CHAPTER 19 INTEGRATION IN ELEMENTARY TERMS

Every computation of a derivative yields, according to the Second Fundamental Theorem of Calculus, a formula about integrals. For example,

$$\text{if } F(x) = x(\log x) - x \quad \text{then } F'(x) = \log x;$$

consequently,

$$\int_a^b \log x\, dx = F(b) - F(a) = b(\log b) - b - [a(\log a) - a], \quad 0 < a, b.$$

Formulas of this sort are simplified considerably if we adopt the notation

$$F(x)\Big|_a^b = F(b) - F(a).$$

We may then write

$$\int_a^b \log x\, dx = x(\log x) - x\Big|_a^b.$$

This evaluation of $\int_a^b \log x\, dx$ depended on the lucky guess that log is the derivative of the function $F(x) = x(\log x) - x$. In general, a function F satisfying $F' = f$ is called a **primitive** of f. Of course, **a continuous function f always has a primitive**, namely,

$$F(x) = \int_a^x f,$$

but in this chapter we will try to find a primitive which can be written in terms of familiar functions like sin, log, etc. A function which can be written in this way is called an elementary function. To be precise,* an **elementary function** is one which can be obtained by addition, multiplication, division, and composition from the rational functions, the trigonometric functions and their inverses, and the functions log and exp.

It should be stated at the very outset that elementary primitives usually cannot be found. For example, there is no *elementary* function F such that

$$F'(x) = e^{-x^2} \quad \text{for all } x$$

(this is not merely a report on the present state of mathematical ignorance; it is a (difficult) theorem that no such function exists). And, what is even worse, you

*The definition which we will give is precise, but not really accurate, or at least not quite standard. Usually the elementary functions are defined to include "algebraic" functions, that is, functions g satisfying an equation

$$(g(x))^n + f_{n-1}(x)(g(x))^{n-1} + \cdots + f_0(x) = 0,$$

where the f_i are rational functions. But for our purposes these functions can be ignored.

will have no way of knowing whether or not an elementary primitive *can* be found (you will just have to hope that the problems for this chapter contain no misprints). Because the search for elementary primitives is so uncertain, finding one is often peculiarly satisfying. If we observe that the function

$$F(x) = x \arctan x - \frac{\log(1+x^2)}{2}$$

satisfies

$$F'(x) = \arctan x$$

(just how we would ever be led to such an observation is quite another matter), so that

$$\int_a^b \arctan x \, dx = x \arctan x - \left.\frac{\log(1+x^2)}{2}\right|_a^b,$$

then we may feel that we have "really" evaluated $\int_a^b \arctan x \, dx$.

This chapter consists of little more than methods for finding elementary primitives of given elementary functions (a process known simply as "integration"), together with some notation, abbreviations, and conventions designed to facilitate this procedure. This preoccupation with elementary functions can be justified by three considerations:

(1) Integration is a standard topic in calculus, and everyone should know about it.
(2) Every once in a while you might actually need to evaluate an integral, under conditions which do not allow you to consult any of the standard integral tables (for example, you might take a (physics) course in which you are expected to be able to integrate).
(3) The most useful "methods" of integration are actually very important theorems (that apply to all functions, not just elementary ones).

Naturally, the last reason is the crucial one. Even if you intend to forget how to integrate (and you probably will forget some details the first time through), you must never forget the basic methods.

These basic methods are theorems which allow us to express primitives of one function in terms of primitives of other functions. To begin integrating we will therefore need a list of primitives for *some* functions; such a list can be obtained simply by differentiating various well-known functions. The list given below makes use of a standard symbol which requires some explanation. The symbol

$$\int f \quad \text{or} \quad \int f(x)\, dx$$

means "a primitive of f" or, more precisely, "the collection of all primitives of f." The symbol $\int f$ will often by used in stating theorems, while $\int f(x)\,dx$ is most useful in formulas like the following:

$$\int x^3 \, dx = \frac{x^4}{4}.$$

This "equation" means that the function $F(x) = x^4/4$ satisfies $F'(x) = x^3$. It cannot be interpreted literally because the right side is a number, not a function, but in this one context we will allow such discrepancies; our aim is to make the integration process as mechanical as possible, and we will resort to any possible device. Another feature of the equation deserves mention. Most people write

$$\int x^3\,dx = \frac{x^4}{4} + C$$

to emphasize that the primitives of $f(x) = x^3$ are precisely the functions of the form $F(x) = x^4/4 + C$ for some number C. Although it is possible (Problem 14) to obtain contradictions if this point is disregarded, in practice such difficulties do not arise, and concern for this constant is merely an annoyance.

There is one important convention accompanying this notation: the letter appearing on the right side of the equation should match with the letter appearing after the "d" on the left side—thus

$$\int u^3\,du = \frac{u^4}{4},$$

$$\int tx\,dx = \frac{tx^2}{2},$$

$$\int tx\,dt = \frac{xt^2}{2}.$$

A function in $\int f(x)\,dx$, i.e., a primitive of f, is often called an "indefinite integral" of f, while $\int_a^b f(x)\,dx$ is called, by way of contrast, a "definite integral." This suggestive notation works out quite well in practice, but it is important not to be led astray. At the risk of boring you, the following fact is emphasized once again: the integral $\int_a^b f(x)\,dx$ is *not* defined as "$F(b) - F(a)$, where F is an indefinite integral of f" (if you do not find this statement repetitious, it is time to reread Chapter 13).

We can verify the formulas in the following short table of indefinite integrals simply by differentiating the functions indicated on the right side.

$$\int a\,dx = ax$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1}, \quad n \neq -1$$

$$\int \frac{1}{x}\,dx = \log x$$ $\left(\int \frac{1}{x}\,dx\right.$ is often written $\int \frac{dx}{x}$ for convenience; similar abbreviations are used in the last two examples of this table.)

$$\int e^x\,dx = e^x$$

$$\int \sin x\,dx = -\cos x$$

$$\int \cos x\,dx = \sin x$$

$$\int \sec^2 x\,dx = \tan x$$

$$\int \sec x \tan x\,dx = \sec x$$

$$\int \frac{dx}{1+x^2} = \arctan x$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x$$

Two general formulas of the same nature are consequences of theorems about differentiation:

$$\int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx,$$

$$\int c \cdot f(x)\,dx = c \cdot \int f(x)\,dx.$$

These equations should be interpreted as meaning that a primitive of $f+g$ can be obtained by adding a primitive of f to a primitive of g, while a primitive of $c \cdot f$ can be obtained by multiplying a primitive of f by c.

Notice the consequences of these formulas for definite integrals: If f and g are continuous, then

$$\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx,$$

$$\int_a^b c \cdot f(x)\,dx = c \cdot \int_a^b f(x)\,dx.$$

These follow from the previous formulas, since each definite integral may be written as the difference of the values at a and b of a corresponding primitive. Continuity is required in order to know that these primitives exist. (Of course, the formulas are also true when f and g are merely integrable, but recall how much more difficult the proofs are in this case.)

The product formula for the derivative yields a more interesting theorem, which will be written in several different ways.

THEOREM 1 (INTEGRATION BY PARTS) If f' and g' are continuous, then

$$\int fg' = fg - \int f'g,$$

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx,$$

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x)\,dx.$$

(Notice that in the second equation $f(x)g(x)$ denotes the *function* $f \cdot g$.)

PROOF The formula

$$(fg)' = f'g + fg'$$

can be written

$$fg' = (fg)' - f'g.$$

Thus

$$\int fg' = \int (fg)' - \int f'g,$$

and fg can be chosen as one of the functions denoted by $\int (fg)'$. This proves the first formula.

The second formula is merely a restatement of the first, and the third formula follows immediately from either of the first two. ∎

As the following examples illustrate, integration by parts is useful when the function to be integrated can be considered as a product of a function f, whose derivative is simpler than f, and another function which is obviously of the form g'.

$$\int \underset{\substack{\downarrow\\ f}}{x}\,\underset{\substack{\downarrow\\ g'}}{e^x}\,dx = \underset{\substack{\downarrow\\ f}}{x}\,\underset{\substack{\downarrow\\ g}}{e^x} - \int \underset{\substack{\downarrow\\ f'}}{1}\cdot \underset{\substack{\downarrow\\ g}}{e^x}\,dx$$
$$= xe^x - e^x$$

$$\int \underset{\substack{\downarrow\\ f}}{x}\,\underset{\substack{\downarrow\\ g'}}{\sin x}\,dx = \underset{\substack{\downarrow\\ f}}{x}\cdot\underset{\substack{\downarrow\\ g}}{(-\cos x)} - \int \underset{\substack{\downarrow\\ f'}}{1}\cdot\underset{\substack{\downarrow\\ g}}{(-\cos x)}\,dx$$
$$= -x\cos x + \sin x$$

There are two special tricks which often work with integration by parts. The first is to consider the function g' to be the factor 1, which can always be written in.

$$\int \log x\,dx = \int \underset{\substack{\downarrow\\ g'}}{1}\cdot\underset{\substack{\downarrow\\ f}}{\log x}\,dx = \underset{\substack{\downarrow\\ g}}{x}\,\underset{\substack{\downarrow\\ f}}{\log x} - \int \underset{\substack{\downarrow\\ g}}{x}\cdot\underset{\substack{\downarrow\\ f'}}{(1/x)}\,dx$$
$$= x(\log x) - x.$$

The second trick is to use integration by parts to find $\int h$ in terms of $\int h$ again, and then solve for $\int h$. A simple example is the calculation

$$\int \underset{\substack{\downarrow\\ g'}}{(1/x)}\cdot\underset{\substack{\downarrow\\ f}}{\log x}\,dx = \underset{\substack{\downarrow\\ g}}{\log x}\cdot\underset{\substack{\downarrow\\ f}}{\log x} - \int \underset{\substack{\downarrow\\ f'}}{(1/x)}\cdot\underset{\substack{\downarrow\\ g}}{\log x}\,dx,$$

which implies that

$$2\int \frac{1}{x}\log x\,dx = (\log x)^2$$

or

$$\int \frac{1}{x} \log x \, dx = \frac{(\log x)^2}{2}.$$

A more complicated calculation is often required:

$$\begin{aligned}
\int \underset{\overset{\downarrow}{f}}{e^x} \underset{\overset{\downarrow}{g'}}{\sin x} \, dx &= \underset{\overset{\downarrow}{f}}{e^x} \cdot \underset{\overset{\downarrow}{g}}{(-\cos x)} - \int \underset{\overset{\downarrow}{f'}}{e^x} \cdot \underset{\overset{\downarrow}{g}}{(-\cos x)} \, dx \\
&= -e^x \cos x + \int \underset{\overset{\downarrow}{u}}{e^x} \underset{\overset{\downarrow}{v'}}{\cos x} \, dx \\
&= -e^x \cos x + [\underset{\overset{\downarrow}{u}}{e^x} \cdot \underset{\overset{\downarrow}{v}}{(\sin x)} - \int \underset{\overset{\downarrow}{u'}}{e^x} \underset{\overset{\downarrow}{v}}{(\sin x)} \, dx];
\end{aligned}$$

therefore,

$$2 \int e^x \sin x \, dx = e^x (\sin x - \cos x)$$

or

$$\int e^x \sin x \, dx = \frac{e^x (\sin x - \cos x)}{2}.$$

Since integration by parts depends upon recognizing that a function is of the form g', the more functions you can already integrate, the greater your chances for success. It is frequently reasonable to do a preliminary integration before tackling the main problem. For example, we can use parts to integrate

$$\int (\log x)^2 \, dx = \int \underset{\overset{\downarrow}{f}}{(\log x)} \underset{\overset{\downarrow}{g'}}{(\log x)} \, dx$$

if we recall that $\int \log x \, dx = x(\log x) - x$ (this formula was itself derived by integration by parts); we have

$$\begin{aligned}
\int \underset{\overset{\downarrow}{f}}{(\log x)} \underset{\overset{\downarrow}{g'}}{(\log x)} \, dx &= \underset{\overset{\downarrow}{f}}{(\log x)} \underset{\overset{\downarrow}{g}}{[x(\log x) - x]} - \int \underset{\overset{\downarrow}{f'}}{(1/x)} \underset{\overset{\downarrow}{g}}{[x(\log x) - x]} \, dx \\
&= (\log x)[x(\log x) - x] - \int [\log x - 1] \, dx \\
&= (\log x)[x(\log x) - x] - \int \log x \, dx + \int 1 \, dx \\
&= (\log x)[x(\log x) - x] - [x(\log x) - x] + x \\
&= x(\log x)^2 - 2x(\log x) + 2x.
\end{aligned}$$

The most important method of integration is a consequence of the Chain Rule. The use of this method requires considerably more ingenuity than integrating by parts, and even the explanation of the method is more difficult. We will therefore develop this method in stages, stating the theorem for definite integrals first, and saving the treatment of indefinite integrals for later.

