CHAPTER 20 APPROXIMATION BY POLYNOMIAL FUNCTIONS

There is one sense in which the "elementary functions" are not elementary at all. If p is a polynomial function,

$$p(x) = a_0 + a_1x + \cdots + a_nx^n,$$

then $p(x)$ can be computed easily for any number x. This is not at all true for functions like sin, log, or exp. At present, to find $\log x = \int_1^x 1/t\,dt$ approximately, we must compute some upper or lower sums, and make certain that the error involved in accepting such a sum for $\log x$ is not too great. Computing $e^x = \log^{-1}(x)$ would be even more difficult: we would have to compute $\log a$ for many values of a until we found a number a such that $\log a$ is approximately x—then a would be approximately e^x.

In this chapter we will obtain important theoretical results which reduce the computation of $f(x)$, for many functions f, to the evaluation of polynomial functions. The method depends on finding polynomial functions which are close approximations to f. In order to guess a polynomial which is appropriate, it is useful to first examine polynomial functions themselves more thoroughly.

Suppose that

$$p(x) = a_0 + a_1x + \cdots + a_nx^n.$$

It is interesting, and for our purposes very important, to note that the coefficients a_i can be expressed in terms of the value of p and its various derivatives at 0. To begin with, note that

$$p(0) = a_0.$$

Differentiating the original expression for $p(x)$ yields

$$p'(x) = a_1 + 2a_2x + \cdots + na_nx^{n-1}.$$

Therefore,

$$p'(0) = p^{(1)}(0) = a_1.$$

Differentiating again we obtain

$$p''(x) = 2a_2 + 3 \cdot 2 \cdot a_3x + \cdots + n(n-1) \cdot a_nx^{n-2}.$$

Therefore,

$$p''(0) = p^{(2)}(0) = 2a_2.$$

In general, we will have

$$p^{(k)}(0) = k!\,a_k \quad \text{or} \quad a_k = \frac{p^{(k)}(0)}{k!}.$$

If we agree to define $0! = 1$, and recall the notation $p^{(0)} = p$, then this formula holds for $k = 0$ also.

If we had begun with a function p that was written as a "polynomial in $(x-a)$,"

$$p(x) = a_0 + a_1(x-a) + \cdots + a_n(x-a)^n,$$

then a similar argument would show that

$$a_k = \frac{p^{(k)}(a)}{k!}.$$

Suppose now that f is a function (not necessarily a polynomial) such that

$$f^{(1)}(a), \ldots, f^{(n)}(a)$$

all exist. Let

$$a_k = \frac{f^{(k)}(a)}{k!}, \qquad 0 \le k \le n,$$

and define

$$P_{n,a}(x) = a_0 + a_1(x-a) + \cdots + a_n(x-a)^n.$$

The polynomial $P_{n,a}$ is called the **Taylor polynomial of degree n for f at a.** (Strictly speaking, we should use an even more complicated expression, like $P_{n,a,f}$, to indicate the dependence on f; at times this more precise notation will be useful.) The Taylor polynomial has been defined so that

$$P_{n,a}{}^{(k)}(a) = f^{(k)}(a) \quad \text{for } 0 \le k \le n;$$

in fact, it is clearly the only polynomial *of degree* $\le n$ with this property.

Although the coefficients of $P_{n,a,f}$ seem to depend upon f in a fairly complicated way, the most important elementary functions have extremely simple Taylor polynomials. Consider first the function sin. We have

$$\begin{aligned}
\sin(0) &= 0,\\
\sin'(0) &= \cos 0 = 1,\\
\sin''(0) &= -\sin 0 = 0,\\
\sin'''(0) &= -\cos 0 = -1,\\
\sin^{(4)}(0) &= \sin 0 = 0.
\end{aligned}$$

From this point on, the derivatives repeat in a cycle of 4. The numbers

$$a_k = \frac{\sin^{(k)}(0)}{k!}$$

are

$$0,\ 1,\ 0,\ -\frac{1}{3!},\ 0,\ \frac{1}{5!},\ 0,\ -\frac{1}{7!},\ 0,\ \frac{1}{9!},\ \ldots.$$

Therefore the Taylor polynomial $P_{2n+1,0}$ of degree $2n+1$ for sin at 0 is

$$P_{2n+1,0}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

(Of course, $P_{2n+1,0} = P_{2n+2,0}$).

The Taylor polynomial $P_{2n,0}$ of degree $2n$ for cos at 0 is (the computations are left to you)

$$P_{2n,0}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!}.$$

The Taylor polynomial for exp is especially easy to compute. Since $\exp^{(k)}(0) = \exp(0) = 1$ for all k, the Taylor polynomial of degree n at 0 is

$$P_{n,0}(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!}.$$

The Taylor polynomial for log must be computed at some point $a \neq 0$, since log is not even defined at 0. The standard choice is $a = 1$. Then

$$\begin{aligned}
\log'(x) &= \frac{1}{x}, & \log'(1) &= 1;\\
\log''(x) &= -\frac{1}{x^2}, & \log''(1) &= -1;\\
\log'''(x) &= \frac{2}{x^3}, & \log'''(1) &= 2;
\end{aligned}$$

in general

$$\log^{(k)}(x) = \frac{(-1)^{k-1}(k-1)!}{x^k}, \qquad \log^{(k)}(1) = (-1)^{k-1}(k-1)!.$$

Therefore the Taylor polynomial of degree n for log at 1 is

$$P_{n,1}(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} + \cdots + \frac{(-1)^{n-1}(x-1)^n}{n}.$$

It is often more convenient to consider the function $f(x) = \log(1+x)$. In this case we can choose $a = 0$. We have

$$f^{(k)}(x) = \log^{(k)}(1+x),$$

so

$$f^{(k)}(0) = \log^{(k)}(1) = (-1)^{k-1}(k-1)!.$$

Therefore the Taylor polynomial of degree n for f at 0 is

$$P_{n,0}(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{(-1)^{n-1}x^n}{n}.$$

There is one other elementary function whose Taylor polynomial is important—arctan. The computations of the derivatives begin

$$\begin{aligned}
\arctan'(x) &= \frac{1}{1+x^2} & \arctan'(0) &= 1;\\
\arctan''(x) &= \frac{-2x}{(1+x^2)^2}, & \arctan''(0) &= 0;\\
\arctan'''(x) &= \frac{(1+x^2)^2 \cdot (-2) + 2x \cdot 2(1+x^2) \cdot 2x}{(1+x^2)^4}, & \arctan'''(0) &= -2.
\end{aligned}$$

It is clear that this brute force computation will never do. However, the Taylor polynomials of arctan will be easy to find after we have examined the properties of Taylor polynomials more closely—although the Taylor polynomial $P_{n,a,f}$ was simply defined so as to have the same first n derivatives at a as f, the connection between f and $P_{n,a,f}$ will actually turn out to be much deeper.

One line of evidence for a closer connection between f and the Taylor polynomials for f may be uncovered by examining the Taylor polynomial of degree 1, which is

$$P_{1,a}(x) = f(a) + f'(a)(x-a).$$

Notice that

$$\frac{f(x) - P_{1,a}(x)}{x-a} = \frac{f(x) - f(a)}{x-a} - f'(a).$$

Now, by the definition of $f'(a)$ we have

$$\lim_{x \to a} \frac{f(x) - P_{1,a}(x)}{x-a} = 0.$$

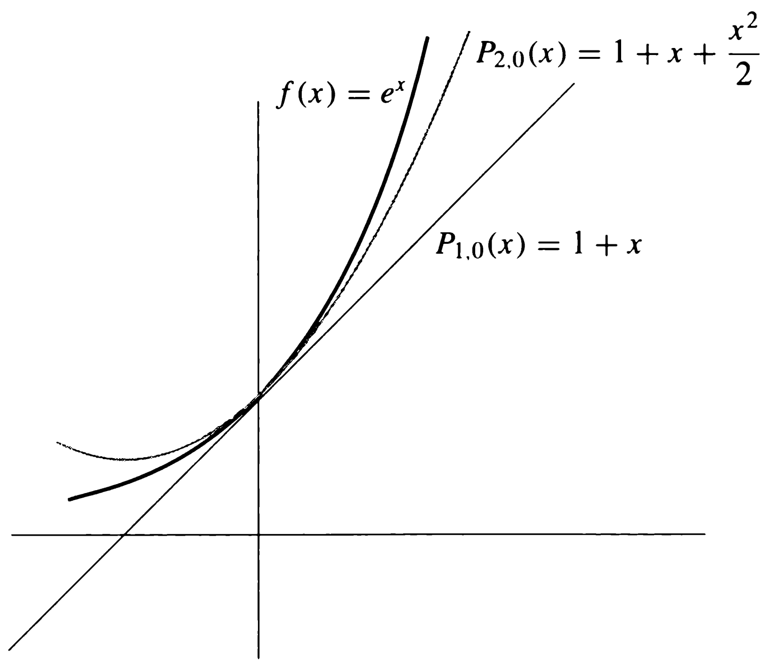


FIGURE 1

In other words, as x approaches a the difference $f(x) - P_{1,a}(x)$ not only becomes small, but actually becomes small even compared to $x - a$. Figure 1 illustrates the graph of $f(x) = e^x$ and of

$$P_{1,0}(x) = f(0) + f'(0)x = 1 + x,$$

which is the Taylor polynomial of degree 1 for f at 0. The diagram also shows the graph of

$$P_{2,0}(x) = f(0) + f'(0) + \frac{f''(0)}{2!}x^2 = 1 + x + \frac{x^2}{2},$$

which is the Taylor polynomial of degree 2 for f at 0. As x approaches 0, the difference $f(x) - P_{2,0}(x)$ seems to be getting small even faster than the difference

