



### Generalities

**Exercise 1.** Comment if the following statements are true or false? And give a valid reason for saying so.

- All functions are continuous in their domain.
- All continuous functions on  $(0, 1)$  are bounded.
- The sum of two monotone functions is monotone.
- If  $f$  and  $g$  are discontinuous at  $x_0$ , then so is  $f + g$ .
- Every odd degree polynomial has at least one real zero.
- It is impossible for a function to be discontinuous at every real number.



### Functions

**Exercise 2.** Find the domain of each function using interval notation:

$$1) f(x) = \sqrt{\frac{x^2 + 2x + 3}{x^2 - 1}}, \quad 2) f(x) = (1 + \ln(x))^{\frac{1}{x}}, \quad 3) f(x) = \frac{1}{\sqrt{\sin x}}, \quad 4) f(x) = \frac{x^3 + 3}{1 - |x|},$$

$$5) f(x) = \frac{\cos x}{1 + \sin 2x}, \quad 6) f(x) = \sqrt{2 \cos x - 1}, \quad 7) f(x) = \frac{e^{\sqrt{x}} - 2 \ln(2x - 3)}{\sqrt{x^2 - 1}(e^{-x} - 2)}.$$

**Exercise 3.** I) Examine the parity of each of the given function:

$$1) f(x) = \ln(x + \sqrt{1 + x^2}), \quad 2) f(x) = \frac{\tan x - x}{x^3 \cos x}, \quad 3) f(x) = x \frac{a^x - 1}{a^x + 1} \quad (a > 0).$$

II) Show that the following functions are periodic and compute their smallest period:

$$1) f(x) = \sin^2 x, \quad 2) f(x) = \frac{\cos 5x}{\sin 3x}.$$

III) Study the parity and periodicity of the function:  $f(x) = \ln(|\sin(\frac{\pi}{2}x)|)$ .

### Limits

**Exercise 4.** Calculate the following limits:

$$1) \lim_{x \rightarrow +\infty} \left( \frac{x-1}{x+1} \right)^x, \quad 2) \lim_{x \rightarrow 0} \frac{\tan x - \sin x}{x^3}, \quad 3) \lim_{x \rightarrow 0} \frac{x}{a} \left[ \frac{b}{x} \right] \quad (a, b) \in \mathbb{R}^* \times \mathbb{R},$$

$$4) \lim_{x \rightarrow -\infty} \frac{x^4 + 1}{\coth \frac{1}{x}}, \quad 5) \lim_{x \rightarrow +\infty} \sin(\sqrt{x+1}) - \sin(\sqrt{x}), \quad 6) \lim_{x \rightarrow 0} \sin x \left( x - \left[ \frac{1}{x} \right] \right)$$

$$7) \lim_{x \rightarrow 0} \frac{\ln \sqrt{x}}{\sqrt[3]{x}}, \quad 8) \lim_{x \rightarrow \frac{\pi}{4}} \frac{\sin(2x) \sin(x - \frac{\pi}{4})}{\sin x - \cos x}, \quad 9) \lim_{x \rightarrow 1} (1-x) \tan\left(\frac{\pi}{2}x\right),$$

$$10) \lim_{x \rightarrow 1} \frac{\ln x}{x^3 - 1}, \quad 11) \lim_{x \rightarrow a} \frac{\sqrt{x} - \sqrt{a} + \sqrt{x-a}}{\sqrt{x^2 - a^2}}, \quad 12) \lim_{x \rightarrow 0} \frac{(x+h)^n - h^n}{x}.$$

**Exercise 5.** I) Prove the following limits using their precise definition:

$$1) \lim_{x \rightarrow -2} \frac{x^2 + x + 1}{1 - x} = 1, \quad 2) \lim_{x \rightarrow 1} \frac{x+2}{x-1} = -\infty, \quad 3) \lim_{x \rightarrow -\infty} \frac{1}{x^3} = 0.$$

II) Choose two ways to prove that the following functions have no limits at  $x_0$ .

$$1) f(x) = \sin(x), \quad x_0 = +\infty, \quad 2) f(x) = x \sqrt{1 + \frac{1}{x^2}}, \quad x_0 = 0.$$

## Continuity

**Exercise 6.** I) Study the continuity of the following functions:

$$1) f(x) = [x] + (x - [x])^2, \forall x \in \mathbb{R}, \quad 2) g(x) = (x + [\cos(x)])^2, \forall x \in [0, \pi],$$

$$3) f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Z}; \\ x & \text{if } x \notin \mathbb{Z}. \end{cases} \quad 4) f(x) = \begin{cases} x^p \sin \frac{1}{x} & \text{if } x \neq 0 \quad (p \in \mathbb{N}); \\ \alpha & \text{if } x = 0 \quad (\alpha \in \mathbb{R}). \end{cases}$$

II) Find the values of  $a$  and  $b$  that makes each function continuous over the given interval.

$$f(x) = \begin{cases} (x-1)^3 & \text{if } x \leq 0; \\ ax+b & \text{if } 0 < x < 1; \\ \sqrt{x} & \text{if } x \geq 1. \end{cases} \quad f(x) = \begin{cases} -2 \sin x & \text{if } x \leq -\frac{\pi}{2}; \\ a \sin x + b & \text{if } -\frac{\pi}{2} < x < \frac{\pi}{2}; \\ \cos x & \text{if } x \geq \frac{\pi}{2}. \end{cases}$$

**Exercise 7.** Determine if the given functions can be extended to the continuous functions:

$$1) f(x) = \frac{1}{x} \sin\left(\frac{1}{x}\right), \quad 2) f(x) = \sin(x+1) \ln|x+1|, \quad 3) f(x) = \frac{|\sin(x)|}{x}, \quad 4) f(x) = \sin x \sin \frac{1}{x}.$$

**Exercise 8 (Course).** Let  $f$  be the function given by:

$$f(x) = \begin{cases} \frac{\sin(\frac{1}{x})}{\ln x} & \text{if } x > 0; \\ \frac{\sin x}{x^2 - \pi^2} & \text{if } x \leq 0. \end{cases}$$

1. Determine  $\mathcal{D}_f$ , the domain of  $f$ .

2. Study the continuity of  $f$  on  $\mathcal{D}_f$ .

3. Does the function  $f$  admit a continuous extension at  $x = -\pi$ ? If yes, set it.



## Intermediate Value Theorem (IVT)

**Exercise 9.** I) Consider the function  $f$  defined by:

$$f(x) = x \ln(x) - 1, \forall x \in [1, 2].$$

Prove that  $f$  has a root between 1 and 2.

II) Let  $a < c < d < b$  and  $f$  be a function defined on  $[a, b]$ . Show that :

If  $f$  is continuous on  $[a, b]$  and  $f(c) + f(d) = k$ , then there exists  $y_0 \in [a, b]$  such that  $f(y_0) = \frac{k}{2}$ .