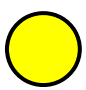
Binomial Queues

- Binomial queues support all three priority queue operations Merge, Insert and DeleteMin in O(log N) time
- Idea: Maintain a collection of heap-ordered trees
  - *Forest of binomial trees*
- Recursive Definition of Binomial Tree (based on height k):
  - Only one binomial tree for a given height
  - Binomial tree of height 0 = single root node
  - Binomial tree of height k = B_k = Attach B_{k-1} to root of another B_{k-1}

Building a Binomial Tree

- To construct a binomial tree B_k of height k:
  1. Take the binomial tree B_{k-1} of height k-1
  2. Place another copy of B_{k-1} one level below the first
  3. Attach the root nodes
- Binomial tree of height k has exactly 2^k nodes (by induction)

B_0 B_1 B_2 B_3



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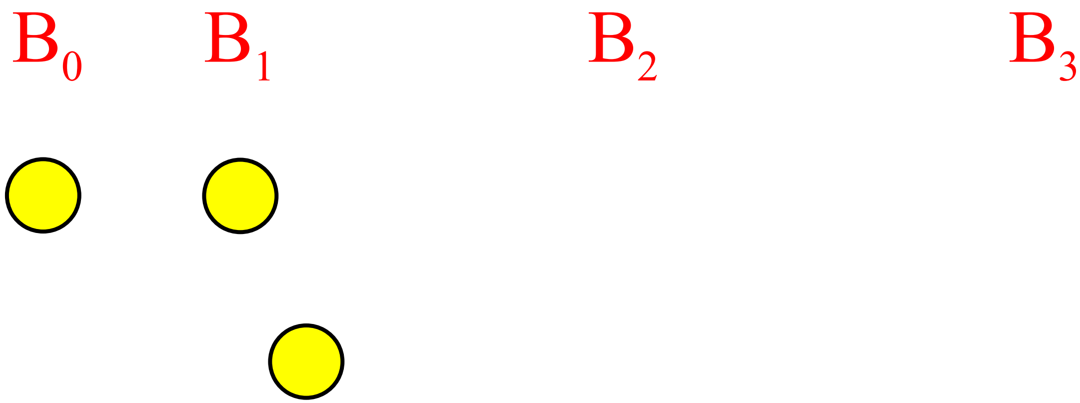

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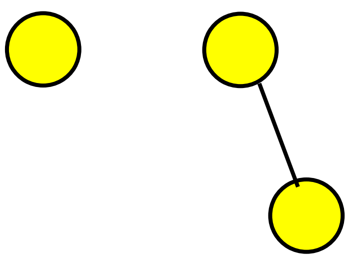

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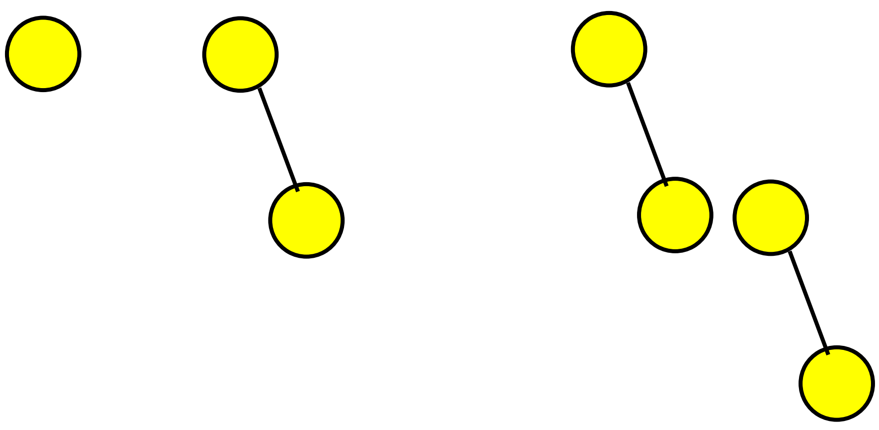

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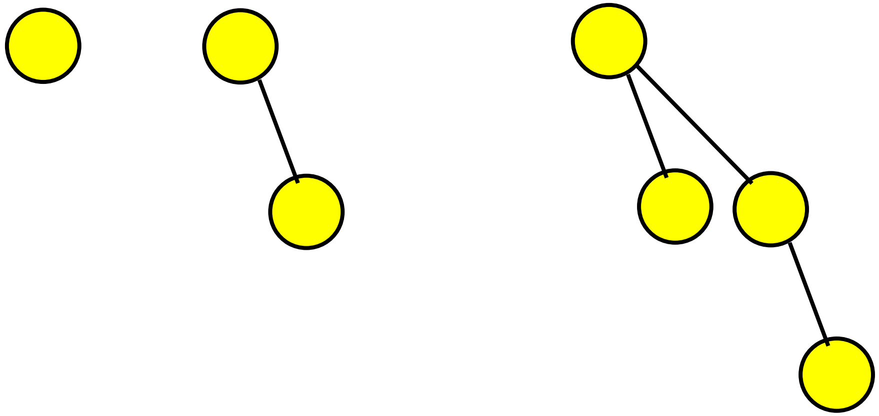

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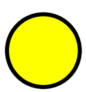
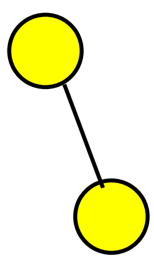
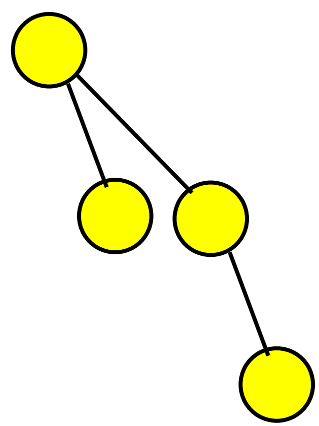