$f(x) - P_{1,0}(x)$. As it stands, this assertion is not very precise, but we are now prepared to give it a definite meaning. We have just noted that in general

$$\lim_{x \to a} \frac{f(x) - P_{1,a}(x)}{x - a} = 0.$$

For $f(x) = e^x$ and $a = 0$ this means that

$$\lim_{x \to 0} \frac{f(x) - P_{1,0}(x)}{x} = \lim_{x \to 0} \frac{e^x - 1 - x}{x} = 0.$$

On the other hand, an easy double application of l'Hôpital's Rule shows that

$$\lim_{x \to 0} \frac{e^x - 1 - x}{x^2} = \frac{1}{2} \neq 0.$$

Thus, although $f(x) - P_{1,0}(x)$ becomes small compared to x, as x approaches 0, it does *not* become small compared to x^2. For $P_{2,0}(x)$ the situation is quite different; the extra term $x^2/2$ provides just the right compensation:

$$\begin{aligned} \lim_{x \to 0} \frac{e^x - 1 - x - \dfrac{x^2}{2}}{x^2} &= \lim_{x \to 0} \frac{e^x - 1 - x}{2x} \\ &= \lim_{x \to 0} \frac{e^x - 1}{2} = 0. \end{aligned}$$

This result holds in general—if $f'(a)$ and $f''(a)$ exist, then

$$\lim_{x \to a} \frac{f(x) - P_{2,a}(x)}{(x - a)^2} = 0;$$

in fact, the analogous assertion for $P_{n,a}$ is also true.

THEOREM 1 Suppose that f is a function for which

$$f'(a), \ldots, f^{(n)}(a)$$

all exist. Let

$$a_k = \frac{f^{(k)}(a)}{k!}, \quad 0 \leq k \leq n,$$

and define

$$P_{n,a}(x) = a_0 + a_1(x - a) + \cdots + a_n(x - a)^n.$$

Then

$$\lim_{x \to a} \frac{f(x) - P_{n,a}(x)}{(x - a)^n} = 0.$$

PROOF Writing out $P_{n,a}(x)$ explicitly, we obtain

$$\frac{f(x)-P_{n,a}(x)}{(x-a)^n} = \frac{f(x)-\displaystyle\sum_{i=0}^{n-1}\frac{f^{(i)}(a)}{i!}(x-a)^i}{(x-a)^n} - \frac{f^{(n)}(a)}{n!}.$$

It will help to introduce the new functions

$$Q(x) = \sum_{i=0}^{n-1}\frac{f^{(i)}(a)}{i!}(x-a)^i \quad \text{and} \quad g(x) = (x-a)^n;$$

now we must prove that

$$\lim_{x\to a}\frac{f(x)-Q(x)}{g(x)} = \frac{f^{(n)}(a)}{n!}.$$

Notice that

$$Q^{(k)}(a) = f^{(k)}(a), \quad k \le n-1,$$
$$g^{(k)}(x) = n!(x-a)^{n-k}/(n-k)!.$$

Thus

$$\lim_{x\to a}[f(x)-Q(x)] = f(a)-Q(a) = 0,$$
$$\lim_{x\to a}[f'(x)-Q'(x)] = f'(a)-Q'(a) = 0,$$
$$\vdots$$
$$\lim_{x\to a}[f^{(n-2)}(x)-Q^{(n-2)}(x)] = f^{(n-2)}(a)-Q^{(n-2)}(a) = 0.$$

and

$$\lim_{x\to a} g(x) = \lim_{x\to a} g'(x) = \cdots = \lim_{x\to a} g^{(n-2)}(x) = 0.$$

We may therefore apply l'Hôpital's Rule $n-1$ times to obtain

$$\lim_{x\to a}\frac{f(x)-Q(x)}{(x-a)^n} = \lim_{x\to a}\frac{f^{(n-1)}(x)-Q^{(n-1)}(x)}{n!\,(x-a)}.$$

Since Q is a polynomial of degree $n-1$, its $(n-1)$st derivative is a constant; in fact, $Q^{(n-1)}(x) = f^{(n-1)}(a)$. Thus

$$\lim_{x\to a}\frac{f(x)-Q(x)}{(x-a)^n} = \lim_{x\to a}\frac{f^{(n-1)}(x)-f^{(n-1)}(a)}{n!\,(x-a)}$$

and this last limit is $f^{(n)}(a)/n!$ by definition of $f^{(n)}(a)$. ▮

One simple consequence of Theorem 1 allows us to perfect the test for local maxima and minima which was developed in Chapter 11. If a is a critical point of f, then, according to Theorem 11-5, the function f has a local minimum at a if $f''(a) > 0$, and a local maximum at a if $f''(a) < 0$. If $f''(a) = 0$ no conclusion was possible, but it is conceivable that the sign of $f'''(a)$ might give further information; and if $f'''(a) = 0$, then the sign of $f^{(4)}(a) = 0$ might be significant. Even more generally, we can ask what happens when

$$(*) \qquad \begin{aligned} &f'(a) = f''(a) = \cdots = f^{(n-1)}(a) = 0, \\ &f^{(n)}(a) \neq 0. \end{aligned}$$

The situation in this case can be guessed by examining the functions

$$\begin{aligned} f(x) &= (x-a)^n, \\ g(x) &= -(x-a)^n, \end{aligned}$$

which satisfy $(*)$. Notice (Figure 2) that if n is odd, then a is neither a local maximum nor a local minimum point for f or g. On the other hand, if n is even, then f, with a positive nth derivative, has a local minimum at a, while g, with a negative nth derivative, has a local maximum at a. Of all functions satisfying $(*)$, these are about the simplest available; nevertheless they indicate the general situation exactly. In fact, the whole point of the next proof is that any function satisfying $(*)$ looks very much like one of these functions, in a sense that is made precise by Theorem 1.

THEOREM 2 Suppose that

$$\begin{aligned} &f'(a) = \cdots = f^{(n-1)}(a) = 0, \\ &f^{(n)}(a) \neq 0. \end{aligned}$$

(1) If n is even and $f^{(n)}(a) > 0$, then f has a local minimum at a.
(2) If n is even and $f^{(n)}(a) < 0$, then f has a local maximum at a.
(3) If n is odd, then f has neither a local maximum nor a local minimum at a.

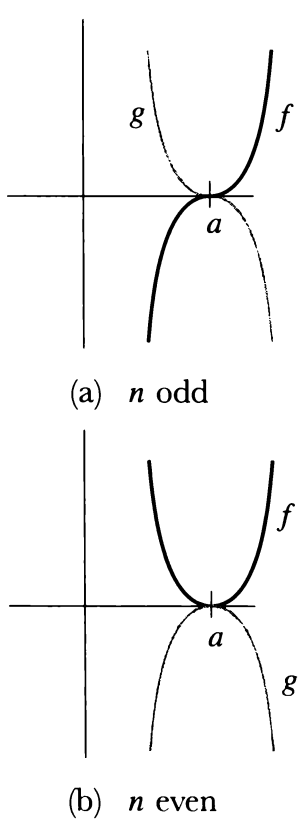


FIGURE 2

PROOF There is clearly no loss of generality in assuming that $f(a) = 0$, since neither the hypotheses nor the conclusion are affected if f is replaced by $f - f(a)$. Then, since the first $n-1$ derivatives of f at a are 0, the Taylor polynomial $P_{n,a}$ of f is

$$\begin{aligned} P_{n,a}(x) &= f(a) + \frac{f'(a)}{1!}(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= \frac{f^{(n)}(a)}{n!}(x-a)^n. \end{aligned}$$

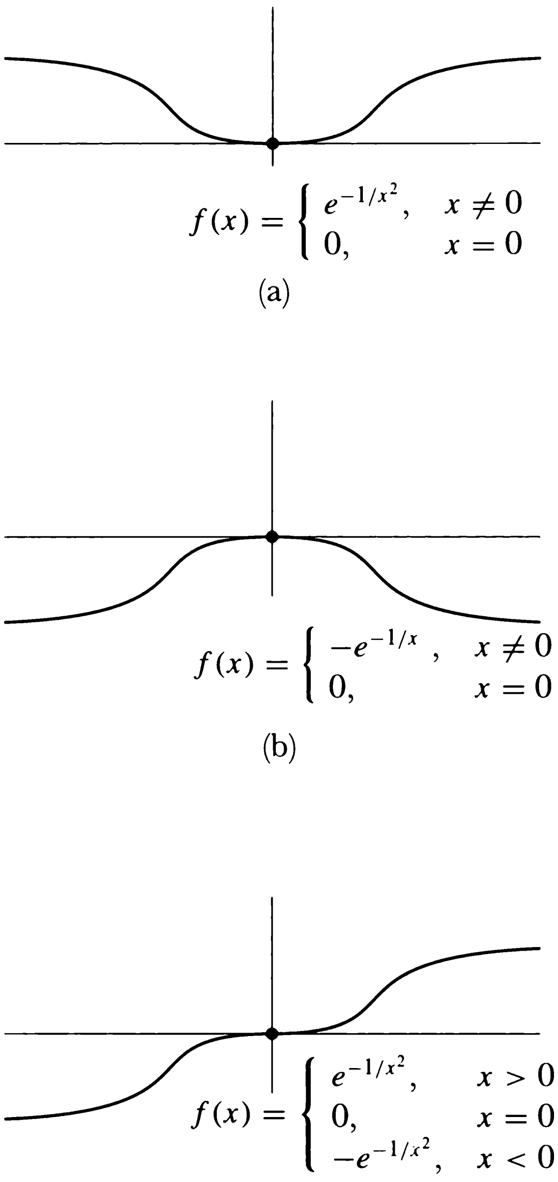
Thus, Theorem 1 states that

$$0 = \lim_{x\to a} \frac{f(x) - P_{n,a}(x)}{(x-a)^n} = \lim_{x\to a} \left[\frac{f(x)}{(x-a)^n} - \frac{f^{(n)}(a)}{n!} \right].$$

