CHAPTER 31

Electromagnetic Oscillations and Alternating Current

31-1 *LC* OSCILLATIONS

Learning Objectives

After reading this module, you should be able to . . .

31.01 Sketch an LC oscillator and explain which quantities oscillate and what constitutes one period of the oscillation.

31.02 For an LC oscillator, sketch graphs of the potential difference across the capacitor and the current through the inductor as functions of time, and indicate the period T on each graph.

31.03 Explain the analogy between a block–spring oscillator and an LC oscillator.

31.04 For an LC oscillator, apply the relationships between the angular frequency ω (and the related frequency f and period T) and the values of the inductance and capacitance.

31.05 Starting with the energy of a block–spring system, explain the derivation of the differential equation for charge q in an LC oscillator and then identify the solution for $q(t)$.

31.06 For an LC oscillator, calculate the charge q on the capacitor for any given time and identify the amplitude Q of the charge oscillations.

31.07 Starting from the equation giving the charge $q(t)$ on the capacitor in an LC oscillator, find the current $i(t)$ in the inductor as a function of time.

31.08 For an LC oscillator, calculate the current i in the inductor for any given time and identify the amplitude I of the current oscillations.

31.09 For an LC oscillator, apply the relationship between the charge amplitude Q, the current amplitude I, and the angular frequency ω.

31.10 From the expressions for the charge q and the current i in an LC oscillator, find the magnetic field energy $U_B(t)$ and the electric field energy $U_E(t)$ and the total energy.

31.11 For an LC oscillator, sketch graphs of the magnetic field energy $U_B(t)$, the electric field energy $U_E(t)$, and the total energy, all as functions of time.

31.12 Calculate the maximum values of the magnetic field energy U_B and the electric field energy U_E and also calculate the total energy.

Key Ideas

- In an oscillating LC circuit, energy is shuttled periodically between the electric field of the capacitor and the magnetic field of the inductor; instantaneous values of the two forms of energy are

$$U_E = \frac{q^2}{2C} \quad \text{and} \quad U_B = \frac{Li^2}{2},$$

where q is the instantaneous charge on the capacitor and i is the instantaneous current through the inductor.

- The total energy $U\,(= U_E + U_B)$ remains constant.
- The principle of conservation of energy leads to

$$L\frac{d^2q}{dt^2} + \frac{1}{C}q = 0 \quad (LC \text{ oscillations})$$

as the differential equation of LC oscillations (with no resistance).

- The solution of this differential equation is

$$q = Q\cos(\omega t + \phi) \quad \text{(charge)},$$

in which Q is the charge amplitude (maximum charge on the capacitor) and the angular frequency ω of the oscillations is

$$\omega = \frac{1}{\sqrt{LC}}.$$

- The phase constant ϕ is determined by the initial conditions (at $t = 0$) of the system.
- The current i in the system at any time t is

$$i = -\omega Q\sin(\omega t + \phi) \quad \text{(current)},$$

in which ωQ is the current amplitude I.

What Is Physics?

We have explored the basic physics of electric and magnetic fields and how energy can be stored in capacitors and inductors. We next turn to the associated applied physics, in which the energy stored in one location can be transferred to another location so that it can be put to use. For example, energy produced at a power plant can show up at your home to run a computer. The total worth of this applied physics is now so high that its estimation is almost impossible. Indeed, modern civilization would be impossible without this applied physics.

In most parts of the world, electrical energy is transferred not as a direct current but as a sinusoidally oscillating current (alternating current, or ac). The challenge to both physicists and engineers is to design ac systems that transfer energy efficiently and to build appliances that make use of that energy. Our first step here is to study the oscillations in a circuit with inductance L and capacitance C.

LC Oscillations, Qualitatively

Of the three circuit elements, resistance R, capacitance C, and inductance L, we have so far discussed the series combinations RC (in Module 27-4) and RL (in Module 30-6). In these two kinds of circuit we found that the charge, current, and potential difference grow and decay exponentially. The time scale of the growth or decay is given by a *time constant* τ, which is either capacitive or inductive.

We now examine the remaining two-element circuit combination LC. You will see that in this case the charge, current, and potential difference do not decay exponentially with time but vary sinusoidally (with period T and angular frequency ω). The resulting oscillations of the capacitor's electric field and the inductor's magnetic field are said to be **electromagnetic oscillations.** Such a circuit is said to oscillate.

Parts a through h of Fig. 31-1 show succeeding stages of the oscillations in a simple LC circuit. From Eq. 25-21, the energy stored in the electric field of the

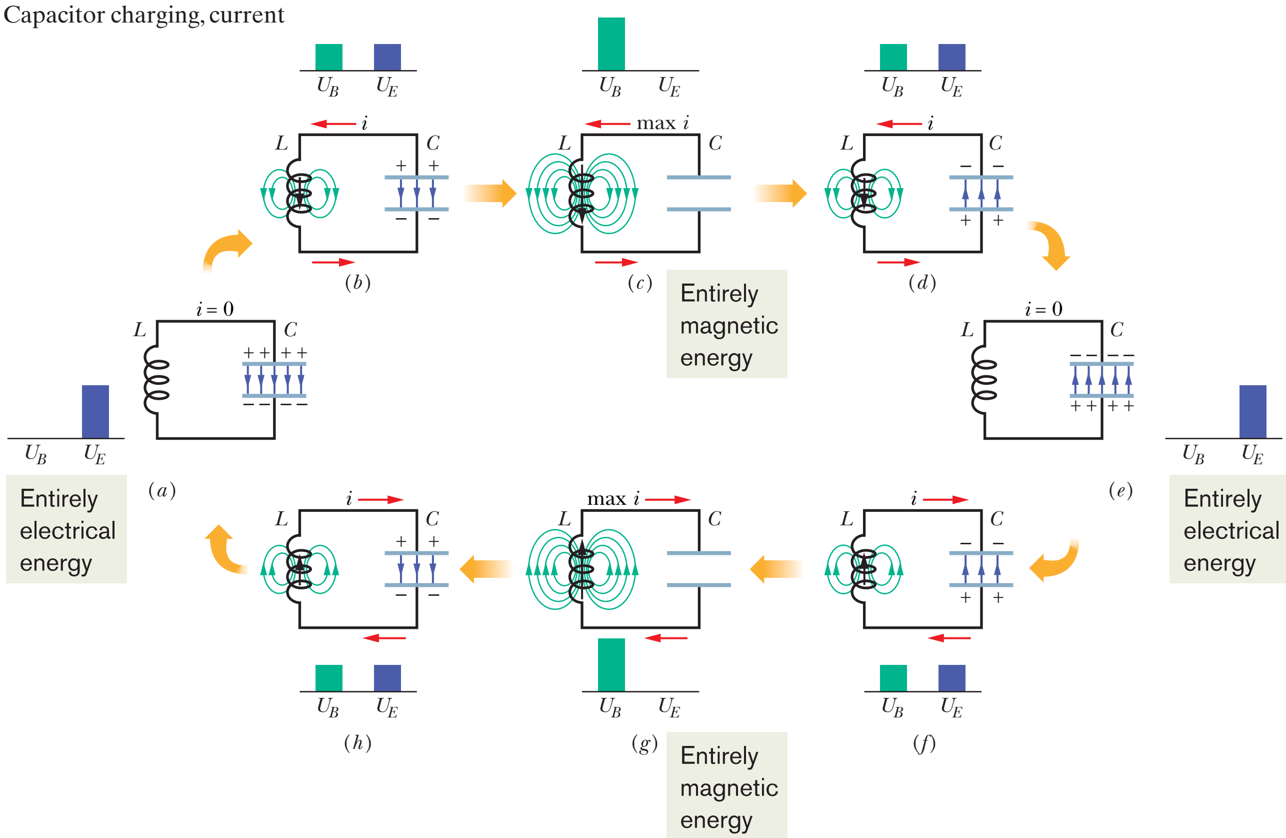


Figure 31-1 Eight stages in a single cycle of oscillation of a resistanceless LC circuit. The bar graphs by each figure show the stored magnetic and electrical energies. The magnetic field lines of the inductor and the electric field lines of the capacitor are shown. (a) Capacitor with maximum charge, no current. (b) Capacitor discharging, current increasing. (c) Capacitor fully discharged, current maximum. (d) Capacitor charging but with polarity opposite that in (a), current decreasing. (e) Capacitor with maximum charge having polarity opposite that in (a), no current. (f) Capacitor discharging, current increasing with direction opposite that in (b). (g) Capacitor fully discharged, current maximum. (h) Capacitor charging, current decreasing.

capacitor at any time is

$$U_E = \frac{q^2}{2C}, \tag{31-1}$$

where q is the charge on the capacitor at that time. From Eq. 30-49, the energy stored in the magnetic field of the inductor at any time is

$$U_B = \frac{Li^2}{2}, \tag{31-2}$$

where i is the current through the inductor at that time.

We now adopt the convention of representing *instantaneous values* of the electrical quantities of a sinusoidally oscillating circuit with small letters, such as q, and the *amplitudes* of those quantities with capital letters, such as Q. With this convention in mind, let us assume that initially the charge q on the capacitor in Fig. 31-1 is at its maximum value Q and that the current i through the inductor is zero. This initial state of the circuit is shown in Fig. 31-1*a*. The bar graphs for energy included there indicate that at this instant, with zero current through the inductor and maximum charge on the capacitor, the energy U_B of the magnetic field is zero and the energy U_E of the electric field is a maximum. As the circuit oscillates, energy shifts back and forth from one type of stored energy to the other, but the total amount is conserved.

The capacitor now starts to discharge through the inductor, positive charge carriers moving counterclockwise, as shown in Fig. 31-1*b*. This means that a current i, given by dq/dt and pointing down in the inductor, is established. As the capacitor's charge decreases, the energy stored in the electric field within the capacitor also decreases. This energy is transferred to the magnetic field that appears around the inductor because of the current i that is building up there. Thus, the electric field decreases and the magnetic field builds up as energy is transferred from the electric field to the magnetic field.

The capacitor eventually loses all its charge (Fig. 31-1*c*) and thus also loses its electric field and the energy stored in that field. The energy has then been fully transferred to the magnetic field of the inductor. The magnetic field is then at its maximum magnitude, and the current through the inductor is then at its maximum value I.

Although the charge on the capacitor is now zero, the counterclockwise current must continue because the inductor does not allow it to change suddenly to zero. The current continues to transfer positive charge from the top plate to the bottom plate through the circuit (Fig. 31-1*d*). Energy now flows from the inductor back to the capacitor as the electric field within the capacitor builds up again. The current gradually decreases during this energy transfer. When, eventually, the energy has been transferred completely back to the capacitor (Fig. 31-1*e*), the current has decreased to zero (momentarily). The situation of Fig. 31-1*e* is like the initial situation, except that the capacitor is now charged oppositely.

The capacitor then starts to discharge again but now with a clockwise current (Fig. 31-1*f*). Reasoning as before, we see that the clockwise current builds to a maximum (Fig. 31-1*g*) and then decreases (Fig. 31-1*h*), until the circuit eventually returns to its initial situation (Fig. 31-1*a*). The process then repeats at some frequency f and thus at an angular frequency $\omega = 2\pi f$. In the ideal LC circuit with no resistance, there are no energy transfers other than that between the electric field of the capacitor and the magnetic field of the inductor. Because of the conservation of energy, the oscillations continue indefinitely. The oscillations need not begin with the energy all in the electric field; the initial situation could be any other stage of the oscillation.

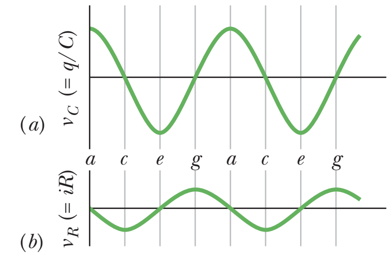


Figure 31-2 (*a*) The potential difference across the capacitor in the circuit of Fig. 31-1 as a function of time. This quantity is proportional to the charge on the capacitor. (*b*) A potential proportional to the current in the circuit of Fig. 31-1. The letters refer to the correspondingly labeled oscillation stages in Fig. 31-1.

To determine the charge q on the capacitor as a function of time, we can put in a voltmeter to measure the time-varying potential difference (or *voltage*) v_C that exists across the capacitor C. From Eq. 25-1 we can write

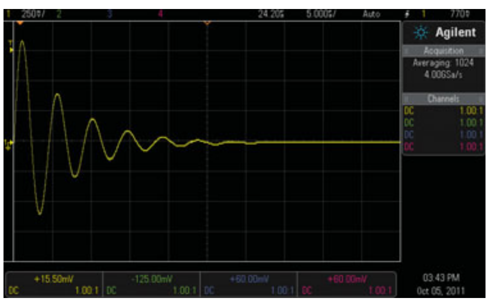
$$v_C = \left(\frac{1}{C}\right) q,$$

which allows us to find q. To measure the current, we can connect a small resistance R in series with the capacitor and inductor and measure the time-varying potential difference v_R across it; v_R is proportional to i through the relation

$$v_R = iR.$$

We assume here that R is so small that its effect on the behavior of the circuit is negligible. The variations in time of v_C and v_R, and thus of q and i, are shown in Fig. 31-2. All four quantities vary sinusoidally.

In an actual LC circuit, the oscillations will not continue indefinitely because there is always some resistance present that will drain energy from the electric and magnetic fields and dissipate it as thermal energy (the circuit may become warmer). The oscillations, once started, will die away as Fig. 31-3 suggests. Compare this figure with Fig. 15-17, which shows the decay of mechanical oscillations caused by frictional damping in a block–spring system.



Courtesy Agilent Technologies

Figure 31-3 An oscilloscope trace showing how the oscillations in an RLC circuit actually die away because energy is dissipated in the resistor as thermal energy.

Checkpoint 1

A charged capacitor and an inductor are connected in series at time $t = 0$. In terms of the period T of the resulting oscillations, determine how much later the following reach their maximum value: (a) the charge on the capacitor; (b) the voltage across the capacitor, with its original polarity; (c) the energy stored in the electric field; and (d) the current.

The Electrical–Mechanical Analogy

Let us look a little closer at the analogy between the oscillating LC system of Fig. 31-1 and an oscillating block–spring system. Two kinds of energy are involved in the block–spring system. One is potential energy of the compressed or extended spring; the other is kinetic energy of the moving block. These two energies are given by the formulas in the first energy column in Table 31-1.

Table 31-1 Comparison of the Energy in Two Oscillating Systems

Block–Spring System		*LC* Oscillator	
Element	Energy	Element	Energy
Spring	Potential, $\frac{1}{2}kx^2$	Capacitor	Electrical, $\frac{1}{2}(1/C)q^2$
Block	Kinetic, $\frac{1}{2}mv^2$	Inductor	Magnetic, $\frac{1}{2}Li^2$
$v = dx/dt$		$i = dq/dt$	

The table also shows, in the second energy column, the two kinds of energy involved in LC oscillations. By looking across the table, we can see an analogy between the forms of the two pairs of energies—the mechanical energies of the block–spring system and the electromagnetic energies of the LC oscillator. The equations for v and i at the bottom of the table help us see the details of the analogy. They tell us that q corresponds to x and i corresponds to v (in both equations, the former is differentiated to obtain the latter). These correspondences then suggest that, in the energy expressions, $1/C$ corresponds to k and L corresponds to m. Thus,

$$\begin{array}{lcl} q \text{ corresponds to } x, & & 1/C \text{ corresponds to } k, \\ i \text{ corresponds to } v, & \text{and} & L \text{ corresponds to } m. \end{array}$$

These correspondences suggest that in an LC oscillator, the capacitor is mathematically like the spring in a block–spring system and the inductor is like the block.

In Module 15-1 we saw that the angular frequency of oscillation of a (frictionless) block–spring system is

$$\omega = \sqrt{\frac{k}{m}} \quad \text{(block–spring system).} \tag{31-3}$$

The correspondences listed above suggest that to find the angular frequency of oscillation for an ideal (resistanceless) LC circuit, k should be replaced by $1/C$ and m by L, yielding

$$\omega = \frac{1}{\sqrt{LC}} \quad (LC \text{ circuit}). \tag{31-4}$$

LC Oscillations, Quantitatively

Here we want to show explicitly that Eq. 31-4 for the angular frequency of LC oscillations is correct. At the same time, we want to examine even more closely the analogy between LC oscillations and block–spring oscillations. We start by extending somewhat our earlier treatment of the mechanical block–spring oscillator.

The Block–Spring Oscillator

We analyzed block–spring oscillations in Chapter 15 in terms of energy transfers and did not—at that early stage—derive the fundamental differential equation that governs those oscillations. We do so now.

We can write, for the total energy U of a block–spring oscillator at any instant,

$$U = U_b + U_s = \tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2, \tag{31-5}$$

where U_b and U_s are, respectively, the kinetic energy of the moving block and the potential energy of the stretched or compressed spring. If there is no friction—which we assume—the total energy U remains constant with time, even though v and x vary. In more formal language, $dU/dt = 0$. This leads to

$$\frac{dU}{dt} = \frac{d}{dt}\left(\tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2\right) = mv\frac{dv}{dt} + kx\frac{dx}{dt} = 0. \tag{31-6}$$

Substituting $v = dx/dt$ and $dv/dt = d^2x/dt^2$, we find

$$m\frac{d^2x}{dt^2} + kx = 0 \quad \text{(block–spring oscillations).} \tag{31-7}$$

Equation 31-7 is the fundamental *differential equation* that governs the frictionless block–spring oscillations.

The general solution to Eq. 31-7 is (as we saw in Eq. 15-3)

$$x = X\cos(\omega t + \phi) \quad \text{(displacement),} \tag{31-8}$$

in which X is the amplitude of the mechanical oscillations (x_m in Chapter 15), ω is the angular frequency of the oscillations, and ϕ is a phase constant.

The *LC* Oscillator

Now let us analyze the oscillations of a resistanceless LC circuit, proceeding exactly as we just did for the block–spring oscillator. The total energy U present at any instant in an oscillating LC circuit is given by

$$U = U_B + U_E = \frac{Li^2}{2} + \frac{q^2}{2C}, \tag{31-9}$$

in which U_B is the energy stored in the magnetic field of the inductor and U_E is the energy stored in the electric field of the capacitor. Since we have assumed the circuit resistance to be zero, no energy is transferred to thermal energy and U remains constant with time. In more formal language, dU/dt must be zero. This leads to

$$\frac{dU}{dt} = \frac{d}{dt}\left(\frac{Li^2}{2} + \frac{q^2}{2C}\right) = Li\frac{di}{dt} + \frac{q}{C}\frac{dq}{dt} = 0. \tag{31-10}$$

However, $i = dq/dt$ and $di/dt = d^2q/dt^2$. With these substitutions, Eq. 31-10 becomes

$$L\frac{d^2q}{dt^2} + \frac{1}{C}q = 0 \quad (LC \text{ oscillations}). \tag{31-11}$$

This is the *differential equation* that describes the oscillations of a resistanceless LC circuit. Equations 31-11 and 31-7 are exactly of the same mathematical form.

Charge and Current Oscillations

Since the differential equations are mathematically identical, their solutions must also be mathematically identical. Because q corresponds to x, we can write the general solution of Eq. 31-11, by analogy to Eq. 31-8, as

$$q = Q\cos(\omega t + \phi) \quad \text{(charge)}, \tag{31-12}$$

where Q is the amplitude of the charge variations, ω is the angular frequency of the electromagnetic oscillations, and ϕ is the phase constant. Taking the first derivative of Eq. 31-12 with respect to time gives us the current:

$$i = \frac{dq}{dt} = -\omega Q\sin(\omega t + \phi) \quad \text{(current)}. \tag{31-13}$$

The amplitude I of this sinusoidally varying current is

$$I = \omega Q, \tag{31-14}$$

and so we can rewrite Eq. 31-13 as

$$i = -I\sin(\omega t + \phi). \tag{31-15}$$

Angular Frequencies

We can test whether Eq. 31-12 is a solution of Eq. 31-11 by substituting Eq. 31-12 and its second derivative with respect to time into Eq. 31-11. The first derivative of Eq. 31-12 is Eq. 31-13. The second derivative is then

$$\frac{d^2q}{dt^2} = -\omega^2 Q\cos(\omega t + \phi).$$

Substituting for q and d^2q/dt^2 in Eq. 31-11, we obtain

$$-L\omega^2 Q\cos(\omega t + \phi) + \frac{1}{C}Q\cos(\omega t + \phi) = 0.$$

Canceling $Q\cos(\omega t+\phi)$ and rearranging lead to

$$\omega = \frac{1}{\sqrt{LC}}.$$

Thus, Eq. 31-12 is indeed a solution of Eq. 31-11 if ω has the constant value $1/\sqrt{LC}$. Note that this expression for ω is exactly that given by Eq. 31-4.

