CHAPTER 9 DERIVATIVES

The derivative of a function is the first of the two major concepts of this section. Together with the integral, it constitutes the source from which calculus derives its particular flavor. While it is true that the concept of a function is fundamental, that you cannot do anything without limits or continuity, and that least upper bounds are essential, everything we have done until now has been preparation—if adequate, this section will be easier than the preceding ones—for the really exciting ideas to come, the powerful concepts that are truly characteristic of calculus.

Perhaps (some would say "certainly") the interest of the ideas to be introduced in this section stems from the intimate connection between the mathematical concepts and certain physical ideas. Many definitions, and even some theorems, may be described in terms of physical problems, often in a revealing way. In fact, the demands of physics were the original inspiration for these fundamental ideas of calculus, and we shall frequently mention the physical interpretations. But we shall always first define the ideas in precise mathematical form, and discuss their significance in terms of mathematical problems.

The collection of all functions exhibits such diversity that there is almost no hope of discovering any interesting general properties pertaining to all. Because continuous functions form such a restricted class, we might expect to find some nontrivial theorems pertaining to them, and the sudden abundance of theorems after Chapter 6 shows that this expectation is justified. But the most interesting and most powerful results about functions will be obtained only when we restrict our attention even further, to functions which have even greater claim to be called "reasonable," which are even better behaved than most continuous functions.

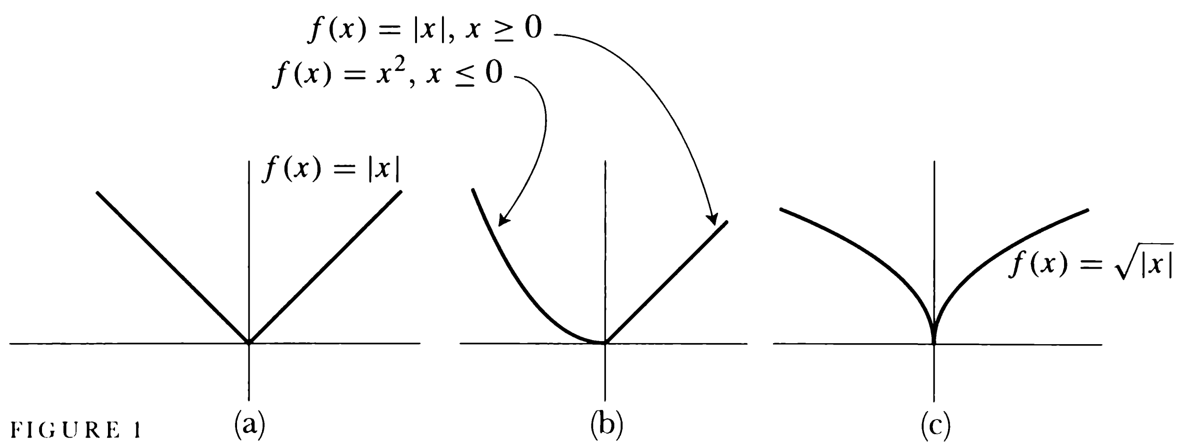


FIGURE 1

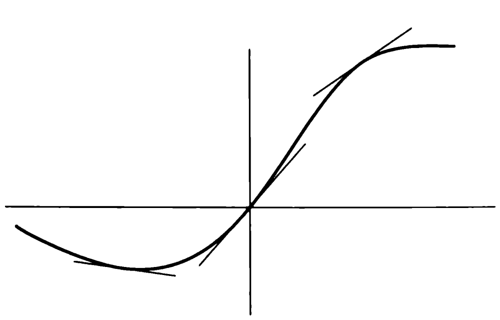


FIGURE 2

Figure 1 illustrates certain types of misbehavior which continuous functions can display. The graphs of these functions are "bent" at $(0, 0)$, unlike the graph of Figure 2, where it is possible to draw a "tangent line" at each point. The quotation marks have been used to avoid the suggestion that we have defined "bent" or

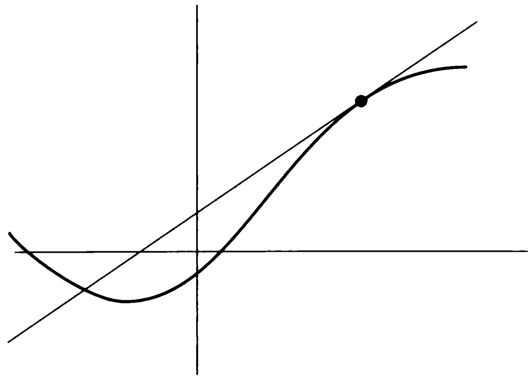


FIGURE 3

"tangent line," although we are suggesting that the graph might be "bent" at a point where a "tangent line" cannot be drawn. You have probably already noticed that a tangent line cannot be defined as a line which intersects the graph only once—such a definition would be both too restrictive and too permissive. With such a definition, the straight line shown in Figure 3 would not be a tangent line to the graph in that picture, while the parabola would have two tangent lines at each point (Figure 4), and the three functions in Figure 5 would have more than one tangent line at the points where they are "bent."

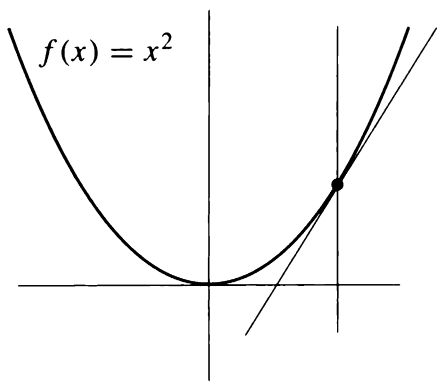


FIGURE 4

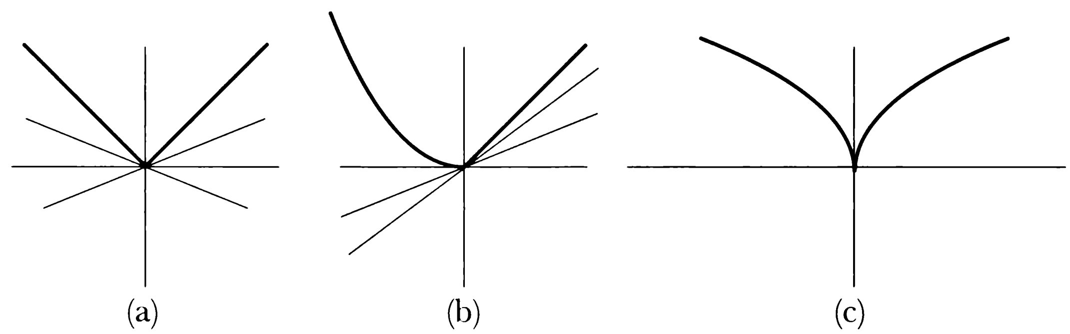


FIGURE 5

A more promising approach to the definition of a tangent line might start with "secant lines," and use the notion of limits. If $h \neq 0$, then the two distinct points $(a, f(a))$ and $(a + h, f(a + h))$ determine, as in Figure 6, a straight line whose slope is

$$\frac{f(a+h) - f(a)}{h}.$$

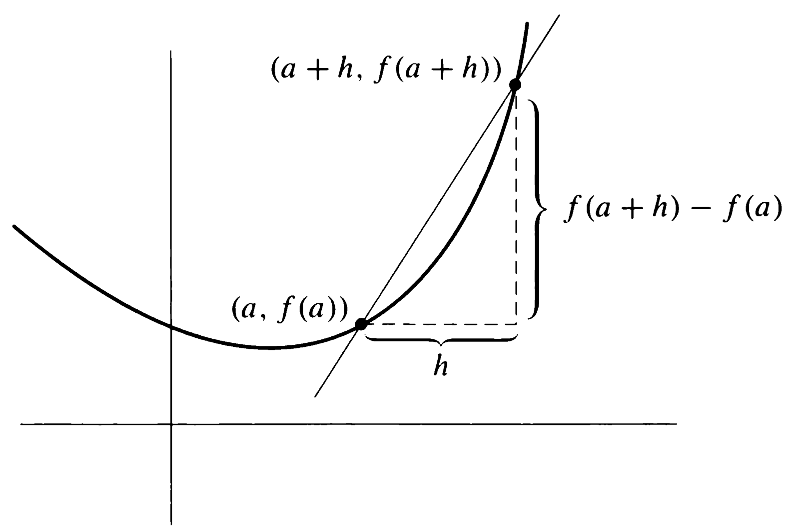


FIGURE 6

As Figure 7 illustrates, the "tangent line" at $(a, f(a))$ seems to be the limit, in some sense, of these "secant lines," as h approaches 0. We have never before talked about a "limit" of lines, but we *can* talk about the limit of their slopes: the slope of the tangent line through $(a, f(a))$ should be

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

We are ready for a definition, and some comments.

