CHAPTER 27 COMPLEX POWER SERIES

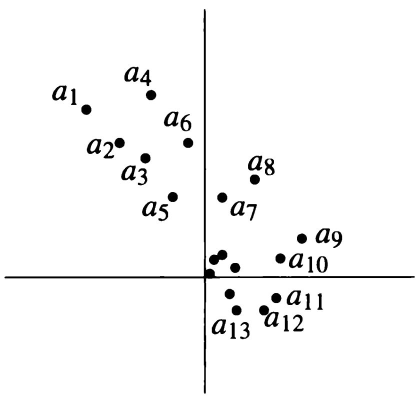

FIGURE 1

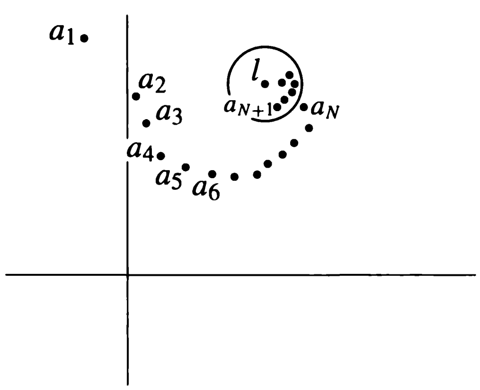

FIGURE 2

If you have not already guessed where differentiable complex functions are going to come from, the title of this chapter should give the secret away: we intend to define functions by means of infinite series. This will necessitate a discussion of infinite sequences of complex numbers, and sums of such sequences, but (as was the case with limits and continuity) the basic definitions are almost exactly the same as for real sequences and series.

An **infinite sequence** of complex numbers is, formally, a complex-valued function whose domain is **N**; the convenient subscript notation for sequences of real numbers will also be used for sequences of complex numbers. A sequence $\{a_n\}$ of complex numbers is most conveniently pictured by labeling the points a_n in the plane (Figure 1).

The sequence shown in Figure 1 converges to 0, "convergence" of complex sequences being defined precisely as for real sequences: the sequence $\{a_n\}$ **converges** to l, in symbols

$$\lim_{n\to\infty} a_n = l,$$

if for every $\varepsilon > 0$ there is a natural number N such that, for all n,

$$\text{if } n > N, \text{ then } |a_n - l| < \varepsilon.$$

This condition means that any circle drawn around l will contain a_n for all sufficiently large n (Figure 2); expressed more colloquially, the sequence is eventually inside any circle drawn around l.

Convergence of complex sequences is not only defined precisely as for real sequences, but can even be reduced to this familiar case.

THEOREM 1 Let

$$a_n = b_n + ic_n \qquad \text{for real } b_n \text{ and } c_n,$$

and let

$$l = \beta + i\gamma \qquad \text{for real } \beta \text{ and } \gamma.$$

Then $\lim_{n\to\infty} a_n = l$ if and only if

$$\lim_{n\to\infty} b_n = \beta \qquad \text{and} \qquad \lim_{n\to\infty} c_n = \gamma.$$

PROOF The proof is left as an easy exercise. If there is any doubt as to how to proceed, consult the similar Theorem 1 of Chapter 26. ∎

The **sum** of a sequence $\{a_n\}$ is defined, once again, as $\lim_{n\to\infty} s_n$, where

$$s_n = a_1 + \cdots + a_n.$$

Sequences for which this limit exists are **summable**; alternatively, we may say that the infinite series $\sum_{n=1}^{\infty} a_n$ **converges** if this limit exists, and **diverges** otherwise. It is unnecessary to develop any new tests for convergence of infinite series, because of the following theorem.

THEOREM 2 Let

$$a_n = b_n + ic_n \qquad \text{for real } b_n \text{ and } c_n.$$

Then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} c_n$ both converge, and in this case

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n + i\left(\sum_{n=1}^{\infty} c_n\right).$$

PROOF This is an immediate consequence of Theorem 1 applied to the sequence of partial sums of $\{a_n\}$. ▮

There is also a notion of absolute convergence for complex series: the series $\sum_{n=1}^{\infty} a_n$ **converges absolutely** if the series $\sum_{n=1}^{\infty} |a_n|$ converges (this is a series of real numbers, and consequently one to which our earlier tests may be applied). The following theorem is not quite so easy as the preceding two.

THEOREM 3 Let

$$a_n = b_n + ic_n \qquad \text{for real } b_n \text{ and } c_n.$$

Then $\sum_{n=1}^{\infty} a_n$ converges absolutely if and only if $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} c_n$ both converge absolutely.

PROOF Suppose first that $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} c_n$ both converge absolutely, i.e., that $\sum_{n=1}^{\infty} |b_n|$ and $\sum_{n=1}^{\infty} |c_n|$ both converge. It follows that $\sum_{n=1}^{\infty} |b_n| + |c_n|$ converges. Now,

$$|a_n| = |b_n + ic_n| \le |b_n| + |c_n|.$$

It follows from the comparison test that $\sum_{n=1}^{\infty} |a_n|$ converges (the numbers $|a_n|$ and $|b_n| + |c_n|$ are real and nonnegative). Thus $\sum_{n=1}^{\infty} a_n$ converges absolutely.

Now suppose that $\sum_{n=1}^{\infty} |a_n|$ converges. Since

$$|a_n| = \sqrt{b_n{}^2 + c_n{}^2},$$

it is clear that

$$|b_n| \le |a_n| \quad \text{and} \quad |c_n| \le |a_n|.$$

Once again, the comparison test shows that $\sum_{n=1}^{\infty} |b_n|$ and $\sum_{n=1}^{\infty} |c_n|$ converge. ▮

Two consequences of Theorem 3 are particularly noteworthy. If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} c_n$ also converge absolutely; consequently $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} c_n$ converge, by Theorem 23-5, so $\sum_{n=1}^{\infty} a_n$ converges by Theorem 2. In other words, absolute convergence implies convergence. Similar reasoning shows that any rearrangement of an absolutely convergent series has the same sum. These facts can also be proved directly, without using the corresponding theorems for real numbers, by first establishing an analogue of the Cauchy criterion (see Problem 13).

With these preliminaries safely disposed of, we can now consider **complex power series**, that is, functions of the form

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots.$$

Here the numbers a and a_n are allowed to be complex, and we are naturally interested in the behavior of f for complex z. As in the real case, we shall usually consider power series centered at 0,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n;$$

in this case, if $f(z_0)$ converges, then $f(z)$ will also converge for $|z| < |z_0|$. The proof of this fact will be similar to the proof of Theorem 24-6, but, for reasons that will soon become clear, we will not use all the paraphernalia of uniform convergence and the Weierstrass M-test, even though they have complex analogues. Our next theorem consequently generalizes only a small part of Theorem 24-6.

THEOREM 4 Suppose that

$$\sum_{n=0}^{\infty} a_n z_0{}^n = a_0 + a_1 z_0 + a_2 z_0{}^2 + \cdots$$

converges for some $z_0 \neq 0$. Then if $|z| < |z_0|$, the two series

$$\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots$$

$$\sum_{n=1}^{\infty} n a_n z^{n-1} = a_1 + 2a_2 z + 3a_3 z^2 + \cdots$$

both converge absolutely.

