CHAPTER 13 INTEGRALS

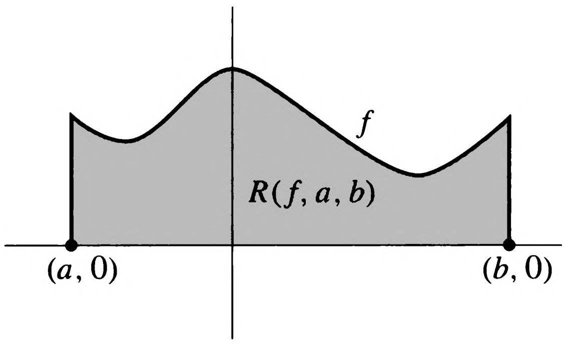


FIGURE 1

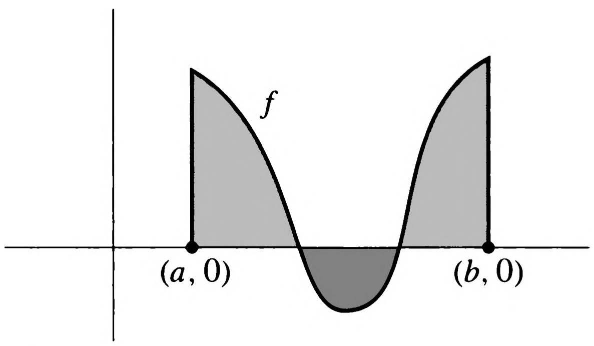


FIGURE 2

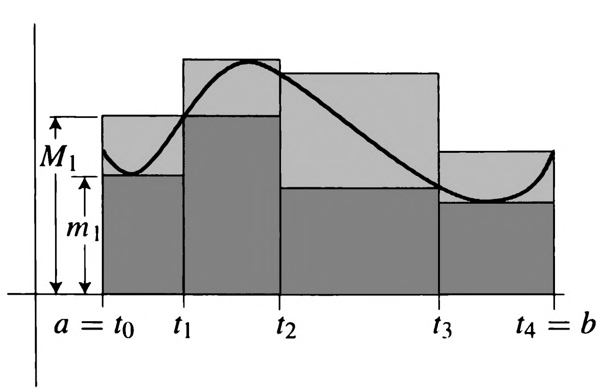


FIGURE 3

The derivative does not display its full strength until allied with the "integral," the second main concept of Part III. At first this topic may seem to be a complete digression—in this chapter derivatives do not appear even once! The study of integrals does require a long preparation, but once this preliminary work has been completed, integrals will be an invaluable tool for creating new functions, and the derivative will reappear in Chapter 14, more powerful than ever.

Although ultimately to be defined in a quite complicated way, the integral formalizes a simple, intuitive concept—that of area. By now it should come as no surprise to learn that the definition of an intuitive concept can present great difficulties—"area" is certainly no exception.

In elementary geometry, formulas are derived for the areas of many plane figures, but a little reflection shows that an acceptable definition of area is seldom given. The area of a region is sometimes defined as the number of squares, with sides of length 1, which fit in the region. But this definition is hopelessly inadequate for any but the simplest regions. For example, a circle of radius 1 supposedly has as area the irrational number π, but it is not at all clear what "π squares" means. Even if we consider a circle of radius $1/\sqrt{\pi}$, which supposedly has area 1, it is hard to say in what way a unit square fits in this circle, since it does not seem possible to divide the unit square into pieces which can be arranged to form a circle.

In this chapter we will only try to define the area of some very special regions (Figure 1)—those which are bounded by the horizontal axis, the vertical lines through $(a, 0)$ and $(b, 0)$, and the graph of a function f such that $f(x) \geq 0$ for all x in $[a, b]$. It is convenient to indicate this region by $R(f, a, b)$. Notice that these regions include rectangles and triangles, as well as many other important geometric figures.

The number which we will eventually assign as the area of $R(f, a, b)$ will be called the *integral* of f on $[a, b]$. Actually, the integral will be defined even for functions f which do not satisfy the condition $f(x) \geq 0$ for all x in $[a, b]$. If f is the function graphed in Figure 2, the integral will represent the difference of the area of the lightly shaded region and the area of the heavily shaded region (the "algebraic area" of $R(f, a, b)$).

The idea behind the prospective definition is indicated in Figure 3. The interval $[a, b]$ has been divided into four subintervals

$$[t_0, t_1] \quad [t_1, t_2] \quad [t_2, t_3] \quad [t_3, t_4]$$

by means of numbers t_0, t_1, t_2, t_3, t_4 with

$$a = t_0 < t_1 < t_2 < t_3 < t_4 = b$$

(the numbering of the subscripts begins with 0 so that the largest subscript will equal the number of subintervals).

On the first interval $[t_0, t_1]$ the function f has the minimum value m_1 and the maximum value M_1; similarly, on the ith interval $[t_{i-1}, t_i]$ let the minimum value of f be m_i and let the maximum value be M_i. The sum

$$s = m_1(t_1 - t_0) + m_2(t_2 - t_1) + m_3(t_3 - t_2) + m_4(t_4 - t_3)$$

represents the total area of rectangles lying inside the region $R(f, a, b)$, while the sum

$$S = M_1(t_1 - t_0) + M_2(t_2 - t_1) + M_3(t_3 - t_2) + M_4(t_4 - t_3)$$

represents the total area of rectangles containing the region $R(f, a, b)$. The guiding principle of our attempt to define the area A of $R(f, a, b)$ is the observation that A should satisfy

$$s \leq A \quad \text{and} \quad A \leq S,$$

and that this should be true, *no matter how the interval* $[a, b]$ *is subdivided.* It is to be hoped that these requirements will determine A. The following definitions begin to formalize, and eliminate some of the implicit assumptions in, this discussion.

DEFINITION

Let $a < b$. A **partition** of the interval $[a, b]$ is a finite collection of points in $[a, b]$, one of which is a, and one of which is b.

The points in a partition can be numbered $t_0, \ldots, t_n$ so that

$$a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b;$$

we shall always assume that such a numbering has been assigned.

DEFINITION

Suppose f is bounded on $[a, b]$ and $P = \{t_0, \ldots, t_n\}$ is a partition of $[a, b]$. Let

$$m_i = \inf\{f(x) : t_{i-1} \leq x \leq t_i\},$$
$$M_i = \sup\{f(x) : t_{i-1} \leq x \leq t_i\}.$$

The **lower sum** of f for P, denoted by $L(f, P)$, is defined as

$$L(f, P) = \sum_{i=1}^{n} m_i(t_i - t_{i-1}).$$

The **upper sum** of f for P, denoted by $U(f, P)$, is defined as

$$U(f, P) = \sum_{i=1}^{n} M_i(t_i - t_{i-1}).$$

The lower and upper sums correspond to the sums s and S in the previous example; they are supposed to represent the total areas of rectangles lying below and above the graph of f. Notice, however, that despite the geometric motivation, these sums have been defined precisely without any appeal to a concept of "area."

Two details of the definition deserve comment. The requirement that f be bounded on $[a, b]$ is essential in order that all the m_i and M_i be defined. Note, also, that it was necessary to define the numbers m_i and M_i as inf's and sup's, rather than as minima and maxima, since f was not assumed continuous.

One thing is clear about lower and upper sums: If P is any partition, then

$$L(f, P) \leq U(f, P),$$

because

$$L(f, P) = \sum_{i=1}^{n} m_i(t_i - t_{i-1}),$$
$$U(f, P) = \sum_{i=1}^{n} M_i(t_i - t_{i-1}),$$

and for each i we have

$$m_i(t_i - t_{i-1}) \leq M_i(t_i - t_{i-1}).$$

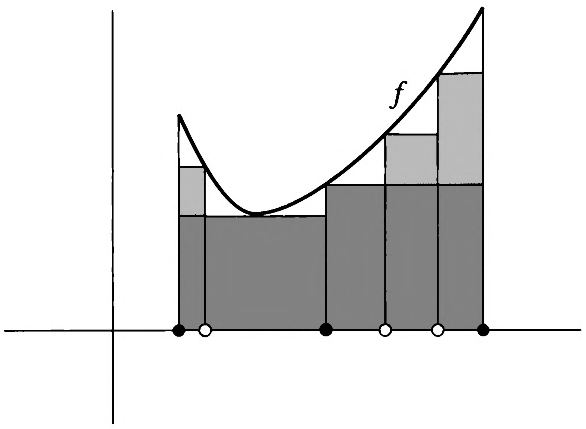


FIGURE 4

On the other hand, something less obvious *ought* to be true: If P_1 and P_2 are *any* two partitions of $[a, b]$, then it should be the case that

$$L(f, P_1) \leq U(f, P_2),$$

because $L(f, P_1)$ should be $\leq$ area $R(f, a, b)$, and $U(f, P_2)$ should be $\geq$ area $R(f, a, b)$. This remark proves nothing (since the "area of $R(f, a, b)$" has not even been defined yet), but it does indicate that if there is to be any hope of defining the area of $R(f, a, b)$, a proof that $L(f, P_1) \leq U(f, P_2)$ should come first. The proof which we are about to give depends upon a lemma which concerns the behavior of lower and upper sums when more points are included in a partition. In Figure 4 the partition P contains the points in black, and Q contains both the points in black and the points in grey. The picture indicates that the rectangles drawn for the partition Q are a better approximation to the region $R(f, a, b)$ than those for the original partition P. To be precise:

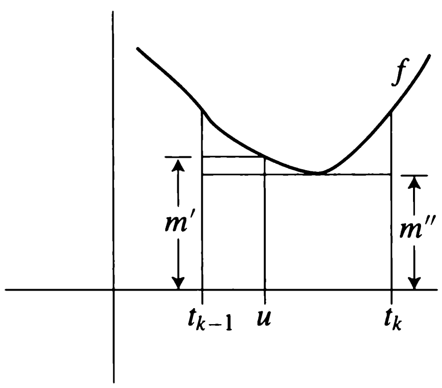


FIGURE 5

LEMMA If Q contains P (i.e., if all points of P are also in Q), then

$$L(f, P) \leq L(f, Q),$$
$$U(f, P) \geq U(f, Q).$$

PROOF Consider first the special case (Figure 5) in which Q contains just one more point than P:

$$P = \{t_0, \ldots, t_n\},$$
$$Q = \{t_0, \ldots, t_{k-1}, u, t_k, \ldots, t_n\},$$

where

$$a = t_0 < t_1 < \cdots < t_{k-1} < u < t_k < \cdots < t_n = b.$$

Let

$$m' = \inf\{f(x) : t_{k-1} \le x \le u\},$$
$$m'' = \inf\{f(x) : u \le x \le t_k\}.$$

Then

$$L(f, P) = \sum_{i=1}^{n} m_i(t_i - t_{i-1}),$$
$$L(f, Q) = \sum_{i=1}^{k-1} m_i(t_i - t_{i-1}) + m'(u - t_{k-1}) + m''(t_k - u) + \sum_{i=k+1}^{n} m_i(t_i - t_{i-1}).$$

To prove that $L(f, P) \le L(f, Q)$ it therefore suffices to show that

$$m_k(t_k - t_{k-1}) \le m'(u - t_{k-1}) + m''(t_k - u).$$

Now the set $\{f(x) : t_{k-1} \le x \le t_k\}$ contains all the numbers in $\{f(x) : t_{k-1} \le x \le u\}$, and possibly some smaller ones, so the greatest lower bound of the first set is *less than or equal to* the greatest lower bound of the second; thus

$$m_k \le m'.$$

Similarly,

$$m_k \le m''.$$

Therefore,

$$m_k(t_k - t_{k-1}) = m_k(u - t_{k-1}) + m_k(t_k - u) \le m'(u - t_{k-1}) + m''(t_k - u).$$

This proves, in this special case, that $L(f, P) \le L(f, Q)$. The proof that $U(f, P) \ge U(f, Q)$ is similar, and is left to you as an easy, but valuable, exercise.

