CHAPTER 18 THE LOGARITHM AND EXPONENTIAL FUNCTIONS

In Chapter 15 the integral provided a rigorous formulation for a preliminary definition of the functions sin and cos. In this chapter the integral plays a more essential role. For certain functions even a preliminary definition presents difficulties. For example, consider the function

$$f(x) = 10^x.$$

This function is assumed to be defined for all x and to have an inverse function, defined for positive x, which is the "logarithm to the base 10,"

$$f^{-1}(x) = \log_{10} x.$$

In algebra, 10^x is usually defined only for *rational* x, while the definition for irrational x is quietly ignored. A brief review of the definition for rational x will not only explain this omission, but also recall an important principle behind the definition of 10^x.

The symbol 10^n is first defined for natural numbers n. This notation turns out to be extremely convenient, especially for multiplying very large numbers, because

$$10^n \cdot 10^m = 10^{n+m}.$$

The extension of the definition of 10^x to rational x is motivated by the desire to preserve this equation; this requirement actually forces upon us the customary definition. Since we want the equation

$$10^0 \cdot 10^n = 10^{0+n} = 10^n$$

to be true, we must define $10^0 = 1$; since we want the equation

$$10^{-n} \cdot 10^n = 10^0 = 1$$

to be true, we must define $10^{-n} = 1/10^n$; since we want the equation

$$\underbrace{10^{1/n} \cdot \ldots \cdot 10^{1/n}}_{n \text{ times}} = 10^{\overbrace{}^{}\underbrace{1/n + \cdots + 1/n}_{n \text{ times}}} = 10^1 = 10$$

to be true, we must define $10^{1/n} = \sqrt[n]{10}$; and since we want the equation

$$\underbrace{10^{1/n} \cdot \ldots \cdot 10^{1/n}}_{m \text{ times}} = 10^{\underbrace{1/n + \cdots + 1/n}_{m \text{ times}}} = 10^{m/n}$$

to be true, we must define $10^{m/n} = (\sqrt[n]{10}\,)^m$.

Unfortunately, at this point the program comes to a dead halt. We have been guided by the principle that 10^x should be defined so as to ensure that $10^{x+y} = 10^x 10^y$; but this principle does not suggest any simple algebraic way of defining

10^x for irrational x. For this reason we will try some more sophisticated ways of finding a function f such that

$$(*) \qquad f(x+y) = f(x) \cdot f(y) \quad \text{for all } x \text{ and } y.$$

Of course, we are interested in a function which is not always zero, so we might add the condition $f(1) \neq 0$. If we add the more specific condition $f(1) = 10$, then $(*)$ will imply that $f(x) = 10^x$ for rational x, and 10^x could be *defined* as $f(x)$ for other x; in general $f(x)$ will equal $[f(1)]^x$ for rational x.

One way to find such a function is suggested if we try to solve an apparently more difficult problem: find a *differentiable* function f such that

$$f(x+y) = f(x) \cdot f(y) \quad \text{for all } x \text{ and } y,$$
$$f(1) = 10.$$

Assuming that such a function exists, we can try to find f'—knowing the derivative of f might provide a clue to the definition of f itself. Now

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{f(x) \cdot f(h) - f(x)}{h} \\ &= f(x) \cdot \lim_{h \to 0} \frac{f(h) - 1}{h}. \end{aligned}$$

The answer thus depends on

$$f'(0) = \lim_{h \to 0} \frac{f(h) - 1}{h};$$

for the moment assume this limit exists, and denote it by α. Then

$$f'(x) = \alpha \cdot f(x) \quad \text{for all } x.$$

Even if α could be computed, this approach seems self-defeating. The derivative of f has been expressed in terms of f again.

If we examine the inverse function $f^{-1} = \log_{10}$, the whole situation appears in a new light:

$$\begin{aligned} \log_{10}{}'(x) &= \frac{1}{f'(f^{-1}(x))} \\ &= \frac{1}{\alpha \cdot f(f^{-1}(x))} = \frac{1}{\alpha x}. \end{aligned}$$

The derivative of f^{-1} is about as simple as one could ask! And, what is even more interesting, of all the integrals $\int_a^b x^n \, dx$ examined previously, the integral $\int_a^b x^{-1} \, dx$ is the only one which we cannot evaluate. Since $\log_{10} 1 = 0$ we should have

$$\frac{1}{\alpha} \int_1^x \frac{1}{t} \, dt = \log_{10} x - \log_{10} 1 = \log_{10} x.$$

This suggests that we define $\log_{10} x$ as $(1/\alpha)\displaystyle\int_1^x t^{-1}\,dt$. The difficulty is that α is unknown. One way of evading this difficulty is to define

$$\log x = \int_1^x \frac{1}{t}\,dt,$$

and hope that this integral will be the logarithm to *some* base, which might be determined later. In any case, the function defined in this way is surely more reasonable, from a mathematical point of view, than $\log_{10}$. The usefulness of $\log_{10}$ depends on the important role of the number 10 in arabic notation (and thus ultimately on the fact that we have ten fingers), while the function log provides a notation for an extremely simple integral which cannot be evaluated in terms of any functions already known to us.

DEFINITION If $x > 0$, then

$$\mathbf{log}\, \boldsymbol{x} = \int_1^x \frac{1}{t}\,dt.$$

The graph of log is shown in Figure 1. Notice that if $x > 1$, then $\log x > 0$, and if $0 < x < 1$, then $\log x < 0$, since, by our conventions,

$$\int_1^x \frac{1}{t}\,dt = -\int_x^1 \frac{1}{t}\,dt < 0.$$

For $x \le 0$, a number $\log x$ cannot be defined in this way, because $f(t) = 1/t$ is not bounded on $[x, 1]$.

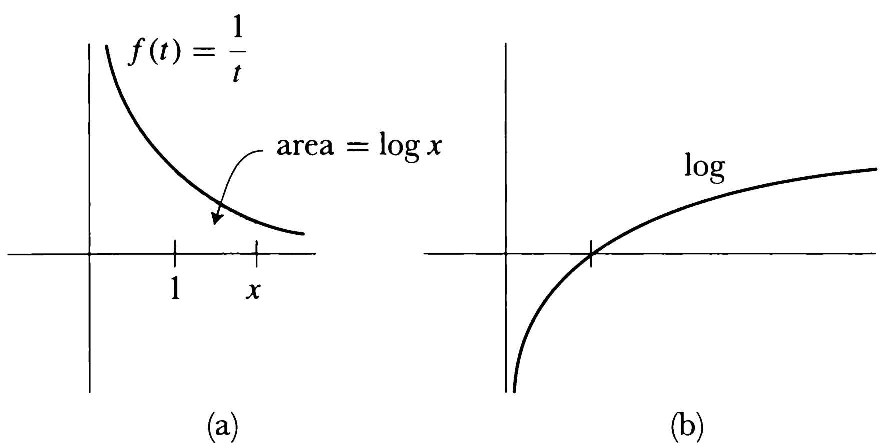

FIGURE 1

The justification for the notation "log" comes from the following theorem.