PROOF As in the proof of Theorem 24-6, we will need only the fact that the set of numbers $a_n {z_0}^n$ is bounded: there is a number M such that

$$|a_n {z_0}^n| \leq M \qquad \text{for all } n.$$

We then have

$$\begin{aligned} |a_n z^n| &= |a_n {z_0}^n| \cdot \left| \frac{z}{z_0} \right|^n \\ &\leq M \left| \frac{z}{z_0} \right|^n, \end{aligned}$$

and, for $z \neq 0$,

$$\begin{aligned} |n a_n z^{n-1}| &= \frac{1}{|z|} n |a_n {z_0}^n| \cdot \left| \frac{z}{z_0} \right|^n \\ &\leq \frac{M}{|z|} n \left| \frac{z}{z_0} \right|^n. \end{aligned}$$

Since the series $\sum_{n=0}^{\infty} |z/z_0|^n$ and $\sum_{n=1}^{\infty} n\, |z/z_0|^n$ converge, this shows that both $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=1}^{\infty} n a_n z^{n-1}$ converge absolutely (the argument for $\sum_{n=1}^{\infty} n a_n z^{n-1}$ assumed that $z \neq 0$, but this series certainly converges for $z = 0$ also). ▮

Theorem 4 evidently restricts greatly the possibilities for the set

$$\left\{ z : \sum_{n=0}^{\infty} a_n z^n \text{ converges} \right\}.$$

For example, the shaded set A in Figure 3 cannot be the set of all z where $\sum_{n=0}^{\infty} a_n z^n$ converges, since it contains z, but not the number w satisfying $|w| < |z|$.

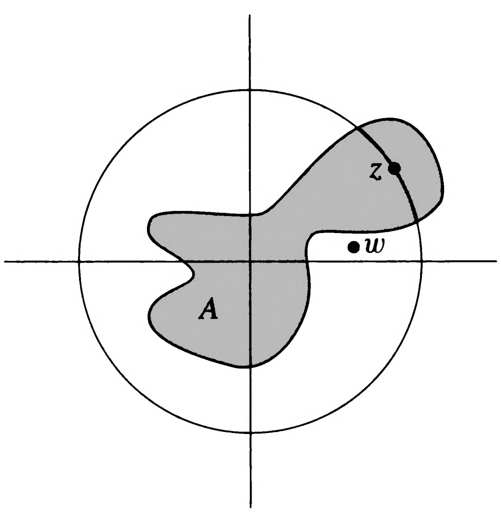

FIGURE 3

It seems quite unlikely that the set of points where a power series converges could be anything except the set of points inside a circle. If we allow "circles of radius 0" (when the power series converges only at 0) and "circles of radius ∞" (when the power series converges at all points), then this assertion is true (with one complication which we will soon mention); the proof requires only Theorem 4 and a knack for good organization.

THEOREM 5 For any power series

$$\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots$$

one of the following three possibilities must be true:

(1) $\sum_{n=0}^{\infty} a_n z^n$ converges only for $z = 0$.

(2) $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely for all z in $\mathbf{C}$.

(3) There is a number $R > 0$ such that $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely if $|z| < R$ and diverges if $|z| > R$. (Notice that we do not mention what happens when $|z| = R$.)

PROOF Let

$$S = \left\{ x \text{ in } \mathbf{R} : \sum_{n=0}^{\infty} a_n w^n \text{ converges for some } w \text{ with } |w| = x \right\}.$$

Suppose first that S is unbounded. Then for any complex number z, there is a number x in S such that $|z| < x$. By definition of S, this means that $\sum_{n=0}^{\infty} a_n w^n$ converges for some w with $|w| = x > |z|$. It follows from Theorem 4 that $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely. Thus, in this case possibility (2) is true.

Now suppose that S is bounded, and let R be the least upper bound of S. If $R = 0$, then $\sum_{n=0}^{\infty} a_n z^n$ converges only for $z = 0$, so possibility (1) is true. Suppose, on the other hand, that $R > 0$. Then if z is a complex number with $|z| < R$, there is a number x in S with $|z| < x$. Once again, this means that $\sum_{n=0}^{\infty} a_n w^n$ converges for some w with $|z| < |w|$, so that $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely. Moreover, if $|z| > R$, then $\sum_{n=0}^{\infty} a_n z^n$ does not converge, since $|z|$ is not in S. ▮

The number R which occurs in case (3) is called the **radius of convergence** of $\sum_{n=0}^{\infty} a_n z^n$. In cases (1) and (2) it is customary to say that the radius of convergence is 0 and ∞, respectively. When $0 < R < \infty$, the circle $\{z : |z| = R\}$ is called the **circle of convergence** of $\sum_{n=0}^{\infty} a_n z^n$. If z is outside the circle, then, of course, $\sum_{n=0}^{\infty} a_n z^n$ does not converge, but actually a much stronger statement can be made: the terms $a_n z^n$ are not even bounded. To prove this, let w be any number with $|z| > |w| > R$; if the terms $a_n z^n$ were bounded, then the proof of Theorem 4 would show that $\sum_{n=0}^{\infty} a_n w^n$ converges, which is false. Thus (Figure 4), inside the circle of convergence the series $\sum_{n=0}^{\infty} a_n z^n$ converges in the best possible way (absolutely) and outside the circle the series diverges in the worst possible way (the terms $a_n z^n$ are not bounded).

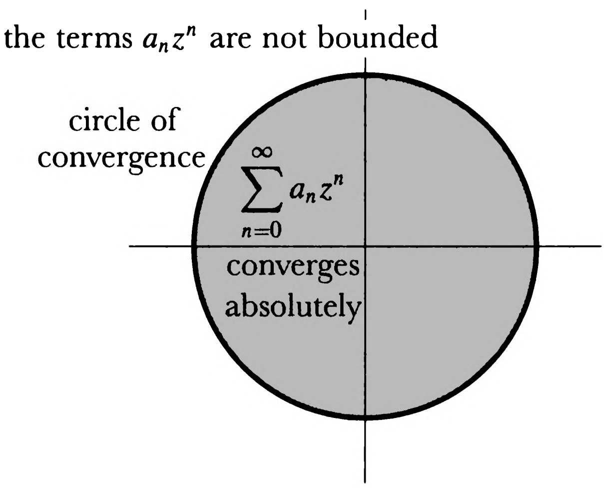

FIGURE 4

What happens *on* the circle of convergence is a much more difficult question. We will not consider that question at all, except to mention that there are power series which converge everywhere on the circle of convergence, power series which converge nowhere on the circle of convergence, and power series that do just about anything in between. (See Problem 5.)

Algebraic manipulations on complex power series can be justified just as in the real case. Thus, if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ both have radius of convergence $\geq R$, then $h(z) = \sum_{n=0}^{\infty} (a_n + b_n) z^n$ also has radius of convergence $\geq R$ and $h = f + g$ inside the circle of radius R. Similarly, the Cauchy product $h(z) = \sum_{n=0}^{\infty} c_n z^n$, for $c_n = \sum_{k=0}^{n} a_k b_{n-k}$, has radius of convergence $\geq R$ and $h = fg$ inside the circle of radius R. And if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has radius of convergence > 0 and $a_0 \neq 0$, then we can find a power series $\sum_{n=0}^{\infty} b_n z^n$ with radius of convergence > 0 which represents $1/f$ inside its circle of convergence.

But our real goal in this chapter is to produce differentiable functions. We therefore want to generalize the result proved for real power series in Chapter 24, that a function defined by a power series can be differentiated term-by-term inside the circle of convergence. At this point we can no longer imitate the proof of Chapter 24, even if we were willing to introduce uniform convergence, because no analogue of Theorem 24-3 seems available. Instead we will use a direct argument (which could also have been used in Chapter 24). Before beginning the proof, we notice that at least there is no problem about the convergence of the series produced by term-by-term differentiation. If the series $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence R, then Theorem 4 immediately implies that the series $\sum_{n=1}^{\infty} n a_n z^{n-1}$ also converges for $|z| < R$. Moreover, if $|z| > R$, so that the terms $a_n z^n$ are unbounded, then the terms $na_n z^{n-1}$ are surely unbounded, so $\sum_{n=1}^{\infty} na_n z^{n-1}$ does not converge.

