CHAPTER 24 UNIFORM CONVERGENCE AND POWER SERIES

The considerations at the end of the previous chapter suggest an entirely new way of looking at infinite series. Our attention will shift from particular infinite sums to equations like

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$$

which concern sums of quantities that depend on x. In other words, we are interested in *functions* defined by equations of the form

$$f(x) = f_1(x) + f_2(x) + f_3(x) + \cdots$$

(in the previous example $f_n(x) = x^{n-1}/(n-1)!$). In such a situation $\{f_n\}$ will be some sequence of functions; for each x we obtain a sequence of numbers $\{f_n(x)\}$, and $f(x)$ is the sum of this sequence. In order to analyze such functions it will certainly be necessary to remember that each sum

$$f_1(x) + f_2(x) + f_3(x) + \cdots$$

is, by definition, the limit of the sequence

$$f_1(x),\ f_1(x) + f_2(x),\ f_1(x) + f_2(x) + f_3(x),\ \ldots.$$

If we define a new sequence of functions $\{s_n\}$ by

$$s_n = f_1 + \cdots + f_n,$$

then we can express this fact more succinctly by writing

$$f(x) = \lim_{n\to\infty} s_n(x).$$

For some time we shall therefore concentrate on functions defined as limits,

$$f(x) = \lim_{n\to\infty} f_n(x),$$

rather than on functions defined as infinite sums. The total body of results about such functions can be summed up very easily: nothing one would hope to be true actually is—instead we have a splendid collection of counterexamples. The first of these shows that even if each f_n is continuous, the function f may not be! Contrary to what you may expect, the functions f_n will be very simple. Figure 1 shows the graphs of the functions

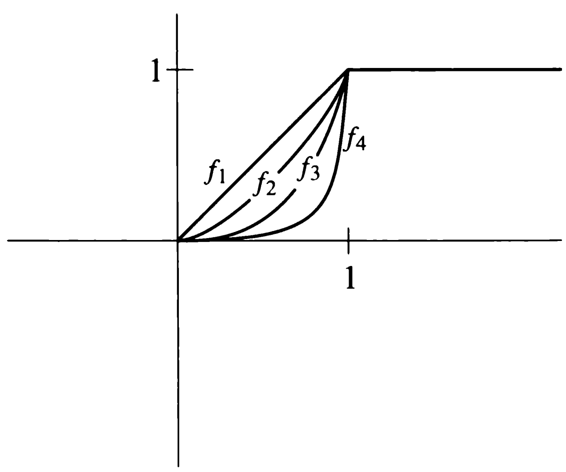


FIGURE 1

$$f_n(x) = \begin{cases} x^n, & 0 \le x \le 1 \\ 1, & x \ge 1. \end{cases}$$

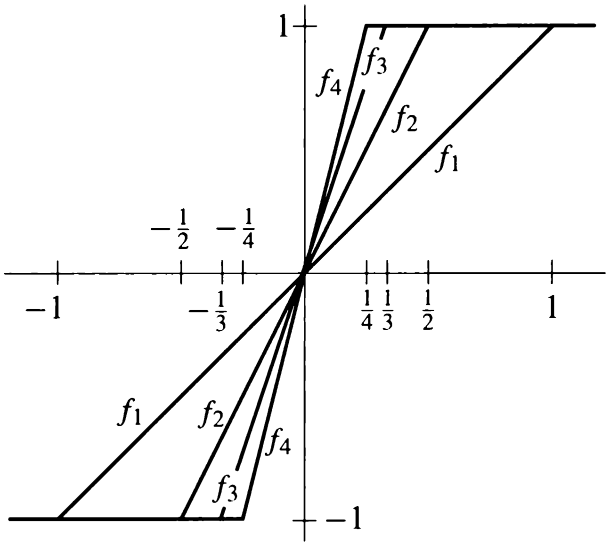


FIGURE 2

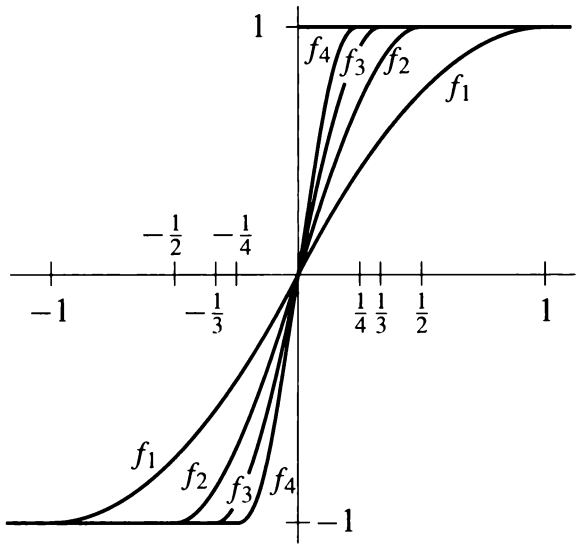


FIGURE 3

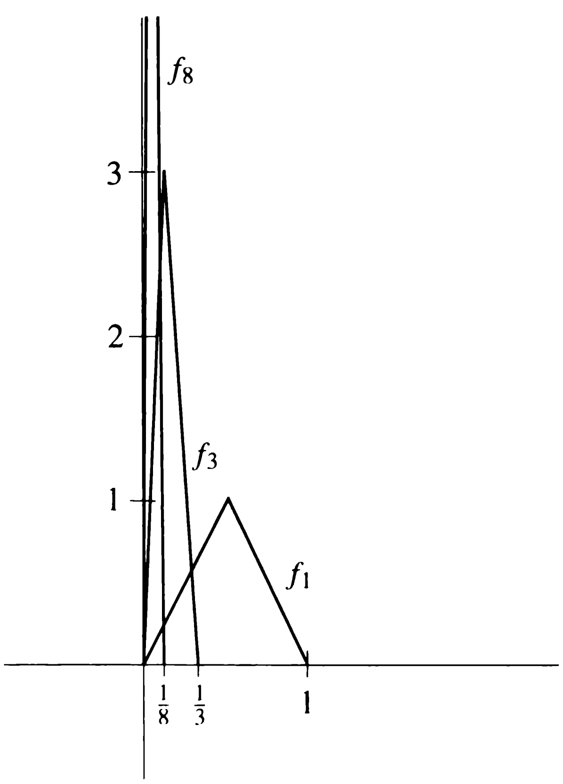


FIGURE 4

These functions are all continuous, but the function $f(x) = \lim_{n\to\infty} f_n(x)$ is not continuous; in fact,

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0, & 0 \le x < 1 \\ 1, & x \ge 1. \end{cases}$$

Another example of this same phenomenon is illustrated in Figure 2; the functions f_n are defined by

$$f_n(x) = \begin{cases} -1, & x \le -\dfrac{1}{n} \\ nx, & -\dfrac{1}{n} \le x \le \dfrac{1}{n} \\ 1, & \dfrac{1}{n} \le x. \end{cases}$$

In this case, if $x < 0$, then $f_n(x)$ is eventually (i.e., for large enough n) equal to -1, and if $x > 0$, then $f_n(x)$ is eventually 1, while $f_n(0) = 0$ for all n. Thus

$$\lim_{n\to\infty} f_n(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0; \end{cases}$$

so, once again, the function $f(x) = \lim_{n\to\infty} f_n(x)$ is not continuous.

By rounding off the corners in the previous examples it is even possible to produce a sequence of *differentiable* functions $\{f_n\}$ for which the function $f(x) = \lim_{n\to\infty} f_n(x)$ is not continuous. One such sequence is easy to define explicitly:

$$f_n(x) = \begin{cases} -1, & x \le -\dfrac{1}{n} \\ \sin\left(\dfrac{n\pi x}{2}\right), & -\dfrac{1}{n} \le x \le \dfrac{1}{n} \\ 1, & \dfrac{1}{n} \le x. \end{cases}$$

These functions are differentiable (Figure 3), but we still have

$$\lim_{n\to\infty} f_n(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0. \end{cases}$$

Continuity and differentiability are, moreover, not the only properties for which problems arise. Another difficulty is illustrated by the sequence $\{f_n\}$ shown in Figure 4; on the interval $[0, 1/n]$ the graph of f_n forms an isosceles triangle of altitude n, while $f_n(x) = 0$ for $x \ge 1/n$. These functions may be defined explicitly as follows:

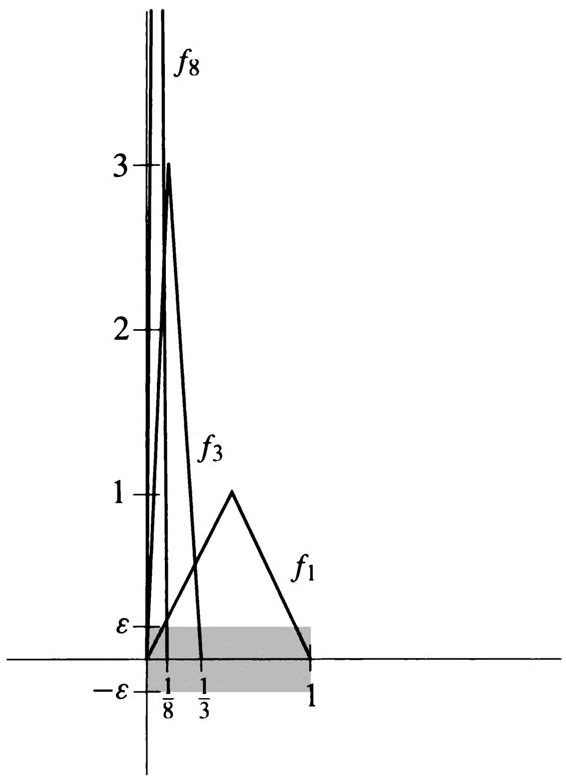


FIGURE 5

$$f_n(x) = \begin{cases} 2n^2x, & 0 \le x \le \dfrac{1}{2n} \\ 2n - 2n^2x, & \dfrac{1}{2n} \le x \le \dfrac{1}{n} \\ 0, & \dfrac{1}{n} \le x \le 1. \end{cases}$$

Because this sequence varies so erratically near 0, our primitive mathematical instincts might suggest that $\lim\limits_{n\to\infty} f_n(x)$ does not always exist. Nevertheless, this limit does exist for all x, and the function $f(x) = \lim\limits_{n\to\infty} f_n(x)$ is even continuous. In fact, if $x > 0$, then $f_n(x)$ is eventually 0, so $\lim\limits_{n\to\infty} f_n(x) = 0$; moreover, $f_n(0) = 0$ for all n, so that we certainly have $\lim\limits_{n\to\infty} f_n(0) = 0$. In other words, $f(x) = \lim\limits_{n\to\infty} f_n(x) = 0$ for all x. On the other hand, the integral quickly reveals the strange behavior of this sequence; we have

$$\int_0^1 f_n(x)\,dx = \tfrac{1}{2},$$

but

$$\int_0^1 f(x)\,dx = 0.$$

Thus,

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx \ne \int_0^1 \lim_{n\to\infty} f_n(x)\,dx.$$

This particular sequence of functions behaves in a way that we really never imagined when we first considered functions defined by limits. Although it is true that

$$f(x) = \lim_{n\to\infty} f_n(x) \qquad \text{for each } x \text{ in } [0, 1],$$

the graphs of the functions f_n do not "approach" the graph of f in the sense of lying close to it—if, as in Figure 5, we draw a strip around f of total width 2ε (allowing a width of ε above and below), then the graphs of f_n do not lie completely within this strip, no matter how large an n we choose. Of course, for each x there is some N such that the point $(x, f_n(x))$ lies in this strip for $n > N$; this assertion just amounts to the fact that $\lim\limits_{n\to\infty} f_n(x) = f(x)$. But it is necessary to choose larger and larger N's as x is chosen closer and closer to 0, and no one N will work for all x at once.

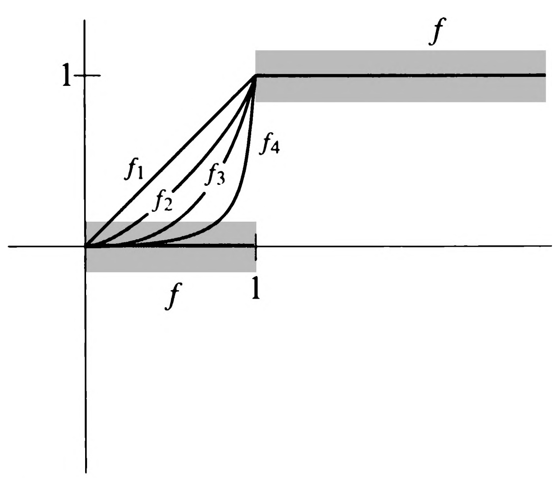


FIGURE 6

The same situation actually occurs, though less blatantly, for each of the other examples given previously. Figure 6 illustrates this point for the sequence

$$f_n(x) = \begin{cases} x^n, & 0 \le x \le 1 \\ 1, & x \ge 1. \end{cases}$$

A strip of total width 2ε has been drawn around the graph of $f(x) = \lim\limits_{n\to\infty} f_n(x)$. If $\varepsilon < \frac{1}{2}$, this strip consists of two pieces, which contain no points with second

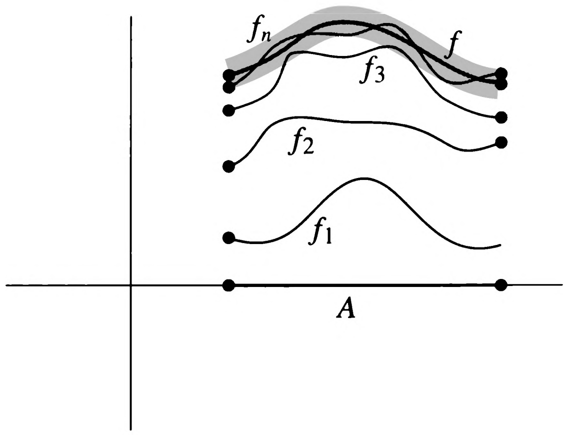


FIGURE 7

coordinate equal to $\frac{1}{2}$; since each function f_n takes on the value $\frac{1}{2}$, the graph of each f_n fails to lie within this strip. Once again, for each point x there is some N such that $(x, f_n(x))$ lies in the strip for $n > N$; but it is not possible to pick one N which works for all x at once.

It is easy to check that precisely the same situation occurs for each of the other examples. In each case we have a function f, and a sequence of functions $\{f_n\}$, all defined on some set A, such that

$$f(x) = \lim_{n\to\infty} f_n(x) \qquad \text{for all } x \text{ in } A.$$

This means that

> for all $\varepsilon > 0$, and for all x in A, there is some N such that if $n > N$, then $|f(x) - f_n(x)| < \varepsilon$.

But in each case different N's must be chosen for different x's, and in each case it is *not* true that

> for all $\varepsilon > 0$ there is some N such that for all x in A, if $n > N$, then $|f(x) - f_n(x)| < \varepsilon$.

Although this condition differs from the first only by a minor displacement of the phrase "for all x in A," it has a totally different significance. If a sequence $\{f_n\}$ satisfies this second condition, then the graphs of f_n eventually lie close to the graph of f, as illustrated in Figure 7. This condition turns out to be just the one which makes the study of limit functions feasible.

DEFINITION

Let $\{f_n\}$ be a sequence of functions defined on A, and let f be a function which is also defined on A. Then f is called the **uniform limit of** $\{f_n\}$ **on** A if for every $\varepsilon > 0$ there is some N such that for all x in A,

$$\text{if } n > N, \text{ then } |f(x) - f_n(x)| < \varepsilon.$$

We also say that $\{f_n\}$ **converges uniformly to f on A**, or that f_n **approaches f uniformly on A**.

As a contrast to this definition, if we know only that

$$f(x) = \lim_{n\to\infty} f_n(x) \qquad \text{for each } x \text{ in } A,$$

then we say that $\{f_n\}$ **converges pointwise to f on A**. Clearly, uniform convergence implies pointwise convergence (but not conversely!).

Evidence for the usefulness of uniform convergence is not at all difficult to amass. Integrals represent a particularly easy topic; Figure 7 makes it almost obvious that if $\{f_n\}$ converges uniformly to f, then the integral of f_n can be made as close to the integral of f as desired. Expressed more precisely, we have the following theorem.

THEOREM 1 Suppose that $\{f_n\}$ is a sequence of functions which are integrable on $[a, b]$, and that $\{f_n\}$ converges uniformly on $[a, b]$ to a function f which is integrable on $[a, b]$. Then

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n.$$

PROOF Let $\varepsilon > 0$. There is some N such that for all $n > N$ we have

$$|f(x) - f_n(x)| < \varepsilon \qquad \text{for all } x \text{ in } [a, b].$$

Thus, if $n > N$ we have

$$\begin{aligned} \left| \int_a^b f(x)\,dx - \int_a^b f_n(x)\,dx \right| &= \left| \int_a^b [f(x) - f_n(x)]\,dx \right| \\ &\le \int_a^b |f(x) - f_n(x)|\,dx \\ &\le \int_a^b \varepsilon\,dx \\ &= \varepsilon(b-a). \end{aligned}$$

Since this is true for any $\varepsilon > 0$, it follows that

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n. \blacksquare$$

The treatment of continuity is only a little bit more difficult, involving an "$\varepsilon/3$-argument," a three-step estimate of $|f(x) - f(x+h)|$. If $\{f_n\}$ is a sequence of continuous functions which converges uniformly to f, then there is some n such that

$$(1) \qquad |f(x) - f_n(x)| < \frac{\varepsilon}{3},$$

$$(2) \qquad |f(x+h) - f_n(x+h)| < \frac{\varepsilon}{3}.$$

Moreover, since f_n is continuous, for sufficiently small h we have

$$(3) \qquad |f_n(x) - f_n(x+h)| < \frac{\varepsilon}{3}.$$

It will follow from (1), (2), and (3) that $|f(x) - f(x+h)| < \varepsilon$. In order to obtain (3), however, we must restrict the size of $|h|$ in a way that cannot be predicted until n has already been chosen; it is therefore quite essential that there be some fixed n which makes (2) true, no matter how small $|h|$ may be—it is precisely at this point that uniform convergence enters the proof.