THEOREM 2 (THE SUBSTITUTION FORMULA) If f and g' are continuous, then

$$\int_{g(a)}^{g(b)} f = \int_a^b (f \circ g) \cdot g'$$

$$\int_{g(a)}^{g(b)} f(u)\,du = \int_a^b f(g(x)) \cdot g'(x)\,dx.$$

PROOF If F is a primitive of f, then the left side is $F(g(b)) - F(g(a))$. On the other hand,

$$(F \circ g)' = (F' \circ g) \cdot g' = (f \circ g) \cdot g',$$

so $F \circ g$ is a primitive of $(f \circ g) \cdot g'$ and the right side is

$$(F \circ g)(b) - (F \circ g)(a) = F(g(b)) - F(g(a)). \blacksquare$$

The simplest uses of the substitution formula depend upon recognizing that a given function is of the form $(f \circ g) \cdot g'$. For example, the integration of

$$\int_a^b \sin^5 x \cos x\,dx \quad \left(= \int_a^b (\sin x)^5 \cos x\,dx \right)$$

is facilitated by the appearance of the factor $\cos x$, which will be the factor $g'(x)$ for $g(x) = \sin x$; the remaining expression, $(\sin x)^5$, can be written as $(g(x))^5 = f(g(x))$, for $f(u) = u^5$. Thus

$$\begin{aligned} &\int_a^b \sin^5 x \cos x\,dx \qquad \begin{bmatrix} g(x) = \sin x \\ f(u) = u^5 \end{bmatrix} \\ &= \int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du \\ &= \int_{\sin a}^{\sin b} u^5\,du = \frac{\sin^6 b}{6} - \frac{\sin^6 a}{6}. \end{aligned}$$

The integration of $\int_a^b \tan x\,dx$ can be treated similarly if we write

$$\int_a^b \tan x\,dx = -\int_a^b \frac{-\sin x}{\cos x}\,dx.$$

In this case the factor $-\sin x$ is $g'(x)$, where $g(x) = \cos x$; the remaining factor $1/\cos x$ can then be written $f(\cos x)$ for $f(u) = 1/u$. Hence

$$\begin{aligned} &\int_a^b \tan x\,dx \qquad \begin{bmatrix} g(x) = \cos x \\ f(u) = \dfrac{1}{u} \end{bmatrix} \\ &= -\int_a^b f(g(x))g'(x)\,dx = -\int_{g(a)}^{g(b)} f(u)\,du \\ &= -\int_{\cos a}^{\cos b} \frac{1}{u}\,du = \log(\cos a) - \log(\cos b). \end{aligned}$$

Finally, to find

$$\int_a^b \frac{1}{x\log x}\,dx,$$

notice that $1/x = g'(x)$ where $g(x) = \log x$, and that $1/\log x = f(g(x))$ for $f(u) = 1/u$. Thus

$$\begin{aligned}\int_a^b \frac{1}{x\log x}\,dx \qquad &\left[\begin{aligned} g(x) &= \log x \\ f(u) &= \frac{1}{u}\end{aligned}\right] \\ &= \int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du \\ &= \int_{\log a}^{\log b} \frac{1}{u}\,du = \log(\log b) - \log(\log a).\end{aligned}$$

Fortunately, these uses of the substitution formula can be shortened considerably. The intermediate steps, which involve writing

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du,$$

can easily be eliminated by noticing the following: To go from the left side to the right side,

$$\text{substitute} \begin{cases} u \text{ for } g(x) \\ du \text{ for } g'(x)\,dx \end{cases}$$

(and change the limits of integration);

the substitutions can be performed directly on the original function (accounting for the name of this theorem). For example,

$$\int_a^b \sin^5 x\cos x\,dx \left[\text{substitute} \begin{array}{c} u \text{ for } \sin x \\ du \text{ for } \cos x\,dx \end{array}\right] = \int_{\sin a}^{\sin b} u^5\,du,$$

and similarly

$$\int_a^b \frac{-\sin x}{\cos x}\,dx \left[\text{substitute} \begin{array}{c} u \text{ for } \cos x \\ du \text{ for } -\sin x\,dx \end{array}\right] = \int_{\cos a}^{\cos b} \frac{1}{u}\,du.$$

Usually we abbreviate this method even more, and say simply:

$$\text{“Let } \begin{aligned} u &= g(x) \\ du &= g'(x)\,dx.\text{”}\end{aligned}$$

Thus

$$\int_a^b \frac{1}{x\log x}\,dx \left[\begin{aligned} \text{let } \ u &= \log x \\ du &= \frac{1}{x}\,dx \end{aligned}\right] = \int_{\log a}^{\log b} \frac{1}{u}\,du.$$

In this chapter we are usually interested in primitives rather than definite integrals, but if we can find $\int_a^b f(x)\,dx$ for all a and b, then we can certainly find

$\int f(x)\,dx$. For example, since

$$\int_a^b \sin^5 x \cos x\,dx = \frac{\sin^6 b}{6} - \frac{\sin^6 a}{6},$$

it follows that

$$\int \sin^5 x \cos x\,dx = \frac{\sin^6 x}{6}.$$

Similarly,

$$\int \tan x\,dx = -\log \cos x,$$

$$\int \frac{1}{x \log x}\,dx = \log(\log x).$$

It is quite uneconomical to obtain primitives from the substitution formula by first finding definite integrals. Instead, the two steps can be combined, to yield the following procedure:

(1) Let

$$u = g(x),$$
$$du = g'(x)\,dx;$$

(after this manipulation only the letter u should appear, *not* the letter x).

(2) Find a primitive (as an expression involving u).

(3) Substitute $g(x)$ back for u.

Thus, to find

$$\int \sin^5 x \cos x\,dx,$$

(1) let

$$u = \sin x,$$
$$du = \cos x\,dx$$

so that we obtain

$$\int u^5\,du;$$

(2) evaluate

$$\int u^5\,du = \frac{u^6}{6};$$

(3) remember to substitute $\sin x$ back for u, so that

$$\int \sin^5 x \cos x\,dx = \frac{\sin^6 x}{6}.$$

Similarly, if

$$u = \log x,$$
$$du = \frac{1}{x}\,dx,$$

then

$$\int \frac{1}{x\log x}\,dx \quad \text{becomes} \quad \int \frac{1}{u}\,du = \log u,$$

so that

$$\int \frac{1}{x\log x}\,dx = \log(\log x).$$

To evaluate

$$\int \frac{x}{1+x^2}\,dx,$$

let

$$u = 1 + x^2,$$
$$du = 2x\,dx;$$

the factor 2 which has just popped up causes no problem—the integral becomes

$$\frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\log u,$$

so

$$\int \frac{x}{1+x^2}\,dx = \frac{1}{2}\log(1+x^2).$$

(This result may be combined with integration by parts to yield

$$\begin{aligned}\int 1\cdot\arctan x\,dx &= x\arctan x - \int \frac{x}{1+x^2}\,dx\\ &= x\arctan x - \tfrac{1}{2}\log(1+x^2),\end{aligned}$$

a formula that has already been mentioned.)

These applications of the substitution formula* illustrate the most straightforward and least interesting types—once the suitable factor $g'(x)$ is recognized, the whole problem may even become simple enough to do mentally. The following three problems require only the information provided by the short table of indefinite integrals at the beginning of the chapter and, of course, the right substitution

*The substitution formula is often written in the form

$$\int f(u)\,du = \int f(g(x))g'(x)\,dx, \quad u = g(x).$$

This formula cannot be taken literally (after all, $\int f(u)\,du$ should mean a primitive of f and the symbol $\int f(g(x))g'(x)\,dx$ should mean a primitive of $(f \circ g)\cdot g'$; these are certainly not equal). However, it may be regarded as a symbolic summary of the procedure which we have developed. If we use Leibniz's notation, and a little fudging, the formula reads particularly well:

$$\int f(u)\,du = \int f(u)\frac{du}{dx}\,dx.$$

(the third problem has been disguised a little by some algebraic chicanery).

$$\int \sec^2 x \tan^5 x \, dx,$$
$$\int (\cos x) e^{\sin x} \, dx,$$
$$\int \frac{e^x}{\sqrt{1-e^{2x}}} \, dx.$$

If you have not succeeded in finding the right substitutions, you should be able to guess them from the answers, which are $(\tan^6 x)/6$, $e^{\sin x}$, and $\arcsin e^x$. At first you may find these problems too hard to do in your head, but at least when g is of the very simple form $g(x) = ax + b$ you should not have to waste time writing out the substitution. The following integrations should all be clear. (The only worrisome detail is the proper positioning of the constant—should the answer to the second be $e^{3x}/3$ or $3e^{3x}$? I always take care of these problems as follows. Clearly $\int e^{3x} \, dx = e^{3x} \cdot$ (something). Now if I differentiate $F(x) = e^{3x}$, I get $F'(x) = 3e^{3x}$, so the "something" must be $\frac{1}{3}$, to cancel the 3.)

$$\int \frac{dx}{x+3} = \log(x+3),$$
$$\int e^{3x} \, dx = \frac{e^{3x}}{3},$$
$$\int \cos 4x \, dx = \frac{\sin 4x}{4},$$
$$\int \sin(2x+1) \, dx = \frac{-\cos(2x+1)}{2},$$
$$\int \frac{dx}{1+4x^2} = \frac{\arctan 2x}{2}.$$

More interesting uses of the substitution formula occur when the factor $g'(x)$ does *not* appear. There are two main types of substitutions where this happens. Consider first

$$\int \frac{1+e^x}{1-e^x} \, dx.$$

The prominent appearance of the expression e^x suggests the simplifying substitution

$$u = e^x,$$
$$du = e^x \, dx.$$

Although the expression $e^x \, dx$ does not appear, it can always be put in:

$$\int \frac{1+e^x}{1-e^x} \, dx = \int \frac{1+e^x}{1-e^x} \cdot \frac{1}{e^x} \cdot e^x \, dx.$$

We therefore obtain

$$\int \frac{1+u}{1-u} \cdot \frac{1}{u} \, du,$$

which can be evaluated by the algebraic trick

$$\int \frac{1+u}{1-u}\cdot\frac{1}{u}\,du = \int \frac{2}{1-u}+\frac{1}{u}\,du = -2\log(1-u)+\log u,$$

so that

$$\int \frac{1+e^x}{1-e^x}\,dx = -2\log(1-e^x)+\log e^x = -2\log(1-e^x)+x.$$

There is an alternative and preferable way of handling this problem, which does not require multiplying and dividing by e^x. If we write

$$u = e^x, \qquad x = \log u,$$
$$dx = \frac{1}{u}\,du,$$

then

$$\int \frac{1+e^x}{1-e^x}\,dx \quad \text{immediately becomes} \quad \int \frac{1+u}{1-u}\cdot\frac{1}{u}\,du.$$

Most substitution problems are much easier if one resorts to this trick of expressing x in terms of u, and dx in terms of du, instead of vice versa. It is not hard to see why this trick always works (as long as the function expressing u in terms of x is one-one for all x under consideration): If we apply the substitution

$$u = g(x), \qquad x = g^{-1}(u)$$
$$dx = (g^{-1})'(u)\,du$$

to the integral

$$\int f(g(x))\,dx,$$

we obtain

$$(1) \qquad \int f(u)(g^{-1})'(u)\,du.$$

On the other hand, if we apply the straightforward substitution

$$u = g(x)$$
$$du = g'(x)\,dx$$

to the same integral,

$$\int f(g(x))\,dx = \int f(g(x))\cdot\frac{1}{g'(x)}\cdot g'(x)\,dx,$$

we obtain

$$(2) \qquad \int f(u)\cdot\frac{1}{g'(g^{-1}(u))}\,du.$$

The integrals (1) and (2) are identical, since $(g^{-1})'(u) = 1/g'(g^{-1}(u))$.

As another concrete example, consider

$$\int \frac{e^{2x}}{\sqrt{e^x+1}}\,dx.$$

In this case we will go the whole hog and replace the entire expression $\sqrt{e^x+1}$ by one letter. Thus we choose the substitution

$$\begin{aligned} u &= \sqrt{e^x+1}, \\ u^2 &= e^x+1, \\ u^2-1 = e^x, \quad x &= \log(u^2-1), \\ dx &= \frac{2u}{u^2-1}\,du. \end{aligned}$$

The integral then becomes

$$\int \frac{(u^2-1)^2}{u}\cdot\frac{2u}{u^2-1}\,du = 2\int u^2-1\,du = \frac{2u^3}{3}-2u.$$

Thus

$$\int \frac{e^{2x}}{\sqrt{e^x+1}}\,dx = \frac{2}{3}(e^x+1)^{3/2}-2(e^x+1)^{1/2}.$$

Another example, which illustrates the second main type of substitution that can occur, is the integral

$$\int \sqrt{1-x^2}\,dx.$$

In this case, instead of replacing a complicated expression by a simpler one, we will replace x by $\sin u$, because $\sqrt{1-\sin^2 u} = \cos u$. This really means that we are using the substitution $u = \arcsin x$, but it is the expression for x in terms of u which helps us find the expression to be substituted for dx. Thus,

$$\begin{aligned} \text{let} \quad x &= \sin u, \quad [u = \arcsin x] \\ dx &= \cos u\,du; \end{aligned}$$

then the integral becomes

$$\int \sqrt{1-\sin^2 u}\,\cos u\,du = \int \cos^2 u\,du.$$

The evaluation of this integral depends on the equation

$$\cos^2 u = \frac{1+\cos 2u}{2}$$

(see the discussion of trigonometric functions below) so that

$$\int \cos^2 u\,du = \int \frac{1+\cos 2u}{2}\,du = \frac{u}{2}+\frac{\sin 2u}{4},$$

and

$$\begin{aligned} \int \sqrt{1-x^2}\,dx &= \frac{\arcsin x}{2}+\frac{\sin(2\arcsin x)}{4} \\ &= \frac{\arcsin x}{2}+\frac{1}{2}\sin(\arcsin x)\cdot\cos(\arcsin x) \\ &= \frac{\arcsin x}{2}+\frac{1}{2}x\sqrt{1-x^2}. \end{aligned}$$

Substitution and integration by parts are the only fundamental methods which you have to learn; with their aid primitives can be found for a large number of functions. Nevertheless, as some of our examples reveal, success often depends upon some additional tricks. The most important are listed below. Using these you should be able to integrate all the functions in Problems 1 to 10 (a few other interesting tricks are explained in some of the remaining problems).

1. TRIGONOMETRIC FUNCTIONS

Since

$$\sin^2 x + \cos^2 x = 1$$

and

$$\cos 2x = \cos^2 x - \sin^2 x,$$

we obtain

$$\begin{aligned}\cos 2x &= \cos^2 x - (1 - \cos^2 x) = 2\cos^2 x - 1,\\ \cos 2x &= (1 - \sin^2 x) - \sin^2 x = 1 - 2\sin^2 x,\end{aligned}$$

or

$$\begin{aligned}\sin^2 x &= \frac{1 - \cos 2x}{2},\\ \cos^2 x &= \frac{1 + \cos 2x}{2}.\end{aligned}$$

These formulas may be used to integrate

$$\int \sin^n x\, dx,$$
$$\int \cos^n x\, dx,$$

if n is even. Substituting

$$\frac{(1 - \cos 2x)}{2} \quad \text{or} \quad \frac{(1 + \cos 2x)}{2}$$

for $\sin^2 x$ or $\cos^2 x$ yields a sum of terms involving lower powers of cos. For example,

$$\int \sin^4 x\, dx = \int \left(\frac{1 - \cos 2x}{2}\right)^2 dx = \int \frac{1}{4}\, dx - \frac{1}{2}\int \cos 2x\, dx + \frac{1}{4}\int \cos^2 2x\, dx$$

and

$$\int \cos^2 2x\, dx = \int \frac{1 + \cos 4x}{2}\, dx.$$

If n is odd, $n = 2k + 1$, then

$$\int \sin^n x\, dx = \int \sin x (1 - \cos^2 x)^k\, dx;$$

the latter expression, multiplied out, involves terms of the form $\sin x \cos^l x$, all of which can be integrated easily. The integral for $\cos^n x$ is treated similarly. An integral

$$\int \sin^n x \cos^m x \, dx$$

is handled the same way if n or m is odd. If n and m are both even, use the formulas for $\sin^2 x$ and $\cos^2 x$.

