28. There is another kind of "improper integral" in which the interval is bounded, but the *function* is unbounded:

(a) If $a > 0$, find $\lim_{\varepsilon \to 0^+} \int_\varepsilon^a 1/\sqrt{x}\,dx$. This limit is denoted by $\int_0^a 1/\sqrt{x}\,dx$, even though the function $f(x) = 1/\sqrt{x}$ is not bounded on $[0, a]$, no matter how we define $f(0)$.

(b) Find $\int_0^a x^r\,dx$ if $-1 < r < 0$.

(c) Use Problem 13-15 to show that $\int_0^a x^{-1}\,dx$ does not make sense, even as a limit.

(d) Invent a reasonable definition of $\int_a^0 |x|^r\,dx$ for $a < 0$ and compute it for $-1 < r < 0$.

(e) Invent a reasonable definition of $\int_{-1}^1 (1 - x^2)^{-1/2}\,dx$, as a sum of two limits, and show that the limits exist. Hint: Why does $\int_{-1}^0 (1 + x)^{-1/2}\,dx$ exist? How does $(1 + x)^{-1/2}$ compare with $(1 - x^2)^{-1/2}$ for $-1 < x < 0$?

29. (a) If f is continuous on $[0, 1]$, compute $\lim_{x \to 0^+} x \displaystyle\int_x^1 \frac{f(t)}{t}\,dt$.

*(b) If f is integrable on $[0, 1]$ and continuous at 0, compute

$$\lim_{x \to 0^+} x \int_x^1 \frac{f(t)}{t^2}\,dt.$$

30. It is possible, finally, to combine the two possible extensions of the notion of the integral.

(a) If $f(x) = 1/\sqrt{x}$ for $0 \le x \le 1$ and $f(x) = 1/x^2$ for $x \ge 1$, find $\displaystyle\int_0^\infty f(x)\,dx$ (after deciding what this should mean).

(b) Show that $\displaystyle\int_0^\infty x^r\,dx$ never makes sense. (Distinguish the cases $-1 < r < 0$ and $r < -1$. In one case things go wrong at 0, in the other case at ∞; for $r = -1$ things go wrong at both places.)

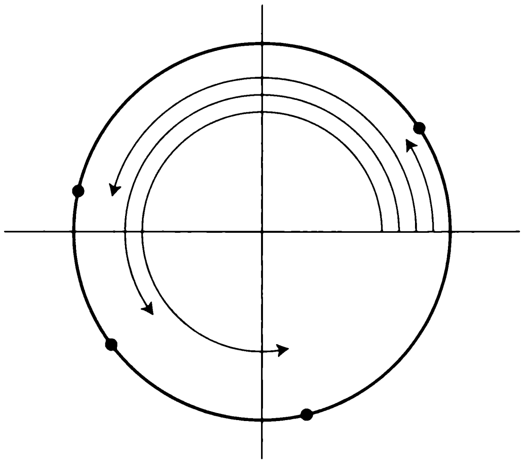

FIGURE 4

The sine and cosine of a directed angle can now be defined as follows (Figure 5): a directed angle is determined by a point (x, y) with $x^2 + y^2 = 1$; the sine of the angle is defined as y, and the cosine as x.

Despite the aura of precision surrounding the previous paragraph, we are not yet finished with the definitions of sin and cos. Indeed, we have barely begun. What we have defined is the sine and cosine of a directed angle; what we *want* to define is $\sin x$ and $\cos x$ for each *number* x. The usual procedure for doing this depends on associating an angle to every number. The oldest method is to "measure angles in degrees." An angle "all the way around" is associated to 360, an angle "half-way around" is associated to 180, an angle "a quarter way around" to 90, etc. (Figure 6). The angle associated, in this manner, to the number x, is called "the angle of x degrees." The angle of 0 degrees is the same as the angle of 360 degrees, and this ambiguity is purposely extended further, so that an angle of 90 degrees is also an angle of $360 + 90$ degrees, etc. One can now define a function, which we will denote by $\sin^\circ$, as follows:

$$\sin^\circ(x) = \text{sine of the angle of } x \text{ degrees.}$$

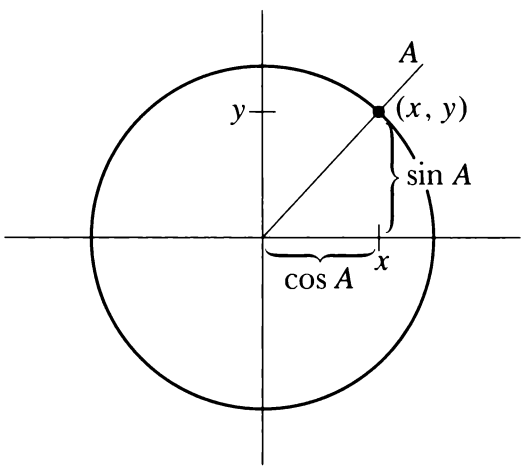

FIGURE 5

There are two difficulties with this approach. Although it may be clear what we mean by an angle of 90 or 45 degrees, it is not quite clear what an angle of $\sqrt{2}$ degrees is, for example. Even if this difficulty could be circumvented, it is unlikely that this system, depending as it does on the arbitrary choice of 360, will lead to elegant results—it would be sheer luck if the function $\sin^\circ$ had mathematically pleasing properties.

"Radian measure" appears to offer a remedy for both these defects. Given any number x, choose a point P on the unit circle such that x is the length of the arc of the circle beginning at $(1, 0)$ and running counterclockwise to P (Figure 7). The directed angle determined by P is called "the angle of x radians." Since the length of the whole circle is 2π, the angle of x radians and the angle of $2\pi + x$ radians are identical. A function $\sin^r$ can now be defined as follows:

$$\sin^r(x) = \text{sine of the angle of } x \text{ radians.}$$

This same method can easily be adopted to define $\sin^\circ$; since we want to have $\sin^\circ 360 = \sin^r 2\pi$, we can define

$$\sin^\circ x = \sin^r \frac{2\pi x}{360} = \sin^r \frac{\pi x}{180}.$$

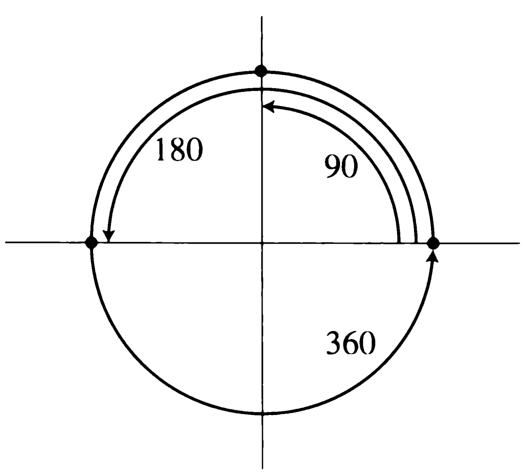

FIGURE 6

We shall soon drop the superscript r in $\sin^r$, since $\sin^r$ (and not $\sin^\circ$) is the only function which will interest us; before we do, a few words of warning are advisable.

The expressions $\sin^\circ x$ and $\sin^r x$ are sometimes written

$$\sin x^\circ$$
$$\sin x \text{ radians,}$$

but this notation is quite misleading; a number x is simply a number—it does not carry a banner indicating that it is "in degrees" or "in radians." If the meaning of the notation "sin x" is in doubt one usually asks:

"Is x in degrees or radians?"

but what one means is:

"Do you mean 'sin°' or 'sin^r'?"

Even for mathematicians, addicted to precision, these remarks might be dispensable, were it not for the fact that failure to take them into account will lead to incorrect answers to certain problems (an example is given in Problem 19).

