CHAPTER 14 THE FUNDAMENTAL THEOREM OF CALCULUS

From the hints given in the previous chapter you may have already guessed the first theorem of this chapter. We know that if f is integrable, then $F(x) = \int_a^x f$ is continuous; it is only fitting that we ask what happens when the original function f is continuous. It turns out that F is differentiable (and its derivative is especially simple).

THEOREM 1 (THE FIRST FUNDAMENTAL THEOREM OF CALCULUS)

Let f be integrable on $[a, b]$, and define F on $[a, b]$ by

$$F(x) = \int_a^x f.$$

If f is continuous at c in $[a, b]$, then F is differentiable at c, and

$$F'(c) = f(c).$$

(If $c = a$ or b, then $F'(c)$ is understood to mean the right- or left-hand derivative of F.)

PROOF

We will assume that c is in (a, b); the easy modifications for $c = a$ or b may be supplied by the reader. By definition,

$$F'(c) = \lim_{h \to 0} \frac{F(c+h) - F(c)}{h}.$$

Suppose first that $h > 0$. Then

$$F(c+h) - F(c) = \int_c^{c+h} f.$$

Define m_h and M_h as follows (Figure 1):

$$m_h = \inf\{f(x) : c \leq x \leq c+h\},$$
$$M_h = \sup\{f(x) : c \leq x \leq c+h\}.$$

It follows from Theorem 13-7 that

$$m_h \cdot h \leq \int_c^{c+h} f \leq M_h \cdot h.$$

Therefore

$$m_h \leq \frac{F(c+h) - F(c)}{h} \leq M_h.$$

If $h < 0$, only a few details of the argument have to be changed. Let

$$m_h = \inf\{f(x) : c+h \leq x \leq c\},$$
$$M_h = \sup\{f(x) : c+h \leq x \leq c\}.$$

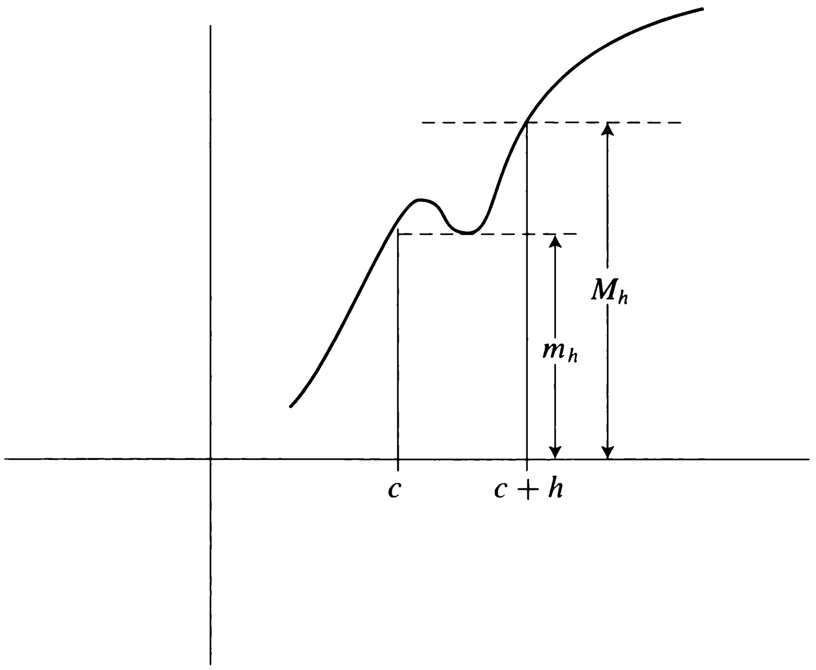

FIGURE 1

Then

$$m_h \cdot (-h) \leq \int_{c+h}^{c} f \leq M_h \cdot (-h).$$

Since

$$F(c+h) - F(c) = \int_{c}^{c+h} f = -\int_{c+h}^{c} f$$

this yields

$$m_h \cdot h \geq F(c+h) - F(c) \geq M_h \cdot h.$$

Since $h < 0$, dividing by h reverses the inequality again, yielding the same result as before:

$$m_h \leq \frac{F(c+h) - F(c)}{h} \leq M_h.$$

This inequality is true for any integrable function, continuous or not. Since f is continuous at c, however,

$$\lim_{h \to 0} m_h = \lim_{h \to 0} M_h = f(c),$$

and this proves that

$$F'(c) = \lim_{h \to 0} \frac{F(c+h) - F(c)}{h} = f(c). \blacksquare$$

Although Theorem 1 deals only with the function obtained by varying the upper limit of integration, a simple trick shows what happens when the lower limit is varied. If G is defined by

$$G(x) = \int_{x}^{b} f,$$

then

$$G(x) = \int_a^b f - \int_a^x f.$$

Consequently, if f is continuous at c, then

$$G'(c) = -f(c).$$

The minus sign appearing here is very fortunate, and allows us to extend Theorem 1 to the situation where the function

$$F(x) = \int_a^x f$$

is defined even for $x < a$. In this case we can write

$$F(x) = -\int_x^a f,$$

so if $c < a$ we have

$$F'(c) = -(-f(c)) = f(c),$$

exactly as before.

Notice that in either case, differentiability of F at c is ensured by continuity of f at c alone. Nevertheless, Theorem 1 is most interesting when f is continuous at all points in $[a, b]$. In this case F is differentiable at all points in $[a, b]$ and

$$F' = f.$$

In general, it is extremely difficult to decide whether a given function f is the derivative of some other function; for this reason Theorem 11-7 and Problems 11-60 and 11-61 are particularly interesting, since they reveal certain properties which f must have. If f is continuous, however, there is no problem at all—according to Theorem 1, f *is* the derivative of some function, namely the function

$$F(x) = \int_a^x f.$$

Theorem 1 has a simple corollary which frequently reduces computations of integrals to a triviality.

