*CHAPTER 16 π IS IRRATIONAL

This short chapter, diverging from the main stream of the book, is included to demonstrate that we are already in a position to do some sophisticated mathematics. This entire chapter is devoted to an elementary proof that π is irrational. Like many "elementary" proofs of deep theorems, the motivation for many steps in our proof cannot be supplied; nevertheless, it is still quite possible to follow the proof step-by-step.

Two observations must be made before the proof. The first concerns the function

$$f_n(x) = \frac{x^n(1-x)^n}{n!},$$

which clearly satisfies

$$0 < f_n(x) < \frac{1}{n!} \quad \text{for } 0 < x < 1.$$

An important property of the function f_n is revealed by considering the expression obtained by actually multiplying out $x^n(1-x)^n$. The lowest power of x appearing will be n and the highest power will be $2n$. Thus f_n can be written in the form

$$f_n(x) = \frac{1}{n!}\sum_{i=n}^{2n} c_i x^i,$$

where the numbers c_i are integers. It is clear from this expression that

$$f_n{}^{(k)}(0) = 0 \quad \text{if } k < n \text{ or } k > 2n.$$

Moreover,

$$f_n{}^{(n)}(x) = \frac{1}{n!}[n!\,c_n + \text{ terms involving } x]$$

$$f_n{}^{(n+1)}(x) = \frac{1}{n!}[(n+1)!\,c_{n+1} + \text{ terms involving } x]$$

$$\vdots$$

$$f_n{}^{(2n)}(x) = \frac{1}{n!}[(2n)!\,c_{2n}].$$

This means that

$$
\begin{aligned}
f_n{}^{(n)}(0) &= c_n, \\
f_n{}^{(n+1)}(0) &= (n+1)c_{n+1} \\
&\vdots \\
f_n{}^{(2n)}(0) &= (2n)(2n-1)\cdot\ldots\cdot(n+1)c_{2n},
\end{aligned}
$$

where the numbers on the right are all integers. Thus

$$f_n{}^{(k)}(0) \text{ is an } \textit{integer} \text{ for all } k.$$

The relation

$$f_n(x) = f_n(1-x)$$

implies that

$$f_n{}^{(k)}(x) = (-1)^k f_n{}^{(k)}(1-x);$$

therefore,

$$f_n{}^{(k)}(1) \text{ is also an } \textit{integer} \text{ for all } k.$$

The proof that π is irrational requires one further observation: if a is any positive number, and $\varepsilon > 0$, then for sufficiently large n we will have

$$\frac{a^n}{n!} < \varepsilon.$$

To prove this, notice that if $n \geq 2a$, then

$$\frac{a^{n+1}}{(n+1)!} = \frac{a}{n+1}\cdot\frac{a^n}{n!} < \frac{1}{2}\cdot\frac{a^n}{n!}.$$

Now let n_0 be any natural number with $n_0 \geq 2a$. Then, whatever value

$$\frac{a^{n_0}}{(n_0)!}$$

may have, the succeeding values satisfy

$$
\begin{aligned}
\frac{a^{(n_0+1)}}{(n_0+1)!} &< \frac{1}{2}\cdot\frac{a^{n_0}}{(n_0)!} \\
\frac{a^{(n_0+2)}}{(n_0+2)!} &< \frac{1}{2}\cdot\frac{a^{(n_0+1)}}{(n_0+1)!} < \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{a^{n_0}}{(n_0)!} \\
&\vdots \\
\frac{a^{(n_0+k)}}{(n_0+k)!} &< \frac{1}{2^k}\cdot\frac{a^{n_0}}{(n_0)!}.
\end{aligned}
$$

If k is so large that $\dfrac{a^{n_0}}{(n_0)!\,\varepsilon} < 2^k$, then

$$\frac{a^{(n_0+k)}}{(n_0+k)!} < \varepsilon,$$

which is the desired result. Having made these observations, we are ready for the one theorem in this chapter.

THEOREM 1 The number π is irrational; in fact, π^2 is irrational. (Notice that the irrationality of π^2 implies the irrationality of π, for if π were rational, then π^2 certainly would be.)

PROOF Suppose π^2 were rational, so that

$$\pi^2 = \frac{a}{b}$$

for some positive integers a and b. Let

(1) $$G(x) = b^n[\pi^{2n} f_n(x) - \pi^{2n-2} f_n{}''(x) + \pi^{2n-4} f_n{}^{(4)}(x) - \cdots + (-1)^n f_n{}^{(2n)}(x)].$$

Notice that each of the factors

$$b^n \pi^{2n-2k} = b^n(\pi^2)^{n-k} = b^n\left(\frac{a}{b}\right)^{n-k} = a^{n-k}b^k$$

is an integer. Since $f_n{}^{(k)}(0)$ and $f_n{}^{(k)}(1)$ are integers, this shows that

$$G(0) \text{ and } G(1) \text{ are integers.}$$

Differentiating G twice yields

(2) $$G''(x) = b^n[\pi^{2n} f_n{}''(x) - \pi^{2n-2} f_n{}^{(4)}(x) + \cdots + (-1)^n f_n{}^{(2n+2)}(x)].$$

The last term, $(-1)^n f_n{}^{(2n+2)}(x)$, is zero. Thus, adding (1) and (2) gives

(3) $$G''(x) + \pi^2 G(x) = b^n \pi^{2n+2} f_n(x) = \pi^2 a^n f_n(x).$$

Now let

$$H(x) = G'(x)\sin \pi x - \pi G(x) \cos \pi x.$$

Then

$$\begin{aligned} H'(x) &= \pi G'(x)\cos\pi x + G''(x)\sin\pi x - \pi G'(x)\cos\pi x + \pi^2 G(x)\sin\pi x \\ &= [G''(x) + \pi^2 G(x)]\sin\pi x \\ &= \pi^2 a^n f_n(x)\sin\pi x, \text{ by (3).} \end{aligned}$$