The phase constant ϕ in Eq. 31-12 is determined by the conditions that exist at any certain time—say, $t=0$. If the conditions yield $\phi=0$ at $t=0$, Eq. 31-12 requires that $q=Q$ and Eq. 31-13 requires that $i=0$; these are the initial conditions represented by Fig. 31-1*a*.

The electrical and magnetic energies vary but the total is constant.

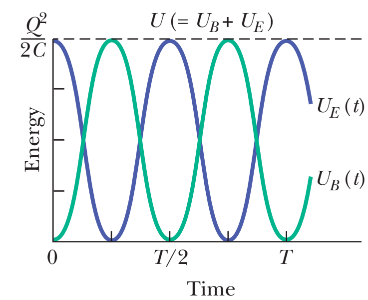


Figure 31-4 The stored magnetic energy and electrical energy in the circuit of Fig. 31-1 as a function of time. Note that their sum remains constant. T is the period of oscillation.

Electrical and Magnetic Energy Oscillations

The electrical energy stored in the *LC* circuit at time *t* is, from Eqs. 31-1 and 31-12,

$$U_E = \frac{q^2}{2C} = \frac{Q^2}{2C}\cos^2(\omega t+\phi). \tag{31-16}$$

The magnetic energy is, from Eqs. 31-2 and 31-13,

$$U_B = \tfrac{1}{2}Li^2 = \tfrac{1}{2}L\omega^2Q^2\sin^2(\omega t+\phi).$$

Substituting for ω from Eq. 31-4 then gives us

$$U_B = \frac{Q^2}{2C}\sin^2(\omega t+\phi). \tag{31-17}$$

Figure 31-4 shows plots of $U_E(t)$ and $U_B(t)$ for the case of $\phi=0$. Note that

1. The maximum values of U_E and U_B are both $Q^2/2C$.
2. At any instant the sum of U_E and U_B is equal to $Q^2/2C$, a constant.
3. When U_E is maximum, U_B is zero, and conversely.

Checkpoint 2

A capacitor in an *LC* oscillator has a maximum potential difference of 17 V and a maximum energy of 160 μJ. When the capacitor has a potential difference of 5 V and an energy of 10 μJ, what are (a) the emf across the inductor and (b) the energy stored in the magnetic field?

Sample Problem 31.01 *LC* oscillator: potential change, rate of current change

A 1.5 μF capacitor is charged to 57 V by a battery, which is then removed. At time $t=0$, a 12 mH coil is connected in series with the capacitor to form an *LC* oscillator (Fig. 31-1).

(a) What is the potential difference $v_L(t)$ across the inductor as a function of time?

KEY IDEAS

(1) The current and potential differences of the circuit (both the potential difference of the capacitor and the potential difference of the coil) undergo sinusoidal oscillations. (2) We can still apply the loop rule to these oscillating potential differences, just as we did for the nonoscillating circuits of Chapter 27.

Calculations: At any time *t* during the oscillations, the loop rule and Fig. 31-1 give us

$$v_L(t) = v_C(t); \tag{31-18}$$

that is, the potential difference v_L across the inductor must always be equal to the potential difference v_C across the capacitor, so that the net potential difference around the circuit is zero. Thus, we will find $v_L(t)$ if we can find $v_C(t)$, and we can find $v_C(t)$ from $q(t)$ with Eq. 25-1 ($q=CV$).

Because the potential difference $v_C(t)$ is maximum when the oscillations begin at time $t=0$, the charge q on the capacitor must also be maximum then. Thus, phase constant ϕ must be zero; so Eq. 31-12 gives us

$$q = Q\cos\omega t. \tag{31-19}$$

(Note that this cosine function does indeed yield maximum q (= Q) when $t = 0$.) To get the potential difference $v_C(t)$, we divide both sides of Eq. 31-19 by C to write

$$\frac{q}{C} = \frac{Q}{C}\cos \omega t,$$

and then use Eq. 25-1 to write

$$v_C = V_C \cos \omega t. \tag{31-20}$$

Here, V_C is the amplitude of the oscillations in the potential difference v_C across the capacitor.

Next, substituting $v_C = v_L$ from Eq. 31-18, we find

$$v_L = V_C \cos \omega t. \tag{31-21}$$

We can evaluate the right side of this equation by first noting that the amplitude V_C is equal to the initial (maximum) potential difference of 57 V across the capacitor. Then we find ω with Eq. 31-4:

$$\omega = \frac{1}{\sqrt{LC}} = \frac{1}{[(0.012\text{ H})(1.5 \times 10^{-6}\text{ F})]^{0.5}}$$
$$= 7454\text{ rad/s} \approx 7500\text{ rad/s}.$$

Thus, Eq. 31-21 becomes

$$v_L = (57\text{ V})\cos(7500\text{ rad/s})t. \qquad \text{(Answer)}$$

(b) What is the maximum rate $(di/dt)_{max}$ at which the current i changes in the circuit?

KEY IDEA

With the charge on the capacitor oscillating as in Eq. 31-12, the current is in the form of Eq. 31-13. Because $\phi = 0$, that equation gives us

$$i = -\omega Q \sin \omega t.$$

Calculations: Taking the derivative, we have

$$\frac{di}{dt} = \frac{d}{dt}(-\omega Q \sin \omega t) = -\omega^2 Q \cos \omega t.$$

We can simplify this equation by substituting CV_C for Q (because we know C and V_C but not Q) and $1/\sqrt{LC}$ for ω according to Eq. 31-4. We get

$$\frac{di}{dt} = -\frac{1}{LC}CV_C \cos \omega t = -\frac{V_C}{L}\cos \omega t.$$

This tells us that the current changes at a varying (sinusoidal) rate, with its maximum rate of change being

$$\frac{V_C}{L} = \frac{57\text{ V}}{0.012\text{ H}} = 4750\text{ A/s} \approx 4800\text{ A/s}. \qquad \text{(Answer)}$$

WILEY PLUS Additional examples, video, and practice available at *WileyPLUS*

31-2 DAMPED OSCILLATIONS IN AN *RLC* CIRCUIT

Learning Objectives

After reading this module, you should be able to . . .

31.13 Draw the schematic of a damped RLC circuit and explain why the oscillations are damped.

31.14 Starting with the expressions for the field energies and the rate of energy loss in a damped RLC circuit, write the differential equation for the charge q on the capacitor.

31.15 For a damped RLC circuit, apply the expression for charge $q(t)$.

31.16 Identify that in a damped RLC circuit, the charge amplitude and the amplitude of the electric field energy decrease exponentially with time.

31.17 Apply the relationship between the angular frequency ω' of a given damped RLC oscillator and the angular frequency ω of the circuit if R is removed.

31.18 For a damped RLC circuit, apply the expression for the electric field energy U_E as a function of time.

Key Ideas

- Oscillations in an LC circuit are damped when a dissipative element R is also present in the circuit. Then

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = 0 \quad (RLC\text{ circuit}).$$

- The solution of this differential equation is

$$q = Qe^{-Rt/2L}\cos(\omega' t + \phi),$$

where $\omega' = \sqrt{\omega^2 - (R/2L)^2}$.

We consider only situations with small R and thus small damping; then $\omega' \approx \omega$.

Damped Oscillations in an *RLC* Circuit

A circuit containing resistance, inductance, and capacitance is called an *RLC circuit.* We shall here discuss only *series RLC circuits* like that shown in Fig. 31-5. With a resistance R present, the total *electromagnetic energy* U of the circuit (the sum of the electrical energy and magnetic energy) is no longer constant; instead, it decreases with time as energy is transferred to thermal energy in the resistance. Because of this loss of energy, the oscillations of charge, current, and potential difference continuously decrease in amplitude, and the oscillations are said to be *damped,* just as with the damped block–spring oscillator of Module 15-5.

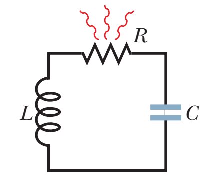


Figure 31-5 A series RLC circuit. As the charge contained in the circuit oscillates back and forth through the resistance, electromagnetic energy is dissipated as thermal energy, damping (decreasing the amplitude of) the oscillations.

To analyze the oscillations of this circuit, we write an equation for the total electromagnetic energy U in the circuit at any instant. Because the resistance does not store electromagnetic energy, we can use Eq. 31-9:

$$U = U_B + U_E = \frac{Li^2}{2} + \frac{q^2}{2C}. \tag{31-22}$$

Now, however, this total energy decreases as energy is transferred to thermal energy. The rate of that transfer is, from Eq. 26-27,

$$\frac{dU}{dt} = -i^2R, \tag{31-23}$$

where the minus sign indicates that U decreases. By differentiating Eq. 31-22 with respect to time and then substituting the result in Eq. 31-23, we obtain

$$\frac{dU}{dt} = Li\frac{di}{dt} + \frac{q}{C}\frac{dq}{dt} = -i^2R.$$

Substituting dq/dt for i and d^2q/dt^2 for di/dt, we obtain

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = 0 \quad (RLC\text{ circuit}), \tag{31-24}$$

which is the differential equation for damped oscillations in an RLC circuit.

Charge Decay. The solution to Eq. 31-24 is

$$q = Qe^{-Rt/2L}\cos(\omega' t + \phi), \tag{31-25}$$

in which

$$\omega' = \sqrt{\omega^2 - (R/2L)^2}, \tag{31-26}$$

where $\omega = 1/\sqrt{LC}$, as with an undamped oscillator. Equation 31-25 tells us how the charge on the capacitor oscillates in a damped RLC circuit; that equation is the electromagnetic counterpart of Eq. 15-42, which gives the displacement of a damped block–spring oscillator.

Equation 31-25 describes a sinusoidal oscillation (the cosine function) with an *exponentially decaying amplitude* $Qe^{-Rt/2L}$ (the factor that multiplies the cosine). The angular frequency ω' of the damped oscillations is always less than the angular frequency ω of the undamped oscillations; however, we shall here consider only situations in which R is small enough for us to replace ω' with ω.

Energy Decay. Let us next find an expression for the total electromagnetic energy U of the circuit as a function of time. One way to do so is to monitor the energy of the electric field in the capacitor, which is given by Eq. 31-1 ($U_E = q^2/2C$). By substituting Eq. 31-25 into Eq. 31-1, we obtain

$$U_E = \frac{q^2}{2C} = \frac{[Qe^{-Rt/2L}\cos(\omega' t + \phi)]^2}{2C} = \frac{Q^2}{2C}e^{-Rt/L}\cos^2(\omega' t + \phi). \tag{31-27}$$

Thus, the energy of the electric field oscillates according to a cosine-squared term, and the amplitude of that oscillation decreases exponentially with time.

Sample Problem 31.02 Damped *RLC* circuit: charge amplitude

A series *RLC* circuit has inductance $L = 12$ mH, capacitance $C = 1.6\ \mu$F, and resistance $R = 1.5\ \Omega$ and begins to oscillate at time $t = 0$.

(a) At what time t will the amplitude of the charge oscillations in the circuit be 50% of its initial value? (Note that we do not know that initial value.)

KEY IDEA

The amplitude of the charge oscillations decreases exponentially with time t: According to Eq. 31-25, the charge amplitude at any time t is $Qe^{-Rt/2L}$, in which Q is the amplitude at time $t = 0$.

Calculations: We want the time when the charge amplitude has decreased to $0.50Q$—that is, when

$$Qe^{-Rt/2L} = 0.50Q.$$

We can now cancel Q (which also means that we can answer the question without knowing the initial charge). Taking the natural logarithms of both sides (to eliminate the exponential function), we have

$$-\frac{Rt}{2L} = \ln 0.50.$$

Solving for t and then substituting given data yield

$$t = -\frac{2L}{R}\ln 0.50 = -\frac{(2)(12 \times 10^{-3}\ \text{H})(\ln 0.50)}{1.5\ \Omega}$$

$$= 0.0111\ \text{s} \approx 11\ \text{ms}. \qquad \text{(Answer)}$$

(b) How many oscillations are completed within this time?

KEY IDEA

The time for one complete oscillation is the period $T = 2\pi/\omega$, where the angular frequency for *LC* oscillations is given by Eq. 31-4 ($\omega = 1/\sqrt{LC}$).

Calculation: In the time interval $\Delta t = 0.0111$ s, the number of complete oscillations is

$$\frac{\Delta t}{T} = \frac{\Delta t}{2\pi\sqrt{LC}}$$

$$= \frac{0.0111\ \text{s}}{2\pi[(12 \times 10^{-3}\ \text{H})(1.6 \times 10^{-6}\ \text{F})]^{1/2}} \approx 13. \qquad \text{(Answer)}$$

Thus, the amplitude decays by 50% in about 13 complete oscillations. This damping is less severe than that shown in Fig. 31-3, where the amplitude decays by a little more than 50% in one oscillation.

31-3 FORCED OSCILLATIONS OF THREE SIMPLE CIRCUITS

Learning Objectives

After reading this module, you should be able to . . .

31.19 Distinguish alternating current from direct current.

31.20 For an ac generator, write the emf as a function of time, identifying the emf amplitude and driving angular frequency.

31.21 For an ac generator, write the current as a function of time, identifying its amplitude and its phase constant with respect to the emf.

31.22 Draw a schematic diagram of a (series) *RLC* circuit that is driven by a generator.

31.23 Distinguish driving angular frequency ω_d from natural angular frequency ω.

31.24 In a driven (series) *RLC* circuit, identify the conditions for resonance and the effect of resonance on the current amplitude.

31.25 For each of the three basic circuits (purely resistive load, purely capacitive load, and purely inductive load), draw the circuit and sketch graphs and phasor diagrams for voltage $v(t)$ and current $i(t)$.

31.26 For the three basic circuits, apply equations for voltage $v(t)$ and current $i(t)$.

31.27 On a phasor diagram for each of the basic circuits, identify angular speed, amplitude, projection on the vertical axis, and rotation angle.

31.28 For each basic circuit, identify the phase constant, and interpret it in terms of the relative orientations of the current phasor and voltage phasor and also in terms of leading and lagging.

31.29 Apply the mnemonic "*ELI* positively is the *ICE* man."

31.30 For each basic circuit, apply the relationships between the voltage amplitude V and the current amplitude I.

31.31 Calculate capacitive reactance X_C and inductive reactance X_L.

Key Ideas

- A series RLC circuit may be set into forced oscillation at a driving angular frequency ω_d by an external alternating emf

$$\mathscr{E} = \mathscr{E}_m \sin \omega_d t.$$

- The current driven in the circuit is

$$i = I \sin(\omega_d t - \phi),$$

where ϕ is the phase constant of the current.

- The alternating potential difference across a resistor has amplitude $V_R = IR$; the current is in phase with the potential difference.
- For a capacitor, $V_C = IX_C$, in which $X_C = 1/\omega_d C$ is the capacitive reactance; the current here leads the potential difference by $90°$ ($\phi = -90° = -\pi/2$ rad).
- For an inductor, $V_L = IX_L$, in which $X_L = \omega_d L$ is the inductive reactance; the current here lags the potential difference by $90°$ ($\phi = +90° = +\pi/2$ rad).

Alternating Current

The oscillations in an RLC circuit will not damp out if an external emf device supplies enough energy to make up for the energy dissipated as thermal energy in the resistance R. Circuits in homes, offices, and factories, including countless RLC circuits, receive such energy from local power companies. In most countries the energy is supplied via oscillating emfs and currents—the current is said to be an **alternating current,** or **ac** for short. (The nonoscillating current from a battery is said to be a **direct current,** or **dc**.) These oscillating emfs and currents vary sinusoidally with time, reversing direction (in North America) 120 times per second and thus having frequency $f = 60$ Hz.

Electron Oscillations. At first sight this may seem to be a strange arrangement. We have seen that the drift speed of the conduction electrons in household wiring may typically be 4×10^{-5} m/s. If we now reverse their direction every $\frac{1}{120}$ s, such electrons can move only about 3×10^{-7} m in a half-cycle. At this rate, a typical electron can drift past no more than about 10 atoms in the wiring before it is required to reverse its direction. How, you may wonder, can the electron ever get anywhere?

Although this question may be worrisome, it is a needless concern. The conduction electrons do not have to "get anywhere." When we say that the current in a wire is one ampere, we mean that charge passes through any plane cutting across that wire at the rate of one coulomb per second. The speed at which the charge carriers cross that plane does not matter directly; one ampere may correspond to many charge carriers moving very slowly or to a few moving very rapidly. Furthermore, the signal to the electrons to reverse directions—which originates in the alternating emf provided by the power company's generator—is propagated along the conductor at a speed close to that of light. All electrons, no matter where they are located, get their reversal instructions at about the same instant. Finally, we note that for many devices, such as lightbulbs and toasters, the direction of motion is unimportant as long as the electrons do move so as to transfer energy to the device via collisions with atoms in the device.

Why ac? The basic advantage of alternating current is this: *As the current alternates, so does the magnetic field that surrounds the conductor.* This makes possible the use of Faraday's law of induction, which, among other things, means that we can step up (increase) or step down (decrease) the magnitude of an alternating potential difference at will, using a device called a transformer, as we shall discuss later. Moreover, alternating current is more readily adaptable to rotating machinery such as generators and motors than is (nonalternating) direct current.

Emf and Current. Figure 31-6 shows a simple model of an ac generator. As the conducting loop is forced to rotate through the external magnetic field $\vec{B}$, a sinusoidally oscillating emf $\mathscr{E}$ is induced in the loop:

$$\mathscr{E} = \mathscr{E}_m \sin \omega_d t. \tag{31-28}$$

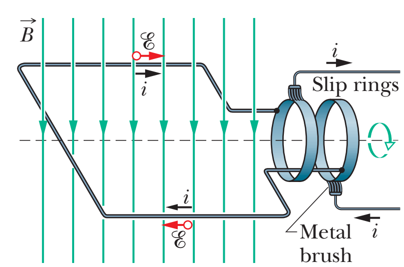


Figure 31-6 The basic mechanism of an alternating-current generator is a conducting loop rotated in an external magnetic field. In practice, the alternating emf induced in a coil of many turns of wire is made accessible by means of slip rings attached to the rotating loop. Each ring is connected to one end of the loop wire and is electrically connected to the rest of the generator circuit by a conducting brush against which the ring slips as the loop (and it) rotates.

The *angular frequency* ω_d of the emf is equal to the angular speed with which the loop rotates in the magnetic field, the *phase* of the emf is $\omega_d t$, and the *amplitude* of the emf is $\mathscr{E}_m$ (where the subscript stands for maximum). When the rotating loop is part of a closed conducting path, this emf produces (*drives*) a sinusoidal (alternating) current along the path with the same angular frequency ω_d, which then is called the **driving angular frequency.** We can write the current as

$$i = I \sin(\omega_d t - \phi), \tag{31-29}$$

in which I is the amplitude of the driven current. (The phase $\omega_d t - \phi$ of the current is traditionally written with a minus sign instead of as $\omega_d t + \phi$.) We include a phase constant ϕ in Eq. 31-29 because the current i may not be in phase with the emf $\mathscr{E}$. (As you will see, the phase constant depends on the circuit to which the generator is connected.) We can also write the current i in terms of the **driving frequency** f_d of the emf, by substituting $2\pi f_d$ for ω_d in Eq. 31-29.

Forced Oscillations

We have seen that once started, the charge, potential difference, and current in both undamped LC circuits and damped RLC circuits (with small enough R) oscillate at angular frequency $\omega = 1/\sqrt{LC}$. Such oscillations are said to be *free oscillations* (free of any external emf), and the angular frequency ω is said to be the circuit's **natural angular frequency.**

When the external alternating emf of Eq. 31-28 is connected to an RLC circuit, the oscillations of charge, potential difference, and current are said to be *driven oscillations* or *forced oscillations.* These oscillations always occur at the driving angular frequency ω_d:

> Whatever the natural angular frequency ω of a circuit may be, forced oscillations of charge, current, and potential difference in the circuit always occur at the driving angular frequency ω_d.

However, as you will see in Module 31-4, the amplitudes of the oscillations very much depend on how close ω_d is to ω. When the two angular frequencies match—a condition known as **resonance**—the amplitude I of the current in the circuit is maximum.

