Chapter 9

Group Representations

A tremendous effort has been made by mathematicians for more than a century to clear up the chaos in group theory. Still, we cannot answer some of the simplest questions.

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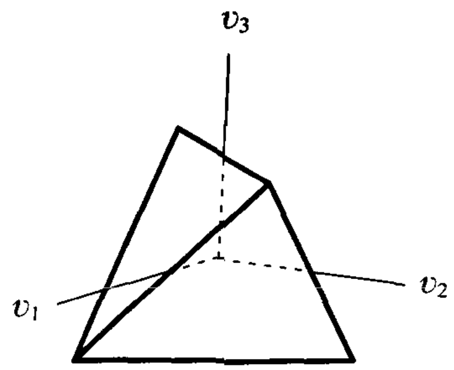
1. DEFINITION OF A GROUP REPRESENTATION

Operations of a group on an arbitrary set were studied in Chapter 5. In this chapter we consider the case that the group elements act as linear operators on a vector space. Such an operation defines a homomorphism from G to the general linear group. A homomorphism to the general linear group is called a *matrix representation.*

The finite rotation groups are good examples to keep in mind. The group T of rotations of a tetrahedron, for example, operates on a three-dimensional space V by rotations. We didn't write down the matrices which represent this action explicitly in Chapter 5; let us do so now. A natural choice of basis has the coordinate axes passing through the midpoints of three of the edges, as illustrated below:



(1.1) **Figure.**

Let $y_i \in T$ denote the rotation by π around an edge, and let $x \in T$ denote rotation by $2\pi/3$ around the front vertex. The matrices representing these operations are

$$(1.2)\qquad R_{y_1} = \begin{bmatrix} 1 & & \\ & -1 & \\ & & -1 \end{bmatrix}, \quad R_{y_2} = \begin{bmatrix} -1 & & \\ & 1 & \\ & & -1 \end{bmatrix}, \quad R_{y_3} = \begin{bmatrix} -1 & & \\ & -1 & \\ & & 1 \end{bmatrix},$$
$$R_x = \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}.$$

The rotations $\{y_i, x\}$ generate the group T, and the matrices $\{R_{y_i}, R_x\}$ generate an isomorphic group of matrices.

It is also easy to write down matrices which represent the actions of C_n, D_n, and O explicitly, but I is fairly complicated.

An n-dimensional *matrix representation* of a group G is a homomorphism

$$(1.3)\qquad R: G \longrightarrow GL_n(F),$$

where F is a field. We will use the notation R_g for the image of g. So each R_g is an invertible matrix, and multiplication in G carries over to matrix multiplication; that is, $R_{gh} = R_g R_h$. The matrices (1.2) describe a three-dimensional matrix representation of T. It happens to be *faithful,* meaning that R is an injection and therefore maps T isomorphically to its image, a subgroup of $GL_3(\mathbb{R})$. Matrix representations are not required to be faithful.

When we study representations, it is essential to work as much as possible without fixing a basis, and to facilitate this, we introduce the concept of a representation of a group on a finite-dimensional vector space V. We denote by

$$(1.4)\qquad GL(V)$$

the group of invertible linear operators on V, the multiplication law being, as always, composition of functions. The choice of a basis of V defines an isomorphism of this group with the group of invertible matrices:

$$(1.5)\qquad \begin{array}{ccc} GL(V) & \longrightarrow & GL_n(F) \\ T & \rightsquigarrow & \text{matrix of } T. \end{array}$$

By a *representation of G on V*, we mean a homomorphism

$$(1.6)\qquad \rho: G \longrightarrow GL(V).$$

The *dimension* of the representation ρ is defined to be the dimension of the vector space V. We will study only representations on *finite-dimensional* vector spaces.

Matrix representations can be thought of as representations of G on the space F^n of column vectors.

Let ρ be a representation. We will denote the image of an element g in $GL(V)$ by ρ_g. Thus ρ_g is a linear operator on V, and $\rho_{gh} = \rho_g \rho_h$. If a basis $\mathbf{B} = (v_1, \dots, v_n)$

is given, the representation ρ defines a matrix representation R by the rule

(1.7) $$R_g = \text{matrix of } \rho_g.$$

We may write this matrix symbolically, as in Chapter 4 (3.1), as

(1.8) $$\rho_g(\mathbf{B}) = \mathbf{B}R_g.$$

If X is the coordinate vector of a vector $v \in V$, that is, if $v = \mathbf{B}X$, then

(1.9) $$R_g X \text{ is the coordinate vector of } \rho_g(v).$$

The rotation groups are examples of representations on a real vector space V without regard to a choice of basis. The rotations are linear operators in $GL(V)$. In (1.1) we chose a basis for V, thereby realizing the elements of T as the matrices (1.2) and obtaining a matrix representation.

So all representations of G on finite-dimensional vector spaces can be reduced to matrix representations if we are willing to choose a basis. We may need to choose one in order to make explicit calculations, but then we must study what happens when we change our basis, which properties are independent of the choice of basis, and which choices are the good ones.

A change of basis in V given by a matrix P changes a matrix representation R to a *conjugate representation* $R' = PRP^{-1}$, that is,

(1.10) $$R_g' = PR_g P^{-1} \quad \text{for every } g.$$

This follows from rule (3.4) in Chapter 4 for change of basis.

There is an equivalent concept, namely that of *operation* of a group G on a vector space V. When we speak of an operation on a vector space, we always mean one which is compatible with the vector space structure—otherwise we shouldn't be thinking of V as a vector space. So such an operation is a group operation in the usual sense [Chapter 5 (5.1)]:

(1.11) $$1v = v \quad \text{and} \quad (gh)v = g(hv),$$

for all $g, h \in G$ and all $v \in V$. In addition, every group element is required to act on V as a *linear operator*. Writing out what this means, we obtain the rules

(1.12) $$g(v + v') = gv + gv' \quad \text{and} \quad g(cv) = cgv$$

which, when added to (1.11), give a complete list of axioms for an operation of G on the vector space V. Since G does operate on the underlying set of V, we can speak of orbits and stabilizers as before.

The two concepts "operation of G on V" and "representation of G on V" are equivalent for the same reason that an operation of a group G on a set S is equivalent to a permutation representation (Chapter 5, Section 8): Given a representation ρ of G on V, we define an operation by the rule

(1.13) $$gv = \rho_g(v),$$

and conversely, given an operation, the same formula can be used to define the operator ρ_g for all $g \in G$. It is a linear operator because of (1.12), and the associative law (1.11) shows that $\rho_g \rho_h = \rho_{gh}$.

Thus we have two notations (1.13) for the action of g on v, and we will use them interchangeably. The notation gv is more compact, so we use it when possible.

In order to focus our attention, and because they are the easiest to handle, we will concentrate on *complex* representations for the rest of this chapter. Therefore the vector spaces V which occur are to be interpreted as complex vector spaces, and GL_n will denote the complex general linear group $GL_n(\mathbb{C})$. Every real matrix representation, such as the three-dimensional representation (1.2) of the rotation group T, can be used to define a complex representation, simply by interpreting the real matrices as complex matrices. We will do this without further comment.

2. G-INVARIANT FORMS AND UNITARY REPRESENTATIONS

A matrix representation $R: G \longrightarrow GL_n$ is called *unitary* if all the matrices R_g are unitary, that is, if the image of the homomorphism R is contained in the unitary group. In other words, a unitary representation is a homomorphism

(2.1) $$R: G \longrightarrow U_n$$

from G to the unitary group.

In this section we prove the following remarkable fact about representations of finite groups.

(2.2) **Theorem.**

(a) Every finite subgroup of GL_n is conjugate to a subgroup of U_n.

(b) Every matrix representation $R: G \longrightarrow GL_n$ of a finite group G is conjugate to a unitary representation. In other words, given R, there is a matrix $P \in GL_n$ such that $PR_gP^{-1} \in U_n$ for every $g \in G$.

(2.3) **Corollary.**

(a) Let A be an invertible matrix of finite order in GL_n, that is, such that $A^r = I$ for some r. Then A is diagonalizable: There is a $P \in GL_n$ so that PAP^{-1} is diagonal.

(b) Let $R: G \longrightarrow GL_n$ be a representation of a finite group G. Then for every $g \in G$, R_g is a diagonalizable matrix.

Proof of the corollary. (a) The matrix A generates a finite subgroup of GL_n. By Theorem (2.2), this subgroup is conjugate to a subgroup of the unitary group. Hence A is conjugate to a unitary matrix. The Spectral Theorem for normal operators [Chapter 7 (7.3)] tells us that every unitary matrix is diagonalizable. Hence A is diagonalizable.

(b) The second part of the corollary follows from the first, because every element g of a finite group has finite order. Since R is a homomorphism, R_g has finite order too. □

The two parts of Theorem (2.2) are more or less the same. We can derive (a) from (b) by considering the inclusion map of a finite subgroup into GL_n as a matrix representation of the group. Conversely, (b) follows by applying (a) to the image of R.

In order to prove part (b), we restate it in basis-free terminology. Consider a hermitian vector space V (a complex vector space together with a positive definite hermitian form $\langle\ ,\ \rangle$). A linear operator T on V is unitary if $\langle v, w\rangle = \langle T(v), T(w)\rangle$ for all $v, w \in V$ [Chapter 7 (5.2)]. Therefore it is natural to call a representation $\rho\colon G \longrightarrow GL(V)$ *unitary* if ρ_g is a unitary operator for all $g \in G$, that is, if

$$\langle v, w\rangle = \langle \rho_g(v), \rho_g(w)\rangle, \tag{2.4}$$

for all $v, w \in V$ and all $g \in G$. The matrix representation R (1.7) associated to a unitary representation ρ will be unitary in the sense of (2.1), provided that the basis is orthonormal. This follows from Chapter 7 (5.2b).

To simplify notation, we will write condition (2.4) as

$$\langle v, w\rangle = \langle gv, gw\rangle. \tag{2.5}$$

We now turn this formula around and view it as a condition on the form instead of on the operation. Given a representation ρ of G on a vector space V, a form $\langle\ ,\ \rangle$ on V is called *G-invariant* if (2.4), or equivalently, (2.5) holds.

(2.6) **Theorem.** Let ρ be a representation of a finite group G on a complex vector space V. There exists a G-invariant, positive definite hermitian form $\langle\ ,\ \rangle$ on V.

Proof. We start with an arbitrary positive definite hermitian form on V; say we denote it by $\{\ ,\ \}$. We will use this form to define a G-invariant form, by *averaging over the group*. Averaging over G is a general method which will be used again. It was already used in Chapter 5 (3.2) to find a fixed point of a finite group operation on the plane. The form $\langle\ ,\ \rangle$ we want is defined by the rule

$$\langle v, w\rangle = \frac{1}{N}\sum_{g\in G}\{gv, gw\}, \tag{2.7}$$

where $N = |G|$ is the order of G. The normalization factor $1/N$ is customary but unimportant. Theorem (2.6) follows from this lemma:

(2.8) **Lemma.** The form (2.7) is hermitian, positive definite, and G-invariant.

Proof. The verification of the first two properties is completely routine. For example,

$$\{gv, g(w + w')\} = \{gv, gw + gw'\} = \{gv, gw\} + \{gv, gw'\}.$$

Therefore

$$\langle v, w + w'\rangle = \frac{1}{N}\sum_{g\in G}\{gv, g(w + w')\} = \frac{1}{N}\sum_{g\in G}\{gv, gw\} + \frac{1}{N}\sum_{g\in G}\{gv, gw'\}$$
$$= \langle v, w\rangle + \langle v, w'\rangle.$$

To show that the form $\langle\, ,\rangle$ is G-invariant, let g_0 be an element of G. We must show that $\langle g_0v, g_0w\rangle = \langle v, w\rangle$ for all $v, w \in V$. By definition,

$$\langle g_0v, g_0w\rangle = \frac{1}{N}\sum_{g\in G}\{gg_0v, gg_0w\}.$$

There is an important trick for analyzing such a summation, based on the fact that right multiplication by g_0 is a bijective map $G \longrightarrow G$. As g runs over the group, the products gg_0 do too, in a different order. We change notation, substituting g' for gg_0. Then in the sum, g' runs over the group. So we may as well write the sum as being over $g' \in G$ rather than over $g \in G$. This merely changes the order in which the sum is taken. Then

$$\langle g_0v, g_0w\rangle = \frac{1}{N}\sum_{g\in G}\{gg_0v, gg_0w\} = \frac{1}{N}\sum_{g'\in G}\{g'v, g'w\} = \langle v, w\rangle,$$

as required. Please think this reindexing trick through and understand it. □

Theorem (2.2) follows easily from Theorem (2.6). Any homomorphism $R: G \longrightarrow GL_n$ is the matrix representation associated to a representation (with $V = \mathbb{C}^n$ and $\mathbf{B} = \mathbf{E}$). By Theorem (2.6), there is a G-invariant form $\langle\, ,\rangle$ on V, and we choose an orthonormal basis for V with respect to this form. The matrix representation R' obtained via this basis is conjugate to R (1.10) and unitary [Chapter 7 (5.2)]. □

(2.9) **Example.** The matrix $A = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$ has order 3, and therefore it defines a matrix representation $\{I, A, A^2\}$ of the cyclic group G of order 3. The averaging process (2.7) will produce a G-invariant form from the standard hermitian product X^*Y on $\mathbb{C}^2$. It is

(2.10) $$\langle X, Y\rangle = \frac{1}{3}[X^*Y + (AX)^*(AY) + (A^2X)^*(A^2Y)] = X^*BX,$$

where

(2.11) $$B = \frac{1}{3}[I + A^*A + (A^2)^*(A^2)] = \frac{2}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

3. COMPACT GROUPS

A linear group is called *compact* if it is a closed and bounded subset of the space of matrices [Appendix (3.8)]. The most important compact groups are the orthogonal and unitary groups:

(3.1) **Proposition.** The orthogonal and unitary groups are compact.

