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Faculty of Sciences
Department of Mathematics
First-Year C S and C E

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Duration: 1h30

Solution of Exam (Analysis 01)

Exercise 1 (4 marks). *Let $\mathcal{A}$ be a nonempty subset of real numbers defined by:*

$$\mathcal{A} = \left\{ a_n = \frac{n}{n+1} + 1,\ n \in \mathbb{N} \right\}.$$

1. *Recall the characterization of the supremum theorem.*

   Theorem 1 (Characterization of the supremum). *Let $\mathcal{A} \subset \mathbb{R}$ be nonempty and bounded above. Then*

   $$M = \sup(\mathcal{A}) \Leftrightarrow \forall \varepsilon > 0, \exists x_\varepsilon \in \mathcal{A};\ M - \varepsilon < x_\varepsilon \le M. \qquad \textbf{(01pt)}$$

2. *Show that $\mathcal{A}$ is a bounded set: Let $a_n = \dfrac{n}{n+1} + 1 = 2 - \dfrac{1}{n+1}$, $n \in \mathbb{N}$, we have*

   $$-1 \le -\frac{1}{n+1} < 0 \Rightarrow 1 \le a_n < 2. \text{ So, the set } \mathcal{A} \text{ is bounded} \qquad \textbf{(01pt)}$$

3. *Find, if there exist, the supremum, the infimum, the maximum, and the minimum of $\mathcal{A}$. Justify your answer.*

   - *The set of lower bounds of $\mathcal{A}$ is $(-\infty, 1]$, So, $\inf(\mathcal{A}) = 1 \in \mathcal{A} \Rightarrow \min(\mathcal{A}) = 1$.* **(0.5pt)**
   - *The set of upper bounds of $\mathcal{A}$ is $[2, +\infty)$, So, $\sup(\mathcal{A}) = 2 \notin \mathcal{A} \Rightarrow \max(\mathcal{A})\ \nexists$.* **(0.5pt)**

   ▸ *Characterization of $\sup(\mathcal{A}) = 2$.*

   $$2 = \sup(\mathcal{A}) \Leftrightarrow \begin{cases} \forall n \in \mathbb{N}, a_n < 2, \\ \forall \varepsilon > 0, \exists a_{n_\varepsilon} \in \mathcal{A};\ 2 - \varepsilon < 2 - \dfrac{1}{n_\varepsilon + 1}. \end{cases} \qquad \textbf{(0.25pt)}$$

   Then, $2 - \varepsilon < 2 - \dfrac{1}{n_\varepsilon + 1} \Rightarrow n_\varepsilon > \frac{1}{\varepsilon} - 1.$ **(0.5pt)**

   Therefore, it is enough to take $n_\varepsilon = \left[\frac{1}{\varepsilon}\right] \in \mathbb{N}$. **(0.25pt)**

Exercise 2 (7 marks). *We consider the recursice sequence $(u_n)_{n\in\mathbb{N}}$ defined by:*

$$\begin{cases} u_0 = a, \quad a \in \mathbb{R}^+, \\ u_{n+1} = \dfrac{1}{2} u_n (u_n + 1), \quad n \in \mathbb{N}. \end{cases}$$

I) Suppose that: $a = \frac{1}{2}$.

1. *Prove that: $\forall\, n \in \mathbb{N}:\ 0 < u_n < 1$. By induction* **(01pt)**

   - *Fo $n = 0$, we have: $u_0 = \frac{1}{2}$ and $0 < \frac{1}{2} \le 1$ is true.*
   - *Assume that $0 < u_n < 1$, and we must show that $0 < u_{n+1} < 1$, $\forall n \in \mathbb{N}$.*

   $$0 < u_n < 1 \Longrightarrow \begin{cases} 0 < \frac{1}{2} u_n \le \frac{1}{2} \\ 1 < u_n + 1 < 2 \end{cases} \Longrightarrow 0 < \frac{1}{2} u_n (u_n + 1) < 1.$$

   Then, $\forall\, n \in \mathbb{N}:\ 0 < u_n < 1$.

2. *Studying the monotonicity of $(u_n)_n$.* **(01pt)**

   We have $u_{n+1} - u_n = \dfrac{1}{2} u_n (u_n + 1) - u_n = \dfrac{1}{2} u_n (u_n - 1)$. On the other hand,

   $$0 < u_n < 1 \Longrightarrow \begin{cases} u_n > 0 \\ u_n - 1 < 0 \end{cases} \Longrightarrow u_{n+1} - u_n < 0.$$

   Hence, the sequence $(u_n)_n$ is strictly decreasing.