A final important trigonometric integral is

$$\int \frac{1}{\cos x} \, dx = \int \sec x \, dx = \log(\sec x + \tan x).$$

Although there are several ways of "deriving" this result, by means of the methods already at our disposal (Problem 13), it is simplest to check this formula by differentiating the right side, and to memorize it.

2. REDUCTION FORMULAS

Integration by parts yields (Problem 21)

$$\int \sin^n x \, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx,$$

$$\int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx,$$

$$\int \frac{1}{(x^2+1)^n} \, dx = \frac{1}{2n-2} \frac{x}{(x^2+1)^{n-1}} + \frac{2n-3}{2n-2} \int \frac{1}{(x^2+1)^{n-1}} \, dx$$

and many similar formulas. The first two, used repeatedly, give a different method for evaluating primitives of $\sin^n$ or $\cos^n$. The third is very important for integrating a large general class of functions, which will complete our discussion.

3. RATIONAL FUNCTIONS

Consider a rational function p/q where

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0,$$
$$q(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_0.$$

We might as well assume that $a_n = b_m = 1$. Moreover, we can assume that $n < m$, for otherwise we may express p/q as a polynomial function plus a rational function which *is* of this form by dividing (the calculation

$$\frac{u^2}{u-1} = u + 1 + \frac{1}{u-1}$$

is a simple example). The integration of an arbitrary rational function depends on two facts; the first follows from the "Fundamental Theorem of Algebra" (see Chapter 26, Theorem 2 and Problem 26-3), but the second will not be proved in this book.

THEOREM Every polynomial function

$$q(x) = x^m + b_{m-1}x^{m-1} + \cdots + b_0$$

can be written as a product

$$q(x) = (x-\alpha_1)^{r_1} \cdot \ldots \cdot (x-\alpha_k)^{r_k}(x^2+\beta_1 x+\gamma_1)^{s_1} \cdot \ldots \cdot (x^2+\beta_l x+\gamma_l)^{s_l}$$

(where $r_1 + \cdots + r_k + 2(s_1 + \cdots + s_l) = m$).

(In this expression, identical factors have been collected together, so that all $x - \alpha_i$ and $x^2 + \beta_i x + \gamma_i$ may be assumed distinct. Moreover, we assume that each quadratic factor cannot be factored further. This means that

$$\beta_i{}^2 - 4\gamma_i < 0,$$

since otherwise we can factor

$$x^2 + \beta_i x + \gamma_i = \left[x - \left(\frac{-\beta_i + \sqrt{\beta_i{}^2 - 4\gamma_i}}{2}\right)\right] \cdot \left[x - \left(\frac{-\beta_i - \sqrt{\beta_i{}^2 - 4\gamma_i}}{2}\right)\right]$$

into linear factors.)

THEOREM If $n < m$ and

$$\begin{aligned} p(x) &= x^n + a_{n-1}x^{n-1} + \cdots + a_0, \\ q(x) &= x^m + b_{m-1}x^{m-1} + \cdots + b_0 \\ &= (x-\alpha_1)^{r_1} \cdot \ldots \cdot (x-\alpha_k)^{r_k}(x^2+\beta_1 x+\gamma_1)^{s_1} \cdot \ldots \cdot (x^2+\beta_l x+\gamma_l)^{s_l}, \end{aligned}$$

then $p(x)/q(x)$ can be written in the form

$$\begin{aligned} \frac{p(x)}{q(x)} = \left[\frac{a_{1,1}}{(x-\alpha_1)} + \cdots + \frac{a_{1,r_1}}{(x-\alpha_1)^{r_1}}\right] + \cdots \\ + \left[\frac{\alpha_{k,1}}{(x-\alpha_k)} + \cdots + \frac{\alpha_{k,r_k}}{(x-\alpha_k)^{r_k}}\right] \\ + \left[\frac{b_{1,1}x + c_{1,1}}{(x^2+\beta_1 x+\gamma_1)} + \cdots + \frac{b_{1,s_1}x + c_{1,s_1}}{(x^2+\beta_1 x+\gamma_1)^{s_1}}\right] + \cdots \\ + \left[\frac{b_{l,1}x + c_{l,1}}{(x^2+\beta_l x+\gamma_l)} + \cdots + \frac{b_{l,s_l}x + c_{l,s_l}}{(x^2+\beta_l x+\gamma_l)^{s_l}}\right]. \end{aligned}$$

This expression, known as the "partial fraction decomposition" of $p(x)/q(x)$, is so complicated that it is simpler to examine the following example, which illustrates such an expression and shows how to find it. According to the theorem, it is possible to write

$$\begin{aligned} &\frac{2x^7 + 8x^6 + 13x^5 + 20x^4 + 15x^3 + 16x^2 + 7x + 10}{(x^2+x+1)^2(x^2+2x+2)(x-1)^2} \\ &\qquad = \frac{a}{x-1} + \frac{b}{(x-1)^2} + \frac{cx+d}{x^2+2x+2} + \frac{ex+f}{x^2+x+1} + \frac{gx+h}{(x^2+x+1)^2}. \end{aligned}$$

To find the numbers a, b, c, d, e, f, g, and h, write the right side as a polynomial over the common denominator $(x^2+x+1)^2(x^2+2x+3)(x-1)^2$; the numerator becomes

$$\begin{aligned}&a(x-1)(x^2+2x+2)(x^2+x+1)^2+b(x^2+2x+2)(x^2+x+1)^2\\&+(cx+d)(x-1)^2(x^2+x+1)^2+(ex+f)(x-1)^2(x^2+2x+2)(x^2+x+1)\\&\qquad\qquad\qquad\qquad+(gx+h)(x-1)^2(x^2+2x+2).\end{aligned}$$

Actually multiplying this out (!) we obtain a polynomial of degree 8, whose coefficients are combinations of $a, \dots, h$. Equating these coefficients with the coefficients of $2x^7+8x^6+13x^5+20x^4+15x^3+16x^2+7x+10$ (the coefficient of x^8 is 0) we obtain 8 equations in the eight unknowns $a, \dots, h$. After heroic calculations these can be solved to give

$$\begin{array}{llll} a=1, & b=2, & c=1, & d=3, \\ e=0, & f=0, & g=0, & h=1. \end{array}$$

Thus

$$\begin{aligned}&\int \frac{2x^7+5x^6+13x^5+20x^4+17x^3+16x^2+7x+7}{(x^2+x+1)^2(x^2+2x+2)(x-1)^2}\,dx\\&=\int\frac{1}{(x-1)}\,dx+\int\frac{2}{(x-1)^2}\,dx+\int\frac{1}{(x^2+x+1)^2}\,dx+\int\frac{x+3}{x^2+2x+2}\,dx.\end{aligned}$$

(In simpler cases the requisite calculations may actually be feasible. I obtained this particular example by *starting* with the partial fraction decomposition and converting it into one fraction.)

We are already in a position to find each of the integrals appearing in the above expression; the calculations will illustrate all the difficulties which arise in integrating rational functions.

The first two integrals are simple:

$$\int\frac{1}{x-1}\,dx=\log(x-1),$$
$$\int\frac{2}{(x-1)^2}\,dx=\frac{-2}{x-1}.$$

The third integration depends on "completing the square":

$$\begin{aligned}x^2+x+1&=(x+\tfrac{1}{2})^2+\tfrac{3}{4}\\&=\frac{3}{4}\left[\left(\frac{x+\frac{1}{2}}{\sqrt{\frac{3}{4}}}\right)^2+1\right].\end{aligned}$$

(If we had obtained $-\frac{3}{4}$ instead of $\frac{3}{4}$ we could not take the square root, but in this case our original quadratic factor could have been factored into linear factors.) We can now write

$$\int \frac{1}{(x^2+x+1)^2}\,dx = \frac{16}{9}\int \frac{1}{\left[\left(\dfrac{x+\frac{1}{2}}{\sqrt{\frac{3}{4}}}\right)+1\right]^2}\,dx.$$

The substitution

$$u = \frac{x+\frac{1}{2}}{\sqrt{\frac{3}{4}}},$$
$$du = \frac{1}{\sqrt{\frac{3}{4}}}\,dx,$$

changes this integral to

$$\frac{16}{9}\int \frac{\sqrt{\frac{3}{4}}}{(u^2+1)^2}\,du,$$

which can be computed using the third reduction formula given above.

Finally, to evaluate

$$\int \frac{x+3}{(x^2+2x+2)}\,dx$$

we write

$$\int \frac{x+3}{x^2+2x+2}\,dx = \frac{1}{2}\int \frac{2x+2}{x^2+2x+2}\,dx + \int \frac{2}{(x+1)^2+1}\,dx.$$

The first integral on the right side has been purposely constructed so that we can evaluate it by using the substitution

$$u = x^2+2x+2,$$
$$du = (2x+2)\,dx$$

The second integral on the right, which is just the difference of the other two, is simply $2\arctan(x+1)$. If the original integral were

$$\int \frac{x+3}{(x^2+2x+2)^n}\,dx = \frac{1}{2}\int \frac{2x+2}{(x^2+2x+2)^n}\,dx + \int \frac{2}{[(x+1)^2+1]^n}\,dx,$$

the first integral on the right would still be evaluated by the same substitution. The second integral would be evaluated by means of a reduction formula.

This example has probably convinced you that integration of rational functions is a theoretical curiosity only, especially since it is necessary to find the factorization of $q(x)$ before you can even begin. This is only partly true. We have already seen that simple rational functions sometimes arise, as in the integration

$$\int \frac{1+e^x}{1-e^x}\,dx;$$

another important example is the integral

$$\int \frac{1}{x^2-1}\,dx = \int \frac{\frac{1}{2}}{x-1} - \frac{\frac{1}{2}}{x+1}\,dx = \frac{1}{2}\log(x-1) - \frac{1}{2}\log(x+1).$$

Moreover, if a problem has been reduced to the integration of a rational function, it is then certain that an elementary primitive exists, even when the difficulty or impossibility of finding the factors of the denominator may preclude writing this primitive explicitly.

PROBLEMS

1. This problem contains some integrals which require little more than algebraic manipulation, and consequently test your ability to discover algebraic tricks, rather than your understanding of the integration processes. Nevertheless, any one of these tricks might be an important preliminary step in an honest integration problem. Moreover, you want to have some feel for which integrals are easy, so that you can see when the end of an integration process is in sight. The answer section, if you resort to it, will only reveal what algebra you should have used.

(i) $\displaystyle\int \frac{\sqrt[5]{x^3}+\sqrt[6]{x}}{\sqrt{x}}\,dx.$

(ii) $\displaystyle\int \frac{dx}{\sqrt{x-1}+\sqrt{x+1}}.$

(iii) $\displaystyle\int \frac{e^x+e^{2x}+e^{3x}}{e^{4x}}\,dx.$

(iv) $\displaystyle\int \frac{a^x}{b^x}\,dx.$

(v) $\displaystyle\int \tan^2 x\,dx.$ (Trigonometric integrals are always very touchy, because there are so many trigonometric identities that an easy problem can easily look hard.)

(vi) $\displaystyle\int \frac{dx}{a^2+x^2}.$

(vii) $\displaystyle\int \frac{dx}{\sqrt{a^2-x^2}}.$

(viii) $\displaystyle\int \frac{dx}{1+\sin x}.$

(ix) $\displaystyle\int \frac{8x^2+6x+4}{x+1}\,dx.$

(x) $\displaystyle\int \frac{1}{\sqrt{2x-x^2}}\,dx.$

2. The following integrations involve simple substitutions, most of which you should be able to do in your head.

(i) $\displaystyle\int e^x \sin e^x\,dx.$

(ii) $\displaystyle\int xe^{-x^2}\,dx.$

(iii) $\displaystyle\int \frac{\log x}{x}\,dx.$ (In the text this was done by parts.)

(iv) $\displaystyle\int \frac{e^x\,dx}{e^{2x}+2e^x+1}.$

(v) $\displaystyle\int e^{e^x}e^x\,dx.$

(vi) $\displaystyle\int \frac{x\,dx}{\sqrt{1-x^4}}.$

(vii) $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx.$

(viii) $\displaystyle\int x\sqrt{1-x^2}\,dx.$

(ix) $\displaystyle\int \log(\cos x)\tan x\,dx.$

(x) $\displaystyle\int \frac{\log(\log x)}{x\log x}\,dx.$

3. Integration by parts.

(i) $\displaystyle\int x^2e^x\,dx.$

(ii) $\displaystyle\int x^3e^{x^2}\,dx.$

(iii) $\displaystyle\int e^{ax}\sin bx\,dx.$

(iv) $\displaystyle\int x^2\sin x\,dx.$

(v) $\displaystyle\int (\log x)^3\,dx.$

(vi) $\displaystyle\int \frac{\log(\log x)}{x}\,dx.$

(vii) $\displaystyle\int \sec^3 x\,dx.$ (This is a tricky and important integral that often comes up. If you do not succeed in evaluating it, be sure to consult the answers.)

(viii) $\displaystyle\int \cos(\log x)\,dx.$

(ix) $\displaystyle\int \sqrt{x}\log x\,dx.$

(x) $\displaystyle\int x(\log x)^2\,dx.$

4. The following integrations can all be done with substitutions of the form $x = \sin u$, $x = \cos u$, etc. To do some of these you will need to remember that

$$\int \sec x\,dx = \log(\sec x + \tan x)$$

as well as the following formula, which can also be checked by differentiation:

$$\int \csc x\,dx = -\log(\csc x + \cot x).$$

In addition, at this point the derivatives of all the trigonometric functions should be kept handy.

(i) $\displaystyle\int \frac{dx}{\sqrt{1-x^2}}$. (You already know this integral, but use the substitution $x = \sin u$ anyway, just to see how it works out.)

(ii) $\displaystyle\int \frac{dx}{\sqrt{1+x^2}}$. (Since $\tan^2 u + 1 = \sec^2 u$, you want to use the substitution $x = \tan u$.)

(iii) $\displaystyle\int \frac{dx}{\sqrt{x^2-1}}$.

(iv) $\displaystyle\int \frac{dx}{x\sqrt{x^2-1}}$. (The answer will be a certain inverse function that was given short shrift in the text.)

(v) $\displaystyle\int \frac{dx}{x\sqrt{1-x^2}}$.

