\theta)$$

means that

$$\begin{aligned} x &= |z|\cos\theta, \\ y &= |z|\sin\theta. \end{aligned}$$

So, for example, if $x > 0$ we can take $\theta = \arctan y/x$; if $x = 0$, we can take $\theta = \pi/2$ when $y > 0$ and $\theta = 3\pi/2$ when $y < 0$.)

Now the product of two nonzero complex numbers

$$\begin{aligned} z &= r(\cos\theta + i\sin\theta), \\ w &= s(\cos\phi + i\sin\phi), \end{aligned}$$

is

$$\begin{aligned} z \cdot w &= rs(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) \\ &= rs[(\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\sin\theta\cos\phi + \cos\theta\sin\phi)] \\ &= rs[\cos(\theta+\phi) + i\sin(\theta+\phi)]. \end{aligned}$$

Thus, the absolute value of a product is the product of the absolute values of the factors, while the sum of any argument for each of the factors will be an argument for the product. For a nonzero complex number

$$z = r(\cos\theta + i\sin\theta)$$

it is now an easy matter to prove by induction the following very important formula (sometimes known as De Moivre's Theorem):

$$z^n = |z|^n(\cos n\theta + i\sin n\theta), \text{ for any argument } \theta \text{ of } z.$$

This formula describes z^n so explicitly that it is easy to decide just when $z^n = w$:

THEOREM 2 Every nonzero complex number has exactly n complex nth roots.

More precisely, for any complex number $w \neq 0$, and any natural number n, there are precisely n different complex numbers z satisfying $z^n = w$.

PROOF Let

$$w = s(\cos\phi + i\sin\phi)$$

for $s = |w|$ and some number ϕ. Then a complex number

$$z = r(\cos\theta + i\sin\theta)$$

satisfies $z^n = w$ if and only if

$$r^n(\cos n\theta + i\sin n\theta) = s(\cos\phi + i\sin\phi),$$

which happens if and only if

$$\begin{aligned} r^n &= s, \\ \cos n\theta + i\sin n\theta &= \cos\phi + i\sin\phi. \end{aligned}$$

From the first equation it follows that

$$r = \sqrt[n]{s},$$

where $\sqrt[n]{s}$ denotes the positive real nth root of s. From the second equation it follows that for some integer k we have

$$\theta = \theta_k = \frac{\phi}{n} + \frac{2k\pi}{n}.$$

Conversely, if we choose $r = \sqrt[n]{s}$ and $\theta = \theta_k$ for some k, then the number $z = r(\cos\theta + i\sin\theta)$ will satisfy $z^n = w$. To determine the number of nth roots of w, it is therefore only necessary to determine which such z are distinct. Now any integer k can be written

$$k = nq + k'$$

for some integer q, and some integer k' between 0 and $n-1$. Then

$$\cos\theta_k + i\sin\theta_k = \cos\theta_{k'} + i\sin\theta_{k'}.$$

This shows that every z satisfying $z^n = w$ can be written

$$z = \sqrt[n]{s}\,(\cos\theta_k + i\sin\theta_k) \quad k = 0, \dots, n-1.$$

Moreover, it is easy to see that these numbers are all different, since any two θ_k for $k = 0, \dots, n-1$ differ by less than 2π. ∎

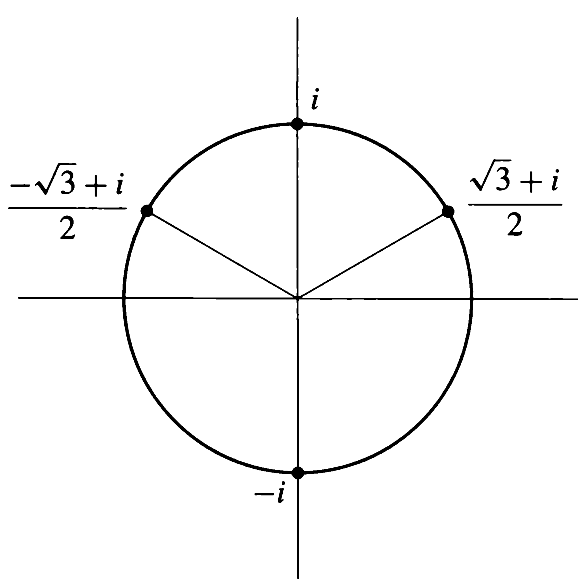

FIGURE 4

In the course of proving Theorem 2, we have actually developed a method for finding the nth roots of a complex number. For example, to find the cube roots of i (Figure 4) note that $|i| = 1$ and that $\pi/2$ is an argument for i. The cube roots of i are therefore

