CHAPTER 7 THREE HARD THEOREMS

This chapter is devoted to three theorems about continuous functions, and some of their consequences. The proofs of the three theorems themselves will not be given until the next chapter, for reasons which are explained at the end of this chapter.

THEOREM 1 If f is continuous on $[a, b]$ and $f(a) < 0 < f(b)$, then there is some x in $[a, b]$ such that $f(x) = 0$.

(Geometrically, this means that the graph of a continuous function which starts below the horizontal axis and ends above it must cross this axis at some point, as in Figure 1.)

THEOREM 2 If f is continuous on $[a, b]$, then f is bounded above on $[a, b]$, that is, there is some number N such that $f(x) \leq N$ for all x in $[a, b]$.

(Geometrically, this theorem means that the graph of f lies below some line parallel to the horizontal axis, as in Figure 2.)

THEOREM 3 If f is continuous on $[a, b]$, then there is some number y in $[a, b]$ such that $f(y) \geq f(x)$ for all x in $[a, b]$ (Figure 3).

These three theorems differ markedly from the theorems of Chapter 6. The hypotheses of those theorems always involved continuity at a single point, while

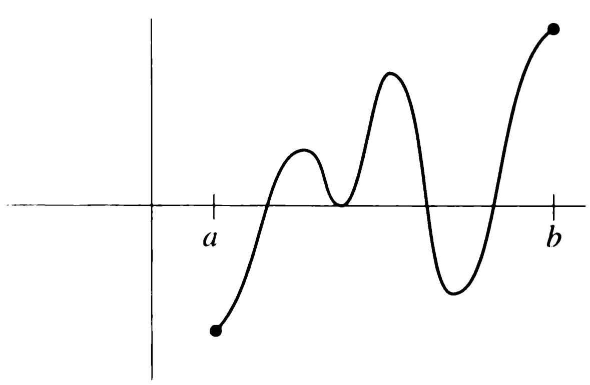

FIGURE 1

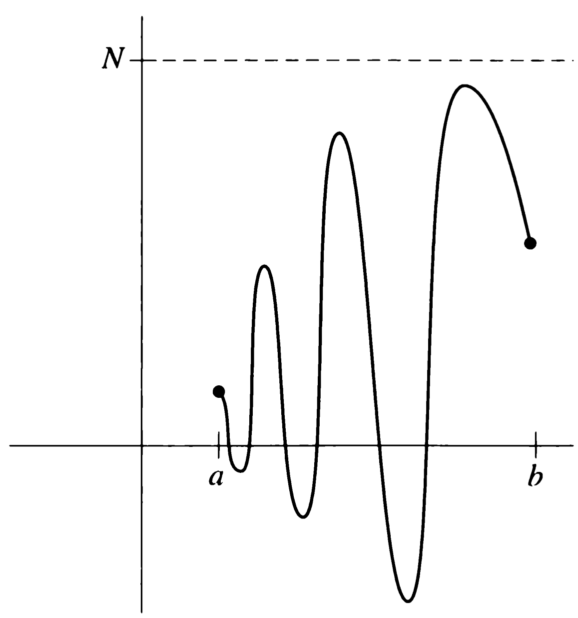

FIGURE 2

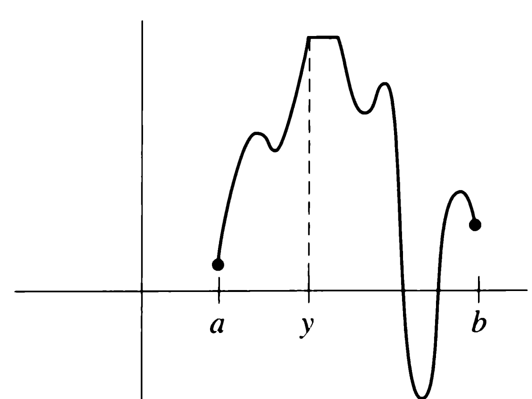

FIGURE 3

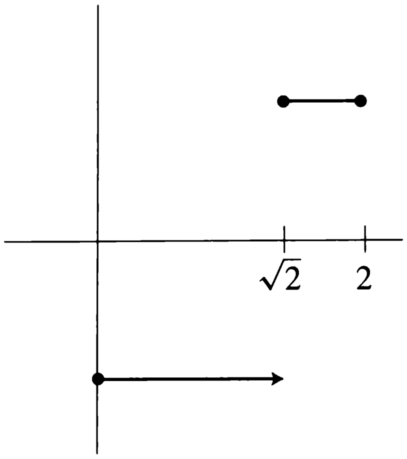

FIGURE 4

the hypotheses of the present theorems require continuity on a whole interval $[a, b]$—if continuity fails to hold at a single point, the conclusions may fail. For example, let f be the function shown in Figure 4,

$$f(x) = \begin{cases} -1, & 0 \le x < \sqrt{2} \\ 1, & \sqrt{2} \le x \le 2. \end{cases}$$

Then f is continuous at every point of $[0, 2]$ except $\sqrt{2}$, and $f(0) < 0 < f(2)$, but there is no point x in $[0, 2]$ such that $f(x) = 0$; the discontinuity at the single point $\sqrt{2}$ is sufficient to destroy the conclusion of Theorem 1.