Consequently, if x is sufficiently close to a, then

$$\frac{f(x)}{(x-a)^n} \quad \text{has the same sign as} \quad \frac{f^{(n)}(a)}{n!}.$$

Suppose now that n is even. In this case $(x-a)^n > 0$ for all $x \neq a$. Since $f(x)/(x-a)^n$ has the same sign as $f^{(n)}(a)/n!$ for x sufficiently close to a, it follows



$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

(a)

$$f(x) = \begin{cases} -e^{-1/x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

(b)

$$f(x) = \begin{cases} e^{-1/x^2}, & x > 0 \\ 0, & x = 0 \\ -e^{-1/x^2}, & x < 0 \end{cases}$$

(c)

FIGURE 3

that $f(x)$ itself has the same sign as $f^n(a)/n!$ for x sufficiently close to a. If $f^{(n)}(a) > 0$, this means that

$$f(x) > 0 = f(a)$$

for x close to a. Consequently, f has a local minimum at a. A similar proof works for the case $f^{(n)}(a) < 0$.

Now suppose that n is odd. The same argument as before shows that if x is sufficiently close to a, then

$$\frac{f(x)}{(x-a)^n} \quad \text{always has the same sign.}$$

But $(x-a)^n > 0$ for $x > a$ and $(x-a)^n < 0$ for $x < a$. Therefore $f(x)$ has *different* signs for $x > a$ and $x < a$. This proves that f has neither a local maximum nor a local minimum at a. ▮

Although Theorem 2 will settle the question of local maxima and minima for just about any function which arises in practice, it does have some theoretical limitations, because $f^{(k)}(a)$ may be 0 for *all* k. This happens (Figure 3(a)) for the function

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0, \end{cases}$$

which has a minimum at 0, and also for the negative of this function (Figure 3(b)), which has a maximum at 0. Moreover (Figure 3(c)), if

$$f(x) = \begin{cases} e^{-1/x^2}, & x > 0 \\ 0, & x = 0 \\ -e^{-1/x^2}, & x < 0, \end{cases}$$

then $f^{(k)}(0) = 0$ for all k, but f has neither a local minimum nor a local maximum at 0.

The conclusion of Theorem 1 is often expressed in terms of an important concept of "order of equality." Two functions f and g are **equal up to order n at a** if

$$\lim_{x \to a} \frac{f(x) - g(x)}{(x-a)^n} = 0.$$

In the language of this definition, Theorem 1 says that the Taylor polynomial $P_{n,a,f}$ equals f up to order n at a. The Taylor polynomial might very well have been designed to make this fact true, because there is at most one polynomial of degree $\leq n$ with this property. This assertion is a consequence of the following elementary theorem.

THEOREM 3 Let P and Q be two polynomials in $(x-a)$, of degree $\leq n$, and suppose that P and Q are equal up to order n at a. Then $P = Q$.

PROOF Let $R = P - Q$. Since R is a polynomial of degree $\leq n$, it is only necessary to prove that if

$$R(x) = b_0 + \cdots + b_n(x-a)^n$$

satisfies

$$\lim_{x\to a} \frac{R(x)}{(x-a)^n} = 0,$$

then $R = 0$. Now the hypothesis on R surely imply that

$$\lim_{x\to a} \frac{R(x)}{(x-a)^i} = 0 \quad \text{for } 0 \le i \le n.$$

For $i = 0$ this condition reads simply $\lim\limits_{x\to a} R(x) = 0$; on the other hand,

$$\begin{aligned} \lim_{x\to a} R(x) &= \lim_{x\to a} [b_0 + b_1(x-a) + \cdots + b_n(x-a)^n] \\ &= b_0. \end{aligned}$$

Thus $b_0 = 0$ and

$$R(x) = b_1(x-a) + \cdots + b_n(x-a)^n.$$

Therefore,

$$\frac{R(x)}{x-a} = b_1 + b_2(x-a) + \cdots + b_n(x-a)^{n-1}$$

and

$$\lim_{x\to a} \frac{R(x)}{x-a} = b_1.$$

Thus $b_1 = 0$ and

$$R(x) = b_2(x-a)^2 + \cdots + b_n(x-a)^n.$$

Continuing in this way, we find that

$$b_0 = \cdots = b_n = 0. \blacksquare$$

COROLLARY Let f be n-times differentiable at a, and suppose that P is a polynomial in $(x-a)$ of degree $\le n$, which equals f up to order n at a. Then $P = P_{n,a,f}$.

PROOF Since P and $P_{n,a,f}$ both equal f up to order n at a, it is easy to see that P equals $P_{n,a,f}$ up to order n at a. Consequently, $P = P_{n,a,f}$ by the Theorem. $\blacksquare$

At first sight this corollary appears to have unnecessarily complicated hypotheses; it might seem that the existence of the polynomial P would automatically imply that f is sufficiently differentiable for $P_{n,a,f}$ to exist. But in fact this is not so. For example (Figure 4), suppose that

$$f(x) = \begin{cases} x^{n+1}, & x \text{ irrational} \\ 0, & x \text{ rational.} \end{cases}$$

If $P(x) = 0$, then P is certainly a polynomial of degree $\le n$ which equals f up to order n at 0. On the other hand, $f'(a)$ does not exist for any $a \ne 0$, so $f''(0)$ is undefined.

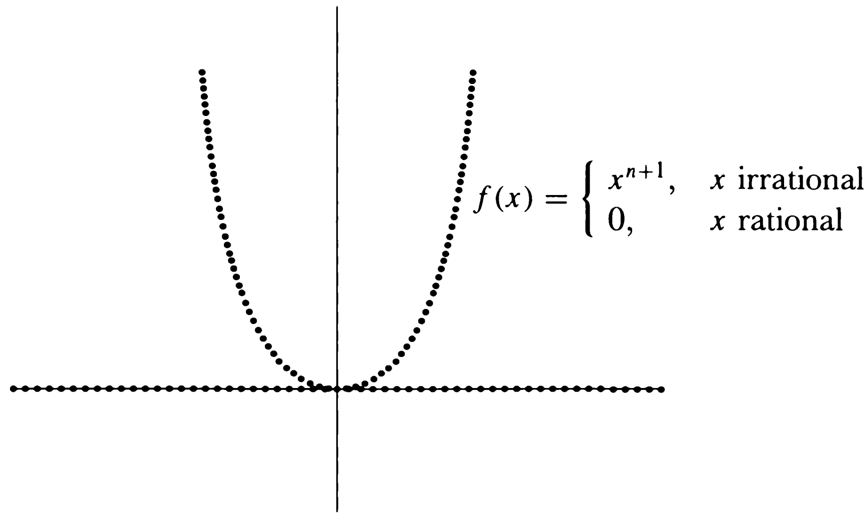


FIGURE 4

When f does have n derivatives at a, however, the corollary may provide a useful method for finding the Taylor polynomial of f. In particular, remember that our first attempt to find the Taylor polynomial for arctan ended in failure. The equation

$$\arctan x = \int_0^x \frac{1}{1+t^2}\,dt$$

suggests a promising method of finding a polynomial close to arctan—divide 1 by $1 + t^2$, to obtain a polynomial plus a remainder:

$$\frac{1}{1+t^2} = 1 - t^2 + t^4 - t^6 + \cdots + (-1)^n t^{2n} + \frac{(-1)^{n+1} t^{2n+2}}{1+t^2}.$$

This formula, which can be checked easily by multiplying both sides by $1 + t^2$, shows that

$$\begin{aligned} \arctan x &= \int_0^x 1 - t^2 + t^4 - \cdots + (-1)^n t^{2n}\,dt + (-1)^{n+1} \int_0^x \frac{t^{2n+2}}{1+t^2}\,dt \\ &= x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + (-1)^{n+1} \int_0^x \frac{t^{2n+2}}{1+t^2}\,dt. \end{aligned}$$

According to our corollary, the polynomial which appears here will be the Taylor polynomial of degree $2n + 1$ for arctan at 0, provided that

$$\lim_{x \to 0} \frac{\displaystyle\int_0^x \frac{t^{2n+2}}{1+t^2}\,dt}{x^{2n+1}} = 0.$$

Since

$$\left| \int_0^x \frac{t^{2n+2}}{1+t^2}\,dt \right| \le \left| \int_0^x t^{2n+2}\,dt \right| = \frac{|x|^{2n+3}}{2n+3},$$

this is clearly true. Thus we have found that the Taylor polynomial of degree $2n + 1$ for arctan at 0 is

$$P_{2n+1,0}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1}.$$

By the way, now that we have discovered the Taylor polynomials of arctan, it is possible to work backwards and find $\arctan^{(k)}(0)$ for all k: Since

$$P_{2n+1,0}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1},$$

and since this polynomial is, by definition,

$$\arctan^{(0)}(0) + \arctan^{(1)}(0) + \frac{\arctan^{(2)}(0)}{2!}x^2 + \cdots + \frac{\arctan^{(2n+1)}(0)}{(2n+1)!}x^{2n+1},$$

we can find $\arctan^{(k)}(0)$ by simply equating the coefficients of x^k in these two polynomials:

$$\frac{\arctan^{(k)}(0)}{k!} = 0 \quad \text{if } k \text{ is even,}$$

$$\frac{\arctan^{(2l+1)}(0)}{(2l+1)!} = \frac{(-1)^l}{2l+1} \quad \text{or} \quad \arctan^{(2l+1)}(0) = (-1)^l \cdot (2l)!\,.$$