Although the function $\sin^r$ is the function which we wish to denote simply by sin (and use exclusively henceforth), there is a difficulty involved even in the definition of $\sin^r$. Our proposed definition depends on the concept of the length of a curve. Although the length of a curve has been defined in several problems, it is also easy to reformulate the definition in terms of areas. (A treatment in terms of length is outlined in Problem 28.)

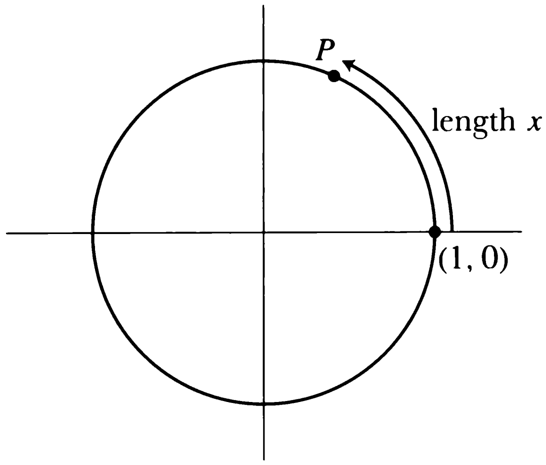

FIGURE 7

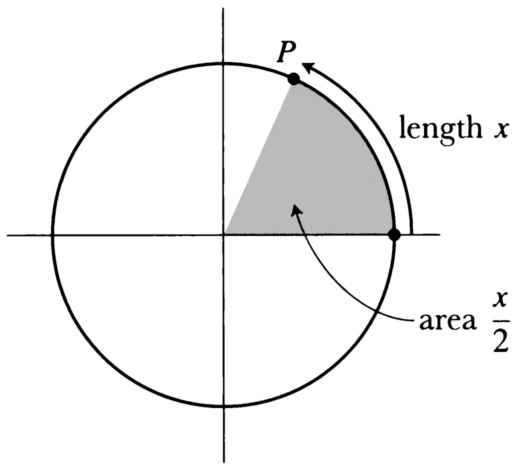

FIGURE 8

Suppose that x is the length of the arc of the unit circle from $(1, 0)$ to P; this arc thus contains $x/2\pi$ of the total length 2π of the circumference of the unit circle. Let S denote the "sector" shown in Figure 8; S is bounded by the unit circle, the horizontal axis, and the half-line through $(0, 0)$ and P. The area of S should be $x/2\pi$ times the area inside the unit circle, which we expect to be π; thus S should have area

$$\frac{x}{2\pi}\cdot\pi = \frac{x}{2}.$$

We can therefore define $\cos x$ and $\sin x$ as the coordinates of the point P which determines a sector of area $x/2$.

With these remarks as background, the rigorous definition of the functions sin and cos now begins. The first definition identifies π as the area of the unit circle—more precisely, as twice the area of a semicircle (Figure 9).

DEFINITION

$$\boxed{\pi = 2\cdot\int_{-1}^{1}\sqrt{1-x^2}\,dx.}$$

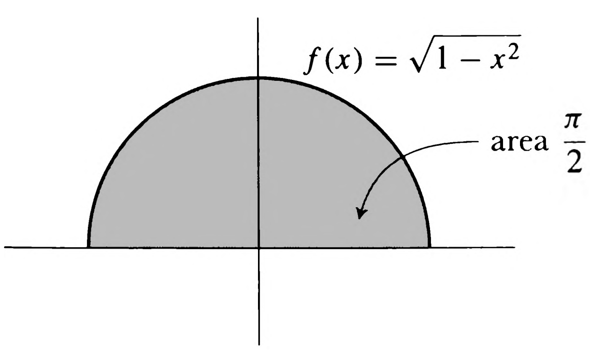

FIGURE 9

(This definition is not offered simply as an embellishment; to define the trigonometric functions it will be necessary to first define $\sin x$ and $\cos x$ only for $0 \le x \le \pi$.)

The second definition is meant to describe, for $-1 \le x \le 1$, the area $A(x)$ of the sector bounded by the unit circle, the horizontal axis, and the half-line through $(x, \sqrt{1-x^2})$. If $0 \le x \le 1$, this area can be expressed (Figure 10) as the sum of the area of a triangle and the area of a region under the unit circle:

$$\frac{x\sqrt{1-x^2}}{2} + \int_x^1 \sqrt{1-t^2}\,dt.$$

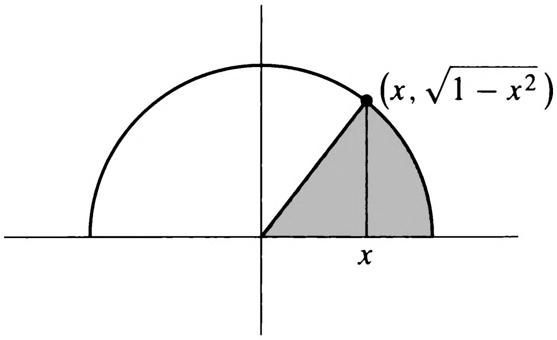

FIGURE 10

This same formula happens to work for $-1 \le x \le 0$ also. In this case (Figure 11), the term

$$\frac{x\sqrt{1-x^2}}{2}$$

is negative, and represents the area of the triangle which must be subtracted from the term

$$\int_x^1 \sqrt{1-t^2}\,dt.$$

DEFINITION

If $-1 \le x \le 1$, then

$$\boldsymbol{A(x)} = \frac{x\sqrt{1-x^2}}{2} + \int_x^1 \sqrt{1-t^2}\,dt.$$

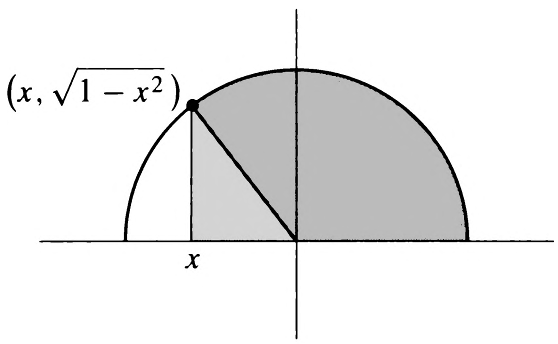

FIGURE 11

Notice that if $-1 < x < 1$, then A is differentiable at x and (using the Fundamental Theorem of Calculus),

$$\begin{aligned}
A'(x) &= \frac{1}{2}\left[x \cdot \frac{-2x}{2\sqrt{1-x^2}} + \sqrt{1-x^2}\right] - \sqrt{1-x^2} \\
&= \frac{1}{2}\left[\frac{-x^2 + (1-x^2)}{\sqrt{1-x^2}}\right] - \sqrt{1-x^2} \\
&= \frac{1-2x^2}{2\sqrt{1-x^2}} - \sqrt{1-x^2} \\
&= \frac{1-2x^2-2(1-x^2)}{2\sqrt{1-x^2}} \\
&= \frac{-1}{2\sqrt{1-x^2}}.
\end{aligned}$$

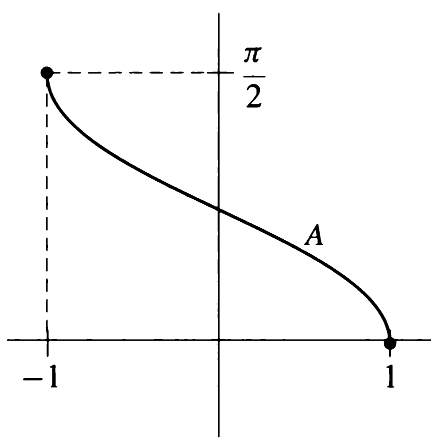

FIGURE 12

Notice also (Figure 12) that on the interval $[-1, 1]$ the function A decreases from

$$A(-1) = 0 + \int_{-1}^1 \sqrt{1-t^2}\,dt = \frac{\pi}{2}$$

to $A(1) = 0$. This follows directly from the definition of A, and also from the fact that its derivative is negative on $(-1, 1)$.