$$1\cdot\left[\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right],$$

$$1\cdot\left[\cos\left(\frac{\pi}{6}+\frac{2\pi}{3}\right) + i\sin\left(\frac{\pi}{6}+\frac{2\pi}{3}\right)\right] = \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6},$$

$$1\cdot\left[\cos\left(\frac{\pi}{6}+\frac{4\pi}{3}\right) + i\sin\left(\frac{\pi}{6}+\frac{4\pi}{3}\right)\right] = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}.$$

Since

$$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}, \qquad \sin\frac{\pi}{6} = \frac{1}{2},$$
$$\cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}, \qquad \sin\frac{5\pi}{6} = \frac{1}{2},$$
$$\cos\frac{3\pi}{2} = 0, \qquad \sin\frac{3\pi}{2} = -1,$$

the cube roots of i are

$$\frac{\sqrt{3}+i}{2}, \quad \frac{-\sqrt{3}+i}{2}, \quad -i.$$

In general, we cannot expect to obtain such simple results. For example, to find the cube roots of $2 + 11i$, note that $|2 + 11i| = \sqrt{2^2 + 11^2} = \sqrt{125}$ and that $\arctan\frac{11}{2}$ is an argument for $2+11i$. One of the cube roots of $2+11i$ is therefore

$$\sqrt[6]{125}\left[\cos\left(\frac{\arctan\frac{11}{2}}{3}\right) + i\sin\left(\frac{\arctan\frac{11}{2}}{3}\right)\right]$$
$$= \sqrt{5}\left[\cos\left(\frac{\arctan\frac{11}{2}}{3}\right) + i\sin\left(\frac{\arctan\frac{11}{2}}{3}\right)\right].$$

Previously we noted that $2 + i$ is also a cube root of $2 + 11i$. Since $|2 + i| = \sqrt{2^2 + 1^2} = \sqrt{5}$, and since $\arctan\frac{1}{2}$ is an argument of $2 + i$, we can write this cube root as

$$2 + i = \sqrt{5}(\cos\arctan\tfrac{1}{2} + i\sin\arctan\tfrac{1}{2}).$$

These two cube roots are actually the same number, because

$$\frac{\arctan\frac{11}{2}}{3} = \arctan\frac{1}{2}$$

(you can check this by using the formula in Problem 15-9), but this is hardly the sort of thing one might notice!

The fact that every complex number has an nth root for all n is just a special case of a very important theorem. The number i was originally introduced in order to provide a solution for the equation $x^2 + 1 = 0$. The *Fundamental Theorem of Algebra* states the remarkable fact that this one addition automatically provides solutions for all other polynomial equations: every equation

$$z^n + a_{n-1}z^{n-1} + \cdots + a_0 = 0 \qquad a_0, \ldots, a_{n-1} \text{ in } \mathbf{C}$$

has a complex root!

In the next chapter we shall give an almost complete proof of the Fundamental Theorem of Algebra; the slight gap left in the text can be filled in as an exercise (Problem 26-5). The proof of the theorem will rely on several new concepts which come up quite naturally in a more thorough investigation of complex numbers.

PROBLEMS

1. Find the absolute value and argument(s) of each of the following.

(i) $3 + 4i$.
(ii) $(3 + 4i)^{-1}$.
(iii) $(1 + i)^5$.
(iv) $\sqrt[7]{3 + 4i}$.
(v) $|3 + 4i|$.

2. Solve the following equations.

(i) $x^2 + ix + 1 = 0$.
(ii) $x^4 + x^2 + 1 = 0$.
(iii) $x^2 + 2ix - 1 = 0$.
(iv) $\begin{cases} ix - (1+i)y = 3, \\ (2+i)x + iy = 4 \end{cases}$.
(v) $x^3 - x^2 - x - 2 = 0$.