A much more interesting fact emerges if we go back to the original equation

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + (-1)^{n+1} \int_0^x \frac{t^{2n+2}}{1+t^2}\,dt,$$

and remember the estimate

$$\left| \int_0^x \frac{t^{2n+2}}{1+t^2}\,dt \right| \le \frac{|x|^{2n+3}}{2n+3}.$$

When $|x| \le 1$, this expression is at most $1/(2n+3)$, and we can make this as small as we like simply by choosing n large enough. In other words, for $|x| \le 1$ *we can use the Taylor polynomials for* arctan *to compute* $\arctan x$ *as accurately as we like.* The most important theorems about Taylor polynomials extend this isolated result to other functions, and the Taylor polynomials will soon play quite a new role. The theorems proved so far have always examined the behavior of the Taylor polynomial $P_{n,a}$ for *fixed* n, as x approaches a. Henceforth we will compare Taylor polynomials $P_{n,a}$ for *fixed* x, and different n. In anticipation of the coming theorem we introduce some new notation.

If f is a function for which $P_{n,a}(x)$ exists, we define the **remainder term** $R_{n,a}(x)$ by

$$\begin{aligned} f(x) &= P_{n,a}(x) + R_{n,a}(x) \\ &= f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_{n,a}(x). \end{aligned}$$

We would like to have an expression for $R_{n,a}(x)$ whose size is easy to estimate. There is such an expression, involving an integral, just as in the case for arctan. One way to guess this expression is to begin with the case $n = 0$:

$$f(x) = f(a) + R_{0,a}(x).$$

The Fundamental Theorem of Calculus enables us to write

$$f(x) = f(a) + \int_a^x f'(t)\,dt,$$

so that

$$R_{0,a}(x) = \int_a^x f'(t)\,dt.$$

A similar expression for $R_{1,a}(x)$ can be derived from this formula using integration by parts in a rather tricky way: Let

$$u(t) = f'(t) \quad \text{and} \quad v(t) = t - x$$

(notice that x represents some fixed number in the expression for $v(t)$, so $v'(t) = 1$); then

$$\begin{aligned}\int_a^x f'(t)\,dt &= \int_a^x \underset{\substack{\downarrow \\ u(t)}}{f'(t)} \cdot \underset{\substack{\downarrow \\ v'(t)}}{1}\,dt \\ &= u(t)v(t)\Big|_a^x - \int_a^x \underset{\substack{\downarrow \\ u'(t)}}{f''(t)}\underset{\substack{\downarrow \\ v(t)}}{(t-x)}\,dt.\end{aligned}$$

Since $v(x) = 0$, we obtain

$$\begin{aligned}f(x) &= f(a) + \int_a^x f'(t)\,dt \\ &= f(a) - u(a)v(a) + \int_a^x f''(t)(x-t)\,dt \\ &= f(a) + f'(a)(x-a) + \int_a^x f''(t)(x-t)\,dt.\end{aligned}$$

Thus

$$R_{1,a}(x) = \int_a^x f''(t)(x-t)\,dt.$$

It is hard to give any motivation for choosing $v(t) = t - x$, rather than $v(t) = t$. It just happens to be the choice which works out, the sort of thing one might discover after sufficiently many similar but futile manipulations. However, it is now easy to guess the formula for $R_{2,a}(x)$. If

$$u(t) = f''(t) \quad \text{and} \quad v(t) = \frac{-(x-t)^2}{2},$$

then $v'(t) = (x - t)$, so

$$\begin{aligned}\int_a^x f''(t)(x-t)\,dt &= u(t)v(t)\Big|_a^x - \int_a^x f'''(t) \cdot \frac{-(x-t)^2}{2}\,dt \\ &= \frac{f''(a)(x-a)^2}{2} + \int_a^x \frac{f'''(t)}{2}(x-t)^2\,dt.\end{aligned}$$

This shows that

$$R_{2,a}(x) = \int_a^x \frac{f^{(3)}(t)}{2}(x-t)^2\,dt.$$

You should now have little difficulty giving a rigorous proof, by induction, that if $f^{(n+1)}$ is continuous on $[a, x]$, then

$$R_{n,a}(x) = \int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n\,dt.$$

This formula is called the integral form of the remainder, and we can easily estimate it in terms of estimates for $f^{(n+1)}/n!$ on $[a, x]$. If m and M are the minimum and maximum of $f^{(n+1)}/n!$ on $[a, x]$, then $R_{n,a}(x)$ satisfies

$$m\int_a^x (x-t)^n\,dt \le R_{n,a}(x) \le M\int_a^x (x-t)\,dt,$$

so we can write

$$R_{n,a}(x) = \alpha \cdot \frac{(x-a)^{n+1}}{n+1}$$

for some number α between m and M. Since we've assumed that $f^{(n+1)}$ is continuous, this means that for some t in (a, x) we can also write

$$R_{n,a}(x) = \frac{f^{(n+1)}(t)}{n!}\frac{(x-a)^{n+1}}{n+1} = \frac{f^{(n+1)}}{(n+1)!}(x-a)^{n+1},$$

which is called the Lagrange form of the remainder (these manipulations will look familiar to those who have done Problem 13-23).

The Lagrange form of the remainder is the one we will need in almost all cases, and we can even give a proof that doesn't require $f^{(n+1)}$ to be continuous (a refinement admittedly of little importance in most applications, where we often assume that f has derivatives of all orders). This is the form of the remainder that we will choose in our statement of the next theorem (Taylor's Theorem).

LEMMA Suppose that the function R is $(n+1)$-times differentiable on $[a, b]$, and

$$R^{(k)}(a) = 0 \quad \text{for } k = 0, 1, 2, \ldots, n.$$

Then for any x in $(a, b]$ we have

$$\frac{R(x)}{(x-a)^{n+1}} = \frac{R^{(n+1)}(t)}{(n+1)!} \quad \text{for some } t \text{ in } (a, x).$$

PROOF For $n = 0$, this is just the Mean Value Theorem, and we will prove the theorem for all n by induction on n. To do this we use the Cauchy Mean Value Theorem to write

$$\frac{R(x)}{(x-a)^{n+2}} = \frac{R'(z)}{(n+2)(z-a)^{n+1}} = \frac{1}{n+2}\frac{R'(z)}{(z-a)^{n+1}} \quad \text{for some } z \text{ in } (a, x),$$

and then apply the induction hypothesis *to R' on the interval* $[a, z]$ to get

$$\begin{aligned}\frac{R(x)}{(x-a)^{n+2}} &= \frac{1}{n+2}\frac{(R')^{(n+1)}(t)}{(n+1)!} \quad \text{for some } t \text{ in } (a, z)\\ &= \frac{R^{(n+2)}(t)}{(n+2)!}. \ \blacksquare\end{aligned}$$

THEOREM 4 (TAYLOR'S THEOREM) Suppose that $f', \ldots, f^{(n+1)}$ are defined on $[a, x]$, and that $R_{n,a}(x)$ is defined by

$$f(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_{n,a}(x).$$

Then

$$R_{n,a}(x) = \frac{f^{(n+1)}(t)}{(n+1)!}(x-a)^{n+1} \quad \text{for some } t \text{ in } (a, x)$$

(Lagrange form of the remainder).

PROOF The function $R_{n,a}$ satisfies the conditions of the Lemma by the very definition of the Taylor polynomial, so

$$\frac{R_{n,a}(x)}{(x-a)^{n+1}} = \frac{R_{n,a}{}^{(n+1)}(t)}{(n+1)!}$$

for some t in (a, x). But

$$R_{n,a}{}^{(n+1)} = f^{(n+1)},$$

since $R_{n,a} - f$ is a polynomial of degree n. ▮

Applying Taylor's Theorem to the functions sin, cos, and exp, with $a = 0$, we obtain the following formulas:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \frac{\sin^{(2n+2)}(t)}{(2n+2)!}x^{2n+2}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \frac{\cos^{(2n+1)}(t)}{(2n+1)!}x^{2n+1}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \frac{e^t}{(n+1)!}x^{n+1}$$

(of course, we could actually go one power higher in the remainder terms for sin and cos).