Proof. The columns of an orthogonal matrix P form an orthonormal basis, so they have length 1. Hence all of the matrix entries have absolute value ≤ 1. This shows that O_n is contained in the box defined by the inequalities $|p_{ij}| \leq 1$. So it is a bounded set. Because it is defined as the common zeros of a set of continuous functions, it is closed too, hence compact. The proof for the unitary group is the same. □

The main theorems (2.2, 2.6) of Section 2 carry over to compact linear groups without major change. We will work out the case of the circle group $G = SO_2$ as an example. The rotation of the plane through the angle θ was denoted by ρ_θ in Chapter 5. Here we will consider an arbitrary representation of G. To avoid confusion, we denote the element

(3.2)
$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \in SO_2$$

by its angle θ, rather than by ρ_θ. Formula (3.2) defines a particular matrix representation of our group, but there are others.

Suppose we are given a *continuous* representation σ of G on a finite-dimensional space V, not necessarily the representation (3.2). Since the group law is addition of angles, the rule for working with σ is $\sigma_{\theta+\eta} = \sigma_\theta\sigma_\eta$. To say that the operation is continuous means that if we choose a basis for V, thereby representing the operation of θ on V by some matrix S_θ, then the entries of S are continuous functions of θ.

Let us try to copy the proof of (2.6). To average over the infinite group G, we replace summation by an integral. We choose any positive definite hermitian form $\{,\}$ on V and define a new form by the rule

(3.3)
$$\langle v, w\rangle = \frac{1}{2\pi}\int_0^{2\pi} \{\sigma_\theta v, \sigma_\theta w\}\, d\theta.$$

This form has the required properties. To check G-invariance, fix any element $\theta_0 \in G$, and let $\eta = \theta + \theta_0$. Then $d\eta = d\theta$. Hence

(3.4)
$$\begin{aligned}\langle \sigma_{\theta_0} v, \sigma_{\theta_0} w\rangle &= \frac{1}{2\pi}\int_0^{2\pi} \{\sigma_\theta\sigma_{\theta_0} v, \sigma_\theta\sigma_{\theta_0} w\}\, d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \{\sigma_\eta v, \sigma_\eta w\}\, d\eta = \langle v, w\rangle,\end{aligned}$$

as required.

We will not carry the proof through for general groups because some serious work has to be done to find a suitable volume element analogous to $d\theta$ in a given compact group G. In the computation (3.4), it is crucial that $d\theta = d(\theta + \theta_0)$, and we were lucky that the obvious integral was the one to use.

For any compact group G there is a volume element dg called *Haar measure,* which has the property of being translation invariant: If $g_0 \in G$ is fixed and

$g' = gg_0$, then

(3.5) $$dg = dg'.$$

Using this measure, the proof carries over. We will not prove the existence of a Haar measure, but assuming one exists, the same reasoning as in (2.8) proves the following analogue of (2.6) and (2.2):

(3.6) **Corollary.** Let G be a compact subgroup of GL_n. Then

(a) Let σ be a representation of G on a finite-dimensional vector space V. There is a G-invariant, positive definite hermitian form $\langle\, ,\rangle$ on V.

(b) Every continuous matrix representation R of G is conjugate to a unitary representation.

(c) Every compact subgroup G of GL_n is conjugate to a subgroup of U_n. □

4. G-INVARIANT SUBSPACES AND IRREDUCIBLE REPRESENTATIONS

Given a representation of a finite group G on a vector space V, Corollary (2.3) tells us that for each group element g there is a basis of V so that the matrix of the operator ρ_g is diagonal. Obviously, it would be very convenient to have a single basis which would diagonalize ρ_g for all group elements g at the same time. But such a basis doesn't exist very often, because any two diagonal matrices commute with each other. In order to diagonalize the matrices of all ρ_g at the same time, these operators must commute. It follows that any group G which has a faithful representation by diagonal matrices is abelian. We will see later (Section 8) that the converse is also true. If G is a finite abelian group, then every matrix representation R of G is diagonalizable; that is, there is a single matrix P so that PR_gP^{-1} is diagonal for all $g \in G$. In this section we discuss what can be done for finite groups in general.

Let ρ be a representation of a group G on a vector space V. A subspace of V is called *G-invariant* if

(4.1) $$gw \in W, \quad \textit{for all } w \in W \textit{ and } g \in G.$$

So the operation by every group element g must carry W to itself, that is, $gW \subset W$. This is an extension of the concept of T-invariant subspace introduced in Section 3 of Chapter 4. In a representation, the elements of G represent linear operators on V, and we ask that W be an invariant subspace for each of these operators. If W is G-invariant, the operation of G on V will restrict to an operation on W.

As an example, consider the three-dimensional representation of the dihedral group defined by the symmetries of an n-gon Δ [Chapter 5 (9.1)]. So $G = D_n$. There are two proper G-invariant subspaces: The plane containing Δ and the line perpendicular to Δ. On the other hand, there is no proper T-invariant subspace for the representation (1.2) of the tetrahedral group T, because there is no line or plane which is carried to itself by *every* element of T.

If a representation ρ of a group G on a nonzero vector space V has no proper G-invariant subspace, it is called an *irreducible* representation. If there is a proper invariant subspace, ρ is said to be *reducible*. The standard three-dimensional representation of T is irreducible.

When V is the direct sum of G-invariant subspaces: $V = W_1 \oplus W_2$, the representation ρ on V is said to be the *direct sum* of its restrictions ρ_i to W_i, and we write

(4.2)
$$\rho = \rho_1 \oplus \rho_2.$$

Suppose this is the case. Choose bases $\mathbf{B}_1, \mathbf{B}_2$ of W_1, W_2, and let $\mathbf{B} = (\mathbf{B}_1, \mathbf{B}_2)$ be the basis of V obtained by listing these two bases in order [Chapter 3 (6.6)]. Then the matrix R_g of ρ_g will have the block form

(4.3)
$$R_g = \left[\begin{array}{c|c} A_g & 0 \\ \hline 0 & B_g \end{array}\right],$$

where A_g is the matrix of ρ_{1g} with respect to $\mathbf{B}_1$ and B_g is the matrix of ρ_{2g} with respect to $\mathbf{B}_2$. Conversely, if the matrices R_g have such a block form, then the representation is a direct sum.

For example, consider the rotation group $G = D_n$ operating on $\mathbb{R}^3$ by symmetries of an n-gon Δ. If we choose an orthonormal basis $\mathbf{B}$ so that v_1 is perpendicular to the plane of Δ and v_2 passes through a vertex, then the rotations corresponding to our standard generators x, y [Chapter 5 (3.6)] are represented by the matrices

(4.4)
$$R_x = \begin{bmatrix} 1 & & \\ & c_n & -s_n \\ & s_n & c_n \end{bmatrix}, \quad R_y = \begin{bmatrix} -1 & & \\ & 1 & \\ & & -1 \end{bmatrix},$$

where $c_n = \cos(2\pi/n)$ and $s_n = \sin(2\pi/n)$. So R is a direct sum of a one-dimensional representation A,

(4.5)
$$A_x = [1], \quad A_y = [-1],$$

and a two-dimensional representation B,

(4.6)
$$B_x = \begin{bmatrix} c_n & -s_n \\ s_n & c_n \end{bmatrix}, \quad B_y = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}.$$

The representation B is the basic two-dimensional representation of D_n as symmetries of Δ in the plane.

On the other hand, even if a representation ρ is reducible, the matrices R_g will not have a block form unless the given basis for V is compatible with the direct sum decomposition. Until we have made a further analysis, it will be difficult to tell that a representation is reducible, when it is presented using the wrong basis.

(4.7) **Proposition.** Let ρ be a unitary representation of G on a hermitian vector space V, and let W be a G-invariant subspace. The orthogonal complement $W^\perp$ is also G-invariant, and ρ is a direct sum of its restrictions to W and $W^\perp$.

Proof. Let $v \in W^{\perp}$, so that $v \perp W$. Since the operators ρ_g are unitary, they preserve orthogonality [Chapter 7 (5.2)], so $gv \perp gW$. Since W is G-invariant, $W = gW$, so $gv \perp W$. Therefore $gv \in W^{\perp}$. This shows that $W^{\perp}$ is G-invariant. We know that $V = W \oplus W^{\perp}$ by Chapter 7 (2.7). □

This proposition allows us to decompose a representation as a direct sum, provided that there is a proper invariant subspace. Together with induction, this gives us the following corollary:

(4.8) **Corollary.** Every unitary representation $\rho: G \longrightarrow GL(V)$ on a hermitian vector space V is a direct sum of irreducible representations. □

Combining this corollary with (2.2), we obtain the following:

(4.9) **Corollary.** *Maschke's Theorem:* Every representation of a finite group G is a direct sum of irreducible representations. □

5. CHARACTERS

Two representations $\rho: G \longrightarrow GL(V)$ and $\rho': G \longrightarrow GL(V')$ of a group G are called *isomorphic*, or *equivalent*, if there is an isomorphism of vector spaces $T: V \longrightarrow V'$ which is compatible with the operation of G:

(5.1) $$gT(v) = T(gv) \quad \text{or} \quad \rho_g' T(v) = T(\rho_g(v)),$$

for all $v \in V$ and $g \in G$. If $\mathbf{B}$ is a basis for V and if $\mathbf{B}' = T(\mathbf{B})$ is the corresponding basis of V', then the associated matrix representations R_g and R_g' will be *equal*.

For the next four sections, we restrict our attention to representations of finite groups. We will see that there are relatively few isomorphism classes of irreducible representations of a finite group. However, each representation has a complicated description in terms of matrices. The secret to understanding representations is not to write down the matrices explicitly unless absolutely necessary. So to facilitate classification we will throw out most of the information contained in a representation ρ, keeping only an essential part. What we will work with is the trace, called the *character*, of ρ. Characters are usually denoted by χ.

The *character* χ of a representation ρ is the map $\chi: G \longrightarrow \mathbb{C}$ defined by

(5.2) $$\chi(g) = \operatorname{trace}(\rho_g).$$

If R is the matrix representation obtained from ρ by a choice of basis for V, then

(5.3) $$\chi(g) = \operatorname{trace}(R_g) = \lambda_1 + \cdots + \lambda_n,$$

where λ_i are the eigenvalues of R_g, or of ρ_g.

The *dimension* of a character χ is defined to be the dimension of the representation ρ. The character of an irreducible representation is called an *irreducible character*.

Here are some basic properties of the character:

(5.4) **Proposition.** Let χ be the character of a representation ρ of a finite group G on a vector space V.

(a) $\chi(1)$ is the dimension of the character [the dimension of V].
(b) $\chi(g) = \chi(hgh^{-1})$ for all $g, h \in G$. In other words, the character is constant on each conjugacy class.
(c) $\chi(g^{-1}) = \overline{\chi(g)}$ [the complex conjugate of $\chi(g)$].
(d) If χ' is the character of another representation ρ', then the character of the direct sum $\rho \oplus \rho'$ is $\chi + \chi'$.

Proof. The symbol 1 in assertion (a) denotes the identity element of G. This property is trivial: $\chi(1) = \text{trace } I = \dim V$. Property (b) is true because the matrix representation R associated to ρ is a homomorphism, which shows that $R_{hgh^{-1}} = R_h R_g R_h^{-1}$, and because trace $(R_h R_g R_h^{-1}) =$ trace R_g [Chapter 4 (4.18)]. Property (d) is also clear, because the trace of the block matrix (4.3) is the sum of the traces of A_g and B_g.