3. *Deduce that the sequence $(u_n)_n$ is convergent and give its limit.*
$(u_n)_n$ *is* $\searrow$ *and bounded below* $\Longrightarrow (u_n)_n$ *is convergent, i.e.,* $\lim\limits_{n\to\infty}(u_n)_n = l.$ **(0.75pt)**
Moreover, $u_{n+1} = f(u_n)$, *such that* $f : x \mapsto \frac{1}{2}x(x+1)$ *is continuous. Thus* l *verifies:*

$$f(l) = l \Rightarrow \frac{1}{2}l(l+1) = l \Rightarrow \frac{1}{2}l(l-1) = 0 \Rightarrow l = 0 \text{ or } l = 1,$$

since $(u_n)_n \searrow$ *and* $0 < u_n < 1$, *then* $l = 0$. **(01pt)**

II) Suppose that $a > 1$.

1. *Prove that:* $\forall\, n \in \mathbb{N}:\ u_n > 1$. ***By induction*** **(0.75pt)**
   - *For* $n = 0$, *we have:* $u_0 = a > 1$ *is true.*
   - *Assume that* $u_n > 1$ *and we must show* $u_{n+1} > 1, \forall n \in \mathbb{N}$.

$$u_n > 1 \Longrightarrow \begin{cases} \frac{1}{2}u_n > \frac{1}{2} \\ u_n + 1 > 2 \end{cases} \Longrightarrow \frac{1}{2}u_n(u_n+1) > 1, \text{i.e., } u_{n+1} > 1.$$

2. *Show that* $(u_n)_{n\in\mathbb{N}}$ *is strictly increasing and* $u_n \geq \left(\frac{1+a}{2}\right)^n\ \forall n \in \mathbb{N}$.
   - *Since* $u_n > 0, \forall n \in \mathbb{N}$, *thus* $\frac{u_{n+1}}{u_n} = \frac{1}{2}(u_n+1) > 1$.
   That is, $(u_n)_n$ *is strictly increasing.* **(0.75pt)**
   - *Puve that,* $\forall n \in \mathbb{N};\ u_n \geq \left(\frac{1+a}{2}\right)^n$. ***By induction*** **(0.75pt)**
   ► *For* $n = 0$, $u_0 = a > 1 = \left(\frac{1+a}{2}\right)^0$.
   Now, assume that $u_n \geq \left(\frac{1+a}{2}\right)^n$ *and we must show that* $u_{n+1} \geq \left(\frac{1+a}{2}\right)^{n+1}$.
   According to $u_0 = a > 1$ *and* $(u_n)_n$ *is* $\nearrow$, *we obtain* $u_n \geq u_0 = a, \forall n \in \mathbb{N}$ *and*

$$u_{n+1} = \frac{1}{2}u_n(u_n+1) \geq \frac{1}{2}\left(\frac{1+a}{2}\right)^n(a+1) \geq \left(\frac{1+a}{2}\right)^{n+1}.$$

3. *Deduce the nature of* $(u_n)_n$. **(01pt)**
   We have $\lim\limits_{n\to+\infty} u_{n+1} \geq \lim\limits_{n\to+\infty}\left(\frac{1+a}{2}\right)^{n+1} = +\infty$. *Hence* $\lim\limits_{n\to+\infty} u_n = +\infty$, *i.e.,* $(u_n)_n$ *is D.V.*

Exercise 3 (5 marks). *Let* f *be a function defined by:*

$$f(x) = \begin{cases} e^x(-x + \frac{4}{\pi}), & x \leq 0; \\ \dfrac{\sin(x) - \cos(x)}{x - \frac{\pi}{4}}, & 0 < x < \frac{\pi}{4}; \\ \sqrt{2} + \frac{\pi}{4} - x, & x > \frac{\pi}{4}. \end{cases}$$

1. *Determine the set* $\mathcal{D}_f$. $D_f = \left]-\infty, \frac{\pi}{4}\right[\cup \left]\frac{\pi}{4}, +\infty\right[= \mathbb{R} \setminus \left\{\frac{\pi}{4}\right\}$. **(0.5pt)**

2. *Study the continuity and the differentiability of* f *on* D_f.
   On $]-\infty, 0[$: $f(x) = e^x(-x + \frac{4}{\pi})$ *is continuous and differentiable because it is a combination of continuous and differentiable functions on* $\mathbb{R}$. **(0.25pt)**
   On $]0, \frac{\pi}{4}[$: $f(x) = \dfrac{\sin(x) - \cos(x)}{x - \frac{\pi}{4}}$ *is continuous and differentiable because it is a fraction of continuous and differentiable functions on* $\mathbb{R}$. **(0.25 pt)**
   On $]\frac{\pi}{4}, +\infty[$: $f(x) = \sqrt{2} + \frac{\pi}{4} - x$ *is continuous and differentiable because it is a polynomial function.* **(0.25 pt)**
   At $x_0 = 0$. **(0.75pt)**

► f *is continuous at* $x_0 = 0 \Longleftrightarrow \lim\limits_{x\to0^+} f(x) = \lim\limits_{x\to0^-} f(x) = f(0)$.