For $0 \le x \le \pi$ we wish to define $\cos x$ and $\sin x$ as the coordinates of a point $P = (\cos x, \sin x)$ on the unit circle which determines a sector whose area is $x/2$ (Figure 13). In other words:

DEFINITION

If $0 \le x \le \pi$, then $\mathbf{cos}\ \boldsymbol{x}$ is the unique number in $[-1, 1]$ such that

$$A(\cos x) = \frac{x}{2};$$

and

$$\mathbf{sin}\ \boldsymbol{x} = \sqrt{1-(\cos x)^2}.$$

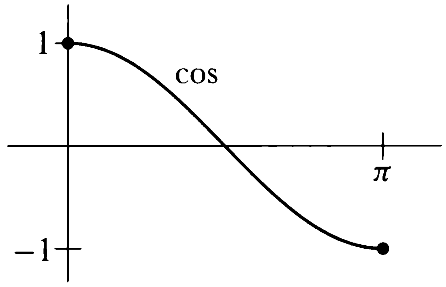

FIGURE 14

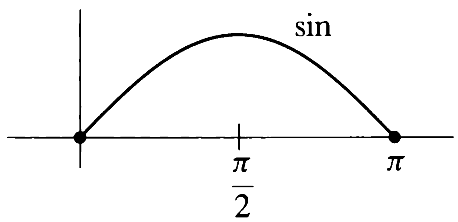

FIGURE 15

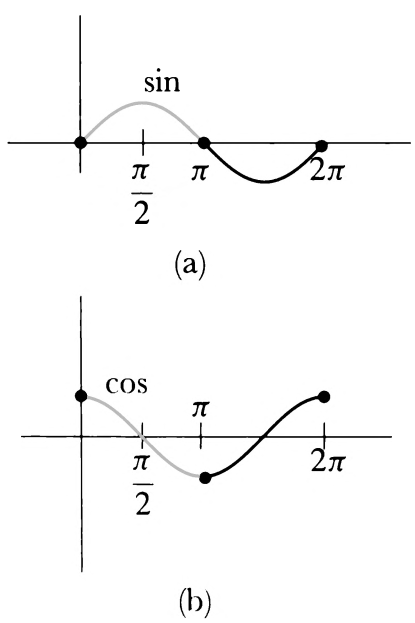

FIGURE 16

sin and cos on the interval $[0, \pi]$. Since

$$\cos'(x) = -\sin x < 0, \quad 0 < x < \pi,$$

the function cos decreases from $\cos 0 = 1$ to $\cos\pi = -1$ (Figure 14). Consequently, $\cos y = 0$ for a unique y in $[0, \pi]$. To find y, we note that the definition of cos,

$$A(\cos x) = \frac{x}{2},$$

means that

$$A(0) = \frac{y}{2},$$

so

$$y = 2\int_0^1 \sqrt{1 - t^2}\, dt.$$

It is easy to see that

$$\int_{-1}^0 \sqrt{1 - t^2}\, dt = \int_0^1 \sqrt{1 - t^2}\, dt$$

so we can also write

$$y = \int_{-1}^1 \sqrt{1 - t^2}\, dt = \frac{\pi}{2}.$$

Now we have

$$\sin'(x) = \cos x \begin{cases} > 0, & 0 < x < \pi/2 \\ < 0, & \pi/2 < x < \pi, \end{cases}$$

so sin increases on $[0, \pi/2]$ from $\sin 0 = 0$ to $\sin\pi/2 = 1$, and then decreases on $[\pi/2, \pi]$ to $\sin\pi = 0$ (Figure 15).

The values of $\sin x$ and $\cos x$ for x not in $[0, \pi]$ are most easily defined by a two-step piecing together process:

(1) If $\pi \le x \le 2\pi$, then

$$\sin x = -\sin(2\pi - x),$$
$$\cos x = \cos(2\pi - x).$$

Figure 16 shows the graphs of sin and cos on $[0, 2\pi]$.

(2) If $x = 2\pi k + x'$ for some integer k, and some x' in $[0, 2\pi]$, then

$$\sin x = \sin x',$$
$$\cos x = \cos x'.$$

Figure 17 shows the graphs of sin and cos, now defined on all of **R**.

Having extended the functions sin and cos to **R**, we must now check that the basic properties of these functions continue to hold. In most cases this is easy. For example, it is clear that the equation

$$\sin^2 x + \cos^2 x = 1$$

This definition actually requires a few words of justification. In order to know that there *is* a number y satisfying $A(y) = x/2$, we use the fact that A is continuous, and that A takes on the values 0 and $\pi/2$. This tacit appeal to the Intermediate Value Theorem is crucial, if we want to make our preliminary definition precise. Having made, and justified, our definition, we can now proceed quite rapidly.

THEOREM 1 If $0 < x < \pi$, then

$$\begin{aligned}\cos'(x) &= -\sin x,\\ \sin'(x) &= \cos x.\end{aligned}$$

PROOF If $B = 2A$, then the definition $A(\cos x) = x/2$ can be written

$$B(\cos x) = x;$$

in other words, cos is just the inverse of B. We have already computed that

$$A'(x) = -\frac{1}{2\sqrt{1-x^2}},$$

from which we conclude that

$$B'(x) = -\frac{1}{\sqrt{1-x^2}}.$$

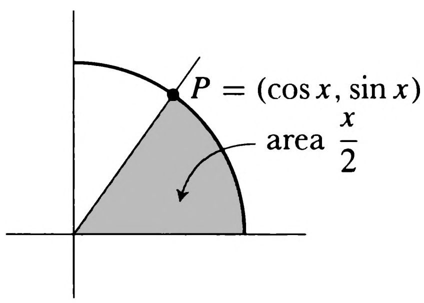

FIGURE 13

Consequently,

$$\begin{aligned}\cos'(x) &= (B^{-1})'(x)\\ &= \frac{1}{B'(B^{-1}(x))}\\ &= \frac{1}{-\dfrac{1}{\sqrt{1-[B^{-1}(x)]^2}}}\\ &= -\sqrt{1-(\cos x)^2}\\ &= -\sin x.\end{aligned}$$

Since

$$\sin x = \sqrt{1-(\cos x)^2},$$

we also obtain

$$\begin{aligned}\sin'(x) &= \frac{1}{2}\cdot\frac{-2\cos x\cdot\cos'(x)}{\sqrt{1-(\cos x)^2}}\\ &= \frac{\cos x \sin x}{\sin x}\\ &= \cos x. \ \blacksquare\end{aligned}$$