(vi) $\displaystyle\int \frac{dx}{x\sqrt{1+x^2}}$.

(vii) $\displaystyle\int x^3\sqrt{1-x^2}\,dx$.

(viii) $\displaystyle\int \sqrt{1-x^2}\,dx$.

$\Big\}$ You will need to remember the methods for integrating powers of sin and cos.

(ix) $\displaystyle\int \sqrt{1+x^2}\,dx$.

(x) $\displaystyle\int \sqrt{x^2-1}\,dx$.

5. The following integrations involve substitutions of various types. There is no substitute for cleverness, but there is a general rule to follow: substitute for an expression which appears frequently or prominently; if two different troublesome expressions appear, try to express them both in terms of some new expression. And don't forget that it usually helps to express x directly in terms of u, to find out the proper expression to substitute for dx.

(i) $\displaystyle\int \frac{dx}{1+\sqrt{x+1}}$.

(ii) $\displaystyle\int \frac{dx}{1+e^x}$.

(iii) $\displaystyle\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}}$.

(iv) $\displaystyle\int \frac{dx}{\sqrt{1+e^x}}$. (The substitution $u = e^x$ leads to an integral requiring yet another substitution; this is all right, but both substitutions can be done at once.)

(v) $\displaystyle\int \frac{dx}{2+\tan x}$.

(vi) $\displaystyle\int \frac{dx}{\sqrt{\sqrt{x}+1}}$. (Another place where one substitution can be made to do the work of two.)

(vii) $\displaystyle\int \frac{4^x+1}{2^x+1}\,dx$.

(viii) $\displaystyle\int e^{\sqrt{x}}\,dx$.

(ix) $\displaystyle\int \frac{\sqrt{1-x}}{1-\sqrt{x}}\,dx$. (In this case two successive substitutions work out best; there are two obvious candidates for the first substitution, and either will work.)

*(x) $\displaystyle\int \sqrt{\frac{x-1}{x+1}}\cdot\frac{1}{x^2}\,dx$.

6. The previous problem provided gratis a haphazard selection of rational functions to be integrated. Here is a more systematic selection.

(i) $\displaystyle\int \frac{2x^2+7x-1}{x^3+x^2-x-1}\,dx$.

(ii) $\displaystyle\int \frac{2x+1}{x^3-3x^2+3x-1}\,dx$.

(iii) $\displaystyle\int \frac{x^3+7x^2-5x+5}{(x-1)^2(x+1)^3}\,dx$.

(iv) $\displaystyle\int \frac{2x^2+x+1}{(x+3)(x-1)^2}\,dx$.

(v) $\displaystyle\int \frac{x+4}{x^2+1}\,dx$.

(vi) $\displaystyle\int \frac{x^3+x+2}{x^4+2x^2+1}\,dx$.

(vii) $\displaystyle\int \frac{3x^2+3x+1}{x^3+2x^2+2x+1}\,dx$.

(viii) $\displaystyle\int \frac{dx}{x^4+1}$.

(ix) $\displaystyle\int \frac{2x}{(x^2+x+1)^2}\,dx$.

(x) $\displaystyle\int \frac{3x}{(x^2+x+1)^3}\,dx$.

***7.** Find $\displaystyle\int \frac{dx}{\sqrt{x^n - x^2}}$, which looks a little different from any of the previous problems. Hint: It helps to write $(x^n - x^2)^{1/2} = x(x^{n-2} - 1)^{1/2}$. Extra Hint 1: Use a substitution of the form $u^2 = \dots$ to obtain an answer involving arctan. Extra Hint 2: Use a substitution of the form $y = x^\alpha$ to obtain an answer involving arcsin.

***8.** Potpourri. (No holds barred.) The following integrations involve all the methods of the previous problems

(i) $\displaystyle\int \frac{\arctan x}{1+x^2}\,dx.$

(ii) $\displaystyle\int \frac{x \arctan x}{(1+x^2)^2}\,dx.$

(iii) $\displaystyle\int \log\sqrt{1+x^2}\,dx.$

(iv) $\displaystyle\int x\log\sqrt{1+x^2}\,dx.$

(v) $\displaystyle\int \frac{x^2-1}{x^2+1}\cdot\frac{1}{\sqrt{1+x^4}}\,dx.$

(vi) $\displaystyle\int \arcsin\sqrt{x}\,dx.$

(vii) $\displaystyle\int \frac{x}{1+\sin x}\,dx.$

(viii) $\displaystyle\int e^{\sin x}\cdot\frac{x\cos^3 x - \sin x}{\cos^2 x}\,dx.$

(ix) $\displaystyle\int \sqrt{\tan x}\,dx.$

(x) $\displaystyle\int \frac{dx}{x^6+1}$. (To factor x^6+1, first factor y^3+1, using Problem 1-1.)

The following two problems provide still more practice at integration, if you need it (and can bear it). Problem 9 involves algebraic and trigonometric manipulations and integration by parts, while Problem 10 involves substitutions. (Of course, in many cases the resulting integrals will require still further manipulations.)

9. Find the following integrals.

(i) $\displaystyle\int \log(a^2+x^2)\,dx.$

(ii) $\displaystyle\int \frac{1+\cos x}{\sin^2 x}\,dx.$

(iii) $\displaystyle\int \frac{x+1}{\sqrt{4-x^2}}\,dx.$

(iv) $\displaystyle\int x\arctan x\,dx.$

(v) $\displaystyle\int \sin^3 x\,dx.$

(vi) $\displaystyle\int \frac{\sin^3 x}{\cos^2 x}\,dx.$

(vii) $\displaystyle\int x^2 \arctan x\,dx.$

(viii) $\displaystyle\int \frac{x\,dx}{\sqrt{x^2-2x+2}}.$

(ix) $\displaystyle\int \sec^3 x \tan x\,dx.$

(x) $\displaystyle\int x \tan^2 x\,dx.$

10. Find the following integrals.

(i) $\displaystyle\int \frac{dx}{(a^2+x^2)^2}.$

(ii) $\displaystyle\int \sqrt{1-\sin x}\,dx.$

(iii) $\displaystyle\int \arctan \sqrt{x}\,dx.$

(iv) $\displaystyle\int \sin \sqrt{x+1}\,dx.$

(v) $\displaystyle\int \frac{\sqrt{x^3-2}}{x}\,dx.$

(vi) $\displaystyle\int \log\left(x+\sqrt{x^2-1}\right)dx.$

(vii) $\displaystyle\int \log(x+\sqrt{x}\,)\,dx.$

There are obvious substitutions to try, but integration by parts is much easier. Comparing the answers obtained is, perhaps, instructive.

(viii) $\displaystyle\int \frac{dx}{x-x^{3/5}}.$

(ix) $\displaystyle\int (\arcsin x)^2\,dx.$

(x) $\displaystyle\int x^5 \arctan(x^2)\,dx.$

11. If you have done Problem 18-10, the integrals (ii) and (iii) in Problem 4 will look very familiar. In general, the substitution $x = \cosh u$ often works for integrals involving $\sqrt{x^2-1}$, while $x = \sinh u$ is the thing to try for integrals involving $\sqrt{x^2+1}$. Try these substitutions on the other integrals in Problem 4. (The method is not really recommended; it is easier to stick with trigonometric substitutions.)

***12.** The world's sneakiest substitution is undoubtedly

$$t = \tan\frac{x}{2}, \qquad x = 2\arctan t, \qquad dx = \frac{2}{1+t^2}\,dt.$$

As we found in Problem 15-17, this substitution leads to the expressions

$$\sin x = \frac{2t}{1+t^2}, \qquad \cos x = \frac{1-t^2}{1+t^2}.$$

This substitution thus transforms any integral which involves only sin and cos, combined by addition, multiplication, and division, into the integral of a rational function. Find

(i) $\displaystyle\int \frac{dx}{1+\sin x}$. (Compare your answer with Problem 1(viii).)

(ii) $\displaystyle\int \frac{dx}{1-\sin^2 x}$. (In this case it is better to let $t = \tan x$. Why?)

(iii) $\displaystyle\int \frac{dx}{a\sin x + b\cos x}$. (There is also another way to do this, using Problem 15-8.)

(iv) $\displaystyle\int \sin^2 x\, dx$. (An exercise to convince you that this substitution should be used only as a last resort.)

(v) $\displaystyle\int \frac{dx}{3+5\sin x}$. (A last resort.)

***13.** Derive the formula for $\int \sec x\, dx$ in the following two ways:

(a) By writing

$$\begin{aligned}\frac{1}{\cos x} &= \frac{\cos x}{\cos^2 x}\\ &= \frac{\cos x}{1-\sin^2 x}\\ &= \frac{1}{2}\left[\frac{\cos x}{1+\sin x} + \frac{\cos x}{1-\sin x}\right],\end{aligned}$$

an expression obviously inspired by partial fraction decompositions. Be sure to note that $\int \cos x/(1-\sin x)\, dx = -\log(1-\sin x)$; the minus sign is very important. And remember that $\frac{1}{2}\log\alpha = \log\sqrt{\alpha}$. From there on, keep doing algebra, and trust to luck.

(b) By using the substitution $t = \tan x/2$. One again, quite a bit of manipulation is required to put the answer in the desired form; the expression $\tan x/2$ can be attacked by using Problem 15-9, or both answers can be expressed in terms of t. There is another expression for $\int \sec x\, dx$, which is less cumbersome than $\log(\sec x + \tan x)$; using Problem 15-9, we obtain

$$\int \sec x\, dx = \log\left(\frac{1+\tan\dfrac{x}{2}}{1-\tan\dfrac{x}{2}}\right) = \log\left(\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right).$$

This last expression was actually the one first discovered, and was due, not to any mathematician's cleverness, but to a curious historical accident: In 1599 Wright computed nautical tables that amounted to definite integrals of sec. When the first tables for the logarithms of tangents were produced, the correspondence between the two tables was immediately noticed (but remained unexplained until the invention of calculus).

14. The derivation of $\int e^x \sin x\,dx$ given in the text seems to prove that the only primitive of $f(x) = e^x \sin x$ is $F(x) = e^x(\sin x - \cos x)/2$, whereas $F(x) = e^x(\sin x - \cos x)/2 + C$ is also a primitive for any number C. Where does C come from? (What is the meaning of the equation

$$\int e^x \sin x\,dx = e^x \sin x - e^x \cos x - \int e^x \sin x\,dx?)$$

15. Suppose that f'' is continuous and that

$$\int_0^{\pi} [f(x) + f''(x)] \sin x\,dx = 2.$$

Given that $f(\pi) = 1$, compute $f(0)$.

16. (a) Find $\int \arcsin x\,dx$, using the same trick that worked for log and arctan.

*(b) Generalize this trick: Find $\int f^{-1}(x)\,dx$ in terms of $\int f(x)\,dx$. Compare with Problems 12-21 and 14-14.

17. (a) Find $\int \sin^4 x\,dx$ in two different ways: first using the reduction formula, and then using the formula for $\sin^2 x$.

(b) Combine your answers to obtain an impressive trigonometric identity.

18. Express $\int \log(\log x)\,dx$ in terms of $\int (\log x)^{-1}\,dx$. (Neither is expressible in terms of elementary functions.)

19. Express $\int x^2 e^{-x^2}\,dx$ in terms of $\int e^{-x^2}\,dx$.

20. Prove that the function $f(x) = e^x/(e^{5x} + e^x + 1)$ has an elementary primitive. (Do not try to find it!)

21. Prove the reduction formulas in the text. For the third one write

$$\int \frac{dx}{(x^2+1)^n} = \int \frac{dx}{(x^2+1)^{n-1}} - \int \frac{x^2\,dx}{(x^2+1)^n}$$

and work on the last integral. (Another possibility is to use the substitution $x = \tan u$.)

22. Find a reduction formula for

(a) $\displaystyle\int x^n e^x\,dx$

(b) $\displaystyle\int (\log x)^n\,dx$.

***23.** Prove that

$$\int_1^{\cosh x} \sqrt{t^2 - 1}\,dt = \frac{\cosh x \sinh x}{2} - \frac{x}{2}.$$

(See Problem 18-7 for the significance of this computation.)

24. Prove that

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx.$$

(A geometric interpretation makes this clear, but it is also a good exercise in the handling of limits of integration during a substitution.)

25. Prove that the area of a circle of radius r is πr^2. (Naturally you must remember that π is defined as the area of the unit circle.)

26. Let ϕ be a nonnegative integrable function such that $\phi(x) = 0$ for $|x| \geq 1$ and such that $\int_{-1}^{1} \phi = 1$. For $h > 0$, let

$$\phi_h(x) = \frac{1}{h}\phi(x/h).$$

(a) Show that $\phi_h(x) = 0$ for $|x| \geq h$ and that $\int_{-h}^{h} \phi_h = 1$.

(b) Let f be integrable on $[-1, 1]$ and continuous at 0. Show that

$$\lim_{h\to 0^+} \int_{-1}^{1} \phi_h f = \lim_{h\to 0^+} \int_{-h}^{h} \phi_h f = f(0).$$

(c) Show that

$$\lim_{h\to 0^+} \int_{-1}^{1} \frac{h}{h^2+x^2}\,dx = \pi.$$

The final part of this problem might appear, at first sight, to be an exact analogue of part (b), but it actually requires more careful argument.

(d) Let f be integrable on $[-1, 1]$ and continuous at 0. Show that

$$\lim_{h\to 0^+} \int_{-1}^{1} \frac{h}{h^2+x^2} f(x)\,dx = \pi f(0).$$

Hint: If h is small, then $h/(h^2+x^2)$ will be small on most of $[-1, 1]$.

The next two problems use the formula

$$\frac{1}{2}\int_{\theta_0}^{\theta_1} f(\theta)^2\,d\theta,$$

derived in Problem 13-24, for the area of a region bounded by the graph of f in polar coordinates.

27. For each of the following functions, find the area bounded by the graphs in polar coordinates. (Be careful about the proper range for θ, or you will get nonsensical results!)

(i) $f(\theta) = a\sin\theta$.
(ii) $f(\theta) = 2 + \cos\theta$.
(iii) $f(\theta)^2 = 2a^2\cos 2\theta$.
(iv) $f(\theta) = a\cos 2\theta$.