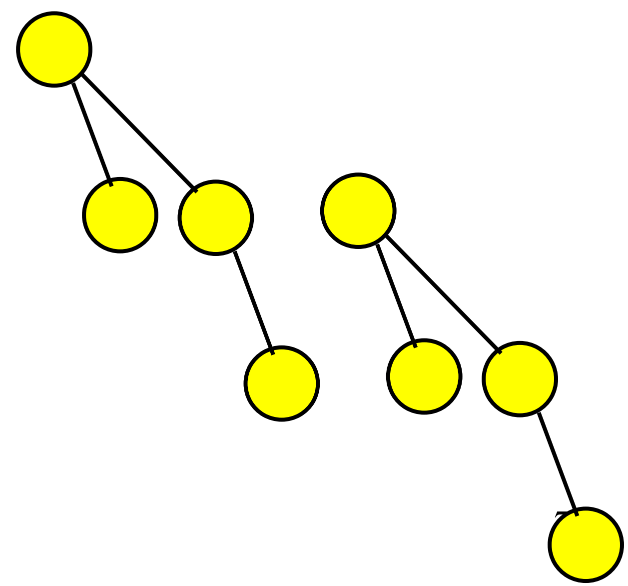
B_1



B_2



B_3



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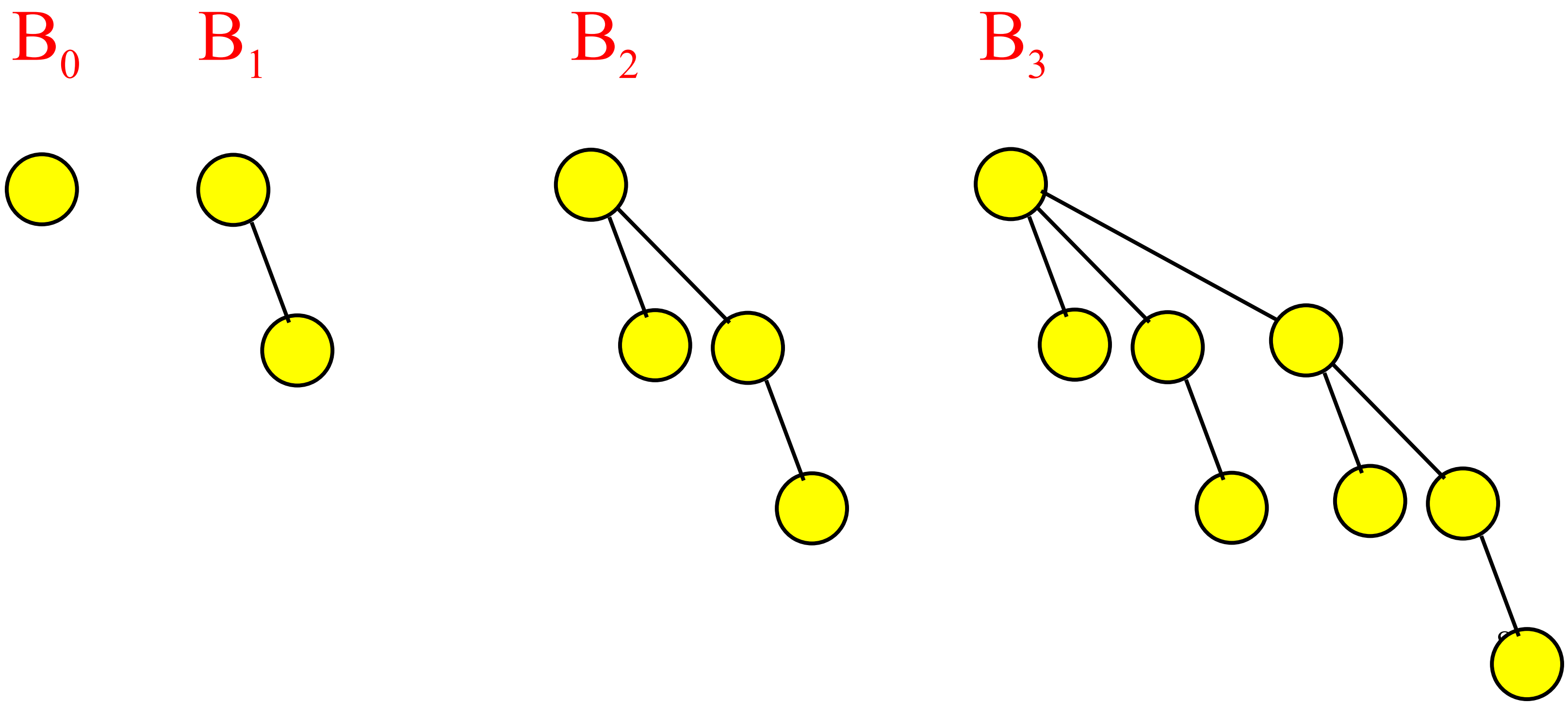
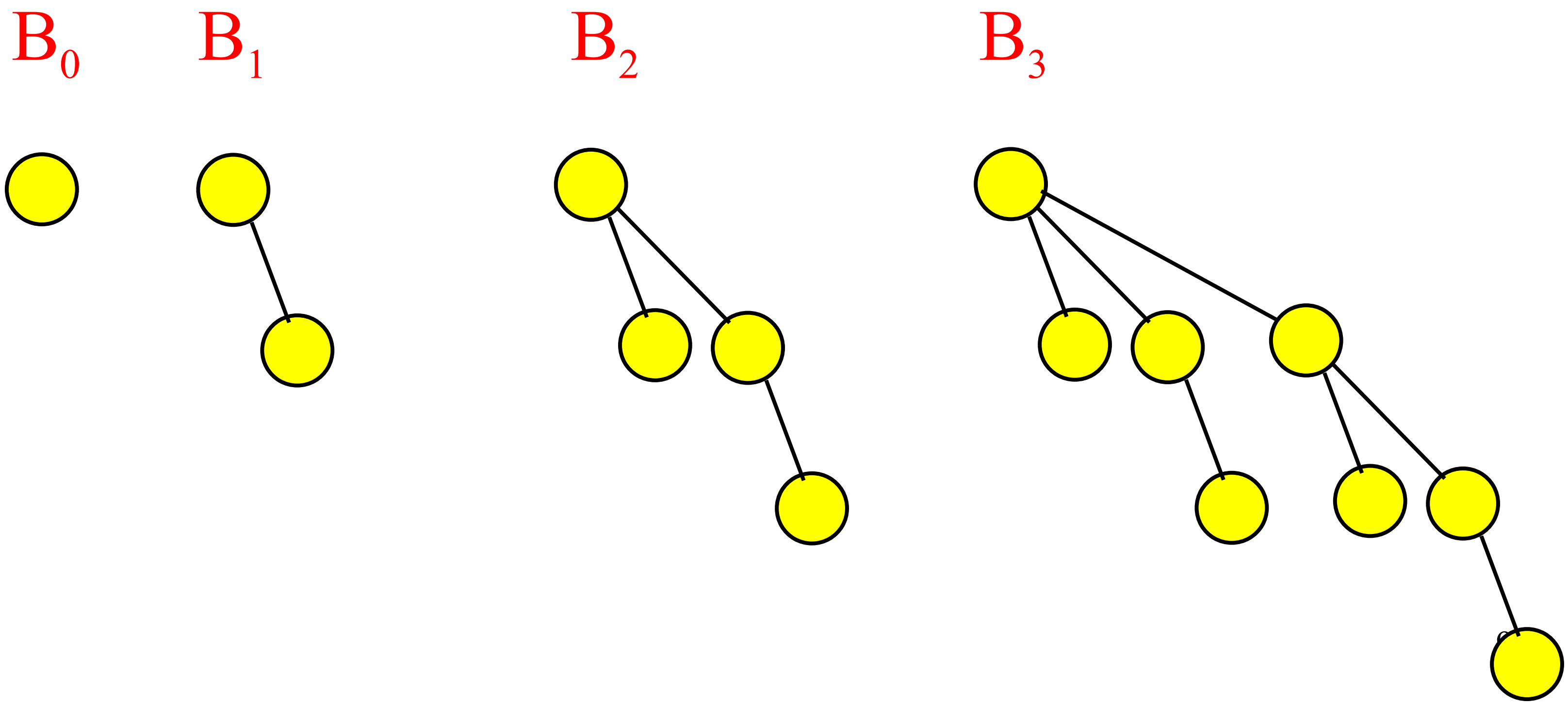