Similarly, suppose that f is the function shown in Figure 5,

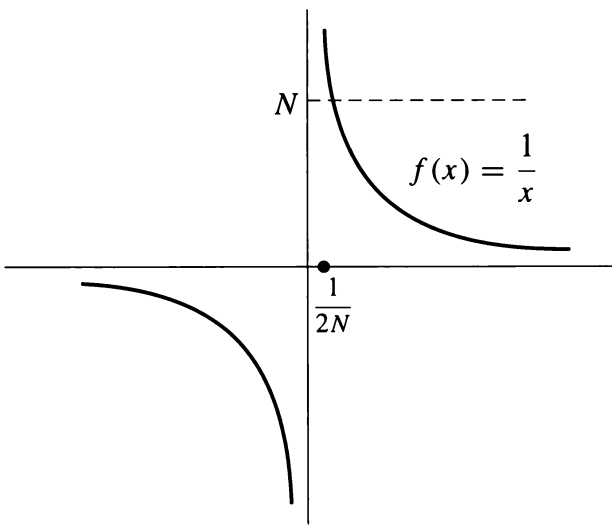

FIGURE 5

$$f(x) = \begin{cases} 1/x, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Then f is continuous at every point of $[0, 1]$ except 0, but f is not bounded above on $[0, 1]$. In fact, for any number $N > 0$ we have $f(1/2N) = 2N > N$.

This example also shows that the closed interval $[a, b]$ in Theorem 2 cannot be replaced by the open interval (a, b), for the function f is continuous on $(0, 1)$, but is not bounded there.

Finally, consider the function shown in Figure 6,

$$f(x) = \begin{cases} x^2, & x < 1 \\ 0, & x \ge 1. \end{cases}$$

On the interval $[0, 1]$ the function f is bounded above, so f does satisfy the conclusion of Theorem 2, even though f is not continuous on $[0, 1]$. But f does not satisfy the conclusion of Theorem 3—there is no y in $[0, 1]$ such that $f(y) \ge f(x)$ for all x in $[0, 1]$; in fact, it is certainly not true that $f(1) \ge f(x)$ for all x in $[0, 1]$ so we cannot choose $y = 1$, nor can we choose $0 \le y < 1$ because $f(y) < f(x)$ if x is any number with $y < x < 1$.

This example shows that Theorem 3 is considerably stronger than Theorem 2. Theorem 3 is often paraphrased by saying that a continuous function on a closed interval "takes on its maximum value" on that interval.

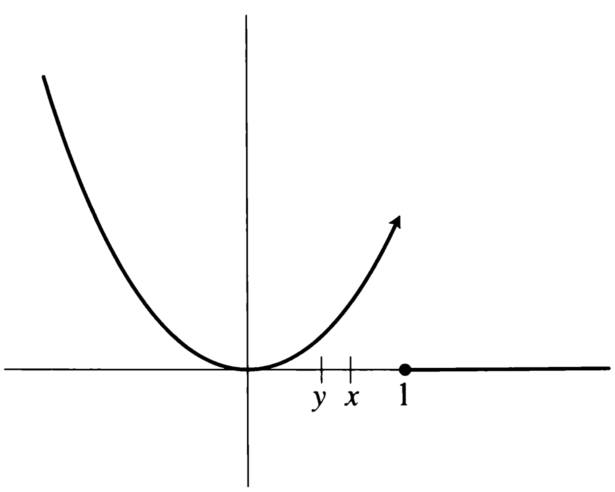

FIGURE 6

As a compensation for the stringency of the hypotheses of our three theorems, the conclusions are of a totally different order than those of previous theorems. They describe the behavior of a function, not just near a point, but on a whole interval; such "global" properties of a function are always significantly more difficult to prove than "local" properties, and are correspondingly of much greater power. To illustrate the usefulness of Theorems 1, 2, and 3, we will soon deduce some important consequences, but it will help to first mention some simple generalizations of these theorems.

THEOREM 4 If f is continuous on $[a, b]$ and $f(a) < c < f(b)$, then there is some x in $[a, b]$ such that $f(x) = c$.

PROOF Let $g = f - c$. Then g is continuous, and $g(a) < 0 < g(b)$. By Theorem 1, there is some x in $[a, b]$ such that $g(x) = 0$. But this means that $f(x) = c$. ▮

THEOREM 5 If f is continuous on $[a, b]$ and $f(a) > c > f(b)$, then there is some x in $[a, b]$ such that $f(x) = c$.

PROOF The function $-f$ is continuous on $[a, b]$ and $-f(a) < -c < -f(b)$. By Theorem 4 there is some x in $[a, b]$ such that $-f(x) = -c$, which means that $f(x) = c$. ▮

Theorems 4 and 5 together show that f takes on any value between $f(a)$ and $f(b)$. We can do even better than this: if c and d are in $[a, b]$, then f takes on any value between $f(c)$ and $f(d)$. The proof is simple: if, for example, $c < d$, then just apply Theorems 4 and 5 to the interval $[c, d]$. Summarizing, if a continuous function on an interval takes on two values, it takes on every value in between; this slight generalization of Theorem 1 is often called the Intermediate Value Theorem.

THEOREM 6 If f is continuous on $[a, b]$, then f is bounded below on $[a, b]$, that is, there is some number N such that $f(x) \geq N$ for all x in $[a, b]$.