THEOREM 2 Suppose that $\{f_n\}$ is a sequence of functions which are continuous on $[a, b]$, and that $\{f_n\}$ converges uniformly on $[a, b]$ to f. Then f is also continuous on $[a, b]$.

PROOF For each x in $[a, b]$ we must prove that f is continuous at x. We will deal only with x in (a, b); the cases $x = a$ and $x = b$ require the usual simple modifications.

Let $\varepsilon > 0$. Since $\{f_n\}$ converges uniformly to f on $[a, b]$, there is some n such that

$$|f(y) - f_n(y)| < \frac{\varepsilon}{3} \qquad \text{for all } y \text{ in } [a, b].$$

In particular, for all h such that $x + h$ is in $[a, b]$, we have

$$(1) \qquad |f(x) - f_n(x)| < \frac{\varepsilon}{3},$$

$$(2) \qquad |f(x+h) - f_n(x+h)| < \frac{\varepsilon}{3}.$$

Now f_n is continuous, so there is some $\delta > 0$ such that for $|h| < \delta$ we have

$$(3) \qquad |f_n(x) - f_n(x+h)| < \frac{\varepsilon}{3}.$$

Thus, if $|h| < \delta$, then

$$\begin{aligned}
&|f(x+h) - f(x)| \\
&\qquad = |f(x+h) - f_n(x+h) + f_n(x+h) - f_n(x) + f_n(x) - f(x)| \\
&\qquad \le |f(x+h) - f_n(x+h)| + |f_n(x+h) - f_n(x)| + |f_n(x) - f(x)| \\
&\qquad < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&\qquad = \varepsilon.
\end{aligned}$$

This proves that f is continuous at x. ▮

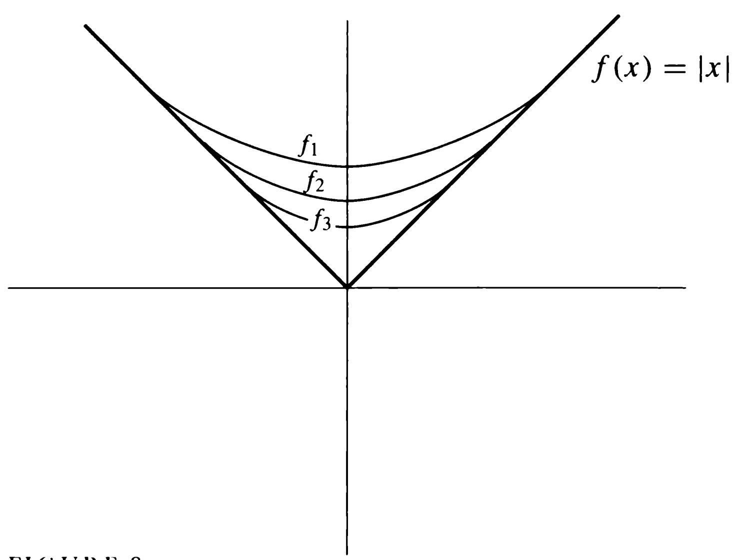


FIGURE 8

After the two noteworthy successes provided by Theorem 1 and Theorem 2, the situation for differentiability turns out to be very disappointing. If each f_n is differentiable, and if $\{f_n\}$ converges uniformly to f, it is still not necessarily true that f is differentiable. For example, Figure 8 shows that there is a sequence of differentiable functions $\{f_n\}$ which converges uniformly to the function $f(x) = |x|$.

Even if f *is* differentiable, it may not be true that

$$f'(x) = \lim_{n\to\infty} f_n'(x);$$

this is not at all surprising if we reflect that a smooth function can be approximated by very rapidly oscillating functions. For example (Figure 9), if

$$f_n(x) = \frac{1}{n}\sin(n^2x),$$

then $\{f_n\}$ converges uniformly to the function $f(x) = 0$, but

$$f_n'(x) = n\cos(n^2x),$$

and $\lim_{n\to\infty} n\cos(n^2x)$ does not always exist (for example, it does not exist if $x = 0$).

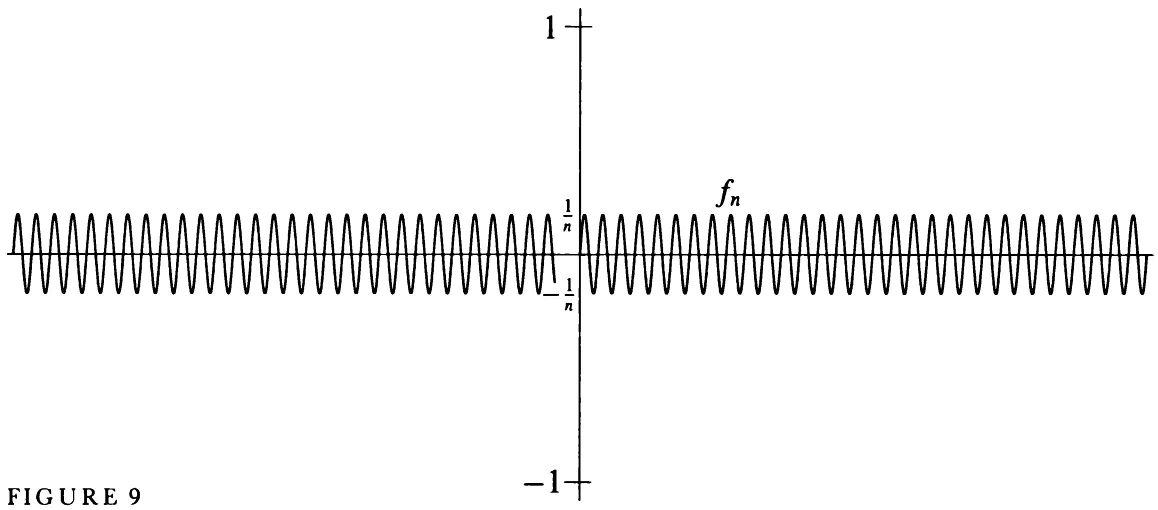


FIGURE 9

Despite such examples, the Fundamental Theorem of Calculus practically guarantees that some sort of theorem about derivatives will be a consequence of Theorem 1; the crucial hypothesis is that $\{f_n'\}$ converges uniformly (to *some* continuous function).

THEOREM 3 Suppose that $\{f_n\}$ is a sequence of functions which are differentiable on $[a,b]$, with integrable derivatives f_n', and that $\{f_n\}$ converges (pointwise) to f. Suppose, moreover, that $\{f_n'\}$ converges uniformly on $[a,b]$ to some continuous function g. Then f is differentiable and

$$f'(x) = \lim_{n\to\infty} f_n'(x).$$

PROOF Applying Theorem 1 to the interval $[a,x]$, we see that for each x we have

$$\begin{aligned}\int_a^x g &= \lim_{n\to\infty}\int_a^x f_n' \\ &= \lim_{n\to\infty}[f_n(x) - f_n(a)] \\ &= f(x) - f(a).\end{aligned}$$

Since g is continuous, it follows that $f'(x) = g(x) = \lim_{n\to\infty} f_n'(x)$ for all x in the interval $[a, b]$. ▮

Now that the basic facts about uniform limits have been established, it is clear how to treat functions defined as infinite sums,

$$f(x) = f_1(x) + f_2(x) + f_3(x) + \cdots .$$

This equation means that

$$f(x) = \lim_{n\to\infty} f_1(x) + \cdots + f_n(x);$$

our previous theorems apply when the new sequence

$$f_1,\ f_1 + f_2,\ f_1 + f_2 + f_3,\ \ldots$$

converges uniformly to f. Since this is the only case we shall ever be interested in, we single it out with a definition.

DEFINITION

The series $\sum_{n=1}^{\infty} f_n$ **converges uniformly** (more formally: the sequence $\{f_n\}$ is **uniformly summable**) **to f on A**, if the sequence

$$f_1,\ f_1 + f_2,\ f_1 + f_2 + f_3,\ \ldots$$

converges uniformly to f on A.

We can now apply each of Theorems 1, 2, and 3 to uniformly convergent series; the results may be stated in one common corollary.