THEOREM 1 If $x, y > 0$, then

$$\log(xy) = \log x + \log y.$$

PROOF Notice first that $\log'(x) = 1/x$, by the Fundamental Theorem of Calculus. Now choose a number $y > 0$ and let

$$f(x) = \log(xy).$$

Then

$$f'(x) = \log'(xy) \cdot y = \frac{1}{xy} \cdot y = \frac{1}{x}.$$

Thus $f' = \log'$. This means that there is a number c such that

$$f(x) = \log x + c \quad \text{for all } x > 0,$$

that is,

$$\log(xy) = \log x + c \quad \text{for all } x > 0.$$

The number c can be evaluated by noting that when $x = 1$ we obtain

$$\begin{aligned} \log(1 \cdot y) &= \log 1 + c \\ &= c. \end{aligned}$$

Thus

$$\log(xy) = \log x + \log y \quad \text{for all } x.$$

Since this is true for all $y > 0$, the theorem is proved. ▮

COROLLARY 1 If n is a natural number and $x > 0$, then

$$\log(x^n) = n \log x.$$

PROOF Let to you (use induction). ▮

COROLLARY 2 If $x, y > 0$, then

$$\log\left(\frac{x}{y}\right) = \log x - \log y.$$

PROOF This follows from the equations

$$\log x = \log\left(\frac{x}{y} \cdot y\right) = \log\left(\frac{x}{y}\right) + \log y. \ ▮$$

Theorem 1 provides some important information about the graph of log. The function log is clearly increasing, but since $\log'(x) = 1/x$, the derivative becomes very small as x becomes large, and log consequently grows more and more slowly. It is not immediately clear whether log is bounded or unbounded on $\mathbf{R}$. Observe, however, that for a natural number n,

$$\log(2^n) = n \log 2 \quad (\text{and } \log 2 > 0);$$

it follows that log is, in fact, not bounded above. Similarly,

$$\log\left(\frac{1}{2^n}\right) = \log 1 - \log 2^n = -n \log 2;$$

therefore log is not bounded below on $(0, 1)$. Since log is continuous, it actually takes on all values. Therefore $\mathbf{R}$ is the domain of the function $\log^{-1}$. This important function has a special name, whose appropriateness will soon become clear.

DEFINITION The "exponential function," **exp**, is defined as $\log^{-1}$.

The graph of exp is shown in Figure 2. Since $\log x$ is defined only for $x > 0$, we always have $\exp(x) > 0$. The derivative of the function exp is easy to determine.

THEOREM 2 For all numbers x,

$$\exp'(x) = \exp(x).$$

PROOF

$$\begin{aligned}\exp'(x) = (\log^{-1})'(x) &= \frac{1}{\log'(\log^{-1}(x))}\\ &= \frac{1}{\dfrac{1}{\log^{-1}(x)}}\\ &= \log^{-1}(x) = \exp(x). \blacksquare\end{aligned}$$

A second important property of exp is an easy consequence of Theorem 1.

THEOREM 3 If x and y are any two numbers, then

$$\exp(x + y) = \exp(x) \cdot \exp(y).$$

PROOF Let $x' = \exp(x)$ and $y' = \exp(y)$, so that

$$\begin{aligned} x &= \log x',\\ y &= \log y'.\end{aligned}$$

Then

$$x + y = \log x' + \log y' = \log(x'y').$$

This means that

$$\exp(x + y) = x'y' = \exp(x) \cdot \exp(y). \blacksquare$$

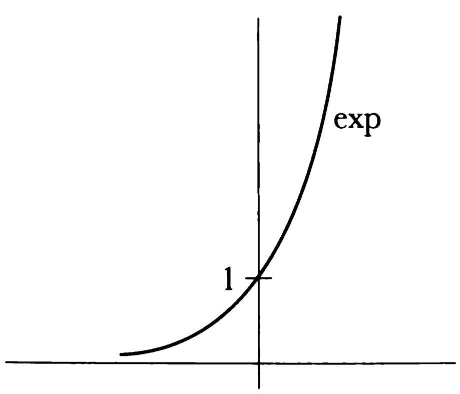

FIGURE 2

This theorem, and the discussion at the beginning of this chapter, suggest that $\exp(1)$ is particularly important. There is, in fact, a special symbol for this number.

DEFINITION $\boldsymbol{e} = \exp(1).$

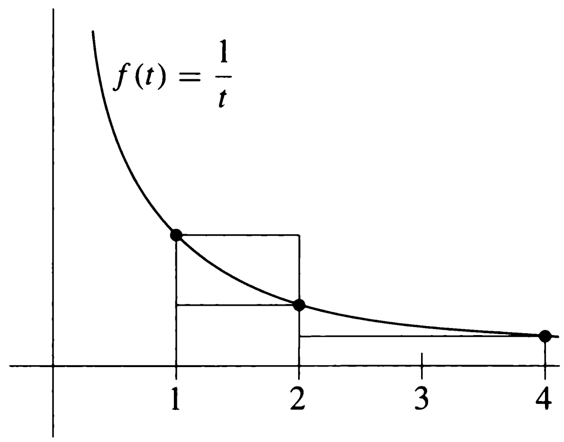

FIGURE 3

This definition is equivalent to the equation

$$1 = \log e = \int_1^e \frac{1}{t}\,dt.$$

As illustrated in Figure 3,

$$\int_1^2 \frac{1}{t}\,dt < 1, \quad \text{since } 1 \cdot (2-1) \text{ is an upper sum for } f(t) = 1/t \text{ on } [1,2],$$

and

$$\int_1^4 \frac{1}{t}\,dt > 1, \quad \text{since } \tfrac{1}{2} \cdot (2-1) + \tfrac{1}{4} \cdot (4-2) = 1 \text{ is a lower sum for } f(t) = 1/t \text{ on } [1,4].$$

Thus

$$\int_1^2 \frac{1}{t}\,dt < \int_1^e \frac{1}{t}\,dt < \int_1^4 \frac{1}{t}\,dt,$$

which shows that

$$2 < e < 4.$$

In Chapter 20 we will find much better approximations for e, and also prove that e is irrational (the proof is much easier than the proof that π is irrational!).

As we remarked at the beginning of the chapter, the equation

$$\exp(x+y) = \exp(x) \cdot \exp(y)$$

implies that

$$\begin{aligned} \exp(x) &= [\exp(1)]^x \\ &= e^x, \text{ for all } \textit{rational } x. \end{aligned}$$

Since exp is defined for all x and $\exp(x) = e^x$ for rational x, it is consistent with our earlier use of the exponential notation to *define* e^x as $\exp(x)$ for all x.

DEFINITION

For any number x,

$$e^x = \exp(x).$$

The terminology "exponential function" should now be clear. We have succeeded in defining e^x for an arbitrary (even irrational) exponent x. We have not yet defined a^x, if $a \neq e$, but there is a reasonable principle to guide us in the attempt. If x is *rational*, then

$$a^x = (e^{\log a})^x = e^{x \log a}.$$

But the last expression is defined for *all* x, so we can use it to define a^x.

DEFINITION

If $a > 0$, then, for any real number x,

$$\boldsymbol{a^x} = e^{x \log a}.$$

(If $a = e$ this definition clearly agrees with the previous one.)

The requirement $a > 0$ is necessary, in order that $\log a$ be defined. This is not unduly restrictive since, for example, we would not even expect

$$(-1)^{1/2} \stackrel{?}{=} \sqrt{-1}$$

to be defined. (Of course, for certain rational x, the symbol a^x will make sense, according to the old definition; for example,

$$(-1)^{1/3} = \sqrt[3]{-1} = -1.)$$

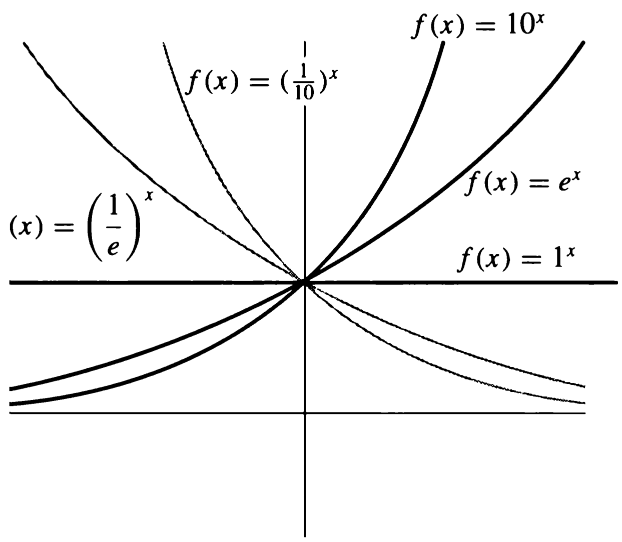

FIGURE 4

Our definition of a^x was designed to ensure that

$$(e^x)^y = e^{xy} \quad \text{for all } x \text{ and } y.$$

As we would hope, this equation turns out to be true when e is replaced by any number $a > 0$. The proof is a moderately involved unraveling of terminology. At the same time we will prove the other important properties of a^x.