Estimates for the first two are especially easy. Since

$$|\sin^{(2n+2)}(t)| \le 1 \quad \text{for all } t,$$

we have

$$\left| \frac{\sin^{(2n+2)}(t)}{(2n+2)!}x^{2n+2} \right| \le \frac{|x|^{2n+2}}{(2n+2)!}.$$

Similarly, we can show that

$$\left| \frac{\cos^{(2n+1)}(t)}{(2n+1)!}x^{2n+1} \right| \le \frac{|x|^{2n+1}}{(2n+1)!}.$$

Estimating the remainder for e^x is only slightly harder. For simplicity assume that $x \geq 0$ (the estimates for $x \leq 0$ are obtained in Problem 15). On the interval $[0, x]$ the maximum value of e^t is e^x, since exp is increasing, so

$$R_{n,0} \leq \frac{e^x x^{n+1}}{(n+1)!}.$$

Since we already know that $e < 4$, we have

$$\frac{e^x x^{n+1}}{(n+1)!} < \frac{4^x x^{n+1}}{(n+1)!},$$

which can be made as small as desired by choosing n sufficiently large. How large n must be will depend on x (and the factor 4^x will make things more difficult). Once again, the estimates are easier for small x. If $0 \leq x \leq 1$, then

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + R, \quad \text{where } 0 < R < \frac{4}{(n+1)!}.$$

In particular, if $n = 4$, then

$$0 < R < \frac{4}{5!} < \frac{1}{10},$$

so

$$\begin{aligned} e = e^1 &= 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + R, \quad \text{where } 0 < R < \frac{1}{10} \\ &= 2 + \frac{17}{24} \end{aligned}$$

which shows that

$$2 < e < 3.$$

(This then shows that

$$0 < R < \frac{3^x x^{n+1}}{(n+1)!},$$

allowing us to improve our estimate of R slightly.) By taking $n = 7$ you can compute that the first 3 decimals for e are

$$e = 2.718\ldots$$

(you should check that $n = 7$ does give this degree of accuracy, but it would be cruel to insist that you actually do the computations).

The function arctan is also important but, as you may recall, an expression for $\arctan^{(k)}(x)$ is hopelessly complicated, so that our expressions for the remainder are pretty useless. On the other hand, our derivation of the Taylor polynomial for arctan automatically provided a formula for the remainder:

$$\arctan x = x - \frac{x^3}{3} + \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \int_0^x \frac{(-1)^{n+1} t^{2n+2}}{1+t^2}\, dt.$$

These estimates are particularly interesting, because (as proved in Chapter 16) for any $\varepsilon > 0$ we can make

$$\frac{x^n}{n!} < \varepsilon$$

by choosing n large enough (how large n must be will depend on x). This enables us to compute $\sin x$ to any degree of accuracy desired simply by evaluating the proper Taylor polynomial $P_{n,0}(x)$. For example, suppose we wish to compute $\sin 2$ with an error of less than 10^{-4}. Since

$$\sin 2 = P_{2n+1,0}(2) + R, \quad \text{where } |R| \leq \frac{2^{2n+2}}{(2n+2)!},$$

we can use $P_{2n+1,0}(2)$ as our answer, provided that

$$\frac{2^{2n+2}}{(2n+2)!} < 10^{-4}.$$

A number n with this property can be found by a straightforward search—it obviously helps to have a table of values for $n!$ and 2^n (see page 432). In this case it happens that $n = 5$ works, so that

$$\begin{aligned} \sin 2 &= P_{11,0}(2) + R \\ &= 2 - \frac{2^3}{3!} + \frac{2^5}{5!} - \frac{2^7}{7!} + \frac{2^9}{9!} - \frac{2^{11}}{11!} + R, \\ &\qquad \text{where } |R| < 10^{-4}. \end{aligned}$$

It is even easier to calculate $\sin 1$ approximately, since

$$\sin 1 = P_{2n+1,0}(1) + R, \quad \text{where } |R| < \frac{1}{(2n+2)!}.$$

To obtain an error less than ε we need only find an n such that

$$\frac{1}{(2n+2)!} < \varepsilon,$$

and this requires only a brief glance at a table of factorials. (Moreover, the individual terms of $P_{2n+1,0}(1)$ will be easier to handle.)

For very small x the estimates will be even easier. For example,

$$\sin \frac{1}{10} = P_{2n+1,0}\left(\frac{1}{10}\right) + R, \quad \text{where } |R| < \frac{1}{10^{2n+2}(2n+2)!}.$$

To obtain $|R| < 10^{-10}$ we can clearly take $n = 4$ (and we could even get away with $n = 3$). These methods can actually be used to compute tables of sin and cos; a high-speed computer can compute $P_{2n+1,0}(x)$ for many different x in almost no time at all. Nowadays, computers, and even cheap calculators, determine the values of such functions "on-the-fly", though by specialized methods that are even faster.

As we have already estimated,

$$\left|\int_0^x \frac{(-1)^{n+1}t^{2n+2}}{1+t^2}\,dt\right| \le \left|\int_0^x t^{2n+2}\,dt\right| = \frac{|x|^{2n+3}}{2n+3}.$$

For the moment we will consider only numbers x with $|x| \le 1$. In this case, the remainder term can clearly be made as small as desired by choosing n sufficiently large. In particular,

$$\arctan 1 = 1 - \frac{1}{3} + \frac{1}{5} - \cdots + \frac{(-1)^n}{2n+1} + R, \quad \text{where } |R| < \frac{1}{2n+3}.$$

With this estimate it is easy to find an n which will make the remainder less than any preassigned number; on the other hand, n will usually have to be so large as to make computations hopelessly long. To obtain a remainder $< 10^{-4}$, for example, we must take $n > (10^4 - 3)/2$. This is really a shame, because $\arctan 1 = \pi/4$, so the Taylor polynomial for arctan should allow us to compute π. Fortunately, there are some clever tricks which enable us to surmount these difficulties. Since

$$|R_{2n+1,0}(x)| < \frac{|x|^{2n+3}}{2n+3},$$

much smaller n's will work for only somewhat smaller x's. The trick for computing π is to express arctan 1 in terms of $\arctan x$ for smaller x; Problem 6 shows how this can be done in a convenient way.

From the calculations on page 413, we see that for $x \ge 0$ we have

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{(-1)^{n-1}x^n}{n} + \frac{(-1)^n}{n+1}t^{n+1}$$

where

$$\left|\frac{(-1)^n}{n+1}t^{n+1}\right| \le \frac{x^{n+1}}{n+1}$$

and there is a slightly more complicated estimate when $-1 < x < 0$ (Problem 16). For this function the remainder term can be made as small as desired by choosing n sufficiently large, provided that $-1 < x \le 1$.

The behavior of the remainder terms for arctan and $f(x) = \log(x+1)$ is quite another matter when $|x| > 1$. In this case, the estimates

$$|R_{2n+1,0}(x)| < \frac{|x|^{2n+3}}{2n+3} \quad \text{for arctan},$$

$$|R_{n,0}(x)| < \frac{x^{n+1}}{n+1} \quad (x > 0) \text{ for } f,$$

are of no use, because when $|x| > 1$ the bounds x^m/m become large as m becomes large. This predicament is unavoidable, and is not just a deficiency of our estimates. It is easy to get estimates in the other direction which show that the remainders actually do remain large. To obtain such an estimate for arctan, note that if t is in $[0, x]$ (or in $[x, 0]$ if $x < 0$), then

$$1 + t^2 \leq 1 + x^2 \leq 2x^2, \quad \text{if } |x| \geq 1,$$

so

$$\left| \int_0^x \frac{t^{2n+2}}{1+t^2}\, dt \right| \geq \frac{1}{2x^2} \left| \int_0^x t^{2n+2}\, dt \right| = \frac{|x|^{2n+1}}{4n+6}.$$

To get a similar estimate for $\log(1+x)$, we can use the formula

$$\frac{1}{1+t} = 1 - t + t^2 - \cdots + (-1)^{n-1}t^{n-1} + \frac{(-1)^n t^n}{1+t};$$

to get

$$\log(1+x) = \int_0^x \frac{1}{1+t}\, dt = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n} + (-1)^n \int_0^x \frac{t^n}{1+t}\, dt.$$

If $x > 0$, then for t in $[0, x]$ we have

$$1 + t \leq 1 + x \leq 2x, \quad \text{if } x \geq 1,$$

so

$$\int_0^x \frac{t^n}{t+1}\, dt \geq \frac{1}{2x} \int_0^x t^n\, dt = \frac{x^n}{2n+2}.$$

These estimates show that if $|x| > 1$, then the remainder terms become large as n becomes large. In other words, for $|x| > 1$, the Taylor polynomials for arctan and f *are of no use whatsoever in computing* $\arctan x$ *and* $\log(x+1)$. This is no tragedy, because the values of these functions can be found for any x once they are known for all x with $|x| < 1$.

This same situation occurs in a spectacular way for the function

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

We have already seen that $f^{(k)}(0) = 0$ for every natural number k. This means that the Taylor polynomial $P_{n,0}$ for f is

$$\begin{aligned} P_{n,0}(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n \\ &= 0. \end{aligned}$$

In other words, the remainder term $R_{n,0}(x)$ always equals $f(x)$, and the Taylor polynomial is useless for computing $f(x)$, except for $x = 0$. Eventually we will be able to offer some explanation for the behavior of this function, which is such a disconcerting illustration of the limitations of Taylor's Theorem.

The word "compute" has been used so often in connection with our estimates for the remainder term, that the significance of Taylor's Theorem might be misconstrued. It is true that Taylor's Theorem can be used as a computational aid (despite its ignominious failure in the previous example), but it has even more important theoretical consequences. Most of these will be developed in succeeding chapters, but two proofs will illustrate some ways in which Taylor's Theorem may be used. The first illustration will be particularly impressive to those who have waded through the proof, in Chapter 16, that π is irrational.

THEOREM 5 e is irrational.