DEFINITION

> The function f is **differentiable at a** if
>
> $$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \text{ exists.}$$
>
> In this case the limit is denoted by $\boldsymbol{f'(a)}$ and is called the **derivative of f at a.** (We also say that f is **differentiable** if f is differentiable at a for every a in the domain of f.)

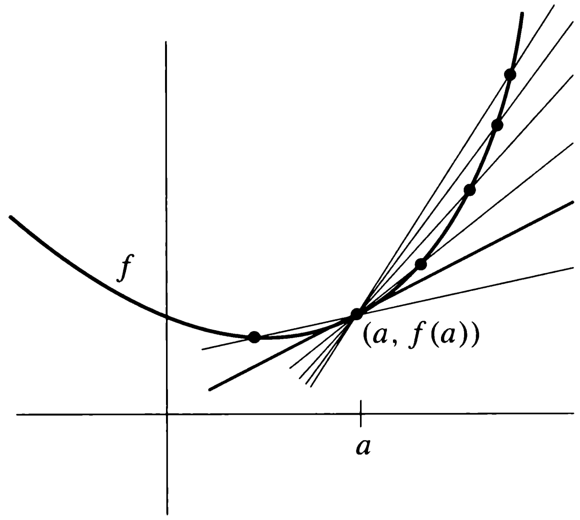


FIGURE 7

The first comment on our definition is really an addendum; we define the **tangent line** to the graph of f at $(a, f(a))$ to be the line through $(a, f(a))$ with slope $f'(a)$. This means that the tangent line at $(a, f(a))$ is defined only if f is differentiable at a.

The second comment refers to notation. The symbol $f'(a)$ is certainly reminiscent of functional notation. In fact, for any function f, we denote by $\boldsymbol{f'}$ the function whose domain is the set of all numbers a such that f is differentiable at a, and whose value at such a number a is

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

(To be very precise: f' is the collection of all pairs

$$\left(a, \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}\right)$$

for which $\lim_{h \to 0} [f(a+h) - f(a)]/h$ exists.) The function f' is called the **derivative** of f.

Our third comment, somewhat longer than the previous two, refers to the physical interpretation of the derivative. Consider a particle which is moving along a straight line (Figure 8(a)) on which we have chosen an "origin" point O, and a direction in which distances from O shall be written as positive numbers, the distance from O of points in the other direction being written as negative numbers. Let $s(t)$ denote the distance of the particle from O, at time t. The suggestive notation $s(t)$ has been chosen purposely; since a distance $s(t)$ is determined for each

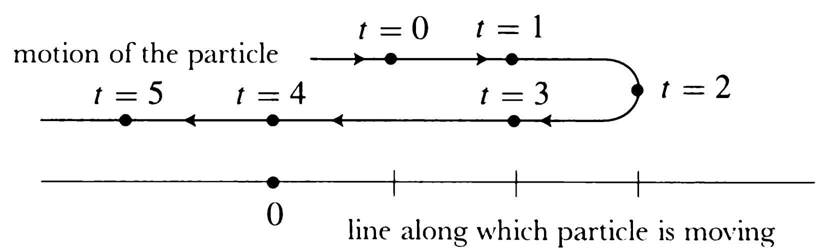


FIGURE 8(a)

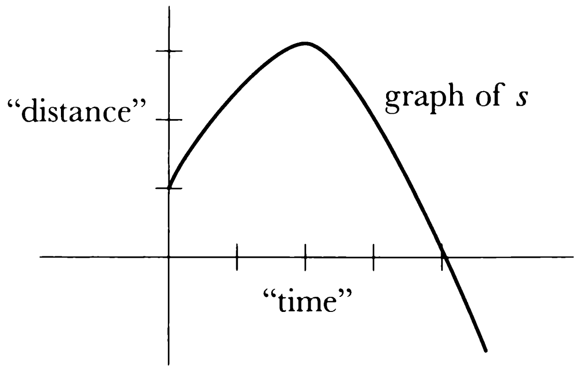


FIGURE 8(b)

number t, the physical situation automatically supplies us with a certain function s. The graph of s indicates the distance of the particle from O, on the vertical axis, in terms of the time, indicated on the horizontal axis (Figure 8(b)).

The quotient

$$\frac{s(a+h)-s(a)}{h}$$

has a natural physical interpretation. It is the "average velocity" of the particle during the time interval from a to $a+h$. For any particular a, this average speed depends on h, of course. On the other hand, the limit

$$\lim_{h\to 0}\frac{s(a+h)-s(a)}{h}$$

depends only on a (as well as the particular function s) and there are important physical reasons for considering this limit. We would like to speak of the "velocity of the particle at time a," but the usual definition of velocity is really a definition of average velocity; the only reasonable definition of "velocity at time a" (so-called "instantaneous velocity") is the limit

$$\lim_{h\to 0}\frac{s(a+h)-s(a)}{h}.$$

Thus we *define* the (**instantaneous**) **velocity** of the particle at a to be $s'(a)$. Notice that $s'(a)$ could easily be negative; the absolute value $|s'(a)|$ is sometimes called the (**instantaneous**) **speed**.

It is important to realize that instantaneous velocity is a theoretical concept, an abstraction which does not correspond precisely to any observable quantity. While it would not be fair to say that instantaneous velocity has nothing to do with average velocity, remember that $s'(t)$ is not

$$\frac{s(t+h)-s(t)}{h}$$

for any particular h, but merely the limit of these average velocities as h approaches 0. Thus, when velocities are measured in physics, what a physicist really measures is an average velocity over some (very small) time interval; such a procedure cannot be expected to give an exact answer, but this is really no defect, because physical measurements can never be exact anyway.

The velocity of a particle is often called the "rate of change of its position." This notion of the derivative, as a rate of change, applies to any other physical situation in which some quantity varies with time. For example, the "rate of change of mass" of a growing object means the derivative of the function m, where $m(t)$ is the mass at time t.

In order to become familiar with the basic definitions of this chapter, we will spend quite some time examining the derivatives of particular functions. Before proving the important theoretical results of Chapter 11, we want to have a good idea of what the derivative of a function looks like. The next chapter is devoted exclusively to one aspect of this problem—calculating the derivative of complicated functions. In this chapter we will emphasize the concepts, rather than the calculations, by considering a few simple examples. Simplest of all is a constant function, $f(x) = c$. In this case

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{c-c}{h} = 0.$$

Thus f is differentiable at a for every number a, and $f'(a) = 0$. This means that the tangent line to the graph of f always has slope 0, so the tangent line always coincides with the graph.

Constant functions are not the only ones whose graphs coincide with their tangent lines—this happens for any linear function $f(x) = cx + d$. Indeed

$$\begin{aligned} f'(a) &= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \\ &= \lim_{h\to 0} \frac{c(a+h)+d-[ca+d]}{h} \\ &= \lim_{h\to 0} \frac{ch}{h} = c; \end{aligned}$$

the slope of the tangent line is c, the same as the slope of the graph of f.

