CHAPTER 23 INFINITE SERIES

Infinite sequences were introduced in the previous chapter with the specific intention of considering their "sums"

$$a_1 + a_2 + a_3 + \cdots$$

in this chapter. This is not an entirely straightforward matter, for the sum of infinitely many numbers is as yet completely undefined. What can be defined are the "partial sums"

$$s_n = a_1 + \cdots + a_n,$$

and the infinite sum must presumably be defined in terms of these partial sums. Fortunately, the mechanism for formulating this definition has already been developed in the previous chapter. If there is to be any hope of computing the infinite sum $a_1 + a_2 + a_3 + \cdots$, the partial sums s_n should represent closer and closer approximations as n is chosen larger and larger. This last assertion amounts to little more than a sloppy definition of limits: the "infinite sum" $a_1 + a_2 + a_3 + \cdots$ ought to be $\lim_{n\to\infty} s_n$. This approach will necessarily leave the "sum" of many sequences undefined, since the sequence $\{s_n\}$ may easily fail to have a limit. For example, the sequence

$$1,\ -1,\ 1,\ -1,\ \ldots$$

with $a_n = (-1)^{n+1}$ yields the new sequence

$$\begin{aligned}
s_1 &= a_1 = 1,\\
s_2 &= a_1 + a_2 = 0,\\
s_3 &= a_1 + a_2 + a_3 = 1,\\
s_4 &= a_1 + a_2 + a_3 + a_4 = 0,\\
&\ldots,
\end{aligned}$$

for which $\lim_{n\to\infty} s_n$ does not exist. Although there happen to be some clever extensions of the definition suggested here (see Problems 12 and 24-20) it seems unavoidable that some sequences will have no sum. For this reason, an acceptable definition of the sum of a sequence should contain, as an essential component, terminology which distinguishes sequences for which sums can be defined from less fortunate sequences.

DEFINITION

The sequence $\{a_n\}$ is **summable** if the sequence $\{s_n\}$ converges, where

$$s_n = a_1 + \cdots + a_n.$$

In this case, $\lim_{n\to\infty} s_n$ is denoted by

$$\sum_{n=1}^{\infty} a_n \qquad \text{(or, less formally, } a_1 + a_2 + a_3 + \cdots\text{)}$$

and is called the **sum** of the sequence $\{a_n\}$.

The terminology introduced in this definition is usually replaced by less precise expressions; indeed the title of this chapter is derived from such everyday language. An infinite sum $\sum_{n=1}^{\infty} a_n$ is usually called an *infinite series*, the word "series" emphasizing the connection with the infinite sequence $\{a_n\}$. The statement that $\{a_n\}$ is, or is not, summable is conventionally replaced by the statement that the series $\sum_{n=1}^{\infty} a_n$ does, or does not, converge. This terminology is somewhat peculiar, because at best the symbol $\sum_{n=1}^{\infty} a_n$ denotes a number (so it can't "converge"), and it doesn't denote anything at all unless $\{a_n\}$ is summable. Nevertheless, this informal language is convenient, standard, and unlikely to yield to attacks on logical grounds.

Certain elementary arithmetical operations on infinite series are direct consequences of the definition. It is a simple exercise to show that if $\{a_n\}$ and $\{b_n\}$ are summable, then

$$\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n,$$
$$\sum_{n=1}^{\infty} c \cdot a_n = c \cdot \sum_{n=1}^{\infty} a_n.$$

As yet these equations are not very interesting, since we have no examples of summable sequences (except for the trivial examples in which the terms are eventually all 0). Before we actually exhibit a summable sequence, some general conditions for summability will be recorded.

There is one necessary and sufficient condition for summability which can be stated immediately. The sequence $\{a_n\}$ is summable if and only if the sequence $\{s_n\}$ converges, which happens, according to Theorem 22-3, if and only if $\lim_{m,n\to\infty} s_m - s_n = 0$; this condition can be rephrased in terms of the original sequence as follows.

THE CAUCHY CRITERION The sequence $\{a_n\}$ is summable if and only if

$$\lim_{m,n\to\infty} a_{n+1} + \cdots + a_m = 0.$$

Although the Cauchy criterion is of theoretical importance, it is not very useful for deciding the summability of any particular sequence. However, one simple consequence of the Cauchy criterion provides a *necessary* condition for summability which is too important not to be mentioned explicitly.

THE VANISHING CONDITION If $\{a_n\}$ is summable, then

$$\lim_{n\to\infty} a_n = 0.$$

This condition follows from the Cauchy criterion by taking $m = n + 1$; it can also be proved directly as follows. If $\lim\limits_{n\to\infty} s_n = l$, then

$$\begin{aligned}\lim_{n\to\infty} a_n = \lim_{n\to\infty}(s_n - s_{n-1}) &= \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1}\\ &= l - l = 0.\end{aligned}$$

Unfortunately, this condition is far from sufficient. For example, $\lim\limits_{n\to\infty} 1/n = 0$, but the sequence $\{1/n\}$ is not summable; in fact, the following grouping of the numbers $1/n$ shows that the sequence $\{s_n\}$ is not bounded:

$$1 + \tfrac{1}{2} + \underbrace{\tfrac{1}{3} + \tfrac{1}{4}}_{\substack{\geq \frac{1}{2}\\ \text{(2 terms,}\\ \text{each } \geq \frac{1}{4})}} + \underbrace{\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8}}_{\substack{\geq \frac{1}{2}\\ \text{(4 terms,}\\ \text{each } \geq \frac{1}{8})}} + \underbrace{\tfrac{1}{9} + \cdots + \tfrac{1}{16}}_{\substack{\geq \frac{1}{2}\\ \text{(8 terms,}\\ \text{each } \geq \frac{1}{16})}} + \cdots.$$

The method of proof used in this example, a clever trick which one might never see, reveals the need for some more standard methods for attacking these problems. These methods shall be developed soon (one of them will give an alternate proof that $\sum\limits_{n=1}^{\infty} 1/n$ does not converge) but it will be necessary to first procure a few examples of convergent series.

The most important of all infinite series are the "geometric series"

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \cdots.$$

Only the cases $|r| < 1$ are interesting, since the individual terms do not approach 0 if $|r| \geq 1$. These series can be managed because the partial sums

$$s_n = 1 + r + \cdots + r^n$$

can be evaluated in simple terms. The two equations

$$\begin{aligned} s_n &= 1 + r + r^2 + \cdots + r^n\\ rs_n &= \phantom{1 + {}} r + r^2 + \cdots + r^n + r^{n+1}\end{aligned}$$

Then if $\sum_{n=1}^{\infty} b_n$ converges, so does $\sum_{n=1}^{\infty} a_n$.

PROOF If

$$s_n = a_1 + \cdots + a_n,$$
$$t_n = b_1 + \cdots + b_n,$$

then

$$0 \le s_n \le t_n \qquad \text{for all } n.$$

Now $\{t_n\}$ is bounded, since $\sum_{n=1}^{\infty} b_n$ converges. Therefore $\{s_n\}$ is bounded; consequently, by the boundedness criterion $\sum_{n=1}^{\infty} a_n$ converges. ▮

Quite frequently the comparison test can be used to analyze very complicated looking series in which most of the complication is irrelevant. For example,

$$\sum_{n=1}^{\infty} \frac{2+\sin^3(n+1)}{2^n+n^2}$$

converges because

$$0 \le \frac{2+\sin^3(n+1)}{2^n+n^2} < \frac{3}{2^n},$$

and

$$\sum_{n=1}^{\infty} \frac{3}{2^n} = 3\sum_{n=1}^{\infty} \frac{1}{2^n}$$

is a convergent (geometric) series.

Similarly, we would expect the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n - 1 + \sin^2 n^3}$$

to converge, since the nth term of the series is practically $1/2^n$ for large n, and we would expect the series

$$\sum_{n=1}^{\infty} \frac{n+1}{n^2+1}$$

to diverge, since $(n+1)/(n^2+1)$ is practically $1/n$ for large n. These facts can be derived immediately from the following theorem, another kind of "comparison test."