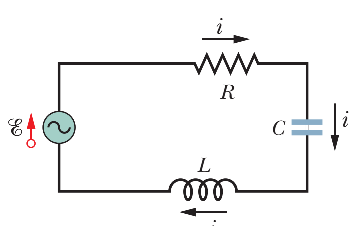


Figure 31-7 A single-loop circuit containing a resistor, a capacitor, and an inductor. A generator, represented by a sine wave in a circle, produces an alternating emf that establishes an alternating current; the directions of the emf and current are indicated here at only one instant.

Three Simple Circuits

Later in this chapter, we shall connect an external alternating emf device to a series RLC circuit as in Fig. 31-7. We shall then find expressions for the amplitude I and phase constant ϕ of the sinusoidally oscillating current in terms of the amplitude $\mathscr{E}_m$ and angular frequency ω_d of the external emf. First, let's consider three simpler circuits, each having an external emf and only one other circuit element: R, C, or L. We start with a resistive element (a purely *resistive load*).

A Resistive Load

Figure 31-8 shows a circuit containing a resistance element of value R and an ac generator with the alternating emf of Eq. 31-28. By the loop rule, we have

$$\mathscr{E} - v_R = 0.$$

With Eq. 31-28, this gives us

$$v_R = \mathscr{E}_m \sin \omega_d t.$$

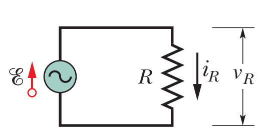


Figure 31-8 A resistor is connected across an alternating-current generator.

Because the amplitude V_R of the alternating potential difference (or voltage) across the resistance is equal to the amplitude $\mathscr{E}_m$ of the alternating emf, we can write this as

$$v_R = V_R \sin \omega_d t. \quad (31\text{-}30)$$

From the definition of resistance ($R = V/i$), we can now write the current i_R in the resistance as

$$i_R = \frac{v_R}{R} = \frac{V_R}{R} \sin \omega_d t. \quad (31\text{-}31)$$

From Eq. 31-29, we can also write this current as

$$i_R = I_R \sin(\omega_d t - \phi), \quad (31\text{-}32)$$

where I_R is the amplitude of the current i_R in the resistance. Comparing Eqs. 31-31 and 31-32, we see that for a purely resistive load the phase constant $\phi = 0°$. We also see that the voltage amplitude and current amplitude are related by

$$V_R = I_R R \quad \text{(resistor).} \quad (31\text{-}33)$$

Although we found this relation for the circuit of Fig. 31-8, it applies to any resistance in any ac circuit.

By comparing Eqs. 31-30 and 31-31, we see that the time-varying quantities v_R and i_R are both functions of $\sin \omega_d t$ with $\phi = 0°$. Thus, these two quantities are *in phase,* which means that their corresponding maxima (and minima) occur at the same times. Figure 31-9*a*, which is a plot of $v_R(t)$ and $i_R(t)$, illustrates this fact. Note that v_R and i_R do not decay here because the generator supplies energy to the circuit to make up for the energy dissipated in R.

The time-varying quantities v_R and i_R can also be represented geometrically by *phasors.* Recall from Module 16-6 that phasors are vectors that rotate around an origin. Those that represent the voltage across and current in the resistor of Fig. 31-8 are shown in Fig. 31-9*b* at an arbitrary time t. Such phasors have the following properties:

Angular speed: Both phasors rotate counterclockwise about the origin with an angular speed equal to the angular frequency ω_d of v_R and i_R.

Length: The length of each phasor represents the amplitude of the alternating quantity: V_R for the voltage and I_R for the current.

Projection: The projection of each phasor on the *vertical* axis represents the value of the alternating quantity at time t: v_R for the voltage and i_R for the current.

Rotation angle: The rotation angle of each phasor is equal to the phase of the

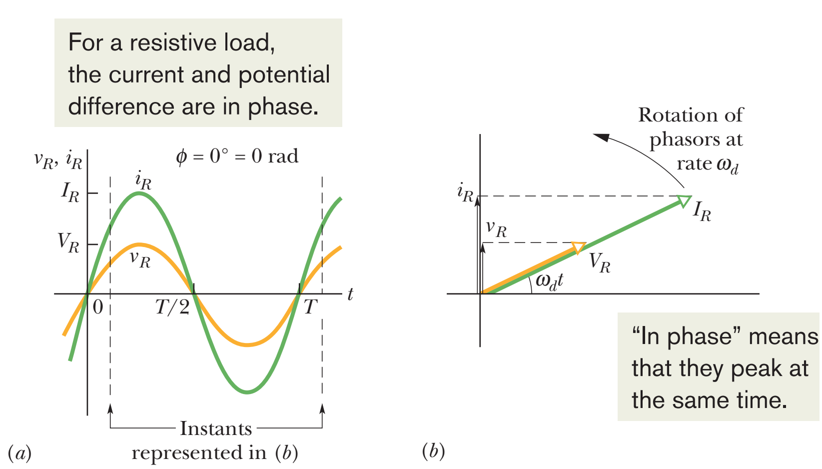


Figure 31-9 (a) The current i_R and the potential difference v_R across the resistor are plotted on the same graph, both versus time t. They are in phase and complete one cycle in one period T. (b) A phasor diagram shows the same thing as (a).

alternating quantity at time t. In Fig. 31-9b, the voltage and current are in phase; so their phasors always have the same phase $\omega_d t$ and the same rotation angle, and thus they rotate together.

Mentally follow the rotation. Can you see that when the phasors have rotated so that $\omega_d t = 90°$ (they point vertically upward), they indicate that just then $v_R = V_R$ and $i_R = I_R$? Equations 31-30 and 31-32 give the same results.

Checkpoint 3

If we increase the driving frequency in a circuit with a purely resistive load, do (a) amplitude V_R and (b) amplitude I_R increase, decrease, or remain the same?

Sample Problem 31.03 Purely resistive load: potential difference and current

In Fig. 31-8, resistance R is 200 Ω and the sinusoidal alternating emf device operates at amplitude $\mathscr{E}_m = 36.0$ V and frequency $f_d = 60.0$ Hz.

(a) What is the potential difference $v_R(t)$ across the resistance as a function of time t, and what is the amplitude V_R of $v_R(t)$?

KEY IDEA

In a circuit with a purely resistive load, the potential difference $v_R(t)$ across the resistance is always equal to the potential difference $\mathscr{E}(t)$ across the emf device.

Calculations: For our situation, $v_R(t) = \mathscr{E}(t)$ and $V_R = \mathscr{E}_m$. Since $\mathscr{E}_m$ is given, we can write

$$V_R = \mathscr{E}_m = 36.0 \text{ V}. \quad \text{(Answer)}$$

To find $v_R(t)$, we use Eq. 31-28 to write

$$v_R(t) = \mathscr{E}(t) = \mathscr{E}_m \sin \omega_d t \tag{31-34}$$

and then substitute $\mathscr{E}_m = 36.0$ V and

$$\omega_d = 2\pi f_d = 2\pi(60 \text{ Hz}) = 120\pi$$

to obtain

$$v_R = (36.0 \text{ V}) \sin(120\pi t). \quad \text{(Answer)}$$

We can leave the argument of the sine in this form for convenience, or we can write it as (377 rad/s)t or as (377 s^{-1})t.

(b) What are the current $i_R(t)$ in the resistance and the amplitude I_R of $i_R(t)$?

KEY IDEA

In an ac circuit with a purely resistive load, the alternating current $i_R(t)$ in the resistance is *in phase* with the alternating potential difference $v_R(t)$ across the resistance; that is, the phase constant ϕ for the current is zero.

Calculations: Here we can write Eq. 31-29 as

$$i_R = I_R \sin(\omega_d t - \phi) = I_R \sin \omega_d t. \tag{31-35}$$

From Eq. 31-33, the amplitude I_R is

$$I_R = \frac{V_R}{R} = \frac{36.0 \text{ V}}{200\ \Omega} = 0.180 \text{ A}. \quad \text{(Answer)}$$

Substituting this and $\omega_d = 2\pi f_d = 120\pi$ into Eq. 31-35, we have

$$i_R = (0.180 \text{ A}) \sin(120\pi t). \quad \text{(Answer)}$$

A Capacitive Load

Figure 31-10 shows a circuit containing a capacitance and a generator with the alternating emf of Eq. 31-28. Using the loop rule and proceeding as we did when we obtained Eq. 31-30, we find that the potential difference across the capacitor is

$$v_C = V_C \sin \omega_d t, \tag{31-36}$$

where V_C is the amplitude of the alternating voltage across the capacitor. From the definition of capacitance we can also write

$$q_C = Cv_C = CV_C \sin \omega_d t. \tag{31-37}$$

Our concern, however, is with the current rather than the charge. Thus, we differ-

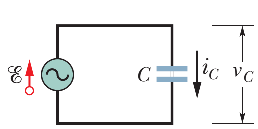


Figure 31-10 A capacitor is connected across an alternating-current generator.

entiate Eq. 31-37 to find

$$i_C = \frac{dq_C}{dt} = \omega_d C V_C \cos \omega_d t. \tag{31-38}$$

We now modify Eq. 31-38 in two ways. First, for reasons of symmetry of notation, we introduce the quantity X_C, called the **capacitive reactance** of a capacitor, defined as

$$X_C = \frac{1}{\omega_d C} \quad \text{(capacitive reactance).} \tag{31-39}$$

Its value depends not only on the capacitance but also on the driving angular frequency ω_d. We know from the definition of the capacitive time constant ($\tau = RC$) that the SI unit for C can be expressed as seconds per ohm. Applying this to Eq. 31-39 shows that the SI unit of X_C is the *ohm,* just as for resistance R.

Second, we replace $\cos \omega_d t$ in Eq. 31-38 with a phase-shifted sine:

$$\cos \omega_d t = \sin(\omega_d t + 90°).$$

You can verify this identity by shifting a sine curve 90° in the negative direction.

With these two modifications, Eq. 31-38 becomes

$$i_C = \left(\frac{V_C}{X_C}\right) \sin(\omega_d t + 90°). \tag{31-40}$$

From Eq. 31-29, we can also write the current i_C in the capacitor of Fig. 31-10 as

$$i_C = I_C \sin(\omega_d t - \phi), \tag{31-41}$$

where I_C is the amplitude of i_C. Comparing Eqs. 31-40 and 31-41, we see that for a purely capacitive load the phase constant ϕ for the current is $-90°$. We also see that the voltage amplitude and current amplitude are related by

$$V_C = I_C X_C \quad \text{(capacitor).} \tag{31-42}$$

Although we found this relation for the circuit of Fig. 31-10, it applies to any capacitance in any ac circuit.

Comparison of Eqs. 31-36 and 31-40, or inspection of Fig. 31-11*a*, shows that the quantities v_C and i_C are 90°, $\pi/2$ rad, or one-quarter cycle, out of phase. Furthermore, we see that i_C *leads* v_C, which means that, if you monitored the current i_C and the potential difference v_C in the circuit of Fig. 31-10, you would find that i_C reaches its maximum *before* v_C does, by one-quarter cycle.

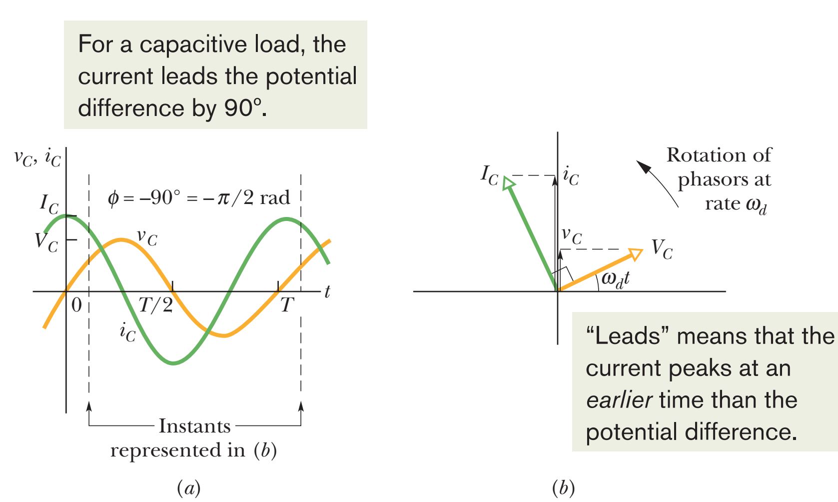
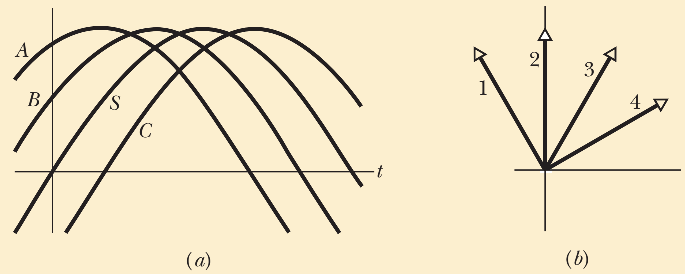


Figure 31-11 (a) The current in the capacitor leads the voltage by 90° (= $\pi/2$ rad). (b) A phasor diagram shows the same thing.

This relation between i_C and v_C is illustrated by the phasor diagram of Fig. 31-11*b*. As the phasors representing these two quantities rotate counterclockwise together, the phasor labeled I_C does indeed lead that labeled V_C, and by an angle of 90°; that is, the phasor I_C coincides with the vertical axis one-quarter cycle before the phasor V_C does. Be sure to convince yourself that the phasor diagram of Fig. 31-11*b* is consistent with Eqs. 31-36 and 31-40.

Checkpoint 4

The figure shows, in (*a*), a sine curve $S(t) = \sin(\omega_d t)$ and three other sinusoidal curves $A(t)$, $B(t)$, and $C(t)$, each of the form $\sin(\omega_d t - \phi)$. (a) Rank the three other curves according to the value of ϕ, most positive first and most negative last. (b) Which curve corresponds to which phasor in (*b*) of the figure? (c) Which curve leads the others?



Sample Problem 31.04 Purely capacitive load: potential difference and current

In Fig. 31-10, capacitance C is 15.0 μF and the sinusoidal alternating emf device operates at amplitude $\mathscr{E}_m = 36.0$ V and frequency $f_d = 60.0$ Hz.

(a) What are the potential difference $v_C(t)$ across the capacitance and the amplitude V_C of $v_C(t)$?

KEY IDEA

In a circuit with a purely capacitive load, the potential difference $v_C(t)$ across the capacitance is always equal to the potential difference $\mathscr{E}(t)$ across the emf device.

Calculations: Here we have $v_C(t) = \mathscr{E}(t)$ and $V_C = \mathscr{E}_m$. Since $\mathscr{E}_m$ is given, we have

$$V_C = \mathscr{E}_m = 36.0 \text{ V}. \qquad \text{(Answer)}$$

To find $v_C(t)$, we use Eq. 31-28 to write

$$v_C(t) = \mathscr{E}(t) = \mathscr{E}_m \sin \omega_d t. \qquad (31\text{-}43)$$

Then, substituting $\mathscr{E}_m = 36.0$ V and $\omega_d = 2\pi f_d = 120\pi$ into Eq. 31-43, we have

$$v_C = (36.0 \text{ V}) \sin(120\pi t). \qquad \text{(Answer)}$$

(b) What are the current $i_C(t)$ in the circuit as a function of time and the amplitude I_C of $i_C(t)$?

KEY IDEA

In an ac circuit with a purely capacitive load, the alternating current $i_C(t)$ in the capacitance leads the alternating potential difference $v_C(t)$ by 90°; that is, the phase constant ϕ for the current is $-90°$, or $-\pi/2$ rad.

Calculations: Thus, we can write Eq. 31-29 as

$$i_C = I_C \sin(\omega_d t - \phi) = I_C \sin(\omega_d t + \pi/2). \qquad (31\text{-}44)$$

We can find the amplitude I_C from Eq. 31-42 ($V_C = I_C X_C$) if we first find the capacitive reactance X_C. From Eq. 31-39 ($X_C = 1/\omega_d C$), with $\omega_d = 2\pi f_d$, we can write

$$X_C = \frac{1}{2\pi f_d C} = \frac{1}{(2\pi)(60.0 \text{ Hz})(15.0 \times 10^{-6} \text{ F})} = 177\ \Omega.$$

Then Eq. 31-42 tells us that the current amplitude is

$$I_C = \frac{V_C}{X_C} = \frac{36.0 \text{ V}}{177\ \Omega} = 0.203 \text{ A}. \qquad \text{(Answer)}$$

Substituting this and $\omega_d = 2\pi f_d = 120\pi$ into Eq. 31-44, we have

$$i_C = (0.203 \text{ A}) \sin(120\pi t + \pi/2). \qquad \text{(Answer)}$$

WileyPLUS Additional examples, video, and practice available at *WileyPLUS*

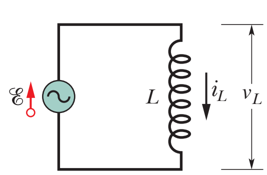


Figure 31-12 An inductor is connected across an alternating-current generator.

An Inductive Load

Figure 31-12 shows a circuit containing an inductance and a generator with the alternating emf of Eq. 31-28. Using the loop rule and proceeding as we did to obtain Eq. 31-30, we find that the potential difference across the inductance is

$$v_L = V_L \sin \omega_d t, \qquad (31\text{-}45)$$

where V_L is the amplitude of v_L. From Eq. 30-35 ($\mathscr{E}_L = -L\, di/dt$), we can write the potential difference across an inductance L in which the current is changing at the rate di_L/dt as

$$v_L = L\frac{di_L}{dt}. \tag{31-46}$$

If we combine Eqs. 31-45 and 31-46, we have

$$\frac{di_L}{dt} = \frac{V_L}{L}\sin\omega_d t. \tag{31-47}$$

Our concern, however, is with the current, so we integrate:

$$i_L = \int di_L = \frac{V_L}{L}\int \sin\omega_d t\, dt = -\left(\frac{V_L}{\omega_d L}\right)\cos\omega_d t. \tag{31-48}$$

We now modify this equation in two ways. First, for reasons of symmetry of notation, we introduce the quantity X_L, called the **inductive reactance** of an inductor, which is defined as

$$X_L = \omega_d L \quad \text{(inductive reactance)}. \tag{31-49}$$

The value of X_L depends on the driving angular frequency ω_d. The unit of the inductive time constant τ_L indicates that the SI unit of X_L is the *ohm*, just as it is for X_C and for R.

Second, we replace $-\cos\omega_d t$ in Eq. 31-48 with a phase-shifted sine:

$$-\cos\omega_d t = \sin(\omega_d t - 90°).$$

You can verify this identity by shifting a sine curve 90° in the positive direction.

With these two changes, Eq. 31-48 becomes

$$i_L = \left(\frac{V_L}{X_L}\right)\sin(\omega_d t - 90°). \tag{31-50}$$

From Eq. 31-29, we can also write this current in the inductance as

$$i_L = I_L\sin(\omega_d t - \phi), \tag{31-51}$$

where I_L is the amplitude of the current i_L. Comparing Eqs. 31-50 and 31-51, we see that for a purely inductive load the phase constant ϕ for the current is $+90°$. We also see that the voltage amplitude and current amplitude are related by

$$V_L = I_L X_L \quad \text{(inductor)}. \tag{31-52}$$

Although we found this relation for the circuit of Fig. 31-12, it applies to any inductance in any ac circuit.

Comparison of Eqs. 31-45 and 31-50, or inspection of Fig. 31-13*a*, shows that the quantities i_L and v_L are 90° out of phase. In this case, however, i_L *lags* v_L; that is, monitoring the current i_L and the potential difference v_L in the circuit of Fig. 31-12 shows that i_L reaches its maximum value *after* v_L does, by one-quarter cycle. The phasor diagram of Fig. 31-13*b* also contains this information. As the phasors rotate counterclockwise in the figure, the phasor labeled I_L does indeed lag that labeled V_L, and by an angle of 90°. Be sure to convince yourself that Fig. 31-13*b* represents Eqs. 31-45 and 31-50.

For an inductive load, the current lags the potential difference by 90°.