A refreshing difference occurs for $f(x) = x^2$. Here

$$\begin{aligned} f'(a) &= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \\ &= \lim_{h\to 0} \frac{(a+h)^2-a^2}{h} \\ &= \lim_{h\to 0} \frac{a^2+2ah+h^2-a^2}{h} \\ &= \lim_{h\to 0} 2a+h \\ &= 2a. \end{aligned}$$

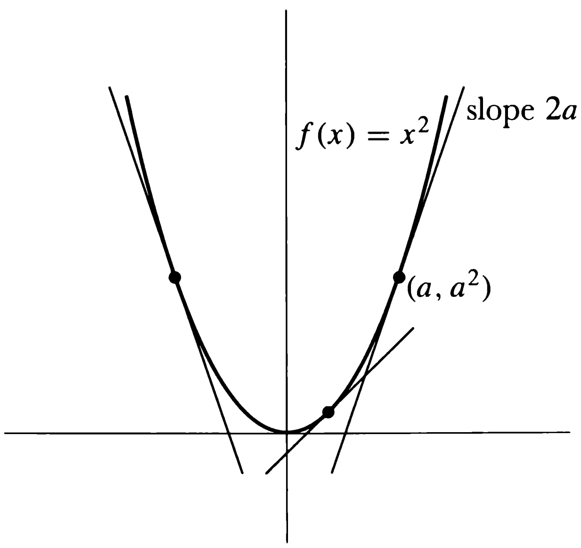


FIGURE 9

Some of the tangent lines to the graph of f are shown in Figure 9. In this picture each tangent line appears to intersect the graph only once, and this fact can be checked fairly easily: Since the tangent line through (a, a^2) has slope $2a$, it is the graph of the function

$$\begin{aligned} g(x) &= 2a(x-a)+a^2 \\ &= 2ax - a^2. \end{aligned}$$

Now, if the graphs of f and g intersect at a point $(x, f(x)) = (x, g(x))$, then

$$\begin{aligned} & x^2 = 2ax - a^2 \\ \text{or}\quad & x^2 - 2ax + a^2 = 0; \\ \text{so}\quad & (x-a)^2 = 0 \\ \text{or}\quad & x = a. \end{aligned}$$

In other words, (a, a^2) is the only point of intersection.

denotes the derivative of the function $f(x) = x^2$. Needless to say, the separate parts of the expression

$$\frac{df(x)}{dx}$$

are not supposed to have any sort of independent existence—the d's are *not* numbers, they *cannot* be canceled, and the entire expression is *not* the quotient of two other numbers "$df(x)$" and "dx." This notation is due to Leibniz (generally considered an independent co-discoverer of calculus, along with Newton), and is affectionately referred to as Leibnizian notation.* Although the notation $df(x)/dx$ seems very complicated, in concrete cases it may be shorter; after all, the symbol dx^2/dx is actually more concise than the phrase "the derivative of the function $f(x) = x^2$."

The following formulas state in standard Leibnizian notation all the information that we have found so far:

$$\begin{aligned} \frac{dc}{dx} &= 0, \\ \frac{d(ax+b)}{dx} &= a, \\ \frac{dx^2}{dx} &= 2x, \\ \frac{dx^3}{dx} &= 3x^2. \end{aligned}$$

Although the meaning of these formulas is clear enough, attempts at literal interpretation are hindered by the reasonable stricture that an equation should not contain a function on one side and a number on the other. For example, if the third equation is to be true, then either $df(x)/dx$ must denote $f'(x)$, rather than f', or else $2x$ must denote, not a number, but the function whose value at x is $2x$. It is really impossible to assert that one or the other of these alternatives is intended; in practice $df(x)/dx$ sometimes means f' and sometimes means $f'(x)$, while $2x$ may denote either a number or a function. Because of this ambiguity, most authors are reluctant to denote $f'(a)$ by

$$\frac{df(x)}{dx}(a);$$

instead $f'(a)$ is usually denoted by the barbaric, but unambiguous, symbol

$$\left.\frac{df(x)}{dx}\right|_{x=a}$$

*Leibniz was led to this symbol by his intuitive notion of the derivative, which he considered to be, not the limit of quotients $[f(x+h)-f(x)]/h$, but the "value" of this quotient when h is an "infinitely small" number. This "infinitely small" quantity was denoted by dx and the corresponding "infinitely small" difference $f(x+dx)-f(x)$ by $df(x)$. Although this point of view is impossible to reconcile with properties (P1)–(P13) of the real numbers, some people find this notion of the derivative congenial.

The function $f(x) = x^2$ happens to be quite special in this regard; usually a tangent line will intersect the graph more than once. Consider, for example, the function $f(x) = x^3$. In this case

$$\begin{aligned} f'(a) &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \\ &= \lim_{h\to 0} \frac{(a+h)^3 - a^3}{h} \\ &= \lim_{h\to 0} \frac{a^3 + 3a^2h + 3ah^2 + h^3 - a^3}{h} \\ &= \lim_{h\to 0} \frac{3a^2h + 3ah^2 + h^3}{h} \\ &= \lim_{h\to 0} 3a^2 + 3ah + h^2 \\ &= 3a^2. \end{aligned}$$

Thus the tangent line to the graph of f at (a, a^3) has slope $3a^2$. This means that the tangent line is the graph of

$$\begin{aligned} g(x) &= 3a^2(x-a) + a^3 \\ &= 3a^2x - 2a^3. \end{aligned}$$

The graphs of f and g intersect at the point $(x, f(x)) = (x, g(x))$ when

$$\begin{aligned} x^3 &= 3a^2x - 2a^3 \\ \text{or} \quad x^3 - 3a^2x + 2a^3 &= 0. \end{aligned}$$

This equation is easily solved if we remember that one solution of the equation has got to be $x = a$, so that $(x - a)$ is a factor of the left side; the other factor can then be found by dividing. We obtain

$$(x-a)(x^2 + ax - 2a^2) = 0.$$

It so happens that $x^2 + ax - 2a^2$ also has $x - a$ as a factor; we obtain finally

$$(x-a)(x-a)(x+2a) = 0.$$

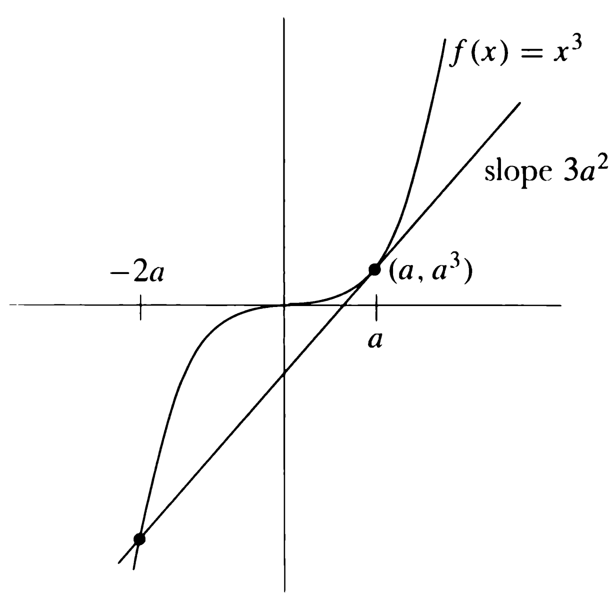


FIGURE 10

Thus, as illustrated in Figure 10, the tangent line through (a, a^3) also intersects the graph at the point $(-2a, -8a^3)$. These two points are always distinct, except when $a = 0$.

We have already found the derivative of sufficiently many functions to illustrate the classical, and still very popular, notation for derivatives. For a given function f, the derivative f' is often denoted by

$$\frac{df(x)}{dx}.$$

For example, the symbol

$$\frac{dx^2}{dx}$$

In addition to these difficulties, Leibnizian notation is associated with one more ambiguity. Although the notation dx^2/dx is absolutely standard, the notation $df(x)/dx$ is often replaced by df/dx. This, of course, is in conformity with the practice of confusing a function with its value at x. So strong is this tendency that functions are often indicated by a phrase like the following: "consider the function $y = x^2$." We will sometimes follow classical practice to the extent of using y as the name of a function, but we will nevertheless carefully distinguish between the function and its values—thus we will always say something like "consider the function (defined by) $y(x) = x^2$."

Despite the many ambiguities of Leibnizian notation, it is used almost exclusively in older mathematical writing, and is still used very frequently today. The staunchest opponents of Leibnizian notation admit that it will be around for quite some time, while its most ardent admirers would say that it will be around forever, and a good thing too! In any case, Leibnizian notation cannot be ignored completely.

The policy adopted in this book is to disallow Leibnizian notation within the text, but to include it in the Problems; several chapters contain a few (immediately recognizable) problems which are expressly designed to illustrate the vagaries of Leibnizian notation. Trusting that these problems will provide ample practice in this notation, we return to our basic task of examining some simple examples of derivatives.