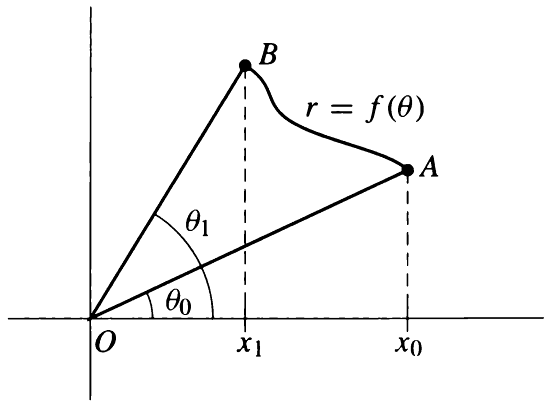


FIGURE 1

28. Figure 1 shows the graph of f in polar coordinates; the region OAB thus has area $\dfrac{1}{2}\displaystyle\int_{\theta_0}^{\theta_1} f(\theta)^2\,d\theta$. Now suppose that this graph also happens to be the ordinary graph of some function g. Then the region OAB also has area

$$\text{area } \Delta Ox_1B + \int_{x_1}^{x_0} g - \text{ area } \Delta Ox_0A.$$

Prove analytically that these two numbers are indeed the same. Hint: The function g is determined by the equations

$$x = f(\theta)\cos\theta, \qquad g(x) = f(\theta)\sin\theta.$$

The next four problems use the formulas, derived in Problems 3 and 4 of the Appendix to Chapter 13, for the length of a curve represented parametrically (and, in particular, as the graph of a function in polar coordinates).

29. Let c be a curve represented parametrically by u and v on $[a,b]$, and let h be an increasing function with $h(\bar{a}) = a$ and $h(\bar{b}) = b$. Then on $[\bar{a},\bar{b}]$ the functions $\bar{u} = u \circ h$, $\bar{v} = v \circ h$ give a parametric representation of another curve $\bar{c}$; intuitively, $\bar{c}$ is just the same curve c traversed at a different rate.

(a) Show, directly from the definition of length, that the length of c on $[a,b]$ equals the length of $\bar{c}$ on $[\bar{a},\bar{b}]$.

(b) Assuming differentiability of any functions required, show that the lengths are equal by using the integral formula for length, and the appropriate substitution.

30. Find the length of the following curves, all described as the graphs of functions, except for (iii), which is represented parametrically.

(i) $f(x) = \dfrac{1}{3}(x^2+2)^{3/2}, \qquad 0 \le x \le 1.$

(ii) $f(x) = x^3 + \dfrac{1}{12x}, \qquad 1 \le x \le 2.$

(iii) $x = a^3\cos^3 t, \qquad y = a^3\sin^3 t, \qquad 0 \le t \le 2\pi.$

(iv) $f(x) = \log(\cos x), \qquad 0 \le x \le \pi/6.$

(v) $f(x) = \log x, \qquad 1 \le x \le e.$

(vi) $f(x) = \arcsin e^x, \qquad -\log 2 \le x \le 0.$

31. For the following functions, find the length of the graph in polar coordinates.

(i) $f(\theta) = a\cos\theta.$

(ii) $f(\theta) = a(1-\cos\theta).$

(iii) $f(\theta) = a\sin^2(\theta/2).$

(iv) $f(\theta) = \theta \qquad 0 \le \theta \le 2\pi.$

(v) $f(\theta) = 3\sec\theta \qquad 0 \le \theta \le \pi/3.$

32. In Problem 8 of the Appendix to Chapter 12 we described the cycloid, which has the parametric representation

$$x = u(t) = a(t - \sin t), \qquad y = v(t) = a(1 - \cos t).$$

(a) Find the length of one arch of the cycloid. [Answer: $8a$.]

(b) Recall that the cycloid is the graph of $v \circ u^{-1}$. Find the area under one arch of the cycloid by using the appropriate substitution in $\int f$ and evaluating the resultant integral. [Answer: $3\pi a^2$.]

33. Use induction and integration by parts to generalize Problem 14-10:

$$\int_0^x \frac{f(u)(x-u)^n}{n!}\,du = \int_0^x \left(\int_0^{u_n} \left(\ldots \left(\int_0^{u_1} f(t)\,dt\right) du_1\right)\ldots\right) du_n.$$

34. If f' is continuous on $[a, b]$, use integration by parts to prove the Riemann-Lebesgue Lemma for f:

$$\lim_{\lambda \to \infty} \int_a^b f(t) \sin(\lambda t)\,dt = 0.$$

This result is just a special case of Problem 15-26, but it can be used to prove the general case (in much the same way that the Riemann-Lebesgue Lemma was derived in Problem 15-26 from the special case in which f is a step function).

35. The Mean Value Theorem for Integrals was introduced in Problem 13-23. The "Second Mean Value Theorem for Integrals" states the following. Suppose that f is integrable on $[a, b]$ and that ϕ is either nondecreasing or nonincreasing on $[a, b]$. Then there is a number ξ in $[a, b]$ such that

$$\int_a^b f(x)\phi(x)\,dx = \phi(a)\int_a^\xi f(x)\,dx + \phi(b)\int_\xi^b f(x)\,dx.$$

In this problem, we will assume that f is continuous and that ϕ is differentiable, with a continuous derivative ϕ'.

(a) Prove that if the result is true for nonincreasing ϕ, then it is also true for nondecreasing ϕ.

(b) Prove that if the result is true for nonincreasing ϕ satisfying $\phi(b) = 0$, then it is true for all nonincreasing ϕ.

Thus, we can assume that ϕ is nonincreasing and $\phi(b) = 0$. In this case, we have to prove that

$$\int_a^b f(x)\phi(x)\,dx = \phi(a)\int_a^\xi f(x)\,dx.$$

(c) Prove this by using integration by parts.

(d) Show that the hypothesis that ϕ is either nondecreasing or nonincreasing is needed.

From this special case of the Second Mean Value Theorem for Integrals, the general case could be derived by some approximation arguments, just as in the case of the Riemann-Lebesgue Lemma. But there is a more instructive way, outlined in the next problem.

36. (a) Given $a_1, \dots, a_n$ and $b_1, \dots, b_n$, let $s_k = a_1 + \cdots + a_k$. Show that

$$(*) \quad a_1b_1 + \cdots + a_nb_n = s_1(b_1 - b_2) + s_2(b_2 - b_3) + \cdots + s_{n-1}(b_{n-1} - b_n) + s_nb_n$$

This disarmingly simple formula is sometimes called "Abel's formula for summation by parts." It may be regarded as an analogue for sums of the integration by parts formula

$$\int_a^b f'(x)g(x)\,dx = f(b)g(b) - f(a)g(a) - \int_a^b f(x)g'(x)\,dx,$$

especially if we use Riemann sums (Chapter 13, Appendix). In fact, for a partition $P = \{t_0, \dots, t_n\}$ of $[a, b]$, the left side is approximately

$$(1) \qquad \sum_{k=1}^{n} f'(t_k)g(t_{k-1})(t_k - t_{k-1}),$$

while the right side is approximately

$$f(b)g(b) - f(a)g(a) - \sum_{k=1}^{n} f(t_k)g'(t_k)(t_k - t_{k-1})$$

which is approximately

$$\begin{aligned} f(b)g(b) - f(a)g(a) &- \sum_{k=1}^{n} f(t_k)\frac{g(t_k) - g(t_{k-1})}{t_k - t_{k-1}}(t_k - t_{k-1}) \\ &= f(b)g(b) - f(a)g(a) + \sum_{k=1}^{n} f(t_k)[g(t_{k-1}) - g(t_k)] \\ &= f(b)g(b) - f(a)g(a) + \sum_{k=1}^{n}[f(t_k) - f(a)] \cdot [g(t_{k-1}) - g(t_k)] \\ &\qquad + f(a)\sum_{k=1}^{n} g(t_{k-1}) - g(t_k). \end{aligned}$$

Since the right-most sum is just $g(a) - g(b)$, this works out to be

$$(2) \qquad [f(b) - f(a)]g(b) + \sum_{k=1}^{n}[f(t_k) - f(a)] \cdot [g(t_{k-1}) - g(t_k)].$$

If we choose

$$a_k = f'(t_k)(t_k - t_{k-1}), \qquad b_k = g(t_{k-1})$$

then

$$(1) \qquad \text{is} \qquad \sum_{k=1}^{n} a_kb_k,$$

which is the left side of $(*)$, while

$$s_k = \sum_{i=1}^{k} f'(t_i)(t_i - t_{i-1}) \quad \text{is approximately} \quad \sum_{i=1}^{k} f(t_i) - f(t_{i-1}) = f(t_k) - f(a),$$

so

$$(2) \qquad \text{is approximately} \qquad s_nb_n + \sum_{k=1}^{n} s_k(b_k - b_{k-1}),$$

which is the right side of $(*)$.

This discussion is not meant to suggest that Abel's formula can actually be derived from the formula for integration by parts, or *vice versa.* But, as we shall see, Abel's formula can often be used as a substitute for integration by parts in situations where the functions in question aren't differentiable.

(b) Suppose that $\{b_n\}$ is nonincreasing, with $b_n \geq 0$ for each n, and that

$$m \leq a_1 + \cdots + a_n \leq M$$

for all n. Prove Abel's Lemma:

$$b_1 m \leq a_1 b_1 + \cdots + a_n b_n \leq b_1 M.$$

(And, moreover,

$$b_k m \leq a_k b_k + \cdots + a_n b_n \leq b_k M,$$

a formula which only looks more general, but really isn't.)

(c) Let f be integrable on $[a, b]$ and let ϕ be nonincreasing on $[a, b]$ with $\phi(b) = 0$. Let $P = \{t_0, \ldots, t_n\}$ be a partition of $[a, b]$. Show that the sum

$$\sum_{i=1}^{n} f(t_{i-1})\phi(t_{i-1})(t_i - t_{i-1})$$

lies between the smallest and the largest of the sums

$$\phi(a) \sum_{i=1}^{k} f(t_{i-1})(t_i - t_{i-1}).$$

Conclude that

$$\int_a^b f(x)\phi(x)\,dx$$

lies between the minimum and the maximum of

$$\phi(a) \int_a^x f(t)\,dt,$$

and that it therefore equals $\phi(a) \displaystyle\int_a^{\xi} f(t)\,dt$ for some ξ in $[a, b]$.

37. (a) Show that the following improper integrals both converge.

(i) $\displaystyle\int_0^1 \sin\left(x + \frac{1}{x}\right) dx.$

(ii) $\displaystyle\int_0^1 \sin^2\left(x + \frac{1}{x}\right) dx.$

(b) Decide which of the following improper integrals converge.

(i) $\displaystyle\int_1^{\infty} \sin\left(\frac{1}{x}\right) dx.$

(ii) $\displaystyle\int_1^{\infty} \sin^2\left(\frac{1}{x}\right) dx.$

38. (a) Compute the (improper) integral $\int_0^1 \log x\, dx$.

(b) Show that the improper integral $\int_0^\pi \log(\sin x)\, dx$ converges.

(c) Use the substitution $x = 2u$ to show that

$$\int_0^\pi \log(\sin x)\, dx = 2\int_0^{\pi/2} \log(\sin x)\, dx + 2\int_0^{\pi/2} \log(\cos x)\, dx + \pi \log 2.$$

(d) Compute $\int_0^{\pi/2} \log(\cos x)\, dx$.

(e) Using the relation $\cos x = \sin(\pi/2 - x)$, compute $\int_0^\pi \log(\sin x)\, dx$.

39. Prove the following version of integration by parts for improper integrals:

$$\int_a^\infty u'(x)v(x)\, dx = u(x)v(x)\Big|_a^\infty - \int_a^\infty u(x)v'(x)\, dx.$$

The first symbol on the right side means, of course,

$$\lim_{x\to\infty} u(x)v(x) - u(a)v(a).$$

***40.** One of the most important functions in analysis is the gamma function,

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}\, dt.$$

(a) Prove that the improper integral $\Gamma(x)$ is defined if $x > 0$.

(b) Use integration by parts (more precisely, the improper integral version in the previous problem) to prove that

$$\Gamma(x+1) = x\Gamma(x).$$

(c) Show that $\Gamma(1) = 1$, and conclude that $\Gamma(n) = (n-1)!$ for all natural numbers n.

The gamma function thus provides a simple example of a continuous function which "interpolates" the values of $n!$ for natural numbers n. Of course there are infinitely many continuous functions f with $f(n) = (n-1)!$; there are even infinitely many continuous functions f with $f(x+1) = xf(x)$ for all $x > 0$. However, the gamma function has the important additional property that $\log \circ \Gamma$ is convex, a condition which expresses the extreme smoothness of this function. A beautiful theorem due to Harold Bohr and Johannes Mollerup states that Γ is the only function f with $\log \circ f$ convex, $f(1) = 1$ and $f(x+1) = xf(x)$. See reference [43] of the Suggested Reading.

***41.** (a) Use the reduction formula for $\int \sin^n x\, dx$ to show that

$$\int_0^{\pi/2} \sin^n x\, dx = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\, dx.$$

(b) Now show that

$$\int_0^{\pi/2} \sin^{2n+1} x\, dx = \frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdot\ldots\cdot\frac{2n}{2n+1},$$

$$\int_0^{\pi/2} \sin^{2n} x\, dx = \frac{\pi}{2}\cdot\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\ldots\cdot\frac{2n-1}{2n},$$

and conclude that

$$\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\ldots\cdot\frac{2n}{2n-1}\cdot\frac{2n}{2n+1}\frac{\displaystyle\int_0^{\pi/2}\sin^{2n} x\, dx}{\displaystyle\int_0^{\pi/2}\sin^{2n+1} x\, dx}.$$

(c) Show that the quotient of the two integrals in this expression is between 1 and $1 + 1/2n$, starting with the inequalities

$$0 < \sin^{2n+1} x \le \sin^{2n} x \le \sin^{2n-1} x \quad \text{for } 0 < x < \pi/2.$$

This result, which shows that the products

$$\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\ldots\cdot\frac{2n}{2n-1}\cdot\frac{2n}{2n+1}$$

can be made as close to $\pi/2$ as desired, is usually written as an infinite product, known as Wallis' product:

$$\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\ldots.$$

(d) Show also that the products

$$\frac{1}{\sqrt{n}}\,\frac{2\cdot 4\cdot 6\cdot\ldots\cdot 2n}{1\cdot 3\cdot 5\cdot\ldots\cdot(2n-1)}$$

can be made as close to $\sqrt{\pi}$ as desired. (This fact is used in the next problem and in Problem 27-19.)