The information contained in Theorem 1 can be used to sketch the graphs of

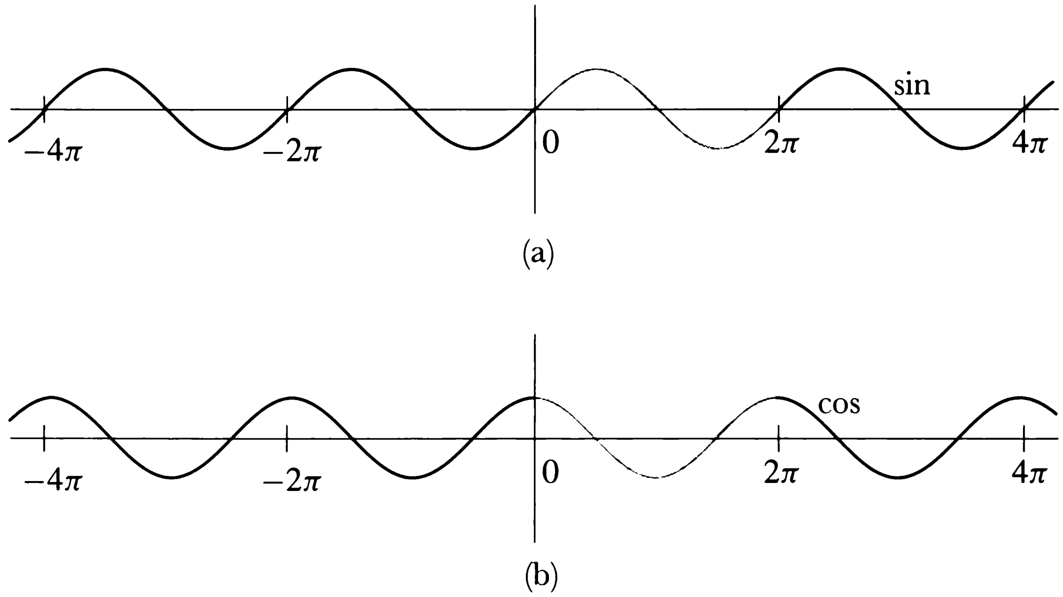

FIGURE 17

holds for all x. It is also not hard to prove that

$$\begin{aligned} \sin'(x) &= \cos x, \\ \cos'(x) &= -\sin x, \end{aligned}$$

if x is not a multiple of π. For example, if $\pi < x < 2\pi$, then

$$\sin x = -\sin(2\pi - x),$$

so

$$\begin{aligned} \sin'(x) &= -\sin'(2\pi - x) \cdot (-1) \\ &= \cos(2\pi - x) \\ &= \cos x. \end{aligned}$$

If x is a multiple of π we resort to a trick; it is only necessary to apply Theorem 11-7 to conclude that the same formulas are true in this case also.

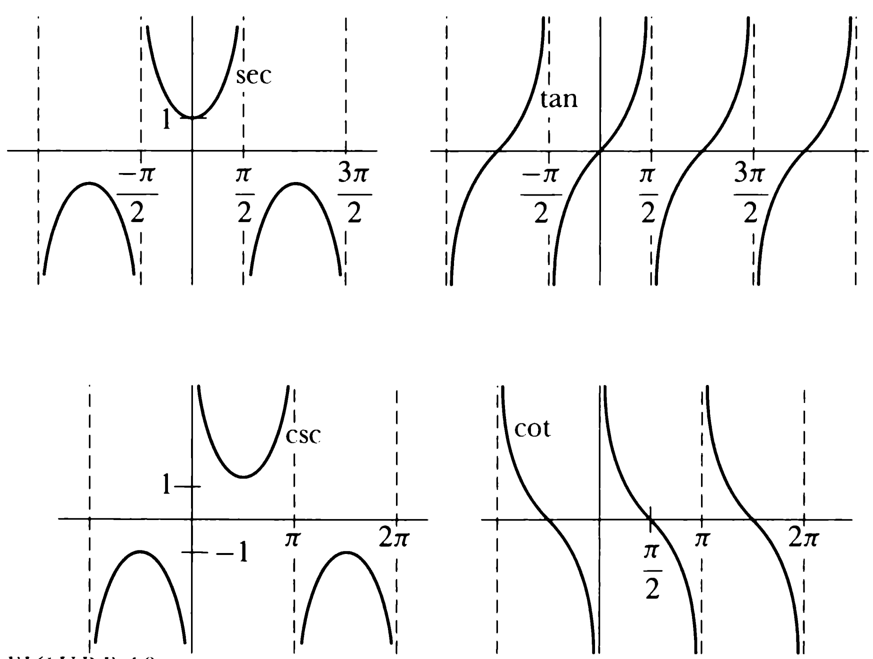

FIGURE 18

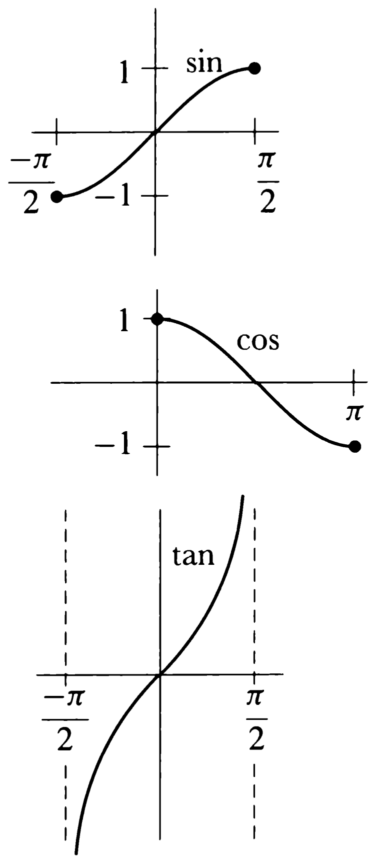

FIGURE 19

The other standard trigonometric functions present no difficulty at all. We define

$$\left.\begin{aligned} \sec x &= \frac{1}{\cos x} \\ \tan x &= \frac{\sin x}{\cos x} \end{aligned}\right\} \quad x \neq k\pi + \pi/2,$$

$$\left.\begin{aligned} \csc x &= \frac{1}{\sin x} \\ \cot x &= \frac{\cos x}{\sin x} \end{aligned}\right\} \quad x \neq k\pi.$$

The graphs are sketched in Figure 18. It is a good idea to convince yourself that the general features of these graphs can be predicted from the derivatives of these functions, which are listed in the next theorem (there is no need to memorize the statement of the theorem, since the results can be rederived whenever needed.)

THEOREM 2 If $x \neq k\pi + \pi/2$, then

$$\begin{aligned} \sec'(x) &= \sec x \tan x, \\ \tan'(x) &= \sec^2 x. \end{aligned}$$

If $x \neq k\pi$, then

$$\begin{aligned} \csc'(x) &= -\csc x \cot x, \\ \cot'(x) &= -\csc^2 x. \end{aligned}$$

PROOF Left to you (a straightforward computation). ▮

The inverses of the trigonometric functions are also easily differentiated. The trigonometric functions are not one-one, so it is first necessary to restrict them to suitable intervals; the largest possible length obtainable is π, and the intervals usually chosen are (Figure 19)

$$\begin{aligned} &[-\pi/2, \pi/2] && \text{for sin,} \\ &[0, \pi] && \text{for cos,} \\ &(-\pi/2, \pi/2) && \text{for tan.} \end{aligned}$$

(The inverses of the other trigonometric functions are so rarely used that they will not even be discussed here.)

The inverse of the function

$$f(x) = \sin x, \qquad -\pi/2 \leq x \leq \pi/2$$

is denoted by **arcsin** (Figure 20); the domain of arcsin is $[-1, 1]$. The notation $\sin^{-1}$ has been avoided because arcsin is not the inverse of sin (which is not one-one), but of the restricted function f; sometimes this function f is denoted by Sin, and arcsin by Sin^{-1}.