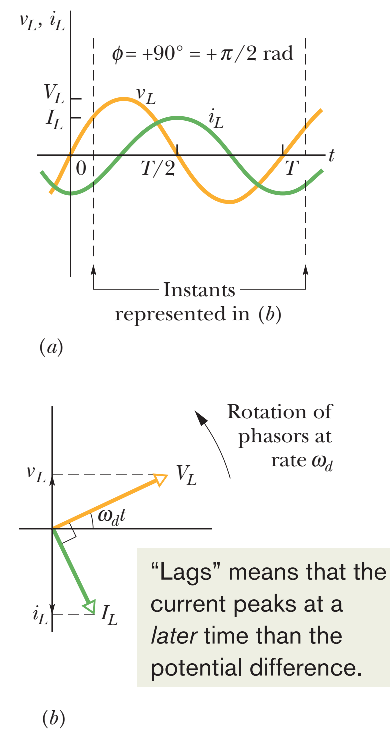


Figure 31-13 (*a*) The current in the inductor lags the voltage by 90° (= $\pi/2$ rad). (*b*) A phasor diagram shows the same thing.

Checkpoint 5

If we increase the driving frequency in a circuit with a purely capacitive load, do (a) amplitude V_C and (b) amplitude I_C increase, decrease, or remain the same? If, instead, the circuit has a purely inductive load, do (c) amplitude V_L and (d) amplitude I_L increase, decrease, or remain the same?

Problem-Solving Tactics

Leading and Lagging in AC Circuits: Table 31-2 summarizes the relations between the current i and the voltage v for each of the three kinds of circuit elements we have considered. When an applied alternating voltage produces an alternating current in these elements, the current is always in phase with the voltage across a resistor, always leads the voltage across a capacitor, and always lags the voltage across an inductor.

Many students remember these results with the mnemonic "*ELI* the *ICE* man." *ELI* contains the letter L (for inductor), and in it the letter I (for current) comes *after* the letter E (for emf or voltage). Thus, for an inductor, the current *lags* (comes after) the voltage. Similarly, *ICE* (which contains a C for capacitor) means that the current *leads* (comes before) the voltage. You might also use the modified mnemonic "*ELI positively* is the *ICE* man" to remember that the phase constant ϕ is positive for an inductor.

If you have difficulty in remembering whether X_C is equal to $\omega_d C$ (wrong) or $1/\omega_d C$ (right), try remembering that C is in the "cellar"—that is, in the denominator.

Table 31-2 Phase and Amplitude Relations for Alternating Currents and Voltages

Circuit Element	Symbol	Resistance or Reactance	Phase of the Current	Phase Constant (or Angle) ϕ	Amplitude Relation
Resistor	R	R	In phase with v_R	$0°$ $(= 0$ rad$)$	$V_R = I_R R$
Capacitor	C	$X_C = 1/\omega_d C$	Leads v_C by $90°$ $(= \pi/2$ rad$)$	$-90°$ $(= -\pi/2$ rad$)$	$V_C = I_C X_C$
Inductor	L	$X_L = \omega_d L$	Lags v_L by $90°$ $(= \pi/2$ rad$)$	$+90°$ $(= +\pi/2$ rad$)$	$V_L = I_L X_L$

Sample Problem 31.05 Purely inductive load: potential difference and current

In Fig. 31-12, inductance L is 230 mH and the sinusoidal alternating emf device operates at amplitude $\mathscr{E}_m = 36.0$ V and frequency $f_d = 60.0$ Hz.

(a) What are the potential difference $v_L(t)$ across the inductance and the amplitude V_L of $v_L(t)$?

KEY IDEA

In a circuit with a purely inductive load, the potential difference $v_L(t)$ across the inductance is always equal to the potential difference $\mathscr{E}(t)$ across the emf device.

Calculations: Here we have $v_L(t) = \mathscr{E}(t)$ and $V_L = \mathscr{E}_m$. Since $\mathscr{E}_m$ is given, we know that

$$V_L = \mathscr{E}_m = 36.0 \text{ V}. \qquad \text{(Answer)}$$

To find $v_L(t)$, we use Eq. 31-28 to write

$$v_L(t) = \mathscr{E}(t) = \mathscr{E}_m \sin \omega_d t. \qquad (31\text{-}53)$$

Then, substituting $\mathscr{E}_m = 36.0$ V and $\omega_d = 2\pi f_d = 120\pi$ into Eq. 31-53, we have

$$v_L = (36.0 \text{ V}) \sin(120\pi t). \qquad \text{(Answer)}$$

(b) What are the current $i_L(t)$ in the circuit as a function of time and the amplitude I_L of $i_L(t)$?

KEY IDEA

In an ac circuit with a purely inductive load, the alternating current $i_L(t)$ in the inductance lags the alternating potential difference $v_L(t)$ by 90°. (In the mnemonic of the problem-solving tactic, this circuit is "positively an *ELI* circuit," which tells us that the emf E leads the current I and that ϕ is *positive.*)

Calculations: Because the phase constant ϕ for the current is $+90°$, or $+\pi/2$ rad, we can write Eq. 31-29 as

$$i_L = I_L \sin(\omega_d t - \phi) = I_L \sin(\omega_d t - \pi/2). \qquad (31\text{-}54)$$

We can find the amplitude I_L from Eq. 31-52 ($V_L = I_L X_L$) if we first find the inductive reactance X_L. From Eq. 31-49 ($X_L = \omega_d L$), with $\omega_d = 2\pi f_d$, we can write

$$X_L = 2\pi f_d L = (2\pi)(60.0 \text{ Hz})(230 \times 10^{-3} \text{ H})$$
$$= 86.7\ \Omega.$$

Then Eq. 31-52 tells us that the current amplitude is

$$I_L = \frac{V_L}{X_L} = \frac{36.0 \text{ V}}{86.7\ \Omega} = 0.415 \text{ A}. \qquad \text{(Answer)}$$

Substituting this and $\omega_d = 2\pi f_d = 120\pi$ into Eq. 31-54, we have

$$i_L = (0.415 \text{ A}) \sin(120\pi t - \pi/2). \qquad \text{(Answer)}$$

31-4 THE SERIES *RLC* CIRCUIT

Learning Objectives

After reading this module, you should be able to . . .

31.32 Draw the schematic diagram of a series RLC circuit.

31.33 Identify the conditions for a mainly inductive circuit, a mainly capacitive circuit, and a resonant circuit.

31.34 For a mainly inductive circuit, a mainly capacitive circuit, and a resonant circuit, sketch graphs for voltage $v(t)$ and current $i(t)$ and sketch phasor diagrams, indicating leading, lagging, or resonance.

31.35 Calculate impedance Z.

31.36 Apply the relationship between current amplitude I, impedance Z, and emf amplitude $\mathscr{E}_m$.

31.37 Apply the relationships between phase constant ϕ and voltage amplitudes V_L and V_C, and also between phase constant ϕ, resistance R, and reactances X_L and X_C.

31.38 Identify the values of the phase constant ϕ corresponding to a mainly inductive circuit, a mainly capacitive circuit, and a resonant circuit.

31.39 For resonance, apply the relationship between the driving angular frequency ω_d, the natural angular frequency ω, the inductance L, and the capacitance C.

31.40 Sketch a graph of current amplitude versus the ratio ω_d/ω, identifying the portions corresponding to a mainly inductive circuit, a mainly capacitive circuit, and a resonant circuit and indicating what happens to the curve for an increase in the resistance.

Key Ideas

- For a series RLC circuit with an external emf given by

$$\mathscr{E} = \mathscr{E}_m \sin \omega_d t,$$

and current given by

$$i = I \sin(\omega_d t - \phi),$$

the current amplitude is given by

$$I = \frac{\mathscr{E}_m}{\sqrt{R^2 + (X_L - X_C)^2}} = \frac{\mathscr{E}_m}{\sqrt{R^2 + (\omega_d L - 1/\omega_d C)^2}} \quad \text{(current amplitude)}.$$

- The phase constant is given by

$$\tan \phi = \frac{X_L - X_C}{R} \quad \text{(phase constant)}.$$

- The impedance Z of the circuit is

$$Z = \sqrt{R^2 + (X_L - X_C)^2} \quad \text{(impedance)}.$$

- We relate the current amplitude and the impedance with

$$I = \mathscr{E}_m/Z.$$

- The current amplitude I is maximum ($I = \mathscr{E}_m/R$) when the driving angular frequency ω_d equals the natural angular frequency ω of the circuit, a condition known as resonance. Then $X_C = X_L$, $\phi = 0$, and the current is in phase with the emf.

The Series *RLC* Circuit

We are now ready to apply the alternating emf of Eq. 31-28,

$$\mathscr{E} = \mathscr{E}_m \sin \omega_d t \quad \text{(applied emf)}, \tag{31-55}$$

to the full RLC circuit of Fig. 31-7. Because R, L, and C are in series, the same current

$$i = I \sin(\omega_d t - \phi) \tag{31-56}$$

is driven in all three of them. We wish to find the current amplitude I and the phase constant ϕ and to investigate how these quantities depend on the driving angular frequency ω_d. The solution is simplified by the use of phasor diagrams as introduced for the three basic circuits of Module 31-3: capacitive load, inductive load, and resistive load. In particular we shall make use of how the voltage phasor is related to the current phasor for each of those basic circuits. We shall find that series RLC circuits can be separated into three types: mainly capacitive circuits, mainly inductive circuits, and circuits that are in resonance.

The Current Amplitude

We start with Fig. 31-14*a*, which shows the phasor representing the current of Eq. 31-56 at an arbitrary time t. The length of the phasor is the current amplitude I, the projection of the phasor on the vertical axis is the current i at time t, and the angle of rotation of the phasor is the phase $\omega_d t - \phi$ of the current at time t.

Figure 31-14*b* shows the phasors representing the voltages across R, L, and C at the same time t. Each phasor is oriented relative to the angle of rotation of current phasor I in Fig. 31-14*a*, based on the information in Table 31-2:

Resistor: Here current and voltage are in phase; so the angle of rotation of voltage phasor V_R is the same as that of phasor I.

Capacitor: Here current leads voltage by 90°; so the angle of rotation of voltage phasor V_C is 90° less than that of phasor I.

Inductor: Here current lags voltage by 90°; so the angle of rotation of voltage phasor v_L is 90° greater than that of phasor I.

Figure 31-14*b* also shows the instantaneous voltages v_R, v_C, and v_L across R, C, and L at time t; those voltages are the projections of the corresponding phasors on the vertical axis of the figure.

Figure 31-14*c* shows the phasor representing the applied emf of Eq. 31-55. The length of the phasor is the emf amplitude $\mathscr{E}_m$, the projection of the phasor on the vertical axis is the emf $\mathscr{E}$ at time t, and the angle of rotation of the phasor is the phase $\omega_d t$ of the emf at time t.

From the loop rule we know that at any instant the sum of the voltages v_R, v_C, and v_L is equal to the applied emf $\mathscr{E}$:

$$\mathscr{E} = v_R + v_C + v_L. \tag{31-57}$$

Thus, at time t the projection $\mathscr{E}$ in Fig. 31-14*c* is equal to the algebraic sum of the projections v_R, v_C, and v_L in Fig. 31-14*b*. In fact, as the phasors rotate together, this equality always holds. This means that phasor $\mathscr{E}_m$ in Fig. 31-14*c* must be equal to the vector sum of the three voltage phasors V_R, V_C, and V_L in Fig. 31-14*b*.

That requirement is indicated in Fig. 31-14*d*, where phasor $\mathscr{E}_m$ is drawn as the sum of phasors V_R, V_L, and V_C. Because phasors V_L and V_C have opposite directions in the figure, we simplify the vector sum by first combining V_L and V_C to form the single phasor $V_L - V_C$. Then we combine that single phasor with V_R to find the net phasor. Again, the net phasor must coincide with phasor $\mathscr{E}_m$, as shown.

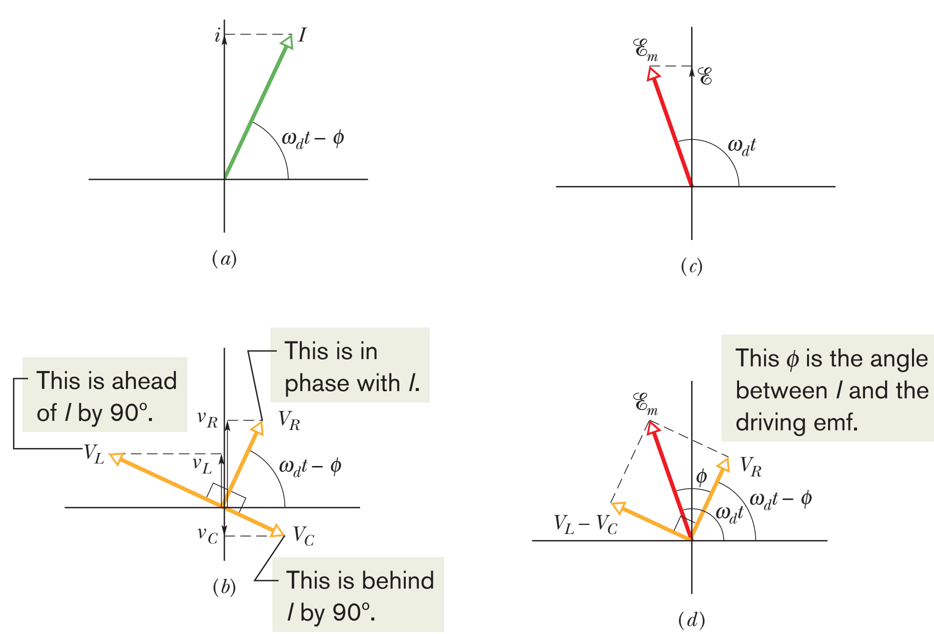


Figure 31-14 (*a*) A phasor representing the alternating current in the driven *RLC* circuit of Fig. 31-7 at time t. The amplitude I, the instantaneous value i, and the phase $(\omega_d t - \phi)$ are shown. (*b*) Phasors representing the voltages across the inductor, resistor, and capacitor, oriented with respect to the current phasor in (*a*). (*c*) A phasor representing the alternating emf that drives the current of (*a*). (*d*) The emf phasor is equal to the vector sum of the three voltage phasors of (*b*). Here, voltage phasors V_L and V_C have been added vectorially to yield their net phasor $(V_L - V_C)$.

Both triangles in Fig. 31-14*d* are right triangles. Applying the Pythagorean theorem to either one yields

$$\mathscr{E}_m^2 = V_R^2 + (V_L - V_C)^2. \tag{31-58}$$

From the voltage amplitude information displayed in the rightmost column of Table 31-2, we can rewrite this as

$$\mathscr{E}_m^2 = (IR)^2 + (IX_L - IX_C)^2, \tag{31-59}$$

and then rearrange it to the form

$$I = \frac{\mathscr{E}_m}{\sqrt{R^2 + (X_L - X_C)^2}}. \tag{31-60}$$

The denominator in Eq. 31-60 is called the **impedance** Z of the circuit for the driving angular frequency ω_d:

$$Z = \sqrt{R^2 + (X_L - X_C)^2} \quad \text{(impedance defined)}. \tag{31-61}$$

We can then write Eq. 31-60 as

$$I = \frac{\mathscr{E}_m}{Z}. \tag{31-62}$$

If we substitute for X_C and X_L from Eqs. 31-39 and 31-49, we can write Eq. 31-60 more explicitly as

$$I = \frac{\mathscr{E}_m}{\sqrt{R^2 + (\omega_d L - 1/\omega_d C)^2}} \quad \text{(current amplitude)}. \tag{31-63}$$

We have now accomplished half our goal: We have obtained an expression for the current amplitude I in terms of the sinusoidal driving emf and the circuit elements in a series *RLC* circuit.

The value of I depends on the difference between $\omega_d L$ and $1/\omega_d C$ in Eq. 31-63 or, equivalently, the difference between X_L and X_C in Eq. 31-60. In either equation, it does not matter which of the two quantities is greater because the difference is always squared.

The current that we have been describing in this module is the *steady-state current* that occurs after the alternating emf has been applied for some time. When the emf is first applied to a circuit, a brief *transient current* occurs. Its duration (before settling down into the steady-state current) is determined by the time constants $\tau_L = L/R$ and $\tau_C = RC$ as the inductive and capacitive elements "turn on." This transient current can, for example, destroy a motor on start-up if it is not properly taken into account in the motor's circuit design.

The Phase Constant

From the right-hand phasor triangle in Fig. 31-14*d* and from Table 31-2 we can write

$$\tan\phi = \frac{V_L - V_C}{V_R} = \frac{IX_L - IX_C}{IR}, \tag{31-64}$$

which gives us

$$\tan\phi = \frac{X_L - X_C}{R} \quad \text{(phase constant)}. \tag{31-65}$$

This is the other half of our goal: an equation for the phase constant ϕ in the sinusoidally driven series *RLC* circuit of Fig. 31-7. In essence, it gives us three different results for the phase constant, depending on the relative values of the reactances X_L and X_C:

$\boldsymbol{X_L > X_C}$**:** The circuit is said to be *more inductive than capacitive.* Equation 31-65 tells us that ϕ is positive for such a circuit, which means that phasor I rotates behind phasor $\mathscr{E}_m$ (Fig. 31-15*a*). A plot of $\mathscr{E}$ and i versus time is like that in Fig. 31-15*b*. (Figures 31-14*c* and *d* were drawn assuming $X_L > X_C$.)

$\boldsymbol{X_C > X_L}$**:** The circuit is said to be *more capacitive than inductive.* Equation 31-65 tells us that ϕ is negative for such a circuit, which means that phasor I rotates ahead of phasor $\mathscr{E}_m$ (Fig. 31-15*c*). A plot of $\mathscr{E}$ and i versus time is like that in Fig. 31-15*d*.

$\boldsymbol{X_C = X_L}$**:** The circuit is said to be in *resonance,* a state that is discussed next. Equation 31-65 tells us that $\phi = 0°$ for such a circuit, which means that phasors $\mathscr{E}_m$ and I rotate together (Fig. 31-15*e*). A plot of $\mathscr{E}$ and i versus time is like that in Fig. 31-15*f*.

As illustration, let us reconsider two extreme circuits: In the *purely inductive circuit* of Fig. 31-12, where X_L is nonzero and $X_C = R = 0$, Eq. 31-65 tells us that the circuit's phase constant is $\phi = +90°$ (the greatest value of ϕ), consistent with Fig. 31-13*b*. In the *purely capacitive circuit* of Fig. 31-10, where X_C is nonzero and $X_L = R = 0$, Eq. 31-65 tells us that the circuit's phase constant is $\phi = -90°$ (the least value of ϕ), consistent with Fig. 31-11*b*.

Resonance

Equation 31-63 gives the current amplitude I in an RLC circuit as a function of the driving angular frequency ω_d of the external alternating emf. For a given resistance R, that amplitude is a maximum when the quantity $\omega_d L - 1/\omega_d C$ in the

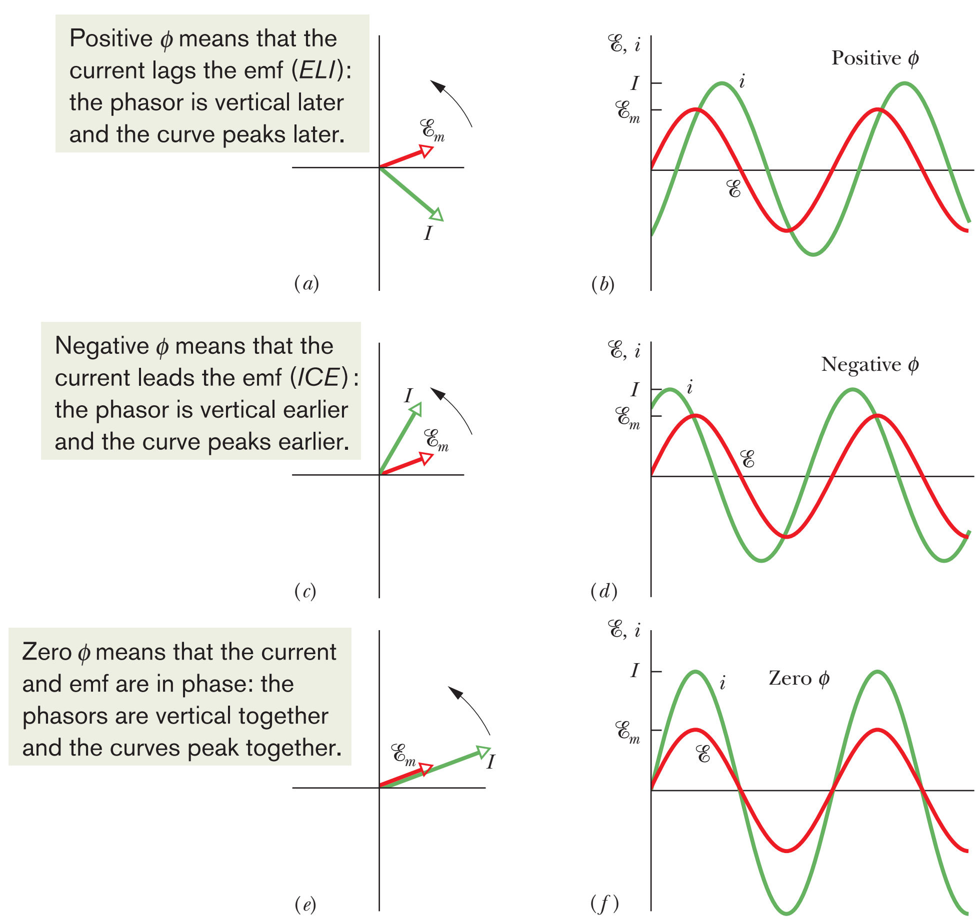


Figure 31-15 Phasor diagrams and graphs of the alternating emf $\mathscr{E}$ and current i for the driven RLC circuit of Fig. 31-7. In the phasor diagram of (*a*) and the graph of (*b*), the current i lags the driving emf $\mathscr{E}$ and the current's phase constant ϕ is positive. In (*c*) and (*d*), the current i leads the driving emf $\mathscr{E}$ and its phase constant ϕ is negative. In (*e*) and (*f*), the current i is in phase with the driving emf $\mathscr{E}$ and its phase constant ϕ is zero.