Property (c) is less obvious. If the eigenvalues of R_g are $\lambda_1, \dots, \lambda_n$, then the eigenvalues of $R_{g^{-1}} = (R_g)^{-1}$ are $\lambda_1^{-1}, \dots, \lambda_n^{-1}$. The assertion of (c) is

$$\chi(g^{-1}) = \lambda_1^{-1} + \cdots + \lambda_n^{-1} = \overline{\lambda}_1 + \cdots + \overline{\lambda}_n = \overline{\chi(g)},$$

and to show this we use the fact that G is a finite group. Every element g of G has finite order. If $g^r = 1$, then R_g is a matrix of order r, so its eigenvalues $\lambda_1, \dots, \lambda_n$ are roots of unity. This implies that $|\lambda_i| = 1$, hence that $\lambda_i^{-1} = \overline{\lambda}_i$ for each i. □

In order to avoid confusing cyclic groups with conjugacy classes, we will denote conjugacy classes by the roman letter C, rather than an italic C, in this chapter. Thus the conjugacy class of an element $g \in G$ will be denoted by C_g.

We shall note two things which simplify the computation of a character. First of all, since the value of χ depends only on the conjugacy class of an element $g \in G$ (5.4b), we need only determine the values of χ on one representative element in each class. Second, since the value of the character $\chi(g)$ is the trace of the operator ρ_g and since the trace doesn't depend on the choice of a basis, we are free to choose a convenient one. Moreover, we may select a convenient basis for each individual group element. There is no need to use the same basis for all elements.

As an example, let us determine the character χ of the rotation representation of the tetrahedral group T defined by (1.2). There are four conjugacy classes in T, and they are represented by the elements $1, x, x^2, y$, where as before x is a rotation by $2\pi/3$ about a vertex and y is a rotation by π about the center of an edge. The values of the character on these representatives can be read off from the matrices (1.2):

$$\chi(1) = 3, \quad \chi(x) = 0, \quad \chi(x^2) = 0, \quad \chi(y) = -1. \tag{5.5}$$

It is sometimes useful to think of a character χ as a vector. We can do this by listing the elements of G in some order: $G = \{g_1, \ldots, g_N\}$; then the vector representing χ will be

$$\chi = (\chi(g_1), \ldots, \chi(g_N))^t. \tag{5.6}$$

Since χ is constant on conjugacy classes, it is natural to list G by listing the conjugacy classes and then running through each conjugacy class in some order. If we do this for the character (5.5), listing C_1, C_x, C_{x^2}, C_y in that order, the vector we obtain is

$$\chi = (3; 0, 0, 0, 0; 0, 0, 0, 0; -1, -1, -1)^t. \tag{5.7}$$

We will not write out such a vector explicitly again.

The main theorem on characters relates them to the hermitian dot product on $\mathbb{C}^N$. This is one of the most beautiful theorems of algebra, both because its statement is intrinsically so elegant and because it simplifies the problem of classifying representations so much. We define

$$\langle \chi, \chi' \rangle = \frac{1}{N} \sum_g \overline{\chi(g)} \chi'(g), \tag{5.8}$$

where $N = |G|$. If χ, χ' are represented by vectors as in (5.7), this is the standard hermitian product, renormalized by the factor $1/N$.

(5.9) **Theorem.** Let G be a group of order N, let $\rho_1, \rho_2, \ldots$ represent the distinct isomorphism classes of irreducible representations of G, and let χ_i be the character of ρ_i.

(a) *Orthogonality Relations:* The characters χ_i are orthonormal. In other words, $\langle \chi_i, \chi_j \rangle = 0$ if $i \neq j$, and $\langle \chi_i, \chi_i \rangle = 1$ for each i.

(b) There are finitely many isomorphism classes of irreducible representations, the same number as the number of conjugacy classes in the group.

(c) Let d_i be the dimension of the irreducible representation ρ_i, and let r be the number of irreducible representations. Then d_i divides N, and

$$N = d_1^2 + \cdots + d_r^2. \tag{5.10}$$

This theorem will be proved in Section 9, with the exception of the assertion that d_i divides N, which we will not prove.

A complex-valued function $\varphi: G \longrightarrow \mathbb{C}$ which is constant on each conjugacy class is called a *class function*. Since a class function is constant on each class, it may also be described as a function on the set of conjugacy classes. The class functions form a complex vector space, which we denote by $\mathscr{C}$. We use the form defined by (5.8) to make $\mathscr{C}$ into a hermitian space.

(5.11) **Corollary.** The irreducible characters form on orthonormal basis of $\mathscr{C}$.

This follows from (5.9a,b). The characters are linearly independent because they are orthogonal, and they span because the dimension of $\mathscr{C}$ is the number of conjugacy classes, which is r. □

The corollary allows us to decompose a given character as a linear combination of the irreducible characters, using the formula for orthogonal projection [Chapter 7 (3.8)]. For let χ be the character of a representation ρ. By Corollary (4.9), ρ is isomorphic to a direct sum of the irreducible representations $\rho_1,\ldots,\rho_r$; say we write this symbolically as $\rho = n_1\rho_1\oplus\cdots\oplus n_r\rho_r$, where n_i are nonnegative integers and where $n\rho$ stands for the direct sum of n copies of the representation ρ. Then $\chi = n_1\chi_1+\cdots+n_r\chi_r$. Since $(\chi_1,\ldots,\chi_r)$ is an orthonormal basis, we have the following:

(5.12) **Corollary.** Let $\chi_1,\ldots,\chi_r$ be the irreducible characters of a finite group G, and let χ be any character. Then $\chi = n_1\chi_1+\cdots+n_r\chi_r$, where $n_i = \langle\chi,\chi_i\rangle$. □

(5.13) **Corollary.** If two representations ρ,ρ' have the same character, they are isomorphic.

For let χ,χ' be the characters of two representations ρ,ρ', where $\rho = n_1\rho_1\oplus\cdots\oplus n_r\rho_r$ and $\rho' = n_1'\rho_1\oplus\cdots\oplus n_r'\rho_r$. Then the characters of these representations are $\chi = n_1\chi_1+\cdots+n_r\chi_r$ and $\chi' = n_1'\chi_1+\cdots+n_r'\chi_r$. Since $\chi_1,\ldots,\chi_r$ are linearly independent, $\chi = \chi'$ implies that $n_i = n_i'$ for each i. □

(5.14) **Corollary.** A character χ has the property $\langle\chi,\chi\rangle = 1$ if and only if it is irreducible.

For if $\chi = n_1\chi_1+\cdots+n_r\chi_r$, then $\langle\chi,\chi\rangle = n_1^2+\cdots+n_r^2$. This gives the value 1 if and only if a single n_i is 1 and the rest are zero. □

The evaluation of $\langle\chi,\chi\rangle$ is a very practical way to check irreducibility of a representation. For example, let χ be the character (5.7) of the representation (1.2). Then $\langle\chi,\chi\rangle = (3^2+1+1+1)/12 = 1$. So χ is irreducible.

Part (c) of Theorem (5.9) should be contrasted with the Class Equation [Chapter 6 (1.7)]. Let $C_1,\ldots,C_r$ be the conjugacy classes in G, and let $c_i = |C_i|$ be the order of the conjugacy class. Then c_i divides N, and $N = c_1 + \cdots + c_r$. Though there is the same number of conjugacy classes as irreducible representations, their exact relationship is very subtle.

As our first example, we will determine the irreducible representations of the dihedral group D_3 [Chapter 5 (3.6)]. There are three conjugacy classes, $C_1 = \{1\}$, $C_2 = \{y, xy, x^2y\}$, $C_3 = \{x, x^2\}$ [Chapter 6 (1.8)], and therefore three irreducible representations. The only solution of equation (5.10) is $6 = 1^2 + 1^2 + 2^2$, so D_3 has two one-dimensional representations ρ_1,ρ_2 and one irreducible two-dimensional representation ρ_3. Every group G has the *trivial* one-dimensional representation

($R_g = 1$ for all g); let us call it ρ_1. The other one-dimensional representation is the *sign representation* of the symmetric group S_3, which is isomorphic to D_3: $R_g = \text{sign}(g) = \pm 1$. This is the representation (4.5); let us call it ρ_2. The two-dimensional representation is defined by (4.6); call it ρ_3.

Rather than listing the characters χ_i as vectors, we usually assemble them into a *character table*. In this table, the three conjugacy classes are represented by the elements $1, y, x$. The orders of the conjugacy classes are given above them. Thus $|C_y| = 3$.

		conjugacy class			
		(1)	(3)	(2)	order of the class
		1	y	x	representative element
irreducible character	χ_1	1	1	1	value of the character
	χ_2	1	-1	1	
	χ_3	2	0	-1	

(5.15) CHARACTER TABLE FOR D_3

In such a table, the top row, corresponding to the trivial character, consists entirely of 1's. The first column contains the dimensions of the representations, because $\chi_i(1) = \dim \rho_i$.

To evaluate the bilinear form (5.8) on the characters, remember that there are three elements in the class of y and two in the class of x. Thus

$$\langle \chi_3, \chi_3 \rangle = \frac{1}{N}\sum_g \overline{\chi_3(g)}\chi_3(g) = (1\cdot(\overline{\chi_3(1)}\chi_3(1)) + 3\cdot(\overline{\chi_3(y)}\chi_3(y)) + 2\cdot(\overline{\chi_3(x)}\chi_3(x)))/6$$

$$= (1\cdot\overline{2}\cdot 2 + 3\cdot\overline{0}\cdot 0 + 2\cdot(\overline{-1})\cdot(-1))/6 = 1.$$

This confirms the fact that ρ_3 is irreducible. □

As another example, consider the cyclic group $C_3 = \{1, x, x^2\}$ of order 3. Since C_3 is abelian, there are three conjugacy classes, each consisting of one element. Theorem (5.9) shows that there are three irreducible representations, and that each has dimension 1. Let $\zeta = \frac{1}{2}(-1 + \sqrt{3}i)$ be a cube root of 1. The three representations are

(5.16) $$\rho_{1_x} = 1, \quad \rho_{2_x} = \zeta, \quad \rho_{3_x} = \zeta^2.$$

	1	x	x^2
χ_1	1	1	1
χ_2	1	ζ	ζ^2
χ_3	1	ζ^2	ζ

(5.17) CHARACTER TABLE FOR C_3

Note that $\overline{\zeta} = \zeta^2$. So

$$\langle \chi_2, \chi_3 \rangle = (\overline{1} \cdot 1 + \overline{\zeta}\zeta^2 + \overline{\zeta}^2\zeta)/3 = (1 + \zeta + \zeta^2)/3 = 0,$$

which agrees with the orthogonality relations.

As a third example, let us determine the character table of the tetrahedral group T. The conjugacy classes C_1, C_x, C_{x^2}, C_y were determined above, and the Class Equation is $12 = 1+4+4+3$. The only solution of (5.10) is $12 = 1^2+1^2+1^2+3^2$, so there are four irreducible representations, of dimensions 1, 1, 1, 3. Now it happens that T has a normal subgroup H of order 4 which is isomorphic to the Klein four group, and such that the quotient $\overline{T} = T/H$ is cyclic of order 3. Any representation $\overline{\rho}$ of $\overline{T}$ will give a representation of T by composition:

$$T \xrightarrow{\pi} \overline{T} \xrightarrow{\overline{\rho}} GL(V).$$

Thus the three one-dimensional representations of the cyclic group determine representations of T. Their characters χ_1, χ_2, χ_3 can be determined from (5.17). The character (5.5) is denoted by χ_4 in the table below.

	(1) 1	(4) x	(4) x^2	(3) y
χ_1	1	1	1	1
χ_2	1	ζ	ζ^2	1
χ_3	1	ζ^2	ζ	1
χ_4	3	0	0	-1

(5.18) CHARACTER TABLE FOR T

Various properties of the group can be read off easily from the character table. Let us forget that this is the character table for T, and suppose that it has been given to us as the character table of an unknown group G. After all, it is conceivable that another isomorphism class of groups has the same characters.