COROLLARY

Let $\sum_{n=1}^{\infty} f_n$ converge uniformly to f on $[a, b]$.

(1) If each f_n is continuous on $[a, b]$, then f is continuous on $[a, b]$.
(2) If f and each f_n is integrable on $[a, b]$, then

$$\int_a^b f = \sum_{n=1}^{\infty} \int_a^b f_n.$$

Moreover, if $\sum_{n=1}^{\infty} f_n$ converges (pointwise) to f on $[a, b]$, each f_n has an integrable derivative f_n' and $\sum_{n=1}^{\infty} f_n'$ converges uniformly on $[a, b]$ to some continuous function, then

(3) $f'(x) = \sum_{n=1}^{\infty} f_n'(x) \qquad$ for all x in $[a, b]$.

PROOF (1) If each f_n is continuous, then so is each $f_1 + \cdots + f_n$, and f is the uniform limit of the sequence $f_1,\ f_1 + f_2,\ f_1 + f_2 + f_3, \ldots$, so f is continuous by Theorem 2.

(2) Since $f_1,\ f_1 + f_2,\ f_1 + f_2 + f_3, \ldots$ converges uniformly to f, it follows from Theorem 1 that

$$\begin{aligned}\int_a^b f &= \lim_{n\to\infty} \int_a^b (f_1 + \cdots + f_n) \\ &= \lim_{n\to\infty} \left(\int_a^b f_1 + \cdots + \int_a^b f_n \right) \\ &= \sum_{n=1}^{\infty} \int_a^b f_n.\end{aligned}$$

(3) Each function $f_1 + \cdots + f_n$ is differentiable, with derivative $f_1{}' + \cdots + f_n{}'$, and $f_1{}',\ f_1{}' + f_2{}',\ f_1{}' + f_2{}' + f_3{}', \ldots$ converges uniformly to a continuous function, by hypothesis. It follows from Theorem 3 that

$$\begin{aligned}f'(x) &= \lim_{n\to\infty} [f_1{}'(x) + \cdots + f_n{}'(x)] \\ &= \sum_{n=1}^{\infty} f_n{}'(x). \ \blacksquare\end{aligned}$$

At the moment this corollary is not very useful, since it seems quite difficult to predict when the sequence $f_1,\ f_1 + f_2,\ f_1 + f_2 + f_3, \ldots$ will converge uniformly. The most important condition which ensures such uniform convergence is provided by the following theorem; the proof is almost a triviality because of the cleverness with which the very simple hypotheses have been chosen.

THEOREM 4 (THE WEIERSTRASS M-TEST) Let $\{f_n\}$ be a sequence of functions defined on A, and suppose that $\{M_n\}$ is a sequence of numbers such that

$$|f_n(x)| \le M_n \qquad \text{for all } x \text{ in } A.$$

Suppose moreover that $\sum_{n=1}^{\infty} M_n$ converges. Then for each x in A the series $\sum_{n=1}^{\infty} f_n(x)$ converges (in fact, it converges absolutely), and $\sum_{n=1}^{\infty} f_n$ converges uniformly on A to the function

$$f(x) = \sum_{n=1}^{\infty} f_n(x).$$

(in order for the second equation to be true it is essential that we choose $h_m = -10^{-m}$ when $a_m = 9$). Now suppose that

$$0.a_{n+1}a_{n+2}a_{n+3}\dots a_m \dots \le \tfrac{1}{2}.$$

Then we also have

$$0.a_{n+1}a_{n+2}a_{n+3}\dots (a_m \pm 1)\dots \le \tfrac{1}{2}$$

(in the special case $m = n + 1$ the second equation is true because we chose $h_m = -10^{-m}$ when $a_m = 4$). This means that

$$\{10^n(a + h_m)\} - \{10^n a\} = \pm 10^{n-m},$$

and exactly the same equation can be derived when $0.a_{n+1}a_{n+2}a_{n+3}\dots > \frac{1}{2}$. Thus, for $n < m$ we have

$$10^{m-n}[\{10^n(a + h_m)\} - \{10^n a\}] = \pm 1.$$

In other words,

$$\frac{f(a + h_m) - f(a)}{h_m}$$

is the sum of $m - 1$ numbers, each of which is ± 1. Now adding $+1$ or -1 to a number changes it from odd to even, and vice versa. The sum of $m - 1$ numbers each ± 1 is therefore an *even integer* if m is odd, and an *odd integer* if m is even. Consequently the sequence of ratios

$$\frac{f(a + h_m) - f(a)}{h_m}$$

cannot possibly converge, since it is a sequence of integers which are alternately odd and even. ▮

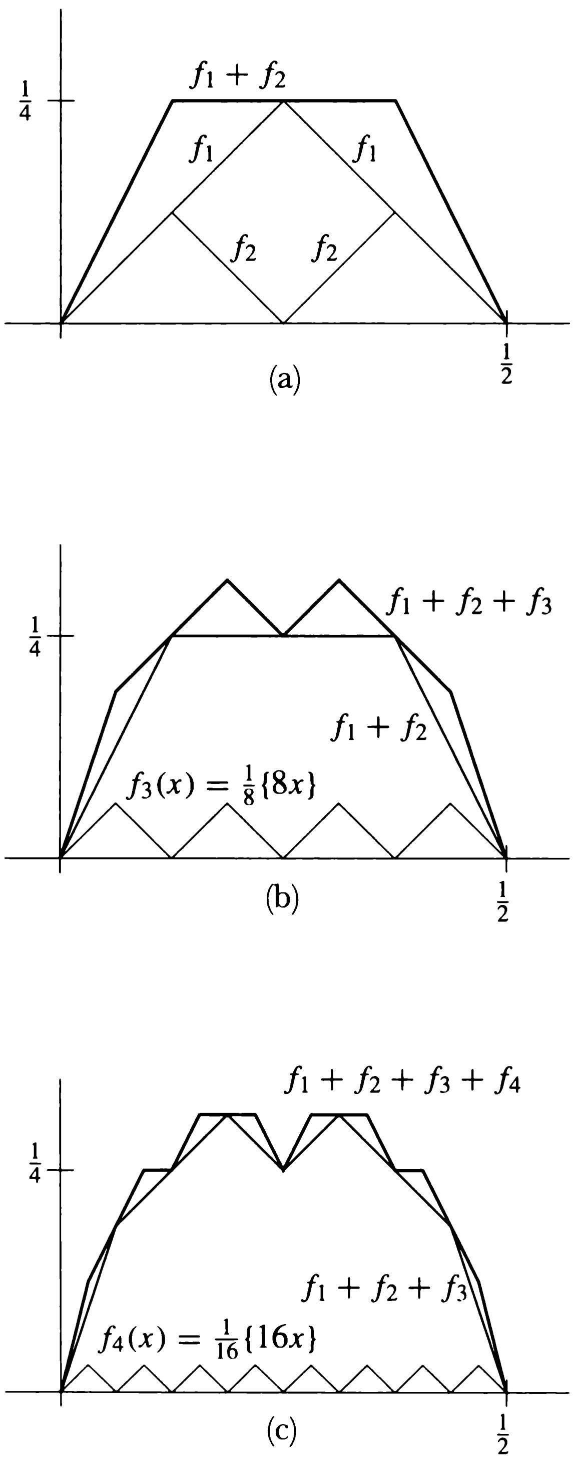


FIGURE 12

In addition to its role in the previous theorem, the Weierstrass M-test is an ideal tool for analyzing functions which are very well behaved. We will give special attention to functions of the form

$$f(x) = \sum_{n=0}^{\infty} a_n(x - a)^n,$$

which can also be described by the equation

$$f(x) = \sum_{n=0}^{\infty} f_n(x),$$

for $f_n(x) = a_n(x - a)^n$. Such an infinite sum, of functions which depend only on powers of $(x - a)$, is called a **power series centered at a**. For the sake of simplicity, we will usually concentrate on power series centered at 0,

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

PROOF For each x in A the series $\sum_{n=1}^{\infty} |f_n(x)|$ converges, by the comparison test; consequently $\sum_{n=1}^{\infty} f_n(x)$ converges (absolutely). Moreover, for all x in A we have

$$\begin{aligned} |f(x) - [f_1(x) + \cdots + f_N(x)]| &= \left| \sum_{n=N+1}^{\infty} f_n(x) \right| \\ &\leq \sum_{n=N+1}^{\infty} |f_n(x)| \\ &\leq \sum_{n=N+1}^{\infty} M_n. \end{aligned}$$

Since $\sum_{n=1}^{\infty} M_n$ converges, the number $\sum_{n=N+1}^{\infty} M_n$ can be made as small as desired, by choosing N sufficiently large. ▮