This shows that the radius of convergence of $\sum_{n=1}^{\infty} na_n z^{n-1}$ is also exactly R.

THEOREM 6 If the power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

has radius of convergence $R > 0$, then f is differentiable at z for all z with $|z| < R$, and

$$f'(z) = \sum_{n=1}^{\infty} na_n z^{n-1}.$$

PROOF We will use another "$\varepsilon/3$ argument." The fact that the theorem is clearly true for polynomial functions suggests writing

$$(*) \quad \left| \frac{f(z+h) - f(z)}{h} - \sum_{n=1}^{\infty} na_n z^{n-1} \right| = \left| \sum_{n=0}^{\infty} a_n \frac{((z+h)^n - z^n)}{h} - \sum_{n=1}^{\infty} na_n z^{n-1} \right|$$
$$\leq \left| \sum_{n=0}^{\infty} a_n \frac{((z+h)^n - z^n)}{h} - \sum_{n=0}^{N} a_n \frac{((z+h)^n - z^n)}{h} \right|$$
$$+ \left| \sum_{n=0}^{N} a_n \frac{((z+h)^n - z^n)}{h} - \sum_{n=1}^{N} na_n z^{n-1} \right|$$
$$+ \left| \sum_{n=1}^{N} na_n z^{n-1} - \sum_{n=1}^{\infty} na_n z^{n-1} \right|.$$

We will show that for any $\varepsilon > 0$, each absolute value on the right side can be made $< \varepsilon/3$ by choosing N sufficiently large and h sufficiently small. This will clearly prove the theorem.

Only the first term in the right side of $(*)$ will present any difficulties. To begin with, choose some z_0 with $|z| < |z_0| < R$; henceforth we will consider only h with $|z+h| \leq |z_0|$. The expression $((z+h)^n - z^n)/h$ can be written in a more convenient way if we remember that

$$\frac{x^n - y^n}{x - y} = x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + y^{n-1}.$$

Applying this to

$$\frac{(z+h)^n - z^n}{h} = \frac{(z+h)^n - z^n}{(z+h) - z},$$

we obtain

$$\frac{(z+h)^n - z^n}{h} = (z+h)^{n-1} + z(z+h)^{n-2} + \cdots + z^{n-1}.$$

Since
$$|(z+h)^{n-1} + z(z+h)^{n-2} + \cdots + z^{n-1}| \leq n|z_0|^{n-1},$$
we have
$$\left| a_n \frac{((z+h)^n - z^n)}{h} \right| \leq n|a_n| \cdot |z_0|^{n-1}.$$
But the series $\sum_{n=1}^{\infty} n|a_n| \cdot |z_0|^{n-1}$ converges, so if N is sufficiently large, then
$$\sum_{n=N+1}^{\infty} n|a_n| \cdot |z_0|^{n-1} < \frac{\varepsilon}{3}.$$
This means that
$$\left| \sum_{n=0}^{\infty} a_n \frac{((z+h)^n - z^n)}{h} - \sum_{n=0}^{N} a_n \frac{((z+h)^n - z^n)}{h} \right|$$
$$= \left| \sum_{n=N+1}^{\infty} a_n \frac{((z+h)^n - z^n)}{h} \right| \leq \sum_{n=N+1}^{\infty} \left| a_n \frac{((z+h)^n - z^n)}{h} \right|$$
$$\leq \sum_{n=N+1}^{\infty} n|a_n| \cdot |z_0|^{n-1} < \frac{\varepsilon}{3}.$$
In short, if N is sufficiently large, then

(1) $$\left| \sum_{n=0}^{\infty} a_n \frac{((z+h)^n - z^n)}{h} - \sum_{n=0}^{N} a_n \frac{((z+h)^n - z^n)}{h} \right| < \frac{\varepsilon}{3},$$

for *all* h with $|z+h| \leq |z_0|$.

It is easy to deal with the third term on the right side of $(*)$: Since $\sum_{n=1}^{\infty} na_n z^{n-1}$ converges, it follows that if N is sufficiently large, then

(2) $$\left| \sum_{n=1}^{\infty} na_n z^{n-1} - \sum_{n=1}^{N} na_n z^{n-1} \right| < \frac{\varepsilon}{3}.$$

Finally, choosing an N such that (1) and (2) are true, we note that
$$\lim_{h \to 0} \sum_{n=0}^{N} a_n \frac{((z+h)^n - z^n)}{h} = \sum_{n=1}^{N} na_n z^{n-1},$$
since the polynomial function $g(z) = \sum_{n=0}^{N} a_n z^n$ is certainly differentiable. Therefore

(3) $$\left| \sum_{n=0}^{N} \frac{a_n((z+h)^n - z^n)}{h} - \sum_{n=1}^{N} na_n z^{n-1} \right| < \frac{\varepsilon}{3}.$$

for sufficiently small h.

As we have already indicated, (1), (2), and (3) prove the theorem. ▮

Theorem 6 has an obvious corollary: a function represented by a power series is infinitely differentiable inside the circle of convergence, and the power series is its Taylor series at 0. It follows, in particular, that f is continuous inside the circle of convergence, since a function differentiable at z is continuous at z (Problem 26-8).

The continuity of a power series inside its circle of convergence helps explain the behavior of certain Taylor series obtained for real functions, and gives the promised answers to the questions raised at the end of Chapter 24. We have already seen that the Taylor series for the function $f(z) = 1/(1+z^2)$, namely,

$$1 - z^2 + z^4 - z^6 + \cdots,$$

converges for real z only when $|z| < 1$, and consequently has radius of convergence 1. It is no accident that the circle of convergence contains the two points i and $-i$ at which f is undefined. If this power series converged in a circle of radius greater than 1, then (Figure 5) it would represent a function which was continuous in that circle, in particular at i and $-i$. But this is impossible, since it equals $1/(1+z^2)$ inside the unit circle, and $1/(1+z^2)$ does not approach a limit as z s approaches i or $-i$ from inside the unit circle.

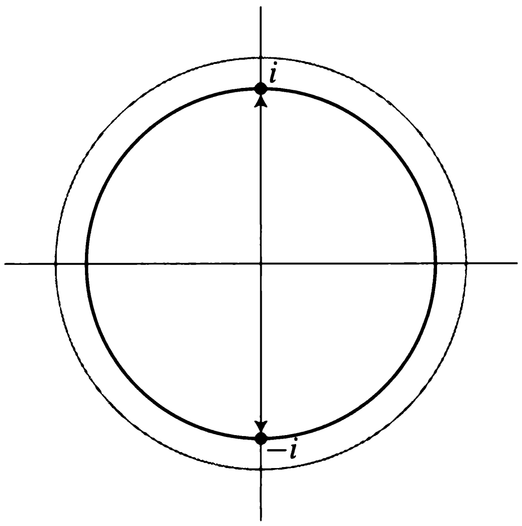

FIGURE 5

The use of complex numbers also sheds some light on the strange behavior of the Taylor series for the function

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Although we have not yet defined e^z for complex z, it will presumably be true that if y is real and unequal to 0, then

$$f(iy) = e^{-1/(iy)^2} = e^{1/y^2}.$$

The interesting fact about this expression is that it becomes large as y becomes small. Thus f will not even be continuous at 0 when defined for complex numbers, so it is hardly surprising that it is equal to its Taylor series only for $z = 0$.