Why Binomial?

- Why are these trees called binomial?
  - Hint: how many nodes at depth d?



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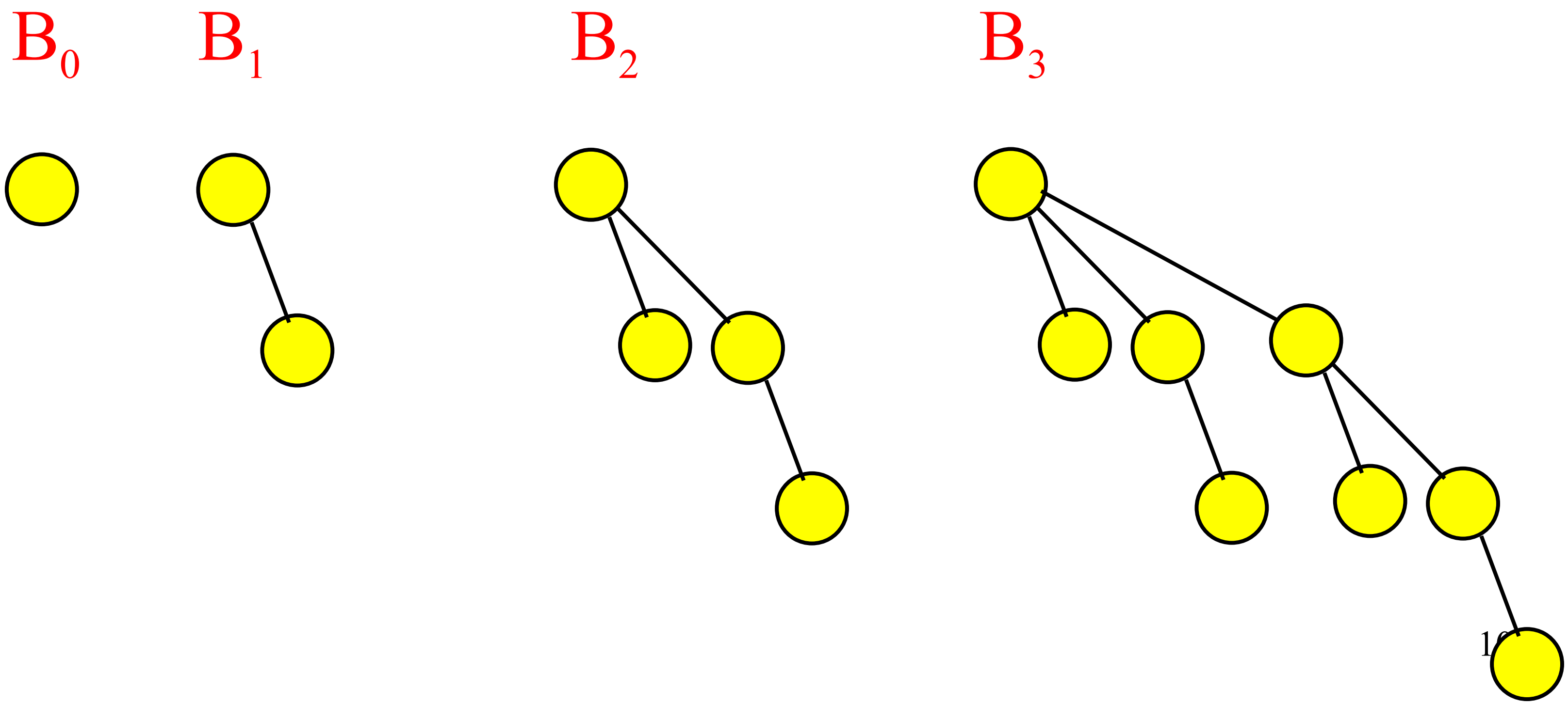
Number of nodes at different depths d for B_k =

[1], [1 1], [1 2 1], [1 3 3 1], …

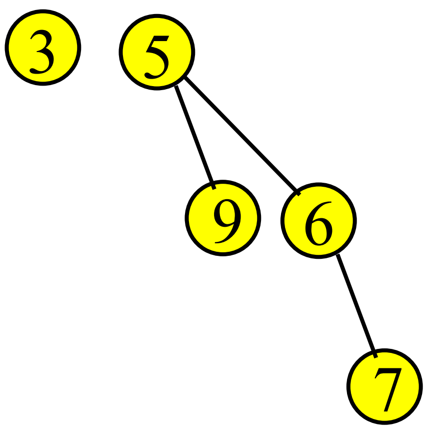
Binomial coefficients of $(a + b)^k = k!/((k-d)!d!)$