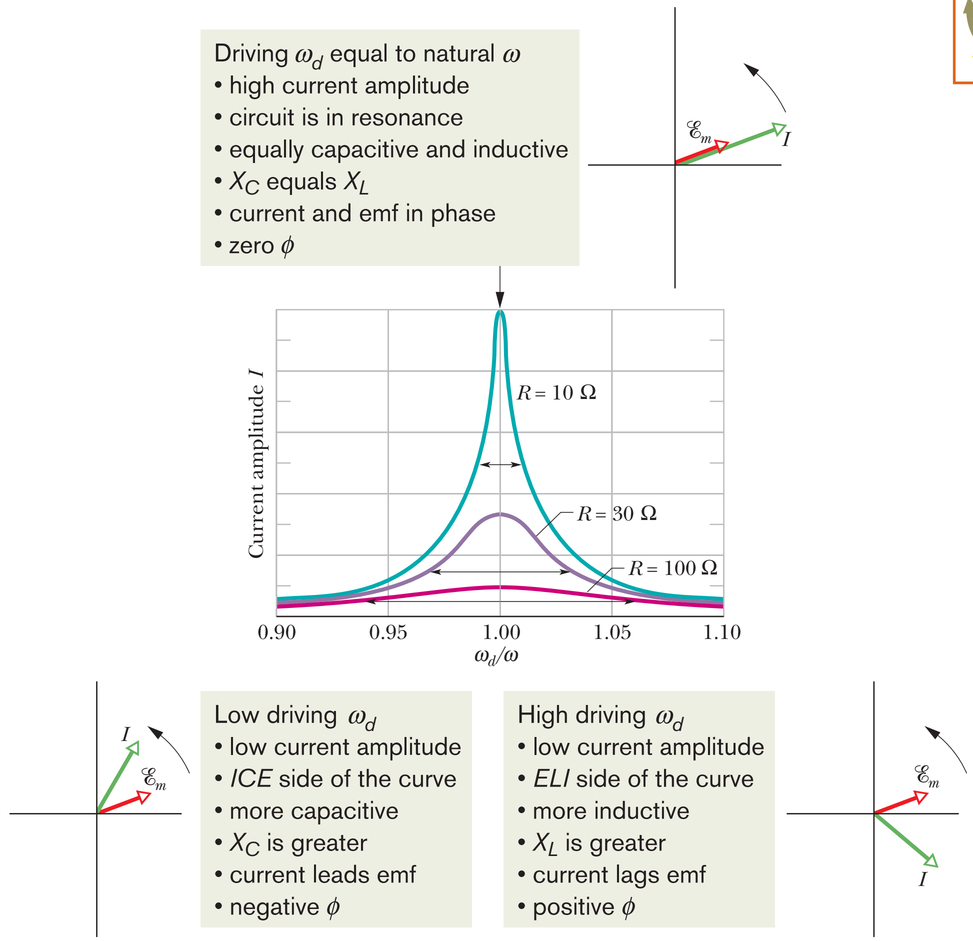


Figure 31-16 *Resonance curves* for the driven *RLC* circuit of Fig. 31-7 with $L = 100\ \mu\text{H}$, $C = 100$ pF, and three values of R. The current amplitude I of the alternating current depends on how close the driving angular frequency ω_d is to the natural angular frequency ω. The horizontal arrow on each curve measures the curve's *half-width,* which is the width at the half-maximum level and is a measure of the sharpness of the resonance. To the left of $\omega_d/\omega = 1.00$, the circuit is mainly capacitive, with $X_C > X_L$; to the right, it is mainly inductive, with $X_L > X_C$.

denominator is zero—that is, when

$$\omega_d L = \frac{1}{\omega_d C}$$

or

$$\omega_d = \frac{1}{\sqrt{LC}} \quad \text{(maximum } I\text{)}. \tag{31-66}$$

Because the natural angular frequency ω of the *RLC* circuit is also equal to $1/\sqrt{LC}$, the maximum value of I occurs when the driving angular frequency matches the natural angular frequency—that is, at resonance. Thus, in an *RLC* circuit, resonance and maximum current amplitude I occur when

$$\omega_d = \omega = \frac{1}{\sqrt{LC}} \quad \text{(resonance)}. \tag{31-67}$$

Resonance Curves. Figure 31-16 shows three *resonance curves* for sinusoidally driven oscillations in three series *RLC* circuits differing only in R. Each curve peaks at its maximum current amplitude I when the ratio ω_d/ω is 1.00, but the maximum value of I decreases with increasing R. (The maximum I is always $\mathscr{E}_m/R$; to see why, combine Eqs. 31-61 and 31-62.) In addition, the curves increase in width (measured in Fig. 31-16 at half the maximum value of I) with increasing R.

To make physical sense of Fig. 31-16, consider how the reactances X_L and X_C change as we increase the driving angular frequency ω_d, starting with a value much less than the natural frequency ω. For small ω_d, reactance X_L $(= \omega_d L)$ is small and reactance X_C $(= 1/\omega_d C)$ is large. Thus, the circuit is mainly capacitive and the impedance is dominated by the large X_C, which keeps the current low.

As we increase ω_d, reactance X_C remains dominant but decreases while reactance X_L increases. The decrease in X_C decreases the impedance, allowing the current to increase, as we see on the left side of any resonance curve in Fig. 31-16. When the increasing X_L and the decreasing X_C reach equal values, the current is greatest and the circuit is in resonance, with $\omega_d = \omega$.

As we continue to increase ω_d, the increasing reactance X_L becomes progressively more dominant over the decreasing reactance X_C. The impedance increases because of X_L and the current decreases, as on the right side of any resonance curve in Fig. 31-16. In summary, then: The low-angular-frequency side of a resonance curve is dominated by the capacitor's reactance, the high-angular-frequency side is dominated by the inductor's reactance, and resonance occurs in the middle.

Checkpoint 6

Here are the capacitive reactance and inductive reactance, respectively, for three sinusoidally driven series *RLC* circuits: (1) 50 Ω, 100 Ω; (2) 100 Ω, 50 Ω; (3) 50 Ω, 50 Ω. (a) For each, does the current lead or lag the applied emf, or are the two in phase? (b) Which circuit is in resonance?

Sample Problem 31.06 Current amplitude, impedance, and phase constant

In Fig. 31-7, let $R = 200\ \Omega$, $C = 15.0\ \mu\text{F}$, $L = 230$ mH, $f_d = 60.0$ Hz, and $\mathscr{E}_m = 36.0$ V. (These parameters are those used in the earlier sample problems.)

(a) What is the current amplitude I?

KEY IDEA

The current amplitude I depends on the amplitude $\mathscr{E}_m$ of the driving emf and on the impedance Z of the circuit, according to Eq. 31-62 $(I = \mathscr{E}_m/Z)$.

Calculations: So, we need to find Z, which depends on resistance R, capacitive reactance X_C, and inductive reactance X_L. The circuit's resistance is the given resistance R. Its capacitive reactance is due to the given capacitance and, from an earlier sample problem, $X_C = 177\ \Omega$. Its inductive reactance is due to the given inductance and, from another sample problem, $X_L = 86.7\ \Omega$. Thus, the circuit's impedance is

$$\begin{aligned} Z &= \sqrt{R^2 + (X_L - X_C)^2} \\ &= \sqrt{(200\ \Omega)^2 + (86.7\ \Omega - 177\ \Omega)^2} \\ &= 219\ \Omega. \end{aligned}$$

We then find

$$I = \frac{\mathscr{E}_m}{Z} = \frac{36.0\ \text{V}}{219\ \Omega} = 0.164\ \text{A}. \quad \text{(Answer)}$$

(b) What is the phase constant ϕ of the current in the circuit relative to the driving emf?

KEY IDEA

The phase constant depends on the inductive reactance, the capacitive reactance, and the resistance of the circuit, according to Eq. 31-65.

Calculation: Solving Eq. 31-65 for ϕ leads to

$$\begin{aligned} \phi &= \tan^{-1}\frac{X_L - X_C}{R} = \tan^{-1}\frac{86.7\ \Omega - 177\ \Omega}{200\ \Omega} \\ &= -24.3° = -0.424\ \text{rad}. \quad \text{(Answer)} \end{aligned}$$

The negative phase constant is consistent with the fact that the load is mainly capacitive; that is, $X_C > X_L$. In the common mnemonic for driven series *RLC* circuits, this circuit is an *ICE* circuit—the current *leads* the driving emf.

31-5 POWER IN ALTERNATING-CURRENT CIRCUITS

Learning Objectives

After reading this module, you should be able to . . .

31.41 For the current, voltage, and emf in an ac circuit, apply the relationship between the rms values and the amplitudes.

31.42 For an alternating emf connected across a capacitor, an inductor, or a resistor, sketch graphs of the sinusoidal variation of the current and voltage and indicate the peak and rms values.

31.43 Apply the relationship between average power P_{avg}, rms current I_{rms}, and resistance R.

31.44 In a driven RLC circuit, calculate the power of each element.

31.45 For a driven RLC circuit in steady state, explain what happens to (a) the value of the average stored energy with time and (b) the energy that the generator puts into the circuit.

31.46 Apply the relationship between the power factor $\cos\phi$, the resistance R, and the impedance Z.

31.47 Apply the relationship between the average power P_{avg}, the rms emf $\mathscr{E}_{rms}$, the rms current I_{rms}, and the power factor $\cos\phi$.

31.48 Identify what power factor is required in order to maximize the rate at which energy is supplied to a resistive load.

Key Ideas

- In a series RLC circuit, the average power P_{avg} of the generator is equal to the production rate of thermal energy in the resistor:

$$P_{avg} = I_{rms}^2 R = \mathscr{E}_{rms} I_{rms} \cos\phi.$$

- The abbreviation rms stands for root-mean-square; the rms quantities are related to the maximum quantities by $I_{rms} = I/\sqrt{2}$, $V_{rms} = V/\sqrt{2}$, and $\mathscr{E}_{rms} = \mathscr{E}_m/\sqrt{2}$. The term $\cos\phi$ is called the power factor of the circuit.

Power in Alternating-Current Circuits

In the RLC circuit of Fig. 31-7, the source of energy is the alternating-current generator. Some of the energy that it provides is stored in the electric field in the capacitor, some is stored in the magnetic field in the inductor, and some is dissipated as thermal energy in the resistor. In steady-state operation, the average stored energy remains constant. The net transfer of energy is thus from the generator to the resistor, where energy is dissipated.

The instantaneous rate at which energy is dissipated in the resistor can be written, with the help of Eqs. 26-27 and 31-29, as

$$P = i^2R = [I\sin(\omega_d t - \phi)]^2 R = I^2R\sin^2(\omega_d t - \phi). \tag{31-68}$$

The *average* rate at which energy is dissipated in the resistor, however, is the average of Eq. 31-68 over time. Over one complete cycle, the average value of $\sin\theta$, where θ is any variable, is zero (Fig. 31-17*a*) but the average value of $\sin^2\theta$ is $\frac{1}{2}$ (Fig. 31-17*b*). (Note in Fig. 31-17*b* how the shaded areas under the curve but above the horizontal line marked $+\frac{1}{2}$ exactly fill in the unshaded spaces below that line.) Thus, we can write, from Eq. 31-68,

$$P_{avg} = \frac{I^2R}{2} = \left(\frac{I}{\sqrt{2}}\right)^2 R. \tag{31-69}$$

The quantity $I/\sqrt{2}$ is called the **root-mean-square,** or **rms,** value of the current i:

$$I_{rms} = \frac{I}{\sqrt{2}} \quad \text{(rms current).} \tag{31-70}$$

We can now rewrite Eq. 31-69 as

$$P_{avg} = I_{rms}^2 R \quad \text{(average power).} \tag{31-71}$$

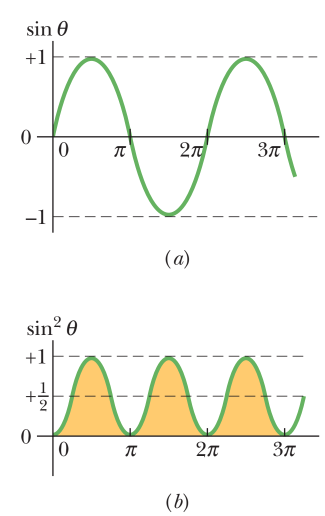


Figure 31-17 (*a*) A plot of $\sin\theta$ versus θ. The average value over one cycle is zero. (*b*) A plot of $\sin^2\theta$ versus θ. The average value over one cycle is $\frac{1}{2}$.

Equation 31-71 has the same mathematical form as Eq. 26-27 ($P = i^2R$); the message here is that if we switch to the rms current, we can compute the average rate of energy dissipation for alternating-current circuits just as for direct-current circuits.

We can also define rms values of voltages and emfs for alternating-current circuits:

$$V_{\text{rms}} = \frac{V}{\sqrt{2}} \quad \text{and} \quad \mathscr{E}_{\text{rms}} = \frac{\mathscr{E}_m}{\sqrt{2}} \quad \text{(rms voltage; rms emf)}. \tag{31-72}$$

Alternating-current instruments, such as ammeters and voltmeters, are usually calibrated to read I_{rms}, V_{rms}, and $\mathscr{E}_{\text{rms}}$. Thus, if you plug an alternating-current voltmeter into a household electrical outlet and it reads 120 V, that is an rms voltage. The *maximum* value of the potential difference at the outlet is $\sqrt{2} \times (120\text{ V})$, or 170 V. Generally scientists and engineers report rms values instead of maximum values.

Because the proportionality factor $1/\sqrt{2}$ in Eqs. 31-70 and 31-72 is the same for all three variables, we can write Eqs. 31-62 and 31-60 as

$$I_{\text{rms}} = \frac{\mathscr{E}_{\text{rms}}}{Z} = \frac{\mathscr{E}_{\text{rms}}}{\sqrt{R^2 + (X_L - X_C)^2}}, \tag{31-73}$$

and, indeed, this is the form that we almost always use.

We can use the relationship $I_{\text{rms}} = \mathscr{E}_{\text{rms}}/Z$ to recast Eq. 31-71 in a useful equivalent way. We write

$$P_{\text{avg}} = \frac{\mathscr{E}_{\text{rms}}}{Z} I_{\text{rms}} R = \mathscr{E}_{\text{rms}} I_{\text{rms}} \frac{R}{Z}. \tag{31-74}$$

From Fig. 31-14*d*, Table 31-2, and Eq. 31-62, however, we see that R/Z is just the cosine of the phase constant ϕ:

$$\cos \phi = \frac{V_R}{\mathscr{E}_m} = \frac{IR}{IZ} = \frac{R}{Z}. \tag{31-75}$$

Equation 31-74 then becomes

$$P_{\text{avg}} = \mathscr{E}_{\text{rms}} I_{\text{rms}} \cos \phi \quad \text{(average power)}, \tag{31-76}$$

in which the term $\cos \phi$ is called the **power factor.** Because $\cos \phi = \cos(-\phi)$, Eq. 31-76 is independent of the sign of the phase constant ϕ.

To maximize the rate at which energy is supplied to a resistive load in an *RLC* circuit, we should keep the power factor $\cos \phi$ as close to unity as possible. This is equivalent to keeping the phase constant ϕ in Eq. 31-29 as close to zero as possible. If, for example, the circuit is highly inductive, it can be made less so by putting more capacitance in the circuit, connected in series. (Recall that putting an additional capacitance into a series of capacitances decreases the equivalent capacitance C_{eq} of the series.) Thus, the resulting decrease in C_{eq} in the circuit reduces the phase constant and increases the power factor in Eq. 31-76. Power companies place series-connected capacitors throughout their transmission systems to get these results.

Checkpoint 7

(a) If the current in a sinusoidally driven series *RLC* circuit leads the emf, would we increase or decrease the capacitance to increase the rate at which energy is supplied to the resistance? (b) Would this change bring the resonant angular frequency of the circuit closer to the angular frequency of the emf or put it farther away?

Sample Problem 31.07 Driven *RLC* circuit: power factor and average power

A series *RLC* circuit, driven with $\mathscr{E}_{rms} = 120$ V at frequency $f_d = 60.0$ Hz, contains a resistance $R = 200\ \Omega$, an inductance with inductive reactance $X_L = 80.0\ \Omega$, and a capacitance with capacitive reactance $X_C = 150\ \Omega$.

(a) What are the power factor $\cos\phi$ and phase constant ϕ of the circuit?

KEY IDEA

The power factor $\cos\phi$ can be found from the resistance R and impedance Z via Eq. 31-75 ($\cos\phi = R/Z$).

Calculations: To calculate Z, we use Eq. 31-61:

$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$
$$= \sqrt{(200\ \Omega)^2 + (80.0\ \Omega - 150\ \Omega)^2} = 211.90\ \Omega.$$

Equation 31-75 then gives us

$$\cos\phi = \frac{R}{Z} = \frac{200\ \Omega}{211.90\ \Omega} = 0.9438 \approx 0.944. \quad \text{(Answer)}$$

Taking the inverse cosine then yields

$$\phi = \cos^{-1} 0.944 = \pm 19.3°.$$

The inverse cosine on a calculator gives only the positive answer here, but both $+19.3°$ and $-19.3°$ have a cosine of 0.944. To determine which sign is correct, we must consider whether the current leads or lags the driving emf. Because $X_C > X_L$, this circuit is mainly capacitive, with the current leading the emf. Thus, ϕ must be negative:

$$\phi = -19.3°. \quad \text{(Answer)}$$

We could, instead, have found ϕ with Eq. 31-65. A calculator would then have given us the answer with the minus sign.

(b) What is the average rate P_{avg} at which energy is dissipated in the resistance?

KEY IDEAS

There are two ways and two ideas to use: (1) Because the circuit is assumed to be in steady-state operation, the rate at which energy is dissipated in the resistance is equal to the rate at which energy is supplied to the circuit, as given by Eq. 31-76 ($P_{avg} = \mathscr{E}_{rms} I_{rms} \cos\phi$). (2) The rate at which energy is dissipated in a resistance R depends on the square of the rms current I_{rms} through it, according to Eq. 31-71 ($P_{avg} = I^2_{rms} R$).

First way: We are given the rms driving emf $\mathscr{E}_{rms}$ and we already know $\cos\phi$ from part (a). The rms current I_{rms} is determined by the rms value of the driving emf and the circuit's impedance Z (which we know), according to Eq. 31-73:

$$I_{rms} = \frac{\mathscr{E}_{rms}}{Z}.$$

Substituting this into Eq. 31-76 then leads to

$$P_{avg} = \mathscr{E}_{rms} I_{rms} \cos\phi = \frac{\mathscr{E}^2_{rms}}{Z} \cos\phi$$
$$= \frac{(120\ \text{V})^2}{211.90\ \Omega}(0.9438) = 64.1\ \text{W}. \quad \text{(Answer)}$$

Second way: Instead, we can write

$$P_{avg} = I^2_{rms} R = \frac{\mathscr{E}^2_{rms}}{Z^2} R$$
$$= \frac{(120\ \text{V})^2}{(211.90\ \Omega)^2}(200\ \Omega) = 64.1\ \text{W}. \quad \text{(Answer)}$$

(c) What new capacitance C_{new} is needed to maximize P_{avg} if the other parameters of the circuit are not changed?