THEOREM 2 (THE LIMIT COMPARISON TEST) If $a_n, b_n > 0$ and $\lim_{n\to\infty} a_n/b_n = c \neq 0$, then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.

lead to

$$s_n(1-r) = 1 - r^{n+1}$$

or

$$s_n = \frac{1-r^{n+1}}{1-r}$$

(division by $1-r$ is valid since we are not considering the case $r = 1$). Now $\lim_{n\to\infty} r^n = 0$, since $|r| < 1$. It follows that

$$\sum_{n=0}^{\infty} r^n = \lim_{n\to\infty} \frac{1-r^{n+1}}{1-r} = \frac{1}{1-r}, \qquad |r| < 1.$$

In particular,

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n - 1 = \frac{1}{1-\frac{1}{2}} - 1 = 1,$$

that is,

$$\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \cdots = 1,$$

an infinite sum which can always be remembered from the picture in Figure 1.

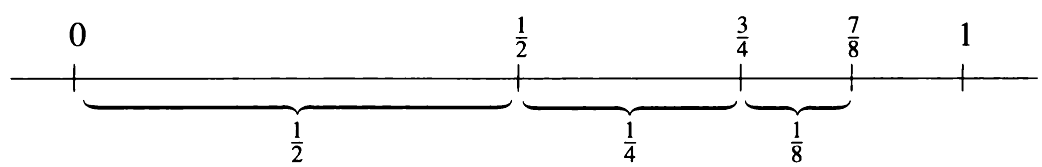


FIGURE 1

Special as they are, geometric series are standard examples from which important tests for summability will be derived.

For a while we shall consider only sequences $\{a_n\}$ with each $a_n \geq 0$; such sequences are called **nonnegative**. If $\{a_n\}$ is a nonnegative sequence, then the sequence $\{s_n\}$ is clearly nondecreasing. This remark, combined with Theorem 22-2, provides a simple-minded test for summability:

THE BOUNDEDNESS CRITERION A nonnegative sequence $\{a_n\}$ is summable if and only if the set of partial sums s_n is bounded.

By itself, this criterion is not very helpful—deciding whether or not the set of all s_n is bounded is just what we are unable to do. On the other hand, if some convergent series are already available for comparison, this criterion can be used to obtain a result whose simplicity belies its importance (it is the basis for almost all other tests).

THEOREM 1 (THE COMPARISON TEST) Suppose that

$$0 \leq a_n \leq b_n \qquad \text{for all } n.$$

PROOF Suppose $\sum_{n=1}^{\infty} b_n$ converges. Since $\lim_{n\to\infty} a_n/b_n = c$, there is some N such that

$$a_n \le 2cb_n \qquad \text{for } n \ge N.$$

But the sequence $2c \sum_{n=N}^{\infty} b_n$ certainly converges. Then Theorem 1 shows that $\sum_{n=N}^{\infty} a_n$ converges, and this implies convergence of the whole series $\sum_{n=1}^{\infty} a_n$, which has only finitely many additional terms.

The converse follows immediately, since we also have $\lim_{n\to\infty} b_n/a_n = 1/c \neq 0$. ▌

The comparison test yields other important tests when we use previously analyzed series as catalysts. Choosing the geometric series $\sum_{n=0}^{\infty} r^n$, the convergent series *par excellence*, we obtain the most important of all tests for summability.

THEOREM 3 (THE RATIO TEST) Let $a_n > 0$ for all n, and suppose that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r.$$

Then $\sum_{n=1}^{\infty} a_n$ converges if $r < 1$. On the other hand, if $r > 1$, then the terms a_n are unbounded, so $\sum_{n=1}^{\infty} a_n$ diverges. (Notice that it is therefore essential to compute $\lim_{n\to\infty} a_{n+1}/a_n$ and not $\lim_{n\to\infty} a_n/a_{n+1}$!)

PROOF Suppose first that $r < 1$. Choose any number s with $r < s < 1$. The hypothesis

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r < 1$$

implies that there is some N such that

$$\frac{a_{n+1}}{a_n} \le s \qquad \text{for } n \ge N.$$

This can be written

$$a_{n+1} \le s a_n \qquad \text{for } n \ge N.$$

Thus

$$\begin{aligned} a_{N+1} &\le s a_N, \\ a_{N+2} &\le s a_{N+1} \le s^2 a_N, \\ &\;\;\vdots \\ a_{N+k} &\le s^k a_N. \end{aligned}$$

Since $\sum_{k=0}^{\infty} a_N s^k = a_N \sum_{k=0}^{\infty} s^k$ converges, the comparison test shows that

$$\sum_{n=N}^{\infty} a_n = \sum_{k=0}^{\infty} a_{N+k}$$

converges. This implies the convergence of the whole series $\sum_{n=1}^{\infty} a_n$.

The case $r > 1$ is even easier. If $1 < s < r$, then there is a number N such that

$$\frac{a_{n+1}}{a_n} \geq s \qquad \text{for } n \geq N,$$

which means that

$$a_{N+k} \geq a_N s^k \qquad k = 0, 1, \ldots,$$

so that the terms are unbounded. ▌

As a simple application of the ratio test, consider the series $\sum_{n=1}^{\infty} 1/n!$. Letting $a_n = 1/n!$ we obtain

$$\frac{a_{n+1}}{a_n} = \frac{\dfrac{1}{(n+1)!}}{\dfrac{1}{n!}} = \frac{n!}{(n+1)!} = \frac{1}{n+1}.$$

Thus

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 0,$$

which shows that the series $\sum_{n=1}^{\infty} 1/n!$ converges. If we consider instead the series $\sum_{n=1}^{\infty} r^n/n!$, where r is some fixed positive number, then

$$\lim_{n \to \infty} \frac{\dfrac{r^{n+1}}{(n+1)!}}{\dfrac{r^n}{n!}} = \lim_{n \to \infty} \frac{r}{n+1} = 0,$$

so $\sum_{n=1}^{\infty} r^n/n!$ converges. It follows that

$$\lim_{n \to \infty} \frac{r^n}{n!} = 0,$$

a result already proved in Chapter 16 (the proof given there was based on the same ideas as those used in the ratio test). Finally, if we consider the series $\sum_{n=1}^{\infty} nr^n$ we have

$$\lim_{n\to\infty} \frac{(n+1)r^{n+1}}{nr^n} = \lim_{n\to\infty} r \cdot \frac{n+1}{n} = r,$$

since $\lim_{n\to\infty} (n+1)/n = 1$. This proves that if $0 \le r < 1$, then $\sum_{n=1}^{\infty} nr^n$ converges, and consequently

$$\lim_{n\to\infty} nr^n = 0.$$

(This result clearly holds for $-1 < r \le 0$, also.) It is a useful exercise to provide a direct proof of this limit, without using the ratio test as an intermediary.

Although the ratio test will be of the utmost theoretical importance, as a practical tool it will frequently be found disappointing. One drawback of the ratio test is the fact that $\lim_{n\to\infty} a_{n+1}/a_n$ may be quite difficult to determine, and may not even exist. A more serious deficiency, which appears with maddening regularity, is the fact that the limit might equal 1. The case $\lim_{n\to\infty} a_{n+1}/a_n = 1$ is precisely the one which is inconclusive: $\{a_n\}$ might not be summable (for example, if $a_n = 1/n$), but then again it might be. In fact, our very next test will show that $\sum_{n=1}^{\infty} (1/n)^2$ converges, even though

$$\lim_{n\to\infty} \frac{\left(\dfrac{1}{n+1}\right)^2}{\left(\dfrac{1}{n}\right)^2} = 1.$$

This test provides a quite different method for determining convergence or divergence of infinite series—like the ratio test, it is an immediate consequence of the comparison test, but the series chosen for comparison is quite novel.

THEOREM 4 (THE INTEGRAL TEST) Suppose that f is positive and decreasing on $[1, \infty)$, and that $f(n) = a_n$ for all n. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if the limit

$$\int_1^{\infty} f = \lim_{A\to\infty} \int_1^A f$$

exists.