The general case can now be deduced quite easily. The partition Q can be obtained from P by adding one point at a time; in other words, there is a sequence of partitions

$$P = P_1, P_2, \ldots, P_\alpha = Q$$

such that P_{j+1} contains just one more point than P_j. Then

$$L(f, P) = L(f, P_1) \le L(f, P_2) \le \cdots \le L(f, P_\alpha) = L(f, Q),$$

and

$$U(f, P) = U(f, P_1) \ge U(f, P_2) \ge \cdots \ge U(f, P_\alpha) = U(f, Q). \blacksquare$$

The theorem we wish to prove is a simple consequence of this lemma.

THEOREM 1 Let P_1 and P_2 be partitions of $[a, b]$, and let f be a function which is bounded on $[a, b]$. Then

$$L(f, P_1) \le U(f, P_2).$$

PROOF There is a partition P which contains both P_1 and P_2 (let P consist of all points in both P_1 and P_2). According to the lemma,

$$L(f, P_1) \leq L(f, P) \leq U(f, P) \leq U(f, P_2). \blacksquare$$

It follows from Theorem 1 that any upper sum $U(f, P')$ is an upper bound for the set of all lower sums $L(f, P)$. Consequently, any upper sum $U(f, P')$ is greater than or equal to the *least* upper bound of all lower sums:

$$\sup\{L(f, P) : P \text{ a partition of } [a, b]\} \leq U(f, P'),$$

for every P'. This, in turn, means that $\sup\{L(f, P)\}$ is a lower bound for the set of all upper sums of f. Consequently,

$$\sup\{L(f, P)\} \leq \inf\{U(f, P)\}.$$

It is clear that both of these numbers are between the lower sum and upper sum of f for *all* partitions:

$$L(f, P') \leq \sup\{L(f, P)\} \leq U(f, P'),$$
$$L(f, P') \leq \inf\{U(f, P)\} \leq U(f, P'),$$

for all partitions P'.

It may well happen that

$$\sup\{L(f, P)\} = \inf\{U(f, P\};$$

in this case, this is the *only* number between the lower sum and upper sum of f for all partitions, and this number is consequently an ideal candidate for the area of $R(f, a, b)$. On the other hand, if

$$\sup\{L(f, P)\} < \inf\{U(f, P)\},$$

then every number x between $\sup\{L(f, P)\}$ and $\inf\{U(f, P)\}$ will satisfy

$$L(f, P') \leq x \leq U(f, P')$$

for all partitions P'.

It is not at all clear just when such an embarrassment of riches will occur. The following two examples, although not as interesting as many which will soon appear, show that both phenomena are possible.

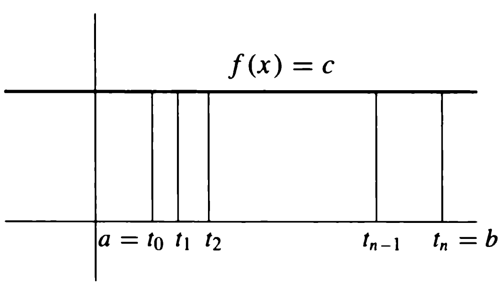


FIGURE 6

Suppose first that $f(x) = c$ for all x in $[a, b]$ (Figure 6). If $P = \{t_0, \ldots, t_n\}$ is any partition of $[a, b]$, then

$$m_i = M_i = c,$$

so

$$L(f, P) = \sum_{i=1}^{n} c(t_i - t_{i-1}) = c(b - a),$$

$$U(f, P) = \sum_{i=1}^{n} c(t_i - t_{i-1}) = c(b - a).$$

In this case, all lower sums and upper sums are equal, and

$$\sup\{L(f, P)\} = \inf\{U(f, P)\} = c(b-a).$$

Now consider (Figure 7) the function f defined by

$$f(x) = \begin{cases} 0, & x \text{ irrational} \\ 1, & x \text{ rational.} \end{cases}$$

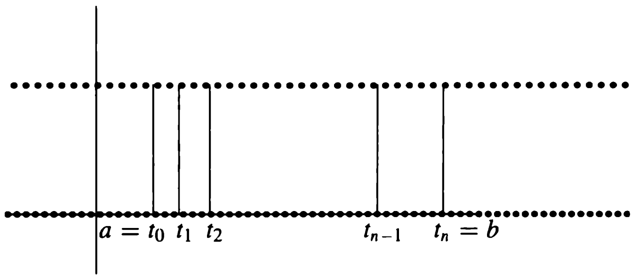


FIGURE 7

If $P = \{t_0, \dots, t_n\}$ is any partition, then

$$m_i = 0, \text{ since there is an irrational number in } [t_{i-1}, t_i],$$

and

$$M_i = 1, \text{ since there is a rational number in } [t_{i-1}, t_i].$$

Therefore,

$$L(f, P) = \sum_{i=1}^{n} 0 \cdot (t_i - t_{i-1}) = 0,$$

$$U(f, P) = \sum_{i=1}^{n} 1 \cdot (t_i - t_{i-1}) = b - a.$$

Thus, in this case it is certainly *not* true that $\sup\{L(f, P)\} = \inf\{U(f, P)\}$. The principle upon which the definition of area was to be based provides insufficient information to determine a specific area for $R(f, a, b)$—any number between 0 and $b - a$ seems equally good. On the other hand, the region $R(f, a, b)$ is so weird that we might with justice refuse to assign it any area at all. In fact, we can maintain, more generally, that whenever

$$\sup\{L(f, P)\} \neq \inf\{U(f, P)\},$$

the region $R(f, a, b)$ is too unreasonable to deserve having an area. As our appeal to the word "unreasonable" suggests, we are about to cloak our ignorance in terminology.

DEFINITION

A function f which is bounded on $[a, b]$ is **integrable** on $[a, b]$ if

$$\sup\{L(f, P) : P \text{ a partition of } [a, b]\} = \inf\{U(f, P) : P \text{ a partition of } [a, b]\}.$$

In this case, this common number is called the **integral** of f on $[a, b]$ and is denoted by

$$\int_a^b f.$$

(The symbol $\int$ is called an *integral sign* and was originally an elongated s, for "sum;" the numbers a and b are called the *lower* and *upper limits of integration.*) The integral $\int_a^b f$ is also called the **area** of $R(f, a, b)$ when $f(x) \geq 0$ for all x in $[a, b]$.

If f is integrable, then according to this definition,

$$L(f, P) \leq \int_a^b f \leq U(f, P) \quad \text{for all partitions } P \text{ of } [a, b].$$

Moreover, $\int_a^b f$ is the *unique* number with this property.

This definition merely pinpoints, and does not solve, the problem discussed before: we do not know which functions are integrable (nor do we know how to find the integral of f on $[a, b]$ when f *is* integrable). At present we know only two examples:

$$\text{(1)} \quad \text{if } f(x) = c, \text{ then } f \text{ is integrable on } [a, b] \text{ and } \int_a^b f = c \cdot (b - a).$$

(Notice that this integral assigns the expected area to a rectangle.)

$$\text{(2)} \quad \text{if } f(x) = \begin{cases} 0, & x \text{ irrational} \\ 1, & x \text{ rational,} \end{cases} \quad \text{then } f \text{ is not integrable on } [a, b].$$

Several more examples will be given before discussing these problems further. Even for these examples, however, it helps to have the following simple criterion for integrability stated explicitly.

THEOREM 2 If f is bounded on $[a, b]$, then f is integrable on $[a, b]$ if and only if for every $\varepsilon > 0$ there is a partition P of $[a, b]$ such that

$$U(f, P) - L(f, P) < \varepsilon.$$

PROOF Suppose first that for every $\varepsilon > 0$ there is a partition P with

$$U(f, P) - L(f, P) < \varepsilon.$$

Since

$$\begin{aligned} \inf\{U(f, P')\} &\leq U(f, P), \\ \sup\{L(f, P')\} &\geq L(f, P), \end{aligned}$$

it follows that

$$\inf\{U(f, P')\} - \sup\{L(f, P')\} < \varepsilon.$$

Since this is true for all $\varepsilon > 0$, it follows that

$$\sup\{L(f, P')\} = \inf\{U(f, P')\};$$

by definition, then, f is integrable. The proof of the converse assertion is similar: If f is integrable, then

$$\sup\{L(f, P)\} = \inf\{U(f, P)\}.$$

This means that for each $\varepsilon > 0$ there are partitions P', P'' with

$$U(f, P'') - L(f, P') < \varepsilon.$$

Since f is integrable, there is only *one* number between all lower and upper sums, namely, the integral of f, so

$$\int_0^2 f = 0.$$

Although the discontinuity of f was responsible for the difficulties in this example, even worse problems arise for very simple continuous functions. For example, let $f(x) = x$, and for simplicity consider an interval $[0, b]$, where $b > 0$. If $P = \{t_0, \ldots, t_n\}$ is a partition of $[0, b]$, then (Figure 9)

$$m_i = t_{i-1} \qquad \text{and} \qquad M_i = t_i$$

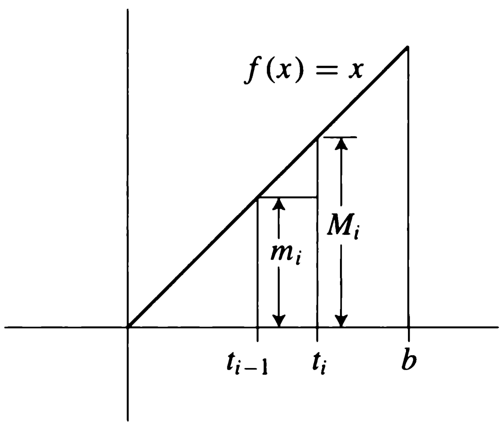


FIGURE 9

and therefore

$$\begin{aligned} L(f, P) &= \sum_{i=1}^{n} t_{i-1}(t_i - t_{i-1}) \\ &= t_0(t_1 - t_0) + t_1(t_2 - t_1) + \cdots + t_{n-1}(t_n - t_{n-1}), \\ U(f, P) &= \sum_{i=1}^{n} t_i(t_i - t_{i-1}) \\ &= t_1(t_1 - t_0) + t_2(t_2 - t_1) + \cdots + t_n(t_n - t_{n-1}). \end{aligned}$$

Neither of these formulas is particularly appealing, but both simplify considerably for partitions $P_n = \{t_0, \ldots, t_n\}$ into n *equal* subintervals. In this case, the length $t_i - t_{i-1}$ of each subinterval is b/n, so

$$\begin{aligned} t_0 &= 0, \\ t_1 &= \frac{b}{n}, \\ t_2 &= \frac{2b}{n}, \text{ etc;} \end{aligned}$$

in general

$$t_i = \frac{ib}{n}.$$

Then

$$\begin{aligned} L(f, P_n) &= \sum_{i=1}^{n} t_{i-1}(t_i - t_{i-1}) \\ &= \sum_{i=1}^{n} \left\{ \frac{(i-1)b}{n} \right\} \cdot \frac{b}{n} \\ &= \left[\sum_{i=1}^{n} (i-1) \right] \frac{b^2}{n^2} \\ &= \left(\sum_{j=0}^{n-1} j \right) \frac{b^2}{n^2}. \end{aligned}$$