COROLLARY If f is continuous on $[a, b]$ and $f = g'$ for some function g, then

$$\int_a^b f = g(b) - g(a).$$

PROOF Let

$$F(x) = \int_a^x f.$$

Then $F' = f = g'$ on $[a, b]$. Consequently, there is a number c such that

$$F = g + c.$$

The number c can be evaluated easily: note that

$$0 = F(a) = g(a) + c,$$

so $c = -g(a)$; thus

$$F(x) = g(x) - g(a).$$

This is true, in particular, for $x = b$. Thus

$$\int_a^b f = F(b) = g(b) - g(a). \blacksquare$$

The proof of this corollary tends, at first sight, to make the corollary seem useless: after all, what good is it to know that

$$\int_a^b f = g(b) - g(a)$$

if g is, for example, $g(x) = \int_a^x f$? The point, of course, is that one might happen to know a quite different function g with this property. For example, if

$$g(x) = \frac{x^3}{3} \quad \text{and} \quad f(x) = x^2,$$

then $g'(x) = f(x)$ so we obtain, without ever computing lower and upper sums:

$$\int_a^b x^2\,dx = \frac{b^3}{3} - \frac{a^3}{3}.$$

One can treat other powers similarly; if n is a natural number and $g(x) = x^{n+1}/(n+1)$, then $g'(x) = x^n$, so

$$\int_a^b x^n\,dx = \frac{b^{n+1}}{n+1} - \frac{a^{n+1}}{n+1}.$$

For any natural number n, the function $f(x) = x^{-n}$ is not bounded on any interval containing 0, but if a and b are both positive or both negative, then

$$\int_a^b x^{-n}\,dx = \frac{b^{-n+1}}{-n+1} - \frac{a^{-n+1}}{-n+1}.$$

Naturally this formula is only true for $n \neq -1$. *We do not know a simple expression for*

$$\int_a^b \frac{1}{x}\,dx.$$

The problem of computing this integral is discussed later, but it provides a good opportunity to warn against a serious error. The conclusion of Corollary 1 is often confused with the definition of integrals—many students think that $\int_a^b f$ is defined as: "$g(b) - g(a)$, where g is a function whose derivative is f." This "definition" is not only wrong—it is useless. One reason is that a function f may be integrable without being the derivative of another function. For example, if $f(x) = 0$ for $x \neq 1$ and $f(1) = 1$, then f is integrable, but f cannot be a derivative (why not?). There is also another reason that is much more important: If f is continuous, then we know that $f = g'$ for some function g; but we know this *only because of Theorem 1.* The function $f(x) = 1/x$ provides an excellent illustration: if $x > 0$, then $f(x) = g'(x)$, where

$$g(x) = \int_1^x \frac{1}{t}\,dt,$$

and we know of no simpler function g with this property.

The corollary to Theorem 1 is so useful that it is frequently called the Second Fundamental Theorem of Calculus. In this book, that name is reserved for a somewhat stronger result (which in practice, however, is not much more useful). As we have just mentioned, a function f might be of the form g' even if f is not continuous. If f is integrable, then it is still true that

$$\int_a^b f = g(b) - g(a).$$

The proof, however, must be entirely different—we cannot use Theorem 1, so we must return to the definition of integrals.

THEOREM 2 (THE SECOND FUNDAMENTAL THEOREM OF CALCULUS) If f is integrable on $[a, b]$ and $f = g'$ for some function g, then

$$\int_a^b f = g(b) - g(a).$$

PROOF Let $P = \{t_0, \ldots, t_n\}$ be any partition of $[a, b]$. By the Mean Value Theorem there is a point x_i in $[t_{i-1}, t_i]$ such that

$$\begin{aligned} g(t_i) - g(t_{i-1}) &= g'(x_i)(t_i - t_{i-1}) \\ &= f(x_i)(t_i - t_{i-1}). \end{aligned}$$

If

$$\begin{aligned} m_i &= \inf\{f(x) : t_{i-1} \le x \le t_i\}, \\ M_i &= \sup\{f(x) : t_{i-1} \le x \le t_i\}, \end{aligned}$$

then clearly

$$m_i(t_i - t_{i-1}) \le f(x_i)(t_i - t_{i-1}) \le M_i(t_i - t_{i-1}),$$

that is,

$$m_i(t_i - t_{i-1}) \le g(t_i) - g(t_{i-1}) \le M_i(t_i - t_{i-1}).$$

Adding these equations for $i = 1, \ldots, n$ we obtain

$$\sum_{i=1}^n m_i(t_i - t_{i-1}) \le g(b) - g(a) \le \sum_{i=1}^n M_i(t_i - t_{i-1})$$

so that

$$L(f, P) \le g(b) - g(a) \le U(f, P)$$

for every partition P. But this means that

$$g(b) - g(a) = \int_a^b f. \blacksquare$$

We have already used the corollary to Theorem 1 (or, equivalently, Theorem 2) to find the integrals of a few elementary functions:

$$\int_a^b x^n\,dx = \frac{b^{n+1}}{n+1} - \frac{a^{n+1}}{n+1}, \quad n \neq -1. \qquad \text{(a and b both positive or both negative if $n > 0$).}$$

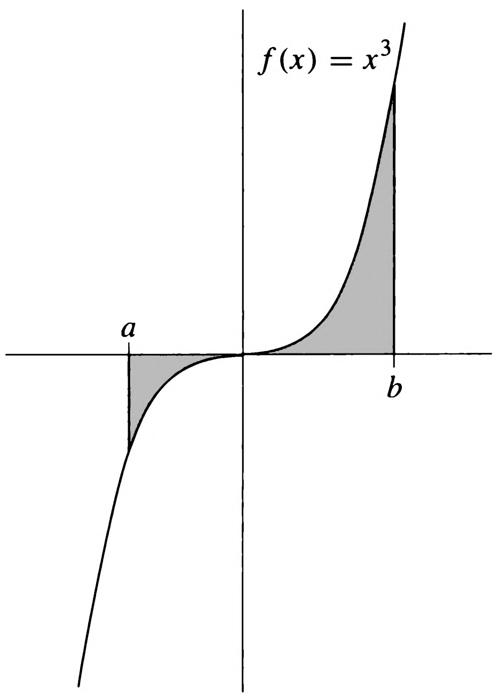

FIGURE 2

As we pointed out in Chapter 13, this integral does not always represent the area bounded by the graph of the function, the horizontal axis, and the vertical lines through $(a, 0)$ and $(b, 0)$. For example, if $a < 0 < b$, then

$$\int_a^b x^3\,dx$$

does not represent the area of the region shown in Figure 2, which is given instead by

$$-\left(\int_a^0 x^3\,dx\right) + \int_0^b x^3\,dx = -\left(\frac{0^4}{4} - \frac{a^4}{4}\right) + \left(\frac{b^4}{4} - \frac{0^4}{4}\right)$$
$$= \frac{a^4}{4} + \frac{b^4}{4}.$$