PROOF The existence of $\lim_{A\to\infty} \int_1^A f$ is equivalent to convergence of the series

$$\int_1^2 f + \int_2^3 f + \int_3^4 f + \cdots .$$

Now, since f is decreasing we have (Figure 2)

$$f(n+1) < \int_n^{n+1} f < f(n).$$

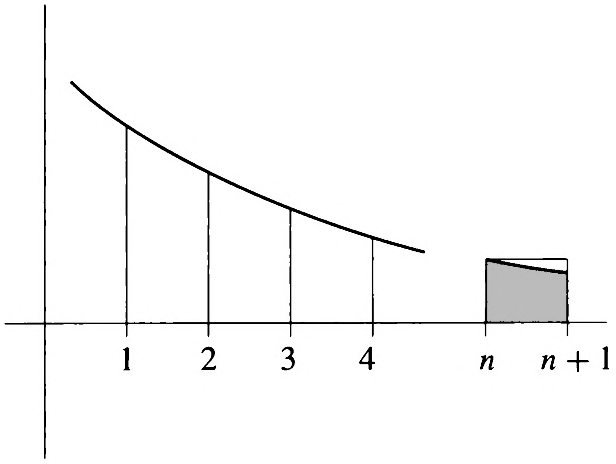


FIGURE 2

The first half of this double inequality shows that the series $\sum_{n=1}^{\infty} a_{n+1}$ may be compared to the series $\sum_{n=1}^{\infty} \int_n^{n+1} f$, proving that $\sum_{n=1}^{\infty} a_{n+1}$ (and hence $\sum_{n=1}^{\infty} a_n$) converges if $\lim_{A\to\infty} \int_1^A$ exists.

The second half of the inequality shows that the series $\sum_{n=1}^{\infty} \int_n^{n+1} f$ may be compared to the series $\sum_{n=1}^{\infty} a_n$, proving that $\lim_{A\to\infty} \int_1^A f$ must exist if $\sum_{n=1}^{\infty} a_n$ converges. ∎

Only one example using the integral test will be given here, but it settles the question of convergence for infinitely many series at once. If $p > 0$, the convergence of $\sum_{n=1}^{\infty} 1/n^p$ is equivalent, by the integral test, to the existence of

$$\int_1^{\infty} \frac{1}{x^p}\,dx.$$

Now

$$\int_1^A \frac{1}{x^p}\,dx = \begin{cases} -\dfrac{1}{(p-1)} \cdot \dfrac{1}{A^{p-1}} + \dfrac{1}{p-1}, & p \neq 1 \\ \log A, & p = 1. \end{cases}$$

This shows that $\lim_{A\to\infty} \int_1^A 1/x^p\,dx$ exists if $p > 1$, but not if $p \le 1$. Thus $\sum_{n=1}^{\infty} 1/n^p$ converges precisely for $p > 1$. In particular, $\sum_{n=1}^{\infty} 1/n$ diverges.

The tests considered so far apply only to nonnegative sequences, but nonpositive sequences may be handled in precisely the same way. In fact, since

$$\sum_{n=1}^{\infty} a_n = -\left(\sum_{n=1}^{\infty} -a_n\right),$$

all considerations about nonpositive sequences can be reduced to questions involving nonnegative sequences. Sequences which contain both positive and negative terms are quite another story.

If $\sum_{n=1}^{\infty} a_n$ is a sequence with both positive and negative terms, one can consider instead the sequence $\sum_{n=1}^{\infty} |a_n|$, all of whose terms are nonnegative. Cheerfully ignoring the possibility that we may have thrown away all the interesting information about the original sequence, we proceed to eulogize those sequences which are converted by this procedure into convergent sequences.

DEFINITION

The series $\sum_{n=1}^{\infty} a_n$ is **absolutely convergent** if the series $\sum_{n=1}^{\infty} |a_n|$ is convergent. (In more formal language, the sequence $\{a_n\}$ is **absolutely summable** if the sequence $\{|a_n|\}$ is summable.)

Although we have no right to expect this definition to be of any interest, it turns out to be exceedingly important. The following theorem shows that the definition is at least not entirely useless.

THEOREM 5 Every absolutely convergent series is convergent. Moreover, a series is absolutely convergent if and only if the series formed from its positive terms and the series formed from its negative terms both converge.

PROOF If $\sum_{n=1}^{\infty} |a_n|$ converges, then, by the Cauchy criterion,

$$\lim_{m,n\to\infty} |a_{n+1}| + \cdots + |a_m| = 0.$$

Since

$$|a_{n+1} + \cdots + a_m| \le |a_{n+1}| + \cdots + |a_m|,$$

it follows that

$$\lim_{m,n\to\infty} a_{n+1} + \cdots + a_m = 0,$$

which shows that $\sum_{n=1}^{\infty} a_n$ converges.

To prove the second part of the theorem, let

$$a_n{}^+ = \begin{cases} a_n, & \text{if } a_n \ge 0 \\ 0, & \text{if } a_n \le 0, \end{cases}$$

$$a_n{}^- = \begin{cases} a_n, & \text{if } a_n \le 0 \\ 0, & \text{if } a_n \ge 0, \end{cases}$$

so that $\sum_{n=1}^{\infty} a_n{}^+$ is the series formed from the positive terms of $\sum_{n=1}^{\infty} a_n$, and $\sum_{n=1}^{\infty} a_n{}^-$ is the series formed from the negative terms.

If $\sum_{n=1}^{\infty} a_n{}^+$ and $\sum_{n=1}^{\infty} a_n{}^-$ both converge, then

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} [a_n{}^+ - (a_n{}^-)] = \sum_{n=1}^{\infty} a_n{}^+ - \sum_{n=1}^{\infty} a_n{}^-$$

also converges, so $\sum_{n=1}^{\infty} a_n$ converges absolutely.

On the other hand, if $\sum_{n=1}^{\infty} |a_n|$ converges, then, as we have just shown, $\sum_{n=1}^{\infty} a_n$ also converges. Therefore

$$\sum_{n=1}^{\infty} a_n{}^+ = \frac{1}{2}\left(\sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} |a_n|\right)$$

and

$$\sum_{n=1}^{\infty} a_n{}^- = \frac{1}{2}\left(\sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} |a_n|\right)$$

both converge. ▮

It follows from Theorem 5 that every convergent series with positive terms can be used to obtain infinitely many other convergent series, simply by putting in minus signs at random. Not every convergent series can be obtained in this way, however—there are series which are convergent but not absolutely convergent (such series are called **conditionally convergent**). In order to prove this statement we need a test for convergence which applies specifically to series with positive and negative terms.

THEOREM 6 (LEIBNIZ'S THEOREM) Suppose that

$$a_1 \geq a_2 \geq a_3 \geq \cdots \geq 0,$$

and that

$$\lim_{n\to\infty} a_n = 0.$$

Then the series

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + a_5 - \cdots$$

converges.

PROOF Figure 3 illustrates relationships between the partial sums which we will establish:

(1) $s_2 \leq s_4 \leq s_6 \leq \cdots,$
(2) $s_1 \geq s_3 \geq s_5 \geq \cdots,$
(3) $s_k \leq s_l$ if k is even and l is odd.

FIGURE 3

To prove the first two inequalities, observe that

$$
\begin{aligned}
(1)\quad s_{2n+2} &= s_{2n} + a_{2n+1} - a_{2n+2} \\
&\geq s_{2n}, \qquad \text{since } a_{2n+1} \geq a_{2n+2} \\
(2)\quad s_{2n+3} &= s_{2n+1} - a_{2n+2} + a_{2n+3} \\
&\leq s_{2n+1}, \qquad \text{since } a_{2n+2} \geq a_{2n+3}.
\end{aligned}
$$

To prove the third inequality, notice first that

$$
\begin{aligned}
s_{2n} &= s_{2n-1} - a_{2n} \\
&\leq s_{2n-1} \qquad \text{since } a_{2n} \geq 0.
\end{aligned}
$$

This proves only a special case of (3), but in conjunction with (1) and (2) the general case is easy: if k is even and l is odd, choose n such that

$$2n \geq k \quad \text{and} \quad 2n - 1 \geq l;$$

then

$$s_k \leq s_{2n} \leq s_{2n-1} \leq s_l,$$

which proves (3).