The method by which we will actually define e^z (as well as $\sin z$ and $\cos z$) for complex z should by now be clear. For real x we know that

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots,$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots,$$
$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$$

For complex z we therefore *define*

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots,$$
$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots,$$
$$\exp(z) = e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots$$

Then $\sin'(z) = \cos z$, $\cos'(z) = -\sin z$, and $\exp'(z) = \exp(z)$ by Theorem 6. Moreover, if we replace z by iz in the series for e^z, and make a rearrangement of the terms (justified by absolute convergence), something particularly interesting happens:

$$\begin{aligned} e^{iz} &= 1 + iz + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \frac{(iz)^4}{4!} + \frac{(iz)^5}{5!} + \cdots \\ &= 1 + iz - \frac{z^2}{2!} - \frac{iz^3}{3!} + \frac{iz^4}{4!} + \frac{iz^5}{5!} + \cdots \\ &= \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots\right) + i\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right), \end{aligned}$$

so

$$e^{iz} = \cos z + i \sin z.$$

It is clear from the definitions (i.e., the power series) that

$$\begin{aligned} \sin(-z) &= -\sin z, \\ \cos(-z) &= \cos z, \end{aligned}$$

so we also have

$$e^{-iz} = \cos z - i \sin z.$$

From the equations for e^{iz} and e^{-iz} we can derive the formulas

$$\begin{aligned} \sin z &= \frac{e^{iz} - e^{-iz}}{2i}, \\ \cos z &= \frac{e^{iz} + e^{-iz}}{2}. \end{aligned}$$

The development of complex power series thus places the exponential function at the very core of the development of the elementary functions—it reveals a connection between the trigonometric and exponential functions which was never imagined when these functions were first defined, and which could never have been discovered without the use of complex numbers. As a by-product of this relationship, we obtain a hitherto unsuspected connection between the numbers e and π: if in the formula

$$e^{iz} = \cos z + i \sin z$$

we take $z = \pi$, we obtain the remarkable result

$$e^{i\pi} = -1.$$

(More generally, $e^{2\pi i/n}$ is an nth root of 1.)

With these remarks we will bring to a close our investigation of complex functions. And yet there are still several basic facts about power series which have not been mentioned. Thus far, we have seldom considered power series centered at a,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n,$$

except for $a = 0$. This omission was adopted partly to simplify the exposition. For power series centered at a there are obvious versions of all the theorems in this chapter (the proofs require only trivial modifications): there is a number R (possibly 0 or "∞") such that the series $\sum_{n=0}^{\infty} a_n(z-a)^n$ converges absolutely for z with $|z-a| < R$, and has unbounded terms for z with $|z-a| > R$; moreover, for all z with $|z-a| < R$ the function

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$$

has derivative

$$f'(z) = \sum_{n=1}^{\infty} na_n(z-a)^{n-1}.$$

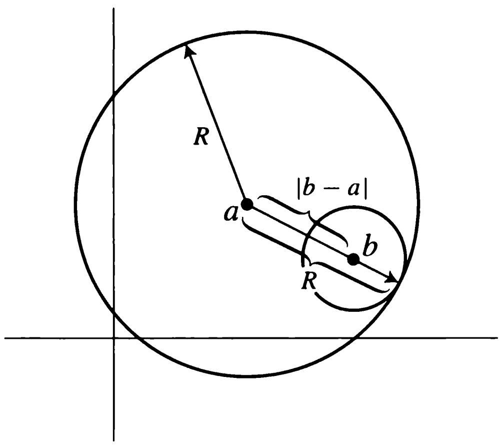

FIGURE 6

It is less straightforward to investigate the possibility of representing a function as a power series centered at b, if it is already written as a power series centered at a. If

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$$

has radius of convergence R, and b is a point with $|b-a| < R$ (Figure 6), then it is true that $f(z)$ can also be written as a power series centered at b,

$$f(z) = \sum_{n=0}^{\infty} b_n(z-b)^n$$

(the numbers b_n are necessarily $f^{(n)}(b)/n!$); moreover, this series has radius of convergence at least $R - |b-a|$ (*it may be larger*).

We will *not* prove the facts mentioned in the previous paragraph, and there are several other important facts we shall not prove. For example, if

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n \qquad \text{and} \qquad g(z) = \sum_{n=0}^{\infty} b_n(z-b)^n,$$

and $g(b) = a$, then we would expect that $f \circ g$ can be written as a power series centered at b. All such facts could be proved now without introducing any basic new ideas, but the proofs would not be as easy as the proofs about sums, products and reciprocals of power series. The possibility of changing a power series centered at a into one centered at b is quite a bit more involved, and the treatment of $f \circ g$ requires still more skill. Rather than end this section with a *tour de force* of computations, we will instead give a preview of "complex analysis," one of the most beautiful branches of mathematics, where all these facts are derived as straightforward consequences of some fundamental results.

Power series were introduced in this chapter in order to provide complex functions which are differentiable. Since these functions are actually infinitely differentiable, it is natural to suppose that we have therefore selected only a very special collection of differentiable complex functions. The basic theorems of complex analysis show that this is not at all true:

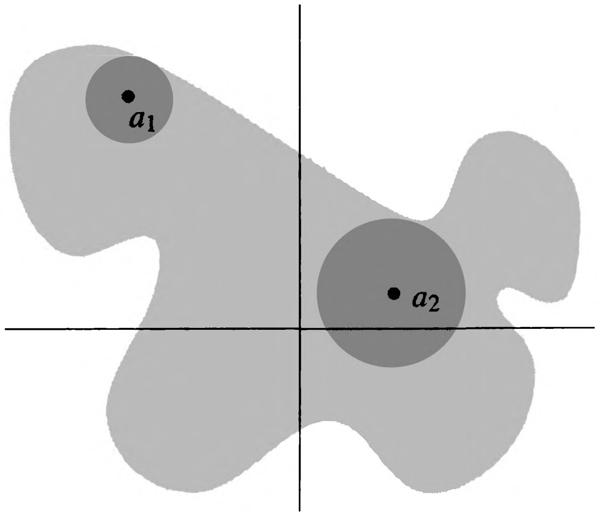

FIGURE 7

> *If a complex function is defined in some region A of the plane and is differentiable in A, then it is automatically infinitely differentiable in A. Moreover, for each point a in A the Taylor series for f at a will converge to f in any circle contained in A (Figure 7).*

These facts are among the first to be proved in complex analysis. It is impossible to give any idea of the proofs themselves—the methods used are quite different from anything in elementary calculus. If these facts are granted, however, then the facts mentioned before can be proved very easily.

Suppose, for example, that f and g are functions which can be written as power series. Then, as we have shown, f and g are differentiable—it then follows from easy general theorems that $f+g$, $f \cdot g$, $1/g$ and $f \circ g$ are also differentiable. Appealing to the results from complex analysis, it follows that they can be written as power series.

