CompStatsProject- Online EM for HMM

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1 Hidden Markov Model classic

1.1 Context

- We want to treat sequential data $\rightarrow$ Markov Chain.

- 1st order Markov chain too constrained $\rightarrow$ Mth order Markov chain.

- Now, number of parameters too large $\rightarrow$ state space model : For each observation x_n, we introduce a corresponding latent variable z_n (which may be of different type or dimensionality to the observed variable). We now assume that it is the latent variables that form a Markov chain.

- If latent variable $\mathbf{z}_n$ is discrete $\rightarrow$ HMM

$$p(\mathbf{X}, \mathbf{Z}|\theta) = p(\mathbf{z}_1|\pi) \prod_{n=2}^{N} p(\mathbf{z}_n|\mathbf{z}_{n-1}, \mathbf{A}) \prod_{m=1}^{N} p(\mathbf{x}_n|\mathbf{z}_n, \phi)$$

where, $\mathbf{X} = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}, \mathbf{Z} = \{\mathbf{z}_1, \ldots, \mathbf{z}_n\}$, and $\theta = \{\pi, \mathbf{A}, \phi\}$.

1.2 Inference

- Direct maximization of likelihood function lead to no closed-form solutions $\rightarrow$ EM

- Start : θ^{old}

- E-step : find $p(\mathbf{Z}|\mathbf{X}, \theta^{old})$ with forward-backward algorithm.

- M-step : Maximise $Q(\theta, \theta^{old}) = \sum_{\mathbf{Z}} p(\mathbf{Z}|\mathbf{X}, \theta^{old}) \ln p(\mathbf{X}, \mathbf{Z}|\theta)$ with respect to $\theta = \{\pi, \mathbf{A}, \phi\}$.

- Amelioration $\rightarrow$ Scaling factors.

1.3 Extension

- The problem of finding the most probable sequence of latent states is not the same as that of finding the set of states that are individually the most probable.

- Finding the set of states that are individually the most probable $\rightarrow$ previous algorithm.

- Finding the most probable sequence of latent states $\rightarrow$ Viterbi algorithm (max-sum algorithm).

2 Personal Notes

2.1 Abstract:

The paper presents an online Expectation-Maximization (EM) algorithm for parameter estimation in Hidden Markov Models (HMMs). It combines EM's reparameterization using sufficient statistics with recursive smoothing based on an auxiliary recursion. While resembling stochastic approximation algorithms, the proposed method is distinct enough that conventional convergence analysis doesn't apply. The algorithm is evaluated through simulations with a noisily observed Markov chain, showing it produces estimation results comparable to the maximum likelihood estimator for large sample sizes.

2.2 Introduction :

This paper addresses online parameter estimation for Hidden Markov Models (HMMs), where data is processed sequentially without being stored. Building on Mongillo and Denève's (2008) work, the paper proposes an online EM algorithm for general HMMs, including those with continuous observations. The key idea is that their recursion is a specific case of recursive smoothing for sum functionals. The paper provides initial results on the algorithm's limiting behavior, linking it to stationary points of a limiting EM mapping. Numerical simulations show the algorithm's effectiveness in estimating parameters of a noisy Markov chain.

2.3 Online EM Algorithm for HMMs:

The online EM algorithm estimates parameters $\theta^\star$ for a stationary Hidden Markov Model (HMM) based on observed sequences $(Y_t)_{t\geq 0}$, where the states $(X_t)_{t\geq 0}$ are latent.

Key Definitions

1. **Stationary and Non-Stationary Models**:

- P_θ: Probability under parameter θ for the stationary process (X_t, Y_t).
- $P_{\nu,\theta}$: Probability under non-stationary initialization ν for $(X_t, Y_t)_{t\geq 0}$.

2. **Transition and Emission Probabilities**:

$$\begin{aligned}&\text{Transition matrix:} \quad q_\theta(x, x') = \mathbb{P}(X_t = x' \mid X_{t-1} = x, \theta),\\&\text{Conditional pdf for emissions:} \quad g_\theta(x, y) = \mathbb{P}(Y_t = y \mid X_t = x, \theta).\end{aligned}$$

3. **Exponential Family Representation**:

$$p_\theta(x_t, y_t \mid x_{t-1}) = h(x_t, y_t) \exp\left(\langle \psi(\theta), s(x_{t-1}, x_t, y_t)\rangle - A(\theta)\right),$$

where:

- $s(x_{t-1}, x_t, y_t)$: Complete-data sufficient statistics,
- $\psi(\theta)$: Natural parameter,
- $A(\theta)$: Log-partition function,
- $\langle \cdot, \cdot \rangle$: Scalar product.

