Lecture I (Chapter 1)

Properties of numbers that all operations must obey:

(P1) If a, b, and c are any numbers, then (associative property):

$$a + (b + c) = (a + b) + c$$

(P2) If a is any number, then (additive identity):

$$a + 0 = 0 + a = a$$

(P3) For every number a, there is a number $-a$ such that (additive inverse):

$$a + (-a) = -a + a = 0$$

Example: Proof: $a + x = a$ when $x = 0$

$$\begin{aligned} \text{(P1)} \ -a + a + x &= a + (-a) \\ \text{(P1)} \ (-a + a) + x &= 0 \\ 0 + x &= 0 \\ \text{(P2)} \ 0 + x &= 0, \quad x = 0 \end{aligned}$$

Note: This can be used to solve any equation.

(P4) If a and b are any numbers, then (commutative property):

$$a + b = b + a$$

These are 4 properties for addition:

(P5) If a, b, and c are any numbers, then (associative property):

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

(P6) If a is any number, then (multiplicative identity):

$$a \cdot 1 = 1 \cdot a = a$$

(P7) For every number $a \neq 0$, there is a number a^{-1} such that (multiplicative inverse):

$$a \cdot a^{-1} = a^{-1} \cdot a = 1$$

Example: If $a \cdot b = a \cdot c$, prove that $b = c \quad (a \neq 0)$

$$\begin{aligned} \text{(P7)} \ a^{-1} \cdot (a \cdot b) &= a^{-1} \cdot (a \cdot c) \\ \text{(P5)} \ (a^{-1} \cdot a) \cdot b &= (a^{-1} \cdot a) \cdot c \\ 1 \cdot b &= 1 \cdot c \\ \text{(P6)} \ b &= c \end{aligned}$$

Example: If $a \cdot b = 0$, prove that $a = 0$ or $b = 0$

Assume $a \neq 0$

Proof: $a^{-1} \cdot (a \cdot b) = 0 \cdot a^{-1}$ will be proved next

Hence, $a^{-1} \cdot a \cdot b = 0$

$1 \cdot b = 0$

$b = 0$

Note: This is used in solving the factorization of quadratic equations.

(P8) If a and b are any numbers, then (commutative property):

$$a \cdot b = b \cdot a$$

Lecture 2

(P9) If a, b, and c are any numbers, then (distributive law):

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

Example: Prove: if $a - b = -a$ when $a = b$

$$\begin{aligned} a - b + b &= b - a + b \\ a &= b + b - a \\ a + a &= b + b - a + a \\ a \cdot 1 + a \cdot 1 &= b \cdot 1 + b \cdot 1 \\ a \cdot (1 + 1) &= b \cdot (1 + 1) \\ a \cdot 2 \cdot 2^{-1} &= b \cdot 2 \cdot 2^{-1} \\ a \cdot 1 &= b \cdot 1 \\ a &= b \end{aligned}$$

Example: Prove $a \cdot 0 = 0$

$$a \cdot 0 + a \cdot 0 = a \cdot (0 + 0)$$
$$a \cdot 0 + a \cdot 0 = a \cdot 0$$
$$a \cdot 0 + a \cdot 0 - a \cdot 0 = a \cdot 0 - a \cdot 0$$
$$a \cdot 0 = 0$$

Example: Prove $(-a) \cdot b = -(a \cdot b)$

$$(-a) \cdot b + a \cdot b = b \cdot ((-a) + a)$$
$$= 0 \cdot b$$
$$= 0$$
$$(-a) \cdot b + a \cdot b - a \cdot b = 0 - (a \cdot b)$$
$$(-a) \cdot b = -(a \cdot b)$$

Example: Prove $(-a) \cdot (-b) = (a \cdot b)$

$$
\begin{aligned}
&(-a) \cdot (-b) + (-(a \cdot b)) = 0 \\
&-(a \cdot b) = (-(-a) \cdot b) = (-a) \cdot (-b) + (-a) \cdot b \\
&= (-a) \cdot ((-b) + b) \\
&= (-a) \cdot 0 \\
&= 0 \\
&(-a) \cdot (-b) + (-(a \cdot b) + a \cdot b = 0 + (a \cdot b) \\
&(-a) \cdot (-b) = (a \cdot b)
\end{aligned}
$$

Example: Prove that $(x-1) \cdot (x-2) = 0$

$$x^2 - 3x + 2 = 0$$

$$
\begin{aligned}
(x-1)(x-2) &= x(x-2) + (-1)(x-2) \\
&= x \cdot x + (-2) \cdot x + (-1) \cdot x + (-1) \cdot (-2) \\
&= x^2 + x(-2 + -1) + 2 \\
&= x^2 + (-3)x + 2 \\
&= x^2 - 3x + 2
\end{aligned}
$$

Note: This strategy is used in the multiplication of any numbers.

Lecture 3

(P10) For every number a, only one of the following holds (Trichotomy Law):

1. $a = 0$
2. a is in P
3. $-a$ is in P

(P11) If a and b are in P, then (closure under addition):

$$a + b \text{ is in } P$$

(P12) If a and b are in P, then (closure under multiplication):

$$a \cdot b \text{ is in } P$$

Definition: $a > b \longleftrightarrow a - b > 0$

Example: If $a < b$ and $b < c$, prove $a < c$

$$
\begin{aligned}
b - a &> 0 \\
c - b &> 0 \\
(b - a) + (c - b) &> 0 \\
c - a &> 0 \\
c &> a
\end{aligned}
$$

Definition: Absolute Value

$$|a| = \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a < 0 \end{cases}$$

Theorem 1: $|a+b| \leq |a| + |b|$ (Triangle Inequality)

Proof:

i) $a > 0, b > 0$

$$a + b = a + b$$

ii) $a < 0, b < 0$

$$-(a+b) = (-a) + (-b)$$

iii) $a > 0, b < 0$

- $*$ Assuming $(a+b)$ is positive

$$\begin{aligned} a + b &< a - b \\ b &< -b \end{aligned}$$

- $*$ Assuming $(a+b)$ is negative

$$\begin{aligned} -(a+b) &= a - b \\ -a - b &= a - b \\ -a &< a \end{aligned}$$

Note: Another proof for the same fact can be used with the definition $|a| = \sqrt{a^2}$