Remembering the formula

$$1 + \cdots + k = \frac{k(k+1)}{2},$$

this can be written

$$\begin{aligned} L(f, P_n) &= \frac{(n-1)(n)}{2} \cdot \frac{b^2}{n^2} \\ &= \frac{n-1}{n} \cdot \frac{b^2}{2}. \end{aligned}$$

Similarly,

$$\begin{aligned} U(f, P_n) &= \sum_{i=1}^{n} t_i (t_i - t_{i-1}) \\ &= \sum_{i=1}^{n} \frac{ib}{n} \cdot \frac{b}{n} \\ &= \frac{n(n+1)}{2} \cdot \frac{b^2}{n^2} \\ &= \frac{n+1}{n} \cdot \frac{b^2}{2}. \end{aligned}$$

If n is very large, both $L(f, P_n)$ and $U(f, P_n)$ are close to $b^2/2$, and this remark makes it easy to show that f is integrable. Notice first that

$$U(f, P_n) - L(f, P_n) = \frac{2}{n} \cdot \frac{b^2}{2}.$$

This shows that there are partitions P_n with $U(f, P_n) - L(f, P_n)$ as small as desired. By Theorem 2 the function f is integrable. Moreover, $\int_0^b f$ may now be found with only a little work. It is clear, first of all, that

$$L(f, P_n) \le \frac{b^2}{2} \le U(f, P_n) \quad \text{for all } n.$$

This inequality shows only that $b^2/2$ lies between certain special upper and lower sums, but we have just seen that $U(f, P_n) - L(f, P_n)$ can be made as small as desired, so there is *only one* number with this property. Since the integral certainly has this property, we can conclude that

$$\int_0^b f = \frac{b^2}{2}.$$

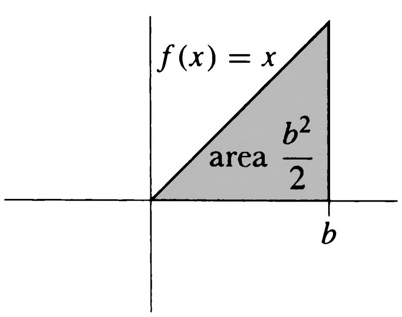


FIGURE 10

Notice that this equation assigns area $b^2/2$ to a right triangle with base and altitude b (Figure 10). Using more involved calculations, or appealing to Theorem 4, it can be shown that

$$\int_a^b f = \frac{b^2}{2} - \frac{a^2}{2}.$$

Let P be a partition which contains both P' and P''. Then, according to the lemma,

$$U(f, P) \leq U(f, P''),$$
$$L(f, P) \geq L(f, P');$$

consequently,

$$U(f, P) - L(f, P) \leq U(f, P'') - L(f, P') < \varepsilon. \blacksquare$$

Although the mechanics of the proof take up a little space, it should be clear that Theorem 2 amounts to nothing more than a restatement of the definition of integrability. Nevertheless, it is a very convenient restatement because there is no mention of sup's and inf's, which are often difficult to work with. The next example illustrates this point, and also serves as a good introduction to the type of reasoning which the complicated definition of the integral necessitates, even in very simple situations.

Let f be defined on $[0, 2]$ by

$$f(x) = \begin{cases} 0, & x \neq 1 \\ 1, & x = 1. \end{cases}$$

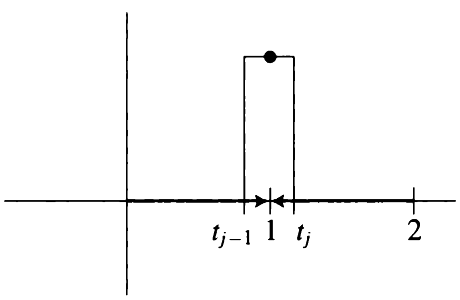


FIGURE 8

Suppose $P = \{t_0, \ldots, t_n\}$ is a partition of $[0, 2]$ with

$$t_{j-1} < 1 < t_j$$

(see Figure 8). Then

$$m_i = M_i = 0 \quad \text{if} \quad i \neq j,$$

but

$$m_j = 0 \qquad \text{and} \qquad M_j = 1.$$

Since

$$L(f, P) = \sum_{i=1}^{j-1} m_i(t_i - t_{i-1}) + m_j(t_j - t_{j-1}) + \sum_{i=j+1}^{n} m_i(t_i - t_{i-1}),$$
$$U(f, P) = \sum_{i=1}^{j-1} M_i(t_i - t_{i-1}) + M_j(t_j - t_{j-1}) + \sum_{i=j+1}^{n} M_i(t_i - t_{i-1}),$$

we have

$$U(f, P) - L(f, P) = t_j - t_{j-1}.$$

This certainly shows that f is integrable: to obtain a partition P with

$$U(f, P) - L(f, P) < \varepsilon,$$

it is only necessary to choose a partition with

$$t_{j-1} < 1 < t_j \quad \text{and} \quad t_j - t_{j-1} < \varepsilon.$$

Moreover, it is clear that

$$L(f, P) \leq 0 \leq U(f, P) \quad \text{for all partitions } P.$$

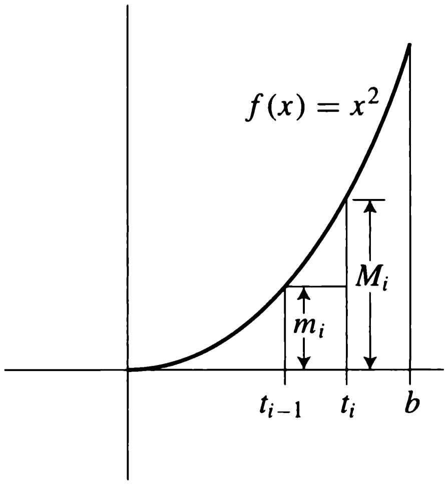


FIGURE 11

The function $f(x) = x^2$ presents even greater difficulties. In this case (Figure 11), if $P = \{t_0, \dots, t_n\}$ is a partition of $[0, b]$, then

$$m_i = f(t_{i-1}) = (t_{i-1})^2 \quad \text{and} \quad M_i = f(t_i) = t_i{}^2.$$

Choosing, once again, a partition $P_n = \{t_0, \dots, t_n\}$ into n equal parts, so that

$$t_i = \frac{i \cdot b}{n}$$

the lower and upper sums become

$$\begin{aligned} L(f, P_n) &= \sum_{i=1}^{n} (t_{i-1})^2 \cdot (t_i - t_{i-1}) \\ &= \sum_{i=1}^{n} (i-1)^2 \frac{b^2}{n^2} \cdot \frac{b}{n} \\ &= \frac{b^3}{n^3} \cdot \sum_{j=0}^{n-1} j^2, \\ U(f, P_n) &= \sum_{i=1}^{n} t_i{}^2 \cdot (t_i - t_{i-1}) \\ &= \sum_{i=1}^{n} i^2 \frac{b^2}{n^2} \cdot \frac{b}{n} \\ &= \frac{b^3}{n^3} \sum_{j=1}^{n} j^2. \end{aligned}$$

Recalling the formula

$$1^2 + \cdots + k^2 = \tfrac{1}{6} k(k+1)(2k+1)$$

from Problem 2-1, these sums can be written as

$$\begin{aligned} L(f, P_n) &= \frac{b^3}{n^3} \cdot \frac{1}{6}(n-1)(n)(2n-1), \\ U(f, P_n) &= \frac{b^3}{n^3} \cdot \frac{1}{6}(n+1)(n)(2n+1). \end{aligned}$$

It is not too hard to show that

$$L(f, P_n) \le \frac{b^3}{3} \le U(f, P_n),$$

and that $U(f, P_n) - L(f, P_n)$ can be made as small as desired, by choosing n sufficiently large. The same sort of reasoning as before then shows that

$$\int_0^b f = \frac{b^3}{3}.$$

This calculation already represents a nontrivial result—the area of the region bounded by a parabola is not usually derived in elementary geometry. Nevertheless, the result was known to Archimedes, who derived it in essentially the same way. The only superiority we can claim is that in the next chapter we will discover a much simpler way to arrive at this result.

Some of our investigations can be summarized as follows:

$$\int_a^b f = c \cdot (b-a) \quad \text{if} \quad f(x) = c \text{ for all } x,$$
$$\int_a^b f = \frac{b^2}{2} - \frac{a^2}{2} \quad \text{if} \quad f(x) = x \text{ for all } x,$$
$$\int_a^b f = \frac{b^3}{3} - \frac{a^3}{3} \quad \text{if} \quad f(x) = x^2 \text{ for all } x.$$

This list already reveals that the notation $\int_a^b f$ suffers from the lack of a convenient notation for naming functions defined by formulas. For this reason an alternative notation,* analogous to the notation $\lim\limits_{x\to a} f(x)$, is also useful:

$$\int_a^b f(x)\,dx \quad \text{means precisely the same as} \quad \int_a^b f.$$

Thus

$$\int_a^b c\,dx = c \cdot (b-a),$$
$$\int_a^b x\,dx = \frac{b^2}{2} - \frac{a^2}{2},$$
$$\int_a^b x^2\,dx = \frac{b^3}{3} - \frac{a^3}{3}.$$

Notice that, as in the notation $\lim\limits_{x\to a} f(x)$, the symbol x can be replaced by any other letter (except f, a, or b, of course):

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(\alpha)\,d\alpha = \int_a^b f(y)\,dy = \int_a^b f(c)\,dc.$$

The symbol dx has no meaning in isolation, any more than the symbol $x \to$ has any meaning, except in the context $\lim\limits_{x\to a} f(x)$. In the equation

$$\int_a^b x^2\,dx = \frac{b^3}{3} - \frac{a^3}{3},$$

*The notation $\int_a^b f(x)\,dx$ is actually the older, and was for many years the only, symbol for the integral. Leibniz used this symbol because he considered the integral to be the sum (denoted by $\int$) of infinitely many rectangles with height $f(x)$ and "infinitely small" width dx. Later writers used $x_0, \dots, x_n$ to denote the points of a partition, and abbreviated $x_i - x_{i-1}$ by Δx_i. The integral was defined as the limit as Δx_i approaches 0 of the sums $\sum\limits_{i=1}^{n} f(x_i)\,\Delta x_i$ (analogous to lower and upper sums). The fact that the limit is obtained by changing $\sum$ to $\int$, $f(x_i)$ to $f(x)$, and Δx_i to dx, delights many people.

the *entire* symbol $x^2\,dx$ may be regarded as an abbreviation for:

$$\text{the function } f \text{ such that } f(x) = x^2 \text{ for all } x.$$

This notation for the integral is as flexible as the notation $\lim_{x\to a} f(x)$. Several examples may aid in the interpretation of various types of formulas which frequently appear; we have made use of Theorems 5 and 6.*

$$(1) \quad \int_a^b (x+y)\,dx = \int_a^b x\,dx + \int_a^b y\,dx = \frac{b^2}{2} - \frac{a^2}{2} + y(b-a).$$

$$(2) \quad \int_a^x (y+t)\,dy = \int_a^x y\,dy + \int_a^x t\,dy = \frac{x^2}{2} - \frac{a^2}{2} + t(x-a).$$

$$\begin{aligned}(3) \quad \int_a^b \left(\int_a^x (1+t)\,dz \right) dx &= \int_a^b (1+t)(x-a)\,dx \\ &= (1+t)\int_a^b (x-a)\,dx \\ &= (1+t)\left[\frac{b^2}{2} - \frac{a^2}{2} - a(b-a)\right].\end{aligned}$$

$$\begin{aligned}(4) \quad \int_a^b \left(\int_c^d (x+y)\,dy \right) dx &= \int_a^b \left[x(d-c) + \frac{d^2}{2} - \frac{c^2}{2} \right] dx \\ &= \left(\frac{d^2}{2} - \frac{c^2}{2}\right)(b-a) + (d-c)\int_a^b x\,dx \\ &= \left(\frac{d^2}{2} - \frac{c^2}{2}\right)(b-a) + (d-c)\left(\frac{b^2}{2} - \frac{a^2}{2}\right).\end{aligned}$$

The computations of $\int_a^b x\,dx$ and $\int_a^b x^2\,dx$ may suggest that evaluating integrals is generally difficult or impossible. As a matter of fact, the integrals of most functions *are* impossible to determine exactly (*although they may be computed to any degree of accuracy desired by calculating lower and upper sums*). Nevertheless, as we shall see in the next chapter, the integral of many functions can be computed very easily.