Similar care must be exercised in finding the areas of regions which are bounded by the graphs of more than one function—a problem which may frequently involve considerable ingenuity in any case. Suppose, to take a simple example first, that we wish to find the area of the region, shown in Figure 3, between the graphs of the functions

$$f(x) = x^2 \quad \text{and} \quad g(x) = x^3$$

on the interval $[0, 1]$. If $0 \le x \le 1$, then $0 \le x^3 \le x^2$, so that the graph of g lies below that of f. The area of the region of interest to us is therefore

$$\text{area } R(f, 0, 1) - \text{ area } R(g, 0, 1),$$

which is

$$\int_0^1 x^2\,dx - \int_0^1 x^3\,dx = \tfrac{1}{3} - \tfrac{1}{4} = \tfrac{1}{12}.$$

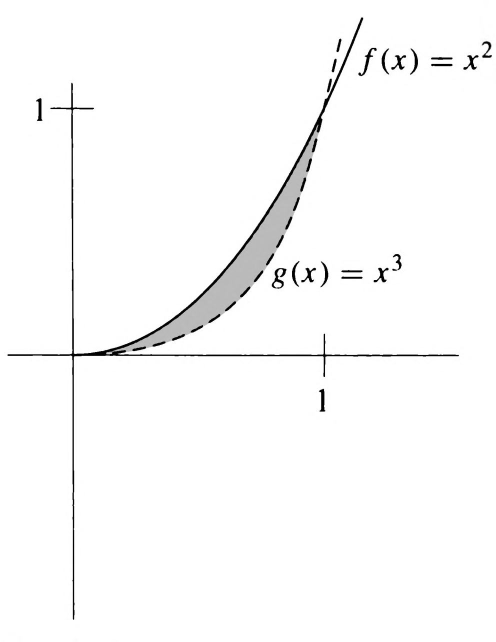

FIGURE 3

This area could have been expressed as

$$\int_a^b (f - g).$$

If $g(x) \le f(x)$ for all x in $[a, b]$, then this integral always gives the area bounded by f and g, *even if f and g are sometimes negative.* The easiest way to see this is shown in Figure 4. If c is a number such that $f + c$ and $g + c$ are nonnegative on $[a, b]$, then the region R_1, bounded by f and g, has the same area as the region R_2, bounded by $f + c$ and $g + c$. Consequently,

$$\begin{aligned}
\text{area } R_1 = \text{area } R_2 &= \int_a^b (f + c) - \int_a^b (g + c) \\
&= \int_a^b [(f + c) - (g + c)] \\
&= \int_a^b (f - g).
\end{aligned}$$

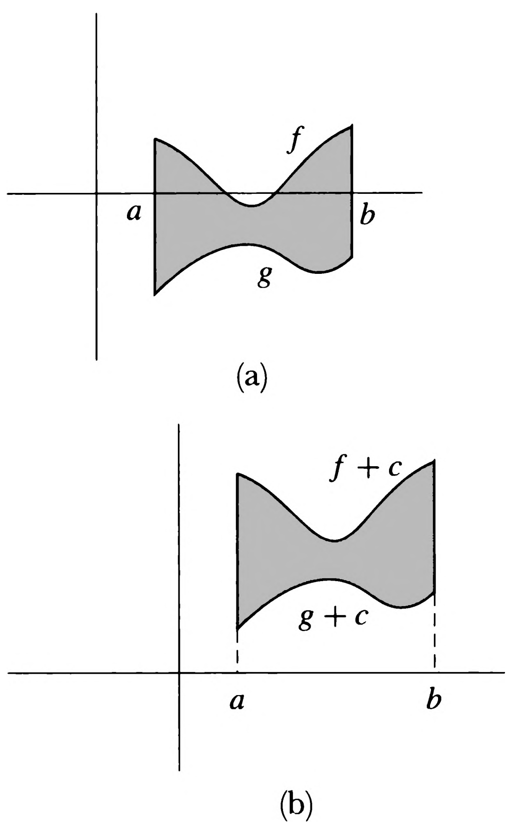

FIGURE 4

This observation is useful in the following problem: Find the area of the region bounded by the graphs of

$$f(x) = x^3 - x \quad \text{and} \quad g(x) = x^2.$$

The first necessity is to determine this region more precisely. The graphs of f and g intersect when

$$\begin{aligned}
& x^3 - x = x^2, \\
\text{or} \quad & x^3 - x^2 - x = 0, \\
\text{or} \quad & x(x^2 - x - 1) = 0, \\
\text{or} \quad & x = 0, \frac{1 + \sqrt{5}}{2}, \frac{1 - \sqrt{5}}{2}.
\end{aligned}$$

On the interval $([1 - \sqrt{5}]/2, 0)$ we have $x^3 - x \geq x^2$ and on the interval $(0, [1 + \sqrt{5}]/2)$ we have $x^2 \geq x^3 - x$. These assertions are apparent from the graphs (Figure 5), but they can also be checked easily, as follows. Since $f(x) = g(x)$ only if $x = 0$, $[1+\sqrt{5}]/2$, or $[1 - \sqrt{5}]/2$, the function $f - g$ does not change sign on the intervals $([1 - \sqrt{5}]/2, 0)$ and $(0, [1+\sqrt{5}]/2)$; it is therefore only necessary to observe, for example, that

$$\begin{aligned}
(-\tfrac{1}{2})^3 - (-\tfrac{1}{2}) - (-\tfrac{1}{2})^2 &= \tfrac{1}{8} > 0, \\
1^3 - 1 - 1^2 &= -1 < 0,
\end{aligned}$$

to conclude that

$$\begin{aligned}
f - g \geq 0 \quad & \text{on } ([1 - \sqrt{5}]/2, 0), \\
f - g \leq 0 \quad & \text{on } (0, [1 + \sqrt{5}]/2).
\end{aligned}$$

The area of the region in question is thus

$$\int_{\frac{1-\sqrt{5}}{2}}^{0} (x^3 - x - x^2)\, dx + \int_0^{\frac{1+\sqrt{5}}{2}} [x^2 - (x^3 - x)]\, dx.$$