We already know how to compute the power series for $f+g$, $f \cdot g$ and $1/g$ from those for f and g. It is also easy to guess how one would compute an expression for $f \circ g$ as a power series in $(z-b)$ when we are given the power series expansions

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$

$$g(z) = \sum_{k=0}^{\infty} b_k (z-b)^k,$$

with $a = g(b) = b_0$, so that

$$g(z) - a = \sum_{k=1}^{\infty} b_k (z-b)^k.$$

First of all, we know how to compute the power series

$$(g(z)-a)^l = \left(\sum_{k=1}^{\infty} b_k (z-b)^k\right)^l,$$

and this power series will begin with $(z-b)^l$. Consequently, the coefficient of z^n in

$$f(g(z)) = \sum_{l=0}^{\infty} a_l (g(z)-a)^l$$

can be calculated as a finite sum, involving only coefficients arising from the first n powers of $g(z)-a$.

Similarly, if

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$

has radius of convergence R, then f is differentiable in the region $A = \{z : |z-a| < R\}$. Thus, if b is in A, it is possible to write f as a power series centered at b, which will converge in the circle of radius $R - |b - a|$. The coefficient of z^n will be $f^{(n)}(b)/n!$. This series may actually converge in a larger circle, because $\sum_{n=0}^{\infty} a_n(z-a)^n$ may be the series for a function differentiable in a larger region than A. For example, suppose that $f(z) = 1/(1+z^2)$. Then f is differentiable, except at i and $-i$, where it is not defined. Thus $f(z)$ can be written as a power series $\sum_{n=0}^{\infty} a_n z^n$ with radius of convergence 1 (as a matter of fact, we know that $a_{2n} = (-1)^n$ and $a_k = 0$ if k is odd). It is also possible to write

$$f(z) = \sum_{n=0}^{\infty} b_n (z - \tfrac{1}{2})^n,$$

where $b_n = f^{(n)}(\frac{1}{2})/n!$. We can easily predict the radius of convergence of this series: it is $\sqrt{1 + (\frac{1}{2})^2}$, the distance from $\frac{1}{2}$ to i or $-i$ (Figure 8).

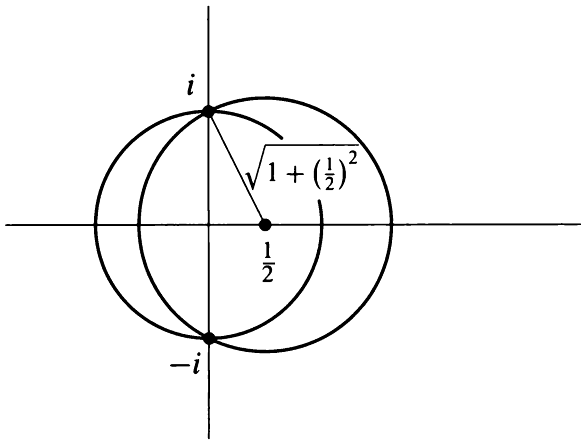

FIGURE 8

As an added incentive to investigate complex analysis further, one more result will be mentioned, which lies quite near the surface, and which will be found in any treatment of the subject.

For real z the values of $\sin z$ always lie between -1 and 1, but for complex z this is not at all true. In fact, if $z = iy$, for y real, then

$$\sin iy = \frac{e^{i(iy)} - e^{-i(iy)}}{2i} = \frac{e^{-y} - e^{y}}{2i}.$$

If y is large, then $\sin iy$ is also large in absolute value. This behavior of sin is typical of functions which are defined and differentiable on the whole complex plane (such functions are called *entire*). A result which comes quite early in complex analysis is the following:

Liouville's Theorem: The only bounded entire functions are the constant functions.

As a simple application of Liouville's Theorem, consider a polynomial function

$$f(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0,$$

where $n > 1$, so that f is not a constant. We already know that $f(z)$ is large for large z, so Liouville's Theorem tells us nothing interesting about f. But consider the function

$$g(z) = \frac{1}{f(z)}.$$

If $f(z)$ were never 0, then g would be entire; since $f(z)$ becomes large for large z, the function g would also be bounded, contradicting Liouville's Theorem. Thus $f(z) = 0$ for some z, and we have proved the Fundamental Theorem of Algebra.

PROBLEMS

1. Decide whether each of the following series converges, and whether it converges absolutely.

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{(1+i)^n}{n!}$.

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{1+2i}{2^n}$.

(iii) $\displaystyle\sum_{n=1}^{\infty} \frac{i^n}{n}$.

(iv) $\displaystyle\sum_{n=1}^{\infty} (\tfrac{1}{2}+\tfrac{1}{2}i)^n$.

(v) $\displaystyle\sum_{n=2}^{\infty} \frac{\log n}{n} + i^n \frac{\log n}{n}$.

2. Use the ratio test to show that the radius of convergence of each of the following power series is 1. (In each case the ratios of successive terms will approach a limit < 1 if $|z| < 1$, but for $|z| > 1$ the ratios will tend to ∞ or to a limit > 1.)

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{n^2}$.

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{n}$.

(iii) $\displaystyle\sum_{n=1}^{\infty} z^n$.

(iv) $\displaystyle\sum_{n=1}^{\infty} (n+2^{-n})z^n$.

(v) $\displaystyle\sum_{n=1}^{\infty} 2^n z^{n!}$.

3. Use the root test (Problem 23-9) to find the radius of convergence of each of the following power series. (In some cases, you will need limits derived in the problems to Chapter 22.)

(i) $\displaystyle \frac{z}{2} + \frac{z^2}{3} + \frac{z^3}{2^2} + \frac{z^4}{3^2} + \frac{z^5}{2^3} + \frac{z^6}{3^3} + \cdots$.

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n} z^n$.

(iii) $\displaystyle\sum_{n=1}^{\infty} \frac{n!\, z^n}{n^n}$.

(iv) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n} z^n$.

(v) $\displaystyle\sum_{n=1}^{\infty} 2^n z^{n!}$.

4. The root test can always be used, in theory at least, to find the radius of convergence of a power series; in fact, a close analysis of the situation leads to a formula for the radius of convergence, known as the "Cauchy-Hadamard formula." Suppose first that the set of numbers $\sqrt[n]{|a_n|}$ is bounded.

(a) Use Problem 23-9 to show that if $\overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|}\,|z| < 1$, then $\sum_{n=0}^{\infty} a_n z^n$ converges.

(b) Also show that if $\overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|}\,|z| > 1$, then $\sum_{n=0}^{\infty} a_n z^n$ has unbounded terms.

(c) Parts (a) and (b) show that the radius of convergence of $\sum_{n=0}^{\infty} a_n z^n$ is $1/\overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|}$ (where "1/0" means "∞"). To complete the formula, define $\overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|} = \infty$ if the set of all $\sqrt[n]{|a_n|}$ is unbounded. Prove that in this case, $\sum_{n=0}^{\infty} a_n z^n$ diverges for $z \neq 0$, so that the radius of convergence is 0 (which may be considered as "$1/\infty$").

5. Consider the following three series from Problem 2:

$$\sum_{n=1}^{\infty} \frac{z^n}{n^2}, \qquad \sum_{n=1}^{\infty} \frac{z^n}{n}, \qquad \sum_{n=1}^{\infty} z^n.$$

Prove that the first series converges everywhere on the unit circle; that the third series converges nowhere on the unit circle; and that the second series converges for at least one point on the unit circle and diverges for at least one point on the unit circle.

6. (a) Prove that $e^z \cdot e^w = e^{z+w}$ for all complex numbers z and w by showing that the infinite series for e^{z+w} is the Cauchy product of the series for e^z and e^w.

(b) Show that $\sin(z+w) = \sin z \cos w + \cos z \sin w$ and $\cos(z+w) = \cos z \cos w - \sin z \sin w$ for all complex z and w.

7. (a) Prove that every complex number of absolute value 1 can be written e^{iy} for some real number y.