The few functions examined so far have all been differentiable. To fully appreciate the significance of the derivative it is equally important to know some examples of functions which are *not* differentiable. The obvious candidates are the three functions first discussed in this chapter, and illustrated in Figure 1; if they turn out to be differentiable at 0 something has clearly gone wrong.

Consider first $f(x) = |x|$. In this case

$$\frac{f(0+h) - f(0)}{h} = \frac{|h|}{h}.$$

Now $|h|/h = 1$ for $h > 0$, and $|h|/h = -1$ for $h < 0$. This shows that

$$\lim_{h \to 0} \frac{f(h) - f(0)}{h} \text{ does not exist.}$$

In fact,

$$\lim_{h \to 0^+} \frac{f(h) - f(0)}{h} = 1$$

$$\text{and} \quad \lim_{h \to 0^-} \frac{f(h) - f(0)}{h} = -1.$$

(These two limits are sometimes called the **right-hand derivative** and the **left-hand derivative**, respectively, of f at 0.)

If $a \neq 0$, then $f'(a)$ does exist. In fact,

$$\begin{aligned} f'(x) &= 1 && \text{if } x > 0, \\ f'(x) &= -1 && \text{if } x < 0. \end{aligned}$$

The proof of this fact is left to you (it is easy if you remember the derivative of a linear function). The graphs of f and of f' are shown in Figure 11.

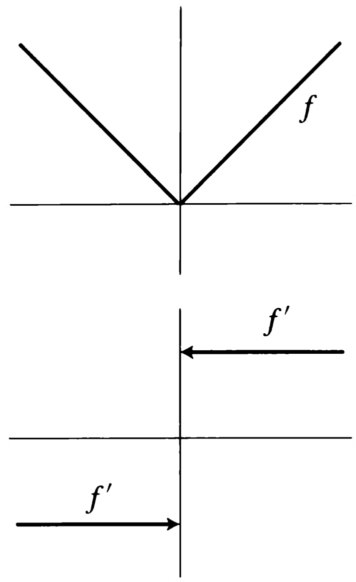


FIGURE 11

For the function

$$f(x) = \begin{cases} x^2, & x \leq 0 \\ x, & x \geq 0, \end{cases}$$

a similar difficulty arises in connection with $f'(0)$. We have

$$\frac{f(h) - f(0)}{h} = \begin{cases} \dfrac{h^2}{h} = h, & h < 0 \\ \dfrac{h}{h} = 1, & h > 0. \end{cases}$$

Therefore,

$$\lim_{h \to 0^-} \frac{f(h) - f(0)}{h} = 0,$$

$$\text{but} \quad \lim_{h \to 0^+} \frac{f(h) - f(0)}{h} = 1.$$

Thus $f'(0)$ does not exist; f is not differentiable at 0. Once again, however, $f'(x)$ exists for $x \neq 0$—it is easy to see that

$$f'(x) = \begin{cases} 2x, & x < 0 \\ 1, & x > 0. \end{cases}$$

The graphs of f and f' are shown in Figure 12.

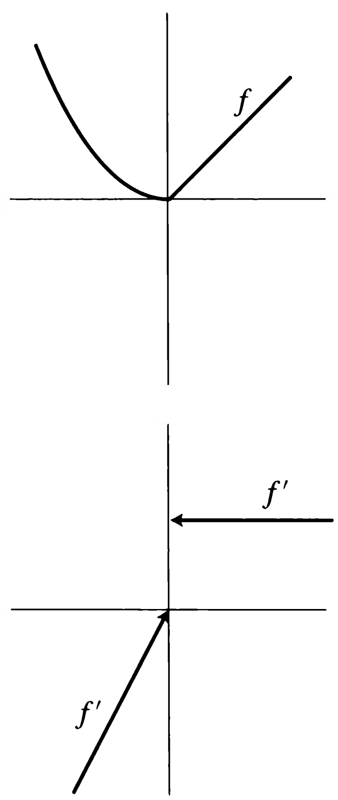


FIGURE 12

Even worse things happen for $f(x) = \sqrt{|x|}$. For this function

$$\frac{f(h) - f(0)}{h} = \begin{cases} \dfrac{\sqrt{h}}{h} = \dfrac{1}{\sqrt{h}}, & h > 0 \\ \dfrac{\sqrt{-h}}{h} = -\dfrac{1}{\sqrt{-h}}, & h < 0. \end{cases}$$

In this case the right-hand limit

$$\lim_{h \to 0^+} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^+} \frac{1}{\sqrt{h}}$$

does not exist; instead $1/\sqrt{h}$ becomes arbitrarily large as h approaches 0. And, what's more, $-1/\sqrt{-h}$ becomes arbitrarily large in absolute value, but *negative* (Figure 13).

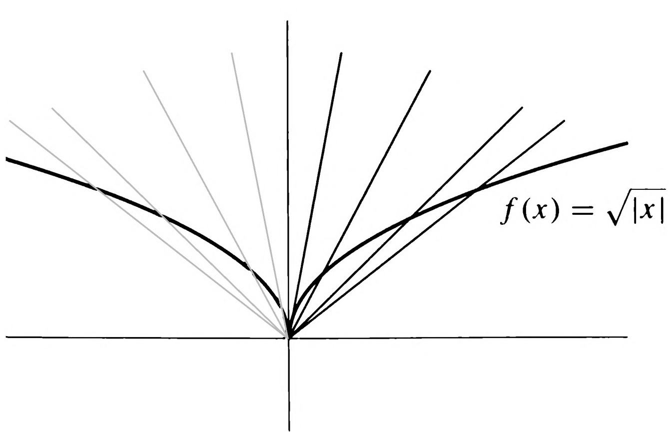


FIGURE 13

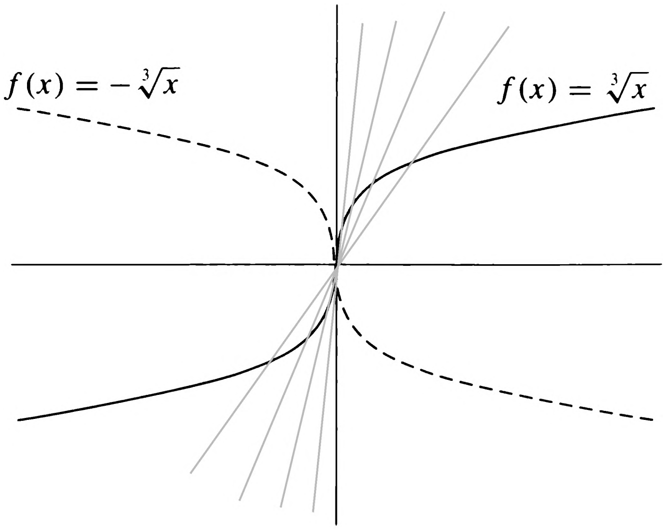


FIGURE 14

The function $f(x) = \sqrt[3]{x}$, although not differentiable at 0, is at least a little better behaved than this. The quotient

$$\frac{f(h) - f(0)}{h} = \frac{\sqrt[3]{h}}{h} = \frac{h^{1/3}}{h} = \frac{1}{h^{2/3}} = \frac{1}{(\sqrt[3]{h})^2}$$

simply becomes arbitrarily large as h goes to 0. Sometimes one says that f has an "infinite" derivative at 0. Geometrically this means that the graph of f has a "tangent line" which is parallel to the vertical axis (Figure 14). Of course, $f(x) = -\sqrt[3]{x}$ has the same geometric property, but one would say that f has a derivative of "negative infinity" at 0.

Remember that differentiability is supposed to be an improvement over mere continuity. This idea is supported by the many examples of functions which are continuous, but not differentiable; however, one important point remains to be noted:

THEOREM 1 If f is differentiable at a, then f is continuous at a.