THEOREM 4

If $a > 0$, then

(1) $(a^b)^c = a^{bc}$ for all b, c.

(Notice that a^b will automatically be positive, so $(a^b)^c$ will be defined);

(2) $a^1 = a$ and $a^{x+y} = a^x \cdot a^y$ for all x, y.

(Notice that (2) implies that this definition of a^x agrees with the old one for all rational x.)

PROOF

$$(1) \quad (a^b)^c = e^{c \log a^b} = e^{c \log(e^{b \log a})} = e^{c(b \log a)} = e^{cb \log a} = a^{bc}.$$

(Each of the steps in this string of equalities depends upon our last definition, or the fact that $\exp = \log^{-1}$.)

$$\begin{aligned} (2) \qquad & a^1 = e^{1 \log a} = e^{\log a} = a, \\ & a^{x+y} = e^{(x+y)\log a} = e^{x \log a + y \log a} = e^{x \log a} \cdot e^{y \log a} = a^x \cdot a^y. \ \blacksquare \end{aligned}$$

Figure 4 shows the graphs of $f(x) = a^x$ for several different a. The behavior of the function depends on whether $a < 1$, $a = 1$, or $a > 1$. If $a = 1$, then

$f(x) = 1^x = 1$. Suppose $a > 1$. In this case $\log a > 0$. Thus,

$$\begin{array}{ll} \text{if} & x < y, \\ \text{then} & x \log a < y \log a, \\ \text{so} & e^{x \log a} < e^{y \log a}, \\ \text{i.e.,} & a^x < a^y. \end{array}$$

Thus the function $f(x) = a^x$ is increasing. On the other hand, if $0 < a < 1$, so that $\log a < 0$, the same sort of reasoning shows that the function $f(x) = a^x$ is decreasing. In either case, if $a > 0$ and $a \neq 1$, then $f(x) = a^x$ is one-one. Since exp takes on every positive value it is also easy to see that a^x takes on every positive value. Thus the inverse function is defined for all positive numbers, and takes on all values. If $f(x) = a^x$, then f^{-1} is the function usually denoted by $\log_a$ (Figure 5).

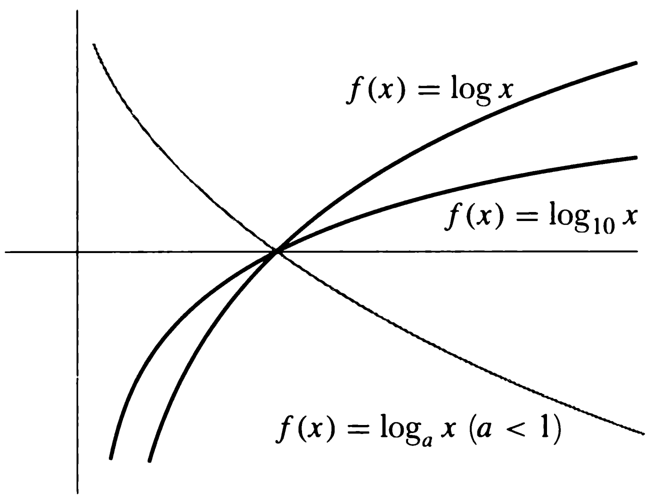

FIGURE 5

Just as a^x can be expressed in terms of exp, so $\log_a$ can be expressed in terms of log. Indeed,

$$\begin{array}{ll} \text{if} & y = \log_a x, \\ \text{then} & x = a^y = e^{y \log a}, \\ \text{so} & \log x = y \log a, \\ \text{or} & y = \dfrac{\log x}{\log a}. \end{array}$$

In other words,

$$\log_a x = \frac{\log x}{\log a}.$$

The derivatives of $f(x) = a^x$ and $g(x) = \log_a x$ are both easy to find:

$$f(x) = e^{x \log a}, \quad \text{so } f'(x) = \log a \cdot e^{x \log a} = \log a \cdot a^x,$$
$$g(x) = \frac{\log x}{\log a}, \quad \text{so } g'(x) = \frac{1}{x \log a}.$$

A more complicated function like

$$f(x) = g(x)^{h(x)}$$

is also easy to differentiate, if you remember that, *by definition*,

$$f(x) = e^{h(x) \log g(x)};$$

it follows from the Chain Rule that

$$\begin{aligned} f'(x) &= e^{h(x) \log g(x)} \cdot \left[h'(x) \log g(x) + h(x) \frac{g'(x)}{g(x)} \right] \\ &= g(x)^{h(x)} \cdot \left[h'(x) \log g(x) + h(x) \frac{g'(x)}{g(x)} \right]. \end{aligned}$$

There is no point in remembering this formula—simply apply the principle behind it in any specific case that arises; it does help, however, to remember that the first factor in the derivative will be $g(x)^{h(x)}$.

There is one special case of the above formula which *is* worth remembering. The function $f(x) = x^a$ was previously defined only for rational a. We can now define and find the derivative of the function $f(x) = x^a$ for any number a; the result is just what we would expect:

$$f(x) = x^a = e^{a \log x}$$

so

$$f'(x) = \frac{a}{x} \cdot e^{a \log x} = \frac{a}{x} \cdot x^a = ax^{a-1}.$$

Algebraic manipulations with the exponential functions will become second nature after a little practice—just remember that all the rules which ought to work actually do. The basic properties of exp are still those stated in Theorems 2 and 3:

$$\exp'(x) = \exp(x),$$
$$\exp(x+y) = \exp(x) \cdot \exp(y).$$

In fact, each of these properties comes close to characterizing the function exp. Naturally, exp is not the only function f satisfying $f' = f$, for if $f = ce^x$, then $f'(x) = ce^x = f(x)$; these functions are the only ones with this property, however.

THEOREM 5 If f is differentiable and

$$f'(x) = f(x) \quad \text{for all } x,$$

then there is a number c such that

$$f(x) = ce^x \quad \text{for all } x.$$

PROOF Let

$$g(x) = \frac{f(x)}{e^x}.$$

(This is permissible, since $e^x \neq 0$ for all x.) Then

$$g'(x) = \frac{e^x f'(x) - f(x)e^x}{(e^x)^2} = 0.$$

Therefore there is a number c such that

$$g(x) = \frac{f(x)}{e^x} = c \quad \text{for all } x. \blacksquare$$

The second basic property of exp requires a more involved discussion. The function exp is clearly not the only function f which satisfies

$$f(x+y) = f(x) \cdot f(y).$$

In fact, $f(x) = 0$ or any function of the form $f(x) = a^x$ also satisfies this equation. But the true story is much more complex than this—there are infinitely many other functions which satisfy this property, but it is impossible, without appealing to more advanced mathematics, to prove that there is even one function other than those already mentioned! It is for this reason that the definition of 10^x is so difficult: there are infinitely many functions f which satisfy

$$f(x+y) = f(x) \cdot f(y),$$
$$f(1) = 10,$$

but which are *not* the function $f(x) = 10^x$! One thing is true however—any *continuous* function f satisfying

$$f(x+y) = f(x) \cdot f(y)$$

must be of the form $f(x) = a^x$ or $f(x) = 0$. (Problem 38 indicates the way to prove this, and also has a few words to say about discontinuous functions with this property.)

In addition to the two basic properties stated in Theorems 2 and 3, the function exp has one further property which is very important—exp "grows faster than any polynomial." In other words,

THEOREM 6 For any natural number n,

$$\lim_{x \to \infty} \frac{e^x}{x^n} = \infty.$$

PROOF The proof consists of several steps.