4. **Sufficient Statistics**:

$$S = \frac{1}{n}\mathbb{E}_{\nu,\theta}\left[\sum_{t=1}^{n} s(X_{t-1}, X_t, Y_t) \mid Y_{0:n}\right].$$

5. **M-Step Equation**:

$$\nabla_\theta \psi(\theta) S - \nabla_\theta A(\theta) = 0.$$

EM Algorithm Steps

1. **E-Step**:

$$S_{k+1} = \frac{1}{n}\mathbb{E}_{\nu,\theta_k}\left[\sum_{t=1}^{n} s(X_{t-1}, X_t, Y_t) \mid Y_{0:n}\right].$$

2. **M-Step**:

$$\theta_{k+1} = \bar{\theta}(S_{k+1}),$$

where $\bar{\theta}(S)$ is the solution to the complete-data maximum likelihood equation:

$$\nabla_\theta \psi(\theta) S - \nabla_\theta A(\theta) = 0.$$

Recursive Form of Smoothing

The recursive smoothing approach allows for the efficient computation of the normalized sum:

$$\frac{1}{n} \mathbb{E}_{\nu,\theta} \left[\sum_{t=1}^{n} s(X_{t-1}, X_t, Y_t) \mid Y_{0:n} \right]$$

using auxiliary recursion methods, originally proposed by Zeitouni and Dembo (1988).

Key Definitions:

1. **Filter** $\phi_{n,\nu,\theta}(x)$:

$$\phi_{n,\nu,\theta}(x) = P_{\nu,\theta}(X_n = x \mid Y_{0:n})$$

2. **Intermediate Quantity** $\rho_{n,\nu,\theta}(x)$:

$$\rho_{n,\nu,\theta}(x) = \frac{1}{n} \mathbb{E}_{\nu,\theta} \left[\sum_{t=1}^{n} s(X_{t-1}, X_t, Y_t) \mid Y_{0:n}, X_n = x \right]$$

Using these definitions, the desired sum can be expressed as:

$$\sum_{x \in \mathcal{X}} \phi_{n,\nu,\theta}(x) \rho_{n,\nu,\theta}(x) = \frac{1}{n} \mathbb{E}_{\nu,\theta} \left[\sum_{t=1}^{n} s(X_{t-1}, X_t, Y_t) \mid Y_{0:n} \right].$$

Recursive Update Equations
Initialization:

$$\phi_{0,\nu,\theta}(x) = \frac{\nu(x) g_\theta(x, Y_0)}{\sum_{x' \in \mathcal{X}} \nu(x') g_\theta(x', Y_0)}, \quad \rho_{0,\nu,\theta}(x) = 0.$$

Recursion for $n \geq 0$: 1. **Filter Update:**

$$\phi_{n+1,\nu,\theta}(x) = \frac{\sum_{x' \in \mathcal{X}} \phi_{n,\nu,\theta}(x') q_\theta(x', x) g_\theta(x, Y_{n+1})}{\sum_{x', x'' \in \mathcal{X}^2} \phi_{n,\nu,\theta}(x') q_\theta(x', x'') g_\theta(x'', Y_{n+1})}.$$

2. **Intermediate Quantity Update:**

$$\rho_{n+1,\nu,\theta}(x) = \sum_{x' \in \mathcal{X}} \left[\frac{1}{n+1} s(x', x, Y_{n+1}) + \left(1 - \frac{1}{n+1}\right) \rho_{n,\nu,\theta}(x') \right] r_{n+1,\nu,\theta}(x' \mid x),$$

where:

$$r_{n+1,\nu,\theta}(x' \mid x) = \frac{\phi_{n,\nu,\theta}(x') q_\theta(x', x)}{\sum_{x'' \in \mathcal{X}} \phi_{n,\nu,\theta}(x'') q_\theta(x'', x)}.$$

Retrospective Probability
The term $r_{n+1,\nu,\theta}(x' \mid x)$ represents the **retrospective probability**:

$$P_{\nu,\theta}(X_n = x' \mid X_{n+1} = x, Y_{0:n}),$$

and is independent of the new observation Y_{n+1}

2.4 Formulating the algo:

The online Expectation-Maximization (EM) algorithm incrementally incorporates observations to update model parameters recursively, making it efficient for large or streaming datasets. It relies on a decreasing step-size sequence to ensure convergence and a minimal number of observations ($n_{\min}$) before starting parameter updates.