As this example reveals, one of the major problems involved in finding the areas of a region may be the exact determination of the region. There are, however, more substantial problems of a logical nature—we have thus far defined the areas of some very special regions only, which do not even include some of the regions whose areas have just been computed! We have simply assumed that area made

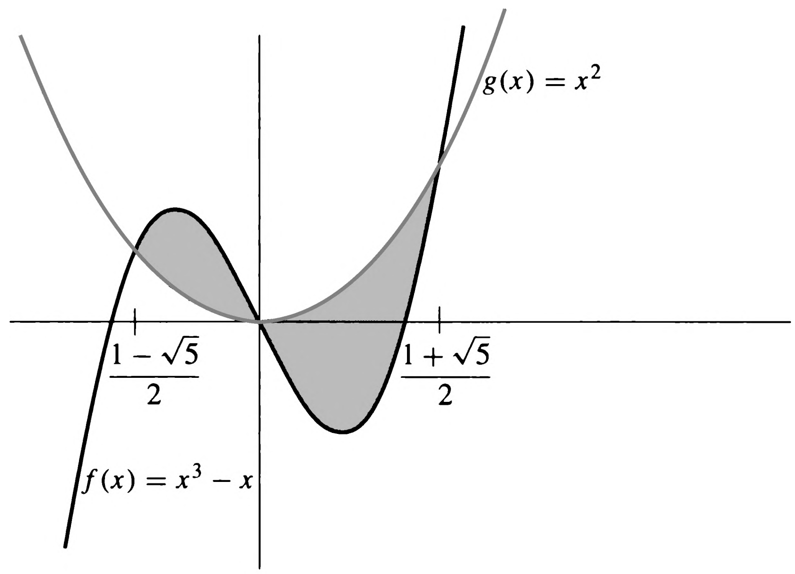

FIGURE 5

sense for these regions, and that certain reasonable properties of "area" do hold. These remarks are not meant to suggest that you should regard exercising ingenuity to compute areas as beneath you, but are meant to indicate that a better approach to the definition of area is available, although its proper place is somewhere in advanced calculus. The desire to define area was the motivation, both in this book and historically, for the definition of the integral, but the integral does not really provide the best method of *defining* areas, although it is frequently the proper tool for *computing* them.

It may be discouraging to learn that integrals are not suitable for the very purpose for which they were invented, but we will soon see how essential they are for other purposes. The most important use of integrals has already been emphasized: if f is continuous, the integral provides a function y such that

$$y'(x) = f(x).$$

This equation is the simplest example of a "differential equation" (an equation for a function y which involves derivatives of y). The Fundamental Theorem of Calculus says that this differential equation has a solution, if f is continuous. In succeeding chapters, and in various problems, we will solve more complicated equations, but the solution almost always depends somehow on the integral; in order to solve a differential equation it is necessary to construct a new function, and the integral is one of the best ways of doing this.

Since the differentiable functions provided by the Fundamental Theorem of Calculus will play such a prominent role in later work, it is very important to realize that these functions may be combined, like less esoteric functions, to yield still more functions, whose derivatives can be found by the Chain Rule.

Suppose, for example, that

$$f(x) = \int_a^{x^3} \frac{1}{1+\sin^2 t}\,dt.$$

Although the notation tends to disguise the fact somewhat, f is the composition of the functions

$$C(x) = x^3 \quad \text{and} \quad F(x) = \int_a^x \frac{1}{1+\sin^2 t}\, dt.$$

In fact, $f(x) = F(C(x))$; in other words, $f = F \circ C$. Therefore, by the Chain Rule,

$$\begin{aligned} f'(x) &= F'(C(x)) \cdot C'(x) \\ &= F'(x^3) \cdot 3x^2 \\ &= \frac{1}{1+\sin^2 x^3} \cdot 3x^2. \end{aligned}$$

If f is defined, instead, as

$$f(x) = \int_{x^3}^a \frac{1}{1+\sin^2 t}\, dt,$$

then

$$f'(x) = -\frac{1}{1+\sin^2 x^3} \cdot 3x^2.$$

If f is defined as the *reverse* composition,

$$f(x) = \left(\int_a^x \frac{1}{1+\sin^2 t}\, dt \right)^3,$$

then

$$\begin{aligned} f'(x) &= C'(F(x)) \cdot F'(x) \\ &= 3 \left(\int_a^x \frac{1}{1+\sin^2 t}\, dt \right)^2 \cdot \frac{1}{1+\sin^2 x}. \end{aligned}$$

Similarly, if

$$\begin{aligned} f(x) &= \int_a^{\sin x} \frac{1}{1+\sin^2 t}\, dt, \\ g(x) &= \int_{\sin x}^a \frac{1}{1+\sin^2 t}\, dt, \\ h(x) &= \sin \left(\int_a^x \frac{1}{1+\sin^2 t}\, dt \right), \end{aligned}$$

then

$$\begin{aligned} f'(x) &= \frac{1}{1+\sin^2(\sin x)} \cdot \cos x, \\ g'(x) &= \frac{-1}{1+\sin^2(\sin x)} \cdot \cos x, \\ h'(x) &= \cos \left(\int_a^x \frac{1}{1+\sin^2 t}\, dt \right) \cdot \frac{1}{1+\sin^2 x}. \end{aligned}$$

The formidable appearing function

$$f(x) = \int_a^{\left(\int_a^x \frac{1}{1+\sin^2 t}\,dt\right)} \frac{1}{1+\sin^2 t}\,dt$$

is also a composition; in fact, $f = F \circ F$. Therefore

$$\begin{aligned} f'(x) &= F'(F(x)) \cdot F'(x) \\ &= \frac{1}{1+\sin^2\left(\displaystyle\int_a^x \frac{1}{1+\sin^2 t}\,dt\right)} \cdot \frac{1}{1+\sin^2 x}. \end{aligned}$$