PROOF The function $-f$ is continuous on $[a, b]$, so by Theorem 2 there is a number M such that $-f(x) \leq M$ for all x in $[a, b]$. But this means that $f(x) \geq -M$ for all x in $[a, b]$, so we can let $N = -M$. ▮

Theorems 2 and 6 together show that a continuous function f on $[a, b]$ is bounded on $[a, b]$, that is, there is a number N such that $|f(x)| \leq N$ for all x in $[a, b]$. In fact, since Theorem 2 ensures the existence of a number N_1 such that $f(x) \leq N_1$ for all x in $[a, b]$, and Theorem 6 ensures the existence of a number N_2 such that $f(x) \geq N_2$ for all x in $[a, b]$, we can take $N = \max(|N_1|, |N_2|)$.

THEOREM 7 If f is continuous on $[a, b]$, then there is some y in $[a, b]$ such that $f(y) \leq f(x)$ for all x in $[a, b]$.
(A continuous function on a closed interval takes on its minimum value on that interval.)

PROOF The function $-f$ is continuous on $[a, b]$; by Theorem 3 there is some y in $[a, b]$ such that $-f(y) \geq -f(x)$ for all x in $[a, b]$, which means that $f(y) \leq f(x)$ for all x in $[a, b]$. ▮

Now that we have derived the trivial consequences of Theorems 1, 2, and 3, we can begin proving a few interesting things.

THEOREM 8 Every positive number has a square root. In other words, if $\alpha > 0$, then there is some number x such that $x^2 = \alpha$.

PROOF Consider the function $f(x) = x^2$, which is certainly continuous. Notice that the statement of the theorem can be expressed in terms of f: "the number α has a square root" means that f takes on the value α. The proof of this fact about f will be an easy consequence of Theorem 4.

There is obviously a number $b > 0$ such that $f(b) > \alpha$ (as illustrated in Figure 7); in fact, if $\alpha > 1$ we can take $b = \alpha$, while if $\alpha < 1$ we can take $b = 1$. Since $f(0) < \alpha < f(b)$, Theorem 4 applied to $[0, b]$ implies that for some x (in $[0, b]$), we have $f(x) = \alpha$, i.e., $x^2 = \alpha$. ▮

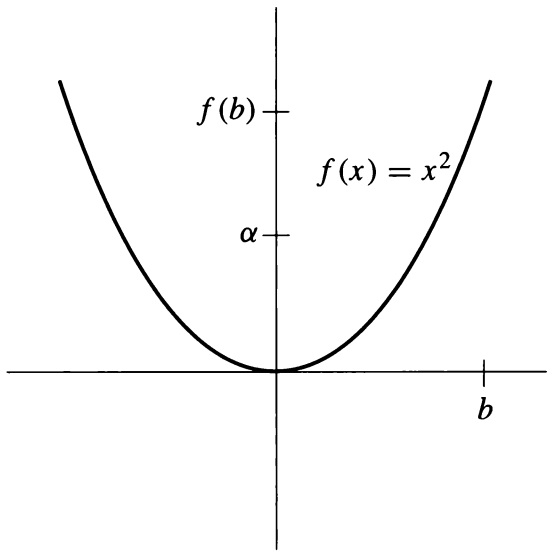

FIGURE 7

Precisely the same argument can be used to prove that a positive number has an nth root, for any natural number n. If n happens to be odd, one can do better: *every* number has an nth root. To prove this we just note that if the positive number α has the nth root x, i.e., if $x^n = \alpha$, then $(-x)^n = -\alpha$ (since n is odd), so $-\alpha$ has the nth root $-x$. The assertion, that for odd n any number α has an nth root, is equivalent to the statement that the equation

$$x^n - \alpha = 0$$

has a root if n is odd. Expressed in this way the result is susceptible of great generalization.

THEOREM 9 If n is odd, then any equation

$$x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$$

has a root.

PROOF We obviously want to consider the function

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0;$$

we would like to prove that f is sometimes positive and sometimes negative. The intuitive idea is that for large $|x|$, the function is very much like $g(x) = x^n$ and, since n is odd, this function is positive for large positive x and negative for large negative x. A little algebra is all we need to make this intuitive idea work.

The proper analysis of the function f depends on writing

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 = x^n\left(1 + \frac{a_{n-1}}{x} + \cdots + \frac{a_0}{x^n}\right).$$

Note that

$$\left|\frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + \cdots + \frac{a_0}{x^n}\right| \le \frac{|a_{n-1}|}{|x|} + \cdots + \frac{|a_0|}{|x^n|}.$$

Consequently, if we choose x satisfying

$$(*) \qquad |x| > 1, 2n|a_{n-1}|, \ldots, 2n|a_0|,$$

then $|x^k| > |x|$ and

$$\frac{|a_{n-k}|}{|x^k|} < \frac{|a_{n-k}|}{|x|} < \frac{|a_{n-k}|}{2n|a_{n-k}|} = \frac{1}{2n},$$

so

$$\left|\frac{a_{n-1}}{x}+\frac{a_{n-2}}{x^2}+\cdots+\frac{a_0}{x^n}\right| \le \underbrace{\frac{1}{2n}+\cdots+\frac{1}{2n}}_{n \text{ terms}} = \frac{1}{2}.$$

