Series $n°$ 2 -Numerical Sequences-

Generalities

Exercise 1. *Comment if the following statements are true or false? And give a valid reason for saying so.*

- *Every bounded sequence is convergent.* ***False***
  Counterexample. *Let* $u_n = (-1)^n$ *be a bounded sequence such that* $|(-1)^n| \le 1$, *but*

$$\lim_{n\to+\infty}(-1)^n = \begin{cases} 1 & \text{if } n = 2k, \\ -1 & \text{if } n = 2k+1. \end{cases}$$

  does not exist. Then $((-1)^n)_n$ *D.V.*

- *Every sequence* $(u_n)_n$ *satisfies* $u_{n+1} \le u_n$ *is decreasing.* ***False***
  Counterexample. *Let* $u_n = (-1)^n$, *we have* $u_1 < u_2$, *but the sequence* $(-1)^n$ *is not decreasing.*

- *The sum of two divergent sequences is a divergent sequence.* ***False***
  Counterexample. *Let* $u_n = (-1)^n$ *and* $v_n = (-1)^{n+1}$ *be two divergent sequences, but*

$$\lim_{n\to+\infty}(u_n + v_n) = \lim_{n\to+\infty}(-1)^n + (-1)^{n+1} = 0 \quad \text{C.V.}$$

- *The product of two divergent sequences is a divergent sequence.* ***False***
  Counterexample. *Let* $u_n = (-1)^n$ *and* $v_n = (-1)^{n+1}$ *be two divergent sequences, but*

$$\lim_{n\to+\infty} u_n v_n = \lim_{n\to+\infty}(-1)^{2n+1} = -1 \quad \text{C.V.}$$

- *If the sequence* $(u_n)_{n\in\mathbb{N}}$ *is increasing and* $u_n \le n$, *then* $(u_n)_n$ *is convergent.* ***False***
  Counterexample. *Let* $u_n = \sqrt{n}$ *be an increasing sequence and* $\sqrt{n} \le n$, *but* $(u_n)_n$ *is divergent.*

- *If* $\lim\limits_{n\to+\infty} u_n^2 = \ell^2$, *then* $\lim\limits_{n\to+\infty} u_n = \ell$. ***False***
  Counterexample. *Let* $u_n = (-1)^n$, *we have*

$$\lim_{n\to+\infty} u_n^2 = \lim_{n\to+\infty}(-1)^{2n} = 1 \not\Longrightarrow \lim_{n\to+\infty} u_n = \lim_{n\to+\infty}(-1)^n = 1,$$

  because $\lim\limits_{n\to+\infty}(-1)^n$ *does not exist.*

- *If* $\lim\limits_{n\to+\infty} |u_n| = |\ell|$, *then* $\lim\limits_{n\to+\infty} u_n = \ell$. ***False***
  Counterexample. *Let* $u_n = (-1)^n$, *we have* $|u_n| \underset{n\to+\infty}{\longrightarrow} 1$, *but according to our knowledge* $(-1)^n$ *D.V.*

Convergent Sequences

Exercise 2. ***I)*** *Consider the real sequence* $(u_n)_{n\in\mathbb{N}^*}$ *defined as:* $u_n = \dfrac{n+1}{2n}$.

$$(u_n)_n \text{ converges to } \ell \Longleftrightarrow (\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N};\ n \ge n_0 \Rightarrow |u_n - \ell| < \varepsilon).$$

Find the absolute value of this inequality $0.49 < u_n < 0.51$, *then deduce* ℓ, ε *and* n_0.
we can write the interval as absolute value,

$$\text{if } a < x < b \Longleftrightarrow |x - x_0| < r_0, \text{ such that } x_0 = \frac{a+b}{2}, \text{ and } r_0 = \frac{b-a}{2}.$$