PROOF We know that, for any n,

$$e = e^1 = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + R_n, \quad \text{where } 0 < R_n < \frac{3}{(n+1)!}.$$

Suppose that e were rational, say $e = a/b$, where a and b are positive integers. Choose $n > b$ and also $n > 3$. Then

$$\frac{a}{b} = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} + R_n,$$

so

$$\frac{n!a}{b} = n! + n! + \frac{n!}{2!} + \cdots + \frac{n!}{n!} + n!R_n.$$

Every term in this equation other than $n!R_n$ is an integer (the left side is an integer because $n > b$). Consequently, $n!R_n$ must be an integer also. But

$$0 < R_n < \frac{3}{(n+1)!},$$

so

$$0 < n!R_n < \frac{3}{n+1} < \frac{3}{4} < 1,$$

which is impossible for an integer. ▮

The second illustration is merely a straightforward demonstration of a fact proved in Chapter 15: If

$$\begin{aligned} f'' + f &= 0, \\ f(0) &= 0, \\ f'(0) &= 0, \end{aligned}$$

then $f = 0$. To prove this, observe first that $f^{(k)}$ exists for every k; in fact

$$\begin{aligned} f^{(3)} &= (f'')' = -f', \\ f^{(4)} &= (f^{(3)})' = (-f')' = -f'' = f, \\ f^{(5)} &= (f^{(4)})' = f', \end{aligned}$$

etc.

This shows, not only that all $f^{(k)}$ exist, but also that there are at most 4 different ones: f, f', $-f$, $-f'$. Since $f(0) = f'(0) = 0$, all $f^{(k)}(0)$ are 0. Now Taylor's Theorem states, for any n, that

$$f(x) = \frac{f^{(n+1)}(t)}{(n+1)!}(x-a)^n$$

for some t in $[0, x]$. Each function $f^{(n+1)}$ is continuous (since $f^{(n+2)}$ exists), so for any particular x there is a number M such that

$$|f^{(n+1)}(t)| \leq M \quad \text{for } 0 \leq t \leq x, \text{ and } \textit{all } n$$

(we can add the phrase "and *all* n" because there are only four different $f^{(k)}$). Thus

$$|f(x)| \leq \frac{M|x|^{n+1}}{(n+1)!}.$$

Since this is true for every n, and since $|x|^n/n!$ can be made as small as desired by choosing n sufficiently large, this shows that $|f(x)| \leq \varepsilon$ for any $\varepsilon > 0$; consequently, $f(x) = 0$.

The other uses to which Taylor's Theorem will be put in succeeding chapters are closely related to the computational considerations which have concerned us for much of this chapter. If the remainder term $R_{n,a}(x)$ can be made as small as desired by choosing n sufficiently large, then $f(x)$ can be computed to any degree of accuracy desired by using the polynomials $P_{n,a}(x)$. As we require greater and greater accuracy we must add on more and more terms. If we are willing to add up infinitely many terms (in theory at least!), then we ought to be able to ignore the remainder completely. There should be "infinite sums" like

$$\begin{aligned}
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots, \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots, \\
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots, \\
\arctan x &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad \text{if } |x| \leq 1, \\
\log(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad \text{if } -1 < x \leq 1.
\end{aligned}$$

We are almost completely prepared for this step. Only one obstacle remains—we have never even defined an infinite sum. Chapters 22 and 23 contain the necessary definitions.

PROBLEMS

1. Find the Taylor polynomials (of the indicated degree, and at the indicated point) for the following functions.

(i) $f(x) = e^{e^x}$; degree 3, at 0.

(ii) $f(x) = e^{\sin x}$ degree 3, at 0.

(iii) sin; degree $2n$, at $\dfrac{\pi}{2}$.

(iv) cos; degree $2n$, at π.

(v) exp; degree n, at 1.

(vi) log; degree n, at 2.

(vii) $f(x) = x^5 + x^3 + x$; degree 4, at 0.

(viii) $f(x) = x^5 + x^3 + x$; degree 4, at 1.

(ix) $f(x) = \dfrac{1}{1+x^2}$; degree $2n+1$, at 0.

(x) $f(x) = \dfrac{1}{1+x}$; degree n, at 0.

2. Write each of the following polynomials in x as a polynomial in $(x-3)$. (It is only necessary to compute the Taylor polynomial at 3, of the same degree as the original polynomial. Why?)

(i) $x^2 - 4x - 9$.

(ii) $x^4 - 12x^3 + 44x^2 + 2x + 1$.

(iii) x^5.

(iv) $ax^2 + bx + c$.

3. Write down a sum (using $\sum$ notation) which equals each of the following numbers to within the specified accuracy. To minimize needless computation, consult the tables for 2^n and $n!$ on the next page.

(i) $\sin 1$; error $< 10^{-17}$.

(ii) $\sin 2$; error $< 10^{-12}$.

(iii) $\sin \frac{1}{2}$; error $< 10^{-20}$.

(iv) e; error $< 10^{-4}$.

(v) e^2; error $< 10^{-5}$.

n	2^n	$n!$
1	2	1
2	4	2
3	8	6
4	16	24
5	32	120
6	64	720
7	128	5,040
8	256	40,320
9	512	362,880
10	1,024	3,628,800
11	2,048	39,916,800
12	4,096	479,001,600
13	8,192	6,227,020,800
14	16,384	87,178,291,200
15	32,768	1,307,674,368,000
16	65,536	20,922,789,888,000
17	131,072	355,687,428,096,000
18	262,144	6,402,373,705,728,000
19	524,888	121,645,100,408,832,000
20	1,048,576	2,432,902,008,176,640,000

***4.** This problem is similar to the previous one, except that the errors demanded are so small that the tables cannot be used. You will have to do a little thinking, and in some cases it may be necessary to consult the proof, in Chapter 16, that $x^n/n!$ can be made small by choosing n large—the proof actually provides a method for finding the appropriate n. In the previous problem it was possible to find rather short sums; in fact, it was possible to find the smallest n which makes the estimate of the remainder given by Taylor's Theorem less than the desired error. But in this problem, finding *any* specific sum is a moral victory (provided you can demonstrate that the sum works).

(i) $\sin 1$; error $< 10^{-(10^{10})}$.
(ii) e; error $< 10^{-1,000}$.
(iii) $\sin 10$; error $< 10^{-20}$.
(iv) e^{10}; error $< 10^{-30}$.
(v) $\arctan \frac{1}{10}$; error $< 10^{-(10^{10})}$.

5. (a) In Problem 11-41 you showed that the equation $x^2 = \cos x$ has precisely two solutions. Use the third degree Taylor polynomial of cos to show that the solutions are approximately $\pm\sqrt{2/3}$, and find bounds on the error. Then use the fifth degree Taylor polynomial to get a better approximation.
(b) Similarly, estimate the solutions of the equation $2x^2 = x \sin x + \cos^2 x$.

6. (a) Prove, using Problem 15-9, that

$$\frac{\pi}{4} = \arctan\frac{1}{2} + \arctan\frac{1}{3},$$
$$\frac{\pi}{4} = 4\arctan\frac{1}{5} - \arctan\frac{1}{239}.$$

(b) Show that $\pi = 3.14159\ldots$. (Every budding mathematician should verify a few decimals of π, but the purpose of this exercise is not to set you off on an immense calculation. If the second expression in part (a) is used, the first 5 decimals for π can be computed with remarkably little work.)

7. Suppose that a_i and b_i are the coefficients in the Taylor polynomials at a of f and g, respectively. In other words, $a_i = f^{(i)}(a)/i!$ and $b_i = g^{(i)}(a)/i!$. Find the coefficients c_i of the Taylor polynomials at a of the following functions, in terms of the a_i's and b_i's.

(i) $f+g$.
(ii) fg.
(iii) f'.
(iv) $h(x) = \displaystyle\int_a^x f(t)\,dt$.
(v) $k(x) = \displaystyle\int_0^x f(t)\,dt$.

8. (a) Prove that the Taylor polynomial of $f(x) = \sin(x^2)$ of degree $4n+2$ at 0 is

$$x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \cdots + (-1)^n\frac{x^{4n+2}}{(2n+1)!}.$$

Hint: If P is the Taylor polynomial of degree $2n+1$ for sin at 0, then $\sin x = P(x) + R(x)$, where $\lim\limits_{x\to 0} R(x)/x^{2n+1} = 0$. What does this imply about $\lim\limits_{x\to 0} R(x^2)/x^{4n+2}$?

(b) Find $f^{(k)}(0)$ for all k.
(c) In general, if $f(x) = g(x^m)$, find $f^{(k)}(0)$ in terms of the derivatives of g at 0.

The ideas in this problem can be extended significantly, in ways that are explored in the next three problems.

9. (a) Problem 7 (i) amounts to the equation

$$P_{n,a,f+g} = P_{n,a,f} + P_{n,a,g}.$$

Give a more direct proof by writing

$$f(x) = P_{n,a,f}(x) + R_{n,a,f}(x)$$
$$g(x) = P_{n,a,g}(x) + R_{n,a,g}(x),$$

and using the obvious fact about $R_{n,a,f} + R_{n,a,g}$.

(b) Similarly, Problem 7 (ii) could be used to show that

$$P_{n,a,fg} = [P_{n,a,f} \cdot P_{n,a,g}]_n,$$

where $[P]_n$ denotes the **truncation** of P to degree n, the sum of all terms of P of degree $\leq n$ [with P written as a polynomial in $x - a$]. Again, give a more direct proof, using obvious facts about products involving terms of the form R_n.