Definition of Binomial Queues

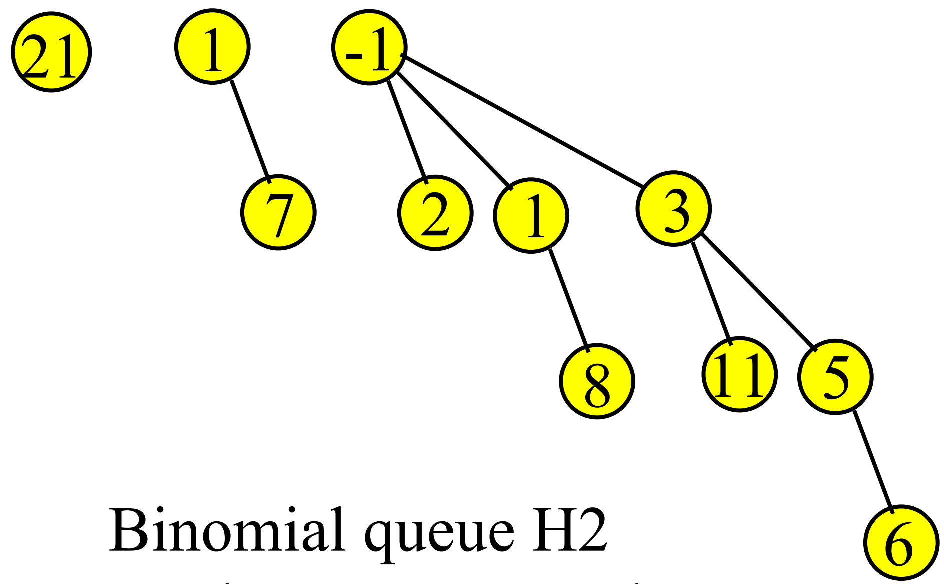
Binomial Queue = “forest” of heap-ordered binomial trees

B_0 B_2



Binomial queue H1
5 elements = 101 base 2
→ $B_2 B_0$

B_0 B_1 B_3



Binomial queue H2
11 elements = 1011 base 2
→ $B_3 B_1 B_0$

Binomial Queue Properties

Suppose you are given a binomial queue of N nodes

1. There is a unique set of binomial trees for N nodes
2. What is the maximum number of trees that can be in an N-node queue?
   - 1 node → 1 tree B_0; 2 nodes → 1 tree B_1; 3 nodes → 2 trees B_0 and B_1; 7 nodes → 3 trees B_0, B_1 and B_2 …
   - Trees $B_0, B_1, \ldots, B_k$ can store up to $2^0 + 2^1 + \ldots + 2^k = 2^{k+1} - 1$ nodes = N.
   - Maximum is when all trees are used. So, solve for (k+1).
   - Number of trees is $\leq \log(N+1) = O(\log N)$

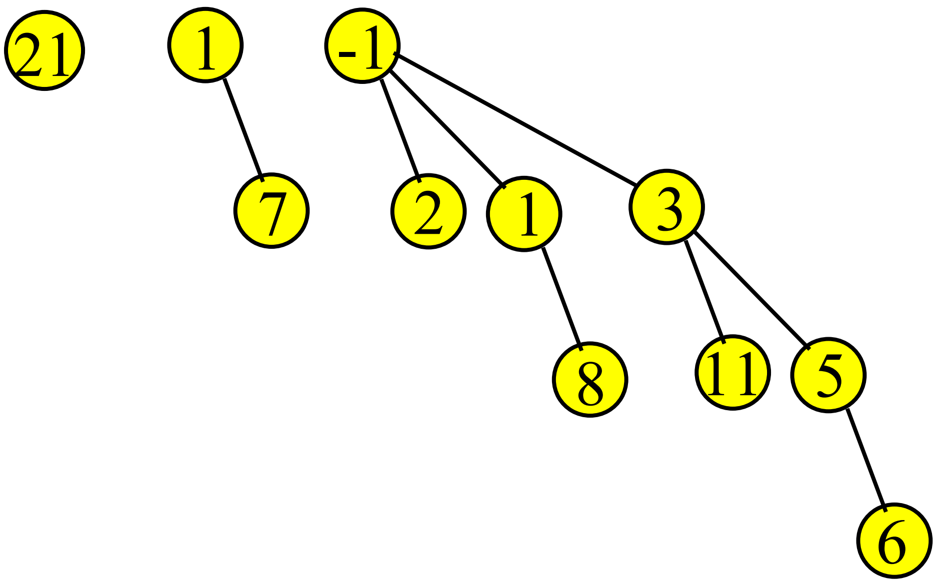
Binomial Queues: Merge

- Main Idea: Merge two binomial queues by merging individual binomial trees
  - Since B_{k+1} is just two B_k's attached together, merging trees is easy
- Steps for creating new queue by merging:

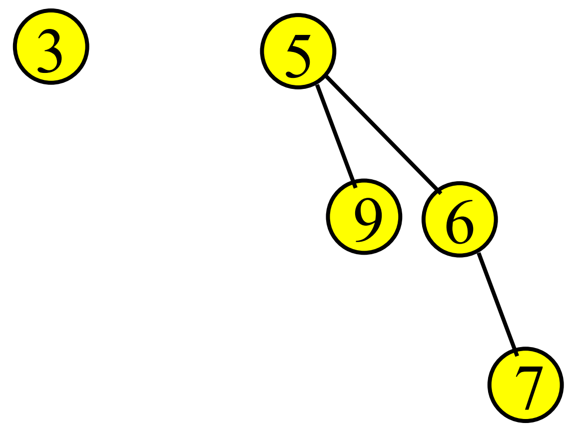
1. Start with B_k for smallest k in either queue.
2. If only one B_k, add B_k to new queue and go to next k.
3. Merge two B_k's to get new B_{k+1} by making larger root the child of smaller root. Go to step 2 with $k = k + 1$.