Wallis' procedure was quite different! He worked with the integral $\int_0^1 (1-x^2)^n\, dx$ (which appears in Problem 42), hoping to recover, from the values obtained for natural numbers n, a formula for

$$\frac{\pi}{4} = \int_0^1 (1-x^2)^{1/2}\, dx.$$

A complete account can be found in reference [49] of the Suggested Reading, but the following summary gives the basic idea. Wallis first obtained the formula

$$\begin{aligned}\int_0^1 (1-x^2)^n\, dx &= \frac{2}{3}\cdot\frac{4}{5}\cdots\frac{2n}{2n+1}\\ &= \frac{(2\cdot 4\cdots 2n)^2}{2\cdot 3\cdot 4\cdots 2n(2n+1)} = \frac{2^n}{2n+1}\frac{(n!)^2}{(2n)!}.\end{aligned}$$

He then reasoned that $\pi/4$ should be

$$\int_0^1 (1-x^2)^{1/2}\, dx = \frac{2^1}{2}\frac{(\frac{1}{2}!)^2}{1!} = (\tfrac{1}{2}!)^2.$$

If we interpret $\frac{1}{2}!$ to mean $\Gamma(1+\frac{1}{2})$, this agrees with Problem 45, but Wallis did not know of the gamma function (which was invented by Euler, guided principally by Wallis' work). Since $(2n)!/(n!)^2$ is the binomial coefficient $\binom{2n}{n}$, Wallis hoped to find $\frac{1}{2}!$ by finding $\binom{p+q}{p}$ for $p=q=1/2$. Now

$$\binom{p+q}{p} = \frac{(p+q)(p+q-1)\cdots(p+1)}{q!}$$

and this makes sense even if p is not a natural number. Wallis therefore decided that

$$\binom{\frac{1}{2}+q}{\frac{1}{2}} = \frac{(\frac{1}{2}+q)\cdots(\frac{3}{2})}{q!}.$$

With this interpretation of $\binom{p+q}{p}$ for $p=1/2$, it is still true that

$$\binom{p+q+1}{p} = \frac{p+q+1}{q+1}\binom{p+q}{p}.$$

Denoting $\binom{\frac{1}{2}+q}{\frac{1}{2}}$ by $W(q)$ this equation can be written

$$W(q+1) = \frac{\frac{1}{2}+q+1}{q+1}W(q) = \frac{2q+3}{2q+2}W(q),$$

which leads to the table

q	1	2	3
$W(q)$	$\frac{3}{2}$	$\frac{3}{2}\cdot\frac{5}{4}$	$\frac{3}{2}\cdot\frac{5}{4}\cdot\frac{7}{6}$

But, since $W(\frac{1}{2})$ should be $4/\pi$, Wallis also constructs the table

q	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$
$W(q)$	$\frac{4}{\pi}$	$\frac{4}{\pi}\cdot\frac{4}{3}$	$\frac{4}{\pi}\cdot\frac{4}{3}\cdot\frac{6}{5}$

Next Wallis notes that if a_1, a_2, a_3, a_4 are 4 successive values $W(q)$, $W(q+1)$, $W(q+2)$, $W(q+3)$, appearing in either of these tables, then

$$\frac{a_2}{a_1} > \frac{a_3}{a_2} > \frac{a_4}{a_3} \qquad \text{since } \frac{2q+3}{2q+2} > \frac{2q+5}{2q+4}\cdot\frac{2q+7}{2q+6}$$

(this says that $\log\circ(1/W)$ is convex, compare the remarks before Problem 41), which implies that

$$\sqrt{\frac{a_3}{a_1}} > \frac{a_3}{a_2} > \sqrt{\frac{a_4}{a_2}}.$$

Wallis then argues that this should still be true when a_1, a_2, a_3, a_4 are four successive values in a combined table where q is given *both* integer and half-integer values! Thus, taking as the four successive values $W(n+\frac{1}{2})$, $W(n)$, $W(n+\frac{3}{2})$, $W(n+1)$, he obtains

$$\sqrt{\frac{\frac{4}{\pi}\cdot\frac{4}{3}\cdot\frac{6}{5}\cdots\frac{2n+4}{2n+3}}{\frac{4}{\pi}\cdot\frac{4}{3}\cdot\frac{6}{5}\cdots\frac{2n+2}{2n+1}}} > \frac{\frac{4}{\pi}\cdot\frac{4}{3}\cdot\frac{6}{5}\cdots\frac{2n+2}{2n+1}}{\frac{3}{2}\cdot\frac{5}{4}\cdot\frac{7}{6}\cdots\frac{2n+1}{2n}} > \sqrt{\frac{\frac{3}{2}\cdot\frac{5}{4}\cdots\frac{2n+3}{2n+2}}{\frac{3}{2}\cdot\frac{5}{4}\cdot\frac{7}{6}\cdots\frac{2n+1}{2n}}}$$

which yields simply

$$\sqrt{\frac{2n+4}{2n+3}} > \frac{4}{\pi}\cdot\left[\frac{2\cdot4\cdot4\cdot6\cdot6\cdots(2n)(2n)(2n+2)}{3\cdot3\cdot5\cdot5\cdots(2n+1)(2n+1)}\right] > \sqrt{\frac{2n+3}{2n+2}},$$

from which Wallis' product follows immediately.

****42.** It is an astonishing fact that improper integrals $\int_0^\infty f(x)\,dx$ can often be computed in cases where ordinary integrals $\int_a^b f(x)\,dx$ cannot. There is no elementary formula for $\int_a^b e^{-x^2}\,dx$, but we can find the value of $\int_0^\infty e^{-x^2}\,dx$ precisely! There are many ways of evaluating this integral, but most require some advanced techniques; the following method involves a fair amount of work, but no facts that you do not already know.

(a) Show that

$$\int_0^1 (1-x^2)^n\,dx = \frac{2}{3}\cdot\frac{4}{5}\cdot\cdots\cdot\frac{2n}{2n+1},$$
$$\int_0^\infty \frac{1}{(1+x^2)^n}\,dx = \frac{\pi}{2}\cdot\frac{1}{2}\cdot\frac{3}{4}\cdot\cdots\cdot\frac{2n-3}{2n-2}.$$

(This can be done using reduction formulas, or by appropriate substitutions, combined with the previous problem.)

(b) Prove, using the derivative, that

$$1-x^2 \le e^{-x^2} \qquad \text{for } 0 \le x \le 1.$$
$$e^{-x^2} \le \frac{1}{1+x^2} \qquad \text{for } 0 \le x.$$

(c) Integrate the nth powers of these inequalities from 0 to 1 and from 0 to ∞, respectively. Then use the substitution $y = \sqrt{n}\,x$ to show that

$$\sqrt{n}\,\frac{2}{3}\cdot\frac{4}{5}\cdot\cdots\cdot\frac{2n}{2n+1}$$
$$\le \int_0^{\sqrt{n}} e^{-y^2}\,dy \le \int_0^\infty e^{-y^2}\,dy$$
$$\le \frac{\pi}{2}\sqrt{n}\,\frac{1}{2}\cdot\frac{3}{4}\cdot\cdots\cdot\frac{2n-3}{2n-2}.$$

(d) Now use Problem 41(d) to show that

$$\int_0^\infty e^{-y^2}\,dy = \frac{\sqrt{\pi}}{2}.$$

****43.** (a) Use integration by parts to show that

$$\int_a^b \frac{\sin x}{x}\,dx = \frac{\cos a}{a} - \frac{\cos b}{b} - \int_a^b \frac{\cos x}{x^2}\,dx,$$

and conclude that $\int_0^\infty (\sin x)/x\,dx$ exists. (Use the left side to investigate the limit as $a \to 0^+$ and the right side for the limit as $b \to \infty$.)

(c) Compute the following integrals:

(i) $$\int_0^{\infty} \frac{e^{-\alpha x} - e^{-\beta x}}{x}\, dx.$$

(ii) $$\int_0^{\infty} \frac{\cos(\alpha x) - \cos(\beta x)}{x}\, dx.$$

In Chapter 13 we said, rather blithely, that integrals may be computed to any degree of accuracy desired by calculating lower and upper sums. But an applied mathematician, who really has to do the calculation, rather than just talking about doing it, may not be overjoyed at the prospect of computing lower sums to evaluate an integral to three decimal places, say (a degree of accuracy that might easily be needed in certain circumstances). The next three problems show how more refined methods can make the calculations much more efficient.

We ought to mention at the outset that computing upper and lower sums might not even be practical, since it might not be possible to compute the quantities m_i and M_i for each interval $[t_{i-1}, t_i]$. It is far more reasonable simply to pick points x_i in $[t_{i-1}, t_i]$ and consider $\displaystyle\sum_{i=1}^{n} f(x_i) \cdot (t_i - t_{i-1})$. This represents the sum of the areas of certain rectangles which partially overlap the graph of f—see Figure 1 in the Appendix to Chapter 13. But we will get a much better result if we instead choose the trapezoids shown in Figure 2.

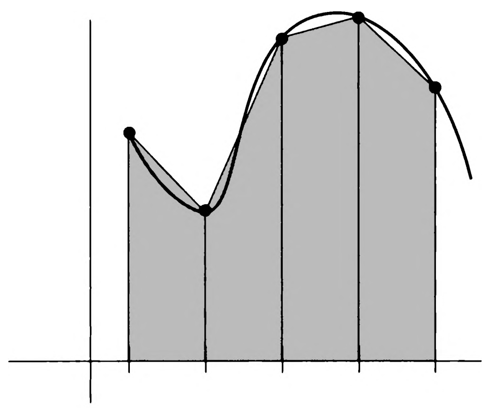


FIGURE 2

Suppose, in particular, that we divide $[a, b]$ into n equal intervals, by means of the points

$$t_i = a + i\left(\frac{b-a}{n}\right) = a + ih.$$

Then the trapezoid with base $[t_{i-1}, t_i]$ has area

$$\frac{f(t_{i-1}) + f(t_i)}{2} \cdot (t_i - t_{i-1})$$

(b) Use Problem 15-33 to show that

$$\int_0^\pi \frac{\sin(n+\frac{1}{2})t}{\sin\dfrac{t}{2}}\,dt = \pi$$

for any natural number n.

(c) Prove that

$$\lim_{\lambda\to\infty}\int_0^\pi \sin(\lambda+\tfrac{1}{2})t\left[\frac{2}{t}-\frac{1}{\sin\dfrac{t}{2}}\right]dt = 0.$$

Hint: The term in brackets is bounded by Problem 15-2(vi); the Riemann-Lebesgue Lemma then applies.

(d) Use the substitution $u = (\lambda + \frac{1}{2})t$ and part (b) to show that

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

44. Given the value of $\displaystyle\int_0^\infty (\sin x)/x\,dx$ from Problem 43, compute

$$\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx$$

by using integration by parts. (As in Problem 38, the formula for $\sin 2x$ will play an important role.)

***45.** (a) Use the substitution $u = t^x$ to show that

$$\Gamma(x) = \frac{1}{x}\int_0^\infty e^{-u^{1/x}}\,du.$$

(b) Find $\Gamma(\frac{1}{2})$.

***46.** (a) Suppose that $\dfrac{f(x)}{x}$ is integrable on every interval $[a,b]$ for $0 < a < b$, and that $\lim_{x\to 0} f(x) = A$ and $\lim_{x\to\infty} f(x) = B$. Prove that for all $\alpha, \beta > 0$ we have

$$\int_0^\infty \frac{f(\alpha x) - f(\beta x)}{x}\,dx = (A - B)\log\frac{\beta}{\alpha}.$$

Hint: To estimate $\displaystyle\int_\varepsilon^N \frac{f(\alpha x) - f(\beta x)}{x}\,dx$ use two different substitutions.

(b) Now suppose instead that $\displaystyle\int_a^\infty \frac{f(x)}{x}\,dx$ converges for all $a > 0$ and that $\lim_{x\to 0} f(x) = A$. Prove that

$$\int_0^\infty \frac{f(\alpha x) - f(\beta x)}{x}\,dx = A\log\frac{\beta}{\alpha}.$$

and the sum of all these areas is simply

$$\begin{aligned}\Sigma_n &= h\left[\frac{f(t_1)+f(a)}{2}+\frac{f(t_2)+f(t_1)}{2}+\cdots+\frac{f(b)+f(t_{n-1})}{2}\right]\\ &= \frac{h}{2}\left[f(a)+2\sum_{i=1}^{n-1} f(a+ih)+f(b)\right], \qquad h=\frac{b-a}{n}.\end{aligned}$$

This method of approximating an integral is called the *trapezoid rule.* Notice that to obtain Σ_{2n} from Σ_n it isn't necessary to recompute the old $f(t_i)$; their contribution to Σ_{2n} is just $\frac{1}{2}\Sigma_n$. So in practice it is best to compute Σ_2, Σ_4, Σ_8, ... to get approximations to $\int_a^b f$. In the next problem we will estimate $\int_a^b f - \Sigma_n$.

47. (a) Suppose that f'' is continuous. Let P_i be the linear function which agrees with f at t_{i-1} and t_i. Using Problem 11-46, show that if n_i and N_i are the minimum and maximum of f'' on $[t_{i-1}, t_i]$ and

$$I = \int_{t_{i-1}}^{t_i} (x-t_{i-1})(x-t_i)\,dx$$

then

$$\frac{n_i I}{2} \geq \int_{t_{i-1}}^{t_i} (f-P_i) \geq \frac{N_i I}{2}.$$

(b) Evaluate I to get

$$-\frac{n_i h^3}{12} \geq \int_{t_{i-1}}^{t_i} (f-P_i) \geq -\frac{N_i h^3}{12}.$$

(c) Conclude that there is some c in (a, b) with

$$\int_a^b f = \Sigma_n - \frac{(b-a)^3}{12n^2} f''(c).$$

Notice that the "error term" $(b-a)^3 f''(c)/12n^2$ varies as $1/n^2$ (while the error obtained using ordinary sums varies as $1/n$).

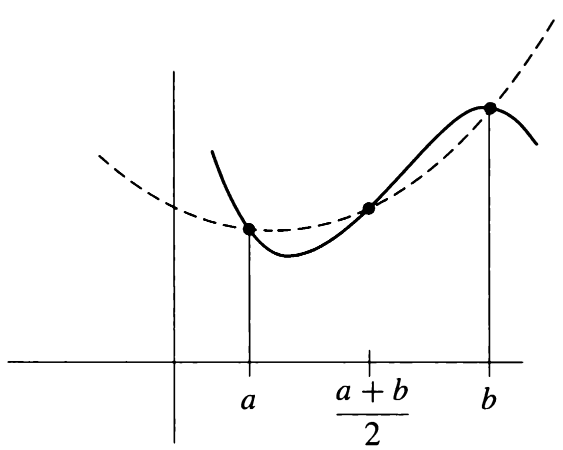


FIGURE 3

We can obtain still more accurate results if we approximate f by quadratic functions rather than by linear functions. We first consider what happens when the interval $[a, b]$ is divided into two equal intervals (Figure 3).