Even though most integrals cannot be computed exactly, it is important at least to know when a function f *is* integrable on $[a,b]$. Although it is possible to say precisely which functions are integrable, the criterion for integrability is a little too difficult to be stated here, and we will have to settle for partial results. The next Theorem gives the most useful result, but the proof given here uses material from the Appendix to Chapter 8. If you prefer, you can wait until the end of the next chapter, when a totally different proof will be given.

*Lest chaos overtake the reader when consulting other books, equation (1) requires an important qualification. This equation interprets $\int_a^b y\,dx$ to mean the integral of the function f such that each value $f(x)$ is the number y. But classical notation often uses y for $y(x)$, so $\int_a^b y\,dx$ might mean the integral of some arbitrary *function* y.

THEOREM 3 If f is continuous on $[a, b]$, then f is integrable on $[a, b]$.

PROOF Notice, first, that f is bounded on $[a, b]$, because it is continuous on $[a, b]$. To prove that f is integrable on $[a, b]$, we want to use Theorem 2, and show that for every $\varepsilon > 0$ there is a partition P of $[a, b]$ such that

$$U(f, P) - L(f, P) < \varepsilon.$$

Now we know, by Theorem 1 of the Appendix to Chapter 8, that f is uniformly continuous on $[a, b]$. So there is some $\delta > 0$ such that for all x and y in $[a, b]$,

$$\text{if } |x - y| < \delta, \text{ then } |f(x) - f(y)| < \frac{\varepsilon}{2(b-a)}.$$

The trick is simply to choose a partition $P = \{t_0, \ldots, t_n\}$ such that each $|t_i - t_{i-1}| < \delta$. Then for each i we have

$$|f(x) - f(y)| < \frac{\varepsilon}{2(b-a)} \qquad \text{for all } x, y \text{ in } [t_{i-1}, t_i],$$

and it follows easily that

$$M_i - m_i \le \frac{\varepsilon}{2(b-a)} < \frac{\varepsilon}{b-a}.$$

Since this is true for all i, we then have

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{i=1}^{n} (M_i - m_i)(t_i - t_{i-1}) \\ &< \frac{\varepsilon}{b-a} \sum_{i=1}^{n} t_i - t_{i-1} \\ &= \frac{\varepsilon}{b-a} \cdot b - a \\ &= \varepsilon, \end{aligned}$$

which is what we wanted. ▮

Although this theorem will provide all the information necessary for the use of integrals in this book, it is more satisfying to have a somewhat larger supply of integrable functions. Several problems treat this question in detail. It will help to know the following three theorems, which show that f is integrable on $[a, b]$, if it is integrable on $[a, c]$ and $[c, b]$; that $f + g$ is integrable if f and g are; and that $c \cdot f$ is integrable if f is integrable and c is any number.

As a simple application of these theorems, recall that if f is 0 except at one point, where its value is 1, then f is integrable. Multiplying this function by c, it follows that the same is true if the value of f at the exceptional point is c. Adding such a function to an integrable function, we see that the value of an integrable function may be changed arbitrarily at one point without destroying integrability. By breaking up the interval into many subintervals, we see that the value can be changed at finitely many points.

The proofs of these theorems usually use the alternative criterion for integrability in Theorem 2; as some of our previous demonstrations illustrate, the details of the argument often conspire to obscure the point of the proof. It is a good idea to attempt proofs of your own, consulting those given here as a last resort, or as a check. This will probably clarify the proofs, and will certainly give good practice in the techniques used in some of the problems.

THEOREM 4 Let $a < c < b$. If f is integrable on $[a, b]$, then f is integrable on $[a, c]$ and on $[c, b]$. Conversely, if f is integrable on $[a, c]$ and on $[c, b]$, then f is integrable on $[a, b]$. Finally, if f is integrable on $[a, b]$, then

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

PROOF Suppose f is integrable on $[a, b]$. If $\varepsilon > 0$, there is a partition $P = \{t_0, \dots, t_n\}$ of $[a, b]$ such that

$$U(f, P) - L(f, P) < \varepsilon.$$

We might as well assume that $c = t_j$ for some j. (Otherwise, let Q be the partition which contains $t_0, \dots, t_n$ and c; then Q contains P, so $U(f, Q) - L(f, Q) \le U(f, P) - L(f, P) < \varepsilon$.)

Now $P' = \{t_0, \dots, t_j\}$ is a partition of $[a, c]$ and $P'' = \{t_j, \dots, t_n\}$ is a partition of $[c, b]$ (Figure 12). Since

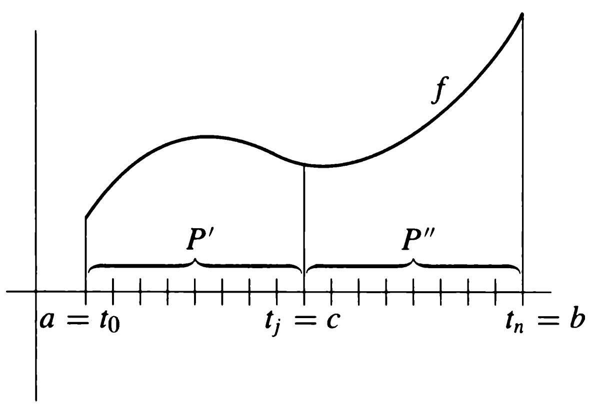


FIGURE 12

$$L(f, P) = L(f, P') + L(f, P''),$$
$$U(f, P) = U(f, P') + U(f, P''),$$

we have

$$[U(f, P') - L(f, P')] + [U(f, P'') - L(f, P'')] = U(f, P) - L(f, P) < \varepsilon.$$

Since each of the terms in brackets is nonnegative, each is less than ε. This shows that f is integrable on $[a, c]$ and $[c, b]$. Note also that

$$L(f, P') \le \int_a^c f \le U(f, P'),$$
$$L(f, P'') \le \int_c^b f \le U(f, P''),$$

so that

$$L(f, P) \le \int_a^c f + \int_c^b f \le U(f, P).$$

Since this is true for any P, this proves that

$$\int_a^c f + \int_c^b f = \int_a^b f.$$

Now suppose that f is integrable on $[a, c]$ and on $[c, b]$. If $\varepsilon > 0$, there is a partition P' of $[a, c]$ and a partition P'' of $[c, b]$ such that

$$U(f, P') - L(f, P') < \varepsilon/2,$$
$$U(f, P'') - L(f, P'') < \varepsilon/2.$$

If P is the partition of $[a, b]$ containing all the points of P' and P'', then

$$L(f, P) = L(f, P') + L(f, P''),$$
$$U(f, P) = U(f, P') + U(f, P'');$$

consequently,

$$U(f, P) - L(f, P) = [U(f, P') - L(f, P')] + [U(f, P'') - L(f, P'')] < \varepsilon. \blacksquare$$

Theorem 4 is the basis for some minor notational conventions. The integral $\int_a^b f$ was defined only for $a < b$. We now add the definitions

$$\int_a^a f = 0 \qquad \text{and} \qquad \int_a^b f = -\int_b^a f \quad \text{if } a > b.$$

With these definitions, the equation $\int_a^c f + \int_c^b f = \int_a^b f$ holds for all a, c, b even if $a < c < b$ is not true (the proof of this assertion is a rather tedious case-by-case check).

THEOREM 5 If f and g are integrable on $[a, b]$, then $f + g$ is integrable on $[a, b]$ and

$$\int_a^b (f + g) = \int_a^b f + \int_a^b g.$$

PROOF Let $P = \{t_0, \dots, t_n\}$ be any partition of $[a, b]$. Let

$$m_i = \inf\{(f + g)(x) : t_{i-1} \le x \le t_i\},$$
$$m_i{}' = \inf\{f(x) : t_{i-1} \le x \le t_i\},$$
$$m_i{}'' = \inf\{g(x) : t_{i-1} \le x \le t_i\},$$

and define M_i, $M_i{}'$, $M_i{}''$ similarly. It is not necessarily true that

$$m_i = m_i{}' + m_i{}'',$$

but it is true (Problem 10) that

$$m_i \ge m_i{}' + m_i{}''.$$

Similarly,

$$M_i \le M_i{}' + M_i{}''.$$

Therefore,

$$L(f, P) + L(g, P) \le L(f + g, P)$$

and

$$U(f + g, P) \le U(f, P) + U(g, P).$$

Thus,

$$L(f, P) + L(g, P) \le L(f + g, P) \le U(f + g, P) \le U(f, P) + U(g, P).$$

Since f and g are integrable, there are partitions P', P'' with

$$U(f, P') - L(f, P') < \varepsilon/2,$$
$$U(g, P'') - L(g, P'') < \varepsilon/2.$$

If P contains both P' and P'', then

$$U(f,P)+U(g,P)-[L(f,P)+L(g,P)]<\varepsilon,$$

and consequently

$$U(f+g,P)-L(f+g,P)<\varepsilon.$$

This proves that $f+g$ is integrable on $[a,b]$. Moreover,

$$\begin{aligned}(1)\qquad L(f,P)+L(g,P)&\le L(f+g,P)\\ &\le \int_a^b (f+g)\\ &\le U(f+g,P)\le U(f,P)+U(g,P);\end{aligned}$$

and also

$$(2)\qquad L(f,P)+L(g,P)\le \int_a^b f+\int_a^b g\le U(f,P)+U(g,P).$$

Since $U(f,P)-L(f,P)$ and $U(g,P)-L(g,P)$ can both be made as small as desired, it follows that

$$U(f,P)+U(g,P)-[L(f,P)+L(g,P)]$$

can also be made as small as desired; it therefore follows from (1) and (2) that

$$\int_a^b (f+g)=\int_a^b f+\int_a^b g. \blacksquare$$

THEOREM 6 If f is integrable on $[a,b]$, then for any number c, the function cf is integrable on $[a,b]$ and

$$\int_a^b cf = c\cdot\int_a^b f.$$

PROOF The proof (which is much easier than that of Theorem 5) is left to you. It is a good idea to treat separately the cases $c \ge 0$ and $c \le 0$. Why? $\blacksquare$

(Theorem 6 is just a special case of the more general theorem that $f\cdot g$ is integrable on $[a,b]$, if f and g are, but this result is quite hard to prove (see Problem 38).)