As these examples reveal, the expression occurring above (or below) the integral sign indicates the function which will appear on the *right* when f is written as a composition. As a final example, consider the triple compositions

$$f(x) = \int_a^{\left(\int_a^{x^3} \frac{1}{1+\sin^2 t}\,dt\right)} \frac{1}{1+\sin^2 t}\,dt, \qquad g(x) = \int_a^{\left[\int_a^{\left(\int_a^x \frac{1}{1+\sin^2 t}\,dt\right)} \frac{1}{1+\sin^2 t}\,dt\right]} \frac{1}{1+\sin^2 t}\,dt,$$

which can be written

$$f = F \circ F \circ C \quad \text{and} \quad g = F \circ F \circ F.$$

Omitting the intermediate steps (which you may supply, if you still feel insecure), we obtain

$$f'(x) = \frac{1}{1+\sin^2\left(\displaystyle\int_a^{x^3} \frac{1}{1+\sin^2 t}\,dt\right)} \cdot \frac{1}{1+\sin^2 x^3} \cdot 3x^2,$$

$$g'(x) = \frac{1}{1+\sin^2\left[\displaystyle\int_a^{\left(\int_a^x \frac{1}{1+\sin^2 t}\,dt\right)} \frac{1}{1+\sin^2 t}\,dt\right]} \cdot \frac{1}{1+\sin^2\left(\displaystyle\int_a^x \frac{1}{1+\sin^2 t}\,dt\right)} \cdot \frac{1}{1+\sin^2 x}.$$

Like the simpler differentiations of Chapter 10, these manipulations should become much easier after the practice provided by some of the problems, and, like the problems of Chapter 10, these differentiations are simply a test of your understanding of the Chain Rule, in the somewhat unfamiliar context provided by the Fundamental Theorem of Calculus.

The powerful uses to which the integral will be put in the following chapters all depend on the Fundamental Theorem of Calculus, yet the proof of that theorem was quite easy—it seems that all the real work went into the definition of the integral. Actually, this is not quite true. In order to apply Theorem 1 to a continuous function we need to know that if f is continuous on $[a, b]$, then f is integrable on $[a, b]$. Although we've already offered one proof of this result, there is a more elementary argument that you might prefer. Like most "elementary" arguments, it's quite tricky, but it has the virtue that it will force a review of the proof of Theorem 1.

If f is any bounded function on $[a, b]$, then

$$\sup\{L(f, P)\} \quad \text{and} \quad \inf\{U(f, P)\}$$

will both exist, even if f is not integrable. These numbers are called the **lower integral** of f on $[a, b]$ and the **upper integral** of f on $[a, b]$, respectively, and will be denoted by

$$\mathbf{L}\int_a^b f \quad \text{and} \quad \mathbf{U}\int_a^b f.$$

The lower and upper integrals both have several properties which the integral possesses. In particular, if $a < c < b$, then

$$\mathbf{L}\int_a^b f = \mathbf{L}\int_a^c f + \mathbf{L}\int_c^b f \quad \text{and} \quad \mathbf{U}\int_a^b f = \mathbf{U}\int_a^c f + \mathbf{U}\int_c^b f,$$

and if $m \le f(x) \le M$ for all x in $[a, b]$, then

$$m(b-a) \le \mathbf{L}\int_a^b f \le \mathbf{U}\int_a^b f \le M(b-a).$$

The proofs of these facts are left as an exercise, since they are quite similar to the corresponding proofs for integrals. The results for integrals are actually a corollary of the results for upper and lower integrals, because f is integrable precisely when

$$\mathbf{L}\int_a^b f = \mathbf{U}\int_a^b f.$$

We will prove that a continuous function f is integrable by showing that this equality always holds for continuous functions. It is actually easier to show that

$$\mathbf{L}\int_a^x f = \mathbf{U}\int_a^x f$$

for all x in $[a, b]$; the trick is to note that most of the proof of Theorem 1 didn't even depend on the fact that f was integrable!

THEOREM 13-3 If f is continuous on $[a, b]$, then f is integrable on $[a, b]$.

PROOF Define functions L and U on $[a, b]$ by

$$L(x) = \mathbf{L}\int_a^x f \quad \text{and} \quad U(x) = \mathbf{U}\int_a^x f.$$

Let x be in (a, b). If $h > 0$ and

$$m_h = \inf\{f(t) : x \le t \le x+h\},$$
$$M_h = \sup\{f(t) : x \le t \le x+h\},$$

then

$$m_h \cdot h \le \mathbf{L}\int_x^{x+h} f \le \mathbf{U}\int_x^{x+h} f \le M_h \cdot h,$$

so

$$m_h \cdot h \le L(x+h) - L(x) \le U(x+h) - U(x) \le M_h \cdot h$$

or

$$m_h \le \frac{L(x+h) - L(x)}{h} \le \frac{U(x+h) - U(x)}{h} \le M_h.$$

If $h < 0$ and

$$\begin{aligned} m_h &= \inf\{f(t) : x+h \le t \le x\}, \\ M_h &= \sup\{f(t) : x+h \le t \le x\}, \end{aligned}$$

one obtains the same inequality, precisely as in the proof of Theorem 1.

Since f is continuous at x, we have

$$\lim_{h\to 0} m_h = \lim_{h\to 0} M_h = f(x),$$

and this proves that

$$L'(x) = U'(x) = f(x) \quad \text{for } x \text{ in } (a,b).$$

This means that there is a number c such that

$$U(x) = L(x) + c \quad \text{for all } x \text{ in } [a,b].$$

Since

$$U(a) = L(a) = 0,$$

the number c must equal 0, so

$$U(x) = L(x) \quad \text{for all } x \text{ in } [a,b].$$

In particular,

$$\mathbf{U}\int_a^b f = U(b) = L(b) = \mathbf{L}\int_a^b f,$$

and this means that f is integrable on $[a,b]$. ▮

PROBLEMS

1. Find the derivatives of each of the following functions.

(i) $F(x) = \displaystyle\int_a^{x^3} \sin^3 t\, dt.$

(ii) $F(x) = \displaystyle\int_3^{\left(\int_1^x \sin^3 t\, dt\right)} \frac{1}{1+\sin^6 t + t^2}\, dt$

(iii) $F(x) = \displaystyle\int_{15}^x \left(\int_8^y \frac{1}{1+t^2+\sin^2 t}\, dt\right) dy.$

(iv) $F(x) = \displaystyle\int_x^b \frac{1}{1+t^2+\sin^2 t}\, dt.$

(v) $F(x) = \displaystyle\int_a^b \frac{x}{1+t^2+\sin^2 t}\,dt.$

(vi) $F(x) = \sin\left(\displaystyle\int_0^x \sin\left(\int_0^y \sin^3 t\,dt\right) dy\right).$

(vii) F^{-1}, where $F(x) = \displaystyle\int_1^x \frac{1}{t}\,dt.$

(viii) F^{-1}, where $F(x) = \displaystyle\int_0^x \frac{1}{\sqrt{1-t^2}}\,dt.$

(Find $(F^{-1})'(x)$ in terms of $F^{-1}(x)$.)