Step 1. $e^x > x$ for all x, and consequently $\lim_{x \to \infty} e^x = \infty$ (this may be considered to be the case $n = 0$).

To prove this statement (which is clear for $x \leq 0$) it suffices to show that

$$x > \log x \quad \text{for all } x > 0.$$

If $x < 1$ this is clearly true, since $\log x < 0$. If $x > 1$, then (Figure 6) $x - 1$ is an upper sum for $f(t) = 1/t$ on $[1, x]$, so $\log x < x - 1 < x$.

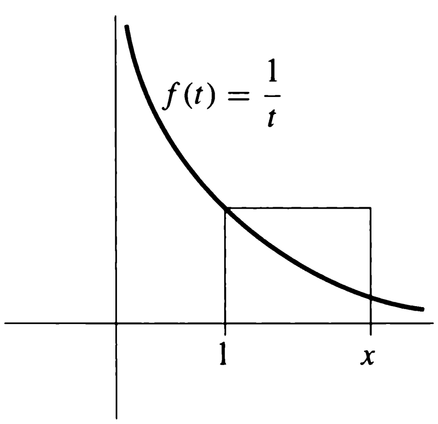

FIGURE 6

Step 2. $\displaystyle\lim_{x \to \infty} \frac{e^x}{x} = \infty.$

To prove this, note that

$$\frac{e^x}{x} = \frac{e^{x/2} \cdot e^{x/2}}{\frac{x}{2} \cdot 2} = \frac{1}{2} \left(\frac{e^{x/2}}{\frac{x}{2}} \right) \cdot e^{x/2}.$$

By Step 1, the expression in parentheses is greater than 1, and $\lim_{x \to \infty} e^{x/2} = \infty$; this shows that $\lim_{x \to \infty} e^x/x = \infty$.

Step 3. $\displaystyle\lim_{x \to \infty} \frac{e^x}{x^n} = \infty.$

Note that

$$\frac{e^x}{x^n} = \frac{(e^{x/n})^n}{\left(\frac{x}{n}\right)^n \cdot n^n} = \frac{1}{n^n} \cdot \left(\frac{e^{x/n}}{\frac{x}{n}} \right)^n.$$

The expression in parentheses becomes arbitrarily large, by Step 2, so the nth power certainly becomes arbitrarily large. ▌

It is now possible to examine carefully the following very interesting function: $f(x) = e^{-1/x^2}$, $x \neq 0$. We have

$$f'(x) = e^{-1/x^2} \cdot \frac{2}{x^3}.$$

Therefore,

$$\begin{aligned} f'(x) &< 0 \quad \text{for } x < 0, \\ f'(x) &> 0 \quad \text{for } x > 0, \end{aligned}$$

so f is decreasing for negative x and increasing for positive x. Moreover, if $|x|$ is large, then x^2 is large, so $-1/x^2$ is close to 0, so e^{-1/x^2} is close to 1 (Figure 7).

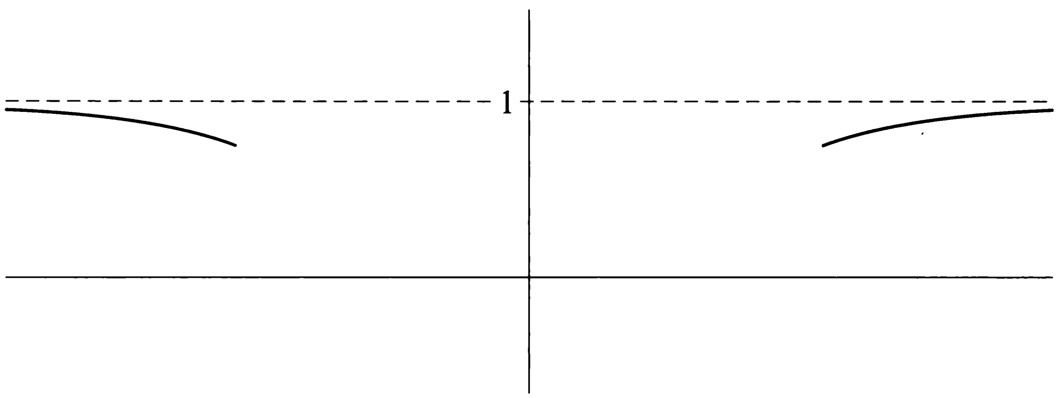

FIGURE 7

The behavior of f near 0 is more interesting. If x is small, then $1/x^2$ is large, so e^{1/x^2} is large, so $e^{-1/x^2} = 1/(e^{1/x^2})$ is small. This argument, suitably stated with ε's and δ's, shows that

$$\lim_{x \to 0} e^{-1/x^2} = 0.$$

Therefore, if we define

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0, \end{cases}$$

then the function f is continuous (Figure 8). In fact, f is actually differentiable

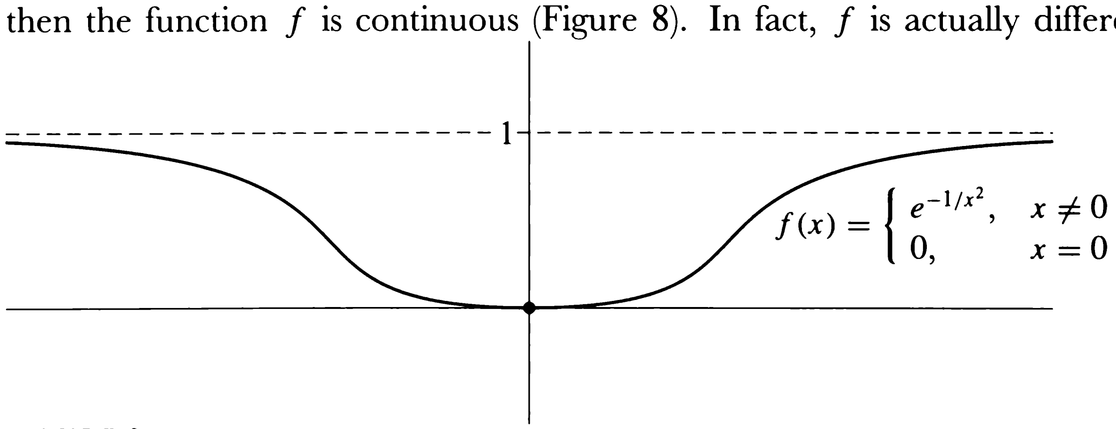

FIGURE 8

at 0: Indeed

$$f'(0) = \lim_{h\to 0} \frac{e^{-1/h^2}}{h} = \lim_{h\to 0} \frac{1/h}{e^{(1/h)^2}},$$

and

$$\lim_{h\to 0^+} \frac{1/h}{e^{(1/h)^2}} = \lim_{x\to\infty} \frac{x}{e^{(x^2)}}, \qquad \text{while} \qquad \lim_{h\to 0^-} \frac{1/h}{e^{(1/h)^2}} = -\lim_{x\to\infty} \frac{x}{e^{(x^2)}}.$$

We already know that

$$\lim_{x\to\infty} \frac{e^x}{x} = \infty;$$

it is all the more true that

$$\lim_{x\to\infty} \frac{e^{(x^2)}}{x} = \infty,$$

and this means that

$$\lim_{x\to\infty} \frac{x}{e^{(x^2)}} = 0.$$

Thus

$$f'(x) = \begin{cases} e^{-1/x^2} \cdot \dfrac{2}{x^3}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

We can now compute that

$$\begin{aligned} f''(0) &= \lim_{h\to 0} \frac{f'(h) - f'(0)}{h} \\ &= \lim_{h\to 0} \frac{e^{-1/h^2} \cdot \dfrac{2}{h^3}}{h} \\ &= \lim_{h\to 0} \frac{2 \cdot e^{-1/h^2}}{h^4} = \lim_{h\to 0} \frac{2 \cdot \dfrac{1}{h^4}}{e^{1/h^2}} = \lim_{x\to\infty} \frac{2x^4}{e^{(x^2)}}; \end{aligned}$$

an argument similar to the one above shows that $f''(0) = 0$. Thus

$$f''(x) = \begin{cases} e^{-1/x^2} \cdot \dfrac{-6}{x^4} + e^{-1/x^2} \cdot \dfrac{4}{x^6}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

This argument can be continued. In fact, using induction it can be shown (Problem 40) that $f^{(k)}(0) = 0$ for *every* k. The function f is *extremely* flat at 0, and approaches 0 so quickly that it can mask many irregularities of other functions. For example (Figure 9), suppose that

$$f(x) = \begin{cases} e^{-1/x^2} \cdot \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

It can be shown (Problem 41) that for this function it is also true that $f^{(k)}(0) = 0$ for all k. This example shows, perhaps more strikingly than any other, just how bad a function can be, and still be infinitely differentiable. In Part IV we will investigate even more restrictive conditions on a function, which will finally rule out behavior of this sort.