In this chapter we have acquired only one complicated definition, a few simple theorems with intricate proofs, and one theorem which required material from the Appendix to Chapter 8. This is not because integrals constitute a more difficult topic than derivatives, but because powerful tools developed in previous chapters have been allowed to remain dormant. The most significant discovery of calculus is the fact that the integral and the derivative are intimately related—once we learn the connection, the integral will become as useful as the derivative, and as easy to use. The connection between derivatives and integrals deserves a separate chapter, but the preparations which we will make in this chapter may serve as a hint. We first state a simple inequality concerning integrals, which plays a role in many important theorems.

THEOREM 7 Suppose f is integrable on $[a, b]$ and that

$$m \leq f(x) \leq M \quad \text{for all } x \text{ in } [a, b].$$

Then

$$m(b-a) \leq \int_a^b f \leq M(b-a).$$

PROOF It is clear that

$$m(b-a) \leq L(f, P) \quad \text{and} \quad U(f, P) \leq M(b-a)$$

for every partition P. Since $\int_a^b f = \sup\{L(f, P)\} = \inf\{U(f, P)\}$, the desired inequality follows immediately. ▮

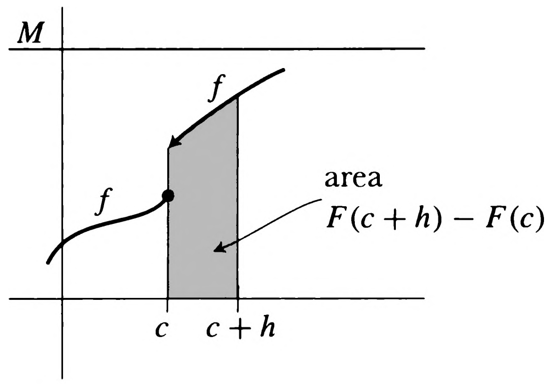


FIGURE 13

Suppose now that f is integrable on $[a, b]$. We can define a new function f on $[a, b]$ by

$$F(x) = \int_a^x f = \int_a^x f(t)\,dt.$$

(This depends on Theorem 4.) We have seen that f may be integrable even if it is not continuous, and the Problems give examples of integrable functions which are quite pathological. The behavior of F is therefore a very pleasant surprise.

THEOREM 8 If f is integrable on $[a, b]$ and F is defined on $[a, b]$ by

$$F(x) = \int_a^x f,$$

then F is continuous on $[a, b]$.

PROOF Suppose c is in $[a, b]$. Since f is integrable on $[a, b]$ it is, by definition, bounded on $[a, b]$; let M be a number such that

$$|f(x)| \leq M \quad \text{for all } x \text{ in } [a, b].$$

If $h > 0$, then (Figure 13)

$$F(c+h) - F(c) = \int_a^{c+h} f - \int_a^c f = \int_c^{c+h} f.$$

Since

$$-M \leq f(x) \leq M \quad \text{for all } x,$$

it follows from Theorem 7 that

$$-M \cdot h \leq \int_c^{c+h} f \leq Mh;$$

in other words,

$$(1) \qquad -M \cdot h \leq F(c+h) - F(c) \leq M \cdot h.$$

If $h < 0$, a similar inequality can be derived: Note that

$$F(c+h) - F(c) = \int_c^{c+h} f = -\int_{c+h}^c f.$$

Applying Theorem 7 to the interval $[c+h, c]$, of length $-h$, we obtain

$$Mh \le \int_{c+h}^{c} f \le -Mh;$$

multiplying by -1, which reverses all the inequalities, we have

$$(2) \qquad Mh \le F(c+h) - F(c) \le -Mh.$$

Inequalities (1) and (2) can be combined:

$$|F(c+h) - F(c)| \le M \cdot |h|.$$

Therefore, if $\varepsilon > 0$, we have

$$|F(c+h) - F(c)| < \varepsilon,$$

provided that $|h| < \varepsilon/M$. This proves that

$$\lim_{h \to 0} F(c+h) = F(c);$$

in other words F is continuous at c. ▮

Figure 14 compares f and $F(x) = \int_a^x f$ for various functions f; it appears that F is always better behaved than f. In the next chapter we will see how true this is.

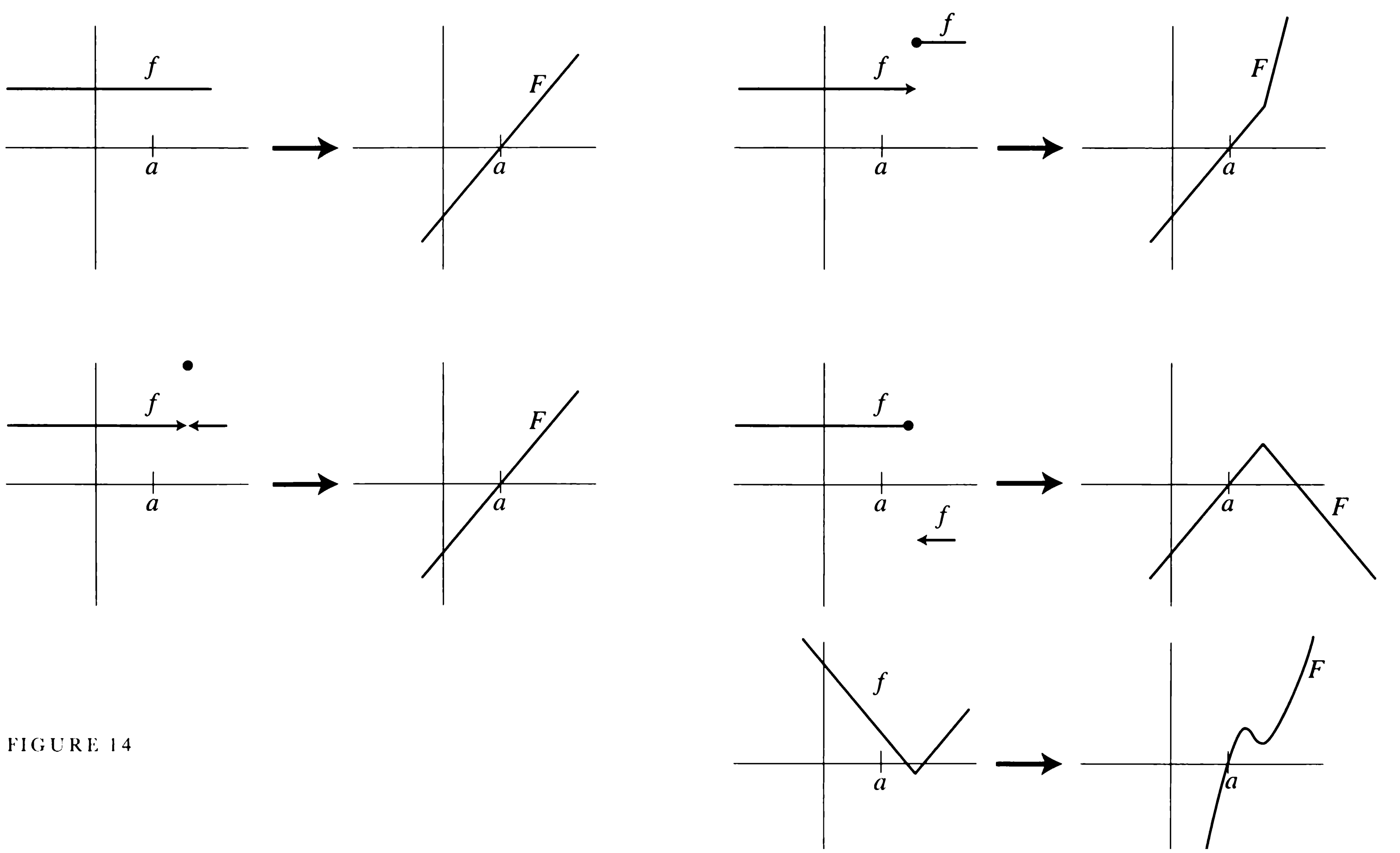


FIGURE 14

PROBLEMS

1. Prove that $\int_0^b x^3\,dx = b^4/4$, by considering partitions into n equal subintervals, using the formula for $\sum_{i=1}^{n} i^3$ which was found in Problem 2-6. This problem requires only a straightforward imitation of calculations in the text, but you should write it up as a formal proof to make certain that all the fine points of the argument are clear.

2. Prove, similarly, that $\int_0^b x^4\,dx = b^5/5$.

***3.** (a) Using Problem 2-7, show that the sum $\sum_{k=1}^{n} k^p/n^{p+1}$ can be made as close to $1/(p+1)$ as desired, by choosing n large enough.

(b) Prove that $\int_0^b x^p\,dx = b^{p+1}/(p+1)$.

***4.** This problem outlines a clever way to find $\displaystyle\int_a^b x^p\,dx$ for $0 < a < b$. (The result for $a = 0$ will then follow by continuity.) The trick is to use partitions $P = \{t_0, \dots, t_n\}$ for which all *ratios* $r = t_i/t_{i-1}$ are equal, instead of using partitions for which all differences $t_i - t_{i-1}$ are equal.

(a) Show that for such a partition P we have

$$t_i = a \cdot c^{i/n} \qquad \text{for } c = \frac{b}{a}.$$

(b) If $f(x) = x^p$, show, using the formula in Problem 2-5, that

$$\begin{aligned} U(f, P) &= a^{p+1}(1 - c^{-1/n}) \sum_{i=1}^{n} (c^{(p+1)/n})^i \\ &= (a^{p+1} - b^{p+1})c^{(p+1)/n}\,\frac{1 - c^{-1/n}}{1 - c^{(p+1)/n}} \\ &= (b^{p+1} - a^{p+1})c^{p/n} \cdot \frac{1}{1 + c^{1/n} + \cdots + c^{p/n}} \end{aligned}$$

and find a similar formula for $L(f, P)$.

(c) Conclude that

$$\int_a^b x^p\,dx = \frac{b^{p+1} - a^{p+1}}{p+1}.$$

(You might find Problem 5-41 useful.)

5. Evaluate without doing any computations:

(i) $\displaystyle\int_{-1}^{1} x^3\sqrt{1 - x^2}\,dx.$

(ii) $\displaystyle\int_{-1}^{1} (x^5 + 3)\sqrt{1 - x^2}\,dx.$

6. Prove that

$$\int_0^x \frac{\sin t}{t+1}\,dt > 0$$

for all $x > 0$.

7. Decide which of the following functions are integrable on $[0, 2]$, and calculate the integral when you can.

(i) $f(x) = \begin{cases} x, & 0 \le x < 1 \\ x-2, & 1 \le x \le 2. \end{cases}$

(ii) $f(x) = \begin{cases} x, & 0 \le x \le 1 \\ x-2, & 1 < x \le 2. \end{cases}$

(iii) $f(x) = x + [x]$.

(iv) $f(x) = \begin{cases} x + [x], & x \text{ rational} \\ 0, & x \text{ irrational.} \end{cases}$

(v) $f(x) = \begin{cases} 1, & x \text{ of the form } a + b\sqrt{2} \text{ for rational } a \text{ and } b \\ 0, & x \text{ not of this form.} \end{cases}$

(vi) $f(x) = \begin{cases} \dfrac{1}{\left[\dfrac{1}{x}\right]}, & 0 < x \le 1 \\ 0, & x = 0 \text{ or } x > 1. \end{cases}$

(vii) f is the function shown in Figure 15.