Now, the sequence $\{s_{2n}\}$ converges, because it is nondecreasing and is bounded above (by s_l for any odd l). Let

$$\alpha = \sup\{s_{2n}\} = \lim_{n\to\infty} s_{2n}.$$

Similarly, let

$$\beta = \inf\{s_{2n+1}\} = \lim_{n\to\infty} s_{2n+1}.$$

It follows from (3) that $\alpha \leq \beta$; since

$$s_{2n+1} - s_{2n} = a_{2n+1} \quad \text{and} \quad \lim_{n\to\infty} a_n = 0$$

it is actually the case that $\alpha = \beta$. This proves that $\alpha = \beta = \lim_{n\to\infty} s_n$. ▮

The standard example derived from Theorem 6 is the series

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \cdots,$$

which is convergent, but *not* absolutely convergent (since $\sum_{n=1}^{\infty} 1/n$ does not converge). If the sum of this series is denoted by x, the following manipulations lead to quite a paradoxical result:

$$
\begin{aligned}
x &= 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \cdots \\
&= 1 - \tfrac{1}{2} - \tfrac{1}{4} + \tfrac{1}{3} - \tfrac{1}{6} - \tfrac{1}{8} + \tfrac{1}{5} - \tfrac{1}{10} - \tfrac{1}{12} + \tfrac{1}{7} - \tfrac{1}{14} - \tfrac{1}{16} + \cdots \\
&\quad \text{(the pattern here is one positive term followed by two negative ones)} \\
&= (1 - \tfrac{1}{2}) - \tfrac{1}{4} + (\tfrac{1}{3} - \tfrac{1}{6}) - \tfrac{1}{8} + (\tfrac{1}{5} - \tfrac{1}{10}) - \tfrac{1}{12} + (\tfrac{1}{7} - \tfrac{1}{14}) - \tfrac{1}{16} + \cdots \\
&= \tfrac{1}{2} - \tfrac{1}{4} + \tfrac{1}{6} - \tfrac{1}{8} + \tfrac{1}{10} - \tfrac{1}{12} + \tfrac{1}{14} - \tfrac{1}{16} + \cdots \\
&= \tfrac{1}{2}(1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \tfrac{1}{7} - \tfrac{1}{8} + \cdots) \\
&= \tfrac{1}{2}x,
\end{aligned}
$$

so $x = x/2$, implying that $x = 0$. On the other hand, it is easy to see that $x \neq 0$: the partial sum s_2 equals $\frac{1}{2}$, and the proof of Leibniz's Theorem shows that $x \geq s_2$.

This contradiction depends on a step which takes for granted that operations valid for finite sums necessarily have analogues for infinite sums. It is true that the sequence

$$\{b_n\} = 1,\ -\tfrac{1}{2},\ -\tfrac{1}{4},\ \tfrac{1}{3},\ -\tfrac{1}{6},\ -\tfrac{1}{8},\ \tfrac{1}{5},\ -\tfrac{1}{10},\ -\tfrac{1}{12},\ \dots$$

contains all the numbers in the sequence

$$\{a_n\} = 1,\ -\tfrac{1}{2},\ \tfrac{1}{3},\ -\tfrac{1}{4},\ \tfrac{1}{5},\ -\tfrac{1}{6},\ \tfrac{1}{7},\ -\tfrac{1}{8},\ \tfrac{1}{9},\ -\tfrac{1}{10},\ \tfrac{1}{11},\ -\tfrac{1}{12},\ \dots.$$

In fact, $\{b_n\}$ is a **rearrangement** of $\{a_n\}$ in the following precise sense: each $b_n = a_{f(n)}$ where f is a certain function which "permutes" the natural numbers, that is, every natural number m is $f(n)$ for precisely one n. In our example

$f(2m+1) = 3m+1$ (the terms $1, \frac{1}{3}, \frac{1}{5}, \dots$ go into the 1st, 4th, 7th, ... places),
$f(4m) = 3m$ (the terms $-\frac{1}{4}, -\frac{1}{8}, -\frac{1}{12}, \dots$ go into the 3rd, 6th, 9th, ... places),
$f(4m+2) = 3m+2$ (the terms $-\frac{1}{2}, -\frac{1}{6}, -\frac{1}{10}, \dots$ go into the 2nd, 5th, 8th, ... places).

Nevertheless, there is no reason to assume that $\sum_{n=1}^{\infty} b_n$ should equal $\sum_{n=1}^{\infty} a_n$: these sums are, by definition, $\lim_{n\to\infty} b_1 + \dots + b_n$ and $\lim_{n\to\infty} a_1 + \dots + a_n$, so the particular order of the terms can quite conceivably matter. The series $\sum_{n=1}^{\infty} (-1)^{n+1}/n$ is not special in this regard; indeed, its behavior is typical of series which are not absolutely convergent—the following result (really more of a grand counterexample than a theorem) shows how bad conditionally convergent series are.

THEOREM 7 If $\sum_{n=1}^{\infty} a_n$ converges, but does not converge absolutely, then for any number α there is a rearrangement $\{b_n\}$ of $\{a_n\}$ such that $\sum_{n=1}^{\infty} b_n = \alpha$.

PROOF Let $\sum_{n=1}^{\infty} p_n$ denote the series formed from the positive terms of $\{a_n\}$ and let $\sum_{n=1}^{\infty} q_n$ denote the series of negative terms. It follows from Theorem 5 that at least one of these series does not converge. As a matter of fact, both must fail to converge, for if one had bounded partial sums, and the other had unbounded partial sums, then the original series $\sum_{n=1}^{\infty} a_n$ would also have unbounded partial sums, contradicting the assumption that it converges.

Now let α be any number. Assume, for simplicity, that $\alpha > 0$ (the proof for $\alpha < 0$ will be a simple modification). Since the series $\sum_{n=1}^{\infty} p_n$ is not convergent, there is a number N such that

$$\sum_{n=1}^{N} p_n > \alpha.$$

We will choose N_1 to be the *smallest* N with this property. This means that

$$(1) \quad \sum_{n=1}^{N_1-1} p_n \leq \alpha,$$

$$\text{but} \quad (2) \quad \sum_{n=1}^{N_1} p_n > \alpha.$$

Then if

$$S_1 = \sum_{n=1}^{N_1} p_n,$$

we have

$$S_1 - \alpha \leq p_{N_1}.$$

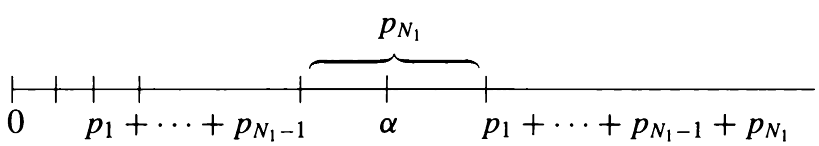


FIGURE 4

This relation, which is clear from Figure 4, follows immediately from equation (1):

$$S_1 - \alpha \leq S_1 - \sum_{n=1}^{N_1-1} p_n = p_{N_1}.$$

To the sum S_1 we now add on just enough negative terms to obtain a new sum T_1 which is less than α. In other words, we choose the smallest integer M_1 for which

$$T_1 = S_1 + \sum_{n=1}^{M_1} q_n < \alpha.$$

As before, we have

$$\alpha - T_1 \leq -q_{M_1}.$$

We now continue this procedure indefinitely, obtaining sums alternately larger and smaller than α, each time choosing the smallest N_k or M_k possible. The sequence

$$p_1, \ldots, p_{N_1}, q_1, \ldots, q_{M_1}, p_{N_1+1}, \ldots, p_{N_2}, \ldots$$

is a rearrangement of $\{a_n\}$. The partial sums of this rearrangement increase to S_1, then decrease to T_1, then increase to S_2, then decrease to T_2, etc. To complete the proof we simply note that $|S_k - \alpha|$ and $|T_k - \alpha|$ are less than or equal to p_{N_k} or $-q_{M_k}$, respectively, and that these terms, being members of the original sequence $\{a_n\}$, must decrease to 0, since $\sum_{n=1}^{\infty} a_n$ converges. ∎

Together with Theorem 7, the next theorem establishes conclusively the distinction between conditionally convergent and absolutely convergent series.