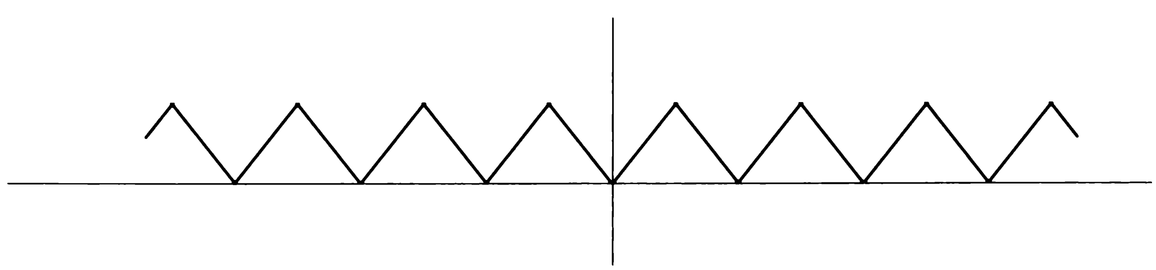
PROOF

$$\begin{aligned}\lim_{h\to 0} f(a+h) - f(a) &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \cdot h \\ &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \cdot \lim_{h\to 0} h \\ &= f'(a) \cdot 0 \\ &= 0.\end{aligned}$$

As we pointed out in Chapter 5, the equation $\lim_{h\to 0} f(a+h) - f(a) = 0$ is equivalent to $\lim_{x\to a} f(x) = f(a)$; thus f is continuous at a. ▮

It is very important to remember Theorem 1, and just as important to remember that the converse is not true. A differentiable function is continuous, but a continuous function need not be differentiable (keep in mind the function $f(x) = |x|$, and you will never forget which statement is true and which false).

The continuous functions examined so far have been differentiable at all points with at most one exception, but it is easy to give examples of continuous functions which are not differentiable at several points, even an infinite number (Figure 15). Actually, one can do much worse than this. There is a function which is *continuous*



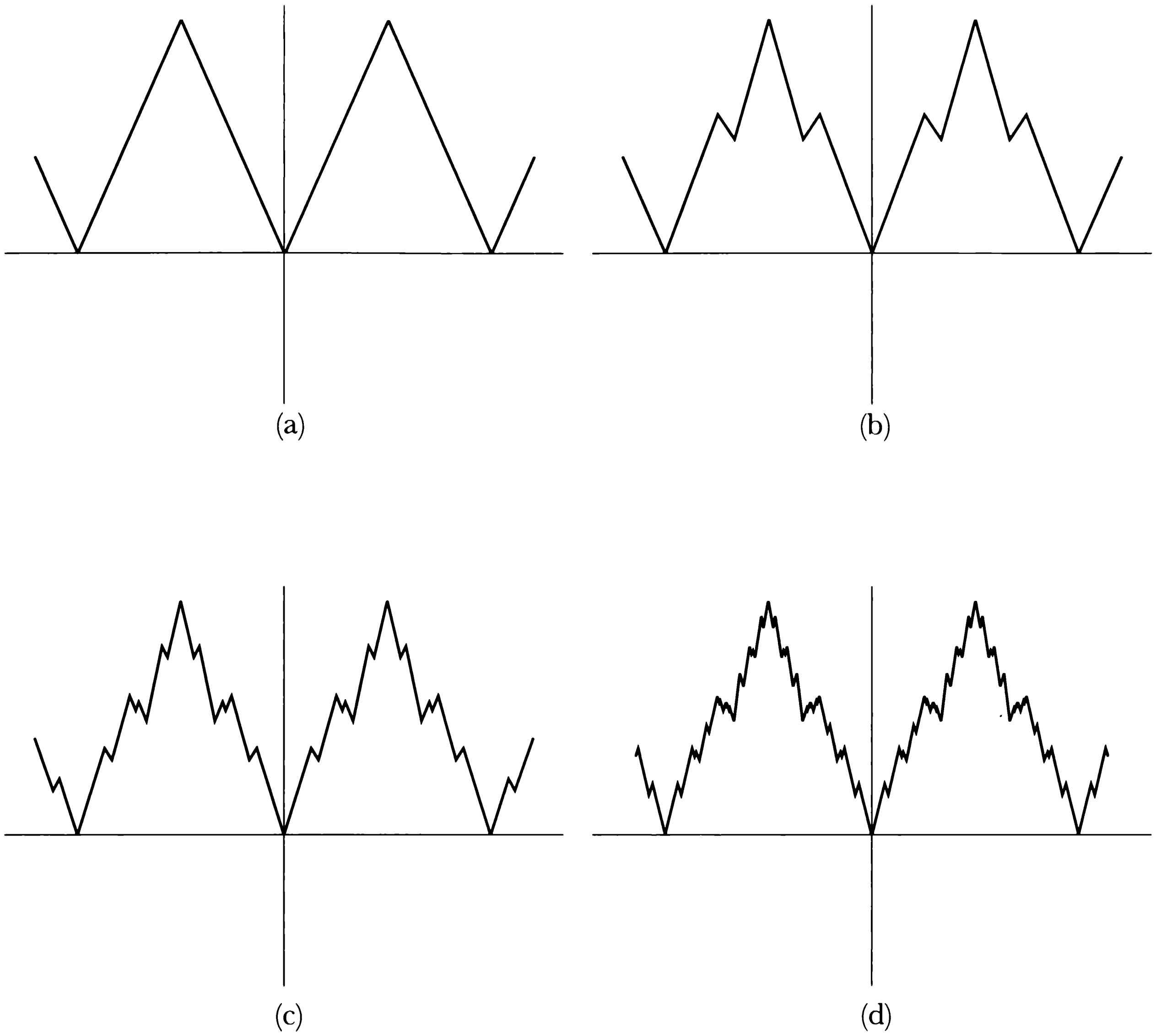


FIGURE 16

everywhere and *differentiable nowhere!* Unfortunately, the definition of this function will be inaccessible to us until Chapter 24, and I have been unable to persuade the artist to draw it (consider carefully what the graph should look like and you will sympathize with her point of view). It is possible to draw some rough approximations to the graph, however; several successively better approximations are shown in Figure 16.

Although such spectacular examples of nondifferentiability must be postponed, we can, with a little ingenuity, find a continuous function which is not differentiable at infinitely many points, *all of which are in* $[0, 1]$. One such function is illustrated in Figure 17. The reader is given the problem of defining it precisely; it is a straight line version of the function

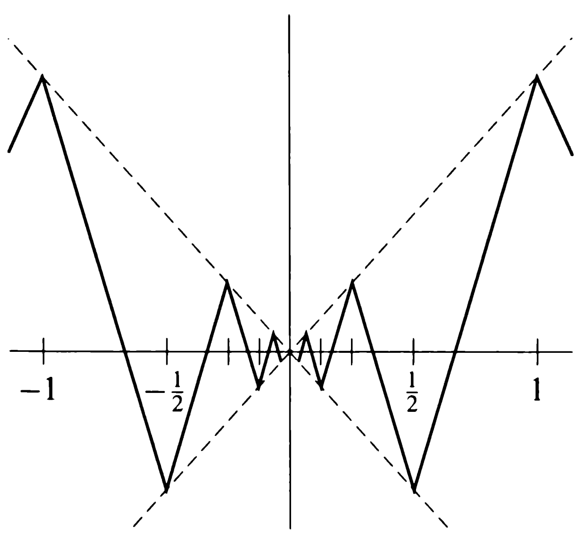


FIGURE 17

$$f(x) = \begin{cases} x \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

This particular function f is itself quite sensitive to the question of differentiability. Indeed, for $h \neq 0$ we have

$$\frac{f(h)-f(0)}{h} = \frac{h\sin\dfrac{1}{h} - 0}{h} = \sin\frac{1}{h}.$$

Long ago we proved that $\lim\limits_{h\to 0} \sin 1/h$ does not exist, so f is not differentiable at 0. Geometrically, one can see that a tangent line cannot exist, by noting that the secant line through $(0,0)$ and $(h, f(h))$ in Figure 18 can have any slope between -1 and 1, no matter how small we require h to be.