In other words,

$$-\frac{1}{2} \le \frac{a_{n-1}}{x}+\cdots+\frac{a_0}{x^n} \le \frac{1}{2},$$

which implies that

$$\frac{1}{2} \le 1+\frac{a_{n-1}}{x}+\cdots+\frac{a_0}{x^n}.$$

Therefore, if we choose an $x_1 > 0$ which satisfies $(*)$, then

$$\frac{(x_1)^n}{2} \le (x_1)^n\left(1+\frac{a_{n-1}}{x_1}+\cdots+\frac{a_0}{(x_1)^n}\right) = f(x_1),$$

so that $f(x_1) > 0$. On the other hand, if $x_2 < 0$ satisfies $(*)$, then $(x_2)^n < 0$ and

$$\frac{(x_2)^n}{2} \ge (x_2)^n\left(1+\frac{a_{n-1}}{x_2}+\cdots+\frac{a_0}{(x_2)^n}\right) = f(x_2),$$

so that $f(x_2) < 0$.

Now applying Theorem 1 to the interval $[x_2, x_1]$ we conclude that there is an x in $[x_2, x_1]$ such that $f(x) = 0$. ▮

Theorem 9 disposes of the problem of odd degree equations so happily that it would be frustrating to leave the problem of even degree equations completely undiscussed. At first sight, however, the problem seems insuperable. Some equations, like $x^2 - 1 = 0$, have a solution, and some, like $x^2 + 1 = 0$, do not—what more is there to say? If we are willing to consider a more general question, however, something interesting *can* be said. Instead of trying to solve the equation

$$x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0,$$

let us ask about the possibility of solving the equations

$$x^n + a_{n-1}x^{n-1} + \cdots + a_0 = c$$

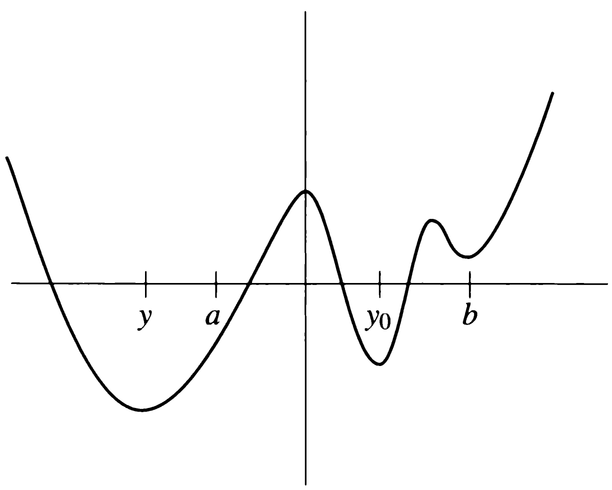

FIGURE 8

for all possible numbers c. This amounts to allowing the constant term a_0 to vary. The information which can be given concerning the solution of these equations depends on a fact which is illustrated in Figure 8.

The graph of the function $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$, with n even, contains, at least the way we have drawn it, a lowest point. In other words, there is a number y such that $f(y) \le f(x)$ for all numbers x—the function f takes on a minimum value, not just on each closed interval, but on the whole line. (Notice that this is false if n is odd.) The proof depends on Theorem 7, but a tricky application will be required. We can apply Theorem 7 to any interval $[a, b]$, and obtain a point y_0 such that $f(y_0)$ is the minimum value of f on $[a, b]$; but if $[a, b]$ happens to be the interval shown in Figure 8, for example, then the point y_0 will not be the place where f has its minimum value for the whole line. In the next and suppose n is even. Then there is a number m such that $(*)$ has a solution for $c \geq m$ and has no solution for $c < m$.

PROOF Let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ (Figure 10).

According to Theorem 10 there is a number y such that $f(y) \leq f(x)$ for all x. Let $m = f(y)$. If $c < m$, then the equation $(*)$ obviously has no solution, since the left side always has a value $\geq m$. If $c = m$, then $(*)$ has y as a solution. Finally, suppose $c > m$. Let b be a number such that $b > y$ and $f(b) > c$. Then $f(y) = m < c < f(b)$. Consequently, by Theorem 4, there is some number x in $[y, b]$ such that $f(x) = c$, so x is a solution of $(*)$. ▌

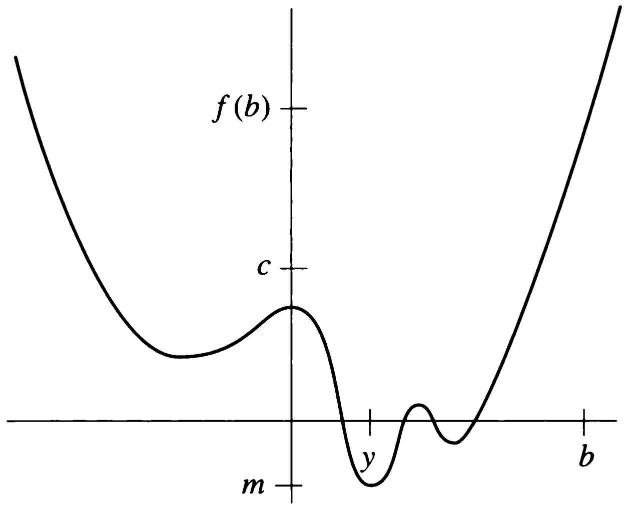

FIGURE 10

These consequences of Theorems 1, 2, and 3 are the only ones we will derive now (these theorems will play a fundamental role in everything we do later, however). Only one task remains—to prove Theorems 1, 2, and 3. Unfortunately, we cannot hope to do this—on the basis of our present knowledge about the real numbers (namely, P1–P12) a proof is *impossible.* There are several ways of convincing ourselves that this gloomy conclusion is actually the case. For example, the proof of Theorem 8 relies only on the proof of Theorem 1; if we could prove Theorem 1, then the proof of Theorem 8 would be complete, and we would have a proof that every positive number has a square root. As pointed out in Part I, it is impossible to prove this on the basis of P1–P12. Again, suppose we consider the function