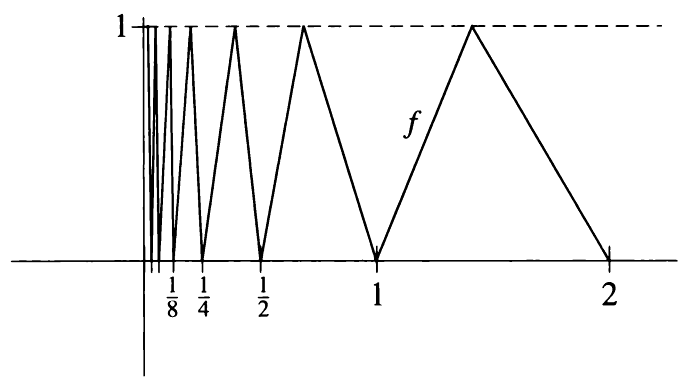


FIGURE 15

8. Find the areas of the regions bounded by

(i) the graphs of $f(x) = x^2$ and $g(x) = \dfrac{x^2}{2} + 2$.

(ii) the graphs of $f(x) = x^2$ and $g(x) = -x^2$ and the vertical lines through $(-1, 0)$ and $(1, 0)$.

(iii) the graphs of $f(x) = x^2$ and $g(x) = 1 - x^2$.

(iv) the graphs of $f(x) = x^2$ and $g(x) = 1 - x^2$ and $h(x) = 2$.

(v) the graphs of $f(x) = x^2$ and $g(x) = x^2 - 2x + 4$ and the vertical axis.

(vi) the graph of $f(x) = \sqrt{x}$, the horizontal axis, and the vertical line through $(2, 0)$. (Don't try to find $\displaystyle\int_0^2 \sqrt{x}\,dx$; you should see a way of guessing the answer, using only integrals that you already know how to evaluate. The questions that this example should suggest are considered in Problem 21.)

9. Find

$$\int_a^b \left(\int_c^d f(x)g(y)\,dy \right) dx$$

in terms of $\int_a^b f$ and $\int_c^d g$. (This problem is an exercise in notation, with a vengeance; it is crucial that you recognize a constant when it appears.)

10. Prove, using the notation of Theorem 5, that

$$m_i{}' + m_i{}'' = \inf\{f(x_1) + g(x_2) : t_{i-1} \le x_1, x_2 \le t_i\} \le m_i.$$

11. (a) Which functions have the property that every lower sum equals every upper sum?

(b) Which functions have the property that some upper sum equals some (other) lower sum?

(c) Which continuous functions have the property that all lower sums are equal?

*(d) Which integrable functions have the property that all lower sums are equal? (Bear in mind that one such function is $f(x) = 0$ for x irrational, $f(x) = 1/q$ for $x = p/q$ in lowest terms.) Hint: You will need the notion of a dense set, introduced in Problem 8-6, as well as the results of Problem 30.

12. If $a < b < c < d$ and f is integrable on $[a, d]$, prove that f is integrable on $[b, c]$. (Don't work hard.)

13. (a) Prove that if f is integrable on $[a, b]$ and $f(x) \ge 0$ for all x in $[a, b]$, then $\displaystyle\int_a^b f \ge 0$.

(b) Prove that if f and g are integrable on $[a, b]$ and $f(x) \ge g(x)$ for all x in $[a, b]$, then $\displaystyle\int_a^b f \ge \int_a^b g$. (By now it should be unnecessary to warn that if you work hard on part (b) you are wasting time.)

14. Prove that

$$\int_a^b f(x)\,dx = \int_{a+c}^{b+c} f(x-c)\,dx.$$

(The geometric interpretation should make this very plausible.) Hint: Every partition $P = \{t_0, \dots, t_n\}$ of $[a, b]$ gives rise to a partition $P' = \{t_0 + c, \dots, t_n + c\}$ of $[a + c, b + c]$, and conversely.

***15.** For $a, b > 1$ prove that

$$\int_1^a \frac{1}{t}\,dt + \int_1^b \frac{1}{t}\,dt = \int_1^{ab} \frac{1}{t}\,dt.$$

Hint: This can be written $\int_1^a 1/t\,dt = \int_b^{ab} 1/t\,dt$. Every partition $P = \{t_0, \dots, t_n\}$ of $[1, a]$ gives rise to a partition $P' = \{bt_0, \dots, bt_n\}$ of $[b, ab]$, and conversely.

***16.** Prove that

$$\int_{ca}^{cb} f(t)\,dt = c\int_a^b f(ct)\,dt.$$

(Notice that Problem 15 is a special case.)

17. Given that the area enclosed by the unit circle, described by the equation $x^2 + y^2 = 1$, is π, use Problem 16 to show that the area enclosed by the ellipse described by the equation $x^2/a^2 + y^2/b^2 = 1$ is πab.

18. This problem outlines yet another way to compute $\int_a^b x^n\,dx$; it was used by Cavalieri, one of the mathematicians working just before the invention of calculus.

(a) Let $c_n = \int_0^1 x^n\,dx$. Use Problem 16 to show that $\int_0^a x^n\,dx = c_n a^{n+1}$.

(b) Problem 14 shows that

$$\int_0^{2a} x^n\,dx = \int_{-a}^{a} (x+a)^n\,dx.$$

Use this formula to prove that

$$2^{n+1}c_n a^{n+1} = 2a^{n+1}\sum_{k\text{ even}}\binom{n}{k}c_k.$$

(c) Now use Problem 2-3 to prove that $c_n = 1/(n+1)$.

19. Suppose that f is bounded on $[a, b]$ and that f is continuous at each point in $[a, b]$ with the exception of x_0 in (a, b). Prove that f is integrable on $[a, b]$. Hint: Imitate one of the examples in the text.

20. Suppose that f is nondecreasing on $[a, b]$. Notice that f is automatically bounded on $[a, b]$, because $f(a) \le f(x) \le f(b)$ for x in $[a, b]$.

(a) If $P = \{t_0, \dots, t_n\}$ is a partition of $[a, b]$, what is $L(f, P)$ and $U(f, P)$?

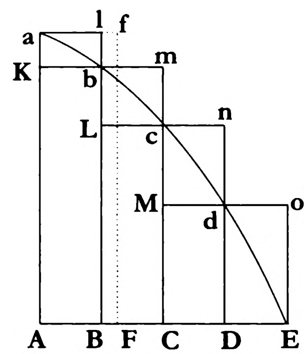
(b) Suppose that $t_i - t_{i-1} = \delta$ for each i. Prove that $U(f, P) - L(f, P) = \delta[f(b) - f(a)]$.

(c) Prove that f is integrable.

(d) Give an example of a nondecreasing function on $[0, 1]$ which is discontinuous at infinitely many points.

It might be of interest to compare this problem with the following extract from Newton's *Principia.**

LEMMA II



If in any figure AacE, *terminated by the right lines* Aa, AE, *and the curve* acE, *there be inscribed any number of parallelograms* Ab, Bc, Cd, *&c., comprehended under equal bases* AB, BC, CD, *&c., and the sides,* Bb, Cc, Dd, *&c., parallel to one side* Aa *of the figure; and the parallelograms* aKbl, bLcm, cMdn, *&c., are completed: then if the breadth of those parallelograms be supposed to be diminished, and their number to be augmented* in infinitum, *I say, that the ultimate ratios which the inscribed figure* AKbLcMdD, *the circumscribed figure* AalbmcndoE, *and curvilinear figure* AabcdE, *will have to one another, are ratios of equality.*

For the difference of the inscribed and circumscribed figures is the sum of the parallelograms Kl, Lm, Mn, Do, that is (from the equality of all their bases), the rectangle under one of their bases Kb and the sum of their altitudes Aa, that is, the rectangle ABla. But this rectangle, because its breadth AB is supposed diminished *in infinitum*, becomes less than any given space. And therefore (by Lem. 1) the figures inscribed and circumscribed become ultimately equal one to the other; and much more will the intermediate curvilinear figure be ultimately equal to either. Q.E.D.

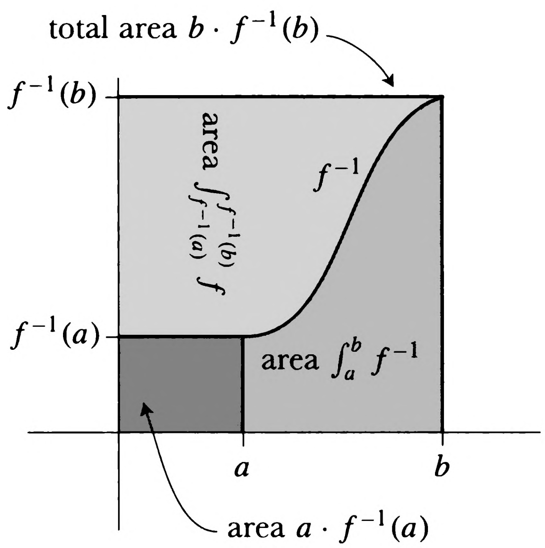


FIGURE 16

***21.** Suppose that f is increasing. Figure 16 suggests that

$$\int_a^b f^{-1} = bf^{-1}(b) - af^{-1}(a) - \int_{f^{-1}(a)}^{f^{-1}(b)} f.$$

(a) If $P = \{t_0, \ldots, t_n\}$ is a partition of $[a, b]$, let $P' = \{f^{-1}(t_0), \ldots, f^{-1}(t_n)\}$. Prove that, as suggested in Figure 17,

$$L(f^{-1}, P) + U(f, P') = bf^{-1}(b) - af^{-1}(a).$$

(b) Now prove the formula stated above.

(c) Find $\displaystyle\int_a^b \sqrt[n]{x}\, dx$ for $0 \leq a < b$.

22. Suppose that f is a continuous increasing function with $f(0) = 0$. Prove that for $a, b > 0$ we have *Young's inequality,*

$$ab \leq \int_0^a f(x)\,dx + \int_0^b f^{-1}(x)\,dx,$$

and that equality holds if and only if $b = f(a)$. Hint: Draw a picture like Figure 16!

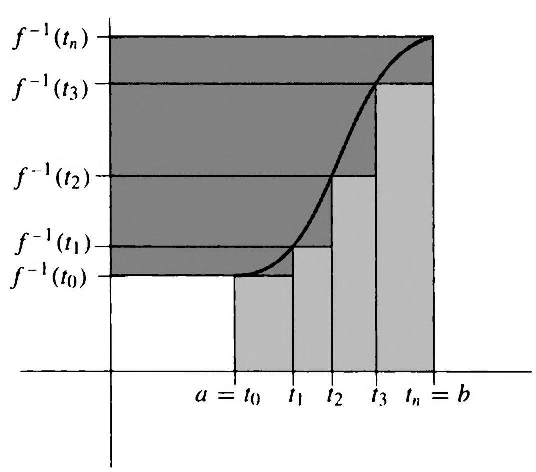


FIGURE 17

* Newton's *Principia*, A Revision of Mott's Translation, by Florian Cajori. University of California Press, Berkeley, California, 1946.

23. (a) Prove that if f is integrable on $[a, b]$ and $m \leq f(x) \leq M$ for all x in $[a, b]$, then

$$\int_a^b f(x)\,dx = (b-a)\mu$$

for some number μ with $m \leq \mu \leq M$.

(b) Prove that if f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = (b-a)f(\xi)$$

for some ξ in $[a, b]$.

(c) Show by an example that continuity is essential.