Example: Binomial Queue Merge

H1:



H2:



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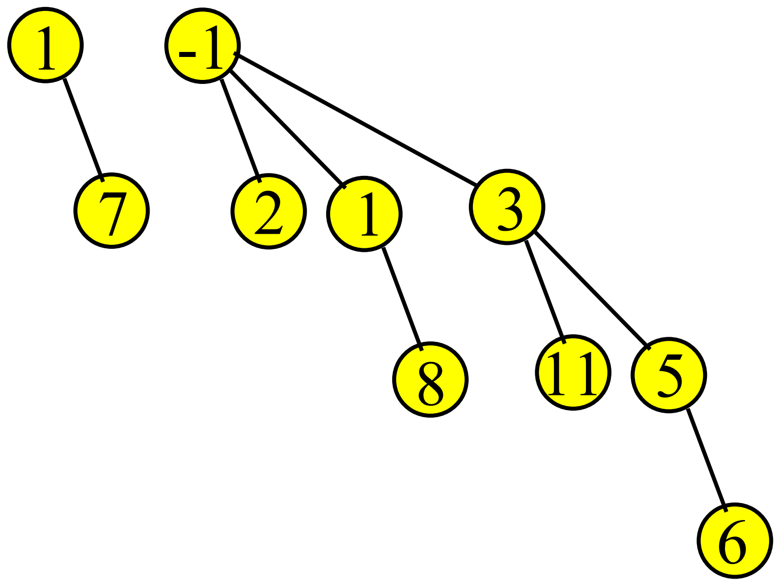
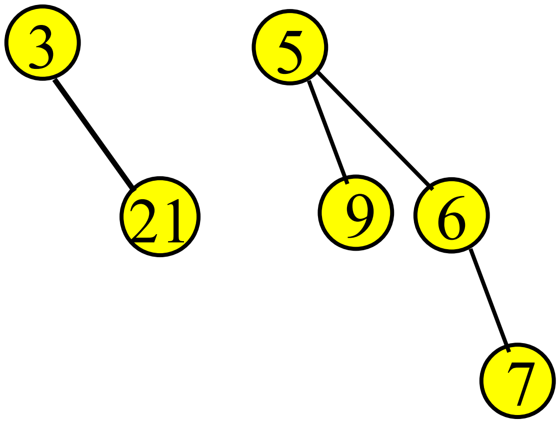


H2:



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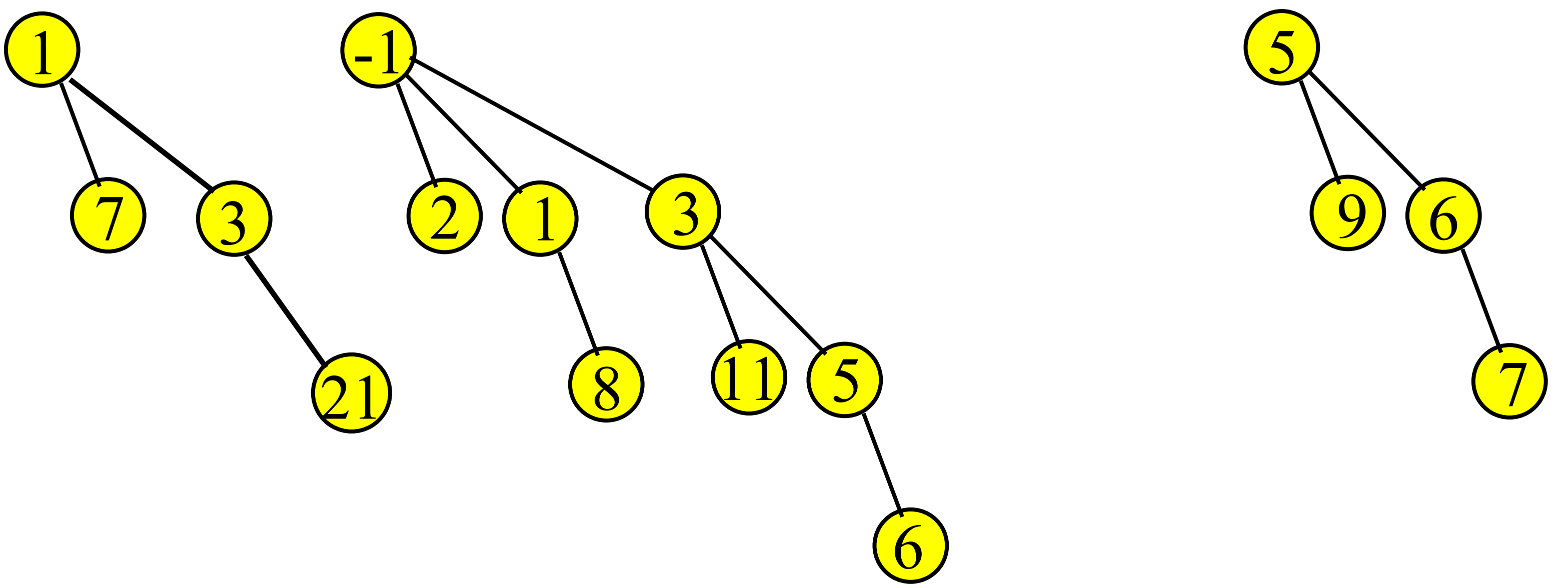

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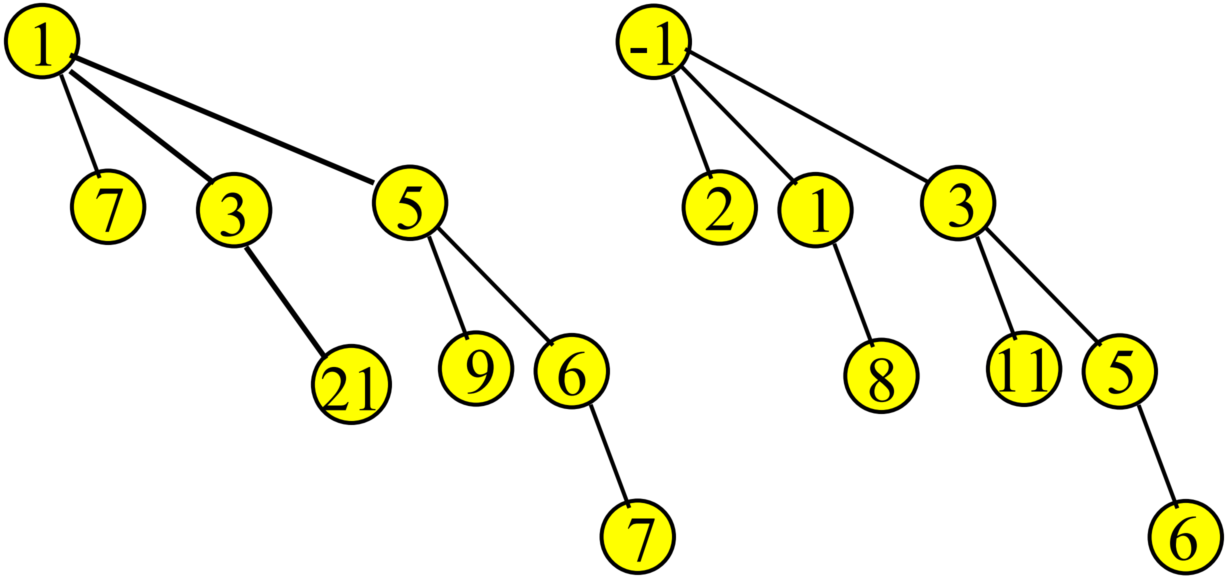