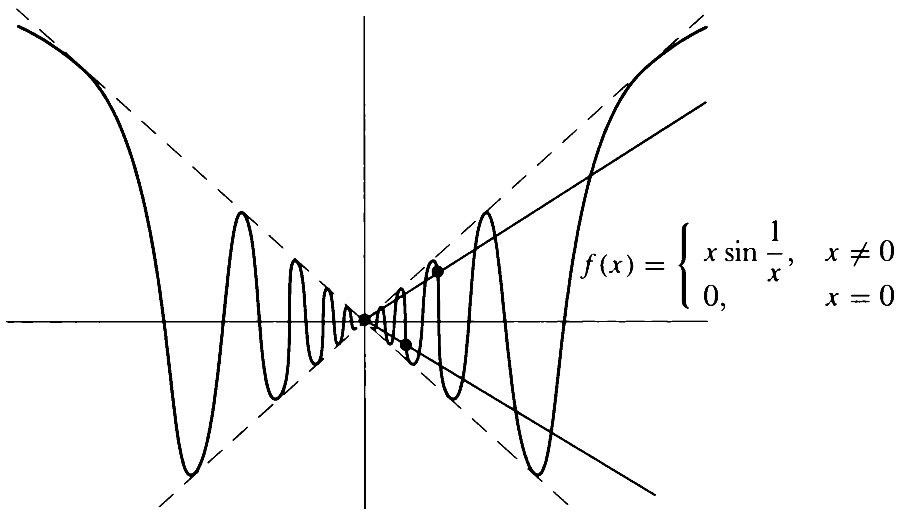


FIGURE 18

This finding represents something of a triumph; although continuous, the function f seems somehow quite unreasonable, and we can now enunciate one mathematically undesirable feature of this function—it is not differentiable at 0. Nevertheless, one should not become too enthusiastic about the criterion of differentiability. For example, the function

$$g(x) = \begin{cases} x^2\sin\dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is differentiable at 0; in fact $g'(0) = 0$:

$$\begin{aligned} \lim_{h\to 0} \frac{g(h)-g(0)}{h} &= \lim_{h\to 0} \frac{h^2\sin\dfrac{1}{h}}{h} \\ &= \lim_{h\to 0} h\sin\frac{1}{h} \\ &= 0. \end{aligned}$$

The tangent line to the graph of g at $(0,0)$ is therefore the horizontal axis (Figure 19).

This example suggests that we should seek even more restrictive conditions on a function than mere differentiability. We can actually use the derivative to formulate such conditions if we introduce another set of definitions, the last of this chapter.

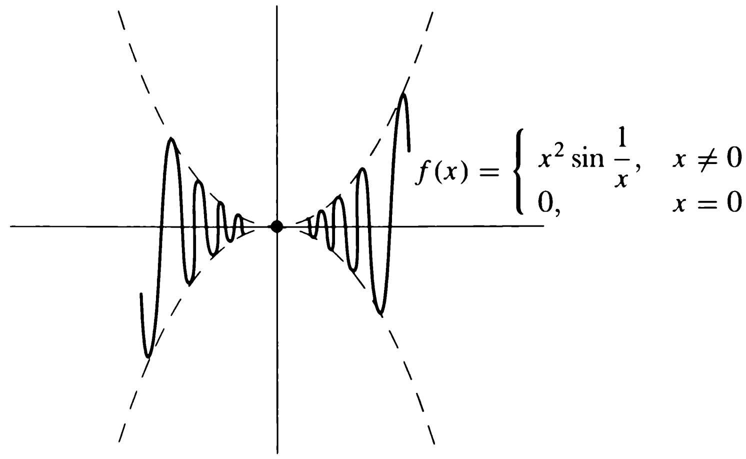


FIGURE 19

For any function f, we obtain, by taking the derivative, a new function f' (whose domain may be considerably smaller than that of f). The notion of differentiability can be applied to the function f', of course, yielding another function $(f')'$, whose domain consists of all points a such that f' is differentiable at a. The function $(f')'$ is usually written simply f'' and is called the **second derivative** of f. If $f''(a)$ exists, then f is said to be 2-times differentiable at a, and the number $f''(a)$ is called the **second derivative of f at a.**

In physics the second derivative is particularly important. If $s(t)$ is the position at time t of a particle moving along a straight line, then $s''(t)$ is called the **acceleration** at time t. Acceleration plays a special role in physics, because, as stated in Newton's laws of motion, the force on a particle is the product of its mass and its acceleration. Consequently you can feel the second derivative when you sit in an accelerating car.

There is no reason to stop at the second derivative—we can define $f''' = (f'')'$, $f'''' = (f''')'$, etc. This notation rapidly becomes unwieldy, so the following abbreviation is usually adopted (it is really a recursive definition):

$$\begin{aligned} f^{(1)} &= f', \\ f^{(k+1)} &= (f^{(k)})'. \end{aligned}$$

Thus

$$\begin{aligned} f^{(1)} &= f' \\ f^{(2)} &= f'' = (f')', \\ f^{(3)} &= f''' = (f'')', \\ f^{(4)} &= f'''' = (f''')', \\ &\text{etc.} \end{aligned}$$

The various functions $f^{(k)}$, for $k \geq 2$, are sometimes called **higher-order derivatives** of f.

Usually, we resort to the notation $f^{(k)}$ only for $k \geq 4$, but it is convenient to have $f^{(k)}$ defined for smaller k also. In fact, a reasonable definition can be made for $f^{(0)}$, namely,

$$f^{(0)} = f.$$

Leibnizian notation for higher-order derivatives should also be mentioned. The natural Leibnizian symbol for $f''(x)$, namely,

$$\frac{d\left(\dfrac{df(x)}{dx}\right)}{dx},$$

is abbreviated to

$$\frac{d^2 f(x)}{(dx)^2}, \qquad \text{or more frequently to} \qquad \frac{d^2 f(x)}{dx^2}.$$

Similar notation is used for $f^{(k)}(x)$.

The following example illustrates the notation $f^{(k)}$, and also shows, in one very simple case, how various higher-order derivatives are related to the original function. Let $f(x) = x^2$. Then, as we have already checked,

$$\begin{aligned} f'(x) &= 2x, \\ f''(x) &= 2, \\ f'''(x) &= 0, \\ f^{(k)}(x) &= 0, \qquad \text{if } k \geq 3. \end{aligned}$$

Figure 20 shows the function f, together with its various derivatives.

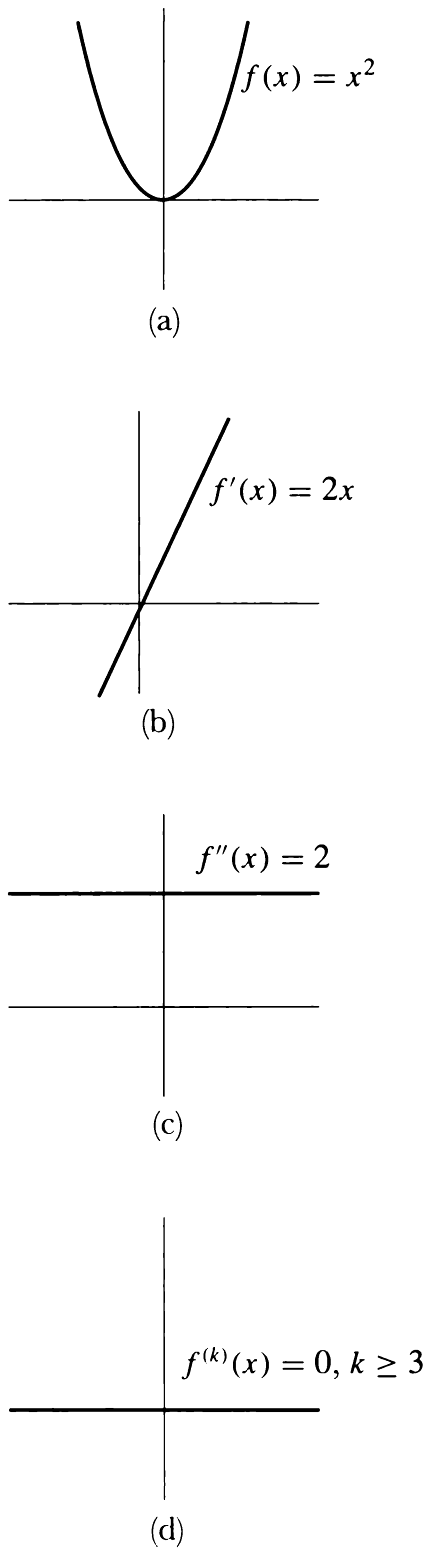


FIGURE 20

A rather more illuminating example is presented by the following function, whose graph is shown in Figure 21(a):

$$f(x) = \begin{cases} x^2, & x \geq 0 \\ -x^2, & x \leq 0. \end{cases}$$

It is easy to see that

$$\begin{aligned} f'(a) &= 2a \qquad \text{if } a > 0, \\ f'(a) &= -2a \qquad \text{if } a < 0. \end{aligned}$$

