

Generalities

Exercise 1. Comment if the following statements are true or false? And give a valid reason for saying so.

- All functions are continuous in their domain.
- All continuous functions on $(0, 1)$ are bounded.
- The sum of two monotone functions is monotone.
- If f and g are discontinuous at x_0 , then so is $f + g$.
- Every odd degree polynomial has at least one real zero.
- It is impossible for a function to be discontinuous at every real number.



Functions

Exercise 2. Find the domain of each function using interval notation:

$$\begin{aligned} 1) f(x) &= \sqrt{\frac{x^2 + 2x + 3}{x^2 - 1}}, \quad 2) f(x) = (1 + \ln(x))^{\frac{1}{x}}, \quad 3) f(x) = \frac{1}{\sqrt{\sin x}}, \quad 4) f(x) = \frac{x^3 + 3}{1 - |x|}, \\ 5) f(x) &= \frac{\cos x}{1 + \sin 2x}, \quad 6) f(x) = \sqrt{2 \cos x - 1}, \quad 7) f(x) = \frac{e^{\sqrt{x}} - 2 \ln(2x - 3)}{\sqrt{x^2 - 1}(e^{-x} - 2)}. \end{aligned}$$

Exercise 3. I) Examine the parity of each of the given function:

$$1) f(x) = \ln(x + \sqrt{1 + x^2}), \quad 2) f(x) = \frac{\tan x - x}{x^3 \cos x}, \quad 3) f(x) = x \frac{a^x - 1}{a^x + 1} \quad (a > 0).$$

II) Show that the following functions are periodic and compute their smallest period:

$$1) f(x) = \sin^2 x, \quad 2) f(x) = \frac{\cos 5x}{\sin 3x}.$$

III) Study the parity and periodicity of the function: $f(x) = \ln(|\sin(\frac{\pi}{2}x)|)$.

Limits

Exercise 4. Calculate the following limits:

$$\begin{array}{lll} 1) \lim_{x \rightarrow +\infty} \left(\frac{x-1}{x+1} \right)^x, & 2) \lim_{x \searrow 0} \frac{\tan x - \sin x}{x^3}, & 3) \lim_{x \rightarrow 0} \frac{x}{a} \left[\frac{b}{x} \right] \quad (a, b) \in \mathbb{R}^* \times \mathbb{R}, \\ 4) \lim_{x \rightarrow -\infty} \frac{x^4 + 1}{\coth \frac{1}{x}}, & 5) \lim_{x \rightarrow +\infty} \sin(\sqrt{x+1}) - \sin(\sqrt{x}), & 6) \lim_{x \rightarrow 0} \sin x \left(x - \left[\frac{1}{x} \right] \right) \\ 7) \lim_{x \rightarrow 0} \frac{\ln \sqrt{x}}{\sqrt[3]{x}}, & 8) \lim_{x \rightarrow \frac{\pi}{4}} \frac{\sin(2x) \sin(x - \frac{\pi}{4})}{\sin x - \cos x}, & 9) \lim_{x \rightarrow 1} (1-x) \tan(\frac{\pi}{2}x), \\ 10) \lim_{x \rightarrow 1} \frac{\ln x}{x^3 - 1}, & 11) \lim_{x \rightarrow a} \frac{\sqrt{x} - \sqrt{a} + \sqrt{x-a}}{\sqrt{x^2 - a^2}}, & 12) \lim_{x \rightarrow 0} \frac{(x+h)^n - h^n}{x}. \end{array}$$

Exercise 5. I) Prove the following limits using their precise definition:

$$1) \lim_{x \rightarrow -2} \frac{x^2 + x + 1}{1 - x} = 1, \quad 2) \lim_{x \searrow 1} \frac{x+2}{x-1} = -\infty, \quad 3) \lim_{x \rightarrow -\infty} \frac{1}{x^3} = 0.$$

II) Choose two ways to prove that the following functions have no limits at x_0 .

$$1) f(x) = \sin(x), \quad x_0 = +\infty, \quad 2) f(x) = x \sqrt{1 + \frac{1}{x^2}}, \quad x_0 = 0.$$

Continuity

Exercise 6. I) Study the continuity of the following functions:

$$1) f(x) = [x] + (x - [x])^2, \forall x \in \mathbb{R}, \quad 2) f(x) = (x + [\cos(x)])^2, \forall x \in [0, \pi],$$

$$3) f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Z}; \\ x & \text{if } x \notin \mathbb{Z}. \end{cases} \quad 4) f(x) = \begin{cases} x^p \sin \frac{1}{x} & \text{if } x \neq 0 \quad (p \in \mathbb{N}); \\ \alpha & \text{if } x = 0 \quad (\alpha \in \mathbb{R}). \end{cases}$$

II) Find the values of a and b that makes each function continuous over the given interval.

$$f(x) = \begin{cases} (x-1)^3 & \text{if } x \leq 0; \\ ax+b & \text{if } 0 < x < 1; \\ \sqrt{x} & \text{if } x \geq 1. \end{cases} \quad f(x) = \begin{cases} -2 \sin x & \text{if } x \leq -\frac{\pi}{2}; \\ a \sin x + b & \text{if } -\frac{\pi}{2} < x < \frac{\pi}{2}; \\ \cos x & \text{if } x \geq \frac{\pi}{2}. \end{cases}$$

Exercise 7. Determine if the given functions can be extended to the continuous functions:

$$1) f(x) = \frac{1}{x} \sin\left(\frac{1}{x}\right), \quad 2) f(x) = \sin(x+1) \ln|x+1|, \quad 3) f(x) = \frac{|\sin(x)|}{x}, \quad 4) f(x) = \sin x \sin\frac{1}{x}.$$

Exercise 8 (Course). Let f be the function given by:

$$f(x) = \begin{cases} \frac{\sin(\frac{1}{x})}{\ln x} & \text{If } x > 0; \\ \frac{\sin x}{x^2 - \pi^2} & \text{If } x \leq 0. \end{cases}$$

1. Determine \mathcal{D}_f , the domain of f .
2. Study the continuity of f on \mathcal{D}_f .
3. Does the function f admit a continuous extension at $x = -\pi$? If yes, set it.



Intermediate Value Theorem (IVT)

Exercise 9. I) Consider the function f defined by:

$$f(x) = x \ln(x) - 1, \forall x \in [1, 2].$$

Prove that f has a root between 1 and 2.

II) Let $a < c < d < b$ and f be a function defined on $[a, b]$. Show that :

If f is continuous on $[a, b]$ and $f(c) + f(d) = k$, then there exists $y_0 \in [a, b]$ such that $f(y_0) = \frac{k}{2}$.