The order of G is 12, the sum of the orders of the conjugacy classes. Next, since the dimension of ρ_2 is 1, $\chi_2(y)$ is the trace of the 1×1 matrix ρ_{2_y}. So the fact that $\chi_2(y) = 1$ shows that $\rho_{2_y} = 1$ too, that is, that y is in the kernel of ρ_2. In fact, the kernel of ρ_2 is identified as the union of the two conjugacy classes $C_1 \cup C_y$. This is a subgroup H of order 4 in G. Moreover, H is the Klein four group. For if H were C_4, its unique element of order 2 would have to be in a conjugacy class by itself. It also follows from the value of $\chi_2(x)$ that the order of x is divisible by 3. Going back to our list [Chapter 6 (5.1)] of groups of order 12, we see that $G \approx T$.

6. PERMUTATION REPRESENTATIONS AND THE REGULAR REPRESENTATION

Let S be a set. We can construct a representation of a group G from an operation of G on S, by passing to the vector space $V = V(S)$ of formal linear combinations [Chapter 3 (3.21)]

$$v = \sum_i a_i s_i, \quad a_i \in \mathbb{C}.$$

An element $g \in G$ operates on vectors by permuting the elements of S, leaving the coefficients alone:

$$gv = \sum_i a_i g s_i. \tag{6.1}$$

If we choose an ordering $s_1, \ldots, s_n$ of S and take the basis $(s_1, \ldots, s_n)$ for V, then R_g is the permutation matrix which describes the operation of g on S.

For example, let $G = T$ and let S be the set of faces of the tetrahedron: $S = (f_1, \ldots, f_4)$. The operation of G on S defines a four-dimensional representation of G. Let x denote the rotation by $2\pi/3$ about a face f_1 and y the rotation by π about an edge as before. Then if the faces are numbered appropriately, we will have

$$R_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad R_y = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \tag{6.2}$$

We will call ρ (or R) the representation *associated* to the operation of G on S and will often refer to ρ as a *permutation representation*, though that expression has a meaning in another context as well (Chapter 5, Section 8).

If we decompose a set on which G operates into orbits, we will obtain a decomposition of the associated representation as a direct sum. This is clear. But there is an important new feature: The fact that linear combinations are available in $V(S)$ allows us to decompose the representation further. Even though S may consist of a single orbit, the associated permutation representation ρ will *never* be irreducible, unless S has only one element. This is because the vector $w = s_1 + \cdots + s_r$ is fixed by every permutation of the basis, and so the one-dimensional subspace $W = \{cw\}$ is G-invariant. The trivial representation is a summand of every permutation representation.

It is easy to compute the character of a permutation representation:

$$\chi(g) = \text{number of elements of } S \text{ fixed by } g, \tag{6.3}$$

because for every index fixed by a permutation, there is a 1 on the diagonal of the associated permutation matrix, and the other diagonal entries are 0. For example, the character χ of the representation of T on the faces of a tetrahedron is

(6.4)

	1	x	x^2	y
χ	4	1	1	0

and the character table (5.18) shows that $\chi = \chi_1 + \chi_4$. Therefore $\rho \approx \rho_1 \oplus \rho_4$ by Corollary (5.13). As another example, the character of the operation of T on the six edges of the tetrahedron is

(6.5)

	1	x	x^2	y
χ	6	0	0	2

and using (5.18) again, we find that $\chi = \chi_1 + \chi_2 + \chi_3 + \chi_4$.

The *regular representation* ρ^{reg} of G is the representation associated to the operation of G on itself by left multiplication. In other words, we let $S = G$, with the operation of left multiplication. This is not an especially interesting operation, but its associated representation is very interesting. Its character χ^{reg} is particularly simple:

$$(6.6) \qquad \chi^{\text{reg}}(1) = N, \quad \text{and} \quad \chi^{\text{reg}}(g) = 0, \quad \text{if } g \neq 1,$$

where $N = G$. The first formula is clear: $\chi(1) = \dim \rho$ for any representation ρ, and ρ^{reg} has dimension N. The second follows from (6.3), because multiplication by g does not fix any element of G, unless $g = 1$.

Because of this formula, it is easy to compute $\langle \chi^{\text{reg}}, \chi \rangle$ for the character χ of any representation ρ by the orthogonal projection formula (5.12). The answer is

$$(6.7) \qquad \langle \chi^{\text{reg}}, \chi \rangle = \dim \rho,$$

because $\chi(1) = \dim \rho$. This allows us to write χ^{reg} as a linear combination of the irreducible characters:

(6.8) **Corollary.** $\chi^{\text{reg}} = d_1\chi_1 + \cdots + d_r\chi_r$, and $\rho^{\text{reg}} \approx d_1\rho_1 \oplus \cdots \oplus d_r\rho_r$, where d_i is the dimension of ρ_i and $d_i\rho_i$ stands for the direct sum of d_i copies of ρ_i. □

Isn't this a nice formula? We can deduce formula (5.10) from (6.8) by counting dimensions. This shows that formula (5.10) of Theorem (5.9) follows from the orthogonality relations.

For instance, for the group D_3, the character of the regular representation is

$$\begin{array}{c|ccc} & 1 & x & y \\ \hline \chi^{\text{reg}} & 6 & 0 & 0 \end{array},$$

and Table (5.15) shows that $\chi^{\text{reg}} = \chi_1 + \chi_2 + 2\chi_3$, as expected.

As another example, consider the regular representation R of the cyclic group $\{1, x, x^2\}$ of order 3. The permutation matrix representing x is

$$R_x = \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}.$$

Its eigenvalues are $1, \zeta, \zeta^2$, where $\zeta = \frac{1}{2}(-1 + \sqrt{3}i)$. Thus R_x is conjugate to

$$R'_x = \begin{bmatrix} 1 & & \\ & \zeta & \\ & & \zeta^2 \end{bmatrix}.$$

This matrix displays the decomposition $\rho^{\text{reg}} \approx \rho_1 \oplus \rho_2 \oplus \rho_3$ of the regular representation into irreducible one-dimensional representations.

7. *THE REPRESENTATIONS OF THE ICOSAHEDRAL GROUP*

In this section we determine the irreducible characters of the icosahedral group. So far, we have seen only its trivial representation ρ_1 and the representation of dimension 3 as a rotation group. Let us denote the rotation representation by ρ_2. There are five conjugacy classes in I [Chapter 6 (2.2)], namely

(7.1)
$$\begin{aligned} C_1 &= \{1\}, \\ C_2 &= 15 \text{ rotations “}x\text{” through the angle } \pi, \\ C_3 &= 20 \text{ rotations “}y\text{” by } 2\pi/3,\ 4\pi/3, \\ C_4 &= 12 \text{ rotations “}z\text{” by } 2\pi/5,\ 8\pi/5, \\ C_5 &= 12 \text{ rotations “}z^2\text{” by } 4\pi/5,\ 6\pi/5, \end{aligned}$$

and therefore there are three more irreducible representations. Given what we know already, the only solution to (5.10) is $d_i = 1, 3, 3, 4, 5$:

$$60 = 1^2+3^2+3^2+4^2+5^2.$$

We denote the remaining representations by ρ_3, ρ_4, ρ_5, where $\dim \rho_3 = 3$, and so on. A good way to find the missing irreducible representations is to decompose some known permutation representations. We know that I operates on a set of five elements [Chapter 6 (2.6)]. This gives us a five-dimensional representation ρ'. As we saw in Section 6, the trivial representation is a summand of ρ'. Its orthogonal complement turns out to be the required irreducible four-dimensional representation: $\rho' = \rho_1 \oplus \rho_4$. Also, I permutes the set of six axes through the centers of opposite faces of the dodecahedron. Let the corresponding six-dimensional representation be ρ''. Then $\rho'' = \rho_1 \oplus \rho_5$. We can check this by computing the characters of ρ_4 and ρ_5 and applying Theorem (5.9). The characters χ_4, χ_5 are computed from χ', χ'' by subtracting $\chi_1 = 1$ from each value (5.4d). For example, ρ' realizes x as an even permutation of $\{\mathbf{1}, \ldots, \mathbf{5}\}$ of order 2, so it is a product of two disjoint transpositions, which fixes one index. Therefore $\chi'(x) = 1$, and $\chi_4(x) = 0$.

The second three-dimensional representation ρ_3 is fairly subtle because it is so similar to ρ_2. It can be obtained this way: Since I is isomorphic to A_5, we may view it as a normal subgroup of the symmetric group S_5. Conjugation by an element p of S_5 which is not in A_5 defines an automorphism σ of A_5. This automorphism interchanges the two conjugacy classes C_4, C_5. The other conjugacy classes are not interchanged, because their elements have different orders. For example, in cycle notation, let $z = (\mathbf{12345})$ and let $p = (\mathbf{2354})$. Then $p^{-1}zp = (\mathbf{4532})(\mathbf{12345})(\mathbf{2354}) = (\mathbf{13524}) = z^2$. The representation ρ_3 is $\rho_2 \circ \sigma$.

The character of ρ_3 is computed from that of ρ_2 by interchanging the values for z, z^2. Once these characters are computed, verification of the relations $\langle \chi_i, \chi_j \rangle = 0$, $\langle \chi_i, \chi_i \rangle = 1$ shows that the representations are irreducible and that our list is correct.

	(1) 1	(15) x	(20) y	(12) z	(12) z^2
χ_1	1	1	1	1	1
χ_2	3	-1	0	α	β
χ_3	3	-1	0	β	α
χ_4	4	0	1	-1	-1
χ_5	5	1	-1	0	0

(7.2) CHARACTER TABLE FOR $I = A_5$

In this table, α is the trace of a three-dimensional rotation through the angle $2\pi/5$, which is

$$\alpha = 1 + 2\cos 2\pi/5 = \tfrac{1}{2}(-1 + \sqrt{5}),$$

and β is computed similarly: $\beta = 1 + 2\cos 4\pi/5 = \tfrac{1}{2}(-1 - \sqrt{5})$.

8. ONE-DIMENSIONAL REPRESENTATIONS

Let ρ be a one-dimensional representation of a group G. So R_g is a 1×1 matrix, and $\chi(g) = R_g$, provided that we identify a 1×1 matrix with its single entry. Therefore in this case the character χ is a homomorphism $\chi\colon G \longrightarrow \mathbb{C}^\times$, that is, it satisfies the rule

(8.1) $$\chi(gh) = \chi(g)\chi(h), \quad \text{if } \dim \rho = 1.$$

Such a character is called *abelian*. Please note that formula (8.1) is not true for characters of dimension >1.

If G is a finite group, the values taken on by an abelian character χ are always roots of 1:

(8.2) $$\chi(g)^r = 1$$

for some r, because the element g has finite order.

The one-dimensional characters form a group under multiplication of functions:

(8.3) $$\chi\chi'(g) = \chi(g)\chi'(g).$$

This group is called the *character group* of G and is often denoted by $\hat{G}$. The character group is especially important when G is abelian, because of the following fact:

(8.4) **Theorem.** If G is a finite abelian group, then every irreducible representation of G is one-dimensional.

Proof. Since G is abelian, every conjugacy class consists of one element. So the number of conjugacy classes is N. By Theorem (5.9), there are N irreducible representations, and $d_1 = d_2 = \cdots = d_r = 1$. □

9. SCHUR'S LEMMA, AND PROOF OF THE ORTHOGONALITY RELATIONS

Let ρ, ρ' be representations of a group G on two vector spaces V, V'. We will call a linear transformation $T\colon V \longrightarrow V'$ *G-invariant* if it is compatible with the two operations of G on V and V', that is, if

(9.1) $$gT(v) = T(gv), \quad \text{or} \quad \rho_g'(T(v)) = T(\rho_g(v)),$$

for all $g \in G$ and $v \in V$. Thus an isomorphism of representations (Section 5) is a bijective G-invariant transformation. We could also write (9.1) as

$$\rho_g' \circ T = T \circ \rho_g, \quad \text{for all } g \in G. \tag{9.2}$$

Let bases $\mathbf{B}, \mathbf{B}'$ for V and V' be given, and let R_g, R_g' and A denote the matrices of ρ_g, ρ_g' and T with respect to these bases. Then (9.2) reads

$$R_g'A = AR_g, \quad \text{for all } g \in G. \tag{9.3}$$

The special case that $\rho = \rho'$ is very important. A *G-invariant* linear operator T on V is one which commutes with ρ_g for every $g \in G$:

$$\rho_g \circ T = T \circ \rho_g \quad \text{or} \quad R_gA = AR_g. \tag{9.4}$$

These formulas just repeat (9.2) and (9.3) when $\rho = \rho'$.