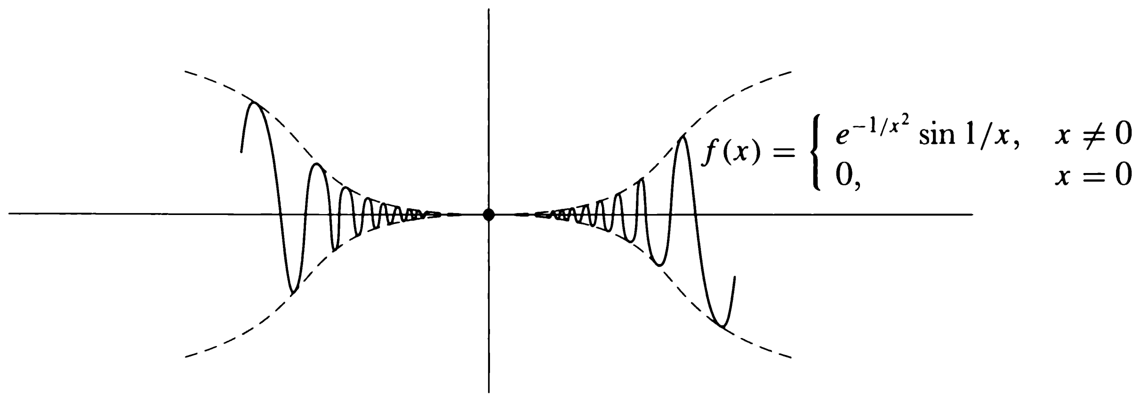

FIGURE 9

PROBLEMS

1. Differentiate each of the following functions (remember that a^{b^c} always denotes $a^{(b^c)}$).

(i) $f(x) = e^{e^{e^{e^x}}}$.

(ii) $f(x) = \log(1 + \log(1 + \log(1 + e^{1+e^{1+x}})))$.

(iii) $f(x) = (\sin x)^{\sin(\sin x)}$.

(iv) $f(x) = e^{\left(\int_0^x e^{-t^2}\,dt\right)}$.

(v) $f(x) = (\sin x)^{(\sin x)^{\sin x}}$.

(vi) $f(x) = \log_{(e^x)} \sin x$.

(vii) $f(x) = \left[\arcsin\left(\dfrac{x}{\sin x}\right)\right]^{\log(\sin e^x)}$.

(viii) $f(x) = (\log(3 + e^4))e^{4x} + (\arcsin x)^{\log 3}$.

(ix) $f(x) = (\log x)^{\log x}$.

(x) $f(x) = x^x$.

(xi) $f(x) = \sin\left(x^{\sin(x^{\sin x})}\right)$.

2. (a) Check that the derivative of $\log \circ f$ is f'/f.

This expression is called the *logarithmic derivative* of f. It is often easier to compute than f', since products and powers in the expression for f become sums and products in the expression for $\log \circ f$. The derivative f' can then be recovered simply by multiplying by f; this process is called *logarithmic differentiation.*

(b) Use logarithmic differentiation to find $f'(x)$ for each of the following.

(i) $f(x) = (1 + x)(1 + e^{x^2})$.

(ii) $f(x) = \dfrac{(3-x)^{1/3}x^2}{(1-x)(3+x)^{2/3}}.$

(iii) $f(x) = (\sin x)^{\cos x} + (\cos x)^{\sin x}.$

(iv) $f(x) = \dfrac{e^x - e^{-x}}{e^{2x}(1+x^3)}.$

3. Find

$$\int_a^b \frac{f'(t)}{f(t)}\,dt$$

(for $f > 0$ on $[a, b]$).

4. Graph each of the following functions.

(a) $f(x) = e^{x+1}.$
(b) $f(x) = e^{\sin x}.$
(c) $f(x) = e^x + e^{-x}.$
(d) $f(x) = e^x - e^{-x}.$
(Compare the graph with the graphs of exp and 1/exp.)

(e) $f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}} = \dfrac{e^{2x}-1}{e^{2x}+1} = 1 - \dfrac{2}{e^{2x}+1}.$

5. Find the following limits by l'Hôpital's Rule.

(i) $\displaystyle\lim_{x\to 0} \frac{e^x - 1 - x - x^2/2}{x^2}.$

(ii) $\displaystyle\lim_{x\to 0} \frac{e^x - 1 - x - x^2/2 - x^3/6}{x^3}.$

(iii) $\displaystyle\lim_{x\to 0} \frac{e^x - 1 - x - x^2/2}{x^3}.$

(iv) $\displaystyle\lim_{x\to 0} \frac{\log(1+x) - x + x^2/2}{x^2}.$

(v) $\displaystyle\lim_{x\to 0} \frac{\log(1+x) - x + x^2/2}{x^3}.$

(vi) $\displaystyle\lim_{x\to 0} \frac{\log(1+x) - x + x^2/2 - x^3/3}{x^3}.$

6. Find the following limits by l'Hôpital's Rule.

(i) $\displaystyle\lim_{x\to 0} (1-x)^{1/x}.$

(ii) $\displaystyle\lim_{x\to \frac{\pi}{4}} (\tan x)^{\tan 2x}.$

(iii) $\displaystyle\lim_{x\to 0} (\cos x)^{1/x^2}.$

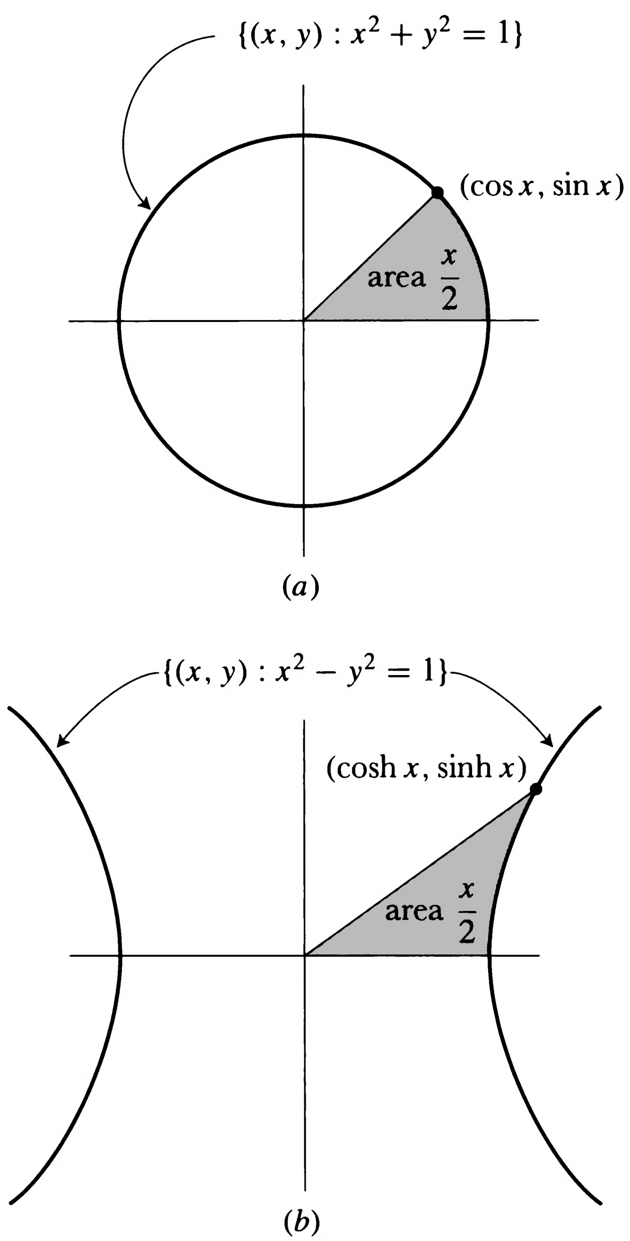

FIGURE 10

7. The functions

$$\sinh x = \frac{e^x - e^{-x}}{2},$$
$$\cosh x = \frac{e^x + e^{-x}}{2},$$
$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = 1 - \frac{2}{e^{2x} + 1},$$