THEOREM 8 If $\sum_{n=1}^{\infty} a_n$ converges absolutely, and $\{b_n\}$ is any rearrangement of $\{a_n\}$, then $\sum_{n=1}^{\infty} b_n$ also converges (absolutely), and

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n.$$

PROOF Let us denote the partial sums of $\{a_n\}$ by s_n, and the partial sums of $\{b_n\}$ by t_n.

Suppose that $\varepsilon > 0$. Since $\sum_{n=1}^{\infty} a_n$ converges, there is some N such that

$$\left| \sum_{n=1}^{\infty} a_n - s_N \right| < \varepsilon.$$

Moreover, since $\sum_{n=1}^{\infty} |a_n|$ converges, we can also choose N so that

$$\sum_{n=1}^{\infty} |a_n| - (|a_1| + \cdots + |a_N|) < \varepsilon,$$

i.e., so that

$$|a_{N+1}| + |a_{N+2}| + |a_{N+3}| + \cdots < \varepsilon.$$

Now choose M so large that each of $a_1, \ldots, a_N$ appear among $b_1, \ldots, b_M$. Then whenever $m > M$, the difference $t_m - s_N$ is the sum of certain a_i, *where* $a_1, \ldots, a_N$ *are definitely excluded.* Consequently,

$$|t_m - s_N| \le |a_{N+1}| + |a_{N+2}| + |a_{N+3}| + \cdots .$$

Suppose that $\{c_n\}$ is some sequence of this sort, containing each product $a_i b_j$ just once. Then we might naively expect to have

$$\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} a_n \cdot \sum_{n=1}^{\infty} b_n.$$

But this *isn't* true (see Problem 10), nor is this really so surprising, since we've said nothing about the specific arrangement of the terms. The next theorem shows that the result does hold when the arrangement of terms is irrelevant.

THEOREM 9 If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge absolutely, and $\{c_n\}$ is any sequence containing the products $a_i b_j$ for each pair (i, j), then

$$\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} a_n \cdot \sum_{n=1}^{\infty} b_n.$$

PROOF Notice first that the sequence

$$p_L = \sum_{i=1}^{L} |a_i| \cdot \sum_{j=1}^{L} |b_j|$$

converges, since $\{a_n\}$ and $\{b_n\}$ are absolutely convergent, and since the limit of a product is the product of the limits. So $\{p_L\}$ is a Cauchy sequence, which means that for any $\varepsilon > 0$, if L and L' are large enough, then

$$\left| \sum_{i=1}^{L'} |a_i| \cdot \sum_{j=1}^{L'} |b_j| - \sum_{i=1}^{L} |a_i| \cdot \sum_{j=1}^{L} |b_j| \right| < \frac{\varepsilon}{2}.$$

It follows that

$$(1) \qquad \sum_{i \text{ or } j > L} |a_i| \cdot |b_j| \le \frac{\varepsilon}{2} < \varepsilon.$$

Now suppose that N is any number so large that the terms c_n for $n \le N$ include every term $a_i b_j$ for $i, j \le L$. Then the difference

$$\sum_{n=1}^{N} c_n - \sum_{i=1}^{L} a_i \cdot \sum_{j=1}^{L} b_j$$

consists of terms $a_i b_j$ with $i > L$ or $j > L$, so

$$(2) \qquad \left| \sum_{n=1}^{N} c_n - \sum_{i=1}^{L} a_i \cdot \sum_{j=1}^{L} b_j \right| \le \sum_{i \text{ or } j > L} |a_i| \cdot |b_j| < \varepsilon \qquad \text{by (1).}$$

But since the limit of a product is the product of the limits, we also have

$$(3) \qquad \left| \sum_{i=1}^{\infty} a_i \cdot \sum_{j=1}^{\infty} b_j - \sum_{i=1}^{L} a_i \cdot \sum_{j=1}^{L} b_j \right| < \varepsilon$$

Thus, if $m > M$, then

$$\begin{aligned}\left|\sum_{n=1}^{\infty} a_n - t_m\right| &= \left|\sum_{n=1}^{\infty} a_n - s_N - (t_m - s_N)\right| \\ &\le \left|\sum_{n=1}^{\infty} a_n - s_N\right| + |t_m - s_N| \\ &< \varepsilon + \varepsilon.\end{aligned}$$

Since this is true for every $\varepsilon > 0$, the series $\sum_{n=1}^{\infty} b_n$ converges to $\sum_{n=1}^{\infty} a_n$.

To show that $\sum_{n=1}^{\infty} b_n$ converges absolutely, note that $\{|b_n|\}$ is a rearrangement of $\{|a_n|\}$; since $\sum_{n=1}^{\infty} |a_n|$ converges absolutely, $\sum_{n=1}^{\infty} |b_n|$ converges by the first part of the theorem. ▮

Absolute convergence is also important when we want to multiply two infinite series. Unlike the situation for addition, where we have the simple formula

$$\sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} (a_n + b_n),$$

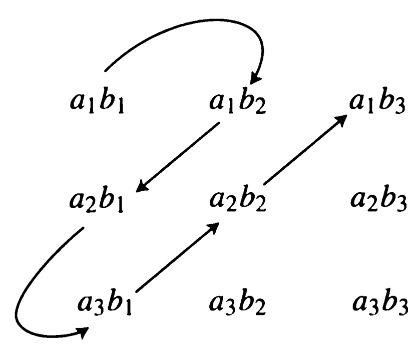
there isn't quite so obvious a candidate for the product

$$\left(\sum_{n=1}^{\infty} a_n\right) \cdot \left(\sum_{n=1}^{\infty} b_n\right) = (a_1 + a_2 + \cdots) \cdot (b_1 + b_2 + \cdots).$$

It would seem that we ought to sum all the products $a_i b_j$. The trouble is that these form a two-dimensional array, rather than a sequence:

$$\begin{matrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & a_3b_3 \\ \vdots & \vdots & \vdots \end{matrix}$$

Nevertheless, all the elements of this array can be arranged in a sequence. The picture below shows one way of doing this, and of course, there are (infinitely) many other ways.



for large enough L. Consequently, if we choose L, and then N, large enough, we will have

$$\begin{aligned}\left|\sum_{i=1}^{\infty} a_i \cdot \sum_{j=1}^{\infty} b_j - \sum_{i=1}^{N} c_n\right| &\le \left|\sum_{i=1}^{\infty} a_i \cdot \sum_{j=1}^{\infty} b_j - \sum_{i=1}^{L} a_i \cdot \sum_{j=1}^{L} b_j\right| \\ &\quad + \left|\sum_{i=1}^{L} a_i \cdot \sum_{j=1}^{L} b_j - \sum_{n=1}^{N} c_n\right| \\ &< 2\varepsilon \qquad \text{by (2) and (3),}\end{aligned}$$

which proves the theorem. ▮

Unlike our previous theorems, which were merely concerned with summability, this result says something about the actual sums. Generally speaking, there is no reason to presume that a given infinite sum can be "evaluated" in any simpler terms. However, many simple expressions can be equated to infinite sums by using Taylor's Theorem. Chapter 20 provides many examples of functions for which

$$f(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i + R_{n,a}(x),$$

where $\lim_{n\to\infty} R_{n,a}(x) = 0$. This is precisely equivalent to

$$f(x) = \lim_{n\to\infty} \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i,$$

which means, in turn, that

$$f(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(a)}{i!}(x-a)^i.$$

As particular examples we have

$$\begin{aligned}\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots, \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots, \\ e^x &= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots, \\ \arctan x &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots, \qquad |x| \le 1, \\ \log(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \cdots, \qquad -1 < x \le 1.\end{aligned}$$

(Notice that the series for $\arctan x$ and $\log(1+x)$ do not even converge for $|x| > 1$; in addition, when $x = -1$, the series for $\log(1+x)$ becomes

$$-1 - \tfrac{1}{2} - \tfrac{1}{3} - \tfrac{1}{4} - \cdots$$

which does not converge.)