(9.5) **Proposition.** *The kernel and image of a G-invariant linear transformation $T: V \rightarrow V'$ are G-invariant subspaces of V and V' respectively.*

Proof. The kernel and image of any linear transformation are subspaces. Let us show that $\ker T$ is G-invariant: We want to show that $gv \in \ker T$ if $v \in \ker T$, or that $T(gv) = 0$ if $T(v) = 0$. Well,

$$T(gv) = gT(v) = g0 = 0.$$

Similarly, if $v' \in \operatorname{im} T$, then $v' = T(v)$ for some $v \in V$. Then

$$gv' = gT(v) = T(gv),$$

so $gv' \in \operatorname{im} T$ too. □

(9.6) **Theorem.** *Schur's Lemma:* Let ρ, ρ' be two irreducible representations of G on vector spaces V, V', and let $T: V \longrightarrow V'$ be a G-invariant transformation.

(a) Either T is an isomorphism, or else $T = 0$.
(b) If $V = V'$ and $\rho = \rho'$, then T is multiplication by a scalar.

Proof. (a) Since ρ is irreducible and since $\ker T$ is a G-invariant subspace, $\ker T = V$ or else $\ker T = 0$. In the first case, $T = 0$. In the second case, T is injective and maps V isomorphically to its image. Then $\operatorname{im} T$ is not zero. Since ρ' is irreducible and $\operatorname{im} T$ is G-invariant, $\operatorname{im} T = V'$. Therefore T is an isomorphism.

(b) Suppose $V = V'$, so that T is a linear operator on V. Choose an eigenvalue λ of T. Then $(T - \lambda I) = T_1$ is also G-invariant. Its kernel is nonzero because it contains an eigenvector. Since ρ is irreducible, $\ker T_1 = V$, which implies that $T_1 = 0$. Therefore $T = \lambda I$. □

The averaging process can be used to create a G-invariant transformation from any linear transformation $T: V \longrightarrow V'$. To do this, we rewrite the condition (9.1) in the form $T(v) = \rho_g'^{-1}(T(\rho_g(v)))$, or

$$(9.7) \qquad T(v) = g^{-1}(T(gv)).$$

The average is the linear operator $\tilde{T}$ defined by

$$(9.8) \qquad \tilde{T}(v) = \frac{1}{N}\sum_g g^{-1}(T(gv)),$$

where $N = |G|$ as before. If bases for V, V' are given and if the matrices for $\rho_g, \rho_g', T, \tilde{T}$ are $R_g, R_g', A, \tilde{A}$ respectively, then

$$(9.9) \qquad \tilde{A} = \frac{1}{N}\sum_g R_g'^{-1}AR_g.$$

Since compositions of linear transformations and sums of linear transformations are again linear, $\tilde{T}$ is a linear transformation. To show that it is G-invariant, we fix an element $h \in G$ and let $g' = gh$. Reindexing as in the proof of Lemma (2.8),

$$h^{-1}\tilde{T}(hv) = \frac{1}{N}\sum_g h^{-1}g^{-1}(T(ghv)) = \frac{1}{N}\sum_{g'} g'^{-1}(T(g'v)) = \tilde{T}(v).$$

Therefore $\tilde{T}(hv) = h\tilde{T}(v)$. Since h is arbitrary, this shows that $\tilde{T}$ is G-invariant. □

It may happen that we end up with the trivial linear transformation, that is, $\tilde{T} = 0$ though T was not zero. In fact, Schur's Lemma tells us that we *must* get $\tilde{T} = 0$ if ρ and ρ' are irreducible but not isomorphic. We will make good use of this seemingly negative fact in the proof of the orthogonality relations.

When $\rho = \rho'$, the average can often be shown to be nonzero by using this proposition.

(9.10) **Proposition.** Let ρ be a representation of a finite group G on a vector space V, and let $T: V \longrightarrow V$ be a linear operator. Define $\tilde{T}$ by formula (9.8). Then trace $\tilde{T}$ = trace T. Thus if the trace of T isn't zero, then $\tilde{T}$ is not zero either.

Proof. We compute as in formula (9.9), with $R' = R$. Since trace A = trace $R_g^{-1}AR_g$, the proposition follows. □

Here is a sample calculation. Let $G = C_3 = \{1, x, x^2\}$, and let $\rho = \rho'$ be the regular representation (Section 6) of G, so that $V = \mathbb{C}^3$ and

$$R_x = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Let T be the linear operator whose matrix is

$$B = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then the matrix of $\tilde{T}$ is

$$\tilde{B} = \frac{1}{3}(IBI + R_x{}^{-1}BR_x + R_x{}^{-2}BR_x{}^2)$$

$$= \frac{1}{3}(B + R_x{}^2BR_x + R_xBR_x{}^2) = \frac{1}{3}\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{bmatrix}.$$

Or, let T be the linear operator whose matrix is the permutation matrix corresponding to the transposition $y = (\mathbf{1}\ \mathbf{2})$. The average over the group is a sum of the three transpositions: $(y + x^{-1}yx + x^{-2}yx)/3 = (y + xy + x^2y)/3$. In this case,

$$P = \frac{1}{3}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \tilde{P} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Note that $\tilde{B}$ and $\tilde{P}$ commute with R_x as claimed [see (9.4)], though the original matrices P and B do not.

We will now prove the orthogonality relations, Theorem (5.9a). We saw in Section 6 that formula (5.10) is a consequence of these relations.

Let χ, χ' be two nonisomorphic irreducible characters, corresponding to representations ρ, ρ' of G on V, V'. Using the rule $\chi'(g^{-1}) = \overline{\chi'(g)}$, we can rewrite the orthogonality $\langle \chi', \chi \rangle = 0$ to be proved as

$$(9.11) \qquad \frac{1}{N}\sum_g \chi'(g^{-1})\chi(g) = 0.$$

Now Schur's Lemma asserts that every G-invariant linear transformation $V \longrightarrow V'$ is zero. In particular, the linear transformation $\tilde{T}$ which we obtain by averaging any linear transformation T is zero. Taking into account formula (9.9), this proves the following lemma:

(9.12) **Lemma.** Let R, R' be nonisomorphic irreducible representations of G. Then

$$\sum_g R_{g^{-1}}{}'AR_g = 0$$

for every matrix A of the appropriate shape. □

Let's warm up by checking orthogonality in the case that ρ and ρ' have dimension 1. In this case, $R_{g'}, R_g'$ are 1×1 matrices, that is, scalars, and $\chi(g) = R_g$. If we set $A = 1$, then except for the factor $1/N$, (9.12) becomes (9.11), and we are done.

Lemma (9.12) also implies orthogonality in higher dimensions, but only after a small computation. Let us denote the entries of a matrix M by $(M)_{ij}$, as we did in Section 7 of Chapter 4. Then $\chi(g) = \operatorname{trace} R_g = \Sigma_i\, (R_g)_{ii}$. So $\langle \chi', \chi \rangle$ expands to

$$(9.13) \qquad \langle \chi', \chi \rangle = \frac{1}{N}\sum_g \sum_{i,j} (R_{g^{-1}}{}')_{ii}(R_g)_{jj}.$$

We may reverse the order of summation. So to prove that $\langle \chi', \chi \rangle = 0$, it suffices to show that for all i, j,

$$\sum_g (R_{g^{-1}}')_{ii}(R_g)_{jj} = 0. \tag{9.14}$$

The proof of the following lemma is elementary:

(9.15) **Lemma.** Let M, N be matrices and let $P = Me_{\alpha\beta}N$, where $e_{\alpha\beta}$ is a matrix unit of suitable size. The entries of P are $(P)_{ij} = (M)_{i\alpha}(N)_{\beta j}$. □

We substitute e_{ij} for A in Lemma (9.12) and apply Lemma (9.15), obtaining

$$0 = (0)_{ij} = \sum_g (R_{g^{-1}}' e_{ij} R_g)_{ij} = \sum_g (R_{g^{-1}}')_{ii}(R_g)_{jj},$$

as required. This shows that $\langle \chi', \chi \rangle = 0$ if χ and χ' are characters of nonisomorphic irreducible representations.

Next, suppose that $\chi = \chi'$. We have to show that $\langle \chi, \chi \rangle = 1$. Averaging A as in (9.9) need not give zero now, but according to Schur's Lemma, it gives a scalar matrix:

$$\frac{1}{N}\sum_g R_{g^{-1}} A R_g = \tilde{A} = aI. \tag{9.16}$$

By Proposition (9.10), trace A = trace $\tilde{A}$, and trace $\tilde{A} = da$, where $d = \dim \rho$. So

$$a = \text{trace } A/d. \tag{9.17}$$

We set $A = e_{ij}$ in (9.16) and apply Lemma (9.15) again, obtaining

$$(aI)_{ij} = \frac{1}{N}\sum_g (R_{g^{-1}} A R_g)_{ij} = \frac{1}{N}\sum_g (R_{g^{-1}})_{ii}(R_g)_{jj}, \tag{9.18}$$

where $a = (\text{trace } e_{ij})/d$. The left-hand side of (9.18) is zero if $i \neq j$ and is equal to $1/d$ if $i = j$. This shows that the terms with $i \neq j$ in (9.13) vanish, and that

$$\langle \chi, \chi \rangle = \frac{1}{N}\sum_g \sum_i (R_{g^{-1}})_{ii}(R_g)_{ii} = \sum_i \left[\frac{1}{N}\sum_g (R_{g^{-1}})_{ii}(R_g)_{ii}\right] = \sum_i 1/d = 1.$$

This completes the proof that the irreducible characters $\chi_1, \chi_2, \ldots$ are orthonormal.

We still have to show that the number of irreducible characters is equal to the number of conjugacy classes, or, equivalently, that the irreducible characters span the space $\mathcal{C}$ of class functions. Let the subspace they span be $\mathcal{X}$. Then [Chapter 7 (2.15)] $\mathcal{C} = \mathcal{X} \oplus \mathcal{X}^{\perp}$. So we must show that $\mathcal{X}^{\perp} = 0$, or that a class function ϕ which is orthogonal to every character is zero.

Assume a class function ϕ is given. So ϕ is a complex-valued function on G which is constant on conjugacy classes. Let χ be the character of a representation ρ, and consider the linear operator T: $V \longrightarrow V$ defined by

$$T = \frac{1}{N}\sum_g \overline{\phi(g)}\, \rho_g. \tag{9.19}$$

Its trace is

(9.20) $$\text{trace } T = \frac{1}{N}\sum_g \overline{\phi(g)}\,\chi(g) = \langle\phi,\chi\rangle = 0,$$

because ϕ is orthogonal to χ.

(9.21) **Lemma.** The operator T defined by (9.19) is G-invariant.

Proof. We have to show (9.2) $\rho_h \circ T = T \circ \rho_h$, or $T = \rho_h^{-1} \circ T \circ \rho_h$, for every $h \in G$. Let $g'' = h^{-1}gh$. Then as g runs over the group G, so does g'', and of course $\rho_h^{-1}\rho_g\rho_h = \rho_{g''}$. Also $\phi(g) = \phi(g'')$ because ϕ is a class function. Therefore

$$\rho_h^{-1}T\rho_h = \frac{1}{N}\sum_g \overline{\phi(g)}\,\rho_h^{-1}\rho_g\rho_h = \frac{1}{N}\sum_{g''} \overline{\phi(g'')}\,\rho_{g''} = T,$$

as required. □

Now if ρ is irreducible as well, then Schur's Lemma (9.6b) applies and shows that $T = cI$. Since trace $T = 0$ (9.20), it follows that $T = 0$. Any representation ρ is a direct sum of irreducible representations, and (9.19) is compatible with direct sums. Therefore $T = 0$ in every case.

We apply this to the case that $\rho = \rho^{\text{reg}}$ is the regular representation. The vector space is $V(G)$. We compute $T(1)$, where 1 denotes the identity element of G. By definition of the regular representation, $\rho_g(1) = g$. So

(9.22) $$0 = T(1) = \frac{1}{N}\sum_g \overline{\phi(g)}\rho_g(1) = \frac{1}{N}\sum_g \overline{\phi(g)}g.$$

Since the elements of G are a basis for $V = V(G)$, this shows that $\overline{\phi(g)} = 0$ for all g, hence that $\phi = 0$. □

10. REPRESENTATIONS OF THE GROUP SU_2

Much of what was done in Sections 6 to 9 carries over without change to *continuous* representations of compact groups G, once a translation-invariant (Haar) measure dg has been found. One just replaces summation by an integral over the group. However, there will be infinitely many irreducible representations if G is not finite.