2. For each of the following f, if $F(x) = \int_0^x f$, at which points x is $F'(x) = f(x)$? (Caution: it might happen that $F'(x) = f(x)$, even if f is not continuous at x.)

(i) $f(x) = 0$ if $x \le 1$, $f(x) = 1$ if $x > 1$.
(ii) $f(x) = 0$ if $x < 1$, $f(x) = 1$ if $x \ge 1$.
(iii) $f(x) = 0$ if $x \ne 1$, $f(x) = 1$ if $x = 1$.
(iv) $f(x) = 0$ if x is irrational, $f(x) = 1/q$ if $x = p/q$ in lowest terms.
(v) $f(x) = 0$ if $x \le 0$, $f(x) = x$ if $x \ge 0$.
(vi) $f(x) = 0$ if $x \le 0$ or $x > 1$, $f(x) = 1/[1/x]$ if $0 < x \le 1$.
(vii) f is the function shown in Figure 6.
(viii) $f(x) = 1$ if $x = 1/n$ for some n in $\mathbf{N}$, $f(x) = 0$ otherwise.

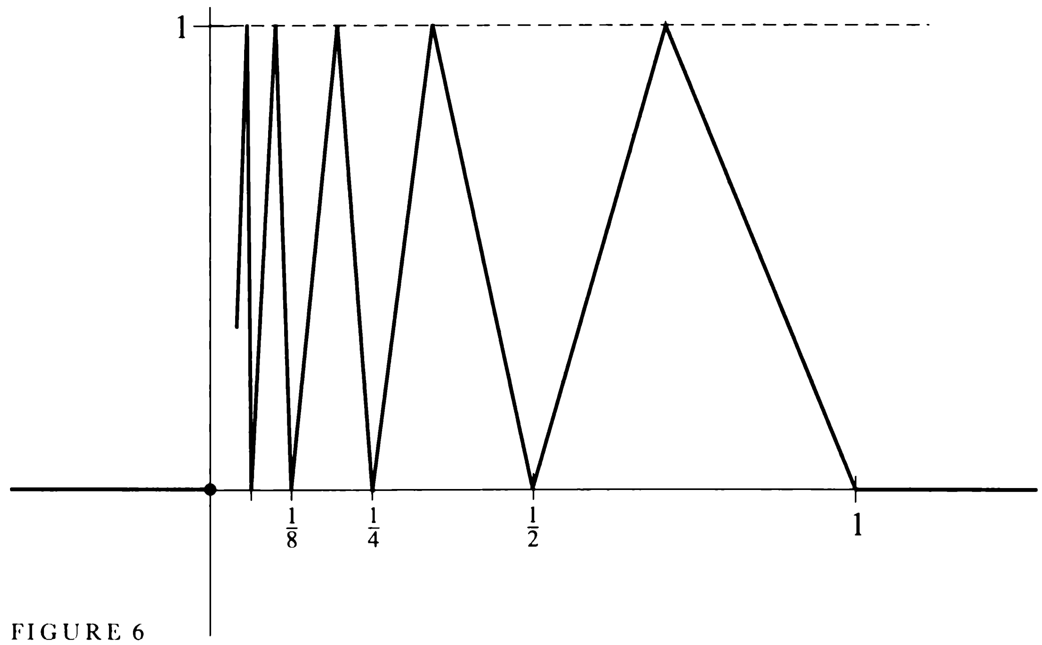

FIGURE 6

3. Show that the values of the following expressions do not depend on x:

(i) $\displaystyle\int_0^x \frac{1}{1+t^2}\,dt + \int_0^{1/x} \frac{1}{1+t^2}\,dt.$

(ii) $\displaystyle\int_{-\cos x}^{\sin x} \frac{1}{\sqrt{1-t^2}}\,dt, \quad x \text{ in } (0, \pi/2).$

4. Find $(f^{-1})'(0)$ if

(i) $f(x) = \int_0^x 1 + \sin(\sin t)\,dt.$

(ii) $f(x) = \int_1^x \cos(\cos t)\,dt.$

(Don't try to evaluate f explicitly.)

5. Find a function g such that

(i) $\int_0^x t g(t)\,dt = x + x^2.$

(ii) $\int_0^{x^2} t g(t)\,dt = x + x^2.$

(Notice that g is not assumed continuous at 0.)

6. (a) Find all continuous functions f satisfying

$$\int_0^x f = (f(x))^2 + C \qquad \text{for some constant } C \neq 0$$

assuming that f has at most one 0.

(b) Also find a solution that is 0 on an interval $(-\infty, b]$ with $0 < b$, but non-zero for $x > b$.

(c) Finally, for $C = 0$ and any interval $[a, b]$ with $a < 0 < b$, find a solution that is 0 on $[a, b]$, but non-zero elsewhere.

7. Use Problem 13-23 to prove that

(i) $\dfrac{1}{7\sqrt{2}} \leq \displaystyle\int_0^1 \frac{x^6}{\sqrt{1+x^2}}\,dx \leq \frac{1}{7}.$

(ii) $\dfrac{3}{8} \leq \displaystyle\int_0^{1/2} \sqrt{\frac{1-x}{1+x}}\,dx \leq \frac{\sqrt{3}}{4}.$

8. Find $F'(x)$ if $F(x) = \int_0^x x f(t)\,dt$. (The answer is *not* $x f(x)$; you should perform an obvious manipulation on the integral before trying to find F'.)

9. Prove that if f is continuous, then

$$\int_0^x f(u)(x-u)\,du = \int_0^x \left(\int_0^u f(t)\,dt \right) du.$$

Hint: Differentiate both sides, making use of Problem 8.

***10.** Use Problem 9 to prove that

$$\int_0^x f(u)(x-u)^2\,du = 2\int_0^x \left(\int_0^{u_2} \left(\int_0^{u_1} f(t)\,dt \right) du_1 \right) du_2.$$

11. Find a function f such that $f'''(x) = 1 \,/\, \sqrt{1 + \sin^2 x}$. (This problem is supposed to be easy; don't misinterpret the word "find.")