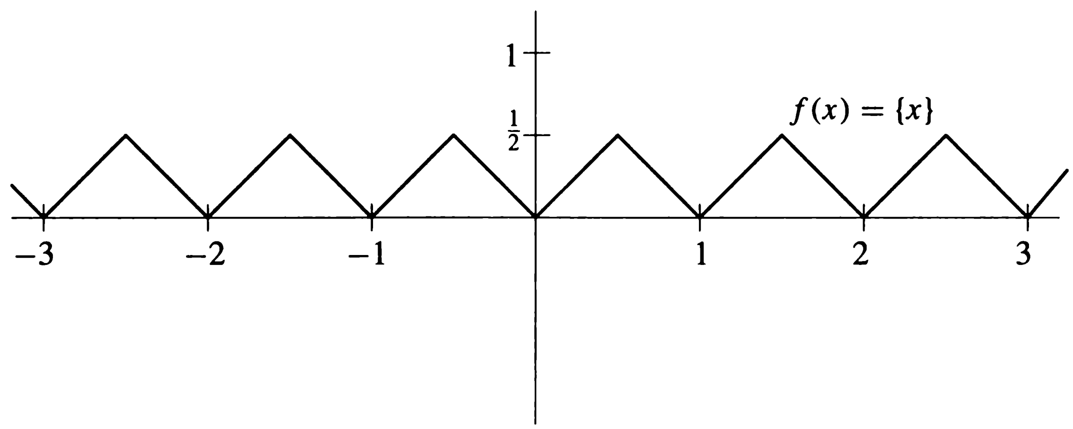
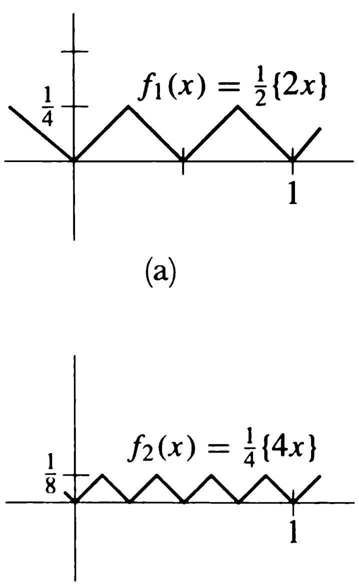


FIGURE 10



(a)

(b)

FIGURE 11

The following sequence $\{f_n\}$ illustrates a simple application of the Weierstrass M-test. Let $\{x\}$ denote the distance from x to the nearest integer (the graph of $f(x) = \{x\}$ is illustrated in Figure 10). Now define

$$f_n(x) = \frac{1}{10^n}\{10^n x\}.$$

The functions f_1 and f_2 are shown in Figure 11 (but to make the drawings simpler, 10^n has been replaced by 2^n). This sequence of functions has been defined so that the Weierstrass M-test automatically applies: clearly

$$|f_n(x)| \leq \frac{1}{10^n} \qquad \text{for all } x,$$

and $\sum_{n=1}^{\infty} 1/10^n$ converges. Thus $\sum_{n=1}^{\infty} f_n$ converges uniformly; since each f_n is continuous, the corollary implies that the function

$$f(x) = \sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} \frac{1}{10^n}\{10^n x\}$$

is also continuous. Figure 12 shows the graph of the first few partial sums $f_1 + \cdots + f_n$. As n increases, the graphs become harder and harder to draw, and the infinite sum $\sum_{n=1}^{\infty} f_n$ is quite undrawable, as shown by the following theorem (included mainly as an interesting sidelight, to be skipped if you find the going too rough).

THEOREM 5 The function

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{10^n}\{10^n x\}$$

is continuous everywhere and differentiable nowhere!

PROOF We have just shown that f is continuous; this is the only part of the proof which uses uniform convergence. We will prove that f is not differentiable at a, for any a, by the straightforward method of exhibiting a particular sequence $\{h_m\}$ approaching 0 for which

$$\lim_{m \to \infty} \frac{f(a + h_m) - f(a)}{h_m}$$

does not exist. It obviously suffices to consider only those numbers a satisfying $0 < a \le 1$.

Suppose that the decimal expansion of a is

$$a = 0.a_1 a_2 a_3 a_4 \ldots.$$

Let $h_m = 10^{-m}$ if $a_m \neq 4$ or 9, but let $h_m = -10^{-m}$ if $a_m = 4$ or 9 (the reason for these two exceptions will appear soon). Then

$$\begin{aligned} \frac{f(a + h_m) - f(a)}{h_m} &= \sum_{n=1}^{\infty} \frac{1}{10^n} \cdot \frac{\{10^n(a + h_m)\} - \{10^n a\}}{\pm 10^{-m}} \\ &= \sum_{n=1}^{\infty} \pm 10^{m-n}[\{10^n(a + h_m)\} - \{10^n a\}]. \end{aligned}$$

This infinite series is really a finite sum, because if $n \ge m$, then $10^n h_m$ is an integer, so

$$\{10^n(a + h_m)\} - \{10^n a\} = 0.$$

On the other hand, for $n < m$ we can write

$$\begin{aligned} 10^n a &= \text{integer} + 0.a_{n+1}a_{n+2}a_{n+3} \ldots a_m \ldots \\ 10^n(a + h_m) &= \text{integer} + 0.a_{n+1}a_{n+2}a_{n+3} \ldots (a_m \pm 1) \ldots \end{aligned}$$

One especially important group of power series are those of the form

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n,$$

where f is some function which has derivatives of all orders at a; this series is called the **Taylor series for f at a**. Of course, it is not necessarily true that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n;$$

this equation holds only when the remainder terms satisfy $\lim_{n\to\infty} R_{n,a}(x) = 0$.

We already know that a power series $\sum_{n=0}^{\infty} a_n x^n$ does not necessarily converge for all x. For example, the power series

$$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

converges only for $|x| \le 1$, while the power series

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \cdots$$

converges only for $-1 < x \le 1$. It is even possible to produce a power series which converges only for $x = 0$. For example, the power series

$$\sum_{n=0}^{\infty} n!\, x^n$$

does not converge for $x \neq 0$; indeed, the ratios

$$\frac{(n+1)!\,(x^{n+1})}{n!\, x^n} = (n+1)x$$

are unbounded for any $x \neq 0$. If a power series $\sum_{n=0}^{\infty} a_n x^n$ does converge for some $x_0 \neq 0$ however, then a great deal can be said about the series $\sum_{n=0}^{\infty} a_n x^n$ for $|x| < |x_0|$.

THEOREM 6 Suppose that the series

$$f(x_0) = \sum_{n=0}^{\infty} a_n {x_0}^n$$

converges, and let a be any number with $0 < a < |x_0|$. Then on $[-a, a]$ the series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

converges uniformly (and absolutely). Moreover, the same is true for the series

$$g(x) = \sum_{n=1}^{\infty} na_n x^{n-1}.$$

Finally, f is differentiable and

$$f'(x) = \sum_{n=1}^{\infty} na_n x^{n-1}$$

for all x with $|x| < |x_0|$.

PROOF Since $\sum_{n=0}^{\infty} a_n x_0{}^n$ converges, the terms $a_n x_0{}^n$ approach 0. Hence they are surely bounded: there is some number M such that

$$|a_n x_0{}^n| = |a_n| \cdot |x_0{}^n| \le M \qquad \text{for all } n.$$

Now if x is in $[-a, a]$, then $|x| \le |a|$, so

$$\begin{aligned} |a_n x^n| &= |a_n| \cdot |x^n| \\ &\le |a_n| \cdot |a^n| \\ &= |a_n| \cdot |x_0|^n \cdot \left| \frac{a}{x_0} \right|^n \qquad \text{(this is the clever step)} \\ &\le M \left| \frac{a}{x_0} \right|^n. \end{aligned}$$

But $|a/x_0| < 1$, so the (geometric) series

$$\sum_{n=0}^{\infty} M \left| \frac{a}{x_0} \right|^n = M \sum_{n=0}^{\infty} \left| \frac{a}{x_0} \right|^n$$

converges. Choosing $M \cdot |a/x_0|^n$ as the number M_n in the Weierstrass M-test, it follows that $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[-a, a]$.

To prove the same assertion for $g(x) = \sum_{n=1}^{\infty} na_n x^{n-1}$ notice that

$$\begin{aligned} |na_n x^{n-1}| &= n|a_n| \cdot |x^{n-1}| \\ &\le n|a_n| \cdot |a^{n-1}| \\ &= \frac{|a_n|}{|a|} \cdot |x_0|^n \, n \left| \frac{a}{x_0} \right|^n \\ &\le \frac{M}{|a|} n \left| \frac{a}{x_0} \right|^n. \end{aligned}$$

Since $|a/x_0| < 1$, the series

$$\sum_{n=1}^{\infty} \frac{M}{|a|} n \left| \frac{a}{x_0} \right|^n = \frac{M}{|a|} \sum_{n=1}^{\infty} n \left| \frac{a}{x_0} \right|^n$$

converges (this fact was proved in Chapter 23 as an application of the ratio test).