Key Components

- **Step-size Sequence $(\gamma_n)_{n\geq 1}$:**

$$\sum_{n\geq 1} \gamma_n = \infty, \quad \sum_{n\geq 1} \gamma_n^2 < \infty.$$

- **Initialization:**

$$\hat{\phi}_0(x) = \frac{\nu(x) g_{\hat{\theta}_0}(x, Y_0)}{\sum_{x'\in\mathcal{X}} \nu(x') g_{\hat{\theta}_0}(x', Y_0)}, \quad \hat{\rho}_0(x) = 0.$$

- **Filter Update (Recursive):**

$$\hat{\phi}_{n+1}(x) = \frac{\sum_{x'\in\mathcal{X}} \hat{\phi}_n(x') q_{\hat{\theta}_n}(x', x) g_{\hat{\theta}_n}(x, Y_{n+1})}{\sum_{x',x''\in\mathcal{X}^2} \hat{\phi}_n(x') q_{\hat{\theta}_n}(x', x'') g_{\hat{\theta}_n}(x'', Y_{n+1})}.$$

- **Intermediate Quantity Update:**

$$\hat{\rho}_{n+1}(x) = \sum_{x'\in\mathcal{X}} \left[\gamma_{n+1} s(x', x, Y_{n+1}) + (1-\gamma_{n+1})\hat{\rho}_n(x')\right] r_n(x' \mid x),$$

  where:

$$r_n(x' \mid x) = \frac{\hat{\phi}_n(x') q_{\hat{\theta}_n}(x', x)}{\sum_{x''\in\mathcal{X}} \hat{\phi}_n(x'') q_{\hat{\theta}_n}(x'', x)}.$$

- **Parameter Update (M-Step):**

$$\hat{\theta}_{n+1} = \begin{cases} \bar{\theta}\left(\sum_{x\in\mathcal{X}} \hat{\rho}_{n+1}(x)\hat{\phi}_{n+1}(x)\right), & n \geq n_{\min}, \\ \hat{\theta}_n, & \text{otherwise.} \end{cases}$$

Notes

- The step-size γ_n balances the influence of current observations against historical data.
- $n_{\min}$ ensures numerical stability during early iterations.

This recursive approach efficiently updates HMM parameters for scenarios with streaming data or where batch computation is infeasible.

2.5 Discussion :

The discussion connects the proposed algorithm with earlier works and discusses its numerical complexity. Key points include:
1. **Generalization**: The algorithm generalizes the online EM algorithm of Cappe and Moulines (2009) for dependent observations. For independent observations, the recursion simplifies:

$$\hat{S}_{n+1} = \gamma_{n+1}\mathbb{E}_{\bar{\theta}(\hat{S}_n)}[s(X_{n+1}, Y_{n+1})|Y_{n+1}] + (1-\gamma_{n+1})\hat{S}_n$$

2. **Auxiliary Recursion**:

$$\hat{\rho}_n \text{ intermediate quantity update}$$

This recursion is essential for computing the conditional expectation of complete-data sufficient statistics.
3. **Finite-Valued HMMs (Mongillo and Deneve (2008))**:
$\hat{\tau}_{n+1}(x) = \gamma_{n+1}\sum_{x'\in X} s(x', x, Y_{n+1})\frac{\hat{\phi}_n(x')q_{\hat{\theta}_n}(x',x)g_{\hat{\theta}_n}(x',Y_{n+1})}{\sum_{x',x''\in X^2}\hat{\phi}_n(x')q_{\hat{\theta}_n}(x',x'')g_{\hat{\theta}_n}(x'',Y_{n+1})}$
$+(1-\gamma_{n+1})\sum_{x'\in X}\hat{\tau}_n(x')\frac{q_{\hat{\theta}_n}(x',x)g_{\hat{\theta}_n}(x',Y_{n+1})}{\sum_{x',x''\in X^2}\hat{\phi}_n(x')q_{\hat{\theta}_n}(x',x'')g_{\hat{\theta}_n}(x'',Y_{n+1})}$
4.**Numerical complexity :** Regarding numerical complexity, the online EM algorithm's complexity in the finite-valued observation case can be reduced from $|X|^4 \times |Y|$ to $|X|^4 + |X|^3 \times |Y|$ per observation. This is compared to $(|X|^2 + |X| \times |Y|)$ per observation *per iteration* for batch EM. The online EM's complexity scales with $|X|^4$ (or $|X|^3$ depending on the implementation), which can be a limitation for models with many states, but is comparable to online gradient methods. Structured transition matrices can reduce this complexity.