KEY IDEAS

(1) The average rate P_{avg} at which energy is supplied and dissipated is maximized if the circuit is brought into resonance with the driving emf. (2) Resonance occurs when $X_C = X_L$.

Calculations: From the given data, we have $X_C > X_L$. Thus, we must decrease X_C to reach resonance. From Eq. 31-39 ($X_C = 1/\omega_d C$), we see that this means we must increase C to the new value C_{new}.

Using Eq. 31-39, we can write the resonance condition $X_C = X_L$ as

$$\frac{1}{\omega_d C_{new}} = X_L.$$

Substituting $2\pi f_d$ for ω_d (because we are given f_d and not ω_d) and then solving for C_{new}, we find

$$C_{new} = \frac{1}{2\pi f_d X_L} = \frac{1}{(2\pi)(60\ \text{Hz})(80.0\ \Omega)}$$
$$= 3.32 \times 10^{-5}\ \text{F} = 33.2\ \mu\text{F}. \quad \text{(Answer)}$$

Following the procedure of part (b), you can show that with C_{new}, the average power of energy dissipation P_{avg} would then be at its maximum value of

$$P_{avg,\max} = 72.0\ \text{W}.$$

31-6 TRANSFORMERS

Learning Objectives

After reading this module, you should be able to . . .

31.49 For power transmission lines, identify why the transmission should be at low current and high voltage.

31.50 Identify the role of transformers at the two ends of a transmission line.

31.51 Calculate the energy dissipation in a transmission line.

31.52 Identify a transformer's primary and secondary.

31.53 Apply the relationship between the voltage and number of turns on the two sides of a transformer.

31.54 Distinguish between a step-down transformer and a step-up transformer.

31.55 Apply the relationship between the current and number of turns on the two sides of a transformer.

31.56 Apply the relationship between the power into and out of an ideal transformer.

31.57 Identify the equivalent resistance as seen from the primary side of a transformer.

31.58 Apply the relationship between the equivalent resistance and the actual resistance.

31.59 Explain the role of a transformer in impedance matching.

Key Ideas

- A transformer (assumed to be ideal) is an iron core on which are wound a primary coil of N_p turns and a secondary coil of N_s turns. If the primary coil is connected across an alternating-current generator, the primary and secondary voltages are related by

$$V_s = V_p \frac{N_s}{N_p} \quad \text{(transformation of voltage).}$$

- The currents through the coils are related by

$$I_s = I_p \frac{N_p}{N_s} \quad \text{(transformation of currents).}$$

- The equivalent resistance of the secondary circuit, as seen by the generator, is

$$R_{eq} = \left(\frac{N_p}{N_s}\right)^2 R,$$

where R is the resistive load in the secondary circuit. The ratio N_p/N_s is called the transformer's turns ratio.

Transformers

Energy Transmission Requirements

When an ac circuit has only a resistive load, the power factor in Eq. 31-76 is $\cos 0° = 1$ and the applied rms emf $\mathscr{E}_{rms}$ is equal to the rms voltage V_{rms} across the load. Thus, with an rms current I_{rms} in the load, energy is supplied and dissipated at the average rate of

$$P_{avg} = \mathscr{E}I = IV. \tag{31-77}$$

(In Eq. 31-77 and the rest of this module, we follow conventional practice and drop the subscripts identifying rms quantities. Engineers and scientists assume that all time-varying currents and voltages are reported as rms values; that is what the meters read.) Equation 31-77 tells us that, to satisfy a given power requirement, we have a range of choices for I and V, provided only that the product IV is as required.

In electrical power distribution systems it is desirable for reasons of safety and for efficient equipment design to deal with relatively low voltages at both the generating end (the electrical power plant) and the receiving end (the home or factory). Nobody wants an electric toaster to operate at, say, 10 kV. However, in the transmission of electrical energy from the generating plant to the consumer, we want the lowest practical current (hence the largest practical voltage) to minimize I^2R losses (often called *ohmic losses*) in the transmission line.

As an example, consider the 735 kV line used to transmit electrical energy from the La Grande 2 hydroelectric plant in Quebec to Montreal, 1000 km away. Suppose that the current is 500 A and the power factor is close to unity. Then from Eq. 31-77, energy is supplied at the average rate

$$P_{avg} = \mathscr{E}I = (7.35 \times 10^5 \text{ V})(500 \text{ A}) = 368 \text{ MW}.$$

The resistance of the transmission line is about 0.220 Ω/km; thus, there is a total resistance of about 220 Ω for the 1000 km stretch. Energy is dissipated due to that resistance at a rate of about

$$P_{\text{avg}} = I^2R = (500 \text{ A})^2(220\ \Omega) = 55.0 \text{ MW},$$

which is nearly 15% of the supply rate.

Imagine what would happen if we doubled the current and halved the voltage. Energy would be supplied by the plant at the same average rate of 368 MW as previously, but now energy would be dissipated at the rate of about

$$P_{\text{avg}} = I^2R = (1000 \text{ A})^2(220\ \Omega) = 220 \text{ MW},$$

which is *almost 60% of the supply rate.* Hence the general energy transmission rule: Transmit at the highest possible voltage and the lowest possible current.

The Ideal Transformer

The transmission rule leads to a fundamental mismatch between the requirement for efficient high-voltage transmission and the need for safe low-voltage generation and consumption. We need a device with which we can raise (for transmission) and lower (for use) the ac voltage in a circuit, keeping the product current × voltage essentially constant. The **transformer** is such a device. It has no moving parts, operates by Faraday's law of induction, and has no simple direct-current counterpart.

The *ideal transformer* in Fig. 31-18 consists of two coils, with different numbers of turns, wound around an iron core. (The coils are insulated from the core.) In use, the primary winding, of N_p turns, is connected to an alternating-current generator whose emf $\mathscr{E}$ at any time t is given by

$$\mathscr{E} = \mathscr{E}_m \sin \omega t. \tag{31-78}$$

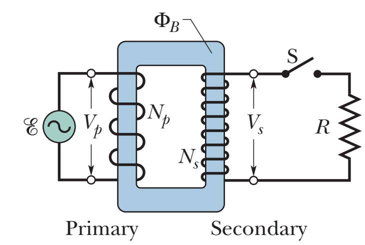


Figure 31-18 An ideal transformer (two coils wound on an iron core) in a basic transformer circuit. An ac generator produces current in the coil at the left (the *primary*). The coil at the right (the *secondary*) is connected to the resistive load R when switch S is closed.

The secondary winding, of N_s turns, is connected to load resistance R, but its circuit is an open circuit as long as switch S is open (which we assume for the present). Thus, there can be no current through the secondary coil. We assume further for this ideal transformer that the resistances of the primary and secondary windings are negligible. Well-designed, high-capacity transformers can have energy losses as low as 1%; so our assumptions are reasonable.

For the assumed conditions, the primary winding (or *primary*) is a pure inductance and the primary circuit is like that in Fig. 31-12. Thus, the (very small) primary current, also called the *magnetizing current* I_{mag}, lags the primary voltage V_p by 90°; the primary's power factor (= cos ϕ in Eq. 31-76) is zero; so no power is delivered from the generator to the transformer.

However, the small sinusoidally changing primary current I_{mag} produces a sinusoidally changing magnetic flux Φ_B in the iron core. The core acts to strengthen the flux and to bring it through the secondary winding (or *secondary*). Because Φ_B varies, it induces an emf $\mathscr{E}_{\text{turn}}$ (= $d\Phi_B/dt$) in each turn of the secondary. In fact, this emf per turn $\mathscr{E}_{\text{turn}}$ is the same in the primary and the secondary. Across the primary, the voltage V_p is the product of $\mathscr{E}_{\text{turn}}$ and the number of turns N_p; that is, $V_p = \mathscr{E}_{\text{turn}}N_p$. Similarly, across the secondary the voltage is $V_s = \mathscr{E}_{\text{turn}}N_s$. Thus, we can write

$$\mathscr{E}_{\text{turn}} = \frac{V_p}{N_p} = \frac{V_s}{N_s},$$

or

$$V_s = V_p \frac{N_s}{N_p} \quad \text{(transformation of voltage).} \tag{31-79}$$

If $N_s > N_p$, the device is a *step-up transformer* because it steps the primary's voltage V_p *up* to a higher voltage V_s. Similarly, if $N_s < N_p$, it is a *step-down transformer.*

With switch S open, no energy is transferred from the generator to the rest of the circuit, but when we close S to connect the secondary to the resistive load R, energy *is* transferred. (In general, the load would also contain inductive and capacitive elements, but here we consider just resistance R.) Here is the process:

1. An alternating current I_s appears in the secondary circuit, with corresponding energy dissipation rate I_s^2R $(= V_s^2/R)$ in the resistive load.
2. This current produces its own alternating magnetic flux in the iron core, and this flux induces an opposing emf in the primary windings.
3. The voltage V_p of the primary, however, cannot change in response to this opposing emf because it must always be equal to the emf $\mathscr{E}$ that is provided by the generator; closing switch S cannot change this fact.
4. To maintain V_p, the generator now produces (in addition to I_{mag}) an alternating current I_p in the primary circuit; the magnitude and phase constant of I_p are just those required for the emf induced by I_p in the primary to exactly cancel the emf induced there by I_s. Because the phase constant of I_p is not 90° like that of I_{mag}, this current I_p can transfer energy to the primary.

Energy Transfers. We want to relate I_s to I_p. However, rather than analyze the foregoing complex process in detail, let us just apply the principle of conservation of energy. The rate at which the generator transfers energy to the primary is equal to I_pV_p. The rate at which the primary then transfers energy to the secondary (via the alternating magnetic field linking the two coils) is I_sV_s. Because we assume that no energy is lost along the way, conservation of energy requires that

$$I_pV_p = I_sV_s.$$

Substituting for V_s from Eq. 31-79, we find that

$$I_s = I_p\frac{N_p}{N_s} \quad \text{(transformation of currents)}. \tag{31-80}$$

This equation tells us that the current I_s in the secondary can differ from the current I_p in the primary, depending on the *turns ratio* N_p/N_s.

Current I_p appears in the primary circuit because of the resistive load R in the secondary circuit. To find I_p, we substitute $I_s = V_s/R$ into Eq. 31-80 and then we substitute for V_s from Eq. 31-79. We find

$$I_p = \frac{1}{R}\left(\frac{N_s}{N_p}\right)^2 V_p. \tag{31-81}$$

This equation has the form $I_p = V_p/R_{\text{eq}}$, where equivalent resistance R_{eq} is

$$R_{\text{eq}} = \left(\frac{N_p}{N_s}\right)^2 R. \tag{31-82}$$

This R_{eq} is the value of the load resistance as "seen" by the generator; the generator produces the current I_p and voltage V_p as if the generator were connected to a resistance R_{eq}.

Impedance Matching

Equation 31-82 suggests still another function for the transformer. For maximum transfer of energy from an emf device to a resistive load, the resistance of the emf device must equal the resistance of the load. The same relation holds for ac circuits except that the *impedance* (rather than just the resistance) of the generator must equal that of the load. Often this condition is not met. For example, in a music-playing system, the amplifier has high impedance and the speaker set has low impedance. We can match the impedances of the two devices by coupling them through a transformer that has a suitable turns ratio N_p/N_s.

Checkpoint 8

An alternating-current emf device in a certain circuit has a smaller resistance than that of the resistive load in the circuit; to increase the transfer of energy from the device to the load, a transformer will be connected between the two. (a) Should N_s be greater than or less than N_p? (b) Will that make it a step-up or step-down transformer?

Sample Problem 31.08 Transformer: turns ratio, average power, rms currents

A transformer on a utility pole operates at $V_p = 8.5$ kV on the primary side and supplies electrical energy to a number of nearby houses at $V_s = 120$ V, both quantities being rms values. Assume an ideal step-down transformer, a purely resistive load, and a power factor of unity.

(a) What is the turns ratio N_p/N_s of the transformer?

KEY IDEA

The turns ratio N_p/N_s is related to the (given) rms primary and secondary voltages via Eq. 31-79 ($V_s = V_p N_s/N_p$).

Calculation: We can write Eq. 31-79 as

$$\frac{V_s}{V_p} = \frac{N_s}{N_p}. \tag{31-83}$$

(Note that the right side of this equation is the *inverse* of the turns ratio.) Inverting both sides of Eq. 31-83 gives us

$$\frac{N_p}{N_s} = \frac{V_p}{V_s} = \frac{8.5 \times 10^3\ \text{V}}{120\ \text{V}} = 70.83 \approx 71. \qquad \text{(Answer)}$$

(b) The average rate of energy consumption (or dissipation) in the houses served by the transformer is 78 kW. What are the rms currents in the primary and secondary of the transformer?

KEY IDEA

For a purely resistive load, the power factor cos ϕ is unity; thus, the average rate at which energy is supplied and dissipated is given by Eq. 31-77 ($P_{avg} = \mathscr{E}I = IV$).

Calculations: In the primary circuit, with $V_p = 8.5$ kV, Eq. 31-77 yields

$$I_p = \frac{P_{avg}}{V_p} = \frac{78 \times 10^3\ \text{W}}{8.5 \times 10^3\ \text{V}} = 9.176\ \text{A} \approx 9.2\ \text{A}. \qquad \text{(Answer)}$$

Similarly, in the secondary circuit,

$$I_s = \frac{P_{avg}}{V_s} = \frac{78 \times 10^3\ \text{W}}{120\ \text{V}} = 650\ \text{A}. \qquad \text{(Answer)}$$

You can check that $I_s = I_p(N_p/N_s)$ as required by Eq. 31-80.

(c) What is the resistive load R_s in the secondary circuit? What is the corresponding resistive load R_p in the primary circuit?

One way: We can use $V = IR$ to relate the resistive load to the rms voltage and current. For the secondary circuit, we find

$$R_s = \frac{V_s}{I_s} = \frac{120\ \text{V}}{650\ \text{A}} = 0.1846\ \Omega \approx 0.18\ \Omega. \qquad \text{(Answer)}$$

Similarly, for the primary circuit we find

$$R_p = \frac{V_p}{I_p} = \frac{8.5 \times 10^3\ \text{V}}{9.176\ \text{A}} = 926\ \Omega \approx 930\ \Omega. \qquad \text{(Answer)}$$

Second way: We use the fact that R_p equals the equivalent resistive load "seen" from the primary side of the transformer, which is a resistance modified by the turns ratio and given by Eq. 31-82 ($R_{eq} = (N_p/N_s)^2 R$). If we substitute R_p for R_{eq} and R_s for R, that equation yields

$$R_p = \left(\frac{N_p}{N_s}\right)^2 R_s = (70.83)^2(0.1846\ \Omega)$$

$$= 926\ \Omega \approx 930\ \Omega. \qquad \text{(Answer)}$$

WILEY PLUS Additional examples, video, and practice available at *WileyPLUS*

Review & Summary

LC Energy Transfers In an oscillating LC circuit, energy is shuttled periodically between the electric field of the capacitor and the magnetic field of the inductor; instantaneous values of the two forms of energy are

$$U_E = \frac{q^2}{2C} \quad \text{and} \quad U_B = \frac{Li^2}{2}, \tag{31-1, 31-2}$$

where q is the instantaneous charge on the capacitor and i is the instantaneous current through the inductor. The total energy $U\ (= U_E + U_B)$ remains constant.

LC Charge and Current Oscillations The principle of conservation of energy leads to

$$L\frac{d^2q}{dt^2} + \frac{1}{C}q = 0 \quad (LC \text{ oscillations}) \tag{31-11}$$

as the differential equation of LC oscillations (with no resistance). The solution of Eq. 31-11 is

$$q = Q\cos(\omega t + \phi) \quad \text{(charge)}, \tag{31-12}$$

in which Q is the *charge amplitude* (maximum charge on the capacitor) and the angular frequency ω of the oscillations is

$$\omega = \frac{1}{\sqrt{LC}}. \tag{31-4}$$

The phase constant ϕ in Eq. 31-12 is determined by the initial conditions (at $t = 0$) of the system.

The current i in the system at any time t is

$$i = -\omega Q\sin(\omega t + \phi) \quad \text{(current)}, \tag{31-13}$$

in which ωQ is the *current amplitude* I.

Damped Oscillations Oscillations in an LC circuit are damped when a dissipative element R is also present in the circuit. Then

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = 0 \quad (RLC \text{ circuit}). \tag{31-24}$$

The solution of this differential equation is

$$q = Qe^{-Rt/2L}\cos(\omega' t + \phi), \tag{31-25}$$

where

$$\omega' = \sqrt{\omega^2 - (R/2L)^2}. \tag{31-26}$$

We consider only situations with small R and thus small damping; then $\omega' \approx \omega$.

Alternating Currents; Forced Oscillations A series RLC circuit may be set into *forced oscillation* at a *driving angular frequency* ω_d by an external alternating emf

$$\mathscr{E} = \mathscr{E}_m \sin\omega_d t. \tag{31-28}$$

The current driven in the circuit is

$$i = I\sin(\omega_d t - \phi), \tag{31-29}$$

where ϕ is the phase constant of the current.

Resonance The current amplitude I in a series RLC circuit driven by a sinusoidal external emf is a maximum ($I = \mathscr{E}_m/R$) when the driving angular frequency ω_d equals the natural angular frequency ω of the circuit (that is, at *resonance*). Then $X_C = X_L$, $\phi = 0$, and the current is in phase with the emf.

Single Circuit Elements The alternating potential difference across a resistor has amplitude $V_R = IR$; the current is in phase with the potential difference.

For a *capacitor,* $V_C = IX_C$, in which $X_C = 1/\omega_d C$ is the **capacitive reactance;** the current here leads the potential difference by 90° ($\phi = -90° = -\pi/2$ rad).

For an *inductor,* $V_L = IX_L$, in which $X_L = \omega_d L$ is the **inductive reactance;** the current here lags the potential difference by 90° ($\phi = +90° = +\pi/2$ rad).

Series *RLC* Circuits For a series RLC circuit with an alternating external emf given by Eq. 31-28 and a resulting alternating current given by Eq. 31-29,

$$I = \frac{\mathscr{E}_m}{\sqrt{R^2 + (X_L - X_C)^2}} = \frac{\mathscr{E}_m}{\sqrt{R^2 + (\omega_d L - 1/\omega_d C)^2}} \quad \text{(current amplitude)} \tag{31-60, 31-63}$$

and

$$\tan\phi = \frac{X_L - X_C}{R} \quad \text{(phase constant)}. \tag{31-65}$$

Defining the impedance Z of the circuit as

$$Z = \sqrt{R^2 + (X_L - X_C)^2} \quad \text{(impedance)} \tag{31-61}$$

allows us to write Eq. 31-60 as $I = \mathscr{E}_m/Z$.

Power In a series RLC circuit, the **average power** P_{avg} of the generator is equal to the production rate of thermal energy in the resistor:

$$P_{avg} = I_{rms}^2 R = \mathscr{E}_{rms} I_{rms}\cos\phi. \tag{31-71, 31-76}$$

Here rms stands for **root-mean-square;** the rms quantities are related to the maximum quantities by $I_{rms} = I/\sqrt{2}$, $V_{rms} = V/\sqrt{2}$, and $\mathscr{E}_{rms} = \mathscr{E}_m/\sqrt{2}$. The term $\cos\phi$ is called the **power factor** of the circuit.

Transformers A *transformer* (assumed to be ideal) is an iron core on which are wound a primary coil of N_p turns and a secondary coil of N_s turns. If the primary coil is connected across an alternating-current generator, the primary and secondary voltages are related by

$$V_s = V_p\frac{N_s}{N_p} \quad \text{(transformation of voltage)}. \tag{31-79}$$

The currents through the coils are related by

$$I_s = I_p\frac{N_p}{N_s} \quad \text{(transformation of currents)}, \tag{31-80}$$

and the equivalent resistance of the secondary circuit, as seen by the generator, is

$$R_{eq} = \left(\frac{N_p}{N_s}\right)^2 R, \tag{31-82}$$

where R is the resistive load in the secondary circuit. The ratio N_p/N_s is called the transformer's *turns ratio.*

Questions

1 Figure 31-19 shows three oscillating LC circuits with identical inductors and capacitors. At a particular time, the charges on the capacitor plates (and thus the electric fields between the plates) are all at their maximum values. Rank the circuits according to the time taken to fully discharge the capacitors during the oscillations, greatest first.

