Series n° 1 -The Real Numbers-

Methods of Proof

Exercise 1.

1. *Calculate the first terms to extract the n^{th} formulas for the following sums, and check them by induction:*

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\cdots+\frac{1}{(n-1)n},$$
$$1+3+\cdots+(2n-1).$$

2. *Let $x, y \in \mathbb{R}$. Prove each of the following:*

$$\forall \varepsilon>0, \quad x<y+\varepsilon \Rightarrow x\leq y,$$
$$\forall \varepsilon>0, \quad |x-y|<\varepsilon \Rightarrow x=y.$$

3. *Prove that $\sqrt{3}\notin\mathbb{Q}$.*

4. *Which are true? Justify your answers.*

   i) *If x and y are rational numbers, then $x+y$ is rational.*
   ii) *If x and y are irrational numbers, then $x+y$ is irrational.*
   iii) *If x and y are irrational numbers, then $x\times y$ is irrational.*
   iv) *If x is a rational number, y is an irrational number, then $x+y$ is irrational.*
   v) *If x is a rational number, y is an irrational number, then $x\times y$ is irrational.*

5. *Show which method can be used to disprove the following statements:*

   i) *Let $n\in\mathbb{N}$ and suppose that n is prime. Then 2^n-1 is prime.*
   ii) *If $a,b\in\mathbb{R}$, $a<b$ then $|a|<|b|$.*

6. *Prove that*

$$\text{if } a>0, \text{ and } b>0, \text{ then } a\leq b \Leftrightarrow a^2\leq b^2.$$

7. *If $a\geq 0$ and $b\geq 0$, show that*

$$\sqrt{ab}\leq\frac{a+b}{2}.$$

8. *If a and b are two rational numbers, $a<b$, prove then that there exists an element $c\in\mathbb{Q}$, such that $a<c<b$.*

Rational numbers

Exercise 2.

1. *Let $A_n=0,\underbrace{20232023\ldots2023}_{n\ times}$. Rewrite A_n as a quotient p/q of two integers $p\in\mathbb{Z}$ and $q\in\mathbb{Z}^*$*

2. *Let $A=0,\overline{2024}$. Rewrite A as a quotient p/q of two integers $p\in\mathbb{Z}$ and $q\in\mathbb{Z}^*$*

3. *The same question for the number*

$$B=0,\bar{1}+0,\bar{2}+\cdots+0,\bar{9}.$$

Absolute value

Exercise 3.

1. *Prove that for every $x, y \in \mathbb{R}$, the following properties hold.*

   a) $|x| \leq a \Leftrightarrow -a \leq x \leq +a$

   b) $|x| \geq a \Leftrightarrow x \in]-\infty, -a] \cup [a, +\infty[$

   c) $|x+y| \leq |x| + |y|$

   d) $||x| - |y|| \leq |x - y|$

2. *Find the solutions of the following equations and inequalities.*

   a) $|x+2| = -1$;

   b) $|x^2 - 1| + |x+2| = |x-3|$;

   c) $|x-3| > -2$;

   d) $|x+2| - |x-3| + |x-4| > 0$.

min, max, sup, inf

Exercise 4. *Consider the following subsets of $\mathbb{R}$.*

$$A = \{n \in \mathbb{N},\ n^2 + 2n \leq 0\};\qquad D = \{x \in \mathbb{R},\ -3 \leq x < 3\};$$

$$B = \left\{x \in \mathbb{R},\ \frac{1}{(x^2+1)} > \frac{1}{2}\right\};\qquad E = \{x \in \mathbb{R}\ x^2 \leq 5\};$$

$$C = \{x \in \mathbb{Z},\ -3 \leq x < 3\};\qquad F = \{x \in \mathbb{Q},\ x^2 < 5\}.$$

1. *Determine which one is an interval or not of the previous sets.*

2. *Find, if there exist, the supremum, the infimum, the maximum and the minimum of the previous sets of $\mathbb{R}$. (Justify your answer).*

Exercise 5. *Let A and B be nonempty bounded subsets of $\mathbb{R}$.*

1. *Prove that*

   i) *If $A \subset B$, then $Inf B \leq Inf A \leq Sup A \leq Sup B$;*

   ii) $Sup(A \cup B) = max(Sup A, Sup B)$;

   iii) $Inf(A \cup B) = min(Inf A, Inf B)$.

2. *For the following sets, find the supremum and the infimum (if they exist) and indicate if they are also a maximum (resp. a minimum).*

$$C = \left\{1 + \frac{1}{2n},\ n \in \mathbb{N}^*\right\};\qquad D = \left\{-1 + \frac{1}{2n+1},\ n \in \mathbb{N}\right\}.$$

3. *Let*

$$E = \left\{(-1)^n + \frac{1}{n},\ n \in \mathbb{N}^*\right\}.$$

   Deduce $\sup E$ and $\inf E$.

The floor of x

Exercise 6. *The floor of x is the largest integer less than or equal to x, that is*

$$[x] = \max\{n \in \mathbb{Z},\ n \leq x\}.$$

1. *Sketch the function's graph $x \mapsto [x]$.*

2. *Determine $[\pi]$, $[-\pi]$ and $[4]$.*

3. *Prove that for every $x, y \in \mathbb{R}$ and $n \in \mathbb{N}^*$, the following properties are satisfied:*

   $x] \leq x n$

   $\left[\frac{n x]}{n}\right] \leq [x]$

   a) $x \leq y \Rightarrow [x] \leq [y]$;

   b) $\left[\frac{[nx]}{n}\right] = [x]$;

   c) $[x] + [y] \leq [x+y] \leq [x] + [y] + 1$;

   d) $[x+1] = [x] + 1$.

4. *Solve in $\mathbb{R}$, the following equations*

   a) $[\sqrt{x}] = 2$,

   b) $[\sqrt{x^2}] = 4$,

   c) $[x[x]] = 1$.