Another appeal to the Weierstrass M-test proves that $\sum_{n=1}^{\infty} na_n x^{n-1}$ converges uniformly on $[-a, a]$.

Finally, our corollary proves, first that g is continuous, and then that

$$f'(x) = g(x) = \sum_{n=1}^{\infty} na_n x^{n-1} \qquad \text{for } x \text{ in } [-a, a].$$

Since we could have chosen any number a with $0 < a < |x_0|$, this result holds for all x with $|x| < |x_0|$. ▮

We are now in a position to manipulate power series with ease. Most algebraic manipulations are fairly straightforward consequences of general theorems about infinite series. For example, suppose that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$, where the two power series both converge for some x_0. Then for $|x| < |x_0|$ we have

$$\sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} (a_n x^n + b_n x^n) = \sum_{n=0}^{\infty} (a_n + b_n) x^n.$$

So the series $h(x) = \sum_{n=0}^{\infty} (a_n + b_n) x^n$ also converges for $|x| < |x_0|$, and $h = f + g$ for these x.

The treatment of products is just a little more involved. If $|x| < |x_0|$, then we know that the series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ converge *absolutely*. So it follows from Theorem 23-9 that the product $\sum_{n=0}^{\infty} a_n x^n \cdot \sum_{n=0}^{\infty} b_n x^n$ is given by

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_i x^i b_j x^j,$$

where the elements $a_i x^i b_j x^j$ are arranged in any order. In particular, we can choose the arrangement

$$a_0 b_0 + (a_0 b_1 + a_1 b_0) x + (a_0 b_2 + a_1 b_1 + a_2 b_0) x^2 + \cdots$$

which can be written as

$$\sum_{n=0}^{\infty} c_n x^n \qquad \text{for } c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

This is the "Cauchy product" that was introduced in Problem 23-10. Thus, the Cauchy product $h(x) = \sum_{n=0}^{\infty} c_n x^n$ also converges for $|x| < |x_0|$ and $h = fg$ for these x.

Finally, suppose that $f(x) = \sum_{n=0}^{\infty} a_n x^n$, where $a_0 \neq 0$, so that $f(0) = a_0 \neq 0$. Then we can try to find a power series $\sum_{n=0}^{\infty} b_n x^n$ which represents $1/f$. This means that we want to have

$$\sum_{n=0}^{\infty} a_n x^n \cdot \sum_{n=0}^{\infty} b_n x^n = 1 = 1 + 0 \cdot x + 0 \cdot x^2 + \cdots .$$

Since the left side of this equation will be given by the Cauchy product, we want to have

$$\begin{aligned}
&a_0 b_0 = 1\\
&a_0 b_1 + a_1 b_0 = 0\\
&a_0 b_2 + a_1 b_1 + a_2 b_0 = 0\\
&\cdots
\end{aligned}$$

Since $a_0 \neq 0$, we can solve the first of these equations for b_0. Then we can solve the second for b_1, etc. Of course, we still have to prove that the new series $\sum_{n=0}^{\infty} b_n x^n$ does converge for some $x \neq 0$. This is left as an exercise (Problem 18).

For derivatives, Theorem 6 gives us all the information we need. In particular, when we apply Theorem 6 to the infinite series

$$\begin{aligned}
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots ,\\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots ,\\
e^x &= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots ,
\end{aligned}$$

we get precisely the results which are expected. Each of these converges for any x_0, hence the conclusions of Theorem 6 apply for any x:

$$\begin{aligned}
\sin'(x) &= 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \cdots = \cos x,\\
\cos'(x) &= -\frac{2x}{2!} + \frac{4x^3}{4!} - \frac{6x^5}{6!} + \cdots = -\sin x,\\
\exp'(x) &= 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \cdots = \exp(x).
\end{aligned}$$

For the functions arctan and $f(x) = \log(1+x)$ the situation is only slightly more complicated. Since the series

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

converges for $x_0 = 1$, it also converges for $|x| < 1$, and

$$\arctan'(x) = 1 - x^2 + x^4 - x^6 + \cdots = \frac{1}{1+x^2} \qquad \text{for } |x| < 1.$$

In this case, the series happens to converge for $x = -1$ also. However, the formula for the derivative is not correct for $x = 1$ or $x = -1$; indeed the series

$$1 - x^2 + x^4 - x^6 + \cdots$$

diverges for $x = 1$ and $x = -1$. Notice that this does not contradict Theorem 6, which proves that the derivative is given by the expected formula only for $|x| < |x_0|$.

Since the series

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots$$

converges for $x_0 = 1$, it also converges for $|x| < 1$, and

$$\frac{1}{1+x} = \log'(1+x) = 1 - x + x^2 - x^3 + \cdots \qquad \text{for } |x| < 1.$$

In this case, the original series does not converge for $x = -1$; moreover, the differentiated series does not converge for $x = 1$.

All the considerations which apply to a power series will automatically apply to its derivative, at the points where the derivative is represented by a power series. If

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

converges for all x in some interval $(-R, R)$, then Theorem 6 implies that

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

for all x in $(-R, R)$. Applying Theorem 6 once again we find that

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2},$$

and proceeding by induction we find that

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1) \cdot \ldots \cdot (n-k+1) a_n x^{n-k}.$$

Thus, a function defined by a power series which converges in some interval $(-R, R)$ is automatically infinitely differentiable in that interval. Moreover, the previous equation implies that

$$f^{(k)}(0) = k!\, a_k,$$

so that

$$a_k = \frac{f^{(k)}(0)}{k!}.$$

In other words, *a convergent power series centered at 0 is always the Taylor series at 0 of the function which it defines.*

On this happy note we could easily end our study of power series and Taylor series. A careful assessment of our situation will reveal some unexplained facts, however.

The Taylor series of sin, cos, and exp are as satisfactory as we could desire; they converge for all x, and can be differentiated term-by-term for all x. The Taylor series of the function $f(x) = \log(1 + x)$ is slightly less pleasing, because it converges only for $-1 < x \le 1$, but this deficiency is a necessary consequence of the basic nature of power series. If the Taylor series for f converged for any x_0 with $|x_0| > 1$, then it would converge on the interval $(-|x_0|, |x_0|)$, and on this interval the function which it defines would be differentiable, and thus continuous. But this is impossible, since it is unbounded on the interval $(-1, 1)$, where it equals $\log(1 + x)$.

The Taylor series for arctan is more difficult to comprehend—there seems to be no possible excuse for the refusal of this series to converge when $|x| > 1$. This mysterious behavior is exemplified even more strikingly by the function $f(x) = 1/(1 + x^2)$, an infinitely differentiable function which is the next best thing to a polynomial function. The Taylor series of f is given by

$$f(x) = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots .$$

If $|x| \ge 1$ the Taylor series does not converge at all. Why? What unseen obstacle prevents the Taylor series from extending past 1 and -1? Asking this sort of question is always dangerous, since we may have to settle for an unsympathetic answer: it happens because it happens—that's the way things are! In this case there does happen to be an explanation, but this explanation is impossible to give at the present time; although the question is about real numbers, it can be answered intelligently only when placed in a broader context. It will therefore be necessary to devote two chapters to quite new material before completing our discussion of Taylor series in Chapter 27.

PROBLEMS

1. For each of the following sequences $\{f_n\}$, determine the pointwise limit of $\{f_n\}$ (if it exists) on the indicated interval, and decide whether $\{f_n\}$ converges uniformly to this function.

(i) $f_n(x) = \sqrt[n]{x}$, on $[0, 1]$.

(ii) $f_n(x) = \begin{cases} 0, & x \le n \\ x - n, & x \ge n, \end{cases}$ on $[a, b]$, and on $\mathbf{R}$.

(iii) $f_n(x) = \dfrac{e^x}{x^n}$, on $(1, \infty)$.

(iv) $f_n(x) = e^{-nx^2}$, on $[-1, 1]$.

(v) $f_n(x) = \frac{e^{-x^2}}{n}$, on $\mathbf{R}$.

2. This problem asks for the same information as in Problem 1, but the functions are not so easy to analyze. Some hints are given at the end.

(i) $f_n(x) = x^n - x^{2n}$ on $[0, 1]$.

(ii) $f_n(x) = \dfrac{nx}{1 + n + x}$ on $[0, \infty)$.

(iii) $f_n(x) = \sqrt{x^2 + \dfrac{1}{n^2}}$ on $[a, \infty)$, $a > 0$.

(iv) $f_n(x) = \sqrt{x^2 + \dfrac{1}{n^2}}$ on $\mathbf{R}$.

(v) $f_n(x) = \sqrt{x + \dfrac{1}{n}} - \sqrt{x}$ on $[a, \infty)$, $a > 0$.

(vi) $f_n(x) = \sqrt{x + \dfrac{1}{n}} - \sqrt{x}$ on $[0, \infty)$.