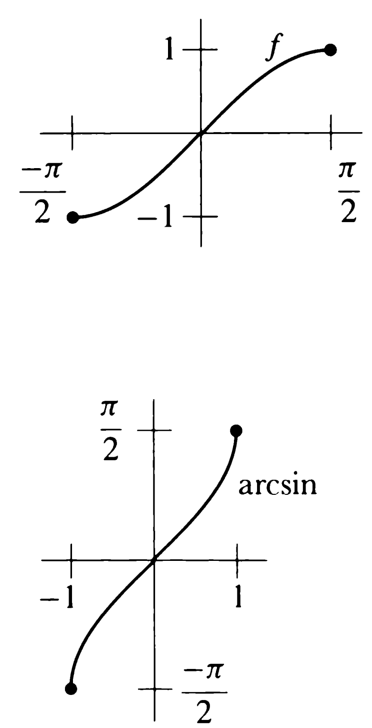

FIGURE 20

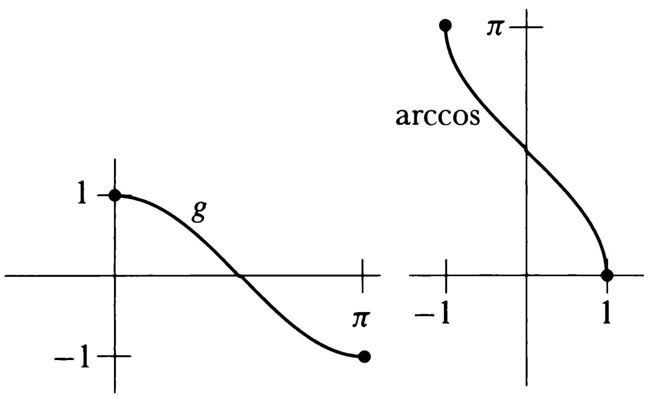

FIGURE 21

The inverse of the function

$$g(x) = \cos x, \qquad 0 \le x \le \pi$$

is denoted by **arccos** (Figure 21); the domain of arccos is $[-1, 1]$. Sometimes g is denoted by Cos, and arccos by Cos^{-1}.

The inverse of the function

$$h(x) = \tan x, \qquad -\pi/2 < x < \pi/2$$

is denoted by **arctan** (Figure 22); arctan is one of the simplest examples of a differentiable function which is bounded even though it is one-one on all of **R**. Sometimes the function h is denoted by Tan, and arctan by Tan^{-1}.

The derivatives of the inverse trigonometric functions are surprisingly simple, and do not involve trigonometric functions at all. Finding the derivatives is a simple matter, but to express them in a suitable form we will have to simplify expressions like

$$\cos(\arcsin x), \qquad \sec(\arctan x).$$

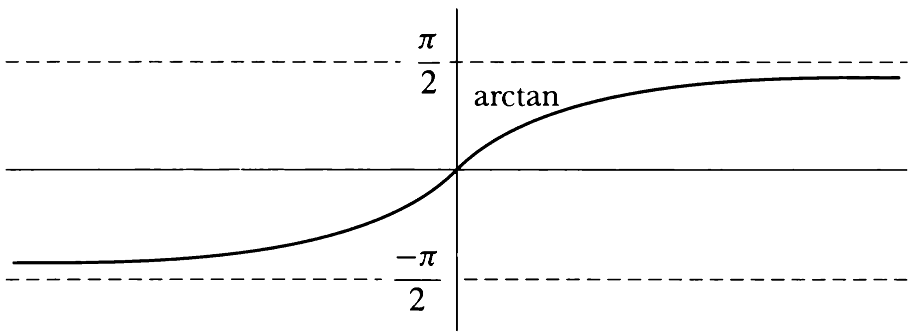

FIGURE 22

A little picture is the best way to remember the correct simplifications. For example, Figure 23 shows a directed angle whose sine is x—the angle shown is thus an angle of $(\arcsin x)$ radians; consequently $\cos(\arcsin x)$ is the length of the other side, namely, $\sqrt{1 - x^2}$. However, in the proof of the next theorem we will not resort to such pictures.

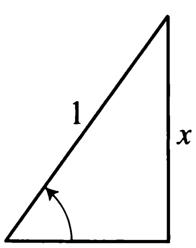

FIGURE 23

THEOREM 3 If $-1 < x < 1$, then

$$\arcsin'(x) = \frac{1}{\sqrt{1 - x^2}},$$
$$\arccos'(x) = \frac{-1}{\sqrt{1 - x^2}}.$$

Moreover, for all x we have

$$\arctan'(x) = \frac{1}{1 + x^2}.$$

the limit

$$\lim_{h \to 0} \frac{\sin h}{h} = 1,$$

and the "addition formula"

$$\sin(x + y) = \sin x \cos y + \cos x \sin y.$$

Both of these formulas can be derived easily now that the derivative of sin and cos are known. The first is just the special case $\sin'(0) = \cos 0$. The second depends on a beautiful characterization of the functions sin and cos. In order to derive this result we need a lemma whose proof involves a clever trick; a more straightforward proof will be supplied in Part IV.

LEMMA Suppose f has a second derivative everywhere and that

$$\begin{aligned} f'' + f &= 0, \\ f(0) &= 0, \\ f'(0) &= 0. \end{aligned}$$

Then $f = 0$.

PROOF Multiplying both sides of the first equation by f' yields

$$f'f'' + ff' = 0.$$

Thus

$$[(f')^2 + f^2]' = 2(f'f'' + ff') = 0,$$

so $(f')^2 + f^2$ is a constant function. From $f(0) = 0$ and $f'(0) = 0$ it follows that the constant is 0; thus

$$[f'(x)]^2 + [f(x)]^2 = 0 \quad \text{for all } x.$$

This implies that

$$f(x) = 0 \quad \text{for all } x. \blacksquare$$

THEOREM 4 If f has a second derivative everywhere and

$$\begin{aligned} f'' + f &= 0, \\ f(0) &= a, \\ f'(0) &= b, \end{aligned}$$

then

$$f = b \cdot \sin + a \cdot \cos.$$

(In particular, if $f(0) = 0$ and $f'(0) = 1$, then $f = \sin$; if $f(0) = 1$ and $f'(0) = 0$, then $f = \cos$.)

PROOF

$$\begin{aligned}\arcsin'(x) &= (f^{-1})'(x)\\ &= \frac{1}{f'(f^{-1}(x))}\\ &= \frac{1}{\sin'(\arcsin x)}\\ &= \frac{1}{\cos(\arcsin x)}.\end{aligned}$$

Now

$$[\sin(\arcsin x)]^2 + [\cos(\arcsin x)]^2 = 1,$$

that is,

$$x^2 + [\cos(\arcsin x)]^2 = 1;$$

therefore,

$$\cos(\arcsin x) = \sqrt{1 - x^2}.$$

(The positive square root is to be taken because $\arcsin x$ is in $(-\pi/2, \pi/2)$, so $\cos(\arcsin x) > 0$.) This proves the first formula.

The second formula has already been established (in the proof of Theorem 1). It is also possible to imitate the proof for the first formula, a valuable exercise if that proof presented any difficulties. The third formula is proved as follows.

$$\begin{aligned}\arctan'(x) &= (h^{-1})'(x)\\ &= \frac{1}{h'(h^{-1}(x))}\\ &= \frac{1}{\tan'(\arctan x)}\\ &= \frac{1}{\sec^2(\arctan x)}\end{aligned}$$

Dividing both sides of the identity

$$\sin^2 a + \cos^2 a = 1$$

by $\cos^2 a$ yields

$$\tan^2 a + 1 = \sec^2 a.$$

It follows that

$$[\tan(\arctan x)]^2 + 1 = \sec^2(\arctan x),$$

or

$$x^2 + 1 = \sec^2(\arctan x),$$

which proves the third formula. ▮

The traditional proof of the formula $\sin'(x) = \cos x$ (quite different from the one given here) is outlined in Problem 27. This proof depends upon first establishing it follows from the Second Fundamental Theorem of Calculus that

$$\arcsin x = \arcsin x - \arcsin 0 = \int_0^x \frac{1}{\sqrt{1-t^2}}\,dt.$$

This equation could have been taken as the *definition* of arcsin. It would follow immediately that

$$\arcsin'(x) = \frac{1}{\sqrt{1-x^2}};$$

the function sin could then be defined as $(\arcsin)^{-1}$ and the formula for the derivative of an inverse function would show that

$$\sin'(x) = \sqrt{1-\sin^2 x},$$

which could be defined as $\cos x$. Eventually, one could show that $A(\cos x) = x/2$, recovering at the very end of the development the definition with which we started. While much of this presentation would proceed more rapidly, the definition would be utterly unmotivated; the reasonableness of the definitions would be known to the author, but not to the student, for whom it was intended! Nevertheless, as we shall see in Chapter 18, an approach of this sort is sometimes very reasonable indeed.