48. (a) Suppose first that $a = 0$ and $b = 2$. Let P be the polynomial function of degree ≤ 2 which agrees with f at 0, 1, and 2 (Problem 3-6). Show that

$$\int_0^2 P = \frac{1}{3}[f(0)+4f(1)+f(2)].$$

(b) Conclude that in the general case

$$\int_a^b P = \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right].$$

(c) Naturally $\int_a^b P = \int_a^b f$ when f is a quadratic polynomial. But, remarkably enough, this same relation holds when f is a cubic polynomial! Prove this, using Problem 11-46; note that f''' is a constant.

The previous problem shows that we do not have to do any new calculations to compute $\int_a^b Q$ when Q is a *cubic* polynomial which agrees with f at a, b, and $\frac{a+b}{2}$: we still have

$$\int_a^b Q = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right].$$

But there is much more lee-way in choosing Q, which we can use to our advantage:

49. (a) Show that there is a cubic polynomial function Q satisfying

$$Q(a) = f(a), \qquad Q(b) = f(b), \qquad Q\left(\frac{a+b}{2}\right) = f\left(\frac{a+b}{2}\right)$$

$$Q'\left(\frac{a+b}{2}\right) = f'\left(\frac{a+b}{2}\right).$$

Hint: Clearly $Q(x) = P(x) + A(x-a)(x-b)\left(x - \frac{a+b}{2}\right)$ for some A.

(b) Prove that if $f^{(4)}$ is defined on $[a,b]$, then for every x in $[a,b]$ we have

$$f(x) - Q(x) = (x-a)\left(x - \frac{a+b}{2}\right)^2 (x-b)\frac{f^{(4)}(\xi)}{4!}$$

for some ξ in (a,b). Hint: Imitate the proof of Problem 11-46.

(c) Conclude that if $f^{(4)}$ is continuous, then

$$\int_a^b f = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{(b-a)^5}{2880}f^{(4)}(c)$$

for some c in (a,b).

(d) Now divide $[a,b]$ into $2n$ intervals by means of the points

$$t_i = a + ih, \qquad h = \frac{b-a}{2n}.$$

Prove *Simpson's rule*:

$$\int_a^b f = \frac{b-a}{6n}\left(f(a) + 4\sum_{i=1}^{n} f(t_{2i-1}) + 2\sum_{i=1}^{n-1} f(t_{2i}) + f(b)\right)$$

$$- \frac{(b-a)^5}{2880n^4}f^{(4)}(\bar{c})$$

for some $\bar{c}$ in (a,b).

APPENDIX. THE COSMOPOLITAN INTEGRAL

We originally introduced integrals in order to find the area under the graph of a function, but the integral is considerably more versatile than that. For example, Problem 13-24 used the integral to express the area of a region of quite another sort. Moreover, Problem 13-25 showed that the integral can also be used to express the lengths of curves—though, as we've seen in Appendix to Chapter 13, a lot of work may be necessary to consider the general case! This result was probably a little more surprising, since the integral seems, at first blush, to be a very two-dimensional creature. Actually, the integral makes its appearance in quite a few geometric formulas, which we will present in this Appendix. To derive these formulas we will assume some results from elementary geometry (and allow a little fudging).

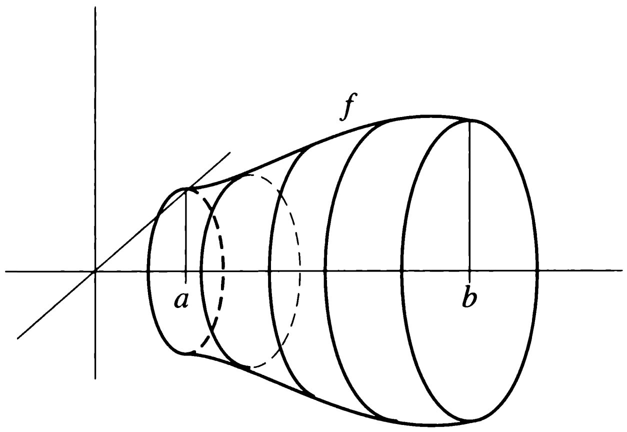


FIGURE 1

Instead of going down to one-dimensional objects, we'll begin by tackling some three-dimensional ones. There are some very special solids whose volumes can be expressed by integrals. The simplest such solid V is a "solid of revolution," obtained by revolving the region under the graph of $f \geq 0$ on $[a, b]$ around the horizontal axis, when we regard the plane as situated in space (Figure 1). If $P = \{t_0, \dots, t_n\}$ is any partition of $[a, b]$, and m_i and M_i have their usual meanings, then

$$\pi m_i{}^2(t_i - t_{i-1})$$

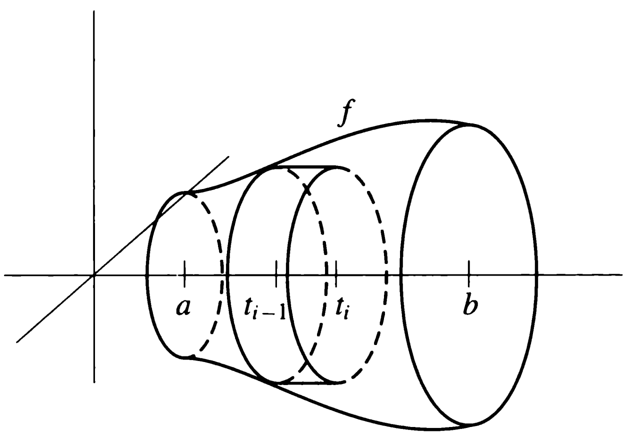


FIGURE 2

is the volume of a disc that lies inside the solid V (Figure 2). Similarly, $\pi M_i{}^2(t_i - t_{i-1})$ is the volume of a disc that contains the part of V between t_{i-1} and t_i. Consequently,

$$\pi \sum_{i=1}^{n} m_i{}^2(t_i - t_{i-1}) \leq \text{ volume } V \leq \pi \sum_{i=1}^{n} M_i{}^2(t_i - t_{i-1}).$$

But the sums on the ends of this inequality are just the lower and upper sums for f^2 on $[a, b]$:

$$\pi \cdot L(f^2, P) \leq \text{volume } V \leq \pi \cdot U(f^2, P).$$

Consequently, the volume of V must be given by

$$\text{volume } V = \pi \int_a^b f(x)^2\, dx.$$

This method of finding volumes is affectionately referred to as the "disc method."

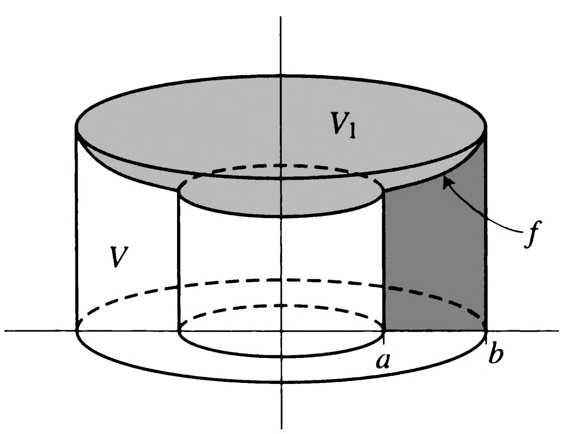


FIGURE 3

Figure 3 shows a more complicated solid V obtained by revolving the region under the graph of f around the *vertical* axis (V is the solid left over when we start with the big cylinder of radius b and take away both the small cylinder of radius a and the solid V_1 sitting right on top of it). In this case we assume $a \geq 0$ as well as $f \geq 0$. Figures 4 and 5 indicate some other possible shapes for V.

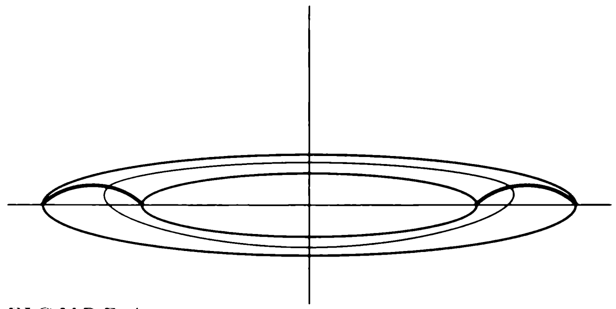


FIGURE 4

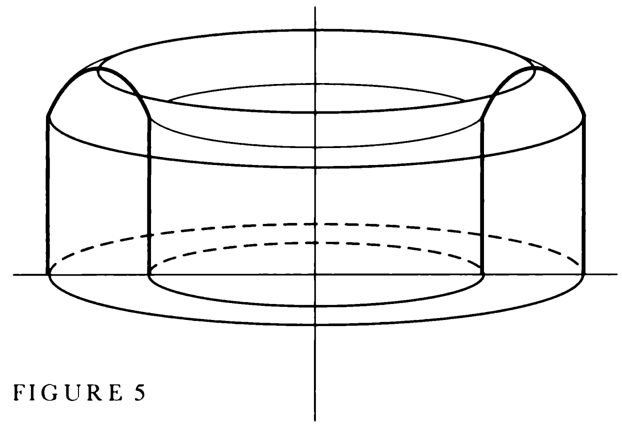


FIGURE 5

For a partition $P = \{t_0, \ldots, t_n\}$ we consider the "shells" obtained by rotating the rectangle with base $[t_{i-1}, t_i]$ and height m_i or M_i (Figure 6). Adding the volumes of these shells we obtain

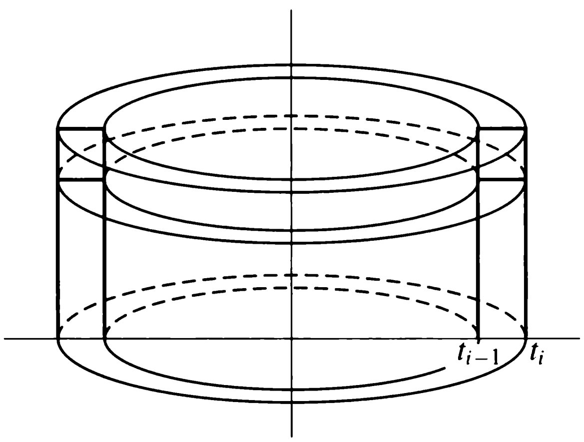


FIGURE 6

$$\pi \sum_{i=1}^{n} m_i(t_i{}^2 - t_{i-1}{}^2) \leq \text{volume } V \leq \pi \sum_{i=1}^{n} M_i(t_i{}^2 - t_{i-1}{}^2),$$

which we can write as

$$\pi \sum_{i=1}^{n} m_i(t_i + t_{i-1})(t_i - t_{i-1}) \leq \text{volume } V \leq \pi \sum_{i=1}^{n} M_i(t_i + t_{i-1})(t_i - t_{i-1}).$$

Now these sums are not lower or upper sums of anything. But Problem 1 of the Appendix to Chapter 13 shows that each sum

$$\sum_{i=1}^{n} m_i t_i(t_i - t_{i-1}) \qquad \text{and} \qquad \sum_{i=1}^{n} m_i t_{i-1}(t_i - t_{i-1})$$

can be made as close as desired to $\int_a^b xf(x)\,dx$ by choosing the lengths $t_i - t_{i-1}$ small enough. The same is true of the sums on the right, so we find that

$$\text{volume } V = 2\pi \int_a^b xf(x)\,dx;$$

this is the so-called "shell method" of finding volumes.

The surface area of certain curved regions can also be expressed in terms of integrals. Before we tackle complicated regions, a little review of elementary geometric formulas may be appreciated here.

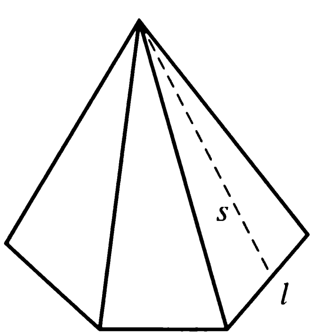


FIGURE 7

Figure 7 shows a right pyramid made up of triangles with bases of length l and altitude s. The total surface area of the sides of the pyramid is thus

$$\frac{1}{2}ps,$$

where p is the perimeter of the base. By choosing the base to be a regular polygon

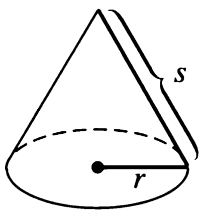


FIGURE 8

with a large number of sides we see that the area of a right circular cone (Figure 8) must be

$$\frac{1}{2}(2\pi r)s = \pi r s,$$

where s is the "slant height." Finally, consider the frustum of a cone with slant height s and radii r_1 and r_2 shown in Figure 9(a). Completing this to a cone, as in Figure 9(b), we have

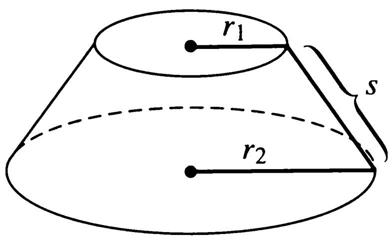
$$\frac{s_1}{r_1} = \frac{s_1 + s}{r_2},$$

so

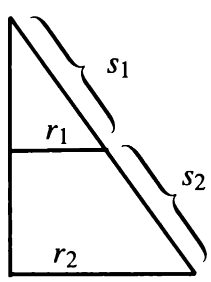
$$s_1 = \frac{r_1 s}{r_2 - r_1}, \qquad s_1 + s = \frac{r_2 s}{r_2 - r_1}.$$

Consequently, the surface area is

$$\pi r_2(s_1 + s) - \pi r_1 s_1 = \pi s \frac{r_2{}^2 - r_1{}^2}{r_2 - r_1} = \pi s(r_1 + r_2).$$



(a)



(b)

FIGURE 9

Now consider the surface formed by revolving the graph of f around the horizontal axis. For a partition $P = \{t_0, \ldots, t_n\}$ we can inscribe a series of frustums of cones, as in Figure 10. The total surface area of these frustums is

$$\pi \sum_{i=1}^{n} [f(t_{i-1}) + f(t_i)]\sqrt{(t_i - t_{i-1})^2 + [f(t_i) - f(t_{i-1})]^2}$$

$$= \pi \sum_{i=1}^{n} [f(t_{i-1}) + f(t_i)]\sqrt{1 + \left(\frac{f(t_i) - f(t_{i-1})}{t_i - t_{i-1}}\right)^2}\,(t_i - t_{i-1}).$$

By the Mean Value Theorem, this is

$$\pi \sum_{i=1}^{n} \big[f(t_{i-1}) + f(t_i)\big]\sqrt{1 + f'(x_i)^2}\,(t_i - t_{i-1})$$

for some x_i in (t_{i-1}, t_i). Appealing to Problem 1 of the Appendix to Chapter 13, we conclude that the surface area is

$$2\pi \int_a^b f(x)\sqrt{1 + f'(x)^2}\,dx.$$

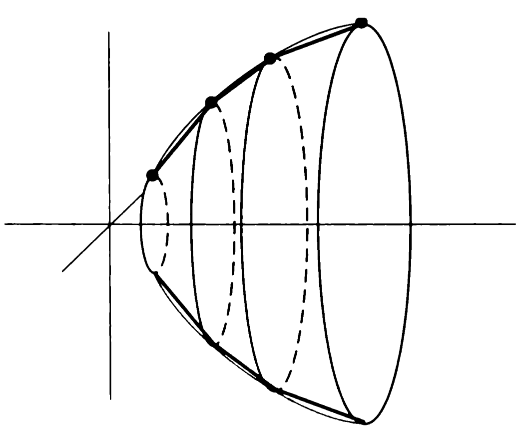


FIGURE 10

PROBLEMS

1. (a) Find the volume of the solid obtained by revolving the region bounded by the graphs of $f(x) = x$ and $f(x) = x^2$ around the horizontal axis.
(b) Find the volume of the solid obtained by revolving this same region around the vertical axis.