3. Describe the set of all complex numbers z such that

(i) $\bar{z} = -z$.
(ii) $\bar{z} = z^{-1}$.
(iii) $|z - a| = |z - b|$.
(iv) $|z - a| + |z - b| = c$.
(v) $|z| < 1 -$ real part of z.

4. Prove that $|z| = |\bar{z}|$, and that the real part of z is $(z+\bar{z})/2$, while the imaginary part is $(z - \bar{z})/2i$.

5. Prove that $|z + w|^2 + |z - w|^2 = 2(|z|^2 + |w|^2)$, and interpret this statement geometrically.

6. What is the pictorial relation between z and $\sqrt{i} \cdot \overline{z\sqrt{-i}}$? (Note that there may be more than one answer, because $\sqrt{i}$ and $\sqrt{-i}$ both have two different possible values.) Hint: Which line goes into the real axis under multiplication by $\sqrt{-i}$?

7. (a) Prove that if $a_0, \ldots, a_{n-1}$ are *real* and $a + bi$ (for a and b real) satisfies the equation $z^n + a_{n-1}z^{n-1} + \cdots + a_0 = 0$, then $a - bi$ also satisfies this equation. (Thus the nonreal roots of such an equation always occur in pairs, and the number of such roots is even.)
(b) Conclude that $z^n + a_{n-1}z^{n-1} + \cdots + a_0$ is divisible by $z^2 - 2az + (a^2 + b^2)$ (whose coefficients are real).

***8.** (a) Let c be an integer which is not the square of another integer. If a and b are integers we define the **conjugate** of $a + b\sqrt{c}$, denoted by $\overline{a + b\sqrt{c}}$, as $a - b\sqrt{c}$. Show that the conjugate is well defined by showing that a number can be written $a + b\sqrt{c}$, for integers a and b, in only one way.

(b) Show that for all α and β of the form $a + b\sqrt{c}$, we have $\bar{\bar{\alpha}} = \alpha$; $\bar{\alpha} = \alpha$ if and only if α is an integer; $\overline{\alpha + \beta} = \bar{\alpha} + \bar{\beta}$; $\overline{-\alpha} = -\bar{\alpha}$; $\overline{\alpha \cdot \beta} = \bar{\alpha} \cdot \bar{\beta}$; and $\overline{\alpha^{-1}} = (\bar{\alpha})^{-1}$ if $\alpha \neq 0$.

(c) Prove that if $a_0, \ldots, a_{n-1}$ are *integers* and $z = a + b\sqrt{c}$ satisfies the equation $z^n + a_{n-1}z^{n-1} + \cdots + a_0 = 0$, then $\bar{z} = a - b\sqrt{c}$ also satisfies this equation.

9. Find all the 4th roots of i; express the one having smallest argument in a form that does not involve any trigonometric functions.

***10.** (a) Prove that if ω is an nth root of 1, then so is ω^k.

(b) A number ω is called a **primitive *n*th root** of 1 if $\{1, \omega, \omega^2, \ldots, \omega^{n-1}\}$ is the set of all nth roots of 1. How many primitive nth roots of 1 are there for $n = 3, 4, 5, 9$?

(c) Let ω be an nth root of 1, with $\omega \neq 1$. Prove that $\displaystyle\sum_{k=0}^{n-1} \omega^k = 0$.

***11.** (a) Prove that if $z_1, \ldots, z_k$ lie on one side of some straight line through 0, then $z_1 + \cdots + z_k \neq 0$. Hint: This is obvious from the geometric interpretation of addition, but an analytic proof is also easy: the assertion is clear if the line is the real axis, and a trick will reduce the general case to this one.

(b) Show further that $z_1{}^{-1}, \ldots, z_k{}^{-1}$ all lie on one side of a straight line through 0, so that $z_1{}^{-1} + \cdots + z_k{}^{-1} \neq 0$.

***12.** Prove that if $|z_1| = |z_2| = |z_3|$ and $z_1 + z_2 + z_3 = 0$, then z_1, z_2, and z_3 are the vertices of an equilateral triangle. Hint: It will help to assume that z_1 is real, and this can be done with no loss of generality. Why?