(vii) $f_n(x) = n\left(\sqrt{x + \dfrac{1}{n}} - \sqrt{x}\right)$ on $[a, \infty)$, $a > 0$.

(viii) $f_n(x) = n\left(\sqrt{x + \dfrac{1}{n}} - \sqrt{x}\right)$ on $[0, \infty)$ and on $(0, \infty)$.

Hints: (i) For each n, find the maximum of $|f - f_n|$ on $[0, 1]$. (ii) For each n, consider $|f(x) - f_n(x)|$ for x large. (iii) Mean Value Theorem. (iv) Give a separate estimate of $|f(x) - f_n(x)|$ for small $|x|$. (vii) Use (v).

3. Find the Taylor series at 0 for each of the following functions.

(i) $f(x) = \dfrac{1}{x - a}$, $a \neq 0$.

(ii) $f(x) = \log(x - a)$, $a < 0$.

(iii) $f(x) = \dfrac{1}{\sqrt{1 - x}} = (1 - x)^{-1/2}$. (Use Problem 20-21.)

(iv) $f(x) = \dfrac{1}{\sqrt{1 - x^2}}$.

(v) $f(x) = \arcsin x$.

4. Find each of the following infinite sums.

(i) $1 - x + \dfrac{x^2}{2!} - \dfrac{x^3}{3!} + \dfrac{x^4}{4!} - \cdots$.

(ii) $1 - x^3 + x^6 - x^9 + \cdots$ for $|x| < 1$.
Hint: What is $1 - x + x^2 - x^3 + \cdots$?

(iii) $\dfrac{x^2}{2} - \dfrac{x^3}{3 \cdot 2} + \dfrac{x^4}{4 \cdot 3} - \dfrac{x^5}{5 \cdot 4} + \cdots$ for $|x| < 1$.
Hint: Differentiate.

5. Evaluate the following infinite sums. (In most cases they are $f(a)$ where a is some obvious number and $f(x)$ is given by some power series. To evaluate the various power series, manipulate them until some well-known power series emerge.)

(i) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} \pi^{2n}}{(2n)!}$.

(ii) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(2n)!}$.

(iii) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2n+1} \left(\frac{1}{2}\right)^{2n+1}$

(iv) $\displaystyle\sum_{n=0}^{\infty} \frac{n}{2^n}$.

(v) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{3^n (n+1)}$.

(vi) $\displaystyle\sum_{n=0}^{\infty} \frac{2n+1}{2^n n!}$.

6. If $f(x) = (\sin x)/x$ for $x \neq 0$ and $f(0) = 1$, find $f^{(k)}(0)$. Hint: Find the power series for f.

7. In this problem we deduce the binomial series $(1+x)^\alpha = \displaystyle\sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$, $|x| < 1$ without all the work of Problem 23-21, although we will use a fact established in part (a) of that problem—the series $f(x) = \displaystyle\sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$ does converge for $|x| < 1$.

(a) Prove that $(1+x)f'(x) = \alpha f(x)$ for $|x| < 1$.

(b) Now show that any function f satisfying part (a) is of the form $f(x) = c(1+x)^\alpha$ for some constant c, and use this fact to establish the binomial series. Hint: Consider $g(x) = f(x)/(1+x)^\alpha$.

8. Suppose that f_n are nonnegative bounded functions on A and let $M_n = \sup f_n$. If $\sum_{n=1}^{\infty} f_n$ converges uniformly on A, does it follow that $\sum_{n=1}^{\infty} M_n$ converges (a converse to the Weierstrass M-test)?

9. Prove that the series

$$\sum_{n=1}^{\infty} \frac{x}{n(1+nx^2)}$$

converges uniformly on **R**.

10. (a) Prove that the series

$$\sum_{n=0}^{\infty} 2^n \sin \frac{1}{3^n x}$$

converges uniformly on $[a, \infty)$ for $a > 0$. Hint: $\lim_{h \to 0} (\sin h)/h = 1$.

(b) By considering the sum from N to ∞ for $x = 2/(\pi 3^N)$, show that the series does not converge uniformly on $(0, \infty)$.

11. (a) Prove that the series

$$f(x) = \sum_{n=0}^{\infty} \frac{nx}{1+n^4x^2}$$

converges uniformly on $[a, \infty)$ for $a > 0$. Hint: First find the maximum of $nx/(1+n^4x^2)$ on $[0, \infty)$.

(b) Show that

$$f\left(\frac{1}{N}\right) \geq \frac{N}{2} \sum_{n \geq \sqrt{N}} \frac{1}{n^3},$$

and by using an integral to estimate the sum, show that $f(1/N^2) \geq 1/4$. Conclude that the series does not converge uniformly on **R**.

(c) What about the series

$$\sum_{n=0}^{\infty} \frac{nx}{1+n^5x^2}?$$

12. (a) Use Problem 15-33 and Abel's Lemma (Problem 19-36) to obtain a "uniform Cauchy condition", showing that for any $\varepsilon > 0$,

$$\left|\sum_{k=m}^{n} \frac{\sin kx}{k}\right|$$

can be made arbitrarily small on the whole interval $[\varepsilon, 2\pi - \varepsilon]$ by choosing m (and n) large enough. Conclude that the series

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n}$$

converges uniformly on $[\varepsilon, 2\pi - \varepsilon]$ for $\varepsilon > 0$.

(b) For $x = \pi/N$, with N large, show that

$$\left|\sum_{k=N}^{2N} \sin kx\right| = \left|\sum_{k=0}^{N} \sin kx\right| \geq \frac{N}{\pi}.$$

Conclude that

$$\left|\sum_{k=N}^{2N} \frac{\sin kx}{k}\right| \geq \frac{1}{2\pi},$$

and that the series does not converge uniformly on $[0, 2\pi]$.

13. (a) Suppose that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges for all x in some interval $(-R, R)$ and that $f(x) = 0$ for all x in $(-R, R)$. Prove that each $a_n = 0$. (If you remember the formula for a_n this is easy.)

(b) Suppose we know only that $f(x_n) = 0$ for some sequence $\{x_n\}$ with $\lim_{n\to\infty} x_n = 0$. Prove again that each $a_n = 0$. Hint: First show that $f(0) = a_0 = 0$; then that $f'(0) = a_1 = 0$, etc.

This result shows that if $f(x) = e^{-1/x^2} \sin 1/x$ for $x \neq 0$, then f cannot possibly be written as a power series. It also shows that a function defined by a power series cannot be 0 for $x \leq 0$ but nonzero for $x > 0$—thus a power series cannot describe the motion of a particle which has remained at rest until time 0, and then begins to move!

(c) Suppose that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ converge for all x in some interval containing 0 and that $f(t_m) = g(t_m)$ for some sequence $\{t_m\}$ converging to 0. Show that $a_n = b_n$ for each n.

14. Prove that if $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is an even function, then $a_n = 0$ for n odd, and if f is an odd function, then $a_n = 0$ for n even.

15. Show that the power series for $f(x) = \log(1-x)$ converges only for $-1 \le x < 1$, and that the power series for $g(x) = \log[(1+x)/(1-x)]$ converges only for x in $(-1, 1)$.

***16.** Recall that the Fibonacci sequence $\{a_n\}$ is defined by $a_1 = a_2 = 1$, $a_{n+1} = a_n + a_{n-1}$.

(a) Show that $a_{n+1}/a_n \le 2$.

(b) Let

$$f(x) = \sum_{n=1}^{\infty} a_n x^{n-1} = 1 + x + 2x^2 + 3x^3 + \cdots .$$

Use the ratio test to prove that $f(x)$ converges if $|x| < 1/2$.

(c) Prove that if $|x| < 1/2$, then

$$f(x) = \frac{-1}{x^2 + x - 1}.$$

Hint: This equation can be written $f(x) - xf(x) - x^2 f(x) = 1$.

(d) Use the partial fraction decomposition for $1/(x^2+x-1)$, and the power series for $1/(x-a)$, to obtain another power series for f.

(e) Since the two power series obtained for f must be the same (they are both the Taylor series of the function), conclude that

$$a_n = \frac{\left(\dfrac{1+\sqrt{5}}{2}\right)^n - \left(\dfrac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}.$$

17. Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$. Suppose we merely knew that $f(x)g(x) = \sum_{n=0}^{\infty} c_n x^n$ for some c_n, but we didn't know how to multiply series in general. Use Leibniz's formula (Problem 10-20) to show directly that this series for fg must indeed be the Cauchy product of the series for f and g.

18. Suppose that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges for some x_0, and that $a_0 \neq 0$; for simplicity, we'll assume that $a_0 = 1$. Let $\{b_n\}$ be the sequence defined recursively by

$$b_0 = 1$$

$$b_n = -\sum_{k=0}^{n-1} b_k a_{n-k}.$$

The aim of this problem is to show that $\sum_{n=0}^{\infty} b_n x^n$ also converges for some $x \neq 0$, so that it represents $1/f$ for small enough $|x|$.