are called the **hyperbolic sine**, **hyperbolic cosine**, and **hyperbolic tangent**, respectively (but usually read 'sinch,' 'cosh,' and 'tanch'). There are many analogies between these functions and their ordinary trigonometric counterparts. One analogy is illustrated in Figure 10; a proof that the region shown in Figure 10(b) really has area $x/2$ is best deferred until the next chapter, when we will develop methods of computing integrals. Other analogies are discussed in the following three problems, but the deepest analogies must wait until Chapter 27. If you have not already done Problem 4, graph the functions sinh, cosh, and tanh.

8. Prove that

(a) $\cosh^2 - \sinh^2 = 1$.
(b) $\tanh^2 + 1/\cosh^2 = 1$.
(c) $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$.
(d) $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$.
(e) $\sinh' = \cosh$.
(f) $\cosh' = \sinh$.
(g) $\tanh' = \dfrac{1}{\cosh^2}$.

9. The functions sinh and tanh are one-one; their inverses $\sinh^{-1}$ and $\tanh^{-1}$, are defined on $\mathbf{R}$ and $(-1, 1)$, respectively. These inverse functions are sometimes denoted by arg sinh and arg tanh (the "argument" of the hyperbolic sine and tangent). If cosh is restricted to $[0, \infty)$ it has an inverse, denoted by arg cosh, or simply $\cosh^{-1}$, which is defined on $[1, \infty)$. Prove, using the information in Problem 8, that

(a) $\sinh(\cosh^{-1} x) = \sqrt{x^2 - 1}$.
(b) $\cosh(\sinh^{-1} x) = \sqrt{1 + x^2}$.
(c) $(\sinh^{-1})'(x) = \dfrac{1}{\sqrt{1 + x^2}}$.
(d) $(\cosh^{-1})'(x) = \dfrac{1}{\sqrt{x^2 - 1}}$ for $x > 1$.
(e) $(\tanh^{-1})'(x) = \dfrac{1}{1 - x^2}$ for $|x| < 1$.

10. (a) Find an explicit formula for $\sinh^{-1}$, $\cosh^{-1}$, and $\tanh^{-1}$ (by solving the equation $y = \sinh^{-1} x$ for x in terms of y, etc.).
(b) Find

$$\int_a^b \frac{1}{\sqrt{1+x^2}}\,dx,$$

$$\int_a^b \frac{1}{\sqrt{x^2-1}}\,dx \quad \text{for } a, b > 1 \text{ or } a, b < -1,$$

$$\int_a^b \frac{1}{1-x^2}\,dx \quad \text{for } |a|, |b| < 1.$$

Compare your answer for the third integral with that obtained by writing

$$\frac{1}{1-x^2} = \frac{1}{2}\left[\frac{1}{1-x} + \frac{1}{1+x}\right].$$

11. Show that

$$F(x) = \int_2^x \frac{1}{\log t}\,dt$$

is not bounded on $[2, \infty)$.

12. Let f be a nondecreasing function on $[1, \infty)$, and define

$$F(x) = \int_1^x \frac{f(t)}{t}\,dt, \qquad x \geq 1.$$

Prove that f is bounded on $[1, \infty)$ if and only if $F/\log$ is bounded on $[1, \infty)$.

13. Find

(a) $\lim\limits_{x\to\infty} a^x$ for $0 < a < 1$. (Remember the definition!)

(b) $\lim\limits_{x\to\infty} \dfrac{x}{(\log x)^n}$.

(c) $\lim\limits_{x\to\infty} \dfrac{(\log x)^n}{x}$.

(d) $\lim\limits_{x\to 0^+} x(\log x)^n$. Hint: $x(\log x)^n = \dfrac{(-1)^n\left(\log \dfrac{1}{x}\right)^n}{\dfrac{1}{x}}$.

(e) $\lim\limits_{x\to 0^+} x^x$.

14. Graph $f(x) = x^x$ for $x > 0$. (Use Problem 13(e).)

15. (a) Find the minimum value of $f(x) = e^x/x^n$ for $x > 0$, and conclude that $f(x) > e^n/n^n$ for $x > n$.
(b) Using the expression $f'(x) = e^x(x-n)/x^{n+1}$, prove that $f'(x) > e^{n+1}/(n+1)^{n+1}$ for $x > n+1$, and thus obtain another proof that $\lim\limits_{x\to\infty} f(x) = \infty$.

16. Graph $f(x) = e^x/x^n$.

17. (a) Find $\lim_{y \to 0} \log(1+y)/y$. (You can use l'Hôpital's Rule, but that would be silly.)

(b) Find $\lim_{x \to \infty} x \log(1+1/x)$.

(c) Prove that $e = \lim_{x \to \infty} (1+1/x)^x$.

(d) Prove that $e^a = \lim_{x \to \infty} (1+a/x)^x$. (It is possible to derive this from part (c) with just a little algebraic fiddling.)

*(e) Prove that $\log b = \lim_{x \to \infty} x(b^{1/x} - 1)$.

18. Graph $f(x) = (1+1/x)^x$ for $x > 0$. (Use Problem 17(c).)

19. If a bank gives a percent interest per annum, then an initial investment I yields $I(1+a/100)$ after 1 year. If the bank compounds the interest (counts the accrued interest as part of the capital for computing interest the next year), then the initial investment grows to $I(1+a/100)^n$ after n years. Now suppose that interest is given twice a year. The final amount after n years is, alas, not $I(1+a/100)^{2n}$, but merely $I(1+a/200)^{2n}$—although interest is awarded twice as often, the interest must be halved in each calculation, since the interest is $a/2$ per half year. This amount is larger than $I(1+a/100)^n$, but not that much larger. Suppose that the bank now compounds the interest continuously, i.e., the bank considers what the investment would yield when compounding k times a year, and then takes the least upper bound of all these numbers. How much will an initial investment of 1 dollar yield after 1 year?

20. (a) Let $f(x) = \log|x|$ for $x \neq 0$. Prove that $f'(x) = 1/x$ for $x \neq 0$.

(b) If $f(x) \neq 0$ for all x, prove that $(\log|f|)' = f'/f$.

21. Suppose that on some interval the function f satisfies $f' = cf$ for some number c.

(a) Assuming that f is never 0, use Problem 20(b) to prove that $|f(x)| = le^{cx}$ for some number l (> 0). It follows that $f(x) = ke^{cx}$ for some k.

(b) Show that this result holds without the added assumption that f is never 0. Hint: Show that f can't be 0 at the endpoint of an open interval on which it is nowhere 0.

(c) Give a simpler proof that $f(x) = ke^{cx}$ for some k by considering the function $g(x) = f(x)/e^{cx}$.

(d) Suppose that $f' = fg'$ for some g. Show that $f(x) = ke^{g(x)}$ for some k.