When we speak of a representation ρ of a compact group, we shall always mean a continuous homomorphism to $GL(V)$, where V is a finite-dimensional complex vector space. The character χ of ρ is then a continuous, complex-valued function on G, which is constant on each conjugacy class. (It is a *class function.*)

For example, the identity map is a two-dimensional representation of SU_2. Its character is the usual trace of 2×2 matrices. We will call this the *standard representation* of SU_2. The conjugacy classes in SU_2 are the sets of matrices with given trace $2c$. They correspond to the latitudes $\{x_1 = c\}$ in the 3-sphere SU_2 [Chapter 8 (2.8)]. Because of this, a class function on SU_2 depends only on x_1. So such a function can be thought of as a continuous function on the interval $[-1, 1]$. In the notation of Chapter 8 (2.5), the character of the standard representation of SU_2 is

$$\chi(P) = \text{trace } P = a + \overline{a} = 2x_1.$$

Let $|G|$ denote the volume of our compact group G with respect to the measure dg:

$$|G| = \int_G 1\, dg. \tag{10.1}$$

Then the hermitian form which replaces (5.8) is

$$\langle \chi, \chi' \rangle = \frac{1}{|G|}\int_G \overline{\chi(g)}\chi'(g)\, dg. \tag{10.2}$$

With this definition, the orthogonality relations carry over. The proofs of the following extensions to compact groups are the same as for finite groups:

(10.3) **Theorem.**

(a) Every finite-dimensional representation of a compact group G is a direct sum of irreducible representations.
(b) *Schur's Lemma:* Let ρ, ρ' be irreducible representations, and let $T: V \longrightarrow V'$ be a G-invariant linear transformation. Then either T is an isomorphism, or else $T = 0$. If $\rho = \rho'$, then T is multiplication by a scalar.
(c) The characters of the irreducible representations are orthogonal with respect to the form (10.2).
(d) If the characters of two representations are equal, then the representations are isomorphic.
(e) A character χ has the property $\langle \chi, \chi \rangle = 1$ if and only if ρ is irreducible.
(f) If G is abelian, then every irreducible representation is one-dimensional. □

However, the other parts of Theorem (5.9) do not carry over directly. The most significant change in the theory is in Section 6. If G is connected, it cannot operate continuously and nontrivially on a finite set, so finite-dimensional representations can not be obtained from actions on sets. In particular, the regular representation is not finite-dimensional. Analytic methods are needed to extend that part of the theory.

Since a Haar measure is easy to find for the groups U_1 and SU_2, we may consider all of (10.3) proved for them.

Representations of the circle group U_1 are easy to describe, but they are fundamental for an understanding of arbitrary compact groups. It will be convenient to use additive and multiplicative notations interchangeably:

$$\begin{aligned} SO_2(\mathbb{R}) &\xrightarrow{\sim} U_1 \\ (\text{rotation by } \theta) &\rightsquigarrow e^{i\theta} = \alpha. \end{aligned} \tag{10.4}$$

(10.5) **Theorem.** The irreducible representations of U_1 are the nth power maps:

$$U_1 \xrightarrow{\text{“}n\text{”}} U_1,$$

sending $\alpha \rightsquigarrow \alpha^n$, or $\theta \rightsquigarrow n\theta$. There is one such representation for every integer n.

Proof. By (10.3f), the irreducible representations are all one-dimensional, and by (3.5), they are conjugate to unitary representations. Since $GL_1 = \mathbb{C}^\times$ is abelian, conjugation is trivial, so a one-dimensional matrix representation is automatically unitary. Hence an irreducible representation of U_1 is a continuous homomorphism from U_1 to itself. We have to show that the only such homomorphisms are the nth power maps.

(10.6) **Lemma.** The continuous homomorphisms $\psi: \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ are multiplication by a scalar: $\psi(x) = cx$, for some $c \in \mathbb{R}$.

Proof. Let $\psi: \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a continuous homomorphism. We will show that $\psi(x) = x\psi(1)$ for all x. This will show that ψ is multiplication by $c = \psi(1)$.

Since ψ is a homomorphism, $\psi(nr) = \psi(r + \cdots + r) = n\psi(r)$, for any real number r and any nonnegative integer n. In particular, $\psi(n) = n\psi(1)$. Also, $\psi(-n) = -\psi(n) = -n\psi(1)$. Therefore $\psi(n) = n\psi(1)$ for every integer n. Next we let $r = m/n$ be a rational number. The $n\psi(r) = \psi(nr) = \psi(m) = m\psi(1)$. Dividing by n, we find $\psi(r) = r\psi(1)$ for every rational number r. Since the rationals are dense in $\mathbb{R}$ and ψ is continuous, $\psi(x) = cx$ for all x. $\square$

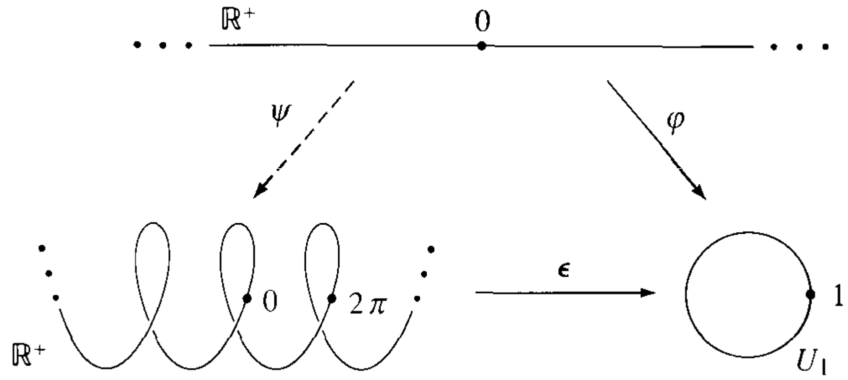
(10.7) **Lemma.** The continuous homomorphisms $\varphi: \mathbb{R}^+ \longrightarrow U_1$ are of the form $\varphi(x) = e^{icx}$ for some $c \in \mathbb{R}$.

Proof. If φ is differentiable, this can be proved using the exponential map of Section 5, Chapter 8. We prove it now for any continuous homomorphism. We consider the exponential homomorphism $\epsilon: \mathbb{R}^+ \longrightarrow U_1$ defined by $\epsilon(x) = e^{ix}$. This homomorphism wraps the real line around the unit circle with period 2π [see Figure (10.8)]. For any continuous function $\varphi: \mathbb{R}^+ \longrightarrow U_1$ such that $\varphi(0) = 1$, there is a unique continuous lifting ψ of this function to the real line such that $\psi(0) = 0$. In other words, we can find a unique continuous function $\psi: \mathbb{R} \longrightarrow \mathbb{R}$ such that $\psi(0) = 0$ and $\varphi(x) = \epsilon(\psi(x))$ for all x. The lifting is constructed starting with the definition $\psi(0) = 0$ and then extending ψ a small interval at a time.

We claim that if φ is a homomorphism, then its lifting ψ is also a homomorphism. If this is shown, then we will conclude that $\psi(x) = cx$ for some c by (10.6), hence that $\varphi(x) = e^{icx}$, as required.

The relation $\varphi(x+y) = \varphi(x)\varphi(y)$ implies that $\epsilon(\psi(x+y)-\psi(x)-\psi(y)) = 1$. Hence $\psi(x + y) - \psi(x)\psi(y) = 2\pi m$ for some integer m which depends continuously on x and y. Varying continuously, m must be constant, and setting $x = y = 0$ shows that $m = 0$. So ψ is a homomorphism, as claimed. $\square$

Now to complete the proof of Theorem (10.5), let $\rho: U_1 \longrightarrow U_1$ be a continuous homomorphism. Then $\varphi = \rho \circ \epsilon: \mathbb{R}^+ \longrightarrow U_1$ is also a continuous homomor-



(10.8) **Figure.**

phism, so $\varphi(x) = e^{icx}$ by (10.7). Moreover, $\varphi(2\pi) = \rho(1)$, which is the case if and only if c is an integer, say n. Then $\rho(e^{ix}) = e^{inx} = (e^{ix})^n$. $\square$

Now let us examine the representations of the group SU_2. Again, there is an infinite family of irreducible representations which arise naturally, and they turn out to form a complete list. Let V_n be the set of homogeneous polynomials of degree n in variables u, v. Such a polynomial will have the form

$$f(u,v) = x_0u^n + x_1u^{n-1}v + \cdots + x_nv^n, \tag{10.9}$$

where the coefficients x_i are complex numbers. Obviously, V_n is a vector space of dimension $n+1$, with basis $(u^n, u^{n-1}v, \ldots, v^n)$. The group $G = GL_2$ operates on V_n in the following way: Let $P \in GL_2$, say

$$P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Let P act on the basis (u,v) of V_1 as usual:

$$(u', v') = (u, v)P = (au + cv, bu + dv);$$

define ρ_{n_P} by the rule

$$u^iv^j \rightsquigarrow u'^iv'^j \quad \text{and} \tag{10.10}$$

$$f(u,v) \rightsquigarrow x_0u'^n + x_1u'^{n-1}v' + \cdots + x_nv'^n.$$

This is a representation

$$\rho_n\colon G \longrightarrow GL(V_n) \approx GL_{n+1}. \tag{10.11}$$

The trivial representation is ρ_0, and the standard representation is ρ_1.

For example, the matrix of ρ_{2_P} is

$$R_{2_P} = \begin{bmatrix} a^2 & ab & b^2 \\ 2ac & ad + bc & 2bd \\ c^2 & cd & d^2 \end{bmatrix}. \tag{10.12}$$

Its first column is the coordinate vector of $\rho_{2_P}(u^2) = (au + cv)^2 = a^2u^2 + 2acuv + c^2v^2$, and so on.

(10.13) **Theorem.** The representations ρ_n ($n = 0, 1, 2, \ldots$) obtained by restricting (10.11) to the subgroup SU_2 are the irreducible representations of SU_2.

Proof. We consider the subgroup T of SU_2 of diagonal matrices

$$\begin{bmatrix} \alpha & \\ & \overline{\alpha} \end{bmatrix} \tag{10.14}$$

where $\alpha = e^{i\theta}$. This group is isomorphic to U_1. The conjugacy class of an arbitrary unitary matrix P contains two diagonal matrices, namely

$$\begin{bmatrix} \lambda & \\ & \overline{\lambda} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \overline{\lambda} & \\ & \lambda \end{bmatrix},$$

where $\lambda, \overline{\lambda}$ are the eigenvalues of P [Chapter 7 (7.4)]. They coincide only when $\lambda = \pm 1$. So every conjugacy class except $\{I\}$ and $\{-I\}$ intersects T in a pair of matrices.

(10.15) **Proposition.**

(a) A class function on SU_2 is determined by its restriction to the subgroup T.

(b) The restriction of a class function φ to T is an *even* function, which means that

$$\varphi(\alpha) = \varphi(\overline{\alpha}) \quad \text{or} \quad \varphi(\theta) = \varphi(-\theta). \ \square$$

Next, any representation ρ of SU_2 restricts to a representation on the subgroup T, and T is isomorphic to U_1. The restriction to T of an irreducible representation of SU_2 will usually be reducible, but it can be decomposed into a direct sum of irreducible representations of T. Therefore the restriction of the character χ to T gives us a sum of irreducible characters on U_1. Theorem (10.5) tells us what the irreducible characters of T are: They are the nth powers $e^{in\theta}$, $n \in \mathbb{Z}$. Therefore we find:

(10.16) **Proposition.** The restriction to T of a character χ on SU_2 is a finite sum of exponential functions $e^{in\theta}$. $\square$

Let us calculate the restriction to T of the character χ_n of ρ_n (10.11). The matrix (10.14) acts on monomials by

$$u^i v^j \rightsquigarrow (\alpha^i u^i)(\overline{\alpha}^j v^j) = \alpha^{i-j} u^i v^j.$$

Therefore, its matrix, acting on the basis $(u^n, u^{n-1}v, \ldots, v^n)$, is the diagonal matrix

$$\begin{bmatrix} \alpha^n & & & \\ & \alpha^{n-2} & & \\ & & \ddots & \\ & & & \alpha^{-n} \end{bmatrix},$$

and the value of the character is

$$\chi_n(\alpha) = \alpha^n+\alpha^{n-2}+\cdots+\alpha^{-n} = e^{in\theta}+e^{i(n-2)\theta}+\cdots+e^{-in\theta}, \tag{10.17}$$

or

$$\begin{aligned} \chi_0 &= 1 \\ \chi_1 &= 2\cos\theta = e^{i\theta} + e^{-i\theta} \\ \chi_2 &= 1+2\cos 2\theta = e^{2i\theta}+1+e^{-2i\theta} \\ \chi_3 &= 2\cos 3\theta + 2\cos\theta \\ &\ \ \vdots \end{aligned} \tag{10.18}$$

Now let χ' be any irreducible character on SU_2. Its restriction to T is even (10.15b) and is a sum of exponentials $e^{in\theta}$ (10.16). To be even, $e^{in\theta}$ and $e^{-in\theta}$ must occur with the same coefficient, so the character is a linear combination of the functions $\cos n\theta = \frac{1}{2}(e^{in\theta}+e^{-in\theta})$. The functions (10.17) form a basis for the vector space spanned by $\{\cos n\theta\}$. Therefore

$$\chi' = \sum_i r_i\chi_i \tag{10.19}$$

where r_i are rational numbers. A priori, this is true on T, but by (10.15a) it is also true on all of SU_2. Clearing denominators and bringing negative terms to the left in (10.19) yields a relation of the form

$$m\chi' + \sum_j n_j\chi_j = \sum_k n_k\chi_k, \tag{10.20}$$

where n_j, n_k are positive integers and the index sets $\{j\}, \{k\}$ are disjoint. This relation implies

$$m\rho' \oplus \sum_j n_j\rho_j = \sum_k n_k\rho_k.$$

Therefore ρ' is one of the representations ρ_k. This completes the proof of Theorem (10.13). □

We leave the obvious generalizations to the reader.