So, $x_0 = \dfrac{0.49+0.51}{2} = \dfrac{1}{2}$ *and* $r_0 = \dfrac{0.51-0.49}{2} = \dfrac{1}{10^2}$.
Therefore, $|u_n - \dfrac{1}{2}| < 10^{-2}$, *we conclude that* $\ell = \dfrac{1}{2}$, $\varepsilon = 10^{-2}$.
Applying the above definition, we get

$$|u_n - \frac{1}{2}| = \left|\frac{n+1}{2n} - \frac{1}{2}\right| = \frac{1}{2n} < 10^{-2}$$
$$\Rightarrow 2n > 100 \Rightarrow n > 50.$$

Then, its enough to take $n_0 = [50] + 1 = 51$.
► *Use the previous definition to prove each of the following:*

- $\lim_{n\to+\infty} \dfrac{2\ln(n+1)}{\ln(n)} = 2$. *Using the definition, we have*

$$\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N};\ n \geq n_0 \Rightarrow \left|\frac{2\ln(n+1)}{\ln(n)} - 2\right| < \varepsilon$$
$$\left|\frac{2\ln(n+1)}{\ln(n)} - 2\right| = \left|\frac{2\ln(n+1) - 2\ln(n)}{\ln(n)}\right|$$
$$= \frac{2\ln(\frac{n+1}{n})}{\ln(n)} \leq \frac{2\ln(2)}{\ln(n)}$$

Since, $1 + \dfrac{1}{n} \leq 2 \Rightarrow \ln(1+\dfrac{1}{n}) \leq \ln(2)$. *Solving for* n *is now easy*

$$\frac{2\ln(2)}{\ln(n)} < \varepsilon \Leftrightarrow \ln(n) > \frac{\ln(4)}{\varepsilon} \Leftrightarrow n > \exp\left(\frac{\ln(4)}{\varepsilon}\right)$$

Then, it's enough to take $n_0 = \left[\exp\left(\dfrac{\ln(4)}{\varepsilon}\right)\right] + 1 \in \mathbb{N}^*$.

- $\lim_{n\to+\infty} \dfrac{5n+3}{7n+1} = \dfrac{5}{7}$. *Using the definition, we have*

$$\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N};\ n \geq n_0 \Rightarrow \left|\frac{5n+3}{7n+1} - \frac{5}{7}\right| < \varepsilon$$
$$\left|\frac{5n+3}{7n+1} - \frac{5}{7}\right| = \left|\frac{16}{49n+7}\right| = \frac{16}{49n+7}$$

Solving for n *is now easy*

$$\frac{16}{49n+7} < \varepsilon \Leftrightarrow \frac{16}{\varepsilon} < 49n + 7 \Leftrightarrow n > \frac{16-7\varepsilon}{49\varepsilon}.$$

Then, its enough to take $n_0 = \left[\dfrac{16-7\varepsilon}{49\varepsilon}\right] + 1 \in \mathbb{N}$.

II) *Study the convergence of the following sequences:*

1) $u_n = \dfrac{\sin(n)}{n}$.
Let $v_n = \sin(n)$ *is bounded, because* $|\sin(n)| \leq 1$, *and* $w_n = \dfrac{1}{n} \underset{n\to+\infty}{\longrightarrow} 0$.
Moreover, $(u_n)_n$ *is the product of two sequences, one of them is bounded and the other is convergent to zero. Then,* (u_n) *converges to zero.*

2) $u_n = \dfrac{1}{n^3}\sum_{k=1}^{n} k$, $n \in \mathbb{N}^*$.
We notice that the numerator is a sum of an arithmetic sequence, so

$$\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Therefore, $$u_n = \frac{n(n+1)}{2} \times \frac{1}{n^3} \underset{n\to+\infty}{\longrightarrow} 0.$$

Then, $(u_n)_n$ *is convergent to* 0.