12. A function f is **periodic**, with **period** a, if $f(x+a) = f(x)$ for all x.

(a) If f is periodic with period a and integrable on $[0, a]$, show that

$$\int_0^a f = \int_b^{b+a} f \quad \text{for all } b.$$

(b) Find a function f such that f is not periodic, but f' is. Hint: Choose a periodic g for which it can be guaranteed that $f(x) = \int_0^x g$ is not periodic.

(c) If f' is periodic with period a and $f(a) = f(0)$, then f is also periodic with period a.

*(d) Conversely, if f' is periodic with period a and f is periodic (with some period not necessarily $= a$), then $f(a) = f(0)$.

13. Find $\int_0^b \sqrt[n]{x}\,dx$, by simply guessing a function f with $f'(x) = \sqrt[n]{x}$, and using the Second Fundamental Theorem of Calculus. Then check with Problem 13-21.

***14.** Use the Fundamental Theorem of Calculus and Problem 13-21 to derive the result stated in Problem 12-21.

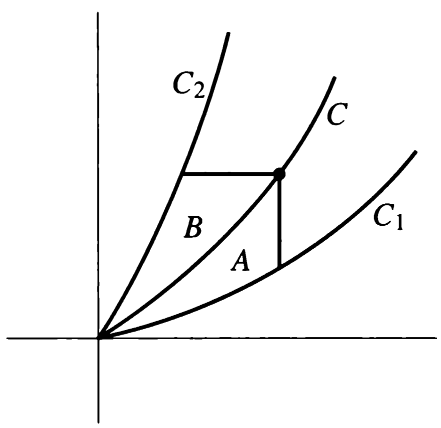

FIGURE 7

***15.** Let C_1, C and C_2 be curves passing through the origin, as shown in Figure 7. Each point on C can be joined to a point of C_1 with a vertical line segment and to a point of C_2 with a horizontal line segment. We will say that C *bisects* C_1 and C_2 if the regions A and B have equal areas for every point on C.

(a) If C_1 is the graph of $f(x) = x^2$, $x \geq 0$ and C is the graph of $f(x) = 2x^2$, $x \geq 0$, find C_2 so that C bisects C_1 and C_2.

(b) More generally, find C_2 if C_1 is the graph of $f(x) = x^m$, and C is the graph of $f(x) = cx^m$ for some $c > 1$.

16. (a) Find the derivatives of $F(x) = \int_1^x 1/t\,dt$ and $G(x) = \int_b^{bx} 1/t\,dt$.

(b) Now give a new proof for Problem 13-15.

***17.** Use the Fundamental Theorem of Calculus and Darboux's Theorem (Problem 11-60) to give another proof of the Intermediate Value Theorem.

18. Prove that if h is continuous, f and g are differentiable, and

$$F(x) = \int_{f(x)}^{g(x)} h(t)\,dt,$$

then $F'(x) = h(g(x)) \cdot g'(x) - h(f(x)) \cdot f'(x)$. Hint: Try to reduce this to the two cases you can already handle, with a constant either as the lower or the upper limit of integration.

19. Let f be integrable on $[a, b]$, let c be in (a, b), and let

$$F(x) = \int_a^x f, \quad a \leq x \leq b.$$

For each of the following statements, give either a proof or a counterexample.

(a) If f is differentiable at c, then F is differentiable at c.

(b) If f is differentiable at c, then F' is continuous at c.

(c) If f' is continuous at c, then F' is continuous at c.

***20.** Let

$$f(x) = \begin{cases} \cos\dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Is the function $F(x) = \int_0^x f$ differentiable at 0? Hint: Stare at page 179.

21. Suppose that f' is integrable on $[0, 1]$ and $f(0) = 0$. Prove that for all x in $[0, 1]$ we have

$$|f(x)| \leq \sqrt{\int_0^1 |f'|^2}.$$

Show also that the hypothesis $f(0) = 0$ is needed. Hint: Problem 13-39.

***22.** Suppose that f is a differentiable function with $f(0) = 0$ and $0 < f' \leq 1$. Prove that for all $x \geq 0$ we have

$$\int_0^x f^3 \leq \left(\int_0^x f\right)^2.$$

***23.** (a) Suppose $G' = g$ and $F' = f$. Prove that if the function y satisfies the differential equation

$$(*) \qquad g(y(x)) \cdot y'(x) = f(x) \quad \text{for all } x \text{ in some interval,}$$

then there is a number c such that

$$(**) \qquad G(y(x)) = F(x) + c \quad \text{for all } x \text{ in this interval.}$$

(b) Show, conversely, that if y satisfies $(**)$, then y is a solution of $(*)$.

(c) Find what condition y must satisfy if

$$y'(x) = \frac{1 + x^2}{1 + y(x)}.$$

(In this case $g(t) = 1 + t$ and $f(t) = 1 + t^2$.) Then "solve" the resulting equations to find all possible solutions y (no solution will have $\mathbf{R}$ as its domain).

(d) Find what condition y must satisfy if

$$y'(x) = \frac{-1}{1 + 5[y(x)]^4}.$$

(An appeal to Problem 12-14 will show that there *are* functions satisfying the resulting equation.)

(e) Find all functions y satisfying

$$y(x)y'(x) = -x.$$

Find the solution y satisfying $y(0) = -1$.

24. In Problem 10-19 we found that the Schwarzian derivative

$$\frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2$$

was 0 for $f(x) = (ax + b)/(cx + d)$. Now suppose that f is any function whose Schwarzian derivative is 0.

(a) f''^2/f'^3 is a constant function.
(b) f is the form $f(x) = (ax + b)/(cx + d)$. Hint: Consider $u = f'$ and apply the previous problem.

25. The limit $\lim\limits_{N\to\infty} \int_a^N f$, if it exists, is denoted by $\int_a^\infty f$ (or $\int_a^\infty f(x)\,dx$), and called an "improper integral."

(a) Determine $\int_1^\infty x^r\,dx$, if $r < -1$.
(b) Use Problem 13-15 to show that $\int_1^\infty 1/x\,dx$ does not exist. Hint: What can you say about $\int_1^{2^n} 1/x\,dx$?
(c) Suppose that $f(x) \ge 0$ for $x \ge 0$ and that $\int_0^\infty f$ exists. Prove that if $0 \le g(x) \le f(x)$ for all $x \ge 0$, and g is integrable on each interval $[0, N]$, then $\int_0^\infty g$ also exists.
(d) Explain why $\int_0^\infty 1/(1 + x^2)\,dx$ exists. Hint: Split this integral up at 1.