Israel Herstein

EXERCISES

1. Definition of a Group Representation

1. Let ρ be a representation of a group G. Show that det ρ is a one-dimensional representation.

2. Suppose that G is a group with a faithful representation by diagonal matrices. Prove that G is abelian.

3. Prove that the rule $S_n \longrightarrow \mathbb{R}^\times$ defined by $p \rightsquigarrow \operatorname{sign} p$ is a one-dimensional representation of the symmetric group.

4. Prove that the only one-dimensional representations of the symmetric group S_5 are the trivial representation defined by $\rho(g) = 1$ for all g and the sign representation.

5. **(a)** Write the standard representation of the octahedral group O by rotations explicitly, choosing a suitable basis for $\mathbb{R}^3$.
(b) Do the same for the dihedral group D_n.
*__(c)__ Do the same for the icosahedral group I.

6. Show that the rule $\sigma(\theta) = \begin{bmatrix} \alpha & \alpha^2 - \alpha \\ 0 & \alpha^2 \end{bmatrix}$, $\alpha = e^{i\theta}$, is a representation of SO_2, when a rotation in SO_2 is represented by its angle.

7. Let H be a subgroup of index 2 of a group G, and let $\rho: G \longrightarrow GL(V)$ be a representation. Define $\rho': G \longrightarrow GL(V)$ by the rule $\rho'(g) = \rho(g)$ if $g \in H$, and $\rho'(g) = -\rho(g)$ if $g \notin H$. Prove that ρ' is a representation of G.

8. Prove that every finite group G has a faithful representation on a finite-dimensional complex vector space.

9. Let N be a normal subgroup of a group G. Relate representations of G/N to representations of G.

10. Choose three axes in $\mathbb{R}^3$ passing through the vertices of a regular tetrahedron centered at the origin. (This is not an orthogonal coordinate system.) Find the coordinates of the fourth vertex, and write the matrix representation of the tetrahedral group T in this coordinate system explicitly.

2. *G*-Invariant Forms and Unitary Representations

1. **(a)** Verify that the form X^*BY (2.10) is G-invariant.
(b) Find an orthonormal basis for this form, and determine the matrix P of change of basis. Verify that PAP^{-1} is unitary.

2. Prove the real analogue of (2.2): Let $R: G \longrightarrow GL_n(\mathbb{R})$ be a representation of a finite group G. There is a $P \in GL_n(\mathbb{R})$ such that PR_gP^{-1} is orthogonal for every $g \in G$.

3. Let $\rho: G \longrightarrow SL_2(\mathbb{R})$ be a faithful representation of a finite group by real 2×2 matrices of determinant 1. Prove that G is a cyclic group.

4. Determine all finite groups which have a faithful real two-dimensional representation.

5. Describe the finite groups G which admit faithful real three-dimensional representations with determinant 1.

6. Let V be a hermitian vector space. Prove that the unitary operators on V form a subgroup $U(V)$ of $GL(V)$, and that a representation ρ on V has image in $U(V)$ if and only if the form $\langle\, ,\rangle$ is G-invariant.

7. Let $\langle\, ,\rangle$ be a nondegenerate skew-symmetric form on a vector space V, and let ρ be a representation of a finite group G on V.
(a) Prove that the averaging process (2.7) produces a G-invariant skew-symmetric form on V.
(b) Does this prove that every finite subgroup of GL_{2n} is conjugate to a subgroup of SP_{2n}?

8. (a) Let R be the standard two-dimensional representation of D_3, with the triangle situated so that the x-axis is a line of reflection. Rewrite this representation in terms of the basis $x' = x$ and $y' = x + y$.
   (b) Use the averaging process to obtain a G-invariant form from dot product in the (x', y')-coordinates.

3. Compact Groups

1. Prove that dx/x is a Haar measure on the multiplicative group $\mathbb{R}^\times$.
2. (a) Let $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$ be a variable 2×2 matrix, and let $dV = dp_{11}dp_{12}dp_{21}dp_{22}$ denote the ordinary volume form on $\mathbb{R}^{2\times 2}$. Show that $(\det P)^{-2}dV$ is a Haar measure on $GL_2(\mathbb{R})$.
   (b) Generalize the results of (a).

*3. Show that the form $\dfrac{dx_2dx_3dx_4}{x_1}$ on the 3-sphere defines a Haar measure on SU_2. What replaces this expression at points where $x_1 = 0$?

4. Take the complex representation of SO_2 in $\mathbb{R}^2$ given by

$$\sigma(\theta) = \begin{bmatrix} \alpha & \alpha^2 - \alpha \\ 0 & \alpha^2 \end{bmatrix}, \quad \alpha = e^{i\theta},$$

and reduce it to a unitary representation by averaging the hermitian product on $\mathbb{R}^2$.

4. G-Invariant Subspaces and Irreducible Representations

1. Prove that the standard three-dimensional representation of the tetrahedral group T is irreducible as a complex representation.
2. Determine all irreducible representations of a cyclic group C_n.
3. Determine the representations of the icosahedral group I which are not faithful.
4. Let ρ be a representation of a finite group G on a vector space V and let $v \in V$.
   (a) Show that averaging gv over G gives a vector $\overline{v} \in V$ which is fixed by G.
   (b) What can you say about this vector if ρ is an irreducible representation?
5. Let $H \subset G$ be a subgroup, let ρ be a representation of G on V, and let $v \in V$. Let $w = \sum_{h \in H} hv$. What can you say about the order of the G-orbit of w?
6. Consider the standard two-dimensional representation of the dihedral group D_n as symmetries of the n-gon. For which values of n is it irreducible as a complex representation?

*7. Let G be the dihedral group D_3, presented as in Chapter 5 (3.6).
   (a) Let ρ be an irreducible unitary representation of dimension 2. Show that there is an orthonormal basis of V such that $R_y = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}$.
   (b) Assume that R_y is as above. Use the defining relations $yx = x^2y$, $x^3 = 1$ to determine the possibilities for R_x.
   (c) Prove that all irreducible two-dimensional representations of G are isomorphic.
   (d) Let ρ be any representation of G, and let $v \in V$ be an eigenvector for the operator ρ_x. Show that v is contained in a G-invariant subspace W of dimension ≤ 2.
   (e) Determine all irreducible representations of G.

5. Characters

1. Corollary (5.11) describes a basis for the space of class functions. Give another basis.
2. Find the decomposition of the standard two-dimensional rotation representation of the cyclic group C_n by rotations into irreducible representations.
3. Prove or disprove: Let χ be a character of a finite group G, and define $\overline{\chi}(g) = \overline{\chi(g)}$. Then $\overline{\chi}$ is also a character of G.
4. Find the dimensions of the irreducible representations of the group O of rotations of a cube, the quaternion group, and the dihedral groups D_4, D_5, and D_6.
5. Describe how to produce a unitary matrix by adjusting the entries of a character table.
6. Compare the character tables for the quaternion group and the dihedral group D_4.
7. Determine the character table for D_6.
8. (a) Determine the character table for the groups C_5 and D_5.
   (b) Decompose the restriction of each irreducible character of D_5 into irreducible characters of C_5.
9. (a) Let ρ be a representation of dimension d, with character χ. Prove that the kernel of ρ is the set of group elements such that $\chi(g) = d$.
   (b) Show that if G has a proper normal subgroup, then there is a representation ρ such that $\ker \rho$ is a proper subgroup.

*10. Let χ be the character of a representation ρ of dimension d. Prove that $|\chi(g)| \leq d$ for all $g \in G$, and that if $|\chi(g)| = d$, then $\rho(g) = \zeta I$, for some root of unity ζ.

11. Let $G' = G/N$ be a quotient group of a finite group G, and let ρ' be an irreducible representation of G'. Prove that the representation of G defined by ρ' is irreducible in two ways: directly, and using Theorem (5.9).
12. Find the missing rows in the character table below:

	(1) 1	(3) a	(6) b	(6) c	(8) d
χ_1	1	1	1	1	1
χ_2	1	1	-1	-1	1
χ_3	3	-1	1	-1	0
χ_4	3	-1	-1	1	0

13. The table below is a partial character table of a finite group, in which $\zeta = \frac{1}{2}(-1 + \sqrt{3}i)$ and $\gamma = \frac{1}{2}(-1 + \sqrt{7}i)$. The conjugacy classes are all there.

	(1)	(3)	(3)	(7)	(7)
χ_1	1	1	1	ζ	$\overline{\zeta}$
χ_2	3	γ	$\overline{\gamma}$	0	0
χ_3	3	$\overline{\gamma}$	γ	0	0

   (a) Determine the order of the group and the number and the dimensions of the irreducible representations.
   (b) Determine the remaining characters.
   (c) Describe the group by generators and relations.

***14.** Describe the commutator subgroup of a group G in terms of the character table.

***15.** Below is a partial character table. One conjugacy class is missing.

	(1) 1	(1) u	(2) v	(2) w	(3) x
χ_1	1	1	1	1	1
χ_2	1	1	1	1	-1
χ_3	1	-1	1	-1	i
χ_4	1	-1	1	-1	$-i$
χ_5	2	-2	-1	-1	0

(a) Complete the table.

(b) Show that u has order 2, x has order 4, w has order 6, and v has order 3. Determine the orders of the elements in the missing conjugacy class.

(c) Show that v generates a normal subgroup.

(d) Describe the group.

***16.** **(a)** Find the missing rows in the character table below.

(b) Show that the group G with this character table has a subgroup H of order 10, and describe this subgroup as a union of conjugacy classes.

(c) Decide whether H is C_{10} or D_5.

(d) Determine the commutator subgroup of G.

(e) Determine all normal subgroups of G.

(f) Determine the orders of the elements a, b, c, d.

(g) Determine the number of Sylow 2-subgroups and the number of Sylow 5-subgroups of this group.

	(1) 1	(4) a	(5) b	(5) c	(5) d
χ_1	1	1	1	1	1
χ_2	1	1	-1	-1	1
χ_3	1	1	$-i$	i	-1
χ_4	1	1	i	$-i$	-1

***17.** In the character table below, $\zeta = \frac{1}{2}(-1 + \sqrt{3}i)$.

	(1) 1	(6) a	(7) b	(7) c	(7) d	(7) e	(7) f
χ_1	1	1	1	1	1	1	1
χ_2	1	1	1	ζ	$\bar{\zeta}$	ζ	$\bar{\zeta}$
χ_3	1	1	1	$\bar{\zeta}$	ζ	$\bar{\zeta}$	ζ
χ_4	1	1	-1	$-\zeta$	$-\bar{\zeta}$	ζ	$\bar{\zeta}$
χ_5	1	1	-1	$-\bar{\zeta}$	$-\zeta$	$\bar{\zeta}$	ζ
χ_6	1	1	-1	-1	-1	1	1
χ_7	6	-1	0	0	0	0	0

(a) Show that G has a normal subgroup N isomorphic to D_7, and determine the structure of G/N.

(b) Decompose the restrictions of each character to N into irreducible N-characters.
(c) Determine the numbers of Sylow p-subgroups, for $p = 2, 3$, and 7.
(d) Determine the orders of the representative elements c, d, e, f.

6. Permutation Representations and the Regular Representation

1. Verify the values of the characters (6.4) and (6.5).
2. Use the orthogonality relations to decompose the character of the regular representation for the tetrahedral group.
3. Show that the dimension of any irreducible representation of a group G of order $N > 1$ is at most $N - 1$.
4. Determine the character tables for the nonabelian groups of order 12.
5. Decompose the regular representation of C_3 into irreducible *real* representations.
6. Prove Corollary (6.8).
7. Let ρ be the permutation representation associated to the operation of D_3 on itself by conjugation. Decompose the character of ρ into irreducible characters.
8. Let S be a G-set, and let ρ be the permutation representation of G on the space $V(S)$. Prove that the orbit decomposition of S induces a direct sum decomposition of ρ.
9. Show that the standard representation of the symmetric group S_n by permutation matrices is the sum of a trivial representation and an irreducible representation.