2.6 Convergence :

Deterministic Limit of EM Updates

The EM update for the parameters converges to a deterministic limit as the number of observations $n \to \infty$, considering the sequential dependence in HMMs.

Theorem 1. Under certain regularity conditions, the EM updates for Hidden Markov Models (HMMs) satisfy the following properties:

1. Convergence of Expected Sufficient Statistics:

$$\frac{1}{n}\mathbb{E}_{\nu,\theta}\left[\sum_{t=1}^{n} s(X_{t-1}, X_t, Y_t) \mid Y_{0:n}\right] \xrightarrow{\text{a.s.}} \mathbb{E}_{\theta^*}\left(\mathbb{E}_{\theta}\left[s(X_{-1}, X_0, Y_0) \mid Y_{-\infty:\infty}\right]\right).$$

2. Fixed Points of the Limiting EM Algorithm: The limiting EM update:

$$\theta_{k+1} = \bar{\theta}\left\{\mathbb{E}_{\theta^*}\left(\mathbb{E}_{\theta_k}\left[s(X_{-1}, X_0, Y_0) \mid Y_{-\infty:\infty}\right]\right)\right\},$$

has fixed points that are the stationary points of the limiting likelihood contrast function $c_{\theta^*}(\theta)$which is defined as $c_{\theta^*}(\theta)$ is defined as:

$$c_{\theta^*}(\theta) = \mathbb{E}_{\theta^*}\big[\log p_\theta(Y_0 \mid Y_{-\infty:-1})\big].$$

Let θ_n denote the parameter estimate at step n. Then, as $n \to \infty$:

$$\theta_{n+1} \to \theta^* \quad \text{such that} \quad \nabla_\theta c_{\theta^*}(\theta^*) = 0,$$

Additionally, Fisher's identity provides a key decomposition for the gradient:

$$\frac{1}{n}\sum_{t=1}^{n}\mathbb{E}_{\nu,\theta}\big[\nabla_\theta \log p_\theta(X_t, Y_t \mid X_{t-1}) \mid Y_{0:n}\big] \to \nabla_\theta c_{\theta^*}(\theta).$$

Assumptions

- **Finite State Space:** X is finite.
- **Compact Parameter Space:** The parameter space Θ is compact, and θ^* lies in its interior.
- **Regular Transition Matrix:** The transition probabilities satisfy:

$$q_\theta(x, x') \geq \epsilon > 0 \quad \forall\, \theta \in \Theta,\, x, x' \in X.$$

- **Bounded Likelihood:** The observation likelihood function $\bar{g}_\theta(y)$ is bounded, and:

$$\sup_\theta \sup_y \bar{g}_\theta(y) < \infty, \quad \mathbb{E}_{\theta^*}\left[|\log \inf_\theta \bar{g}_\theta(Y_0)|\right] < \infty,$$

  where $\bar{g}_\theta(y) = \sum_x g_\theta(x, y)$.
- **Smoothness:** The model functions ψ, A are continuously differentiable on the interior of Θ.

Convergence of Auxiliary Quantity $\hat{\rho}_n(x)$

The auxiliary quantity $\hat{\rho}_n(x)$, computed during the recursive step, converges to a deterministic value independent of x when parameter updates are frozen.

Corollary 1. If $\theta_n = \theta$ for all n, then:

$$\hat{\rho}_n(x) \to \mathbb{E}_{\theta^*}\big[\mathbb{E}_\theta\big[s(X_{-1}, X_0, Y_0) \mid Y_{-\infty:\infty}\big]\big].$$

This ensures stability of $\hat{\rho}_n(x)$, even though other terms like $\hat{\phi}_n(x)$ may depend on observations.

These results highlight the unique challenges of online EM in HMMs, where recursive smoothing requires data dependencies on both the infinite past and future. Extending this analysis to full convergence remains an open research direction.