26. Decide whether or not the following improper integrals exist.

(i) $\displaystyle\int_0^\infty \frac{1}{\sqrt{1+x^3}}\,dx.$

(ii) $\displaystyle\int_0^\infty \frac{x}{1+x^{3/2}}\,dx.$

(iii) $\displaystyle\int_0^\infty \frac{1}{x\sqrt{1+x}}\,dx$ (this is really a type considered in Problem 28).

27. The improper integral $\int_{-\infty}^a f$ is defined in the obvious way, as $\lim\limits_{N\to-\infty} \int_N^a f$. But another kind of improper integral $\int_{-\infty}^\infty f$ is defined in a nonobvious way: it is $\int_0^\infty f + \int_{-\infty}^0 f$, provided these improper integrals both exist.

(a) Explain why $\int_{-\infty}^\infty 1/(1 + x^2)\,dx$ exists.
(b) Explain why $\int_{-\infty}^\infty x\,dx$ does not exist. (But notice that $\lim\limits_{N\to\infty} \int_{-N}^N x\,dx$ *does* exist.)
(c) Prove that if $\int_{-\infty}^\infty f$ exists, then $\lim\limits_{N\to\infty} \int_{-N}^N f$ exists and equals $\int_{-\infty}^\infty f$. Show moreover, that $\lim\limits_{N\to\infty} \int_{-N}^{N+1} f$ and $\lim\limits_{N\to\infty} \int_{-N^2}^N f$ both exist and equal $\int_{-\infty}^\infty f$. Can you state a reasonable generalization of these facts? (If you can't, you will have a miserable time trying to do these special cases!)

CHAPTER 15 THE TRIGONOMETRIC FUNCTIONS

The definitions of the functions sin and cos are considerably more subtle than one might suspect. For this reason, this chapter begins with some informal and intuitive definitions, which should not be scrutinized too carefully, as they shall soon be replaced by the formal definitions which we really intend to use.

In elementary geometry an angle is simply the union of two half-lines with a common initial point (Figure 1).

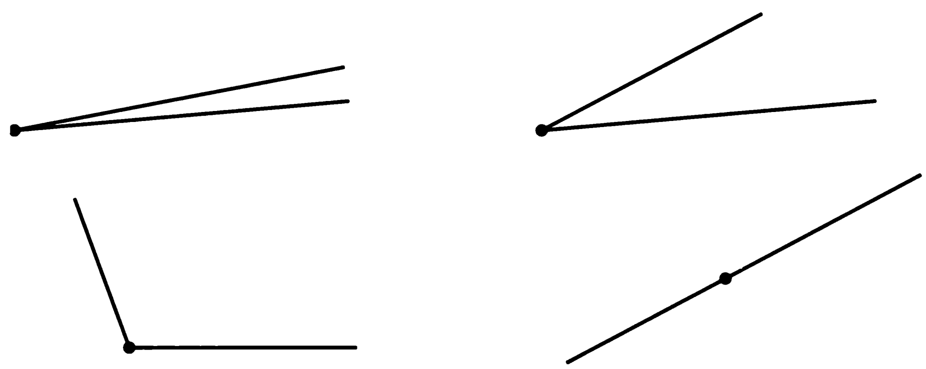

FIGURE 1

More useful for trigonometry are "directed angles," which may be regarded as pairs (l_1, l_2) of half-lines with the same initial point, visualized as in Figure 2.

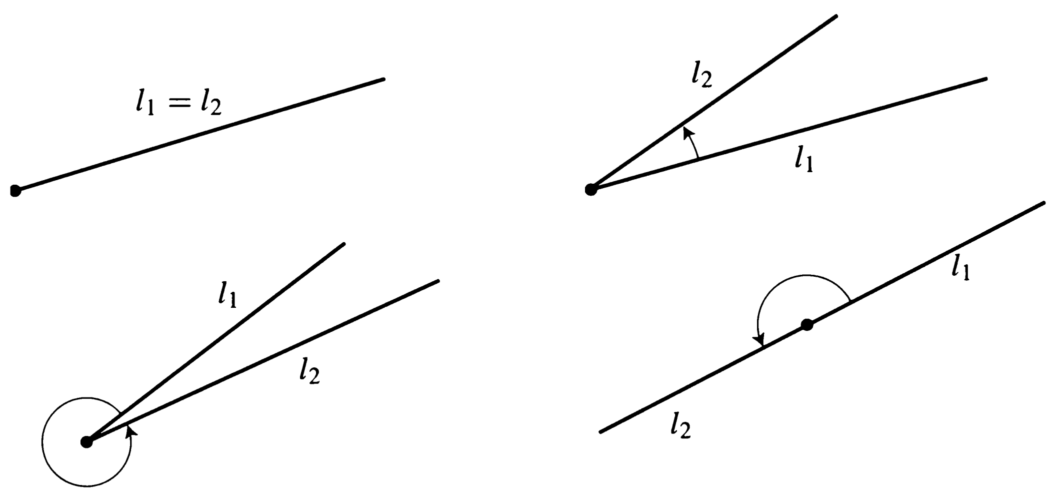

FIGURE 2

If for l_1 we always choose the positive half of the horizontal axis, a directed angle is described completely by the second half-line (Figure 3).

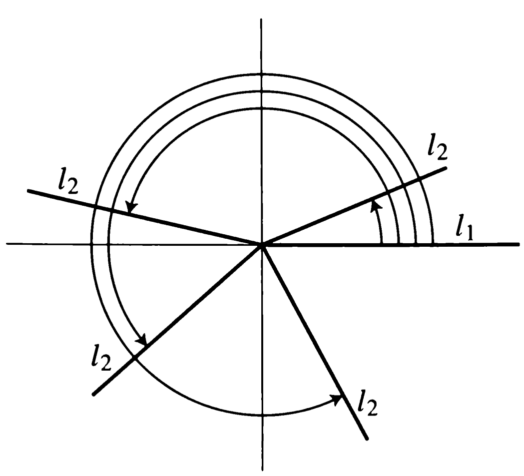

FIGURE 3

Since each half-line intersects the unit circle precisely once, a directed angle is described, even more simply, by a point on the unit circle (Figure 4), that is, by a point (x, y) with $x^2 + y^2 = 1$.