$$f(x) = \frac{1}{x^2 - 2}.$$

If there were no number x with $x^2 = 2$, then f would be continuous, since the denominator would never $= 0$. But f is not bounded on $[0, 2]$. So Theorem 2 depends essentially on the existence of numbers other than rational numbers, and therefore on some property of the real numbers other than P1–P12.

Despite our inability to prove Theorems 1, 2, and 3, they are certainly results which we want to be true. If the pictures we have been drawing have any connection with the mathematics we are doing, if our notion of continuous function corresponds to any degree with our intuitive notion, Theorems 1, 2, and 3 have got to be true. Since a proof of any of these theorems must require some new property of **R** which has so far been overlooked, our present difficulties suggest a way to discover that property: let us try to construct a proof of Theorem 1, for example, and see what goes wrong.

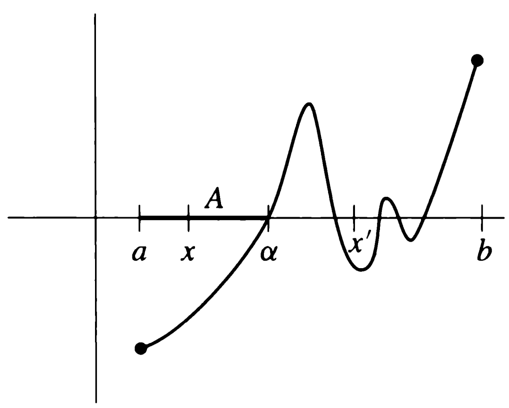

FIGURE 11

One idea which seems promising is to locate the first point where $f(x) = 0$, that is, the smallest x in $[a, b]$ such that $f(x) = 0$. To find this point, first consider the set A which contains all numbers x in $[a, b]$ such that f is negative on $[a, x]$. In Figure 11, x is such a point, while x' is not. The set A itself is indicated by a heavy line. Since f is negative at a, and positive at b, the set A contains some points greater than a, while all points sufficiently close to b are not in A. (We are here using the continuity of f on $[a, b]$, as well as Problem 6-16.)

theorem the entire point of the proof is to choose an interval $[a, b]$ in such a way that this cannot happen.

THEOREM 10 If n is even and $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$, then there is a number y such that $f(y) \le f(x)$ for all x.

PROOF As in the proof of Theorem 9, if

$$M = \max(1, 2n|a_{n-1}|, \dots, 2n|a_0|),$$

then for all x with $|x| \ge M$, we have

$$\frac{1}{2} \le 1 + \frac{a_{n-1}}{x} + \cdots + \frac{a_0}{x^n}.$$

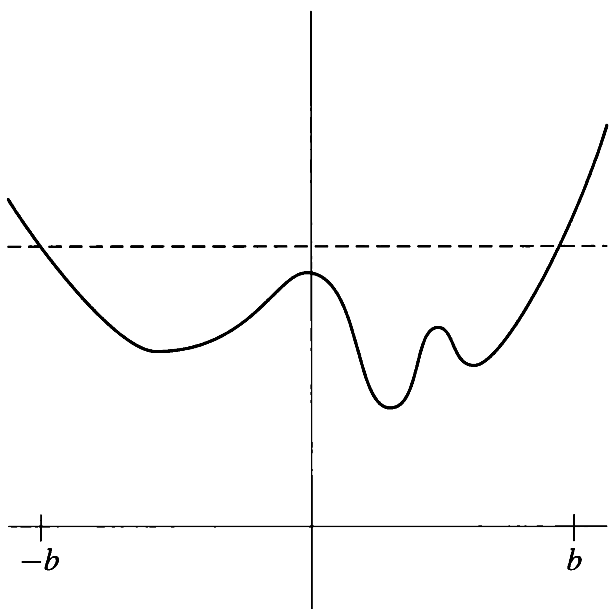

FIGURE 9

Since n is even, $x^n \ge 0$ for all x, so

$$\frac{x^n}{2} \le x^n\left(1 + \frac{a_{n-1}}{x} + \cdots + \frac{a_0}{x^n}\right) = f(x),$$

provided that $|x| \ge M$. Now consider the number $f(0)$. Let $b > 0$ be a number such that $b^n \ge 2f(0)$ and also $b > M$. Then, if $x \ge b$, we have (Figure 9)

$$f(x) \ge \frac{x^n}{2} \ge \frac{b^n}{2} \ge f(0).$$

Similarly, if $x \le -b$, then

$$f(x) \ge \frac{x^n}{2} \ge \frac{(-b)^n}{2} = \frac{b^n}{2} \ge f(0).$$

Summarizing:

$$\text{if } x \ge b \text{ or } x \le -b, \text{ then } f(x) \ge f(0).$$

Now apply Theorem 7 to the function f on the interval $[-b, b]$. We conclude that there is a number y such that

$$(1) \qquad \text{if } -b \le x \le b, \text{ then } f(y) \le f(x).$$

In particular, $f(y) \le f(0)$. Thus

$$(2) \qquad \text{if } x \le -b \text{ or } x \ge b, \text{ then } f(x) \ge f(0) \ge f(y).$$

Combining (1) and (2) we see that $f(y) \le f(x)$ for all x. ▮

Theorem 10 now allows us to prove the following result.