PROBLEMS

1. Differentiate each of the following functions.

(i) $f(x) = \arctan(\arctan(\arctan x))$.
(ii) $f(x) = \arcsin(\arctan(\arccos x))$.
(iii) $f(x) = \arctan(\tan x \arctan x)$.
(iv) $f(x) = \arcsin\left(\dfrac{1}{\sqrt{1+x^2}}\right)$.

2. Find the following limits by l'Hôpital's Rule.

(i) $\displaystyle\lim_{x\to 0} \frac{\sin x - x + x^3/6}{x^3}$.

(ii) $\displaystyle\lim_{x\to 0} \frac{\sin x - x + x^3/6}{x^4}$.

(iii) $\displaystyle\lim_{x\to 0} \frac{\cos x - 1 + x^2/2}{x^2}$.

(iv) $\displaystyle\lim_{x\to 0} \frac{\cos x - 1 + x^2/2}{x^4}$.

(v) $\displaystyle\lim_{x\to 0} \frac{\arctan x - x + x^3/3}{x^3}$.

(vi) $\displaystyle\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{\sin x}\right)$.

PROOF Let

$$g(x) = f(x) - b\sin x - a\cos x.$$

Then

$$\begin{aligned} g'(x) &= f'(x) - b\cos x + a\sin x, \\ g''(x) &= f''(x) + b\sin x + a\cos x. \end{aligned}$$

Consequently,

$$\begin{aligned} g'' + g &= 0, \\ g(0) &= 0, \\ g'(0) &= 0, \end{aligned}$$

which shows that

$$0 = g(x) = f(x) - b\sin x - a\cos x, \qquad \text{for all } x. \blacksquare$$

THEOREM 5 If x and y are any two numbers, then

$$\begin{aligned} \sin(x+y) &= \sin x\cos y + \cos x\sin y, \\ \cos(x+y) &= \cos x\cos y - \sin x\sin y. \end{aligned}$$

PROOF For any particular number y we can define a function f by

$$f(x) = \sin(x+y).$$

Then

$$\begin{aligned} f'(x) &= \cos(x+y) \\ f''(x) &= -\sin(x+y). \end{aligned}$$

Consequently,

$$\begin{aligned} f'' + f &= 0, \\ f(0) &= \sin y, \\ f'(0) &= \cos y. \end{aligned}$$

It follows from Theorem 4 that

$$f = (\cos y)\cdot\sin + (\sin y)\cdot\cos;$$

that is,

$$\sin(x+y) = \cos y\sin x + \sin y\cos x, \qquad \text{for all } x.$$

Since any number y could have been chosen to begin with, this proves the first formula for all x and y.

The second formula is proved similarly. ▮

As a conclusion to this chapter, and as a prelude to Chapter 18, we will mention an alternative approach to the definition of the function sin. Since

$$\arcsin'(x) = \frac{1}{\sqrt{1-x^2}} \quad \text{for } -1 < x < 1,$$

3. Let $f(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0. \end{cases}$

(a) Find $f'(0)$.
(b) Find $f''(0)$.

At this point, you will almost certainly have to use l'Hôpital's Rule, but in Chapter 24 we will be able to find $f^{(k)}(0)$ for all k, with almost no work at all.

4. Graph the following functions.

(a) $f(x) = \sin 2x$.
(b) $f(x) = \sin(x^2)$. (A pretty respectable sketch of this graph can be obtained using only a picture of the graph of sin. Indeed, pure thought is your only hope in this problem, because determining the sign of the derivative $f'(x) = \cos(x^2) \cdot 2x$ is no easier than determining the behavior of f directly. The formula for $f'(x)$ does indicate one important fact, however—$f'(0) = 0$, which must be true since f is even, and which should be clear in your graph.)
(c) $f(x) = \sin x + \sin 2x$. (It will probably be instructive to first draw the graphs of $g(x) = \sin x$ and $h(x) = \sin 2x$ carefully on the same set of axes, from 0 to 2π, and guess what the sum will look like. You can easily find out how many critical points f has on $[0, 2\pi]$ by considering the derivative of f. You can then determine the nature of these critical points by finding out the sign of f at each point; your sketch will probably suggest the answer.)
(d) $f(x) = \tan x - x$. (First determine the behavior of f in $(-\pi/2, \pi/2)$; in the intervals $(k\pi - \pi/2, k\pi + \pi/2)$ the graph of f will look exactly the same, except moved up a certain amount. Why?)
(e) $f(x) = \sin x - x$. (The material in the Appendix to Chapter 11 will be particularly helpful for this function.)
(f) $f(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0. \end{cases}$
(Part (d) should enable you to determine approximately where the zeros of f' are located. Notice that f is even and continuous at 0; also consider the size of f for large x.)
(g) $f(x) = x \sin x$.

***5.** The *hyperbolic spiral* is the graph of the function $f(\theta) = a/\theta$ in polar coordinates (Chapter 4, Appendix 3). Sketch this curve, paying particular attention to its behavior for θ close to 0.

6. Prove the addition formula for cos.

7. (a) From the addition formula for sin and cos derive formulas for $\sin 2x$, $\cos 2x$, $\sin 3x$, and $\cos 3x$.

(b) Use these formulas to find the following values of the trigonometric functions (usually deduced by geometric arguments in elementary trigonometry):

$$\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2},$$
$$\tan\frac{\pi}{4} = 1,$$
$$\sin\frac{\pi}{6} = \frac{1}{2},$$
$$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}.$$

8. (a) Show that $A\sin(x+B)$ can be written as $a\sin x + b\cos x$ for suitable a and b. (One of the theorems in this chapter provides a one-line proof. You should also be able to figure out what a and b are.)

(b) Conversely, given a and b, find numbers A and B such that $a\sin x + b\cos x = A\sin(x+B)$ for all x.

(c) Use part (b) to graph $f(x) = \sqrt{3}\sin x + \cos x$.

9. (a) Prove that

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

provided that x, y, and $x+y$ are not of the form $k\pi + \pi/2$. (Use the addition formulas for sin and cos.)

(b) Prove that

$$\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right),$$

indicating any necessary restrictions on x and y. Hint: Replace x by $\arctan x$ and y by $\arctan y$ in part (a).

10. Prove that

$$\arcsin\alpha + \arcsin\beta = \arcsin\left(\alpha\sqrt{1-\beta^2} + \beta\sqrt{1-\alpha^2}\right),$$

indicating any restrictions on α and β.

11. Prove that if m and n are any numbers, then

$$\begin{aligned}
\sin mx \sin nx &= \tfrac{1}{2}[\cos(m-n)x - \cos(m+n)x],\\
\sin mx \cos nx &= \tfrac{1}{2}[\sin(m+n)x + \sin(m-n)x],\\
\cos mx \cos nx &= \tfrac{1}{2}[\cos(m+n)x + \cos(m-n)x].
\end{aligned}$$

12. Prove that if m and n are natural numbers, then

$$\int_{-\pi}^{\pi} \sin mx \sin nx\, dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n, \end{cases}$$

$$\int_{-\pi}^{\pi} \cos mx \cos nx\, dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n, \end{cases}$$

$$\int_{-\pi}^{\pi} \sin mx \cos nx\, dx = 0.$$

These relations are particularly important in the theory of Fourier series. Although this topic will receive serious attention only in the Suggested Reading (see reference [26]), the next problem provides a hint as to their importance.