Moreover,

$$\begin{aligned} f'(0) &= \lim_{h \to 0} \frac{f(h) - f(0)}{h} \\ &= \lim_{h \to 0} \frac{f(h)}{h}. \end{aligned}$$

Now

$$\lim_{h \to 0^+} \frac{f(h)}{h} = \lim_{h \to 0^+} \frac{h^2}{h} = 0$$

$$\text{and} \quad \lim_{h \to 0^-} \frac{f(h)}{h} = \lim_{h \to 0^-} \frac{-h^2}{h} = 0,$$

so

$$f'(0) = \lim_{h \to 0} \frac{f(h)}{h} = 0.$$

This information can all be summarized as follows:

$$f'(x) = 2|x|.$$

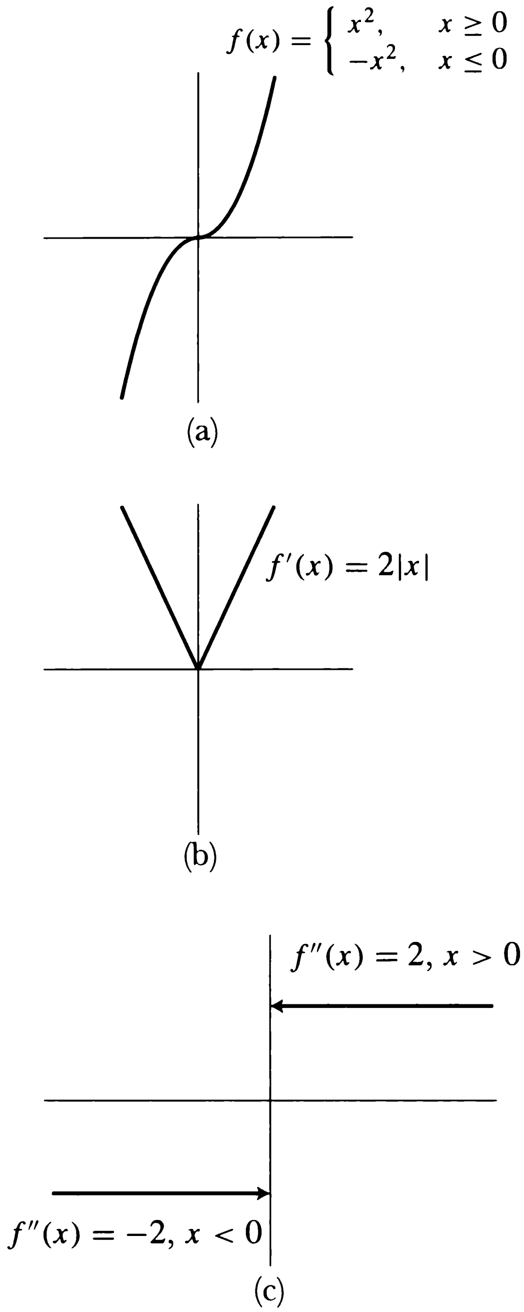


FIGURE 21

It follows that $f''(0)$ does not exist! Existence of the second derivative is thus a rather strong criterion for a function to satisfy. Even a "smooth looking" function like f reveals some irregularity when examined with the second derivative. This suggests that the irregular behavior of the function

$$g(x) = \begin{cases} x^2 \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

might also be revealed by the second derivative. At the moment we know that $g'(0) = 0$, but we do not know $g'(a)$ for any $a \neq 0$, so it is hopeless to begin computing $g''(0)$. We will return to this question at the end of the next chapter, after we have perfected the technique of finding derivatives.

PROBLEMS

1. (a) Prove, working directly from the definition, that if $f(x) = 1/x$, then $f'(a) = -1/a^2$, for $a \neq 0$.
(b) Prove that the tangent line to the graph of f at $(a, 1/a)$ does not intersect the graph of f, except at $(a, 1/a)$.

2. (a) Prove that if $f(x) = 1/x^2$, then $f'(a) = -2/a^3$ for $a \neq 0$.
(b) Prove that the tangent line to f at $(a, 1/a^2)$ intersects f at one other point, which lies on the opposite side of the vertical axis.

3. Prove that if $f(x) = \sqrt{x}$, then $f'(a) = 1/(2\sqrt{a})$, for $a > 0$. (The expression you obtain for $[f(a+h) - f(a)]/h$ will require some algebraic face lifting, but the answer should suggest the right trick.)

4. For each natural number n, let $S_n(x) = x^n$. Remembering that $S_1{}'(x) = 1$, $S_2{}'(x) = 2x$, and $S_3{}'(x) = 3x^2$, conjecture a formula for $S_n{}'(x)$. Prove your conjecture. (The expression $(x + h)^n$ may be expanded by the binomial theorem.)

5. Find f' if $f(x) = [x]$.

6. Prove, starting from the definition (and drawing a picture to illustrate):

(a) if $g(x) = f(x) + c$, then $g'(x) = f'(x)$;
(b) if $g(x) = cf(x)$, then $g'(x) = cf'(x)$.

7. Suppose that $f(x) = x^3$.

(a) What is $f'(9)$, $f'(25)$, $f'(36)$?
(b) What is $f'(3^2)$, $f'(5^2)$, $f'(6^2)$?
(c) What is $f'(a^2)$, $f'(x^2)$?

If you do not find this problem silly, you are missing a very important point: $f'(x^2)$ means the derivative of f at the number which we happen to be calling x^2; it is *not* the derivative at x of the function $g(x) = f(x^2)$. Just to drive the point home:

(d) For $f(x) = x^3$, compare $f'(x^2)$ and $g'(x)$ where $g(x) = f(x^2)$.

8. (a) Suppose $g(x) = f(x+c)$. Prove (starting from the definition) that $g'(x) = f'(x+c)$. Draw a picture to illustrate this. To do this problem you must write out the definitions of $g'(x)$ and $f'(x+c)$ correctly. The purpose of Problem 7 was to convince you that although this problem is easy, it is not an utter triviality, and there is something to prove: you cannot simply put prime marks into the equation $g(x) = f(x+c)$. To emphasize this point:

(b) Prove that if $g(x) = f(cx)$, then $g'(x) = c \cdot f'(cx)$. Try to see pictorially why this should be true, also.

(c) Suppose that f is differentiable and periodic, with period a (i.e., $f(x+a) = f(x)$ for all x). Prove that f' is also periodic.

9. Find $f'(x)$ and also $f'(x+3)$ in the following cases. Be very methodical, or you will surely slip up somewhere. Consult the answers (after you do the problem, naturally).

(i) $f(x) = (x+3)^5$.
(ii) $f(x+3) = x^5$.
(iii) $f(x+3) = (x+5)^7$.

10. Find $f'(x)$ if $f(x) = g(t+x)$, and if $f(t) = g(t+x)$. The answers will *not* be the same.

11. (a) Prove that Galileo was wrong: if a body falls a distance $s(t)$ in t seconds, and s' is proportional to s, then s cannot be a function of the form $s(t) = ct^2$.

(b) Prove that the following facts are true about s if $s(t) = (a/2)t^2$ (the first fact will show why we switched from c to $a/2$):

(i) $s''(t) = a$ (the acceleration is constant).
(ii) $[s'(t)]^2 = 2as(t)$.

(c) If s is measured in feet, the value of a is 32. How many seconds do you have to get out of the way of a chandelier which falls from a 400-foot ceiling? If you don't make it, how fast will the chandelier be going when it hits you? Where was the chandelier when it was moving with half that speed?

12. Imagine a road on which the speed limit is specified at every single point. In other words, there is a certain function L such that the speed limit x miles from the beginning of the road is $L(x)$. Two cars, A and B, are driving along this road; car A's position at time t is $a(t)$, and car B's is $b(t)$.

(a) What equation expresses the fact that car A always travels at the speed limit? (The answer is *not* $a'(t) = L(t)$.)

(b) Suppose that A always goes at the speed limit, and that B's position at time t is A's position at time $t-1$. Show that B is also going at the speed limit at all times.