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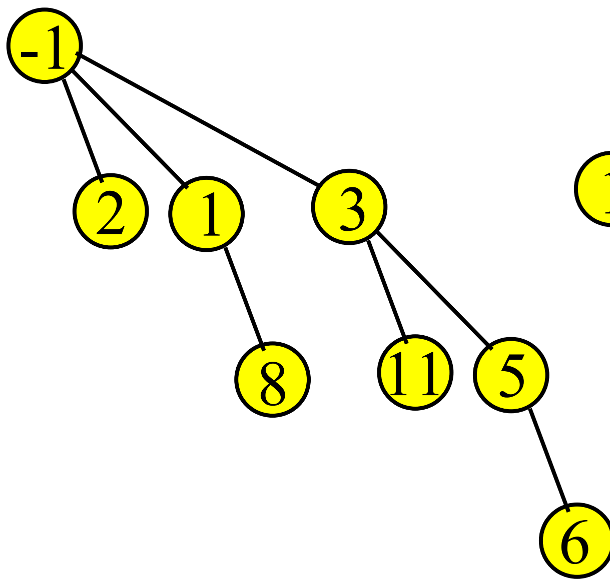
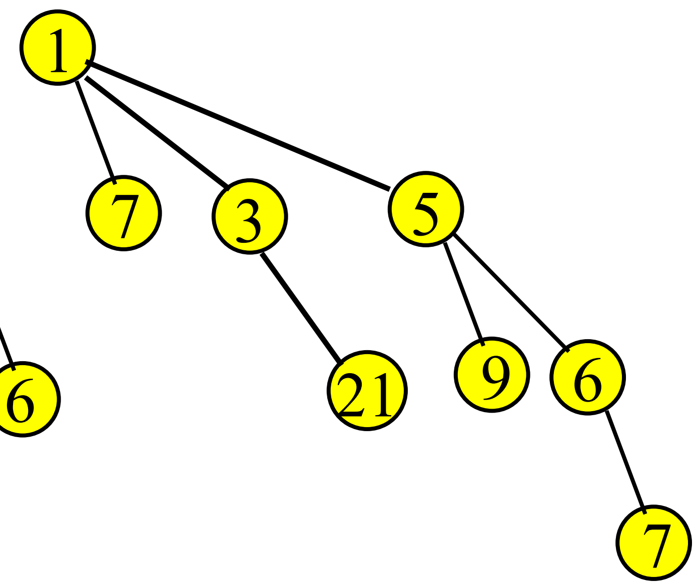


H2:



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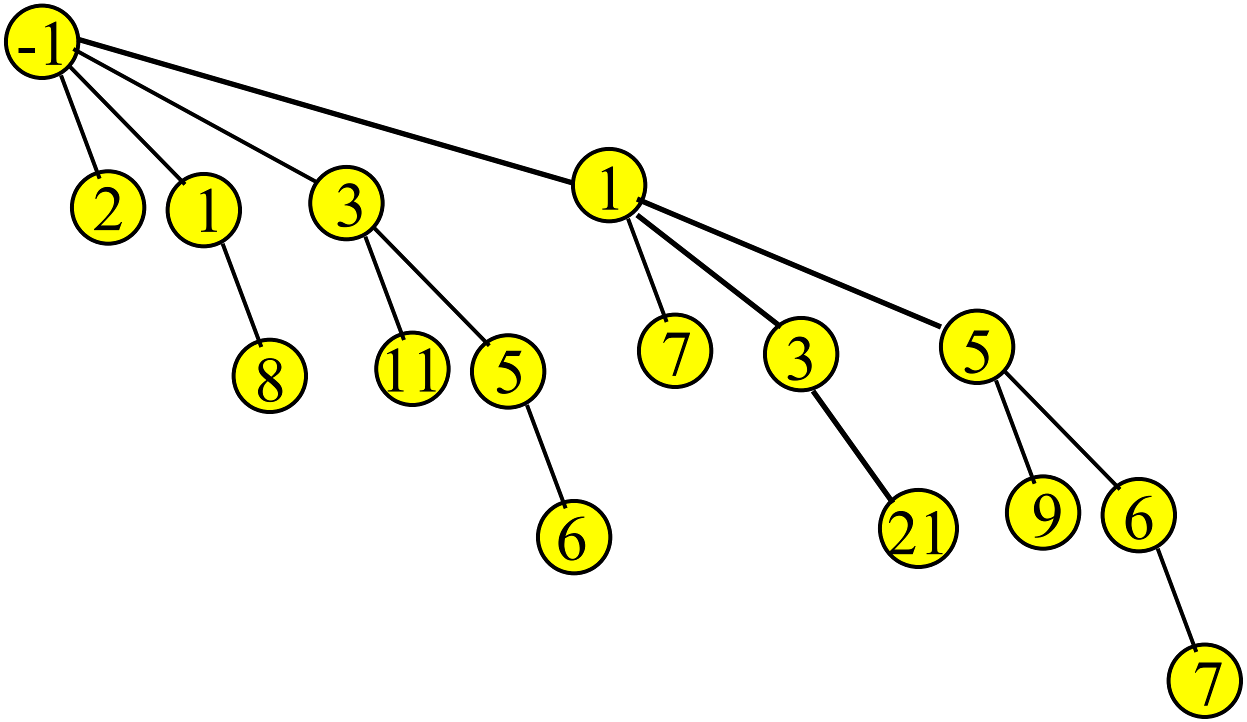