Some pretty impressive results are obtained with particular values of x:

$$\begin{aligned} 0 &= \pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \frac{\pi^7}{7!} + \cdots, \\ e &= 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots, \\ \frac{\pi}{4} &= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots, \\ \log 2 &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots. \end{aligned}$$

More significant developments may be anticipated if we compare the series for $\sin x$ and $\cos x$ a little more carefully. The series for $\cos x$ is just the one we would have obtained if we had enthusiastically differentiated both sides of the equation

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

term-by-term, ignoring the fact that we have never proved anything about the derivatives of infinite sums. Likewise, if we differentiate both sides of the formula for $\cos x$ formally (i.e., without justification) we obtain the formula $\cos'(x) = -\sin x$, and if we differentiate the formula for e^x we obtain $\exp'(x) = \exp(x)$. In the next chapter we shall see that such term-by-term differentiation of infinite sums is indeed valid in certain important cases.

PROBLEMS

1. Decide whether each of the following infinite series is convergent or divergent. The tools which you will need are Leibniz's Theorem and the comparison, ratio, and integral tests. A few examples have been picked with malice aforethought; two series which look quite similar may require different tests (and then again, they may not). The hint below indicates which tests may be used.

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n\theta}{n^2}$.

(ii) $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$.

(iii) $1 - \frac{1}{2} + \frac{2}{3} - \frac{1}{3} + \frac{2}{4} - \frac{1}{4} + \frac{2}{5} - \frac{1}{5} + \cdots$.

(iv) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\log n}{n}$.

(v) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt[3]{n^2-1}}$. (The summation begins with $n = 2$ simply to avoid the meaningless term obtained for $n = 1$).

(vi) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2+1}}$.

(vii) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n!}$.

(viii) $\displaystyle\sum_{n=1}^{\infty} \frac{\log n}{n}$.

(ix) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\log n}$.

(x) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\log n)^k}$.

(xi) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\log n)^n}$.

(xii) $\displaystyle\sum_{n=2}^{\infty} (-1)^n \frac{1}{(\log n)^n}$.

(xiii) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$.

(xiv) $\displaystyle\sum_{n=1}^{\infty} \sin \frac{1}{n}$.

(xv) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \log n}$.

(xvi) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$.

(xvii) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2(\log n)}$.

(xviii) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$.

(xix) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n n!}{n^n}$.

(xx) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n n!}{n^n}$.

Hint: Use the comparison test for (i), (v), (vi), (ix), (x), (xi), (xiii), (xiv), (xvii); the ratio test for (vii), (xviii), (xix), (xx); the integral test for (viii), (xv), (xvi).

The next two problems examine, with hints, some infinite series that require more delicate analysis than those in Problem 1.

***2.** (a) If you have successfully solved examples (xix) and (xx) from Problem 1, it should be clear that $\displaystyle\sum_{n=1}^{\infty} a^n n!/n^n$ converges for $a < e$ and diverges for $a > e$. For $a = e$ the ratio test fails; show that $\displaystyle\sum_{n=1}^{\infty} e^n n!/n^n$ actually diverges, by using Problem 22-13.

(b) Decide when $\displaystyle\sum_{n=1}^{\infty} n^n/a^n n!$ converges, again resorting to Problem 22-13 when the ratio test fails.

***3.** Problem 1 presented the two series $\displaystyle\sum_{n=2}^{\infty} (\log n)^{-k}$ and $\displaystyle\sum_{n=2}^{\infty} (\log n)^{-n}$, of which the first diverges while the second converges. The series

$$\sum_{n=2}^{\infty} \frac{1}{(\log n)^{\log n}},$$

which lies between these two, is analyzed in parts (a) and (b).

(a) Show that $\int_1^{\infty} e^y/y^y \, dy$ exists, by considering the series $\displaystyle\sum_{n=1}^{\infty} (e/n)^n$.

(b) Show that

$$\sum_{n=2}^{\infty} \frac{1}{(\log n)^{\log n}}$$

converges, by using the integral test. Hint: Use an appropriate substitution and part (a).

(c) Show that

$$\sum_{n=2}^{\infty} \frac{1}{(\log n)^{\log(\log n)}}$$

diverges, by using the integral test. Hint: Use the same substitution as in part (b), and show directly that the resulting integral diverges.

4. Decide whether or not $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$ converges.

5. (a) Prove that if $\displaystyle\sum_{n=1}^{\infty} a_n$ converges absolutely, then so does $\displaystyle\sum_{n=1}^{\infty} a_n{}^3$.

*(b) Show that this does not hold for conditional convergence.

6. Let f be a continuous function on an interval around 0, and let $a_n = f(1/n)$ (for large enough n).

(a) Prove that if $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, then $f(0) = 0$.

(b) Prove that if $f'(0)$ exists and $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, then $f'(0) = 0$.

(c) Prove that if $f''(0)$ exists and $f(0) = f'(0) = 0$, then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

(d) Suppose $\displaystyle\sum_{n=1}^{\infty} a_n$ converges. Must $f'(0)$ exist?

(e) Suppose $f(0) = f'(0) = 0$. Must $\displaystyle\sum_{n=1}^{\infty} a_n$ converge?

7. (a) Let $\{a_n\}$ be a sequence of integers with $0 \le a_n \le 9$. Prove that $\displaystyle\sum_{n=1}^{\infty} a_n 10^{-n}$ exists (and lies between 0 and 1). (This, of course, is the number which we usually denote by $0.a_1a_2a_3a_4\ldots$.)

(b) Suppose that $0 \le x \le 1$. Prove that there is a sequence of integers $\{a_n\}$ with $0 \le a_n \le 9$ and $\displaystyle\sum_{n=1}^{\infty} a_n 10^{-n} = x$. Hint: For example, $a_1 = [10x]$ (where $[y]$ denotes the greatest integer which is $\le y$).

(c) Show that if $\{a_n\}$ is repeating, i.e., is of the form $a_1, a_2, \ldots, a_k$, $a_1, a_2, \ldots, a_k, a_1, a_2, \ldots$, then $\displaystyle\sum_{n=1}^{\infty} a_n 10^{-n}$ is a rational number (and find it). The same result naturally holds if $\{a_n\}$ is eventually repeating, i.e., if the sequence $\{a_{N+k}\}$ is repeating for some N.

(d) Prove that if $x = \displaystyle\sum_{n=1}^{\infty} a_n 10^{-n}$ is rational, then $\{a_n\}$ is eventually repeating. (Just look at the process of finding the decimal expansion of p/q—dividing q into p by long division.)

8. Suppose that $\{a_n\}$ satisfies the hypothesis of Leibniz's Theorem. Use the proof of Leibniz's Theorem to obtain the following estimate:

$$\left| \sum_{n=1}^{\infty} (-1)^{n+1} a_n - [a_1 - a_2 + \cdots \pm a_N] \right| < a_{N+1}.$$

9. (a) Prove that if $a_n \geq 0$ and $\lim_{n\to\infty} \sqrt[n]{a_n} = r$, then $\sum_{n=1}^{\infty} a_n$ converges if $r < 1$, and diverges if $r > 1$. (The proof is very similar to that of the ratio test.) This result is known as the "root test." More generally, $\sum_{n=1}^{\infty} a_n$ converges if there is some $s < 1$ such that all but finitely many $\sqrt[n]{a_n}$ are $\leq s$, and $\sum_{n=1}^{\infty} a_n$ diverges if infinitely many $\sqrt[n]{a_n}$ are ≥ 1. This result is known as the "delicate root test" (there is a similar delicate ratio test). It follows, using the notation of Problem 22-27, that $\sum_{n=1}^{\infty} a_n$ converges if $\overline{\lim_{n\to\infty}} \sqrt[n]{a_n} < 1$ and diverges if $\overline{\lim_{n\to\infty}} \sqrt[n]{a_n} > 1$; no conclusion is possible if $\overline{\lim_{n\to\infty}} \sqrt[n]{a_n} = 1$.