***22.** A radioactive substance diminishes at a rate proportional to the amount present (since all atoms have equal probability of disintegrating, the total disintegration is proportional to the number of atoms remaining). If $A(t)$ is the amount at time t, this means that $A'(t) = cA(t)$ for some c (which represents the probability that an atom will disintegrate).

(a) Find $A(t)$ in terms of the amount $A_0 = A(0)$ present at time 0.

(b) Show that there is a number τ (the "half-life" of the radioactive element) with the property that $A(t+\tau) = A(t)/2$.

23. *Newton's law of cooling* states that an object cools at a rate proportional to the difference of its temperature and the temperature of the surrounding medium. Find the temperature $T(t)$ of the object at time t, in terms of its temperature T_0 at time 0, assuming that the temperature of the surrounding medium is kept at a constant, M. Hint: To solve the differential equation expressing Newton's law, remember that $T' = (T - M)'$.

24. Prove that if $f(x) = \int_0^x f(t)\,dt$, then $f = 0$.

25. Find all continuous functions f satisfying

(i) $\displaystyle\int_0^x f = e^x$.

(ii) $\displaystyle\int_0^{x^2} f = 1 - e^{2x^2}$.

26. Find all functions f satisfying $f'(t) = f(t) + \displaystyle\int_0^1 f(t)\,dt$.

27. Find all continuous functions f which satisfy the equation

$$(f(x))^2 = \int_0^x f(t)\frac{t}{1+t^2}\,dt.$$

28. (a) Let f and g be continuous functions on $[a, b]$ with g nonnegative. Suppose that for some C we have

$$f(x) \leq C + \int_a^x fg, \qquad a \leq x \leq b.$$

Prove *Gronwall's inequality*:

$$f(x) \leq Ce^{\int_a^x g}.$$

Hint: Consider the derivative of the function $h(x) = \left(C + \int_a^x fg\right)e^{-\int_a^x g}$.

(b) Let f and g be nonnegative functions with g continuous and f differentiable. Suppose that $f'(x) = g(x)f(x)$ and $f(0) = 0$. Prove that $f = 0$. (Compare Problem 21.)

29. (a) Prove that

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} \leq e^x \quad \text{for } x \geq 0.$$

Hint: Use induction on n, and compare derivatives.

(b) Give a new proof that $\lim\limits_{x\to\infty} e^x/x^n = \infty$.

30. Give yet another proof of this fact, using the appropriate form of l'Hôpital's Rule. (See Problem 11-56.)

31. (a) Evaluate $\lim_{x\to\infty} e^{-x^2} \int_0^x e^{t^2}\,dt$. (You should be able to make an educated guess before doing any calculations.)

(b) Evaluate the following limits.

(i) $\lim_{x\to\infty} e^{-x^2} \int_x^{x+(1/x)} e^{t^2}\,dt$.

(ii) $\lim_{x\to\infty} e^{-x^2} \int_x^{x+(\log x)/x} e^{t^2}\,dt$.

(iii) $\lim_{x\to\infty} e^{-x^2} \int_x^{x+(\log x)/2x} e^{t^2}\,dt$.

32. This problem outlines the classical approach to logarithms and exponentials. To begin with, we will simply assume that the function $f(x) = a^x$, defined in an elementary way for rational x, can somehow be extended to a continuous one-one function, obeying the same algebraic rules, on the whole line. (See Problem 22-29 for a direct proof of this.) The inverse of f will then be denoted by $\log_a$.

(a) Show, directly from the definition, that

$$\begin{aligned}\log_a{}'(x) &= \lim_{h\to 0} \log_a\left(1+\frac{h}{x}\right)^{1/h}\\ &= \frac{1}{x}\cdot\log_a\left(\lim_{k\to 0}(1+k)^{1/k}\right).\end{aligned}$$

Thus, the whole problem has been reduced to the determination of $\lim_{h\to 0}(1+h)^{1/h}$. If we can show that this has a limit e, then $\log_e{}'(x) = \frac{1}{x}\cdot\log_e e = \frac{1}{x}$, and consequently $\exp = \log_e^{-1}$ has derivative $\exp'(x) = \exp(x)$.

(b) Let $a_n = \left(1+\frac{1}{n}\right)^n$ for natural numbers n. Using the binomial theorem, show that

$$a_n = 2 + \sum_{k=2}^{n} \frac{1}{k!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdot\ldots\cdot\left(1-\frac{k-1}{n}\right).$$

Conclude that $a_n < a_{n+1}$.

(c) Using the fact that $1/k! \le 1/2^{k-1}$ for $k \ge 2$, show that all $a_n < 3$. Thus, the set of numbers $\{a_1, a_2, a_3, \ldots\}$ is bounded, and therefore has a least upper bound e. Show that for any $\varepsilon > 0$ we have $e - a_n < \varepsilon$ for large enough n.

(d) If $n \le x \le n+1$, then

$$\left(1+\frac{1}{n+1}\right)^n \le \left(1+\frac{1}{x}\right)^x \le \left(1+\frac{1}{n}\right)^{n+1}$$

Conclude that $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$. Also show that $\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^x = e$, and conclude that $\lim_{h\to 0}(1+h)^{1/h} = e$.

***33.** A point P is moving along a line segment AB of length 10^7 while another point Q moves along an infinite ray (Figure 11). The velocity of P is always equal to the distance from P to B (in other words, if $P(t)$ is the position of P at time t, then $P'(t) = 10^7 - P(t)$), while Q moves with constant velocity $Q'(t) = 10^7$. The distance traveled by Q after time t is defined to be the *Napierian logarithm* of the distance from P to B at time t. Thus

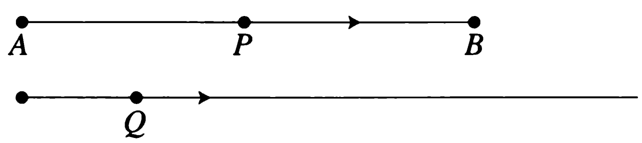

FIGURE 11

$$10^7 t = \text{Nap log}[10^7 - P(t)].$$

This was the definition of logarithms given by Napier (1550–1617) in his publication of 1614, *Mirifici logarithmonum canonis description* (A Description of the Wonderful Law of Logarithms); work which was done *before* the use of exponents was invented! The number 10^7 was chosen because Napier's tables (intended for astronomical and navigational calculations), listed the logarithms of sines of angles, for which the best possible available tables extended to seven decimal places, and Napier wanted to avoid fractions. Prove that

$$\text{Nap log}\, x = 10^7 \log \frac{10^7}{x}.$$

Hint: Use the same trick as in Problem 23 to solve the equation for P.

***34.**
(a) Sketch the graph of $f(x) = (\log x)/x$ (paying particular attention to the behavior near 0 and ∞).
(b) Which is larger, e^π or π^e?
(c) Prove that if $0 < x \le 1$, or $x = e$, then the only number y satisfying $x^y = y^x$ is $y = x$; but if $x > 1$, $x \ne e$, then there is precisely one number $y \ne x$ satisfying $x^y = y^x$; moreover, if $x < e$, then $y > e$, and if $x > e$, then $y < e$. (Interpret these statements in terms of the graph in part (a)!)
(d) Prove that if x and y are natural numbers and $x^y = y^x$, then $x = y$ or $x = 2$, $y = 4$, or $x = 4$, $y = 2$.
(e) Show that the set of all pairs (x, y) with $x^y = y^x$ consists of a curve and a straight line which intersect; find the intersection and draw a rough sketch.
**(f) For $1 < x < e$ let $g(x)$ be the unique number $> e$ with $x^{g(x)} = g(x)^x$. Prove that g is differentiable. (It is a good idea to consider separate functions,

$$f_1(x) = \frac{\log x}{x}, \quad 0 < x < e$$
$$f_2(x) = \frac{\log x}{x}, \quad e < x$$

and write g in terms of f_1 and f_2. You should be able to show that

$$g'(x) = \frac{[g(x)]^2}{1 - \log g(x)} \cdot \frac{1 - \log x}{x^2}$$

if you do this part properly.)