13. (a) If f is integrable on $[-\pi, \pi]$, show that the minimum value of

$$\int_{-\pi}^{\pi} (f(x) - a\cos nx)^2\, dx$$

occurs when

$$a = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx,$$

and the minimum value of

$$\int_{-\pi}^{\pi} (f(x) - a\sin nx)^2\, dx$$

when

$$a = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx.$$

(In each case, bring a outside the integral sign, obtaining a quadratic expression in a.)

(b) Define

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx, \quad n = 0, 1, 2, \dots,$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx, \quad n = 1, 2, 3, \dots.$$

Show that if c_i and d_i are any numbers, then

$$\int_{-\pi}^{\pi} \left(f(x) - \left[\frac{c_0}{2} + \sum_{n=1}^{N} c_n\cos nx + d_n \sin nx\right]\right)^2 dx$$

$$= \int_{-\pi}^{\pi} [f(x)]^2\, dx - 2\pi\left(\frac{a_0c_0}{2} + \sum_{n=1}^{N} a_nc_n + b_nd_n\right) + \pi\left(\frac{{c_0}^2}{2} + \sum_{n=1}^{N} {c_n}^2 + {d_n}^2\right)$$

$$= \int_{-\pi}^{\pi} [f(x)]^2\, dx - \pi\left(\frac{{a_0}^2}{2} + \sum_{n=1}^{N} {a_n}^2 + {b_n}^2\right)$$

$$+ \pi\left(\left(\frac{c_0}{\sqrt{2}} - \frac{a_0}{\sqrt{2}}\right)^2 + \sum_{n=1}^{N} (c_n - a_n)^2 + (d_n - b_n)^2\right),$$

23. (a) Prove that π is the maximum possible length of an interval on which sin is one-one, and that such an interval must be of the form $[2k\pi - \pi/2, 2k\pi + \pi/2]$ or $[2k\pi + \pi/2, 2(k+1)\pi - \pi/2]$.

(b) Suppose we let $g(x) = \sin x$ for x in $(2k\pi - \pi/2, 2k\pi + \pi/2)$. What is $(g^{-1})'$?

24. Let $f(x) = \sec x$ for $0 \le x \le \pi$. Find the domain of f^{-1} and sketch its graph.

25. Prove that $|\sin x - \sin y| < |x - y|$ for all numbers $x \ne y$. Hint: The same statement, with $<$ replaced by $\le$, is a very straightforward consequence of a well-known theorem; simple supplementary considerations then allow $\le$ to be improved to $<$.

***26.** It is an excellent test of intuition to predict the value of

$$\lim_{\lambda\to\infty} \int_a^b f(x)\sin\lambda x\,dx.$$

Continuous functions should be most accessible to intuition, but once you get the right idea for a proof the limit can easily be established for any integrable f.

(a) Show that $\lim_{\lambda\to\infty} \int_c^d \sin\lambda x\,dx = 0$, by computing the integral explicitly.

(b) Show that if s is a step function on $[a, b]$ (terminology from Problem 13-26), then $\lim_{\lambda\to\infty} \int_a^b s(x)\sin\lambda x\,dx = 0$.

(c) Finally, use Problem 13-26 to show that $\lim_{\lambda\to\infty} \int_a^b f(x)\sin\lambda x\,dx = 0$ for any function f which is integrable on $[a, b]$. This result, like Problem 12, plays an important role in the theory of Fourier series; it is known as the Riemann-Lebesgue Lemma.

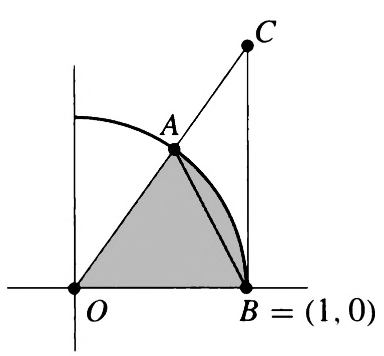

FIGURE 24

27. This problem outlines the classical approach to the trigonometric functions. The shaded sector in Figure 24 has area $x/2$.

(a) By considering the triangles OAB and OCB prove that if $0 < x < \pi/4$, then

$$\frac{\sin x}{2} < \frac{x}{2} < \frac{\sin x}{2\cos x}.$$

(b) Conclude that

$$\cos x < \frac{\sin x}{x} < 1,$$

and prove that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

(c) Use this limit to find

$$\lim_{x\to 0} \frac{1-\cos x}{x}.$$

thus showing that the first integral is smallest when $a_i = c_i$ and $b_i = d_i$. In other words, among all "linear combinations" of the functions $s_n(x) = \sin nx$ and $c_n(x) = \cos nx$ for $1 \le n \le N$, the particular function

$$g(x) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos nx + b_n \sin nx$$

has the "closest fit" to f on $[-\pi, \pi]$.

14. (a) Find a formula for $\sin x + \sin y$. (Notice that this also gives a formula for $\sin x - \sin y$.) Hint: First find a formula for $\sin(a+b) + \sin(a-b)$. What good does that do?
(b) Also find a formula for $\cos x + \cos y$ and $\cos x - \cos y$.

15. (a) Starting from the formula for $\cos 2x$, derive formulas for $\sin^2 x$ and $\cos^2 x$ in terms of $\cos 2x$.
(b) Prove that

$$\cos\frac{x}{2} = \sqrt{\frac{1+\cos x}{2}} \quad \text{and} \quad \sin\frac{x}{2} = \sqrt{\frac{1-\cos x}{2}}$$

for $0 \le x \le \pi/2$.
(c) Use part (a) to find $\int_a^b \sin^2 x\,dx$ and $\int_a^b \cos^2 x\,dx$.
(d) Graph $f(x) = \sin^2 x$.

16. Find $\sin(\arctan x)$ and $\cos(\arctan x)$ as expressions not involving trigonometric functions. Hint: $y = \arctan x$ means that $x = \tan y = \sin y/\cos y = \sin y/\sqrt{1-\sin^2 y}$.

17. If $x = \tan u/2$, express $\sin u$ and $\cos u$ in terms of x. (Use Problem 16; the answers should be very simple expressions.)

18. (a) Prove that $\sin(x + \pi/2) = \cos x$. (All along we have been drawing the graphs of sin and cos as if this were the case.)
(b) What is $\arcsin(\cos x)$ and $\arccos(\sin x)$?

19. (a) Find $\displaystyle\int_0^1 \frac{1}{1+t^2}\,dt$. Hint: The answer is not 45.
(b) Find $\displaystyle\int_0^\infty \frac{1}{1+t^2}\,dt$.

20. Find $\displaystyle\lim_{x\to\infty} x \sin\frac{1}{x}$.

21. (a) Define functions $\sin^\circ$ and $\cos^\circ$ by $\sin^\circ(x) = \sin(\pi x/180)$ and $\cos^\circ(x) = \cos(\pi x/180)$. Find $(\sin^\circ)'$ and $(\cos^\circ)'$ in terms of these same functions.
(b) Find $\displaystyle\lim_{x\to 0} \frac{\sin^\circ x}{x}$ and $\displaystyle\lim_{x\to\infty} x \sin^\circ \frac{1}{x}$.

22. Prove that every point on the unit circle is of the form $(\cos\theta, \sin\theta)$ for at least one (and hence for infinitely many) numbers θ.

(d) Using parts (b) and (c), and the addition formula for sin, find $\sin'(x)$, starting from the definition of the derivative.