(b) Prove that $|e^{x+iy}| = e^x$ for real x and y.

8. (a) Prove that exp takes on every complex value except 0.

(b) Prove that sin takes on every complex value.

9. For each of the following functions, compute the first three nonzero terms of the Taylor series centered at 0 by manipulating power series.

(i) $f(z) = \tan z$.
(ii) $f(z) = z(1-z)^{-1/2}$.

(iii) $f(z) = \dfrac{e^{\sin z} - 1}{z}$.

(iv) $f(z) = \log(1 - z^2)$.

(v) $f(z) = \dfrac{\sin^2 z}{z^2}$.

(vi) $f(z) = \dfrac{\sin(z^2)}{z\cos^2 z}$.

(vii) $f(z) = \dfrac{1}{z^4 - 2z^2 + 3}$.

(viii) $f(z) = \dfrac{1}{z}[e^{(\sqrt{1+z}-1)} - 1]$.

10. (a) Suppose that we write a differentiable complex function f as $f = u + iv$, where u and v are real-valued. Let $\bar{u}$ and $\bar{v}$ denote the restrictions of u and v to the real numbers. In other words, $\bar{u}(x) = u(x)$ for real numbers x (but $\bar{u}$ is not defined for other x). Using Problem 26-9, show that for real x we have

$$f'(x) = \bar{u}'(x) + i\bar{v}'(x),$$

where f' denotes the complex derivative, while $\bar{u}'$ and $\bar{v}'$ denote the ordinary derivatives of these real-valued functions on $\mathbf{R}$.

(b) Show, more generally, that

$$f^{(k)}(x) = \bar{u}^{(k)}(x) + i\bar{v}^{(k)}(x).$$

(c) Suppose that f satisfies the equation

$$(*) \qquad f^{(n)} + a_{n-1}f^{(n-1)} + \cdots + a_0 f = 0,$$

where the a_i are real numbers, and where the $f^{(k)}$ denote higher-order complex derivatives. Show that $\bar{u}$ and $\bar{v}$ satisfy the same equation, where $\bar{u}^{(k)}$ and $\bar{v}^{(k)}$ now denote higher-order derivatives of real-valued functions on $\mathbf{R}$.

(d) Show that if $a = b + ci$ is a complex root of the equation $z^n + a_{n-1}z^{n-1} + \cdots + a_0 = 0$, then $f(x) = e^{bx}\sin cx$ and $f(x) = e^{bx}\cos cx$ are both solutions of $(*)$.

11. (a) Show that exp is *not* one-one on $\mathbf{C}$.

(b) Given $w \neq 0$, show that $e^z = w$ if and only if $z = x + iy$ with $x = \log|w|$ (here log denotes the real logarithm function), and y an argument of w.

*(c) Show that there does not exist a continuous function log defined for nonzero complex numbers, such that $\exp(\log(z)) = z$ for all $z \neq 0$. (Show that log cannot even be defined continuously for $|z| = 1$.)

Since there is no way to define a continuous logarithm function we cannot speak of *the* logarithm of a complex number, but only of "a logarithm for w," meaning one of the infinitely many numbers z with $e^z = w$. And for complex numbers a and b we define a^b to be a *set* of complex numbers, namely the set of all numbers $e^{b \log a}$ or, more precisely, the set of all numbers e^{bz} where z is a logarithm for a.

(d) If m is an integer, then a^m consists of only one number, the one given by the usual elementary definition of a^m.

(e) If m and n are integers, then the set $a^{m/n}$ coincides with the set of values given by the usual elementary definition, namely the set of all b^m where b is an nth root of a.

(f) If a and b are real and b is irrational, then a^b contains infinitely many members, even for $a > 0$.

(g) Find all logarithms of i, and find all values of i^i.

(h) By $(a^b)^c$ we mean the set of all numbers of the form z^c for some number z in the set a^b. Show that $(1^i)^i$ has infinitely many values, while $1^{i \cdot i}$ has only one.

(i) Show that all values of $a^{b \cdot c}$ are also values of $(a^b)^c$. Is $a^{b \cdot c} = (a^b)^c \cap (a^c)^b$?

12. (a) For real x show that we can choose $\log(x+i)$ and $\log(x-i)$ to be

$$\log(x+i) = \log(\sqrt{1+x^2}) + i\left(\frac{\pi}{2} - \arctan x\right),$$
$$\log(x-i) = \log(\sqrt{1+x^2}) - i\left(\frac{\pi}{2} - \arctan x\right).$$

(It will help to note that $\pi/2 - \arctan x = \arctan 1/x$ for $x > 0$.)

(b) The expression

$$\frac{1}{1+x^2} = \frac{1}{2i}\left(\frac{1}{x-i} - \frac{1}{x+i}\right)$$

yields, formally,

$$\int \frac{dx}{1+x^2} = \frac{1}{2i}[\log(x-i) - \log(x+i)].$$

Use part (a) to check that this answer agrees with the usual one.

13. (a) A sequence $\{a_n\}$ of complex numbers is called a **Cauchy sequence** if $\lim_{m,n\to\infty} |a_m - a_n| = 0$. Suppose that $a_n = b_n + ic_n$, where b_n and c_n are real. Prove that $\{a_n\}$ is a Cauchy sequence if and only if $\{b_n\}$ and $\{c_n\}$ are Cauchy sequences.

(b) Prove that every Cauchy sequence of complex numbers converges.

(c) Give direct proofs, without using theorems about real series, that an absolutely convergent series is convergent and that any rearrangement has the same sum. (It is permitted, and in fact advisable, to use the *proofs* of the corresponding theorems for real series.)

14. (a) Prove that

$$\sum_{k=1}^{n} e^{ikx} = e^{ix}\frac{1-e^{inx}}{1-e^{ix}} = \frac{\sin\left(\frac{n}{2}x\right)}{\sin\frac{x}{2}} e^{i(n+1)x/2}.$$

(b) Deduce the formulas for $\sum_{k=1}^{n} \cos kx$ and $\sum_{k=1}^{n} \sin kx$ that are given in Problem 15-33.

15. Let $\{a_n\}$ be the Fibonacci sequence, $a_1 = a_2 = 1, a_{n+2} = a_n + a_{n+1}$.

(a) If $r_n = a_{n+1}/a_n$, show that $r_{n+1} = 1 + 1/r_n$.

(b) Show that if $r = \lim_{n\to\infty} r_n$ exists, then $r = 1 + 1/r$, so that $r = \left(1+\sqrt{5}\right)/2$.

(c) Prove that the limit does exist. Hint: If $r_n < \left(1+\sqrt{5}\right)/2$, then $r_n{}^2 - r_n - 1 < 0$ and $r_n < r_{n+2}$.

(d) Show that $\sum_{n=1}^{\infty} a_n z^n$ has radius of convergence $2/\left(1+\sqrt{5}\right)$. (Using the unproved theorems in this chapter and the fact that $\sum_{n=1}^{\infty} a_n z^n = -1/(z^2 + z - 1)$ from Problem 24-16 we could have predicted that the radius of convergence is the smallest absolute value of the roots of $z^2 + z - 1 = 0$; since the roots are $\left(-1 \pm \sqrt{5}\right)/2$, the radius of convergence should be $\left(-1+\sqrt{5}\right)/2$. Notice that this number is indeed equal to $2/\left(1+\sqrt{5}\right)$.)