(a) If all $|a_n x_0{}^n| \le M$, show that

$$|b_n x_0{}^n| \le M \sum_{k=0}^{n-1} |b_k x_0{}^k|.$$

(b) Choose M so that $|a_n x_0{}^n| \le M$, and also so that $M/(M^2 - 1) \le 1$. Show that

$$|b_n x_0{}^n| \le M^{2n}.$$

(c) Conclude that $\sum_{n=0}^{\infty} b_n x^n$ converges for $|x|$ sufficiently small.

***19.** Suppose that $\sum_{n=0}^{\infty} a_n$ converges. We know that the series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ must converge uniformly on $[-a, a]$ for $0 < a < 1$, but it may not converge uniformly on $[-1, 1]$; in fact, it may not even converge at the point -1 (for example, if $f(x) = \log(1 + x)$). However, a beautiful theorem of Abel shows that the series *does* converge uniformly on $[0, 1]$. Consequently, f is continuous on $[0, 1]$ and, in particular, $\sum_{n=0}^{\infty} a_n = \lim_{x \to 1^-} \sum_{n=0}^{\infty} a_n x^n$. Prove Abel's Theorem by noticing that if $|a_m + \cdots + a_n| < \varepsilon$, then $|a_m x^m + \cdots + a_n x^n| < \varepsilon$, by Abel's Lemma (Problem 19-36).

20. A sequence $\{a_n\}$ is called **Abel summable** if $\lim_{x \to 1^-} \sum_{n=0}^{\infty} a_n x^n$ exists; Problem 19 shows that a summable sequence is necessarily Abel summable. Find a sequence which is Abel summable, but which is not summable. Hint: Look over the list of Taylor series until you find one which does not converge at 1, even though the function it represents is continuous at 1.

21. (a) Using Problem 19, find the following infinite sums.

(i) $\dfrac{1}{2 \cdot 1} - \dfrac{1}{3 \cdot 2} + \dfrac{1}{4 \cdot 3} - \dfrac{1}{5 \cdot 4} + \cdots.$

(ii) $1 - \frac{1}{4} + \frac{1}{7} - \frac{1}{10} + \cdots.$

(b) Let $\sum_{n=0}^{\infty} c_n$ be the Cauchy product of two convergent power series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$, and suppose merely that $\sum_{n=0}^{\infty} c_n$ converges. Prove that, in fact, it converges to the product $\sum_{n=0}^{\infty} a_n \cdot \sum_{n=0}^{\infty} b_n$.

22. (a) Show that the series

$$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1} - \frac{x^{n+1}}{2n+2}$$

converges uniformly to $\frac{1}{2}\log(x+1)$ on $[-a, a]$ for $0 < a < 1$, but that at 1 it converges to $\log 2$. (Why doesn't this contradict Abel's Theorem (Problem 19)?)

23. (a) Suppose that $\{f_n\}$ is a sequence of bounded (not necessarily continuous) functions on $[a, b]$ which converge uniformly to f on $[a, b]$. Prove that f is bounded on $[a, b]$.

(b) Find a sequence of continuous functions on $[a, b]$ which converge pointwise to an unbounded function on $[a, b]$.

24. Suppose that f is differentiable. Prove that the function f' is the pointwise limit of a sequence of continuous functions. (Since we already know examples of discontinuous derivatives, this provides another example where the pointwise limit of continuous functions is not continuous.)

25. Find a sequence of integrable functions $\{f_n\}$ which converges to the (nonintegrable) function f that is 1 on the rationals and 0 on the irrationals. Hint: Each f_n will be 0 except at a few points.

26. (a) Prove that if f is the uniform limit of $\{f_n\}$ on $[a, b]$ and each f_n is integrable on $[a, b]$, then so is f. (So one of the hypotheses in Theorem 1 was unnecessary.)

(b) In Theorem 3 we assumed only that $\{f_n\}$ converges pointwise to f. Show that the remaining hypotheses ensure that $\{f_n\}$ actually converges uniformly to f.

(c) Suppose that in Theorem 3 we do not assume $\{f_n\}$ converges to a function f, but instead assume only that $f_n(x_0)$ converges for some x_0 in $[a, b]$. Show that f_n does converge (uniformly) to some f (with $f' = g$).

(d) Prove that the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{x+n}$$

converges uniformly on $[0, \infty)$.

27. Suppose that f_n are continuous functions on $[0, 1]$ that converge uniformly to f. Prove that

$$\lim_{n\to\infty} \int_0^{1-1/n} f_n = \int_0^1 f.$$

Is this true if the convergence isn't uniform?

*(f) Prove that a continuous function is regulated. Hint: To find a step function s on $[a, b]$ with $|f(x) - s(x)| < \varepsilon$ for all x in $[a, b]$, consider all y for which there is such a step function on $[a, y]$.

(g) Every step function s has the property that $\lim_{x \to a^+} s(x)$ and $\lim_{x \to a^-} s(x)$ exist for all a. Conclude that every regulated function has the same property, and find an integrable function that is not regulated. (It is also true that, conversely, every function f with the property that $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ exist for all a is regulated.)

***31.** Find a sequence $\{f_n\}$ approaching f uniformly on $[0, 1]$ for which we have $\lim_{n \to \infty}$ (length of f_n on $[0, 1]$) $\neq$ length of f on $[0, 1]$. (Length is defined in Problem 13-25, but the simplest example will involve functions the length of whose graphs will be obvious.)

28. (a) Suppose that $\{f_n\}$ is a sequence of continuous functions on $[a, b]$ which approaches 0 pointwise. Suppose moreover that we have $f_n(x) \geq f_{n+1}(x) \geq 0$ for all n and all x in $[a, b]$. Prove that $\{f_n\}$ actually approaches 0 uniformly on $[a, b]$. Hint: Suppose not, choose an appropriate sequence of points x_n in $[a, b]$, and apply the Bolzano-Weierstrass theorem.

(b) Prove Dini's Theorem: If $\{f_n\}$ is a nonincreasing sequence of continuous functions on $[a, b]$ which approaches the continuous function f pointwise, then $\{f_n\}$ also approaches f uniformly on $[a, b]$. (The same result holds if $\{f_n\}$ is a nondecreasing sequence.)

(c) Does Dini's Theorem hold if f isn't continuous? How about if $[a, b]$ is replaced by the open interval (a, b)?

29. (a) Suppose that $\{f_n\}$ is a sequence of continuous functions on $[a, b]$ that converges uniformly to f. Prove that if x_n approaches x, then $f_n(x_n)$ approaches $f(x)$.

(b) Is this statement true without assuming that the f_n are continuous?

(c) Prove the converse of part (a): If f is continuous on $[a, b]$ and $\{f_n\}$ is a sequence with the property that $f_n(x_n)$ approaches $f(x)$ whenever x_n approaches x, then f_n converges uniformly to f on $[a, b]$. Hint: If not, there is an $\varepsilon > 0$ and a sequence x_n with $|f_n(x_n) - f(x_n)| > \varepsilon$ for infinitely many distinct x_n. Then use the Bolzano-Weierstrass theorem.

30. This problem outlines a completely different approach to the integral; consequently, it is unfair to use any facts about integrals learned previously.

(a) Let s be a step function on $[a, b]$, so that s is constant on (t_{i-1}, t_i) for some partition $\{t_0, \ldots, t_n\}$ of $[a, b]$. Define $\displaystyle\int_a^b s$ as $\displaystyle\sum_{i=1}^{n} s_i \cdot (t_i - t_{i-1})$ where s_i is the (constant) value of s on (t_{i-1}, t_i). Show that this definition does not depend on the partition $\{t_0, \ldots, t_n\}$.

(b) A function f is called a **regulated** function on $[a, b]$ if it is the uniform limit of a sequence of step functions $\{s_n\}$ on $[a, b]$. Show that in this case there is, for every $\varepsilon > 0$, some N such that for $m, n > N$ we have $|s_n(x) - s_m(x)| < \varepsilon$ for all x in $[a, b]$.

(c) Show that the sequence of numbers $\left\{\displaystyle\int_a^b s_n\right\}$ will be a Cauchy sequence.

(d) Suppose that $\{t_n\}$ is another sequence of step functions on $[a, b]$ which converges uniformly to f. Show that for every $\varepsilon > 0$ there is an N such that for $n > N$ we have $|s_n(x) - t_n(x)| < \varepsilon$ for x in $[a, b]$.

(e) Conclude that $\displaystyle\lim_{n\to\infty} \int_a^b s_n = \lim_{n\to\infty} \int_a^b t_n$. This means that we can *define* $\displaystyle\int_a^b f$ to be $\displaystyle\lim_{n\to\infty} s_n$ for any sequence of step functions $\{s_n\}$ converging uniformly to f. The only remaining question is: Which functions are regulated?