3) $u_n = \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k}}$.
For every $n \in \mathbb{N}^*$, *we have*

$$\begin{array}{rcl}
1 & \le & k & \le n \\
n^2+1 & \le & n^2+k & \le n^2+n \\
\sqrt{n^2+1} & \le & \sqrt{n^2+k} & \le \sqrt{n^2+n} \\
\sum_{k=1}^{n} \frac{1}{\sqrt{n^2+n}} & \le & u_n & \le \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+1}} \\
\frac{n}{\sqrt{n^2+n}} & \le & u_n & \le \frac{n}{\sqrt{n^2+1}} \\
\lim_{n\to+\infty} \frac{n}{n\sqrt{1+1/n}} & \le & \lim_{n\to+\infty} u_n & \le \lim_{n\to+\infty} \frac{n}{n\sqrt{1+1/n^2}} \\
1 & \le & \lim_{n\to+\infty} u_n & \le 1.
\end{array}$$

Then, by squeeze theorem, the sequence $(u_n)_n$ *converges to* 1.

4) $u_n = \frac{3^n - 7^n}{3^n + 7^n}$. *We have*

$$\lim_{n\to+\infty} u_n = \lim_{n\to+\infty} \frac{\left(\frac{3}{7}\right)^n - 1}{\left(\frac{3}{7}\right)^n + 1} = -1.$$

Since, $\left(\frac{3}{7}\right)^n$ *is a geometric sequence of reason* $\left(\frac{3}{7}\right) < 1$,*then* $\left(\frac{3}{7}\right)^n \underset{n\to+\infty}{\longrightarrow} 0$.
Therefore, $(u_n)_n$ *converges to* -1.

5) $u_n = \frac{1}{n^2} \sum_{k=1}^{n} [kx]$, $n \in \mathbb{N}^*$, *we have*

$$\begin{array}{rcl}
x-1 & < & [x] & \le x \\
2x-1 & < & [2x] & \le 2x \\
3x-1 & < & [3x] & \le 3x \\
& & \vdots & \\
nx-1 & < & [nx] & \le nx
\end{array}$$

By adding all sides, we find

$$\begin{array}{rcl}
(x-1)+(2x-1)+\cdots+nx+1 & < & [x]+[2x]+\cdots+[nx] & \le x+2x+\cdots+nx \\
\frac{1}{n^2}\left[x(1+2+\cdots+n)+(1+1+\cdots+1)\right] & < & u_n & \le \frac{1}{n^2}\left[x(1+2+\cdots+n)\right] \\
\frac{nx(n+1)}{2n^2} - \frac{n}{n^2} & < & u_n & \le \frac{nx(n+1)}{2n^2} \\
\lim_{n\to+\infty}\left[\frac{nx(n+1)}{2n^2} - \frac{n}{n^2}\right] & < & \lim_{n\to+\infty} u_n & \le \lim_{n\to+\infty} \frac{nx(n+1)}{2n^2} \\
x/2 & < & \lim_{n\to+\infty} u_n & \le x/2.
\end{array}$$

Then by squeeze theorem, the sequence $(u_n)_n$ *converges to* $x/2$.

6) $u_n = \sqrt[3]{1+n} - \sqrt[3]{n}$.
We have

$$a^3 - b^3 = (a-b)(a^2+ab+b^2).$$

Let $a = \sqrt[3]{1+n}$ *and* $b = \sqrt[3]{n}$, *then*

$$\begin{aligned}
\lim_{n\to+\infty} u_n &= \lim_{n\to+\infty} \sqrt[3]{1+n} - \sqrt[3]{n} \\
&= \lim_{n\to+\infty} \frac{(\sqrt[3]{1+n})^3 - (\sqrt[3]{n})^3}{(\sqrt[3]{1+n})^2 + \sqrt[3]{n(1+n)} + (\sqrt[3]{n})^2} \\
&= \lim_{n\to+\infty} \frac{1}{(\sqrt[3]{1+n})^2 + \sqrt[3]{n(1+n)} + (\sqrt[3]{n})^2} = 0
\end{aligned}$$

Therefore u_n *C.V.*

7) $u_n = \left(\dfrac{n^2-n+3}{n^2+3n-1}\right)^n$.