*10. Let H be a subgroup of a finite group G. Given an irreducible representation ρ of G, we may decompose its restriction to H into irreducible H-representations. Show that every irreducible representation of H can be obtained in this way.

7. The Representations of the Icosahedral Group

1. Compute the characters χ_2, χ_4, χ_5 of I, and use the orthogonality relations to determine the remaining character χ_3.
2. Decompose the representations of the icosahedral group on the sets of faces, edges, and vertices into irreducible representations.
3. The group S_5 operates by conjugation on its subgroup A_5. How does this action operate on the set of irreducible representations of A_5?

*4. Derive an algorithm for checking that a group is simple by looking at its character table.

5. Use the character table of the icosahedral group to prove that it is a simple group.
6. Let H be a subgroup of index 2 of a group G, and let $\sigma: H \longrightarrow GL(V)$ be a representation. Let a be an element of G not in H. Define a *conjugate* representation $\sigma': H \longrightarrow GL(V)$ by the rule $\sigma'(h) = \sigma(a^{-1}ha)$.
   (a) Prove that σ' is a representation of H.
   (b) Prove that if σ is the restriction to H of a representation of G, then σ' is isomorphic to σ.
   (c) Prove that if b is another element of G not in H, then the representation $\sigma''(h) = \sigma(b^{-1}hb)$ is isomorphic to σ'.
7. (a) Choose coordinates and write the standard three-dimensional matrix representation of the octahedral group O explicitly.

(b) Identify the five conjugacy classes in O, and find the orders of its irreducible representations.

(c) The group O operates on these sets:

(i) six faces of the cube

(ii) three pairs of opposite faces

(iii) eight vertices

(iv) four pairs of opposite vertices

(v) six pairs of opposite edges

(vi) two inscribed tetrahedra

Identify the irreducible representations of O as summands of these representations, and compute the character table for O. Verify the orthogonality relations.

(d) Decompose each of the representations (c) into irreducible representations.

(e) Use the character table to find all normal subgroups of O.

8. (a) The icosahedral group I contains a subgroup T, the stabilizer of one of the cubes [Chapter 6 (6.7)]. Decompose the restrictions to T of the irreducible characters of I.

(b) Do the same thing as (a) with a subgroup D_5 of I.

9. Here is the character table for the group $G = PSL_2(\mathbb{F}_7)$, with $\gamma = \frac{1}{2}(-1 + \sqrt{7}i]$, $\gamma' = \frac{1}{2}(-1 - \sqrt{7}i)$.

	(1) 1	(21) a	(24) b	(24) c	(42) d	(56) e
χ_1	1	1	1	1	1	1
χ_2	3	-1	γ	γ'	1	0
χ_3	3	-1	γ'	γ	1	0
χ_4	6	2	-1	-1	0	0
χ_5	7	-1	0	0	-1	1
χ_6	8	0	1	1	0	-1

(a) Use it to give two different proofs that this group is simple.

(b) Identify, so far as possible, the conjugacy classes of the elements

$$\begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix}, \begin{bmatrix} 2 & \\ & 4 \end{bmatrix},$$

and find matrices which represent the remaining conjugacy classes.

(c) G operates on the set of one-dimensional subspaces of F^2 ($F = \mathbb{F}_7$). Decompose the associated character into irreducible characters.

8. One-dimensional Representations

1. Prove that the abelian characters of a group G form a group.

2. Determine the character group for the Klein four group and for the quaternion group.

3. Let A,B be matrices such that some power of each matrix is the identity and such that A and B commute. Prove that there is an invertible matrix P such that PAP^{-1} and PBP^{-1} are both diagonal.

4. Let G be a finite abelian group. Show that the order of the character group is equal to the order of G.

*5. Prove that the sign representation $p \rightsquigarrow \operatorname{sign} p$ and the trivial representation are the only one-dimensional representations of the symmetric group S_n.

6. Let G be a cyclic group of order n, generated by an element x, and let $\zeta = e^{2\pi i/n}$.
   (a) Prove that the irreducible representations are $\rho_0, \dots, \rho_{n-1}$, where $\rho_k \colon G \rightsquigarrow \mathbb{C}^\times$ is defined by $\rho_k(x) = \zeta^k$.
   (b) Identify the character group of G.
   (c) Verify the orthogonality relations for G explicitly.

7. (a) Let $\varphi \colon G \longrightarrow G'$ be a homomorphism of abelian groups. Define an induced homomorphism $\hat{\varphi} \colon \hat{G}' \longleftarrow \hat{G}$ between their character groups.
   (b) Prove that $\hat{\varphi}$ is surjective if φ is injective, and conversely.

9. Schur's Lemma, and Proof of the Orthogonality Relations

1. Let ρ be a representation of G. Prove or disprove: If the only G-invariant operators on V are multiplication by a scalar, then ρ is irreducible.

2. Let ρ be the standard three-dimensional representation of T, and let ρ' be the permutation representation obtained from the action of T on the four vertices. Prove by averaging that ρ is a summand of ρ'.

3. Let $\rho = \rho'$ be the two-dimensional representation (4.6) of the dihedral group D_3, and let $A = \begin{bmatrix} 1 & 1 \\ & \end{bmatrix}$. Use the averaging process to produce a G-invariant transformation from left multiplication by A.

4. (a) Show that $R_x = \begin{bmatrix} 1 & 1 & -1 \\ & & 1 \\ 1 & & -1 \end{bmatrix}$, $R_y = \begin{bmatrix} & -1 & -1 \\ -1 & & 1 \\ & & -1 \end{bmatrix}$ defines a representation of D_3.
   (b) We may regard the representation ρ_2 of (5.15) as a 1×1 matrix representation. Let T be the linear transformation $\mathbb{C}^1 \longrightarrow \mathbb{C}^3$ whose matrix is $(1,0,0)^t$. Use the averaging method to produce a G-invariant linear transformation from T, using ρ_2 and the representation R defined in (a).
   (c) Do part (b), replacing ρ_2 by ρ_1 and ρ_3.
   (d) Decompose R explicitly into irreducible representations.

10. Representations of the Group SU_2

1. Determine the irreducible representations of the rotation group SO_3.

2. Determine the irreducible representations of the orthogonal group O_2.

3. Prove that the orthogonal representation $SU_2 \longrightarrow SO_3$ is irreducible, and identify its character in the list (10.18).

4. Prove that the functions (10.18) form a basis for the vector space spanned by $\{\cos n\theta\}$.

5. Left multiplication defines a representation of SU_2 on the space $\mathbb{R}^4$ with coordinates $x_1, \dots, x_4$, as in Chapter 8, Section 2. Decompose the associated complex representation into irreducible representations.

6. (a) Calculate the four-dimensional volume of the 4-ball of radius r, $B^4 = \{x_1^2 + x_2^2 + x_3^2 + x_4^2 \le r^2\}$, by slicing with three-dimensional slices.
   (b) Calculate the three-dimensional volume of the 3-sphere S^3, again by slicing. It is

advisable to review the analogous computation of the area of a 2-sphere first. You should find $\frac{d}{dr}$(volume of B^4) = (volume of S^3). If not, try again.

***7.** Prove the orthogonality relations for the irreducible characters (10.17) of SU_2 by integration over S^3.

Miscellaneous Problems

***1.** Prove that a finite simple group which is not of prime order has no nontrivial representation of dimension 2.

***2.** Let H be a subgroup of index 2 of a finite group G, and let a be an element of G not in H, so that aH is the second coset of H in G. Let $S: H \longrightarrow GL_n$ be a matrix representation of H. Define a representation *ind* $S: G \longrightarrow GL_{2n}$ of G, called the *induced* representation, as follows:

$$(ind\ S)_h = \left[\begin{array}{c|c} S_h & \\ \hline & S_{a^{-1}ha} \end{array}\right], \quad (ind\ S)_{ah} = \left[\begin{array}{c|c} & S_{aha} \\ \hline S_h & \end{array}\right].$$

(a) Prove that *ind* S is a representation of G.

(b) Describe the character $\chi_{ind S}$ of *ind* S in terms of the character χ_S of S.

(c) If $R: G \longrightarrow GL_m$ is a representation of G, we may restrict it to H. We denote the restriction by *res* $R: H \longrightarrow GL_n$. Prove that *res* (*ind* S) $\approx S \oplus S'$, where S' is the conjugate representation defined by $S_h' = S_{a^{-1}ha}$.

(d) Prove *Frobenius reciprocity:* $\langle \chi_{ind S}, \chi_R \rangle = \langle \chi_S, \chi_{res R} \rangle$.

(e) Use Frobenius reciprocity to prove that if S and S' are not isomorphic representations, then the induced representation *ind* S of G is irreducible. On the other hand, if $S \approx S'$, then *ind* S is a sum of two irreducible representations R, R'.

***3.** Let H be a subgroup of index 2 of a group G, and let R be a matrix representation of G. Let R' denote the *conjugate* representation, defined by $R_g' = R_g$ if $g \in H$, and $R_g' = -R_g$ otherwise.

(a) Show that R' is isomorphic to R if and only if the character of R is identically zero on the coset gH, where $g \notin H$.

(b) Use Frobenius reciprocity to show that *ind*(*res* R) $\approx R \oplus R'$.

(c) Show that if R is not isomorphic to R', then *res* R is irreducible, and if these two representations are isomorphic, then *res* R is a sum of two irreducible representations of H.

***4.** Using Frobenius reciprocity, derive the character table of S_n from that of A_n when (a) $n = 3$, (b) $n = 4$, (c) $n = 5$.

***5.** Determine the characters of the dihedral group D_n, using representations induced from C_n.

6. (a) Prove that the only element of SU_2 of order 2 is $-I$.

(b) Consider the homomorphism $\varphi: SU_2 \longrightarrow SO_3$. Let A be an element of SU_2 such that $\varphi(A) = \bar{A}$ has finite order $\bar{n}$ in SO_3. Prove that the order n of A is either $\bar{n}$ or $2\bar{n}$. Also prove that if $\bar{n}$ is even, then $n = 2\bar{n}$.

***7.** Let G be a finite subgroup of SU_2, and let $\bar{G} = \varphi(G)$, where $\varphi: SU_2 \longrightarrow SO_3$ is the orthogonal representation (Chapter 8, Section 3). Prove the following.

(a) If $|\bar{G}|$ is even, then $|G| = 2|\bar{G}|$ and $G = \varphi^{-1}(\bar{G})$.

(b) Either $G = \varphi^{-1}(\overline{G})$, or else G is a cyclic group of odd order.

(c) Let G be a cyclic subgroup of SU_2 of order n. Prove that G is conjugate to the group generated by $\begin{bmatrix} \zeta & \\ & \zeta^{-1} \end{bmatrix}$, where $\zeta = e^{2\pi i/n}$.

(d) Show that if $\overline{G}$ is the group D_2, then G is the quaternion group. Determine the matrix representation of the quaternion group H as a subgroup of SU_2 with respect to a suitable orthonormal basis in $\mathbb{C}^2$.

(e) If $\overline{G} = T$, prove that G is a group of order 24 which is not isomorphic to the symmetric group S_4.

***8.** Let ρ be an irreducible representation of a finite group G. How unique is the positive definite G-invariant hermitian form?

***9.** Let G be a finite subgroup of $GL_n(\mathbb{C})$. Prove that if $\Sigma_g \operatorname{tr} g = 0$, then $\Sigma_g\, g = 0$.

***10.** Let $\rho: G \longrightarrow GL(V)$ be a two-dimensional representation of a finite group G, and assume that 1 is an eigenvalue of ρ_g for every $g \in G$. Prove that ρ is a sum of two one-dimensional representations.

***11.** Let $\rho: G \longrightarrow GL_n(\mathbb{C})$ be an irreducible representation of a finite group G. Given any representation $\sigma: GL_n \longrightarrow GL(V)$ of GL_n, we can consider the composition $\sigma \circ \rho$ as a representation of G.

(a) Determine the character of the representation obtained in this way when σ is left multiplication of GL_n on the space $\mathbb{C}^{n\times n}$ of $n \times n$ matrices. Decompose $\sigma \circ \rho$ into irreducible representations in this case.

(b) Find the character of $\sigma \circ \rho$ when σ is the operation of conjugation on $M_n(\mathbb{C})$.