(c) Suppose, instead, that B always stays a constant distance behind A. Under what conditions will B still always travel at the speed limit?

13. Suppose that $f(a) = g(a)$ and that the left-hand derivative of f at a equals the right-hand derivative of g at a. Define $h(x) = f(x)$ for $x \leq a$, and $h(x) = g(x)$ for $x \geq a$. Prove that h is differentiable at a.

14. Let $f(x) = x^2$ if x is rational, and $f(x) = 0$ if x is irrational. Prove that f is differentiable at 0. (Don't be scared by this function. Just write out the definition of $f'(0)$.)

15. (a) Let f be a function such at $|f(x)| \leq x^2$ for all x. Prove that f is differentiable at 0. (If you have done Problem 14 you should be able to do this.)
(b) This result can be generalized if x^2 is replaced by $|g(x)|$, where g has what property?

16. Let $\alpha > 1$. If f satisfies $|f(x)| \leq |x|^\alpha$, prove that f is differentiable at 0.

17. Let $0 < \beta < 1$. Prove that if f satisfies $|f(x)| \geq |x|^\beta$ and $f(0) = 0$, then f is not differentiable at 0.

***18.** Let $f(x) = 0$ for irrational x, and $1/q$ for $x = p/q$ in lowest terms. Prove that f is not differentiable at a for any a. Hint: It obviously suffices to prove this for irrational a. Why? If $a = m.a_1a_2a_3\ldots$ is the decimal expansion of a, consider $[f(a+h) - f(a)]/h$ for h rational, and also for

$$h = -0.00\ldots 0a_{n+1}a_{n+2}\ldots.$$

19. (a) Suppose that $f(a) = g(a) = h(a)$, that $f(x) \leq g(x) \leq h(x)$ for all x, and that $f'(a) = h'(a)$. Prove that g is differentiable at a, and that $f'(a) = g'(a) = h'(a)$. (Begin with the definition of $g'(a)$.)
(b) Show that the conclusion does not follow if we omit the hypothesis $f(a) = g(a) = h(a)$.

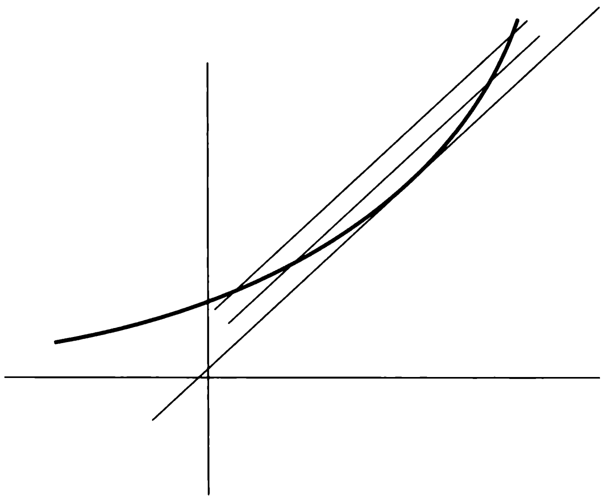


FIGURE 22

20. Let f be any polynomial function; we will see in the next chapter that f is differentiable. The tangent line to f at $(a, f(a))$ is the graph of $g(x) = f'(a)(x-a) + f(a)$. Thus $f(x) - g(x)$ is the polynomial function $d(x) = f(x) - f'(a)(x-a) - f(a)$. We have already seen that if $f(x) = x^2$, then $d(x) = (x-a)^2$, and if $f(x) = x^3$, then $d(x) = (x-a)^2(x+2a)$.

(a) Find $d(x)$ when $f(x) = x^4$, and show that it is divisible by $(x-a)^2$.
(b) There certainly seems to be some evidence that $d(x)$ is always divisible by $(x-a)^2$. Figure 22 provides an intuitive argument: usually, lines parallel to the tangent line will intersect the graph at two points; the tangent line intersects the graph only once near the point, so the intersection should be a "double intersection." To give a rigorous proof, first note that

$$\frac{d(x)}{x-a} = \frac{f(x) - f(a)}{x-a} - f'(a).$$

Now answer the following questions. Why is $f(x) - f(a)$ divisible by $(x-a)$? Why is there a polynomial function h such that $h(x) =$

$d(x)/(x-a)$ for $x \neq a$? Why is $\lim_{x\to a} h(x) = 0$? Why is $h(a) = 0$? Why does this solve the problem?

21. (a) Show that $f'(a) = \lim_{x\to a}[f(x) - f(a)]/(x-a)$. (Nothing deep here.)

(b) Show that derivatives are a "local property": if $f(x) = g(x)$ for all x in some open interval containing a, then $f'(a) = g'(a)$. (This means that in computing $f'(a)$, you can ignore $f(x)$ for any particular $x \neq a$. Of course you can't ignore $f(x)$ for all such x at once!)

22. (a) Suppose that f is differentiable at x. Prove that

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x-h)}{2h}.$$

Hint: Remember an old algebraic trick—a number is not changed if the same quantity is added to and then subtracted from it.

**(b) Prove, more generally, that

$$f'(x) = \lim_{h,k\to 0^+} \frac{f(x+h) - f(x-k)}{h+k}.$$

Although we haven't encountered something like $\lim_{h,k\to 0}$ before, its meaning should be clear, and you should be able to make an appropriate ε-δ definition. The important thing here is that we actually have $\lim_{h,k\to 0^+}$, so that we are only considering positive h and k.

23. Prove that if f is even, then $f'(x) = -f'(-x)$. (In order to minimize confusion, let $g(x) = f(-x)$; find $g'(x)$ and *then* remember what other thing g is.) Draw a picture!

24. Prove that if f is odd, then $f'(x) = f'(-x)$. Once again, draw a picture.

25. Problems 23 and 24 say that f' is even if f is odd, and odd if f is even. What can therefore be said about $f^{(k)}$?

26. Find $f''(x)$ if

(i) $f(x) = x^3$.
(ii) $f(x) = x^5$.
(iii) $f'(x) = x^4$.
(iv) $f(x+3) = x^5$.

27. If $S_n(x) = x^n$, and $0 \le k \le n$, prove that

$$\begin{aligned} S_n{}^{(k)}(x) &= \frac{n!}{(n-k)!}x^{n-k} \\ &= k!\binom{n}{k}x^{n-k}. \end{aligned}$$

28. (a) Find $f'(x)$ if $f(x) = |x|^3$. Find $f''(x)$. Does $f'''(x)$ exist for all x?

(b) Analyze f similarly if $f(x) = x^4$ for $x \ge 0$ and $f(x) = -x^4$ for $x \le 0$.

29. Let $f(x) = x^n$ for $x \geq 0$ and let $f(x) = 0$ for $x \leq 0$. Prove that $f^{(n-1)}$ exists (and find a formula for it), but that $f^{(n)}(0)$ does not exist.

30. Interpret the following specimens of Leibnizian notation; each is a restatement of some fact occurring in a previous problem.

(i) $\dfrac{dx^n}{dx} = nx^{n-1}$

(ii) $\dfrac{dz}{dy} = -\dfrac{1}{y^2}$ if $z = \dfrac{1}{y}$.

(iii) $\dfrac{d[f(x)+c]}{dx} = \dfrac{df(x)}{dx}$.

(iv) $\dfrac{d[cf(x)]}{dx} = c\dfrac{df(x)}{dx}$.

(v) $\dfrac{dz}{dx} = \dfrac{dy}{dx}$ if $z = y + c$.

(vi) $\left.\dfrac{dx^3}{dx}\right|_{x=a^2} = 3a^4$.

(vii) $\left.\dfrac{df(x+a)}{dx}\right|_{x=b} = \left.\dfrac{df(x)}{dx}\right|_{x=b+a}$.

(viii) $\left.\dfrac{df(cx)}{dx}\right|_{x=b} = c \cdot \left.\dfrac{df(x)}{dx}\right|_{x=cb}$.

(ix) $\dfrac{df(cx)}{dx} = c \cdot \left.\dfrac{df(y)}{dy}\right|_{y=cx}$.

(x) $\dfrac{d^k x^n}{dx^k} = k!\dbinom{n}{k}x^{n-k}$.