Binomial Queues: Merge and Insert

- What is the run time for Merge of two O(N) queues?
  - O(number of trees) = O(log N)
- How would you insert a new item into the queue?
  - Create a single node queue B_0 with new item and merge with existing queue
  - Again, O(log N) time
- Example: Insert 1, 2, 3, …,7 into an empty binomial queue

Insert 1,2,…,7

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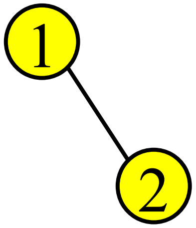

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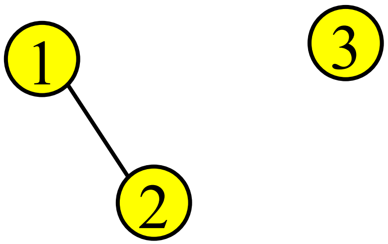

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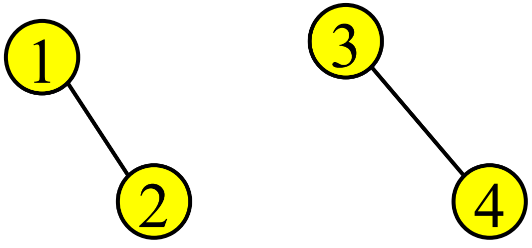

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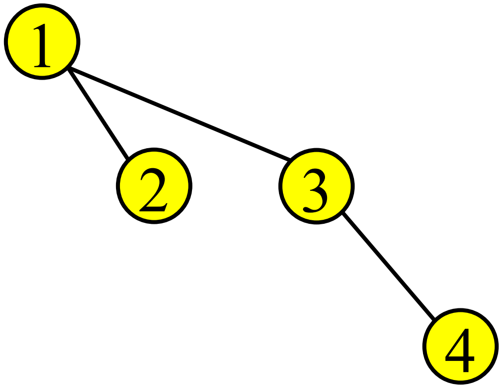

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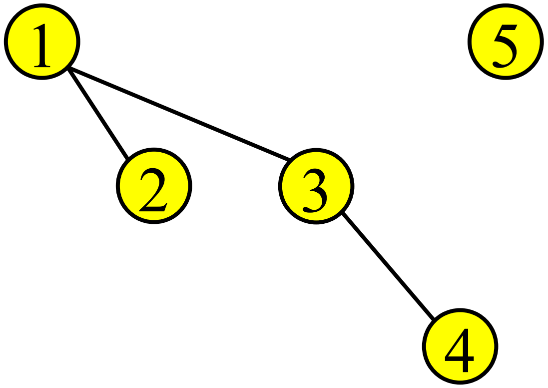

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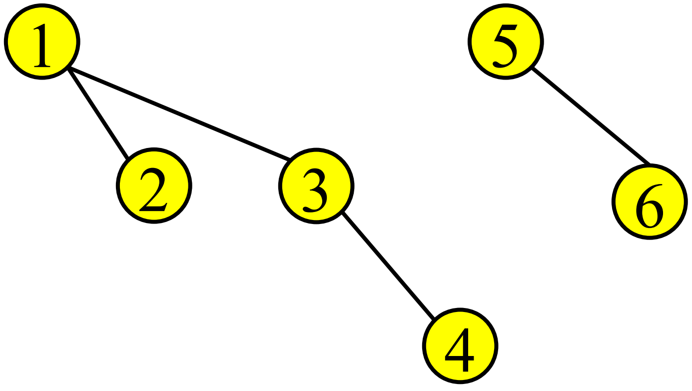

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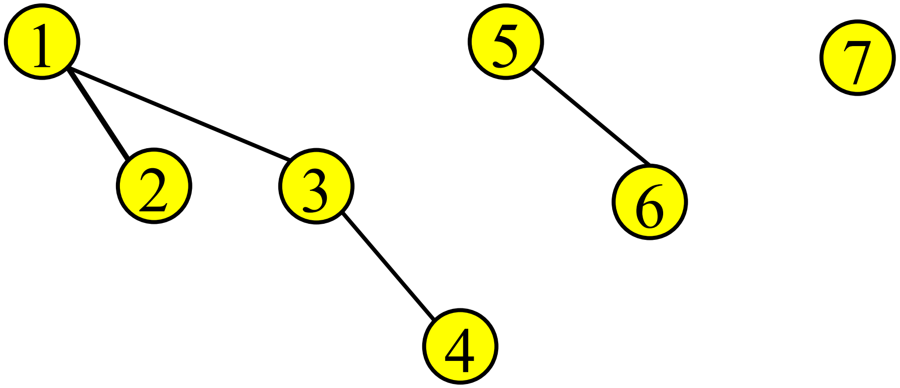
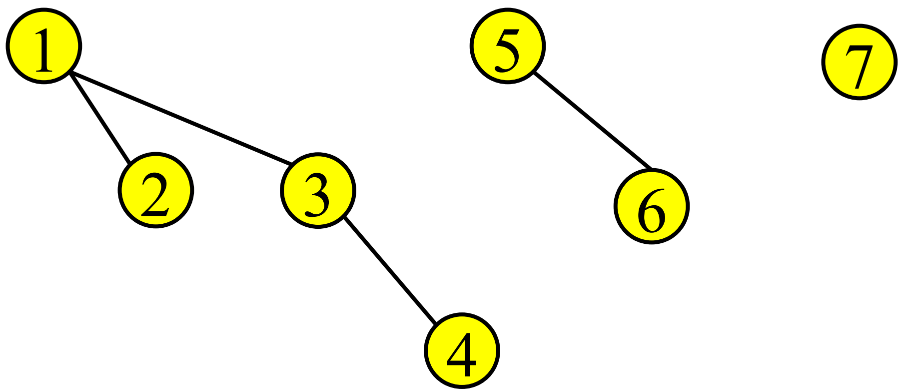