(b) Prove that if the ratio test works, the root test will also. Hint: Use a problem from the previous chapter.

It is easy to construct series for which the ratio test fails, while the root test works. For example, the root test shows that the series

$$\tfrac{1}{2} + \tfrac{1}{3} + (\tfrac{1}{2})^2 + (\tfrac{1}{3})^2 + (\tfrac{1}{2})^3 + (\tfrac{1}{3})^3 + \cdots$$

converges, even though the ratios of successive terms do not approach a limit. Most examples are of this rather artificial nature, but the root test is nevertheless quite an important theoretical tool.

10. For two sequences $\{a_n\}$ and $\{b_n\}$, let $c_n = \sum_{k=1}^{n} a_k b_{n+1-k}$. (Then c_n is the sum of the terms on the nth diagonal in the picture on page 486.) The series $\sum_{n=1}^{\infty} c_n$ is called the *Cauchy product* of $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$. If $a_n = b_n = (-1)^n/\sqrt{n}$, show that $|c_n| \geq 1$, so that the Cauchy product does not converge.

11. (a) Consider the collection A of natural numbers that do *not* contain a 9 in their usual (base 10) representation. Show that the sum of the reciprocals of the numbers in A *converges*. Hint: How many numbers between 1 and 9 are in A?; how many between 10 and 99?; etc.

(b) If B is the collection of natural numbers that do not have *all 10 digits* $0, \ldots, 9$ in their usual representation, then the sum of the reciprocals of the numbers in B converges. (So "most" integers must have all ten digits in their representation.)

12. A sequence $\{a_n\}$ is called **Cesaro summable**, with Cesaro sum l, if

$$\lim_{n\to\infty} \frac{s_1 + \cdots + s_n}{n} = l$$

(where $s_k = a_1 + \cdots + a_k$). Problem 22-16 shows that a summable sequence

(c) Prove, using Problem 15-33, that the series $\sum_{n=1}^{\infty}(\cos nx)/n$ converges if x is not of the form $2k\pi$ for any integer k (in which case it clearly diverges).

(d) Prove *Abel's test*: If $\sum_{n=1}^{\infty} a_n$ converges and $\{b_n\}$ is a sequence which is either nondecreasing or nonincreasing and which is bounded, then $\sum_{n=1}^{\infty} a_n b_n$ converges. Hint: Consider $b_n - b$, where $b = \lim_{n\to\infty} b_n$.

***23.** Suppose $\{a_n\}$ is decreasing and $\lim_{n\to\infty} a_n = 0$. Prove that if $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} 2^n a_{2^n}$ also converges (the "Cauchy Condensation Theorem"). Notice that the divergence of $\sum_{n=1}^{\infty} 1/n$ is a special case, for if $\sum_{n=1}^{\infty} 1/n$ converged, then $\sum_{n=1}^{\infty} 2^n(1/2^n)$ would also converge; this remark may serve as a hint.

***24.** (a) Prove that if $\sum_{n=1}^{\infty} {a_n}^2$ and $\sum_{n=1}^{\infty} {b_n}^2$ converge, then $\sum_{n=1}^{\infty} a_n b_n$ converges.

(b) Prove that if $\sum_{n=1}^{\infty} {a_n}^2$ converges, then $\sum_{n=1}^{\infty} a_n/n^{\alpha}$ converges for any $\alpha > \frac{1}{2}$.

***25.** Suppose $\{a_n\}$ is decreasing and each $a_n > 0$. Prove that if $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} na_n = 0$. Hint: Write down the Cauchy criterion and be sure to use the fact that $\{a_n\}$ is decreasing.

***26.** If $\sum_{n=1}^{\infty} a_n$ converges, then the partial sums s_n are bounded, and $\lim_{n\to\infty} a_n = 0$. It is tempting to conjecture that boundedness of the partial sums, together with the condition $\lim_{n\to\infty} a_n = 0$, implies convergence of the series $\sum_{n=1}^{\infty} a_n$. Find a counterexample to show that this is *not* true. Hint: Notice that some *subsequence* of the partial sums will have to converge; you must somehow allow this to happen, without letting the sequence of partial sums itself converge.

27. Prove that if $a_n \geq 0$ and $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} \frac{a_n}{1 + a_n}$ also diverges. Hint: Compare the partial sums. Does the converse hold?

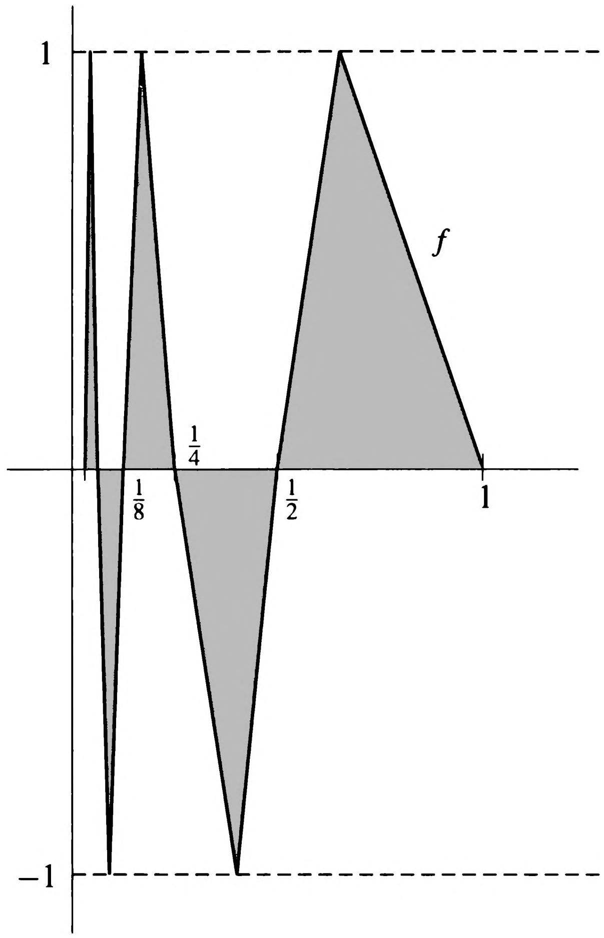


FIGURE 5

***19.** Let $f(x) = x \sin 1/x$ for $0 < x \leq 1$, and let $f(0) = 0$. Recall the definition of $\ell(f, P)$ from Problem 13-25. Show that the set of all $\ell(f, P)$ for P a partition of $[0, 1]$ is not bounded (thus f has "infinite length"). Hint: Try partitions of the form

$$P = \left\{0, \frac{2}{(2n+1)\pi}, \dots, \frac{2}{7\pi}, \frac{2}{5\pi}, \frac{2}{3\pi}, \frac{2}{\pi}, 1\right\}.$$

20. Let f be the function shown in Figure 5. Find $\int_0^1 f$, and also the area of the shaded region in Figure 5.

***21.** In this problem we will establish the "binomial series"

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k, \qquad |x| < 1,$$

for any α, by showing that $\lim_{n\to\infty} R_{n,0}(x) = 0$. The proof is in several steps, and uses the Cauchy and Lagrange forms as found in Problem 20-21.

(a) Use the ratio test to show that the series $\sum_{k=0}^{\infty} \binom{\alpha}{k} r^k$ does indeed converge for $|r| < 1$ (this is not to say that it necessarily converges to $(1+r)^\alpha$). It follows in particular that $\lim_{n\to\infty} \binom{\alpha}{n} r^n = 0$ for $|r| < 1$.