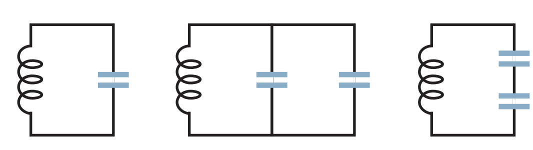


Figure 31-19 Question 1.

2 Figure 31-20 shows graphs of capacitor voltage v_C for LC circuits 1 and 2, which contain identical capacitances and have the same maximum charge Q. Are (a) the inductance L and (b) the maximum current I in circuit 1 greater than, less than, or the same as those in circuit 2?

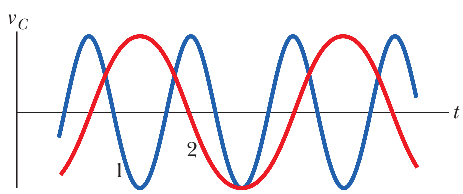


Figure 31-20 Question 2.

3 A charged capacitor and an inductor are connected at time $t = 0$. In terms of the period T of the resulting oscillations, what is the first later time at which the following reach a maximum: (a) U_B, (b) the magnetic flux through the inductor, (c) di/dt, and (d) the emf of the inductor?

4 What values of phase constant ϕ in Eq. 31-12 allow situations (*a*), (*c*), (*e*), and (*g*) of Fig. 31-1 to occur at $t = 0$?

5 Curve a in Fig. 31-21 gives the impedance Z of a driven RC circuit versus the driving angular frequency ω_d. The other two curves are similar but for different values of resistance R and capacitance C. Rank the three curves according to the corresponding value of R, greatest first.

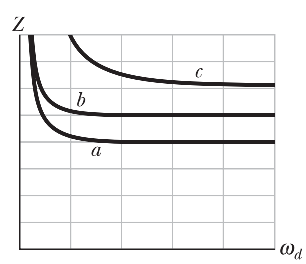


Figure 31-21 Question 5.

6 Charges on the capacitors in three oscillating LC circuits vary as: (1) $q = 2 \cos 4t$, (2) $q = 4 \cos t$, (3) $q = 3 \cos 4t$ (with q in coulombs and t in seconds). Rank the circuits according to (a) the current amplitude and (b) the period, greatest first.

7 An alternating emf source with a certain emf amplitude is connected, in turn, to a resistor, a capacitor, and then an inductor. Once connected to one of the devices, the driving frequency f_d is varied and the amplitude I of the resulting current through the device is measured and plotted. Which of the three plots in Fig. 31-22 corresponds to which of the three devices?

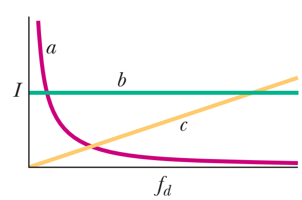


Figure 31-22 Question 7.

8 The values of the phase constant ϕ for four sinusoidally driven series RLC circuits are (1) $-15°$, (2) $+35°$, (3) $\pi/3$ rad, and (4) $-\pi/6$ rad. (a) In which is the load primarily capacitive? (b) In which does the current lag the alternating emf?

9 Figure 31-23 shows the current i and driving emf $\mathscr{E}$ for a series RLC circuit. (a) Is the phase constant positive or negative? (b) To increase the rate at which energy is transferred to the resistive load, should L be increased or decreased? (c) Should, instead, C be increased or decreased?

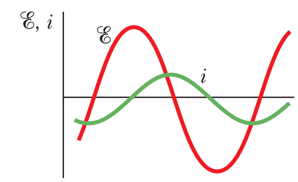


Figure 31-23 Question 9.

10 Figure 31-24 shows three situations like those of Fig. 31-15. Is the driving angular frequency greater than, less than, or equal to the resonant angular frequency of the circuit in (a) situation 1, (b) situation 2, and (c) situation 3?

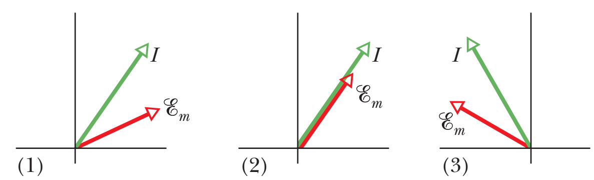


Figure 31-24 Question 10.

11 Figure 31-25 shows the current i and driving emf $\mathscr{E}$ for a series RLC circuit. Relative to the emf curve, does the current curve shift leftward or rightward and does the amplitude of that curve increase or decrease if we slightly increase (a) L, (b) C, and (c) ω_d?

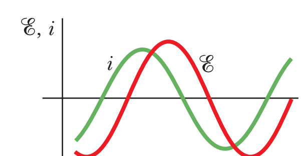


Figure 31-25 Questions 11 and 12.

12 Figure 31-25 shows the current i and driving emf $\mathscr{E}$ for a series RLC circuit. (a) Does the current lead or lag the emf? (b) Is the circuit's load mainly capacitive or mainly inductive? (c) Is the angular frequency ω_d of the emf greater than or less than the natural angular frequency ω?

13 Does the phasor diagram of Fig. 31-26 correspond to an alternating emf source connected to a resistor, a capacitor, or an inductor? (b) If the angular speed of the phasors is increased, does the length of the current phasor increase or decrease when the scale of the diagram is maintained?

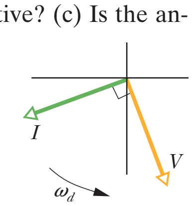


Figure 31-26 Question 13.

Problems

GO Tutoring problem available (at instructor's discretion) in *WileyPLUS* and WebAssign
SSM Worked-out solution available in Student Solutions Manual
• – ••• Number of dots indicates level of problem difficulty
Additional information available in *The Flying Circus of Physics* and at flyingcircusofphysics.com
WWW Worked-out solution is at
ILW Interactive solution is at
http://www.wiley.com/college/halliday

Module 31-1 *LC* Oscillations

•1 An oscillating LC circuit consists of a 75.0 mH inductor and a 3.60 μF capacitor. If the maximum charge on the capacitor is 2.90 μC, what are (a) the total energy in the circuit and (b) the maximum current?

•2 The frequency of oscillation of a certain LC circuit is 200 kHz. At time $t = 0$, plate A of the capacitor has maximum positive charge. At what earliest time $t > 0$ will (a) plate A again have maximum positive charge, (b) the other plate of the capacitor have maximum positive charge, and (c) the inductor have maximum magnetic field?

•3 In a certain oscillating LC circuit, the total energy is converted from electrical energy in the capacitor to magnetic energy in the inductor in 1.50 μs. What are (a) the period of oscillation and (b) the frequency of oscillation? (c) How long after the magnetic energy is a maximum will it be a maximum again?

•4 What is the capacitance of an oscillating LC circuit if the maximum charge on the capacitor is 1.60 μC and the total energy is 140 μJ?

•5 In an oscillating LC circuit, $L = 1.10$ mH and $C = 4.00$ μF. The maximum charge on the capacitor is 3.00 μC. Find the maximum current.

•6 A 0.50 kg body oscillates in SHM on a spring that, when extended 2.0 mm from its equilibrium position, has an 8.0 N restoring force. What are (a) the angular frequency of oscillation, (b) the period of oscillation, and (c) the capacitance of an LC circuit with the same period if L is 5.0 H?

•7 SSM The energy in an oscillating LC circuit containing a 1.25 H inductor is 5.70 μJ. The maximum charge on the capacitor is 175 μC. For a mechanical system with the same period, find the (a) mass, (b) spring constant, (c) maximum displacement, and (d) maximum speed.

•8 A single loop consists of inductors ($L_1, L_2, \ldots$), capacitors ($C_1, C_2, \ldots$), and resistors ($R_1, R_2, \ldots$) connected in series as shown, for example, in Fig. 31-27*a*. Show that regardless of the sequence of these circuit elements in the loop, the behavior of this circuit is identical to that of the simple LC circuit shown in Fig. 31-27*b*. (*Hint:* Consider the loop rule and see Problem 47 in Chapter 30.)

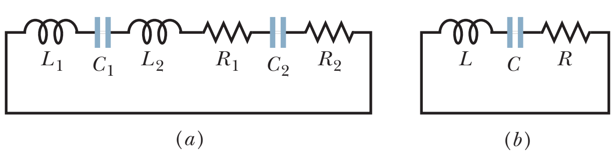


Figure 31-27 Problem 8.

•9 ILW In an oscillating LC circuit with $L = 50$ mH and $C = 4.0$ μF, the current is initially a maximum. How long will it take before the capacitor is fully charged for the first time?

•10 LC oscillators have been used in circuits connected to loudspeakers to create some of the sounds of electronic music. What inductance must be used with a 6.7 μF capacitor to produce a frequency of 10 kHz, which is near the middle of the audible range of frequencies?

••11 SSM WWW A variable capacitor with a range from 10 to 365 pF is used with a coil to form a variable-frequency LC circuit to tune the input to a radio. (a) What is the ratio of maximum frequency to minimum frequency that can be obtained with such a capacitor? If this circuit is to obtain frequencies from 0.54 MHz to 1.60 MHz, the ratio computed in (a) is too large. By adding a capacitor in parallel to the variable capacitor, this range can be adjusted. To obtain the desired frequency range, (b) what capacitance should be added and (c) what inductance should the coil have?

••12 In an oscillating LC circuit, when 75.0% of the total energy is stored in the inductor's magnetic field, (a) what multiple of the maximum charge is on the capacitor and (b) what multiple of the maximum current is in the inductor?

••13 In an oscillating LC circuit, $L = 3.00$ mH and $C = 2.70$ μF. At $t = 0$ the charge on the capacitor is zero and the current is 2.00 A. (a) What is the maximum charge that will appear on the capacitor? (b) At what earliest time $t > 0$ is the rate at which energy is stored in the capacitor greatest, and (c) what is that greatest rate?

••14 To construct an oscillating LC system, you can choose from a 10 mH inductor, a 5.0 μF capacitor, and a 2.0 μF capacitor. What are the (a) smallest, (b) second smallest, (c) second largest, and (d) largest oscillation frequency that can be set up by these elements in various combinations?

••15 ILW An oscillating LC circuit consisting of a 1.0 nF capacitor and a 3.0 mH coil has a maximum voltage of 3.0 V. What are (a) the maximum charge on the capacitor, (b) the maximum current through the circuit, and (c) the maximum energy stored in the magnetic field of the coil?

••16 An inductor is connected across a capacitor whose capacitance can be varied by turning a knob. We wish to make the frequency of oscillation of this LC circuit vary linearly with the angle of rotation of the knob, going from 2×10^5 to 4×10^5 Hz as the knob turns through 180°. If $L = 1.0$ mH, plot the required capacitance C as a function of the angle of rotation of the knob.

••17 GO ILW In Fig. 31-28, $R = 14.0$ Ω, $C = 6.20$ μF, and $L = 54.0$ mH, and the ideal battery has emf $\mathscr{E} = 34.0$ V. The switch is kept at a for a long time and then thrown to position b. What are the (a) frequency and (b) current amplitude of the resulting oscillations?

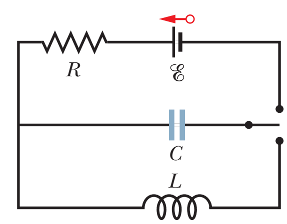


Figure 31-28 Problem 17.

••18 An oscillating LC circuit has a current amplitude of 7.50 mA, a potential amplitude of 250 mV, and a capacitance of 220 nF. What are (a) the period of oscillation, (b) the maximum energy stored in the capacitor, (c) the maximum energy stored in the inductor, (d) the maximum rate at which the current changes, and (e) the maximum rate at which the inductor gains energy?

••19 Using the loop rule, derive the differential equation for an LC circuit (Eq. 31-11).

••20 GO In an oscillating LC circuit in which $C = 4.00$ μF, the maximum potential difference across the capacitor during the oscillations is 1.50 V and the maximum current through the inductor is 50.0 mA. What are (a) the inductance L and (b) the frequency of the oscillations? (c) How much time is required for the charge on the capacitor to rise from zero to its maximum value?

••21 ILW In an oscillating LC circuit with $C = 64.0$ μF, the current is given by $i = (1.60)\sin(2500t + 0.680)$, where t is in seconds, i in amperes, and the phase constant in radians. (a) How soon after $t = 0$ will the current reach its maximum value? What are (b) the inductance L and (c) the total energy?

••22 A series circuit containing inductance L_1 and capacitance C_1 oscillates at angular frequency ω. A second series circuit, containing inductance L_2 and capacitance C_2, oscillates at the same angular frequency. In terms of ω, what is the angular frequency of oscillation of a series circuit containing all four of these elements? Neglect resistance. (*Hint:* Use the formulas for equivalent capacitance and equivalent inductance; see Module 25-3 and Problem 47 in Chapter 30.)

••23 GO In an oscillating LC circuit, $L = 25.0$ mH and $C = 7.80\ \mu$F. At time $t = 0$ the current is 9.20 mA, the charge on the capacitor is 3.80 μC, and the capacitor is charging. What are (a) the total energy in the circuit, (b) the maximum charge on the capacitor, and (c) the maximum current? (d) If the charge on the capacitor is given by $q = Q\cos(\omega t + \phi)$, what is the phase angle ϕ? (e) Suppose the data are the same, except that the capacitor is discharging at $t = 0$. What then is ϕ?

Module 31-2 Damped Oscillations in an *RLC* Circuit

••24 GO A single-loop circuit consists of a 7.20 Ω resistor, a 12.0 H inductor, and a 3.20 μF capacitor. Initially the capacitor has a charge of 6.20 μC and the current is zero. Calculate the charge on the capacitor N complete cycles later for (a) $N = 5$, (b) $N = 10$, and (c) $N = 100$.

••25 ILW What resistance R should be connected in series with an inductance $L = 220$ mH and capacitance $C = 12.0\ \mu$F for the maximum charge on the capacitor to decay to 99.0% of its initial value in 50.0 cycles? (Assume $\omega' \approx \omega$.)

••26 GO In an oscillating series RLC circuit, find the time required for the maximum energy present in the capacitor during an oscillation to fall to half its initial value. Assume $q = Q$ at $t = 0$.

•••27 SSM In an oscillating series RLC circuit, show that $\Delta U/U$, the fraction of the energy lost per cycle of oscillation, is given to a close approximation by $2\pi R/\omega L$. The quantity $\omega L/R$ is often called the Q of the circuit (for *quality*). A high-Q circuit has low resistance and a low fractional energy loss ($= 2\pi/Q$) per cycle.

Module 31-3 Forced Oscillations of Three Simple Circuits

•28 A 1.50 μF capacitor is connected as in Fig. 31-10 to an ac generator with $\mathscr{E}_m = 30.0$ V. What is the amplitude of the resulting alternating current if the frequency of the emf is (a) 1.00 kHz and (b) 8.00 kHz?

•29 ILW A 50.0 mH inductor is connected as in Fig. 31-12 to an ac generator with $\mathscr{E}_m = 30.0$ V. What is the amplitude of the resulting alternating current if the frequency of the emf is (a) 1.00 kHz and (b) 8.00 kHz?

•30 A 50.0 Ω resistor is connected as in Fig. 31-8 to an ac generator with $\mathscr{E}_m = 30.0$ V. What is the amplitude of the resulting alternating current if the frequency of the emf is (a) 1.00 kHz and (b) 8.00 kHz?

•31 (a) At what frequency would a 6.0 mH inductor and a 10 μF capacitor have the same reactance? (b) What would the reactance be? (c) Show that this frequency would be the natural frequency of an oscillating circuit with the same L and C.

••32 GO An ac generator has emf $\mathscr{E} = \mathscr{E}_m \sin \omega_d t$, with $\mathscr{E}_m = 25.0$ V and $\omega_d = 377$ rad/s. It is connected to a 12.7 H inductor. (a) What is the maximum value of the current? (b) When the current is a maximum, what is the emf of the generator? (c) When the emf of the generator is -12.5 V and increasing in magnitude, what is the current?

••33 SSM An ac generator has emf $\mathscr{E} = \mathscr{E}_m \sin(\omega_d t - \pi/4)$, where $\mathscr{E}_m = 30.0$ V and $\omega_d = 350$ rad/s. The current produced in a connected circuit is $i(t) = I\sin(\omega_d t - 3\pi/4)$, where $I = 620$ mA. At what time after $t = 0$ does (a) the generator emf first reach a maximum and (b) the current first reach a maximum? (c) The circuit contains a single element other than the generator. Is it a capacitor, an inductor, or a resistor? Justify your answer. (d) What is the value of the capacitance, inductance, or resistance, as the case may be?

••34 GO An ac generator with emf $\mathscr{E} = \mathscr{E}_m \sin \omega_d t$, where $\mathscr{E}_m = 25.0$ V and $\omega_d = 377$ rad/s, is connected to a 4.15 μF capacitor. (a) What is the maximum value of the current? (b) When the current is a maximum, what is the emf of the generator? (c) When the emf of the generator is -12.5 V and increasing in magnitude, what is the current?

Module 31-4 The Series *RLC* Circuit

•35 ILW A coil of inductance 88 mH and unknown resistance and a 0.94 μF capacitor are connected in series with an alternating emf of frequency 930 Hz. If the phase constant between the applied voltage and the current is 75°, what is the resistance of the coil?

•36 An alternating source with a variable frequency, a capacitor with capacitance C, and a resistor with resistance R are connected in series. Figure 31-29 gives the impedance Z of the circuit versus the driving angular frequency ω_d; the curve reaches an asymptote of 500 Ω, and the horizontal scale is set by $\omega_{ds} = 300$ rad/s. The figure also gives the reactance X_C for the capacitor versus ω_d. What are (a) R and (b) C?

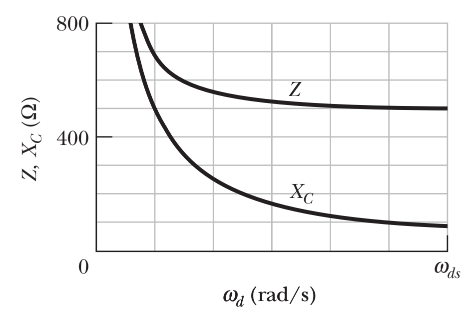


Figure 31-29 Problem 36.

•37 An electric motor has an effective resistance of 32.0 Ω and an inductive reactance of 45.0 Ω when working under load. The voltage amplitude across the alternating source is 420 V. Calculate the current amplitude.

•38 The current amplitude I versus driving angular frequency ω_d for a driven RLC circuit is given in Fig. 31-30, where the vertical axis scale is set by $I_s = 4.00$ A. The inductance is 200 μH, and the emf amplitude is 8.0 V. What are (a) C and (b) R?

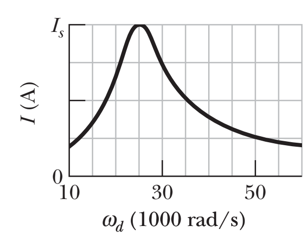


Figure 31-30 Problem 38.

•39 Remove the inductor from the circuit in Fig. 31-7 and set $R = 200$ Ω, $C = 15.0\ \mu$F, $f_d = 60.0$ Hz, and $\mathscr{E}_m = 36.0$ V. What are (a) Z, (b) ϕ, and (c) I? (d) Draw a phasor diagram.

•40 An alternating source drives a series RLC circuit with an emf amplitude of 6.00 V, at a phase angle of +30.0°. When the potential difference across the capacitor reaches its maximum positive value of +5.00 V, what is the potential difference across the inductor (sign included)?

•41 SSM In Fig. 31-7, set $R = 200$ Ω, $C = 70.0\ \mu$F, $L = 230$ mH, $f_d = 60.0$ Hz, and $\mathscr{E}_m = 36.0$ V. What are (a) Z, (b) ϕ, and (c) I? (d) Draw a phasor diagram.

•42 An alternating source with a variable frequency, an inductor with inductance L, and a resistor with resistance R are connected in series. Figure 31-31 gives the impedance Z of the circuit versus the driving angular frequency ω_d, with the horizontal axis scale set by ω_{ds} = 1600 rad/s. The figure also gives the reactance X_L for the inductor versus ω_d. What are (a) R and (b) L?

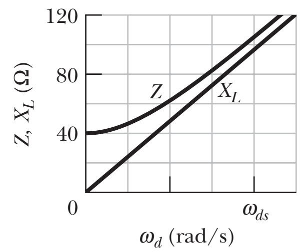


Figure 31-31 Problem 42.