16. Since $(e^z - 1)/z$ can be written as the power series $1 + z/2! + z^2/3! + \cdots$ which is nonzero at 0, it follows that there is a power series

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{b_n}{n!} z^n$$

with nonzero radius of convergence. Using the unproved theorems in this chapter, we can even predict the radius of convergence; it is 2π, since this is the smallest absolute value of the non-zero numbers $z = 2k\pi i$ for which $e^z - 1 = 0$. The numbers b_n appearing here are called the **Bernoulli numbers.***

(a) Clearly $b_0 = 1$. Now show that

$$\frac{z}{e^z - 1} = -\frac{z}{2} + \frac{z}{2} \cdot \frac{e^z + 1}{e^z - 1},$$

$$\frac{e^{-z} + 1}{e^{-z} - 1} = -\frac{e^z + 1}{e^z - 1},$$

and deduce that

$$b_1 = -\tfrac{1}{2}, \quad b_n = 0 \quad \text{if } n \text{ is odd and } n > 1.$$

*Sometimes the numbers $B_n = (-1)^{n-1} b_{2n}$ are called the Bernoulli numbers, because $b_n = 0$ if n is odd and > 1 (see part (a)) and because the numbers b_{2n} alternate in sign, although we will not prove this. Other modifications of this nomenclature are also in use.

(b) By finding the coefficient of z^n in the right side of the equation

$$z = \left(\sum_{k=0}^{\infty} \frac{b_k}{k!} z^k\right)\left(z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right),$$

show that

$$\sum_{i=0}^{n-1} \binom{n}{i} b_i = 0 \quad \text{for } n > 1.$$

This formula allows us to compute any b_k in terms of previous ones, and shows that each is rational. Calculate two or three of the following:

$$b_2 = \tfrac{1}{6}, \quad b_4 = -\tfrac{1}{30}, \quad b_6 = \tfrac{1}{42}, \quad b_8 = -\tfrac{1}{30}.$$

*(c) Part (a) shows that

$$\sum_{n=0}^{\infty} \frac{b_{2n}}{(2n)!} z^{2n} = \frac{z}{2} \cdot \frac{e^z + 1}{e^z - 1} = \frac{z}{2} \cdot \frac{e^{z/2} + e^{-z/2}}{e^{z/2} - e^{-z/2}}.$$

Replace z by $2iz$ and show that

$$z \cot z = \sum_{n=0}^{\infty} \frac{b_{2n}}{(2n)!} (-1)^n 2^{2n} z^{2n}.$$

*(d) Show that

$$\tan z = \cot z - 2 \cot 2z.$$

*(e) Show that

$$\tan z = \sum_{n=1}^{\infty} \frac{b_{2n}}{(2n)!} (-1)^{n-1} 2^{2n} (2^{2n} - 1) z^{2n-1}.$$

(This series converges for $|z| < \pi/2$.)

17. The Bernoulli numbers play an important role in a theorem which is best introduced by some notational nonsense. Let us use D to denote the "differentiation operator," so that Df denotes f'. Then $D^k f$ will mean $f^{(k)}$ and $e^D f$ will mean $\sum_{n=0}^{\infty} f^{(n)}/n!$ (of course this series makes no sense in general, but it will make sense if f is a polynomial function, for example). Finally, let Δ denote the "difference operator" for which $\Delta f(x) = f(x+1) - f(x)$. Now Taylor's Theorem implies, disregarding questions of convergence, that

$$f(x+1) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!},$$

or

$$(*) \quad f(x+1) - f(x) = \sum_{n=1}^{\infty} \frac{f^{(n)}(x)}{n!};$$

we may write this symbolically as $\Delta f = (e^D - 1)f$, where 1 stands for the "identity operator." Even more symbolically this can be written $\Delta = e^D - 1$, which suggests that

$$D = \frac{D}{e^D - 1}\Delta.$$

Thus we obviously ought to have

$$D = \sum_{k=0}^{\infty} \frac{b_k}{k!} D^k \Delta,$$

i.e.,

$$(**) \quad f'(x) = \sum_{k=0}^{\infty} \frac{b_k}{k!}[f^{(k)}(x+1) - f^{(k)}(x)].$$

The beautiful thing about all this nonsense is that it works!

(a) Prove that $(**)$ is literally true if f is a polynomial function (in which case the infinite sum is really a finite sum). Hint: By applying $(*)$ to $f^{(k)}$, find a formula for $f^{(k)}(x+1) - f^{(k)}(x)$; then use the formula in Problem 16(b) to find the coefficient of $f^{(j)}(x)$ in the right side of $(**)$.

(b) Deduce from $(**)$ that

$$f'(0) + \cdots + f'(n) = \sum_{k=0}^{\infty} \frac{b_k}{k!}[f^{(k)}(n+1) - f^{(k)}(0)].$$

(c) Show that for any polynomial function g we have

$$g(0) + \cdots + g(n) = \int_0^{n+1} g(t)\,dt + \sum_{k=1}^{\infty} \frac{b_k}{k!}[g^{(k-1)}(n+1) - g^{(k-1)}(0)].$$

(d) Apply this to $g(x) = x^p$ to show that

$$\sum_{k=1}^{n-1} k^p = \frac{n^{p+1}}{p+1} + \sum_{k=1}^{p} \frac{b_k}{k}\binom{p}{k-1} n^{p-k+1}.$$

Using the fact that $b_1 = -\frac{1}{2}$, show that

$$\sum_{k=1}^{n} k^p = \frac{n^{p+1}}{p+1} + \frac{n^p}{2} + \sum_{k=2}^{p} \frac{b_k}{k}\binom{p}{k-1} n^{p-k+1}.$$

The first ten instances of this formula were written out in Problem 2-7, which offered as a challenge the discovery of the general pattern. This may now seem to be a preposterous suggestion, but the Bernoulli numbers were actually discovered in precisely this way! After writing out these 10 formulas, Bernoulli claims (in his posthumously printed work *Ars Conjectandi*, 1713): "Whoever will examine the series as to their regularity may be able to continue the table." He then writes down the above formula, offering no proof at all, merely noting that the coefficients b_k (which he denoted simply by $A, B, C, \ldots$) satisfy the equation in Problem 16(b). The relation between these numbers and the coefficients in the power series for $z/(e^z - 1)$ was discovered by Euler.

***18.** The formula in Problem 17(c) can be generalized to the case where g is not a polynomial function; the infinite sum must be replaced by a finite sum plus a remainder term. In order to find an expression for the remainder, it is useful to introduce some new functions.

(a) The *Bernoulli polynomials* φ_n are defined by

$$\varphi_n(x) = \sum_{k=0}^{n} \binom{n}{k} b_{n-k} x^k.$$

The first three are

$$\varphi_1(x) = x - \frac{1}{2},$$
$$\varphi_2(x) = x^2 - x + \frac{1}{6},$$
$$\varphi_3(x) = x^3 - \frac{3x^2}{2} + \frac{x}{2}.$$

Show that

$$\begin{aligned} \varphi_n(0) &= b_n, \\ \varphi_n(1) &= b_n \qquad \text{if } n > 1, \\ \varphi_n{}'(x) &= n\varphi_{n-1}(x), \\ \varphi_n(x) &= (-1)^n \varphi_n(1-x). \end{aligned}$$

Hint: Prove the last equation by induction on n.

(b) Let $R_N{}^k(x)$ be the remainder term in Taylor's Theorem for $f^{(k)}$, on the interval $[x, x+1]$, so that

$$(*) \qquad f^{(k)}(x+1) - f^{(k)}(x) = \sum_{n=1}^{N} \frac{f^{(k+n)}(x)}{n!} + R_N{}^k(x).$$

Prove that

$$f'(x) = \sum_{k=0}^{N} \frac{b_k}{k!}[f^{(k)}(x+1) - f^{(k)}(x)] - \sum_{k=0}^{N} \frac{b_k}{k!} R_{N-k}{}^k(x).$$

Hint: Imitate Problem 17(a). Notice the subscript $N - k$ on R.