***28.** This problem gives a treatment of the trigonometric functions in terms of length, and uses Problem 13-25. Let $f(x) = \sqrt{1-x^2}$ for $-1 \le x \le 1$. Define $\mathscr{L}(x)$ to be the length of f on $[x, 1]$.

(a) Show that

$$\mathscr{L}(x) = \int_x^1 \frac{1}{\sqrt{1-t^2}}\, dt.$$

(This is an improper integral, as defined in Problem 14-28, so you must first prove the corresponding assertion for the length on $[x, 1-\varepsilon]$ and then prove that $\mathscr{L}(x)$ is the limit of these lengths as $\varepsilon \to 0^+$.)

(b) Show that

$$\mathscr{L}'(x) = -\frac{1}{\sqrt{1-x^2}} \qquad \text{for } -1 < x < 1.$$

(c) Define π as $\mathscr{L}(-1)$. For $0 \le x \le \pi$, define $\cos x$ by $\mathscr{L}(\cos x) = x$, and define $\sin x = \sqrt{1-\cos^2 x}$. Prove that $\cos'(x) = -\sin x$ and $\sin'(x) = \cos x$ for $0 < x < \pi$.

***29.** Yet another development of the trigonometric functions was briefly mentioned in the text—starting with inverse functions defined by integrals. It is convenient to begin with arctan, since this function is defined for all x. To do this problem, pretend that you have never heard of the trigonometric functions.

(a) Let $\alpha(x) = \int_0^x (1+t^2)^{-1}\, dt$. Prove that α is odd and increasing, and that $\lim_{x\to\infty} \alpha(x)$ and $\lim_{x\to-\infty} \alpha(x)$ both exist, and are negatives of each other. If we define $\pi = 2 \lim_{x\to\infty} \alpha(x)$, then α^{-1} is defined on $(-\pi/2, \pi/2)$.

(b) Show that $(\alpha^{-1})'(x) = 1 + [\alpha^{-1}(x)]^2$.

(c) For $-\pi/2 < x < \pi/2$, define $\tan x = \alpha^{-1}(x)$, and then define $\sin x = \tan x/\sqrt{1+\tan^2 x}$. Show that

(i) $\displaystyle\lim_{x\to\pi/2^-} \sin x = 1$

(ii) $\displaystyle\lim_{x\to-\pi/2^+} \sin x = -1$

(iii) $\sin'(x) = \begin{cases} \dfrac{\sin x}{\tan x}, & -\pi/2 < x < \pi/2 \text{ and } x \ne 0 \\ 1, & x = 0 \end{cases}$

(iv) $\sin''(x) = -\sin x$ for $-\pi/2 < x < \pi 2$.

***30.** If we are willing to assume that certain differential equations have solutions, another approach to the trigonometric functions is possible. Suppose, in particular, that there is some function y_0 which is not always 0 and which satisfies $y_0'' + y_0 = 0$.

(a) Prove that $y_0{}^2 + (y_0{}')^2$ is constant, and conclude that either $y_0(0) \neq 0$ or $y_0{}'(0) \neq 0$.

(b) Prove that there is a function s satisfying $s'' + s = 0$ and $s(0) = 0$ and $s'(0) = 1$. Hint: Try s of the form $ay_0 + by_0{}'$.

If we define $\sin = s$ and $\cos = s'$, then almost all facts about trigonometric functions become trivial. There is one point which requires work, however—producing the number π. This is most easily done using an exercise from the Appendix to Chapter 11:

(c) Use Problem 6 of the Appendix to Chapter 11 to prove that $\cos x$ cannot be positive for all $x > 0$. It follows that there is a smallest $x_0 > 0$ with $\cos x_0 = 0$, and we can define $\pi = 2x_0$.

(d) Prove that $\sin \pi/2 = 1$. (Since $\sin^2 + \cos^2 = 1$, we have $\sin \pi/2 = \pm 1$; the problem is to decide why $\sin \pi/2$ is positive.)

(e) Find $\cos \pi$, $\sin \pi$, $\cos 2\pi$, and $\sin 2\pi$. (Naturally you may use any addition formulas, since these can be derived once we know that $\sin' = \cos$ and $\cos' = -\sin$.)

(f) Prove that cos and sin are periodic with period 2π.

31. (a) After all the work involved in the definition of sin, it would be disconcerting to find that sin is actually a rational function. Prove that it isn't. (There is a simple property of sin which a rational function cannot possibly have.)

(b) Prove that sin isn't even defined implicitly by an algebraic equation; that is, there do not exist rational functions $f_0, \dots, f_{n-1}$ such that

$$(\sin x)^n + f_{n-1}(x)(\sin x)^{n-1} + \cdots + f_0(x) = 0 \quad \text{for all } x.$$

Hint: Prove that $f_0 = 0$, so that $\sin x$ can be factored out. The remaining factor is 0 except perhaps at multiples of π. But this implies that it is 0 for all x. (Why?) You are now set up for a proof by induction.

***32.** Suppose that ϕ_1 and ϕ_2 satisfy

$$\begin{aligned} \phi_1{}'' + g_1\phi_1 &= 0, \\ \phi_2{}'' + g_2\phi_2 &= 0, \end{aligned}$$

and that $g_2 > g_1$.

(a) Show that

$$\phi_1{}''\phi_2 - \phi_2{}''\phi_1 - (g_2 - g_1)\phi_1\phi_2 = 0.$$

(b) Show that if $\phi_1(x) > 0$ and $\phi_2(x) > 0$ for all x in (a, b), then

$$\int_a^b [\phi_1{}''\phi_2 - \phi_2{}''\phi_1] > 0,$$

and conclude that

$$[\phi_1{}'(b)\phi_2(b) - \phi_1{}'(a)\phi_2(a)] - [\phi_1(b)\phi_2{}'(b) - \phi_1(a)\phi_2{}'(a)] > 0.$$

(c) Show that in this case we cannot have $\phi_1(a) = \phi_1(b) = 0$. Hint: Consider the sign of $\phi_1{}'(a)$ and $\phi_1{}'(b)$.

(d) Show that the equations $\phi_1(a) = \phi_1(b) = 0$ are also impossible if $\phi_1 > 0$, $\phi_2 < 0$ or $\phi_1 < 0$, $\phi_2 > 0$, or $\phi_1 < 0$, $\phi_2 < 0$ on (a, b). (You should be able to do this with almost no extra work.)

The net result of this problem may be stated as follows: if a and b are consecutive zeros of ϕ_1, then ϕ_2 must have a zero somewhere between a and b. This result, in a slightly more general form, is known as the Sturm Comparison Theorem. As a particular example, any solution of the differential equation

$$y'' + (x+1)y = 0$$

must have at least one zero in any interval $(n\pi, (n+1)\pi)$.

33. (a) Using the formula for $\sin x - \sin y$ derived in Problem 14, show that

$$\sin(k + \tfrac{1}{2})x - \sin(k - \tfrac{1}{2})x = 2\sin\frac{x}{2}\cos kx.$$

(b) Conclude that

$$\frac{1}{2} + \cos x + \cos 2x + \cdots + \cos nx = \frac{\sin(n+\tfrac{1}{2})x}{2\sin\dfrac{x}{2}}.$$

Like two other results in this problem set, this equation is very important in the study of Fourier series, and we also make use of it in Problems 19-43 and 23-22.

(c) Similarly, derive the formula

$$\sin x + \sin 2x + \cdots + \sin nx = \frac{\sin\left(\dfrac{n+1}{2}x\right)\sin\left(\dfrac{n}{2}x\right)}{\sin\dfrac{x}{2}}.$$

(A more natural derivation of these formulas will be given in Problem 27-14.)

(d) Use parts (b) and (c) to find $\int_0^b \sin x\,dx$ and $\int_0^b \cos x\,dx$ directly from the definition of the integral.