(d) More generally, suppose that f is continuous on $[a, b]$ and that g is integrable and nonnegative on $[a, b]$. Prove that

$$\int_a^b f(x)g(x)\,dx = f(\xi)\int_a^b g(x)\,dx$$

for some ξ in $[a, b]$. This result is called the Mean Value Theorem for Integrals.

(e) Deduce the same result if g is integrable and nonpositive on $[a, b]$.

(f) Show that one of these two hypotheses for g is essential.

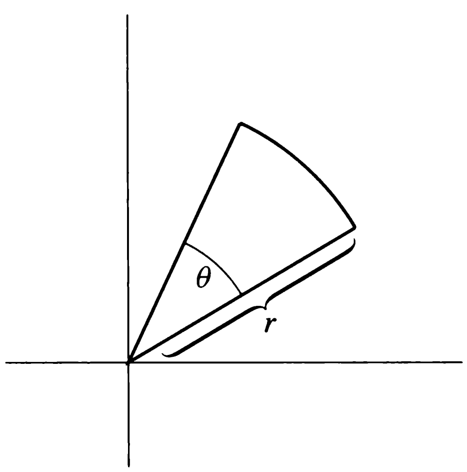


FIGURE 18

24. In this problem we consider the graph of a function in polar coordinates (Chapter 4, Appendix 3). Figure 18 shows a sector of a circle, with central angle θ. When θ is measured in radians, the area of this sector is $r^2 \cdot \dfrac{\theta}{2}$. Now consider the region A shown in Figure 19, where the curve is the graph in polar coordinates of the continuous function f. Show that

$$\text{area } A = \frac{1}{2}\int_{\theta_0}^{\theta_1} f(\theta)^2 d\theta.$$

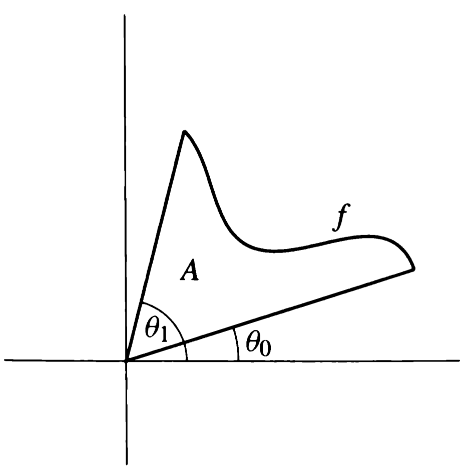


FIGURE 19

***25.** Let f be a continuous function on $[a, b]$. If $P = \{t_0, \dots, t_n\}$ is a partition of $[a, b]$, define

$$\ell(f, P) = \sum_{i=1}^{n} \sqrt{(t_i - t_{i-1})^2 + [f(t_i) - f(t_{i-1})]^2}.$$

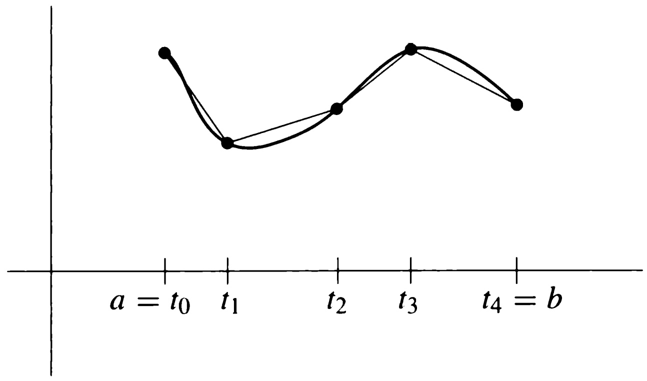


FIGURE 20

The number $\ell(f, P)$ represents the length of a polygonal curve inscribed in the graph of f (see Figure 20). We define the **length** of f on $[a, b]$ to be the least upper bound of all $\ell(f, P)$ for all partitions P (provided that the set of all such $\ell(f, P)$ is bounded above).

(a) If f is a linear function on $[a, b]$, prove that the length of f is the distance from $(a, f(a))$ to $(b, f(b))$.

(b) If f is not linear, prove that there is a partition $P = \{a, t, b\}$ of $[a, b]$ such that $\ell(f, P)$ is greater than the distance from $(a, f(a))$ to $(b, f(b))$. (You will need Problem 4-9.)

(c) Conclude that of all functions f on $[a, b]$ with $f(a) = c$ and $f(b) = d$, the length of the linear function is less than the length of any other. (Or, in conventional but hopelessly muddled terminology: "A straight line is the shortest distance between two points".)

(d) Suppose that f' is bounded on $[a, b]$. If P is any partition of $[a, b]$ show that

$$L\left(\sqrt{1+(f')^2}, P\right) \le \ell(f, P) \le U\left(\sqrt{1+(f')^2}, P\right).$$

Hint: Use the Mean Value Theorem.

(e) Why is $\sup\left\{L(\sqrt{1+(f')^2}, P)\right\} \le \sup\{\ell(f, P)\}$? (This is easy.)

(f) Now show that $\sup\{\ell(f, P)\} \le \inf\left\{U(\sqrt{1+(f')^2}, P)\right\}$, thereby proving that the length of f on $[a, b]$ is $\int_a^b \sqrt{1+(f')^2}$, if $\sqrt{1+(f')^2}$ is integrable on $[a, b]$. Hint: It suffices to show that if P' and P'' are any two partitions, then $\ell(f, P') \le U\left(\sqrt{1+(f')^2}, P''\right)$. If P contains the points of both P' and P'', how does $\ell(f, P')$ compare to $\ell(f, P)$?

(g) Let $\mathscr{L}(x)$ be the length of the graph of f on $[a, x]$, and let $d(x)$ be the length of the straight line segment from $(a, f(a))$ to $(x, f(x))$. Show that if $\sqrt{1+(f')^2}$ is integrable on $[a, b]$ and f' is continuous at a (i.e., if $\lim_{x \to a^+} f'(x) = f'(a)$), then

$$\lim_{x \to a^+} \frac{\mathscr{L}(x)}{d(x)} = 1.$$

Hint: It will help to use a couple of Mean Value Theorems.

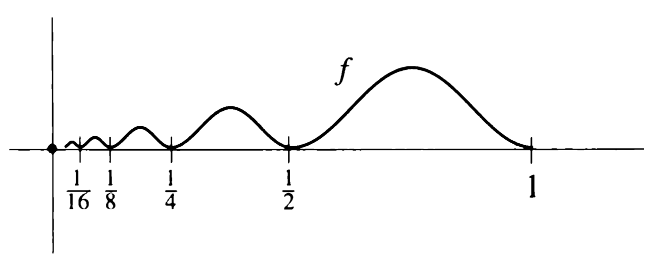


FIGURE 21

(h) In Figure 21, the part of the graph of f between $\frac{1}{4}$ and $\frac{1}{2}$ is just half the size of the part between $\frac{1}{2}$ and 1, the part between $\frac{1}{8}$ and $\frac{1}{4}$ is just half the size of the part between $\frac{1}{4}$ and $\frac{1}{2}$, etc. Show that the conclusion of part (g) does not hold for this f.

26. A function s defined on $[a, b]$ is called a **step function** if there is a partition $P = \{t_0, \dots, t_n\}$ of $[a, b]$ such that s is a constant on each (t_{i-1}, t_i) (the values of s at t_i may be arbitrary).

(a) Prove that if f is integrable on $[a, b]$, then for any $\varepsilon > 0$ there is a step function $s_1 \le f$ with $\int_a^b f - \int_a^b s_1 < \varepsilon$, and also a step function $s_2 \ge f$ with $\int_a^b s_2 - \int_a^b f < \varepsilon$.

(b) Suppose that for all $\varepsilon > 0$ there are step functions $s_1 \le f$ and $s_2 \ge f$ such that $\int_a^b s_2 - \int_a^b s_1 < \varepsilon$. Prove that f is integrable.

(c) Find a function f which is not a step function, but which satisfies $\int_a^b f = L(f, P)$ for some partition P of $[a, b]$.

***27.** Prove that if f is integrable on $[a, b]$, then for any $\varepsilon > 0$ there are continuous functions $g \le f \le h$ with $\int_a^b h - \int_a^b g < \varepsilon$. Hint: First get step functions with this property, and then continuous ones. A picture will help immensely.

28. (a) Show that if s_1 and s_2 are step functions on $[a, b]$, then $s_1 + s_2$ is also.

(b) Prove, without using Theorem 5, that $\int_a^b (s_1 + s_2) = \int_a^b s_1 + \int_a^b s_2$.

(c) Use part (b) (and Problem 26) to give an alternative proof of Theorem 5.

29. Suppose that f is integrable on $[a, b]$. Prove that there is a number x in $[a, b]$ such that $\int_a^x f = \int_x^b f$. Show by example that it is *not* always possible to choose x to be in (a, b).

***30.** The purpose of this problem is to show that if f is integrable on $[a, b]$, then f must be continuous at many points in $[a, b]$.

(a) Let $P = \{t_0, \dots, t_n\}$ be a partition of $[a, b]$ with $U(f, P) - L(f, P) < b - a$. Prove that for some i we have $M_i - m_i < 1$.

(b) Prove that there are numbers a_1 and b_1 with $a < a_1 < b_1 < b$ and $\sup\{f(x) : a_1 \le x \le b_1\} - \inf\{f(x) : a_1 \le x \le b_1\} < 1$. (You can choose $[a_1, b_1] = [t_{i-1}, t_i]$ from part (a) unless $i = 1$ or n; and in these two cases a very simple device solves the problem.)

(c) Prove that there are numbers a_2 and b_2 with $a_1 < a_2 < b_2 < b_1$ and $\sup\{f(x) : a_2 \le x \le b_2\} - \inf\{f(x) : a_2 \le x \le b_2\} < \frac{1}{2}$.

(d) Continue in this way to find a sequence of intervals $I_n = [a_n, b_n]$ such that $\sup\{f(x) : x \text{ in } I_n\} - \inf\{f(x) : x \text{ in } I_n\} < 1/n$. Apply the Nested Intervals Theorem (Problem 8-14) to find a point x at which f is continuous.

(e) Prove that f is continuous at infinitely many points in $[a, b]$.

***31.** Let f be integrable on $[a, b]$. Recall, from Problem 13, that $\int_a^b f \ge 0$ if $f(x) \ge 0$ for all x in $[a, b]$.

(a) Give an example where $f(x) \ge 0$ for all x, and $f(x) > 0$ for some x in $[a, b]$, and $\int_a^b f = 0$.

***38.** Suppose f and g are integrable on $[a,b]$ and $f(x), g(x) \geq 0$ for all x in $[a,b]$. Let P be a partition of $[a,b]$. Let $M_i{}'$ and $m_i{}'$ denote the appropriate sup's and inf's for f, define $M_i{}''$ and $m_i{}''$ similarly for g, and define M_i and m_i similarly for fg.

(a) Prove that $M_i \leq M_i{}'M_i{}''$ and $m_i \geq m_i{}'m_i{}''$.

(b) Show that

$$U(fg, P) - L(fg, P) \leq \sum_{i=1}^{n} [M_i{}'M_i{}'' - m_i{}'m_i{}''](t_i - t_{i-1}).$$

(c) Using the fact that f and g are bounded, so that $|f(x)|, |g(x)| \leq M$ for x in $[a,b]$, show that

$$U(fg, P) - L(fg, P)$$
$$\leq M \left\{ \sum_{i=1}^{n} [M_i{}' - m_i{}'](t_i - t_{i-1}) + \sum_{i=1}^{n} [M_i{}'' - m_i{}''](t_i - t_{i-1}) \right\}.$$

(d) Prove that fg is integrable.