THEOREM 11 Consider the equation

$$(*) \qquad x^n + a_{n-1}x^{n-1} + \cdots + a_0 = c,$$

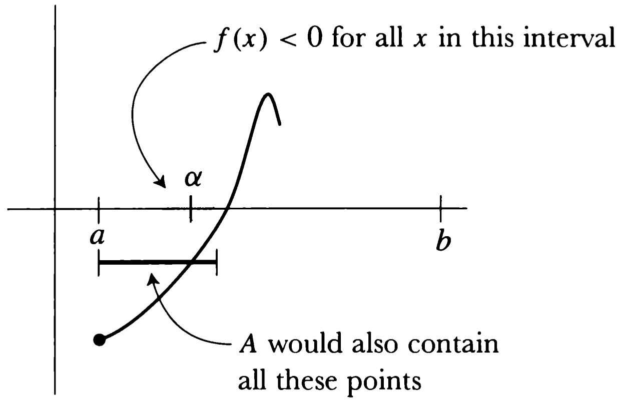

FIGURE 12

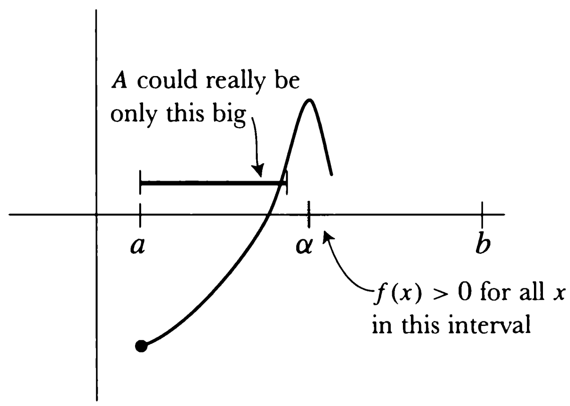

FIGURE 13

Now suppose α is the smallest number which is greater than all members of A; clearly $a < \alpha < b$. We claim that $f(\alpha) = 0$, and to prove this we only have to eliminate the possibilities $f(\alpha) < 0$ and $f(\alpha) > 0$.

Suppose first that $f(\alpha) < 0$. Then, by Theorem 6-3, $f(x)$ would be less than 0 for all x in a small interval containing α, in particular for some numbers bigger than α (Figure 12); but this contradicts the fact that α is bigger than every member of A, since the larger numbers would also be in A. Consequently, $f(\alpha) < 0$ is false.

On the other hand, suppose $f(\alpha) > 0$. Again applying Theorem 6-3, we see that $f(x)$ would be positive for all x in a small interval containing α, in particular for some numbers smaller than α (Figure 13). This means that these smaller numbers are all *not* in A. Consequently, one could have chosen an even smaller α which would be greater than all members of A. Once again we have a contradiction; $f(\alpha) > 0$ is also false. Hence $f(\alpha) = 0$ and, we are tempted to say, Q.E.D.

We know, however, that something must be wrong, since no new properties of $\mathbf{R}$ were ever used, and it does not require much scrutiny to find the dubious point. It is clear that we can choose a number α which is greater than all members of A (for example, we can choose $\alpha = b$), but it is not so clear that we can choose a *smallest* one. In fact, suppose A consists of all numbers $x \geq 0$ such that $x^2 < 2$. If the number $\sqrt{2}$ did not exist, there would not be a least number greater than all the members of A; for any $y > \sqrt{2}$ we chose, we could always choose a still smaller one.

Now that we have discovered the fallacy, it is almost obvious what additional property of the real numbers we need. All we must do is say it properly and use it. That is the business of the next chapter.

PROBLEMS

1. For each of the following functions, decide which are bounded above or below on the indicated interval, and which take on their maximum or minimum value. (Notice that f *might* have these properties even if f is not continuous, and even if the interval is not a closed interval.)

(i) $f(x) = x^2$ on $(-1, 1)$.

(ii) $f(x) = x^3$ on $(-1, 1)$.

(iii) $f(x) = x^2$ on $\mathbf{R}$.

(iv) $f(x) = x^2$ on $[0, \infty)$.

(v) $f(x) = \begin{cases} x^2, & x \leq a \\ a+2, & x > a \end{cases}$ on $(-a-1, a+1)$. (We assume $a > -1$, so that $-a-1 < a+1$; it will be necessary to consider several possibilities for a.)

(vi) $f(x) = \begin{cases} x^2, & x < a \\ a+2, & x \geq a \end{cases}$ on $[-a-1, a+1]$. (Again assume $a > -1$.)

(vii) $f(x) = \begin{cases} 0, & x \text{ irrational} \\ 1/q, & x = p/q \text{ in lowest terms} \end{cases}$ on $[0, 1]$.