(c) Use the integral form of the remainder to show that

$$\sum_{k=0}^{N} \frac{b_k}{k!} R_{N-k}{}^k(x) = \int_x^{x+1} \frac{\varphi_N(x+1-t)}{N!} f^{(N+1)}(t)\, dt.$$

(d) Deduce the "Euler-Maclaurin Summation Formula":

$$g(x)+g(x+1)+\cdots+g(x+n)$$
$$=\int_x^{x+n+1} g(t)\,dt+\sum_{k=1}^{N}\frac{b_k}{k!}[g^{(k-1)}(x+n+1)-g^{(k-1)}(x)]+S_N(x,n),$$

where

$$S_N(x,n)=-\sum_{j=0}^{n}\int_{x+j}^{x+j+1}\frac{\varphi_N(x+j+1-t)}{N!}g^{(N)}(t)\,dt.$$

(e) Let ψ_n be the periodic function, with period 1, which satisfies $\psi_n(t)=\varphi_n(t)$ for $0\le t<1$. (Part (a) implies that if $n>1$, then ψ_n is continuous, since $\varphi_n(1)=\varphi_n(0)$, and also that ψ_n is even if n is even and odd if n is odd.) Show that

$$S_N(x,n)=-\int_x^{x+n+1}\frac{\psi_N(x-t)}{N!}g^{(N)}(t)\,dt$$
$$\left(=(-1)^{N+1}\int_x^{x+n+1}\frac{\psi_N(t)}{N!}g^{(N)}(t)\,dt\qquad\text{if } x \text{ is an integer}\right).$$

Unlike the remainder in Taylor's Theorem, the remainder $S_N(x,n)$ usually does not satisfy $\lim_{N\to\infty} S_N(x,n)=0$, because the Bernoulli numbers and functions become large very rapidly (although the first few examples do not suggest this). Nevertheless, important information can often be obtained from the summation formula. The general situation is best discussed within the context of a specialized study ("asymptotic series"), but the next problem shows one particularly important example.

****19.** (a) Use the Euler-Maclaurin Formula, with $N=2$, to show that

$$\log 1+\cdots+\log(n-1)$$
$$=\int_1^n \log t\,dt-\frac{1}{2}\log n+\frac{1}{12}\left(\frac{1}{n}-1\right)+\int_1^n\frac{\psi_2(t)}{2t^2}\,dt.$$

(b) Show that

$$\log\left(\frac{n!}{n^{n+1/2}e^{-n+1/(12n)}}\right)=\frac{11}{12}+\int_1^n\frac{\psi_2(t)}{2t^2}\,dt.$$

(c) Explain why the improper integral $\beta=\int_1^\infty \psi_2(t)/2t^2\,dt$ exists, and show that if $\alpha=\exp(\beta+11/12)$, then

$$\log\left(\frac{n!}{\alpha n^{n+1/2}e^{-n+1/(12n)}}\right)=-\int_n^\infty\frac{\psi_2(t)}{2t^2}\,dt.$$

(d) Problem 19-41(d) shows that

$$\sqrt{\pi} = \lim_{n\to\infty} \frac{(n!)^2 2^{2n}}{(2n)!\sqrt{n}}.$$

Use part (c) to show that

$$\sqrt{\pi} = \lim_{n\to\infty} \frac{\alpha^2 n^{2n+1} e^{-2n} 2^{2n}}{\alpha (2n)^{2n+1/2} e^{-2n}\sqrt{n}},$$

and conclude that $\alpha = \sqrt{2\pi}$.

(e) Show that

$$\int_0^{1/2} \varphi_2(t)\,dt = \int_0^1 \varphi_2(t)\,dt = 0.$$

(You can do the computations explicitly, but the result also follows immediately from Problem 18(a).) Conclude that

$$\bar{\psi}(x) = \int_0^x \psi_2(t)\,dt \quad \begin{cases} \geq 0 & \text{for } 0 \leq x \leq 1/2 \\ \leq 0 & \text{for } 1/2 \leq x \leq 1, \end{cases}$$

with $\bar{\psi}(n) = 0$ for all n. Hint: Graph $\bar{\psi}$ on $[0, 1]$, paying particular attention to its values at x_0, $\frac{1}{2}$, and x_1, where x_0 and x_1 are the roots of φ_2 (Figure 9).

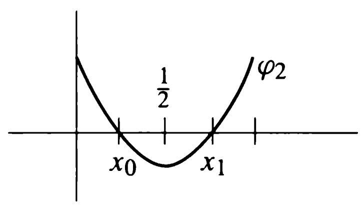

FIGURE 9

(f) Noting that $\bar{\psi}(x) = -\bar{\psi}(1-x)$, show that

$$\bar{\bar{\psi}}(x) = \int_0^x \bar{\psi}(t)\,dt \geq 0 \quad \text{on } [0, 1],$$

and hence everywhere, with $\bar{\bar{\psi}}(n) = 0$ for all n.

(g) Finally, use this information and integration by parts to show that

$$\int_n^\infty \frac{\psi_2(t)}{2t^2}\,dt > 0.$$

(h) Using the fact that the maximum value of $|\varphi_2(x)|$ for x in $[0, 1]$ is $\frac{1}{6}$, conclude that

$$0 < \int_n^\infty \frac{\psi_2(t)}{2t^2}\,dt < \frac{1}{12n}.$$

(i) Finally, conclude that

$$\sqrt{2\pi}\, n^{n+1/2} e^{-n} < n! < \sqrt{2\pi}\, n^{n+1/2} e^{-n+1/(12n)}.$$

The final result of Problem 19, a strong form of Stirling's Formula, shows that $n!$ is approximately $\sqrt{2\pi}\, n^{n+1/2}e^{-n}$, in the sense that this expression differs from $n!$ by an amount which is small compared to n when n is large. For example, for $n = 10$ we obtain 3598696 instead of 3628800, with an error < 1%.

A more general form of Stirling's Formula illustrates the "asymptotic" nature of the summation formula. The same argument which was used in Problem 19 can now be used to show that for $N \geq 2$ we have

$$\log\left(\frac{n!}{\sqrt{2\pi}\, n^{n+1/2}e^{-n}}\right) = \sum_{k=2}^{N} \frac{b_k}{k(k-1)n^{k-1}} \pm \int_n^\infty \frac{\psi_N(t)}{Nt^N}\, dt.$$

Since ψ_N is bounded, we can obtain estimates of the form

$$\left|\int_n^\infty \frac{\psi_N(t)}{Nt^N}\, dt\right| \leq \frac{M_N}{n^{N-1}}.$$

If N is large, the constant M_N will also be large; but for very large n the factor n^{1-N} will make the product very small. Thus, the expression

$$\sqrt{2\pi}\, n^{n+1/2}e^{-n} \cdot \exp\left(\sum_{k=2}^{N} \frac{b_k}{k(k-1)n^{k-1}}\right)$$

may be a very bad approximation for $n!$ when n is small, but for large n (*how* large depends on N) it will be an extremely good one (*how* good depends on N).

PART 5
EPILOGUE

There was a most ingenious Architect
who had contrived a new Method
for building Houses,
by beginning at the Roof, and working
downwards to the Foundation.

JONATHAN SWIFT