(c) Prove that if p and q are polynomials in $x-a$ and $\lim_{x\to 0} R(x)/(x-a)^n = 0$, then

$$p(q(x) + R(x)) = p(q(x)) + \bar{R}(x)$$

where

$$\lim_{x\to 0} \bar{R}(x)/(x-a)^n = 0.$$

Also note that if p is a polynomial in $x - a$ having only terms of degree $> n$, and q is a polynomial in $x - a$ whose constant term is 0, then all terms of $p(q(x - a))$ are of degree $> n$.

(d) If $a = 0$ and $b = g(a) = 0$, then

$$P_{n,a,f\circ g} = [P_{n,b,f} \circ P_{n,a,g}]_n.$$

(Problem 8 is a special case.)

(e) The same result actually holds for all a and any value of $g(a)$. Hint: Consider $F(x) = f(x+g(a))$, $G(x) = g(x+a)$ and $H(x) = G(x)-g(a)$.

(f) If $g(a) = 0$, then

$$P_{n,a,\frac{1}{1-g}} = \left[1 + P_{n,a,g} + (P_{n,a,g})^2 + \cdots + (P_{n,a,g})^n\right]_n.$$

10. For the following applications of Problem 9, we assume $a = 0$ for simplicity, and just write $P_{n,f}$ instead of $P_{n,a,f}$.

(a) For $f(x) = e^x$ and $g(x) = \sin x$, find $P_{5,f+g}(x)$.

(b) For the same f and g, find $P_{5,fg}$.

(c) Find $P_{5,\tan}(x)$. Hint: First use Problem 9 (f) and the value of $P_{5,\cos}(x)$ to find $P_{5,1/\cos}(x)$. (Answer: $x + \frac{x^3}{3} + \frac{2x^5}{15}$)

(d) Find $P_{4,f}$ for $f(x) = e^{2x}\cos x$. (Answer: $1 + 2x + \frac{3}{2}x^2 + \frac{1}{3}x^3 - \frac{7}{24}x^4$)

(e) Find $P_{5,f}$ for $f(x) = \sin x / \cos 2x$. (Answer: $x + \frac{11}{6}x^3 + \frac{361}{120}x^5$)

(f) Find $P_{6,f}$ for $f(x) = x^3/[(1+x^2)e^x]$. (Answer: $x^3 - x^4 - \frac{1}{2}x^5 + \frac{5}{6}x^6$)

11. Calculations of this sort may be used to evaluate limits that we might otherwise try to find through laborious use of l'Hôpital's Rule. Find the following:

(a) $$\lim_{x\to 0} \frac{e^x - 1 - x - \frac{1}{2}x^2}{x - \sin x} = \lim_{x\to 0} \frac{N(x)}{D(x)}.$$

Hint: First find $P_{3,0,N}(x)$ and $P_{3,0,D}(x)$ for the numerator and denominator $N(x)$ and $D(x)$.

(b) $\displaystyle\lim_{x\to 0} \frac{\dfrac{e^x}{1+x} - 1 - \frac{1}{2}x^2}{x - \sin x}.$

Hint: For the term $e^x/(1+x)$, first write $1/(1+x) = 1-x+x^2-x^3+\cdots$.

(c) $\displaystyle\lim_{x\to 0} \left(\frac{1}{\sin^2 x} - \frac{1}{x^2}\right).$

(d) $\displaystyle\lim_{x\to 0} \frac{1-\cos(x^2)}{x^2 \sin^2 x}.$

(e) $\displaystyle\lim_{x\to 0} \frac{1}{\sin^2 x} - \frac{1}{\sin(x^2)}.$

(f) $\displaystyle\lim_{x\to 0} \frac{(\sin x)(\arctan x) - x^2}{1-\cos(x^2)}.$

12. Let

$$f(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0. \end{cases}$$

Starting with the Taylor polynomial of degree $2n+1$ for $\sin x$, together with the estimate for the remainder term derived on page 424, show that

$$f(x) = \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \cdots + (-1)^n \frac{x^{2n}}{(2n+1)!}\right) + R_{2n,0,f}(x)$$

where

$$|R_{2n,0,f}(x)| \leq \frac{|x|^{2n+1}}{(2n+2)!},$$

and use this to conclude that

$$\int_0^1 f \approx \int_0^1 \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!}\right) dx = \frac{1703}{1800} \approx .946$$

with an error of less than 10^{-3}.

13. Let

$$f(x) = \begin{cases} \dfrac{e^x - 1}{x}, & x \neq 0 \\ 1, & x = 0. \end{cases}$$

(a) Find the Taylor polynomial of degree n for f at 0, compute $f^{(k)}(0)$, and give an estimate for the remainder term $R_{n,0,f}$.

(b) Compute $\displaystyle\int_0^1 f$ with an error of less than 10^{-4}.

14. Estimate $\displaystyle\int_0^{0.1} \exp(x^2)\,dx$ with an error of less than 10^{-5}.

15. Prove that if $x \le 0$, then the remainder term $R_{n,0}$ for e^x satisfies

$$|R_{n,0}| \le \frac{|x|^{n+1}}{(n+1)!}.$$

16. Prove that if $-1 < x \le 0$, then the remainder term $R_{n,0}$ for $\log(1+x)$ satisfies

$$|R_{n,0}| \le \frac{|x|^{n+1}}{(1+x)(n+1)}.$$

***17.** (a) Show that if $|g'(x)| \le M|x-a|^n$ for $|x-a| < \delta$, then $|g(x) - g(a)| \le M|x-a|^{n+1}/(n+1)$ for $|x-a| < \delta$.

(b) Use part (a) to show that if $\lim_{x \to a} g'(x)/(x-a)^n = 0$, then

$$\lim_{x \to a} \frac{g(x) - g(a)}{(x-a)^{n+1}} = 0.$$

(c) Show that if $g(x) = f(x) - P_{n,a,f}(x)$, then $g'(x) = f'(x) - P_{n-1,a,f'}(x)$.

(d) Give an inductive proof of Theorem 1, without using l'Hôpital's Rule.

18. Deduce Theorem 1 as a corollary of Taylor's Theorem, with any form of the remainder. (The catch is that it will be necessary to assume one more derivative than in the hypotheses for Theorem 1.)

19. Lagrange's method for proving Taylor's Theorem used the following device. We consider a fixed number x and write

$$(*) \qquad f(x) = f(t) + f'(t)(x-t) + \cdots + \frac{f^{(n)}(t)}{n!}(x-t)^n + S(t)$$

for $S(t) = R_{n,t}(x)$. The notation is a tip-off that we are going to consider the right side as giving the value of some function for a given t, and then write down the fact the derivative of this function is 0, since it equals the constant function whose value is always $f(x)$. To make sure you understand the roles of x and t, check that if

$$g(t) = \frac{f^{(k)}(t)}{k!}(x-t)^k,$$

then

$$\begin{aligned} g'(t) &= \frac{f^{(k)}(t)}{k!}k(x-t)^{k-1}(-1) + \frac{f^{(k+1)}(t)}{k!}(x-t)^k \\ &= -\frac{f^{(k)}(t)}{(k-1)!}(x-t)^{k-1} + \frac{f^{(k+1)}(t)}{k!}(x-t)^k. \end{aligned}$$

(a) Show that

$$
\begin{aligned}
0 = f'(t) &+ \left[-f'(t) + \frac{f''(t)}{1!}(x-t)\right] \\
&+ \left[\frac{-f''(t)}{1!}(x-t) + \frac{f^{(3)}(t)}{2!}(x-t)^2\right] \\
&+ \cdots \\
&\quad + \left[\frac{-f^{(n)}(t)}{(n-1)!}(x-t)^{n-1} + \frac{f^{(n+1)}(t)}{n!}(x-t)^n\right] \\
&\quad\quad + S'(t),
\end{aligned}
$$

and notice that everything collapses to

$$S'(t) = -\frac{f^{(n+1)}(t)}{n!}(x-t)^n.$$

Noting that

$$
\begin{aligned}
S(x) &= R_{n,x}(x) = 0, \\
S(a) &= R_{n,a}(x),
\end{aligned}
$$

apply the Cauchy Mean Value Theorem to the functions S and $h(t) = (x-t)^{n+1}$ on $[a, x]$ to obtain the Lagrange form of the remainder (Lagrange actually handled this part of the argument differently).

(b) Similarly, apply the regular Mean Value Theorem to S to obtain the strange hybrid formula

$$R_{n,a}(x) = \frac{f^{(n+1)}(t)}{n!}(x-t)^n(x-a).$$

This is called the Cauchy form of the remainder.

20. Deduce the Cauchy and Lagrange forms of the remainder from the integral form on page 423, using Problem 13-23. There will be the same catch as in Problem 18.

I know of only one situation where the Cauchy form of the remainder is used. The next problem is preparation for that eventuality.

21. For every number α, and every natural number n, we define the "binomial coefficient"

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdot\ldots\cdot(\alpha-n+1)}{n!},$$

and we define $\binom{\alpha}{0} = 1$, as usual. If α is not an integer, then $\binom{\alpha}{n}$ is never 0, and alternates in sign for $n > \alpha$. Show that the Taylor polynomial of degree n for $f(x) = (1+x)^\alpha$ at 0 is $P_{n,0}(x) = \displaystyle\sum_{k=0}^{n} \binom{\alpha}{k} x^k$, and that the Cauchy and Lagrange forms of the remainder are the following:

Cauchy form:

$$
\begin{aligned}
R_{n,0}(x) &= \frac{\alpha(\alpha-1)\cdot\ldots\cdot(\alpha-n)}{n!}x(x-t)^n(1+t)^{\alpha-n-1} \\
&= \frac{\alpha(\alpha-1)\cdot\ldots\cdot(\alpha-n)}{n!}x(1+t)^{\alpha-1}\left(\frac{x-t}{1+t}\right)^n \\
&= (n+1)\binom{\alpha}{n+1}x(1+t)^{\alpha-1}\left(\frac{x-t}{1+t}\right)^n, \qquad t \text{ in } (0,x) \text{ or } (x,0).
\end{aligned}
$$

Lagrange form:

$$
\begin{aligned}
R_{n,0}(x) &= \frac{\alpha(\alpha-1)\cdot\ldots\cdot(\alpha-n)}{(n+1)!}x^{n+1}(1+t)^{\alpha-n-1} \\
&= \binom{\alpha}{n+1}x^{n+1}(1+t)^{\alpha-n-1}, \qquad t \text{ in } (0,x) \text{ or } (x,0).
\end{aligned}
$$

Estimates for these remainder terms are rather difficult to handle, and are postponed to Problem 23-21.