2. Find the volume of a sphere of radius r.

3. When the ellipse consisting of all points (x, y) with $x^2/a^2 + y^2/b^2 = 1$ is rotated around the horizontal axis we obtain an "ellipsoid of revolution" (Figure 11). Find the volume of the enclosed solid.

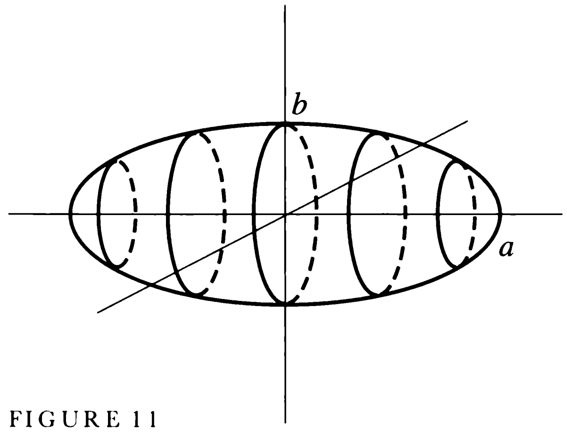


FIGURE 11

4. Find the volume of the "torus" (Figure 12), obtained by rotating the circle $(x - a)^2 + y^2 = b^2$ $(a > b)$ around the vertical axis.

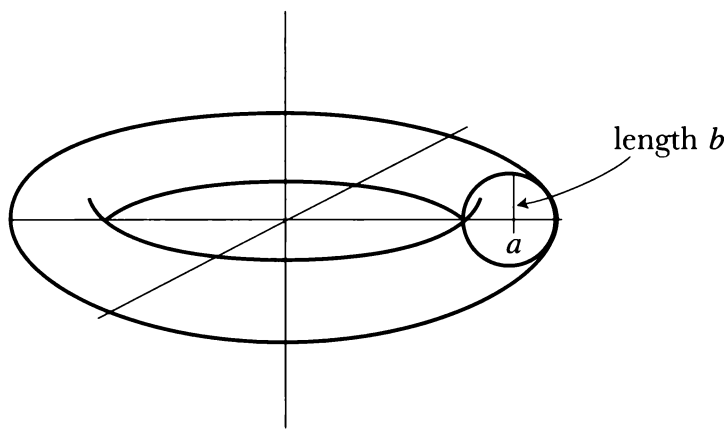


FIGURE 12

5. A cylindrical hole of radius a is bored through the center of a sphere of radius $2a$ (Figure 13). Find the volume of the remaining solid.

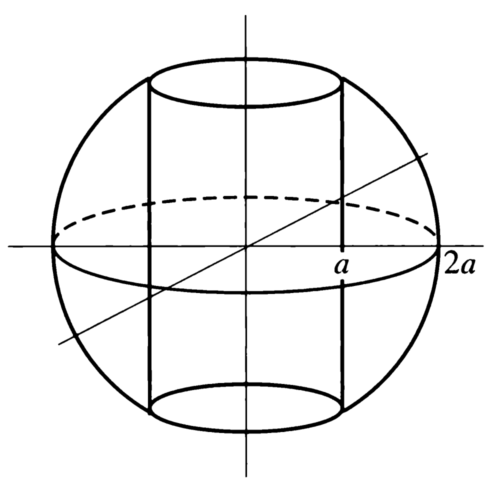


FIGURE 13

6. (a) For the solid shown in Figure 14, find the volume by the shell method.

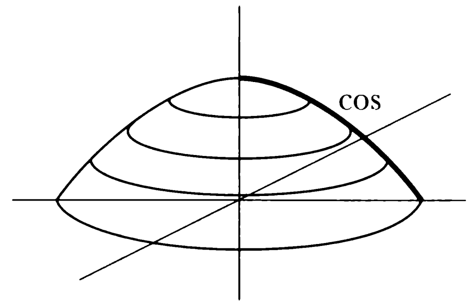


FIGURE 14

(b) This volume can also be evaluated by the disc method. Write down the integral which must be evaluated in this case; notice that it is more complicated. The next problem takes up a question which this might suggest.

7. Figure 15 shows a cylinder of height b and radius $f(b)$, divided into three solids, one of which, V_1, is a cylinder of height a and radius $f(a)$. If f is one-one, then a comparison of the disk method and the shell method of computing volumes leads us to believe that

$$\begin{aligned}\pi b f(b)^2 - \pi a f(a)^2 - \pi \int_a^b f(x)^2\,dx &= \text{volume } V_2\\ &= 2\pi \int_{f(a)}^{f(b)} y f^{-1}(y)\,dy.\end{aligned}$$

Prove this analytically, using the formula for $\int f^{-1}$ from Problem 19-16, or more simply by going through the steps by which this formula was derived.

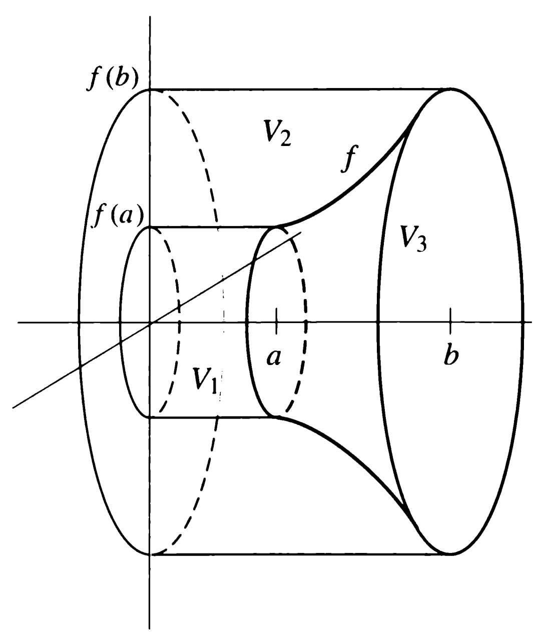


FIGURE 15

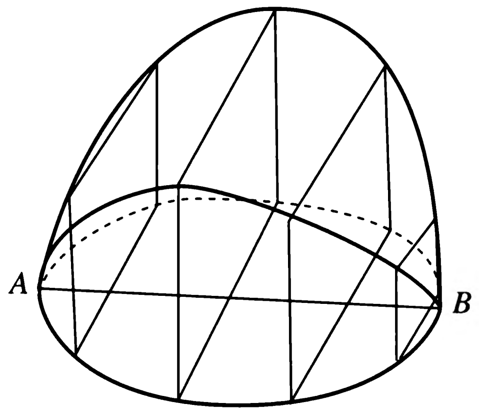


FIGURE 16

8. (a) Figure 16 shows a solid with a circular base of radius a. Each plane perpendicular to the diameter AB intersects the solid in a square. Using arguments similar to those already used in this Appendix, express the volume of the solid as an integral, and evaluate it.
(b) Same problem if each plane intersects the solid in an equilateral triangle.

9. Find the volume of a pyramid (Figure 17) in terms of its height h and the area A of its base.

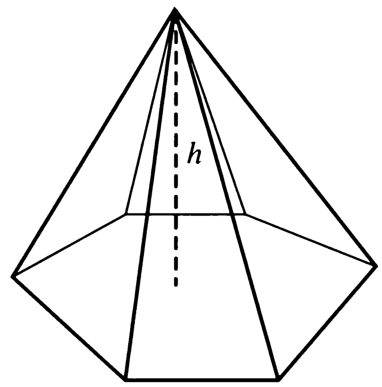


FIGURE 17

10. Find the volume of the solid which is the intersection of the two cylinders in Figure 18. Hint: Find the intersection of this solid with each horizontal plane.

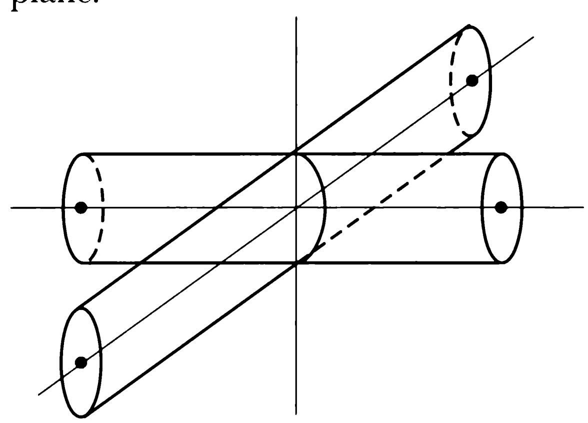


FIGURE 18

11. (a) Prove that the surface area of a sphere of radius r is $4\pi r^2$.
(b) Prove, more generally, that the area of the portion of the sphere shown in Figure 19 is $2\pi rh$. (Notice that this depends only on h, not on the position of the planes.)

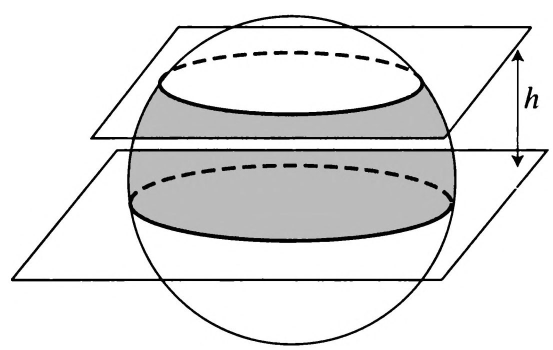
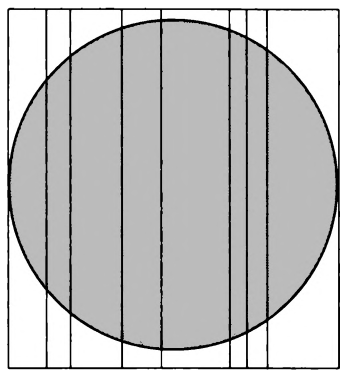


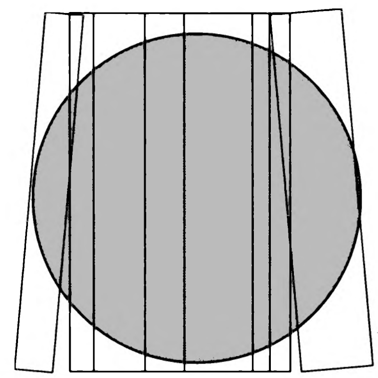
FIGURE 19

PART 4
INFINITE SEQUENCES AND INFINITE SERIES

(c) A circular mud puddle can just be covered by a parallel collection of boards of length at least the radius of the circle, as in Figure 20 (a). Prove that it cannot be covered by the same boards if they are arranged in any non-parallel configuration, as in (b).



(a)



(b)

FIGURE 20

12. (a) Find the surface area of the ellipsoid of revolution in Problem 19-3.
(b) Find the surface area of the torus in Problem 19-4.

13. The graph of $f(x) = 1/x$, $x \geq 1$ is revolved around the horizontal axis (Figure 21).

(a) Find the volume of the enclosed "infinite trumpet."
(b) Show that the surface area is infinite.
(c) Suppose that we fill up the trumpet with the finite amount of paint found in part (a). It would seem that we have thereby coated the infinite inside surface area with only a finite amount of paint. How is this possible?

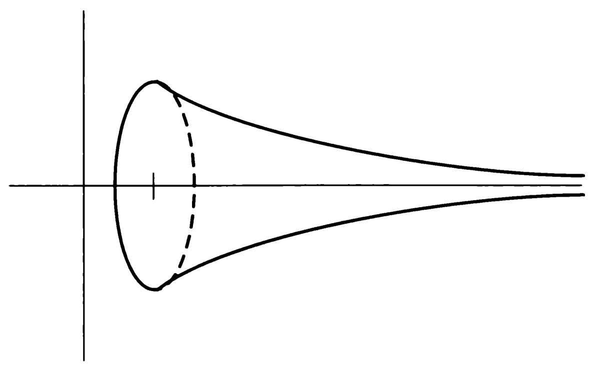


FIGURE 21

One of the most remarkable series of
algebraic analysis is the following:

$$1 + \frac{m}{1}x + \frac{m(m-1)}{1 \cdot 2}x^2 + \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3}x^3 + \cdots + \frac{m(m-1)\cdots[m-(n-1)]}{1 \cdot 2 \cdot \cdots \cdot n}x^n + \cdots$$

When m is a positive whole number
the sum of the series,
which is then finite, can be expressed,
as is known, by $(1 + x)^m$.
When m is not an integer,
the series goes on to infinity, and it will
converge or diverge according
as the quantities
m and x have this or that value.
In this case, one writes the same equality

$$(1 + x)^m = 1 + \frac{m}{1}x + \frac{m(m-1)}{1 \cdot 2}x^2 + \cdots \textit{ etc.}$$

. . . It is assumed that
the numerical equality will always occur
whenever the series is convergent, but
this has never yet been proved.

NIELS HENRIK ABEL