***35.** This problem uses the material from the Appendix to Chapter 11.

(a) Prove that exp is convex and log is concave.

(b) Prove that if $\sum_{i=1}^{n} p_i = 1$ and all $p_i > 0$, then for all $z_i > 0$ we have

$$z_1{}^{p_1} \cdot \ldots \cdot z_n{}^{p_n} < p_1 z_1 + \cdots + p_n z_n.$$

(Use Problem 8 from the Appendix to Chapter 11.)

(c) Deduce another proof that $G_n \le A_n$ (Problem 2-22).

36. (a) Let f be a positive function on $[a, b]$, and let P_n be the partition of $[a, b]$ into n equal intervals. Use Problem 2-22 to show that

$$\frac{1}{b-a} L(\log f, P_n) \le \log\left(\frac{1}{b-a} L(f, P_n)\right).$$

(b) Use the Appendix to Chapter 13 to conclude that for all integrable $f > 0$ we have

$$\frac{1}{b-a} \int_a^b \log f \le \log\left(\frac{1}{b-a} \int_a^b f\right).$$

A more direct approach is illustrated in the next part:

(c) In Problem 35, Problem 2-22 was deduced as a special case of the inequality

$$g\left(\sum_{i=1}^{n} p_i x_i\right) \le \sum_{i=1}^{n} p_i g(x_i)$$

for $p_i > 0$, $\sum_{i=1}^{n} p_i = 1$ and g convex. For g concave we have the reverse inequality

$$\sum_{i=1}^{n} p_i g(x_i) \le g\left(\sum_{i=1}^{n} p_i x_i\right).$$

Apply this with $g = \log$ to prove the result of part (b) directly for any integrable f.

(d) State a general theorem of which part (b) is just a special case.

37. Suppose f satisfies $f' = f$ and $f(x + y) = f(x)f(y)$ for all x and y. Prove that $f = \exp$ or $f = 0$.

***38.** Prove that if f is continuous and $f(x+y) = f(x)f(y)$ for all x and y, then either $f = 0$ or $f(x) = [f(1)]^x$ for all x. Hint: Show that $f(x) = [f(1)]^x$ for rational x, and then use Problem 8-6. This problem is closely related to Problem 8-7, and the information mentioned at the end of Problem 8-7 can be used to show that there are discontinuous functions f satisfying $f(x+y) = f(x)f(y)$.

***39.** Prove that if f is a continuous function defined on the positive real numbers, and $f(xy) = f(x) + f(y)$ for all positive x and y, then $f = 0$ or $f(x) = f(e)\log x$ for all $x > 0$. Hint: Consider $g(x) = f(e^x)$.

***40.** Prove that if $f(x) = e^{-1/x^2}$ for $x \neq 0$, and $f(0) = 0$, then $f^{(k)}(0) = 0$ for all k (you will encounter the same sort of difficulties as in Problem 10-21). Hint: Consider functions $g(x) = e^{-1/x^2}P(1/x)$ for a polynomial function P.

***41.** Prove that if $f(x) = e^{-1/x^2}\sin 1/x$ for $x \neq 0$, and $f(0) = 0$, then $f^{(k)}(0) = 0$ for all k.

42. (a) Prove that if α is a root of the equation

$$(*) \qquad a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0,$$

then the function $y(x) = e^{\alpha x}$ satisfies the differential equation

$$(**) \qquad a_n y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0.$$

*(b) Prove that if α is a double root of $(*)$, then $y(x) = xe^{\alpha x}$ also satisfies $(**)$. Hint: Remember that if α is a double root of a polynomial equation $f(x) = 0$, then $f'(\alpha) = 0$.

*(c) Prove that if α is a root of $(*)$ of order r, then $y(x) = x^k e^{\alpha x}$ is a solution for $0 \le k \le r-1$.

If $(*)$ has n real numbers as roots (counting multiplicities), part (c) gives n solutions $y_1, \dots, y_n$ of $(**)$.

(d) Prove that in this case the function $c_1 y_1 + \cdots + c_n y_n$ also satisfies $(**)$.

It is a theorem that in this case these are the only solutions of $(**)$. Problem 21 and the next two problems prove special cases of this theorem, and the general case is considered in Problem 20-26. In Chapter 27 we will see what to do when $(*)$ does not have n real numbers as roots.

***43.** Suppose that f satisfies $f'' - f = 0$ and $f(0) = f'(0) = 0$. Prove that $f = 0$ as follows.

(a) Show that $f^2 - (f')^2 = 0$.

(b) Suppose that $f(x) \neq 0$ for all x in some interval (a, b). Show that either $f(x) = ce^x$ or else $f(x) = ce^{-x}$ for all x in (a, b), for some constant c.

**(c) If $f(x_0) \neq 0$ for $x_0 > 0$, say, then there would be a number a such that $0 \le a < x_0$ and $f(a) = 0$, while $f(x) \neq 0$ for $a < x < x_0$. Why? Use this fact and part (b) to deduce a contradiction.

(e) Arrange each of the following sets of functions in increasing order of growth (for convenience, we indicate each function simply by giving its value at x):

(i) x^3, e^x, $x^3 + \log(x^3)$, $\log 4x$, $(\log x)^x$, x^x, $x + e^{-5x}$, $x^3 \log x$.
(ii) $x \log^2 x$, e^{5x}, $\log(x^x)$, e^{x^2}, x^x, $x^{\log x}$, $(\log x)^x$.
(iii) e^x, x^e, x^x, e^{x^2}, 2^x, $e^{x/2}$, $(\log x)^{2x}$.

48. Suppose that $g_1, g_2, g_3, \ldots$ are continuous functions. Show that there is a continuous function f which grows faster than each g_i.

49. Prove that $\log_{10} 2$ is irrational.

***44.** (a) Show that if f satisfies $f'' - f = 0$, then $f(x) = ae^x + be^{-x}$ for some a and b. (First figure out what a and b should be in terms of $f(0)$ and $f'(0)$, and then use Problem 43.)

(b) Show also that $f = a\sinh + b\cosh$ for some (other) a and b.

45. Find all functions f satisfying

(a) $f^{(n)} = f^{(n-1)}$.

(b) $f^{(n)} = f^{(n-2)}$.

***46.** This problem, a companion to Problem 15-30, outlines a treatment of the exponential function starting from the assumption that the differential equation $f' = f$ has a nonzero solution.

(a) Suppose there is a function $f \neq 0$ with $f' = f$. Prove that $f(x) \neq 0$ for each x by considering the function $g(x) = f(x_0 + x)f(x_0 - x)$, where $f(x_0) \neq 0$.

(b) Show that there is a function f satisfying $f' = f$ and $f(0) = 1$.

(c) For this f show that $f(x + y) = f(x) \cdot f(y)$ by considering the function $g(x) = f(x + y)/f(x)$.

(d) Prove that f is one-one and that $(f^{-1})'(x) = 1/x$.

47. Let f and g be continuous functions such that $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$. We say that f *grows faster than* g ($f \gg g$) if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty,$$

and we say that f and g *grow at the same rate* ($f \sim g$) if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} \text{ exists and is } \neq 0, \infty.$$

For example, for any polynomial function P with $\lim_{x\to\infty} P(x) = \infty$ (i.e., P is non-constant and has positive leading coefficient) we have $\exp \gg P$ and $P \gg \log^n$ for any positive integer n.

(a) Given f and g, with $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$, is it necessarily true that one of the three conditions $f \gg g$ or $g \gg f$ or $f \sim g$ holds?

(b) If $f \gg g$, then $f + g \sim f$.

(c) If

$$\frac{\log f}{\log g} \geq c > 1$$

for sufficiently large x, then $f \gg g$.

(d) If $f \gg g$ and $F(x) = \int_0^x f$, $G(x) = \int_0^x g$, does it necessarily follow that $F \gg G$?