# 1 Question 1

For directed graphs, we can for example train one embedding based on walks considering outgoing edges, and another embedding based on walks considering incoming edges.
For weighted graphs, rather than uniform sampling, we can use techniques such as biased sampling to prioritize higher-weight edges.

# 2 Question 2

We observe that the second coordinates of the rows of X2 have an opposite sign with respect to those of X1. THis means that X2 =X1.R with :

$$\mathbf{R} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

# 3 Question 3

For a GCN with 2 message passing layers, the receptive field of a node includes nodes up to 2 hops away (distance 2).
In general, for a GCN with k message passing layers, the receptive field will include nodes up to k hops away from the given node.
The maximal number of edges separating node i from the nodes considered in its prediction is k.

# 4 Question 4

: for a complete graph $K_4$ is:

$$\tilde{\mathbf{A}}_{\mathbf{K_4}} = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

and the degree matrix is :

$$\tilde{\mathbf{D}}_{\mathbf{K_4}} = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

and the normalized adjacency matrix $\hat{A}$ is :

$$\hat{\mathbf{A}}_{\mathbf{K_4}} = \begin{bmatrix} \frac{2}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{2}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{2}{5} \end{bmatrix}$$

$$\hat{\mathbf{A}}_{\mathbf{K_4}} = \begin{bmatrix} 0.4 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.4 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.4 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.4 \end{bmatrix}$$

$$Z_0{}^{K_4} = RELU(\hat{A}_{K_4} X W_0) = RELU(\begin{bmatrix}1\\1\\1\\1\end{bmatrix} W_0) = RELU(\begin{bmatrix}-0.8 & 0.5\\-0.8 & 0.5\\-0.8 & 0.5\\-0.8 & 0.5\end{bmatrix}) = \begin{bmatrix}0 & 0.5\\0 & 0.5\\0 & 0.5\\0 & 0.5\end{bmatrix}$$

$$Z_1{}^{K_4} = RELU(\hat{A}_{K_4} Z_0 W_1) = RELU(\begin{bmatrix}0 & 0.5\\0 & 0.5\\0 & 0.5\\0 & 0.5\end{bmatrix} W_1) = RELU(\begin{bmatrix}-0.2 & 0.3 & 0.25\\-0.2 & 0.3 & 0.25\\-0.2 & 0.3 & 0.25\\-0.2 & 0.3 & 0.25\end{bmatrix}) = \begin{bmatrix}0 & 0.3 & 0.25\\0 & 0.3 & 0.25\\0 & 0.3 & 0.25\\0 & 0.3 & 0.25\end{bmatrix}$$

for a star graph $S_4$ is:

$$\tilde{\mathbf{A}}_{\mathbf{S_4}} = \begin{bmatrix}2 & 1 & 1 & 1\\1 & 2 & 0 & 0\\1 & 0 & 2 & 0\\1 & 0 & 0 & 2\end{bmatrix}$$

and the degree matrix is :

$$\tilde{\mathbf{D}}_{\mathbf{S_4}} = \begin{bmatrix}5 & 0 & 0 & 0\\0 & 3 & 0 & 0\\0 & 0 & 3 & 0\\0 & 0 & 0 & 3\end{bmatrix}$$

Then

$$\tilde{\mathbf{D}}_{\mathbf{S_4}}^{\frac{-1}{2}} = \begin{bmatrix}\frac{1}{\sqrt{5}} & 0 & 0 & 0\\0 & \frac{1}{\sqrt{3}} & 0 & 0\\0 & 0 & \frac{1}{\sqrt{3}} & 0\\0 & 0 & 0 & \frac{1}{\sqrt{3}}\end{bmatrix}$$

and the normalized adjacency matrix is :

$$\hat{\mathbf{A}}_{\mathbf{S_4}} = \begin{bmatrix}\frac{2}{5} & \frac{1}{\sqrt{15}} & \frac{1}{\sqrt{15}} & \frac{1}{\sqrt{15}}\\\frac{1}{\sqrt{15}} & \frac{2}{3} & 0 & 0\\\frac{1}{\sqrt{15}} & 0 & \frac{2}{3} & 0\\\frac{1}{\sqrt{15}} & 0 & 0 & \frac{2}{3}\end{bmatrix}$$

$$\hat{\mathbf{A}}_{\mathbf{S_4}} = \begin{bmatrix}0.4 & 0.26 & 0.26 & 0.26\\0.26 & 0.67 & 0 & 0\\0.26 & 0 & 0.67 & 0\\0.26 & 0 & 0 & 0.67\end{bmatrix}$$

$$Z_0{}^{S_4} = RELU(\hat{A}_{S_4} X W_0) = RELU(\begin{bmatrix}1.18\\0.93\\0.93\\0.93\end{bmatrix} W_0) = RELU(\begin{bmatrix}-0.94 & 0.59\\-0.74 & 0.46\\-0.74 & 0.46\\-0.74 & 0.46\end{bmatrix}) = \begin{bmatrix}0 & 0.59\\0 & 0.46\\0 & 0.46\\0 & 0.46\end{bmatrix}$$

$$Z_1{}^{S_4} = RELU(\hat{A}_{S_4} Z_0 W_1) = RELU(\begin{bmatrix}0 & 0.59\\0 & 0.46\\0 & 0.46\\0 & 0.46\end{bmatrix} W_1) = RELU(\begin{bmatrix}-0.24 & 0.35 & 0.29\\-0.18 & 0.28 & 0.23\\-0.18 & 0.28 & 0.23\\-0.18 & 0.28 & 0.23\end{bmatrix}) = \begin{bmatrix}0 & 0.35 & 0.29\\0 & 0.28 & 0.23\\0 & 0.28 & 0.23\\0 & 0.28 & 0.23\end{bmatrix}$$

For $K_4$,all nodes have identical features after $Z_1{}^{K_4}$ due to the symmetry of the graph.
For $S_4$ the central node has a larger feature value compared to the leaf nodes due to its higher degree. If node features X were sampled randomly from a uniform distribution, the resulting matrices $Z_1$ would exhibit non-uniform feature propagation across the graph, reflecting the variability in initial features rather than identical outputs for symmetric structures like $K_4$ or $S_4$ . .