2.7 Application to gaussian HMMs :

HMM with Product Parameterization

- **State Variables (X_t)**: Take values in $\{1, \ldots, m\}$.
- **Parameters (θ)**: Split into:
  1. **State transition matrix** (q_θ).
  2. **State-conditional densities** ($g_\theta(i, \cdot)$), parameterized by:
     - Mean vectors ($\mu_\theta(i)$).
     - Covariance matrices ($\Sigma_\theta(i)$) for Gaussian HMMs.

EM Complete-Data Sufficient Statistics

Two auxiliary functions arise, corresponding to transition and state-conditional updates:

1. ρ_q: Tracks state transitions.
2. ρ_g: Tracks state-conditional density contributions, incorporating the initial term ($t = 0$) for consistency.

Online Algorithm Implementation

1. Auxiliary Statistics Updates:

$$\hat{\rho}_{q,n+1}(i, j, k) = \gamma_{n+1}\delta(j-k)\hat{r}_{n+1}(i|j) + (1-\gamma_{n+1})\sum_{k'}\hat{\rho}_{q,n}(i, j, k')\hat{r}_{n+1}(k'|k),$$

$$\hat{\rho}_{g,n+1}(i, k) = \gamma_{n+1}\delta(i-k)s(Y_{n+1}) + (1-\gamma_{n+1})\sum_{k'}\hat{\rho}_{g,n}(i, k')\hat{r}_{n+1}(k'|k),$$

where δ is the Kronecker delta, and $\hat{r}_{n+1}(i|j)$ is an approximate retrospective conditional probability.

2. M-Step Updates:

- **Transition matrix q_θ:**

$$q_{\hat{\theta}_n}(i,j) = \frac{\hat{S}_{q,n}(i,j)}{\sum_{j'} \hat{S}_{q,n}(i,j')}.$$

- **Gaussian parameters:**
  - Mean vector:

$$\mu_{\hat{\theta}_n}(i) = \frac{\hat{S}_{g,n,1}(i)}{\hat{S}_{g,n,0}(i)}.$$

  - Covariance matrix:

$$\Sigma_{\hat{\theta}_n}(i) = \frac{\hat{S}_{g,n,2}(i)}{\hat{S}_{g,n,0}(i)} - \mu_{\hat{\theta}_n}(i)\mu_{\hat{\theta}_n}(i)^\top.$$

Key Points

- Recursive updates depend on smoothing probabilities, ensuring efficient handling of large datasets.
- The form of $s(Y_t)$ depends on the nature of g_θ; for Gaussian HMMs, $s(Y_t) = \{1, Y_t, Y_t Y_t^\top\}$.
- These updates adapt the algorithm for specific HMM applications like speech processing and other domains.

Markov Chain Observed in Gaussian Noise :

Observation Model

The observations Y_t are modeled as:

$$Y_t = X_t + V_t,$$

where X_t is the hidden Markov chain with transition matrix q, and V_t is scalar Gaussian noise with variance ν.

Sufficient Statistics for the E-Step

The sufficient statistics used in the EM algorithm are defined as:

Transition Statistics:

$$S_q^n(i,j) = \frac{1}{n}\mathbb{E}_{\nu,\theta}\left[\sum_{t=1}^{n} \mathbb{1}\{X_{t-1} = i, X_t = j\} \mid Y_{0:n}\right].$$

State Occupancy and Observation Moments:

$$S_{g,0}^n(i) = \frac{1}{n}\mathbb{E}_{\nu,\theta}\left[\sum_{t=0}^{n} \mathbb{1}\{X_t = i\} \mid Y_{0:n}\right],$$

$$S_{g,1}^n(i) = \frac{1}{n}\mathbb{E}_{\nu,\theta}\left[\sum_{t=0}^{n} \mathbb{1}\{X_t = i\}Y_t \mid Y_{0:n}\right],$$

$$S_{g,2}^n(i) = \frac{1}{n}\mathbb{E}_{\nu,\theta}\left[\sum_{t=0}^{n} \mathbb{1}\{X_t = i\}Y_t^2 \mid Y_{0:n}\right].$$

M-Step Parameter Updates

The parameters $\theta = \{q, \mu, \nu\}$ are updated as follows:

Transition Matrix q:

$$q(i,j) = \frac{S_q^n(i,j)}{\sum_{j'} S_q^n(i,j')}.$$

Means $\mu(i)$:

$$\mu(i) = \frac{S_{g,1}^n(i)}{S_{g,0}^n(i)}.$$

Variance ν:

$$\nu = \frac{\sum_{i=1}^{m}\left(S_{g,2}^n(i) - \mu^2(i)S_{g,0}^n(i)\right)}{\sum_{i=1}^{m} S_{g,0}^n(i)}.$$

Filtering Update

The approximate filtering step updates the probability of each state X_t:

$$\phi_{n+1}(k) = \frac{\sum_{k'} \phi_n(k')q_n(k',k)g_{\theta_n}(k, Y_{n+1})}{\sum_{k',k''} \phi_n(k')q_n(k',k'')g_{\theta_n}(k'', Y_{n+1})},$$

where $g_{\theta_n}(k,y)$ is the Gaussian likelihood:

$$g_{\theta_n}(k,y) = \exp\left(-\frac{(y-\mu_n(k))^2}{2\nu_n}\right).$$

Simplified Variance Update

The second-order statistic $S_{g,2}^n(i)$ can be avoided, as the M-step only requires:

$$\sum_{i=1}^{m} S_{g,2}^n(i) = \frac{1}{n}\sum_{t=0}^{n} Y_t^2.$$

This reduces computational complexity without altering the results.

2.8 Numerical experiments :

The section discusses numerical experiments evaluating the Expectation-Maximization (EM) algorithm for estimating parameters in a two-state model. Key observations are:

- **Slow Convergence**: The batch EM algorithm takes around 50 iterations to provide good estimates. Variability decreases with larger sample sizes, but statistical consistency is only achieved with more iterations.
- **Comparison with Online EM**: Online EM with a slowly decreasing step-size ($\gamma_n = n^{-0.6}$) is more consistent for larger sample sizes compared to batch EM, which has reduced accuracy and higher computational cost.
- **Step-Size Schemes**: The simple step-size scheme $\gamma_n = n^{-1}$ is less robust, and hand-tuned schemes improve performance, though they still don't outperform $\gamma_n = n^{-0.6}$.
- **Computational Efficiency**: Online EM is computationally more efficient, and forward-backward batch EM is more scalable for larger models.
- **Asymptotic Performance**: After Polyak-Ruppert averaging, online EM approaches the maximum likelihood estimator, achieving asymptotic efficiency.

2.9 Key Equations

1. Fisher Information Matrix (estimated by gradient):

$$\frac{1}{n}\nabla q(i,j)\log \mathcal{L}_\theta(Y_{0:n}) = \frac{S_q}{q(i,j)} - \frac{S_q}{q(i,m)}$$

$$\frac{1}{n}\nabla \mu(i)\log \mathcal{L}_\theta(Y_{0:n}) = \frac{S_g}{\mu(i)} - \frac{S_g}{\mu(i)}$$

2. Polyak-Ruppert Averaging:

$$\hat{\theta}_n = \frac{1}{n - n_{\text{avg}}} \sum_{i=n_{\text{avg}}}^{n} \hat{\theta}_i$$

This improves convergence rates, making the algorithm asymptotically efficient.

The experiments show that online EM with well-chosen step-sizes performs better for large sample sizes, while batch EM struggles to provide consistent estimates with fixed iterations.

2.10 Conclusions

The proposed algorithm for online estimation of HMM parameters is based on two key ideas:

1. **Reparameterization in Sufficient Statistics**: Inspired by previous work, this approach approximates the EM recursion using stochastic approximation in the space of sufficient statistics. Theorem 1 shows its potential for HMMs.

2. **Recursive Smoothing Computation**: A specific method for HMMs involves recursively implementing smoothing for sum functionals of the hidden state, as detailed in Proposition 1. This requires approximating an auxiliary quantity, $\rho_{n,\nu,\theta}$, during the algorithm's execution.

Performance encouraging, but several questions raised by this approach :

1. **Theoretical analysis of the convergence**: The first is the theoretical analysis of the convergence of Algorithm 1, which is still missing. Although originally inspired by stochastic approximation ideas, it seems that Algorithm 1 would be difficult to analyze using currently available stochastic approximation results due to the backward kernel operator.

2. **General state-space of the hidden chain**: The proposed algorithm may become less attractive, from a computational point of view, when used in models with many distinct state values. In such cases and, more generally, in cases where the state-space of the hidden chain is no longer finite, a promising approach consists in using some form of Monte Carlo computation.