22. (a) Suppose that f is twice differentiable on $(0,\infty)$ and that $|f(x)| \le M_0$ for all $x > 0$, while $|f''(x)| \le M_2$ for all $x > 0$. Use an appropriate Taylor polynomial to prove that for any $x > 0$ we have

$$|f'(x)| \le \frac{2}{h}M_0 + \frac{h}{2}M_2 \qquad \text{for all } h > 0.$$

(b) Show that for all $x > 0$ we have

$$|f'(x)| \le 2\sqrt{M_0M_2}.$$

Hint: Consider the smallest value of the expression appearing in (a).

(c) If f is twice differentiable on $(0,\infty)$, f'' is bounded, and $f(x)$ approaches 0 as $x \to \infty$, then also $f'(x)$ approaches 0 as $x \to \infty$.

(d) If $\lim_{x\to\infty} f(x)$ exists and $\lim_{x\to\infty} f''(x)$ exists, then $\lim_{x\to\infty} f''(x) = \lim_{x\to\infty} f'(x) = 0$. (Compare Problem 11-34.)

23. (a) Prove that if $f''(a)$ exists, then

$$f''(a) = \lim_{h\to 0}\frac{f(a+h)+f(a-h)-2f(a)}{h^2}.$$

The limit on the right is called the *Schwarz second derivative* of f at a. Hint: Use the Taylor polynomial $P_{2,a}(x)$ with $x = a+h$ and with $x = a-h$.

(b) Let $f(x) = x^2$ for $x \ge 0$, and $-x^2$ for $x \le 0$. Show that

$$\lim_{h\to 0}\frac{f(0+h)+f(0-h)-2f(0)}{h^2}$$

exists, even though $f''(0)$ does not.

(c) Prove that if f has a local maximum at a, and the Schwarz second derivative of f at a exists, then it is ≤ 0.

(d) Prove that if $f'''(a)$ exists, then

$$\frac{f'''(a)}{3} = \lim_{h \to 0} \frac{f(a+h) - f(a-h) - 2hf'(a)}{h^3}.$$

24. Use the Taylor polynomial $P_{1,a,f}$, together with the remainder, to prove a weak form of Theorem 2 of the Appendix to Chapter 11: If $f'' > 0$, then the graph of f always lies above the tangent line of f, except at the point of contact.

***25.** Problem 18-43 presented a rather complicated proof that $f = 0$ if $f'' - f = 0$ and $f(0) = f'(0) = 0$. Give another proof, using Taylor's Theorem. (This problem is really a preliminary skirmish before doing battle with the general case in Problem 26, and is meant to convince you that Taylor's Theorem is a good tool for tackling such problems, even though tricks work out more neatly for special cases.)

****26.** Consider a function f which satisfies the differential equation

$$f^{(n)} = \sum_{j=0}^{n-1} a_j f^{(j)},$$

for certain numbers $a_0, \dots, a_{n-1}$. Several special cases have already received detailed treatment, either in the text or in other problems; in particular, we have found all functions satisfying $f' = f$, or $f'' + f = 0$, or $f'' - f = 0$. The trick in Problem 18-42 enables us to find many solutions for such equations, but doesn't say whether these are the only solutions. This requires a *uniqueness* result, which will be supplied by this problem. At the end you will find some (necessarily sketchy) remarks about the general solution.

(a) Derive the following formula for $f^{(n+1)}$ (let us agree that "a_{-1}" will be 0):

$$f^{(n+1)} = \sum_{j=0}^{n-1} (a_{j-1} + a_{n-1}a_j) f^{(j)}.$$

(b) Deduce a formula for $f^{(n+2)}$.

The formula in part (b) is not going to be used; it was inserted only to convince you that a general formula for $f^{(n+k)}$ is out of the question. On the other hand, as part (c) shows, it is not very hard to obtain estimates on the size of $f^{(n+k)}(x)$.

(c) Let $N = \max(1, |a_0|, \dots, |a_{n-1}|)$. Then $|a_{j-1} + a_{n-1}a_j| \le 2N^2$; this means that

$$f^{(n+1)} = \sum_{j=0}^{n-1} b_j{}^1 f^{(j)}, \quad \text{where } |b_j{}^1| \le 2N^2.$$

Show that

$$f^{(n+2)} = \sum_{j=0}^{n-1} b_j{}^2 f^{(j)}, \quad \text{where } |b_j{}^2| \le 4N^3,$$

and, more generally,

$$f^{(n+k)} = \sum_{j=0}^{n-1} b_j{}^k f^{(j)}, \quad \text{where } |b_j{}^k| \le 2^k N^{k+1}.$$

(d) Conclude from part (c) that, for any particular number x, there is a number M such that

$$|f^{(n+k)}(x)| \le M \cdot 2^k N^{k+1} \quad \text{for all } k.$$

(e) Now suppose that $f(0) = f'(0) = \cdots = f^{(n-1)}(0) = 0$. Show that

$$|f(x)| \le \frac{M \cdot 2^{k+1} N^{k+2} |x|^{n+k+1}}{(n+k+1)!} \le \frac{M \cdot |2Nx|^{n+k+1}}{(n+k+1)!},$$

and conclude that $f = 0$.

(f) Show that if f_1 and f_2 are both solutions of the differential equation

$$f^{(n)} = \sum_{j=0}^{n-1} a_j f^{(j)},$$

and $f_1{}^{(j)}(0) = f_2{}^{(j)}(0)$ for $0 \le j \le n-1$, then $f_1 = f_2$.

In other words, the solutions of this differential equation are determined by the "initial conditions" (the values $f^{(j)}(0)$ for $0 \le j \le n-1$). This means that we can find *all* solutions once we can find enough solutions to obtain any given set of initial conditions. If the equation

$$x^n - a_{n-1}x^{n-1} - \cdots - a_0 = 0$$

has n distinct roots $\alpha_1, \ldots, \alpha_n$, then any function of the form

$$f(x) = c_1 e^{\alpha_1 x} + \cdots + c_n e^{\alpha_n x}$$

is a solution, and

$$\begin{aligned} f(0) &= c_1 + \cdots + c_n, \\ f'(0) &= \alpha_1 c_1 + \cdots + \alpha_n c_n, \\ &\vdots \\ f^{(n-1)}(0) &= \alpha_1{}^{n-1} c_1 + \cdots + \alpha_n{}^{n-1} c_n. \end{aligned}$$

As a matter of fact, every solution is of this form, because we can obtain any set of numbers on the left side by choosing the c's properly, but we will not try to prove this last assertion. (It is a purely algebraic fact, which you can easily check for $n = 2$ or 3.) These remarks are also true if some of the roots are multiple roots, and even in the more general situation considered in Chapter 27.

**27. (a) Suppose that f is a continuous function on $[a,b]$ with $f(a) = f(b)$ and that for all x in (a,b) the Schwarz second derivative of f at x is 0 (Problem 23). Show that f is constant on $[a,b]$. Hint: Suppose that $f(x) > f(a)$ for some x in (a,b). Consider the function

$$g(x) = f(x) - \varepsilon(x-a)(b-x)$$

with $g(a) = g(b) = f(a)$. For sufficiently small $\varepsilon > 0$ we will have $g(x) > g(a)$, so g will have a maximum point y in (a,b). Now use Problem 23(c) (the Schwarz second derivative of $(x-a)(b-x)$ is simply its ordinary second derivative).

(b) If f is a continuous function on $[a,b]$ whose Schwarz second derivative is 0 at all points of (a,b), then f is linear.

*28. (a) Let $f(x) = x^4 \sin 1/x^2$ for $x \neq 0$, and $f(0) = 0$. Show that $f = 0$ up to order 2 at 0, even though $f''(0)$ does not exist.

This example is slightly more complex, but also slightly more impressive, than the example in the text, because both $f'(a)$ and $f''(a)$ exist for $a \neq 0$. Thus, for each number a there is another number $m(a)$ such that

$$(*) \quad f(x) = f(a) + f'(a)(x-a) + \frac{m(a)}{2}(x-a)^2 + R_a(x),$$

$$\text{where } \lim_{x \to a} \frac{R_a(x)}{(x-a)^2} = 0;$$

namely, $m(a) = f''(a)$ for $a \neq 0$, and $m(0) = 0$. Notice that the function m defined in this way is not continuous.

(b) Suppose that f is a differentiable function such that $(*)$ holds for all a, with $m(a) = 0$. Use Problem 27 to show that $f''(a) = m(a) = 0$ for all a.

(c) Now suppose that $(*)$ holds for all a, and that m is continuous. Prove that for all a the second derivative $f''(a)$ exists and equals $m(a)$.