By the Second Fundamental Theorem of Calculus,

$$\begin{aligned} \pi^2 \int_0^1 a^n f_n(x)\sin\pi x\,dx &= H(1) - H(0) \\ &= G'(1)\sin\pi - \pi G(1)\cos\pi - G'(0)\sin 0 + \pi G(0)\cos 0 \\ &= \pi[G(1) + G(0)]. \end{aligned}$$

Thus

$$\pi \int_0^1 a^n f_n(x)\sin\pi x\,dx \quad \text{is an } \textit{integer.}$$

On the other hand, $0 < f_n(x) < 1/n!$ for $0 < x < 1$, so

$$0 < \pi a^n f_n(x) \sin \pi x < \frac{\pi a^n}{n!} \quad \text{for } 0 < x < 1.$$

Consequently,

$$0 < \pi \int_0^1 a^n f_n(x) \sin \pi x \, dx < \frac{\pi a^n}{n!}.$$

This reasoning was completely independent of the value of n. Now if n is large enough, then

$$0 < \pi \int_0^1 a^n f_n(x) \sin \pi x \, dx < \frac{\pi a^n}{n!} < 1.$$

But this is absurd, because the integral is an integer, and there is no integer between 0 and 1. Thus our original assumption must have been incorrect: π^2 is irrational. ▮

This proof is admittedly mysterious; perhaps most mysterious of all is the way that π enters the proof—it almost looks as if we have proved π irrational without ever mentioning a definition of π. A close reexamination of the proof will show that precisely one property of π is essential—

$$\sin(\pi) = 0.$$

The proof really depends on the properties of the function sin, and proves the irrationality of the smallest positive number x with $\sin x = 0$. In fact, very few properties of sin are required, namely,

$$\begin{aligned} \sin' &= \cos, \\ \cos' &= -\sin, \\ \sin(0) &= 0, \\ \cos(0) &= 1. \end{aligned}$$

Even this list could be shortened; as far as the proof is concerned, cos might just as well be defined as $\sin'$. The properties of sin required in the proof may then be written

$$\begin{aligned} \sin'' + \sin &= 0, \\ \sin(0) &= 0, \\ \sin'(0) &= 1. \end{aligned}$$

Of course, this is not really very surprising at all, since, as we have seen in the previous chapter, these properties characterize the function sin completely.

PROBLEMS

1. (a) For the areas of triangles OAB and OAC in Figure 1, with $\angle AOB \le \pi/4$, show that we have

$$\text{area } OAC = \frac{1}{2}\sqrt{\frac{1-\sqrt{1-16(\text{area } OAB)^2}}{2}}.$$

Hint: Solve the equations $xy = 2(\text{area } OAB)$, $x^2 + y^2 = 1$, for y.

(b) Let P_m be the regular polygon of m sides inscribed in the unit circle. If A_m is the area of P_m show that

$$A_{2m} = \frac{m}{2}\sqrt{2-2\sqrt{1-(2A_m/m)^2}}.$$

This result allows one to obtain (more and more complicated) expressions for A_{2^n}, starting with $A_4 = 2$, and thus to compute π as accurately as desired (according to Problem 8-11). Although better methods will appear in Chapter 20, a slight variant of this approach yields a very interesting expression for π:

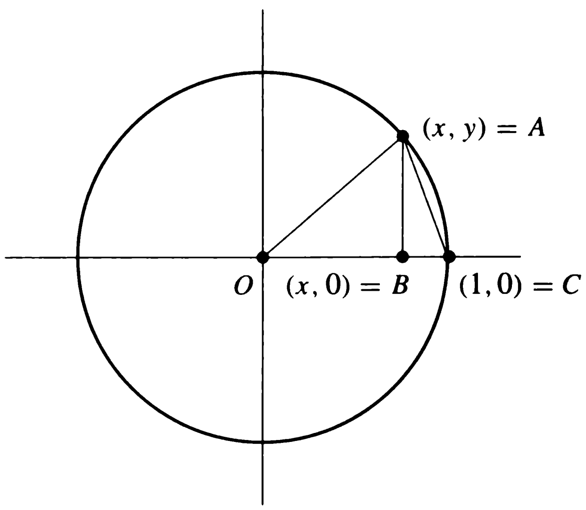

FIGURE 1

2. (a) Using the fact that

$$\frac{\text{area}(OAB)}{\text{area}(OAC)} = OB,$$

show that if α_m is the distance from O to one side of P_m, then

$$\frac{A_m}{A_{2m}} = \alpha_m.$$

(b) Show that

$$\frac{2}{A_{2^k}} = \alpha_4 \cdot \alpha_8 \cdot \ldots \cdot \alpha_{2^{k-1}}.$$

(c) Using the fact that

$$\alpha_m = \cos\frac{\pi}{m},$$

and the formula $\cos x/2 = \sqrt{\dfrac{1+\cos x}{2}}$ (Problem 15-15), prove that

$$\alpha_4 = \sqrt{\frac{1}{2}}$$

$$\alpha_8 = \sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}},$$

$$\alpha_{16} = \sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}},$$

etc.

Together with part (b), this shows that $2/\pi$ can be written as an "infinite product"

$$\frac{2}{\pi} = \sqrt{\frac{1}{2}} \cdot \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}} \cdot \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}}} \cdot \dots ;$$

to be precise, this equation means that the product of the first n factors can be made as close to $2/\pi$ as desired, by choosing n sufficiently large. This product was discovered by François Viète in 1579, and is only one of many fascinating expressions for π, some of which are mentioned later.