We have $f(0) = \frac{4}{\pi}$,

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{\sin(x)-\cos(x)}{x-\frac{\pi}{4}} = \frac{4}{\pi} \text{ and } \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} e^x(-x+\frac{4}{\pi}) = \frac{4}{\pi}.$$

Therefor, f is continuous at $x_0 = 0$, i.e. f is continuous on D_f. **(0.25pt)**

► *f is differentiable at $x_0 = 0 \Longleftrightarrow f'_+(0) = f'_-(0) = f'(0)$.*

We have

$$f'_+(0) = \lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \frac{\frac{\sin(x)-\cos(x)}{x-\frac{\pi}{4}} - \frac{4}{\pi}}{x} \overset{L'hp}{=} \left(1-\frac{4}{\pi}\right)\left(\frac{16}{\pi^2}\right), \quad \textbf{(0.5pt)}$$

$$f'_-(0) = \lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{e^x(-x+\frac{4}{\pi})-\frac{4}{\pi}}{x} \overset{L'hp}{=} \lim_{x\to 0^-} -e^x = -1+\frac{4}{\pi} \quad \textbf{(0.5pt)}$$

We get $f'_+(0) \neq f'_-(0)$, thus f is not differentiable at $x_0 = 0$. **(0.25pt)**

3. *Does f accept a continuous extension? If yes, set it. The function f is not defined at the point $x_0 = \frac{\pi}{4}$.*
   Moreover, $\lim_{x\to\frac{\pi}{4}^-} (-\sqrt{2}+\frac{\pi}{4}-x) = \sqrt{2}$ **(0.25pt)** *and* $\lim_{x\to\frac{\pi}{4}^+} \frac{\sin x - \cos x}{x-\frac{\pi}{4}} = \frac{0}{0}$ *I.F.*
   In this reason, we set $y = x - \frac{\pi}{4}$, if $x \to \frac{\pi}{4}$, then $y \to 0$. Hence

$$\lim_{y\to 0} \frac{\sin(y+\frac{\pi}{4})-\cos(y+\frac{\pi}{4})}{y} = \lim_{y\to 0} \sqrt{2}\frac{\sin y}{y} = \sqrt{2}. \quad \textbf{(0.75pt)}$$

Thus, f admits a continuous extension given by

$$\tilde{f}(x) = \begin{cases} f(x), & x \in \mathbb{R}\setminus\{\frac{\pi}{4}\}; \\ \sqrt{2}, & x = \frac{\pi}{4}. \end{cases} \quad \textbf{(0.5pt)}$$

قسم الرياضيات

Exercise 4 (4 marks).

1. *State the mean value theorem on $[a,b]$.* **(1pt)**

Theorem 2. *Let $f : [a,b] \to \mathbb{R}$. If f is continuous on $[a,b]$ and differentiable on (a,b). Then:*

$$\exists c \in (a,b) : f(b) - f(a) = (b-a)f'(c).$$

2. *Show that:*

$$\forall x > 0; \ \frac{1}{x+1} < \ln(x+1) - \ln(x) < \frac{1}{x}.$$

Consider $f(t) = \ln(t)$, $t \in [x, x+1]$, $x > 0$. **(0.25pt)** *Let's use the mean value theorem:*
We have, $t \mapsto \ln(t)$ is continuous on $[x,x+1]$ and differentiable on $(x,x+1)$. **(0.5pt)** *Then*

$$\exists c \in (x,x+1) \text{ such that } \ln(x+1)-\ln(x) = \frac{1}{c} \quad \textbf{(0.5pt)}$$

On the other hand, we have $x < c < x+1$, so $\frac{1}{x+1} < \frac{1}{c} = \ln(x+1) - \ln(x) < \frac{1}{x}$. **(0.75pt)**

3. *By using the second question, conclude the following limit:* $\lim_{x\to+\infty} \left(1+\frac{1}{x}\right)^x$. **(1pt)**

$$\frac{1}{x+1} < \ln(x+1)-\ln(x) < \frac{1}{x} \Leftrightarrow \frac{1}{x+1} < \ln(1+\frac{1}{x}) < \frac{1}{x} \Leftrightarrow e^{\frac{x}{x+1}} < \left(1+\frac{1}{x}\right)^x < e.$$

Then, by squeeze Theorem, $\lim_{x\to+\infty} \left(1+\frac{1}{x}\right)^x = e.$


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