Binomial Queues: DeleteMin

- Steps:
  1. Find tree B_k with the smallest root
  2. Remove B_k from the queue
  3. Delete root of B_k (return this value); You now have a new queue made up of the forest B_0, B_1, …, B_{k-1}
  4. Merge this queue with remainder of the original (from step 2)
- Run time analysis: Step 1 is O(log N), step 2 and 3 are O(1), and step 4 is O(log N). Total time = O(log N)
- Example: Insert 1, 2, …, 7 into empty queue and DeleteMin

Insert 1,2,…,7



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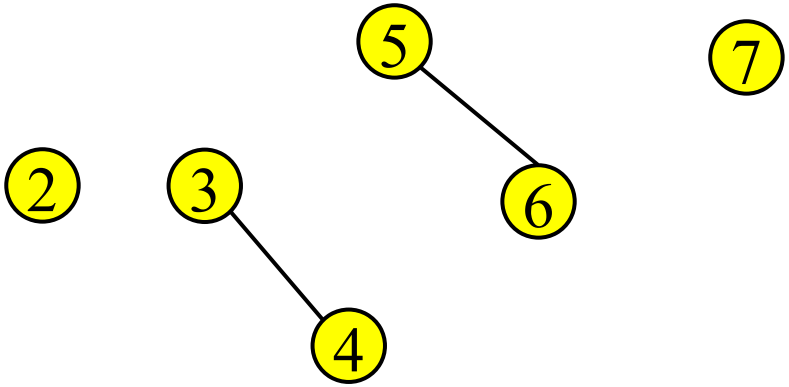
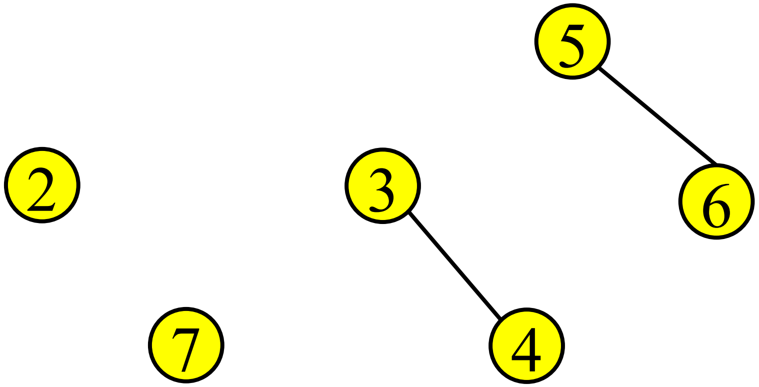


Merge



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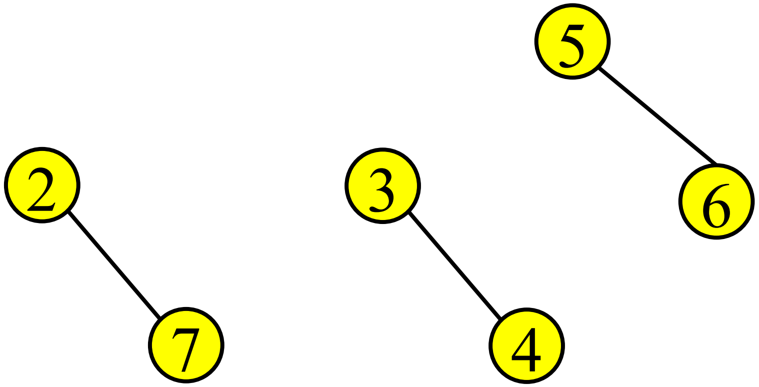

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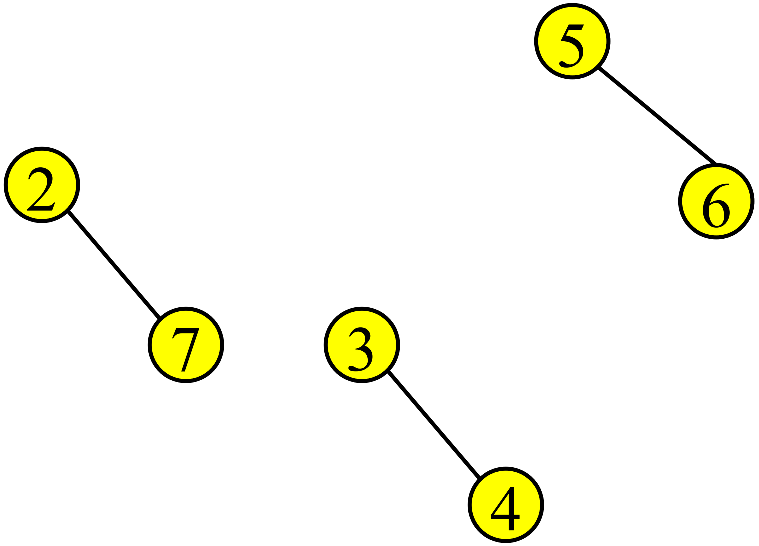

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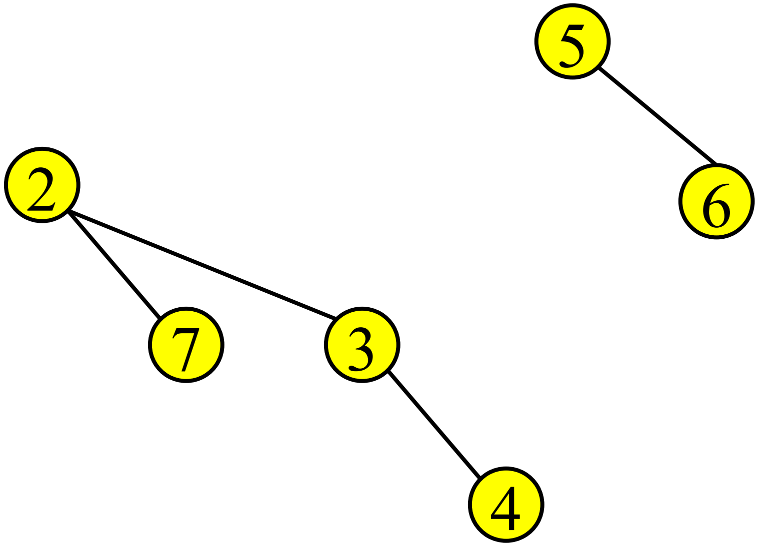

DONE!