(viii) $f(x) = \begin{cases} 1, & x \text{ irrational} \\ 1/q, & x = p/q \text{ in lowest terms} \end{cases}$ on $[0, 1]$.

(ix) $f(x) = \begin{cases} 1, & x \text{ irrational} \\ -1/q, & x = p/q \text{ in lowest terms} \end{cases}$ on $[0, 1]$.

(x) $f(x) = \begin{cases} x, & x \text{ rational} \\ 0, & x \text{ irrational} \end{cases}$ on $[0, a]$.

(xi) $f(x) = \sin^2(\cos x + \sqrt{a + a^2})$ on $[0, a^3]$.

(xii) $f(x) = [x]$ on $[0, a]$.

2. For each of the following polynomial functions f, find an integer n such that $f(x) = 0$ for some x between n and $n + 1$.

(i) $f(x) = x^3 - x + 3$.

(ii) $f(x) = x^5 + 5x^4 + 2x + 1$.

(iii) $f(x) = x^5 + x + 1$.

(iv) $f(x) = 4x^2 - 4x + 1$.

3. Prove that there is some number x such that

(i) $x^{179} + \dfrac{163}{1 + x^2 + \sin^2 x} = 119$.

(ii) $\sin x = x - 1$.

4. This problem is a continuation of Problem 3-7.

(a) If $n - k$ is even, and ≥ 0, find a polynomial function of degree n with exactly k roots.

(b) A root a of the polynomial function f is said to have **multiplicity** m if $f(x) = (x - a)^m g(x)$, where g is a polynomial function that does *not* have a as a root. Let f be a polynomial function of degree n. Suppose that f has k roots, counting multiplicities, i.e., suppose that k is the sum of the multiplicities of all the roots. Show that $n - k$ is even.

5. Suppose that f is continuous on $[a, b]$ and that $f(x)$ is always rational. What can be said about f?

6. Suppose that f is a *continuous* function on $[-1, 1]$ such that $x^2 + (f(x))^2 = 1$ for all x. (This means that $(x, f(x))$ always lies on the unit circle.) Show that either $f(x) = \sqrt{1 - x^2}$ for all x, or else $f(x) = -\sqrt{1 - x^2}$ for all x.

7. How many continuous functions f are there which satisfy $(f(x))^2 = x^2$ for all x?

8. Suppose that f and g are continuous, that $f^2 = g^2$, and that $f(x) \neq 0$ for all x. Prove that either $f(x) = g(x)$ for all x, or else $f(x) = -g(x)$ for all x.

9. (a) Suppose that f is continuous, that $f(x) = 0$ only for $x = a$, and that $f(x) > 0$ for some $x > a$ as well as for some $x < a$. What can be said about $f(x)$ for all $x \neq a$?

***18.** Suppose that f is a continuous function with $f(x) > 0$ for all x, and $\lim_{x\to\infty} f(x) = 0 = \lim_{x\to-\infty} f(x)$. (Draw a picture.) Prove that there is some number y such that $f(y) \geq f(x)$ for all x.

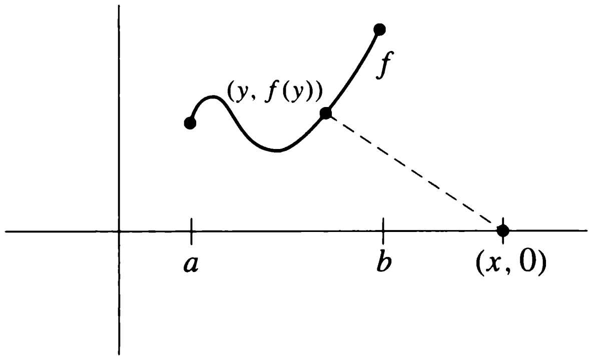

FIGURE 15

***19.** (a) Suppose that f is continuous on $[a, b]$, and let x be any number. Prove that there is a point on the graph of f which is closest to $(x, 0)$; in other words there is some y in $[a, b]$ such that the distance from $(x, 0)$ to $(y, f(y))$ is $\leq$ distance from $(x, 0)$ to $(z, f(z))$ for all z in $[a, b]$. (See Figure 15.)

(b) Show that this same assertion is not necessarily true if $[a, b]$ is replaced by (a, b) throughout.

(c) Show that the assertion *is* true if $[a, b]$ is replaced by $\mathbf{R}$ throughout.

(d) In cases (a) and (c), let $g(x)$ be the minimum distance from $(x, 0)$ to a point on the graph of f. Prove that $g(y) \leq g(x) + |x - y|$, and conclude that g is continuous.

(e) Prove that there are numbers x_0 and x_1 in $[a, b]$ such that the distance from $(x_0, 0)$ to $(x_1, f(x_1))$ is $\leq$ the distance from $(x_0', 0)$ to $(x_1', f(x_1'))$ for any x_0', x_1' in $[a, b]$.

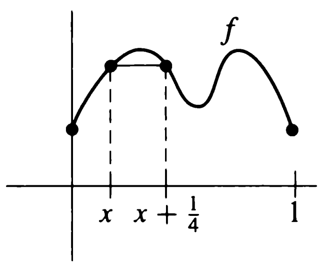

FIGURE 16

20. (a) Suppose that f is continuous on $[0, 1]$ and $f(0) = f(1)$. Let n be any natural number. Prove that there is some number x such that $f(x) = f(x + 1/n)$, as shown in Figure 16 for $n = 4$. Hint: Consider the function $g(x) = f(x) - f(x + 1/n)$; what would be true if $g(x) \neq 0$ for all x?