(b) Suppose first that $0 \leq x < 1$. Show that $\lim_{n\to\infty} R_{n,0}(x) = 0$, by using Lagrange's form of the remainder, noticing that $(1+t)^{\alpha-n-1} \leq 1$ for $n+1 > \alpha$.

(c) Now suppose that $-1 < x < 0$; the number t in Cauchy's form of the remainder satisfies $-1 < x < t \leq 0$. Show that

$$|x(1+t)^{\alpha-1}| \leq |x|M, \qquad \text{where } M = \max(1, (1+x)^{\alpha-1}),$$

and

$$\left|\frac{x-t}{1+t}\right| = |x| \left(\frac{1-t/x}{1+t}\right) \leq |x|.$$

Using Cauchy's form of the remainder, and the fact that

$$(n+1)\binom{\alpha}{n+1} = \alpha\binom{\alpha-1}{n},$$

show that $\lim_{n\to\infty} R_{n,0}(x) = 0$.

22. (a) Suppose that the partial sums of the sequence $\{a_n\}$ are bounded and that $\{b_n\}$ is a sequence with $b_n \geq b_{n+1}$ and $\lim_{n\to\infty} b_n = 0$. Prove that $\sum_{n=1}^{\infty} a_n b_n$ converges. This is known as *Dirichlet's test.* Hint: Use Abel's Lemma (Problem 19-36) to check the Cauchy criterion.

(b) Derive Leibniz's Theorem from this result.

is automatically Cesaro summable, with sum equal to its Cesaro sum. Find a sequence which is *not* summable, but which *is* Cesaro summable.

13. Suppose that $a_n > 0$ and $\{a_n\}$ is Cesaro summable. Suppose also that the sequence $\{na_n\}$ is bounded. Prove that the series $\sum_{n=1}^{\infty} a_n$ converges. Hint: If $s_n = \sum_{i=1}^{n} a_i$ and $\sigma_n = \frac{1}{n}\sum_{i=1}^{n} s_i$, prove that $s_n - \frac{n}{n+1}\sigma_n$ is bounded.

14. This problem outlines an alternative proof of Theorem 8 which does not rely on the Cauchy criterion.

(a) Suppose that $a_n \geq 0$ for each n. Let $\{b_n\}$ be a rearrangement of $\{a_n\}$, and let $s_n = a_1 + \cdots + a_n$ and $t_n = b_1 + \cdots + b_n$. Show that for each n there is some m with $s_n \leq t_m$.

(b) Show that $\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n$ if $\sum_{n=1}^{\infty} b_n$ exists.

(c) Show that $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n$.

(d) Now replace the condition $a_n \geq 0$ by the hypothesis that $\sum_{n=1}^{\infty} a_n$ converges absolutely, using the second part of Theorem 5.

15. (a) Prove that if $\sum_{n=1}^{\infty} a_n$ converges absolutely, and $\{b_n\}$ is any subsequence of $\{a_n\}$, then $\sum_{n=1}^{\infty} b_n$ converges (absolutely).

(b) Show that this is false if $\sum_{n=1}^{\infty} a_n$ does not converge absolutely.

*(c) Prove that if $\sum_{n=1}^{\infty} a_n$ converges absolutely, then

$$\sum_{n=1}^{\infty} a_n = (a_1 + a_3 + a_5 + \cdots) + (a_2 + a_4 + a_6 + \cdots).$$

16. Prove that if $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then $\left|\sum_{n=1}^{\infty} a_n\right| \leq \sum_{n=1}^{\infty} |a_n|$.

***17.** Problem 19-43 shows that the improper integral $\int_0^{\infty} (\sin x)/x\, dx$ converges. Prove that $\int_0^{\infty} |(\sin x)/x|\, dx$ diverges.

***18.** Find a continuous function f with $f(x) \geq 0$ for all x such that $\int_0^{\infty} f(x)\, dx$ exists, but $\lim_{x\to\infty} f(x)$ does not exist.

28. For $b_n > 0$ we say that the infinite product $\prod_{n=1}^{\infty} b_n$ *converges* if the sequence $p_n = \prod_{i=1}^{n} b_i$ converges, and also $\lim_{n\to\infty} p_n \neq 0$.

(a) Prove that if $\prod_{n=1}^{\infty} b_n$ converges, then b_n approaches 1.

(b) Prove that $\prod_{n=1}^{\infty} b_n$ converges if and only if $\sum_{n=1}^{\infty} \log b_n$ converges.

(c) For $a_n \geq 0$, prove that $\prod_{n=1}^{\infty}(1 + a_n)$ converges if and only if $\sum_{n=1}^{\infty} a_n$ converges. Hint: Use Problem 27 for one implication, and a simple estimate for $\log(1 + a)$ for the reverse implication.

The remaining parts of this Problem show that the hypothesis $a_n \geq 0$ is needed.

(d) Use the Taylor series for $\log(1 + x)$ to show that for sufficiently small x we have

$$\tfrac{1}{4}x^2 \leq x - \log(1 + x) \leq \tfrac{3}{4}x^2.$$

Conclude that if all $a_n > -1$ and $\sum_{n=1}^{\infty} a_n$ converges, then the series $\sum_{n=1}^{\infty} \log(1 + a_n)$ converges if and only if $\sum_{n=1}^{\infty} {a_n}^2$ converges. Similarly, if all $a_n > -1$ and $\sum_{n=1}^{\infty} {a_n}^2$ converges, then $\sum_{n=1}^{\infty} \log(1 + a_n)$ converges if and only if $\sum_{n=1}^{\infty} a_n$ converges. Hint: Use the Cauchy criterion.

(e) Show that

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

converges, but

$$\prod_{n=2}^{\infty} \left(1 + \frac{(-1)^n}{\sqrt{n}}\right)$$

diverges.

(f) Consider the sequence

$$\{a_n\} = \underbrace{1, -\tfrac{1}{2},}_{\text{1 pair}}\ \underbrace{\tfrac{1}{3}, -\tfrac{1}{4}, \tfrac{1}{3}, -\tfrac{1}{4}, \tfrac{1}{3}, -\tfrac{1}{4},}_{\text{3 pairs}}\ \underbrace{\tfrac{1}{5}, -\tfrac{1}{6}, \tfrac{1}{5}, -\tfrac{1}{6}, \tfrac{1}{5}, -\tfrac{1}{6}, \ldots}_{\text{5 pairs}}$$

(compare Problem 26). Show that $\sum_{n=1}^{\infty} a_n$ diverges, but

$$\prod_{n=1}^{\infty}(1+a_n) = 1.$$

29. (a) Compute $\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right)$.

(b) Compute $\prod_{n=1}^{\infty}(1+x^{2^n})$ for $|x| < 1$.

30. The divergence of $\sum_{n=1}^{\infty} 1/n$ is related to the following remarkable fact: Any positive rational number x can be written as a *finite* sum of *distinct* numbers of the form $1/n$. The idea of the proof is shown by the following calculation for $\frac{27}{31}$: Since

$$\begin{aligned}
\tfrac{27}{31} - \tfrac{1}{2} &= \tfrac{23}{62}\\
\tfrac{23}{62} - \tfrac{1}{3} &= \tfrac{7}{186}\\
\tfrac{7}{186} &< \tfrac{1}{4}, \dots, \tfrac{1}{26}\\
\tfrac{7}{186} - \tfrac{1}{27} &= \tfrac{1}{1674}
\end{aligned}$$

we have

$$\tfrac{27}{31} = \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{27} + \tfrac{1}{1674}.$$

Notice that the numerators 23, 7, 1 of the differences are decreasing.

(a) Prove that if $1/(n+1) < x < 1/n$ for some n, then the numerator in this sort of calculation must always decrease; conclude that x can be written as a finite sum of distinct numbers $1/k$.

(b) Now prove the result for all x by using the divergence of $\sum_{n=1}^{\infty} 1/n$.