•43 Remove the capacitor from the circuit in Fig. 31-7 and set R = 200 Ω, L = 230 mH, f_d = 60.0 Hz, and $\mathscr{E}_m$ = 36.0 V. What are (a) Z, (b) ϕ, and (c) I? (d) Draw a phasor diagram.

••44 GO An ac generator with emf amplitude $\mathscr{E}_m$ = 220 V and operating at frequency 400 Hz causes oscillations in a series RLC circuit having R = 220 Ω, L = 150 mH, and C = 24.0 μF. Find (a) the capacitive reactance X_C, (b) the impedance Z, and (c) the current amplitude I. A second capacitor of the same capacitance is then connected in series with the other components. Determine whether the values of (d) X_C, (e) Z, and (f) I increase, decrease, or remain the same.

••45 GO ILW (a) In an RLC circuit, can the amplitude of the voltage across an inductor be greater than the amplitude of the generator emf? (b) Consider an RLC circuit with emf amplitude $\mathscr{E}_m$ = 10 V, resistance R = 10 Ω, inductance L = 1.0 H, and capacitance C = 1.0 μF. Find the amplitude of the voltage across the inductor at resonance.

••46 GO An alternating emf source with a variable frequency f_d is connected in series with a 50.0 Ω resistor and a 20.0 μF capacitor. The emf amplitude is 12.0 V. (a) Draw a phasor diagram for phasor V_R (the potential across the resistor) and phasor V_C (the potential across the capacitor). (b) At what driving frequency f_d do the two phasors have the same length? At that driving frequency, what are (c) the phase angle in degrees, (d) the angular speed at which the phasors rotate, and (e) the current amplitude?

••47 SSM WWW An RLC circuit such as that of Fig. 31-7 has R = 5.00 Ω, C = 20.0 μF, L = 1.00 H, and $\mathscr{E}_m$ = 30.0 V. (a) At what angular frequency ω_d will the current amplitude have its maximum value, as in the resonance curves of Fig. 31-16? (b) What is this maximum value? At what (c) lower angular frequency ω_{d1} and (d) higher angular frequency ω_{d2} will the current amplitude be half this maximum value? (e) For the resonance curve for this circuit, what is the fractional half-width $(\omega_{d1} - \omega_{d2})/\omega$?

••48 GO Figure 31-32 shows a driven RLC circuit that contains two identical capacitors and two switches. The emf amplitude is set at 12.0 V, and the driving frequency is set at 60.0 Hz. With both switches open, the current leads the emf by 30.9°. With switch S_1 closed and switch S_2 still open, the emf leads the current by 15.0°. With both switches closed, the current amplitude is 447 mA. What are (a) R, (b) C, and (c) L?

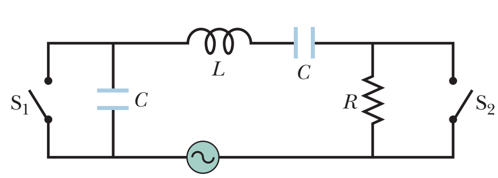


Figure 31-32 Problem 48.

••49 GO In Fig. 31-33, a generator with an adjustable frequency of oscillation is connected to resistance R = 100 Ω, inductances L_1 = 1.70 mH and L_2 = 2.30 mH, and capacitances C_1 = 4.00 μF, C_2 = 2.50 μF, and C_3 = 3.50 μF. (a) What is the resonant frequency of the circuit? (*Hint:* See Problem 47 in Chapter 30.) What happens to the resonant frequency if (b) R is increased, (c) L_1 is increased, and (d) C_3 is removed from the circuit?

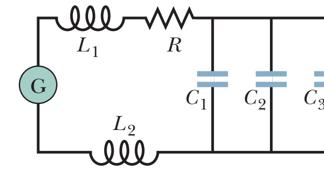


Figure 31-33 Problem 49.

••50 An alternating emf source with a variable frequency f_d is connected in series with an 80.0 Ω resistor and a 40.0 mH inductor. The emf amplitude is 6.00 V. (a) Draw a phasor diagram for phasor V_R (the potential across the resistor) and phasor V_L (the potential across the inductor). (b) At what driving frequency f_d do the two phasors have the same length? At that driving frequency, what are (c) the phase angle in degrees, (d) the angular speed at which the phasors rotate, and (e) the current amplitude?

••51 SSM The fractional half-width $\Delta\omega_d$ of a resonance curve, such as the ones in Fig. 31-16, is the width of the curve at half the maximum value of I. Show that $\Delta\omega_d/\omega = R(3C/L)^{1/2}$, where ω is the angular frequency at resonance. Note that the ratio $\Delta\omega_d/\omega$ increases with R, as Fig. 31-16 shows.

Module 31-5 Power in Alternating-Current Circuits

•52 An ac voltmeter with large impedance is connected in turn across the inductor, the capacitor, and the resistor in a series circuit having an alternating emf of 100 V (rms); the meter gives the same reading in volts in each case. What is this reading?

•53 SSM An air conditioner connected to a 120 V rms ac line is equivalent to a 12.0 Ω resistance and a 1.30 Ω inductive reactance in series. Calculate (a) the impedance of the air conditioner and (b) the average rate at which energy is supplied to the appliance.

•54 What is the maximum value of an ac voltage whose rms value is 100 V?

•55 What direct current will produce the same amount of thermal energy, in a particular resistor, as an alternating current that has a maximum value of 2.60 A?

••56 A typical light dimmer used to dim the stage lights in a theater consists of a variable inductor L (whose inductance is adjustable between zero and L_{max}) connected in series with a lightbulb B, as shown in Fig. 31-34. The electrical supply is 120 V (rms) at 60.0 Hz; the lightbulb is rated at 120 V, 1000 W. (a) What L_{max} is required if the rate of energy dissipation in the lightbulb is to be varied by a factor of 5 from its upper limit of 1000 W? Assume that the resistance of the lightbulb is independent of its temperature. (b) Could one use a variable resistor (adjustable between zero and R_{max}) instead of an inductor? (c) If so, what R_{max} is required? (d) Why isn't this done?

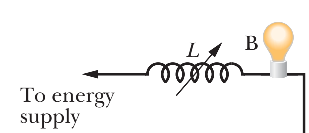


Figure 31-34 Problem 56.

••57 In an RLC circuit such as that of Fig. 31-7 assume that R = 5.00 Ω, L = 60.0 mH, f_d = 60.0 Hz, and $\mathscr{E}_m$ = 30.0 V. For what values of the capacitance would the average rate at which energy is dissipated in the resistance be (a) a maximum and (b) a minimum? What are (c) the maximum dissipation rate and the corresponding

(d) phase angle and (e) power factor? What are (f) the minimum dissipation rate and the corresponding (g) phase angle and (h) power factor?

••58 For Fig. 31-35, show that the average rate at which energy is dissipated in resistance R is a maximum when R is equal to the internal resistance r of the ac generator. (In the text discussion we tacitly assumed that $r = 0$.)

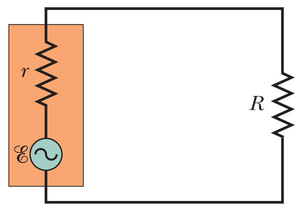


Figure 31-35 Problems 58 and 66.

••59 GO In Fig. 31-7, $R = 15.0\ \Omega$, $C = 4.70\ \mu\text{F}$, and $L = 25.0$ mH. The generator provides an emf with rms voltage 75.0 V and frequency 550 Hz. (a) What is the rms current? What is the rms voltage across (b) R, (c) C, (d) L, (e) C and L together, and (f) R, C, and L together? At what average rate is energy dissipated by (g) R, (h) C, and (i) L?

••60 GO In a series oscillating RLC circuit, $R = 16.0\ \Omega$, $C = 31.2\ \mu\text{F}$, $L = 9.20$ mH, and $\mathscr{E}_m = \mathscr{E}_m \sin \omega_d t$ with $\mathscr{E}_m = 45.0$ V and $\omega_d = 3000$ rad/s. For time $t = 0.442$ ms find (a) the rate P_g at which energy is being supplied by the generator, (b) the rate P_C at which the energy in the capacitor is changing, (c) the rate P_L at which the energy in the inductor is changing, and (d) the rate P_R at which energy is being dissipated in the resistor. (e) Is the sum of P_C, P_L, and P_R greater than, less than, or equal to P_g?

••61 SSM WWW Figure 31-36 shows an ac generator connected to a "black box" through a pair of terminals. The box contains an RLC circuit, possibly even a multiloop circuit, whose elements and connections we do not know. Measurements outside the box reveal that

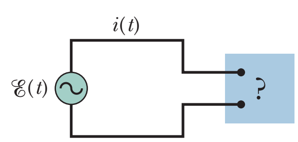


Figure 31-36 Problem 61.

$$\mathscr{E}(t) = (75.0\ \text{V}) \sin \omega_d t$$

and
$$i(t) = (1.20\ \text{A}) \sin(\omega_d t + 42.0°).$$

(a) What is the power factor? (b) Does the current lead or lag the emf? (c) Is the circuit in the box largely inductive or largely capacitive? (d) Is the circuit in the box in resonance? (e) Must there be a capacitor in the box? (f) An inductor? (g) A resistor? (h) At what average rate is energy delivered to the box by the generator? (i) Why don't you need to know ω_d to answer all these questions?

Module 31-6 Transformers

•62 A generator supplies 100 V to a transformer's primary coil, which has 50 turns. If the secondary coil has 500 turns, what is the secondary voltage?

•63 SSM ILW A transformer has 500 primary turns and 10 secondary turns. (a) If V_p is 120 V (rms), what is V_s with an open circuit? If the secondary now has a resistive load of 15 Ω, what is the current in the (b) primary and (c) secondary?

•64 Figure 31-37 shows an "autotransformer." It consists of a single coil (with an iron core). Three taps T_i are provided. Between taps T_1 and T_2 there are 200 turns, and between taps T_2 and T_3 there are 800 turns. Any two taps can be chosen as the primary terminals, and any two taps can be chosen as the secondary terminals. For choices producing a step-up transformer, what are the (a) smallest, (b) second smallest, and (c) largest values of the ratio V_s/V_p? For a step-down transformer, what are the (d) smallest, (e) second smallest, and (f) largest values of V_s/V_p?

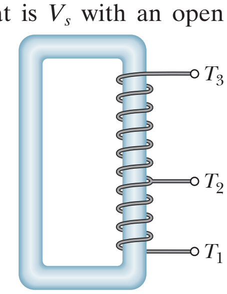


Figure 31-37 Problem 64.

••65 An ac generator provides emf to a resistive load in a remote factory over a two-cable transmission line. At the factory a step-down transformer reduces the voltage from its (rms) transmission value V_t to a much lower value that is safe and convenient for use in the factory. The transmission line resistance is 0.30 Ω/cable, and the power of the generator is 250 kW. If $V_t = 80$ kV, what are (a) the voltage decrease ΔV along the transmission line and (b) the rate P_d at which energy is dissipated in the line as thermal energy? If $V_t = 8.0$ kV, what are (c) ΔV and (d) P_d? If $V_t = 0.80$ kV, what are (e) ΔV and (f) P_d?

Additional Problems

66 In Fig. 31-35, let the rectangular box on the left represent the (high-impedance) output of an audio amplifier, with $r = 1000\ \Omega$. Let $R = 10\ \Omega$ represent the (low-impedance) coil of a loudspeaker. For maximum transfer of energy to the load R we must have $R = r$, and that is not true in this case. However, a transformer can be used to "transform" resistances, making them behave electrically as if they were larger or smaller than they actually are. (a) Sketch the primary and secondary coils of a transformer that can be introduced between the amplifier and the speaker in Fig. 31-35 to match the impedances. (b) What must be the turns ratio?

67 GO An ac generator produces emf $\mathscr{E} = \mathscr{E}_m \sin(\omega_d t - \pi/4)$, where $\mathscr{E}_m = 30.0$ V and $\omega_d = 350$ rad/s. The current in the circuit attached to the generator is $i(t) = I \sin(\omega_d t + \pi/4)$, where $I = 620$ mA. (a) At what time after $t = 0$ does the generator emf first reach a maximum? (b) At what time after $t = 0$ does the current first reach a maximum? (c) The circuit contains a single element other than the generator. Is it a capacitor, an inductor, or a resistor? Justify your answer. (d) What is the value of the capacitance, inductance, or resistance, as the case may be?

68 A series RLC circuit is driven by a generator at a frequency of 2000 Hz and an emf amplitude of 170 V. The inductance is 60.0 mH, the capacitance is 0.400 μF, and the resistance is 200 Ω. (a) What is the phase constant in radians? (b) What is the current amplitude?

69 A generator of frequency 3000 Hz drives a series RLC circuit with an emf amplitude of 120 V. The resistance is 40.0 Ω, the capacitance is 1.60 μF, and the inductance is 850 μH. What are (a) the phase constant in radians and (b) the current amplitude? (c) Is the circuit capacitive, inductive, or in resonance?

70 A 45.0 mH inductor has a reactance of 1.30 kΩ. (a) What is its operating frequency? (b) What is the capacitance of a capacitor with the same reactance at that frequency? If the frequency is doubled, what is the new reactance of (c) the inductor and (d) the capacitor?

71 An RLC circuit is driven by a generator with an emf amplitude of 80.0 V and a current amplitude of 1.25 A. The current leads the emf by 0.650 rad. What are the (a) impedance and (b) resistance of the circuit? (c) Is the circuit inductive, capacitive, or in resonance?

72 A series RLC circuit is driven in such a way that the maximum voltage across the inductor is 1.50 times the maximum voltage across the capacitor and 2.00 times the maximum voltage across the resistor. (a) What is ϕ for the circuit? (b) Is the circuit inductive, capacitive, or in resonance? The resistance is 49.9 Ω, and the current amplitude is 200 mA. (c) What is the amplitude of the driving emf?

73 A capacitor of capacitance 158 μF and an inductor form an LC circuit that oscillates at 8.15 kHz, with a current amplitude of 4.21 mA. What are (a) the inductance, (b) the total energy in the circuit, and (c) the maximum charge on the capacitor?

74 An oscillating LC circuit has an inductance of 3.00 mH and a capacitance of 10.0 μF. Calculate the (a) angular frequency and (b) period of the oscillation. (c) At time $t = 0$, the capacitor is charged to 200 μC and the current is zero. Roughly sketch the charge on the capacitor as a function of time.

75 For a certain driven series RLC circuit, the maximum generator emf is 125 V and the maximum current is 3.20 A. If the current leads the generator emf by 0.982 rad, what are the (a) impedance and (b) resistance of the circuit? (c) Is the circuit predominantly capacitive or inductive?

76 A 1.50 μF capacitor has a capacitive reactance of 12.0 Ω. (a) What must be its operating frequency? (b) What will be the capacitive reactance if the frequency is doubled?

77 SSM In Fig. 31-38, a three-phase generator G produces electrical power that is transmitted by means of three wires. The electric potentials (each relative to a common reference level) are $V_1 = A \sin \omega_d t$ for wire 1, $V_2 = A \sin(\omega_d t - 120°)$ for wire 2, and $V_3 = A \sin(\omega_d t - 240°)$ for wire 3. Some types of industrial equipment (for example, motors) have three terminals and are designed to be connected directly to these three wires. To use a more conventional two-terminal device (for example, a lightbulb), one connects it to any two of the three wires. Show that the potential difference between *any two* of the wires (a) oscillates sinusoidally with angular frequency ω_d and (b) has an amplitude of $A\sqrt{3}$.

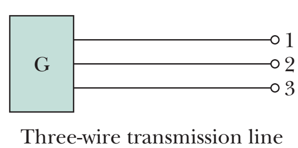


Figure 31-38 Problem 77.

78 An electric motor connected to a 120 V, 60.0 Hz ac outlet does mechanical work at the rate of 0.100 hp (1 hp = 746 W). (a) If the motor draws an rms current of 0.650 A, what is its effective resistance, relative to power transfer? (b) Is this the same as the resistance of the motor's coils, as measured with an ohmmeter with the motor disconnected from the outlet?

79 SSM (a) In an oscillating LC circuit, in terms of the maximum charge Q on the capacitor, what is the charge there when the energy in the electric field is 50.0% of that in the magnetic field? (b) What fraction of a period must elapse following the time the capacitor is fully charged for this condition to occur?

80 A series RLC circuit is driven by an alternating source at a frequency of 400 Hz and an emf amplitude of 90.0 V. The resistance is 20.0 Ω, the capacitance is 12.1 μF, and the inductance is 24.2 mH. What is the rms potential difference across (a) the resistor, (b) the capacitor, and (c) the inductor? (d) What is the average rate at which energy is dissipated?

81 SSM In a certain series RLC circuit being driven at a frequency of 60.0 Hz, the maximum voltage across the inductor is 2.00 times the maximum voltage across the resistor and 2.00 times the maximum voltage across the capacitor. (a) By what angle does the current lag the generator emf? (b) If the maximum generator emf is 30.0 V, what should be the resistance of the circuit to obtain a maximum current of 300 mA?

82 A 1.50 mH inductor in an oscillating LC circuit stores a maximum energy of 10.0 μJ. What is the maximum current?

83 A generator with an adjustable frequency of oscillation is wired in series to an inductor of $L = 2.50$ mH and a capacitor of $C = 3.00$ μF. At what frequency does the generator produce the largest possible current amplitude in the circuit?

84 A series RLC circuit has a resonant frequency of 6.00 kHz. When it is driven at 8.00 kHz, it has an impedance of 1.00 kΩ and a phase constant of 45°. What are (a) R, (b) L, and (c) C for this circuit?

85 SSM An LC circuit oscillates at a frequency of 10.4 kHz. (a) If the capacitance is 340 μF, what is the inductance? (b) If the maximum current is 7.20 mA, what is the total energy in the circuit? (c) What is the maximum charge on the capacitor?

86 When under load and operating at an rms voltage of 220 V, a certain electric motor draws an rms current of 3.00 A. It has a resistance of 24.0 Ω and no capacitive reactance. What is its inductive reactance?

87 The ac generator in Fig. 31-39 supplies 120 V at 60.0 Hz. With the switch open as in the diagram, the current leads the generator emf by 20.0°. With the switch in position 1, the current lags the generator emf by 10.0°. When the switch is in position 2, the current amplitude is 2.00 A. What are (a) R, (b) L, and (c) C?

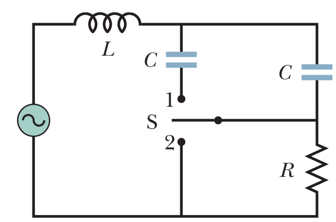


Figure 31-39 Problem 87.

88 In an oscillating LC circuit, $L = 8.00$ mH and $C = 1.40$ μF. At time $t = 0$, the current is maximum at 12.0 mA. (a) What is the maximum charge on the capacitor during the oscillations? (b) At what earliest time $t > 0$ is the rate of change of energy in the capacitor maximum? (c) What is that maximum rate of change?

89 SSM For a sinusoidally driven series RLC circuit, show that over one complete cycle with period T (a) the energy stored in the capacitor does not change; (b) the energy stored in the inductor does not change; (c) the driving emf device supplies energy $(\frac{1}{2}T)\mathscr{E}_m I \cos \phi$; and (d) the resistor dissipates energy $(\frac{1}{2}T)RI^2$. (e) Show that the quantities found in (c) and (d) are equal.

90 What capacitance would you connect across a 1.30 mH inductor to make the resulting oscillator resonate at 3.50 kHz?

91 A series circuit with resistor–inductor–capacitor combination R_1, L_1, C_1 has the same resonant frequency as a second circuit with a different combination R_2, L_2, C_2. You now connect the two combinations in series. Show that this new circuit has the same resonant frequency as the separate circuits.

92 Consider the circuit shown in Fig. 31-40. With switch S_1 closed and the other two switches open, the circuit has a time constant τ_C. With switch S_2 closed and the other two switches open, the circuit has a time constant τ_L. With switch S_3 closed and the other two switches open, the circuit oscillates with a period T. Show that $T = 2\pi\sqrt{\tau_C \tau_L}$.

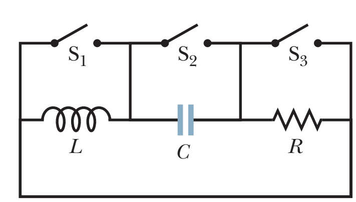


Figure 31-40 Problem 92.

93 When the generator emf in Sample Problem 31.07 is a maximum, what is the voltage across (a) the generator, (b) the resistance, (c) the capacitance, and (d) the inductance? (e) By summing these with appropriate signs, verify that the loop rule is satisfied.