*(b) Suppose $0 < a < 1$, but that a is not equal to $1/n$ for any natural number n. Find a function f which is continuous on $[0, 1]$ and which satisfies $f(0) = f(1)$, but which does not satisfy $f(x) = f(x + a)$ for any x.

***21.** (a) Prove that there does not exist a continuous function f defined on $\mathbf{R}$ which takes on every value exactly twice. Hint: If $f(a) = f(b)$ for $a < b$, then either $f(x) > f(a)$ for all x in (a, b) or $f(x) < f(a)$ for all x in (a, b). Why? In the first case all values close to $f(a)$, but slightly larger than $f(a)$, are taken on somewhere in (a, b); this implies that $f(x) < f(a)$ for $x < a$ and $x > b$.

(b) Refine part (a) by proving that there is no continuous function f which takes on each value either 0 times or 2 times, i.e., which takes on exactly twice each value that it does take on. Hint: The previous hint implies that f has either a maximum or a minimum value (which must be taken on twice). What can be said about values close to the maximum value?

(c) Find a continuous function f which takes on every value exactly 3 times. More generally, find one which takes on every value exactly n times, if n is odd.

(d) Prove that if n is even, then there is no continuous f which takes on every value exactly n times. Hint: To treat the case $n = 4$, for example, let $f(x_1) = f(x_2) = f(x_3) = f(x_4)$. Then either $f(x) > 0$ for all x in two of the three intervals (x_1, x_2), (x_2, x_3), (x_3, x_4), or else $f(x) < 0$ for all x in two of these three intervals.

(b) Again assume that f is continuous and that $f(x) = 0$ only for $x = a$, but suppose, instead, that $f(x) > 0$ for some $x > a$ and $f(x) < 0$ for some $x < a$. Now what can be said about $f(x)$ for $x \neq a$?

*(c) Discuss the sign of $x^3 + x^2y + xy^2 + y^3$ when x and y are not both 0.

10. Suppose f and g are continuous on $[a, b]$ and that $f(a) < g(a)$, but $f(b) > g(b)$. Prove that $f(x) = g(x)$ for some x in $[a, b]$. (If your proof isn't very short, it's not the right one.)

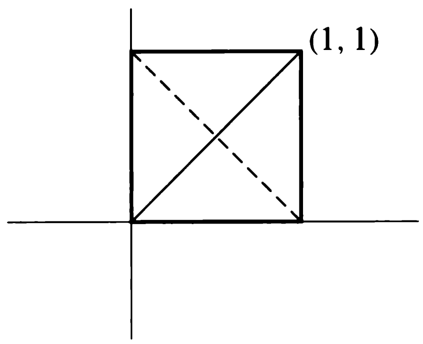

FIGURE 14

11. Suppose that f is a continuous function on $[0, 1]$ and that $f(x)$ is in $[0, 1]$ for each x (draw a picture). Prove that $f(x) = x$ for some number x.

12. (a) Problem 11 shows that f intersects the diagonal of the square in Figure 14 (solid line). Show that f must also intersect the other (dashed) diagonal.

(b) Prove the following more general fact: If g is continuous on $[0, 1]$ and $g(0) = 0$, $g(1) = 1$ or $g(0) = 1$, $g(1) = 0$, then $f(x) = g(x)$ for some x.

13. (a) Let $f(x) = \sin 1/x$ for $x \neq 0$ and let $f(0) = 0$. Is f continuous on $[-1, 1]$? Show that f satisfies the conclusion of the Intermediate Value Theorem on $[-1, 1]$; in other words, if f takes on two values somewhere on $[-1, 1]$, it also takes on every value in between.

*(b) Suppose that f satisfies the conclusion of the Intermediate Value Theorem, and that f takes on each value *only once*. Prove that f is continuous.

*(c) Generalize to the case where f takes on each value only finitely many times.

14. If f is a continuous function on $[0, 1]$, let $\|f\|$ be the maximum value of $|f|$ on $[0, 1]$.

(a) Prove that for any number c we have $\|cf\| = |c| \cdot \|f\|$.

*(b) Prove that $\|f + g\| \leq \|f\| + \|g\|$. Give an example where $\|f + g\| \neq \|f\| + \|g\|$.

(c) Prove that $\|h - f\| \leq \|h - g\| + \|g - f\|$.

***15.** Suppose that ϕ is continuous and $\lim_{x\to\infty} \phi(x)/x^n = 0 = \lim_{x\to-\infty} \phi(x)/x^n$.

(a) Prove that if n is odd, then there is a number x such that $x^n + \phi(x) = 0$.

(b) Prove that if n is even, then there is a number y such that $y^n + \phi(y) \leq x^n + \phi(x)$ for all x.

Hint: Of which proofs does this problem test your understanding?

***16.** (a) Suppose that f is continuous on (a, b) and $\lim_{x\to a^+} f(x) = \lim_{x\to b^-} f(x) = \infty$. Prove that f has a minimum on all of (a, b).

(b) Prove the corresponding result when $a = -\infty$ and/or $b = \infty$.

***17.** Let f be any polynomial function. Prove that there is some number y such that $|f(y)| \leq |f(x)|$ for all x.