(e) Now eliminate the restriction that $f(x), g(x) \geq 0$ for x in $[a,b]$.

39. Suppose that f and g are integrable on $[a,b]$. The *Cauchy-Schwarz inequality* states that

$$\left(\int_a^b fg \right)^2 \leq \left(\int_a^b f^2 \right) \left(\int_a^b g^2 \right).$$

(a) Show that the Schwarz inequality is a special case of the Cauchy-Schwarz inequality.

(b) Give three proofs of the Cauchy-Schwarz inequality by imitating the proofs of the Schwarz inequality in Problem 2-21. (The last one will take some imagination.)

(c) If equality holds, is it necessarily true that $f = \lambda g$ for some λ? What if f and g are continuous?

(d) Prove that $\left(\int_0^1 f \right)^2 \leq \left(\int_0^1 f^2 \right)$. Is this result true if 0 and 1 are replaced by a and b?

***40.** Suppose that f is integrable on $[0, x]$ for all $x > 0$ and $\lim_{x \to \infty} f(x) = a$. Prove that

$$\lim_{x \to \infty} \frac{1}{x} \int_0^x f(t)\, dt = a.$$

Hint: The condition $\lim_{x \to \infty} f(x) = a$ implies that $f(t)$ is close to a for $t \geq$ some N. This means that $\int_N^{N+M} f(t)\, dt$ is close to Ma. If M is large in comparison to N, then $Ma/(N+M)$ is close to a.

(b) Suppose $f(x) \geq 0$ for all x in $[a, b]$ and f is continuous at x_0 in $[a, b]$ and $f(x_0) > 0$. Prove that $\int_a^b f > 0$. Hint: It suffices to find one lower sum $L(f, P)$ which is positive.

(c) Suppose f is integrable on $[a, b]$ and $f(x) > 0$ for all x in $[a, b]$. Prove that $\int_a^b f > 0$. Hint: You will need Problem 30; indeed that was one reason for including Problem 30.

***32.** (a) Suppose that f is continuous on $[a, b]$ and $\int_a^b fg = 0$ for all continuous functions g on $[a, b]$. Prove that $f = 0$. (This is easy; there is an obvious g to choose.)

(b) Suppose f is continuous on $[a, b]$ and that $\int_a^b fg = 0$ for those continuous functions g on $[a, b]$ which satisfy the extra conditions $g(a) = g(b) = 0$. Prove that $f = 0$. (This innocent looking fact is an important lemma in the calculus of variations; see reference [22] of the Suggested Reading.) Hint: Derive a contradiction from the assumption $f(x_0) > 0$ or $f(x_0) < 0$; the g you pick will depend on the behavior of f near x_0.

33. Let $f(x) = x$ for x rational and $f(x) = 0$ for x irrational.

(a) Compute $L(f, P)$ for all partitions P of $[0,1]$.

(b) Find $\inf\{U(f, P) : P \text{ a partition of } [0, 1]\}$.

***34.** Let $f(x) = 0$ for irrational x, and $1/q$ if $x = p/q$ in lowest terms. Show that f is integrable on $[0, 1]$ and that $\int_0^1 f = 0$. (Every lower sum is clearly 0; you must figure out how to make upper sums small.)

***35.** Find two functions f and g which are integrable, but whose composition $g \circ f$ is not. Hint: Problem 34 is relevant.

***36.** Let f be a bounded function on $[a, b]$ and let P be a partition of $[a, b]$. Let M_i and m_i have their usual meanings, and let $M_i{}'$ and $m_i{}'$ have the corresponding meanings for the function $|f|$.

(a) Prove that $M_i{}' - m_i{}' \leq M_i - m_i$.

(b) Prove that if f is integrable on $[a, b]$, then so is $|f|$.

(c) Prove that if f and g are integrable on $[a, b]$, then so are $\max(f, g)$ and $\min(f, g)$.

(d) Prove that f is integrable on $[a, b]$ if and only if its "positive part" $\max(f, 0)$ and its "negative part" $\min(f, 0)$ are integrable on $[a, b]$.

37. Prove that if f is integrable on $[a, b]$, then

$$\left| \int_a^b f(t)\,dt \right| \leq \int_a^b |f(t)|\,dt.$$

Hint: This follows easily from a certain string of inequalities; Problem 1-14 is relevant.

APPENDIX. RIEMANN SUMS

Suppose that $P = \{t_0, \dots, t_n\}$ is a partition of $[a, b]$, and that for each i we choose some point x_i in $[t_{i-1}, t_i]$. Then we clearly have

$$L(f, P) \le \sum_{i=1}^{n} f(x_i)(t_i - t_{i-1}) \le U(f, P).$$

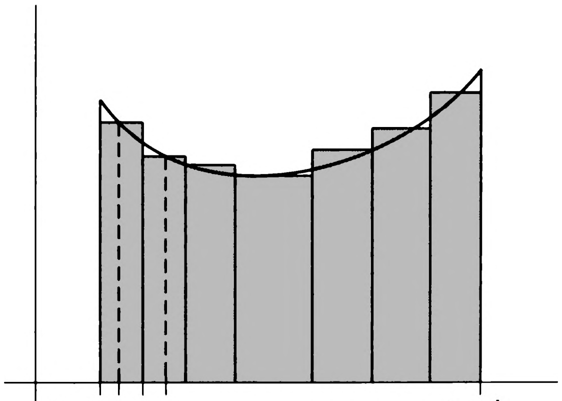


FIGURE 1

Any sum $\sum_{i=1}^{n} f(x_i)(t_i - t_{i-1})$ is called a *Riemann sum* of f for P. Figure 1 shows the geometric interpretation of a Riemann sum; it is the total area of n rectangles that lie partly below the graph of f and partly above it. Because of the arbitrary way in which the heights of the rectangles have been picked, we can't say for sure whether a particular Riemann sum is less than or greater than the integral $\int_a^b f(x)\,dx$. But it does seem that the overlaps shouldn't matter too much; if the bases of all the rectangles are narrow enough, then the Riemann sum ought to be close to the integral. The following theorem states this precisely.

THEOREM 1 Suppose that f is integrable on $[a, b]$. Then for every $\varepsilon > 0$ there is some $\delta > 0$ such that, if $P = \{t_0, \dots, t_n\}$ is any partition of $[a, b]$ with all lengths $t_i - t_{i-1} < \delta$, then

$$\left| \sum_{i=1}^{n} f(x_i)(t_i - t_{i-1}) - \int_a^b f(x)\,dx \right| < \varepsilon,$$

for any Riemann sum formed by choosing x_i in $[t_{i-1}, t_i]$.

PROOF Since the Riemann sum and the integral both lie between $L(f, P)$ and $U(f, P)$, this amounts to showing that for any given ε we can make $U(f, P) - L(f, P) < \varepsilon$ by choosing a δ such that $U(f, P) - L(f, P) < \varepsilon$ for any partition with all lengths $t_i - t_{i-1} < \delta$.

The definition of f being integrable on $[a, b]$ includes the condition that $|f| \le M$ for some M. First choose some particular partition $P^* = \{u_0, , \dots, u_K\}$ for which

$$U(f, P^*) - L(f, P^*) < \varepsilon/2,$$

and then choose a δ such that

$$\delta < \frac{\varepsilon}{4MK}.$$

For any partition P with all $t_i - t_{i-1} < \delta$, we can break the sum

$$U(f, P) - L(f, P) = \sum_{i=1}^{n} (M_i - m_i)(t_i - t_{i-1})$$

into two sums. The first involves those i for which the interval $[t_{i-1}, t_i]$ is completely contained within one of the intervals $[u_{j-1}, u_j]$. This sum is clearly $\le U(f, P^*) - L(f, P^*) < \varepsilon/2$. For all other i we will have $t_{i-1} < u_j < t_i$ for some $j = 1, \dots, K - 1$, so there are at most $K - 1$ of them. Consequently, the sum for these terms is $< (K - 1) \cdot 2M \cdot \delta < \varepsilon/2$. ∎

The moral of this tale is that anything which looks like a good approximation to an integral really is, provided that all the lengths $t_i - t_{i-1}$ of the intervals in the partition are small enough. Some of the following problems should bring home this message with even greater force.

PROBLEMS

1. Suppose that f and g are continuous functions on $[a, b]$. For a partition $P = \{t_0, \dots, t_n\}$ of $[a, b]$ choose a set of points x_i in $[t_{i-1}, t_i]$ and another set of points u_i in $[t_{i-1}, t_i]$. Consider the sum

$$\sum_{i=1}^{n} f(x_i)g(u_i)(t_i - t_{i-1}).$$

Notice that this is *not* a Riemann sum of fg for P. Nevertheless, show that all such sums will be within ε of $\int_a^b fg$ provided that the partition P has all lengths $t_i - t_{i-1}$ small enough. Hint: Estimate the difference between such a sum and a Riemann sum; you will need to use uniform continuity (Chapter 8, Appendix).

2. This problem is similar to, but somewhat harder than, the previous one. Suppose that f and g are continuous nonnegative functions on $[a, b]$. For a partition P, consider sums

$$\sum_{i=1}^{n} \sqrt{f(x_i) + g(u_i)}\,(t_i - t_{i-1}).$$

Show that these sums will be within ε of $\int_a^b \sqrt{f+g}$ if all $t_i - t_{i-1}$ are small enough. Hint: Use the fact that the square-root function is uniformly continuous on a closed interval $[0, M]$.

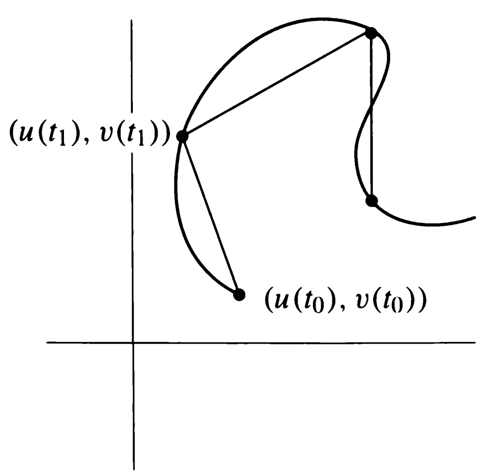


FIGURE 2

3. Finally, we're ready to tackle something big! (Compare Problem 13-25.) Consider a curve c given parametrically by two functions u and v on $[a, b]$. For a partition $P = \{t_0, \dots, t_n\}$ of $[a, b]$ we define

$$\ell(c, P) = \sum_{i=1}^{n} \sqrt{[u(t_i) - u(t_{i-1})]^2 + [v(t_i) - v(t_{i-1})]^2};$$

this represents the length of an inscribed polygonal curve (Figure 2). We define the length of c to be the least upper bound of all $\ell(f, P)$, if it exists.

Prove that if u' and v' are continuous on $[a, b]$, then the length of c is

$$\int_a^b \sqrt{u'^2 + v'^2}.$$

4. Let f' be continuous on the interval $[\theta_0, \theta_1]$. Show that the graph of f in polar coordinates on this interval has the length

$$\int_{\theta_0}^{\theta_1} \sqrt{f^2 + f'^2}.$$

5. Using Theorem 1, show that the Cauchy-Schwarz inequality (Problem 13-39) is a consequence of the Schwarz inequality.