Firstly

$$\begin{aligned}\lim_{n\to+\infty} u_n &= \lim_{n\to+\infty}\left(\frac{n^2-n+3}{n^2+3n-1}\right)^n\\ &= \lim_{n\to+\infty}\left(1-\frac{4n-4}{n^2+3n-1}\right)^n \text{ (using Euclidean division)}\\ &= \lim_{n\to+\infty}\left(1-\frac{4n-4}{n^2+3n-1}\right)^{n\left(\frac{n^2+3n-1}{-4n+4}\right)\left(\frac{-4n+4}{n^2+3n-1}\right)}.\end{aligned}$$

We have $\lim_{n\to+\infty} n\left(\dfrac{-4n+4}{n^2+3n-1}\right) = -4$. *Moreover, we set* $X = \dfrac{-4n+4}{n^2+3n-1}$ *to obtain*

$$n\to+\infty \Longrightarrow X\to 0.$$

Hence

$$\lim_{n\to+\infty} u_n = \lim_{X\to 0}(1+X)^{\frac{-4}{X}} = \lim_{X\to 0} e^{\frac{-4\ln(1+X)}{X}} = e^{-4}.$$

Then, u_n *is C.V.*

8) $u_n = \dfrac{5\times 7\times\cdots\times(2n+5)}{4\times 7\times\cdots(3n+4)}$.

This type is always treated in the same way; u_{n+1} *must be expressed as a function of* u_n, *i.e.,*

$$u_{n+1} = \frac{5\times 7\times\cdots\times(2n+5)\times(2n+7)}{4\times 7\times\cdots\times(3n+4)\times(3n+7)} = u_n\times\frac{2n+7}{3n+7}.$$

If the limit ℓ *exists, then it verifies*

$$\ell = \ell\times\frac{2}{3} \Longrightarrow \ell(1-\frac{2}{3}) = 0 \Longrightarrow \ell = 0.$$

It remains to show that u_n *converges. Clearly,* $u_n > 0$, *i.e.,* u_n *is bounded below, and we note*

$$\frac{u_{n+1}}{u_n} = \frac{2}{3} < 1.$$

Therefore, $(u_n)_n$ *is decreasing and bounded below, so it converges.*
From the above result, the only possible limit is "0".

Adjacent Sequences

Exercise 3. ***I)*** *Let* $a, b\in\mathbb{R},\ 0 < a\leq b$. *Let the numerical sequences* $(u_n)_n$ *and* $(v_n)_n$ *be given by*

$$\begin{cases} u_0 = a,\\ u_{n+1} = \dfrac{2u_n+v_n}{3},\ \forall n\in\mathbb{N}.\end{cases}\qquad\qquad \begin{cases} v_0 = b,\\ v_{n+1} = \dfrac{2v_n+u_n}{3},\ \forall n\in\mathbb{N}.\end{cases}$$

1. *Calculate* u_1 *and* v_1.

$$u_1 = \frac{2u_0+v_0}{3} = \frac{2a+b}{3},\quad v_1 = \frac{2v_0+u_0}{3} = \frac{2b+a}{3}.$$

2. *Prove that:* "$\forall n\in\mathbb{N},\ u_n\leq u_{n+1}\leq v_{n+1}\leq v_n$" $\cdots\ P(n)$
We will show by induction that the property $P(n)$ *is true for* $n\in\mathbb{N}$.
Step 01. *For* $n = 0$, *we have*

$$\bullet\ u_1 = \frac{2a+b}{3} \geq \frac{2a+a}{3} = a = u_0. \text{ Then } u_1\geq u_0.$$

$$\bullet\ v_1 = \frac{a+2b}{3} \leq \frac{b+2b}{3} = b = v_0. \text{ Then } v_1\leq v_0.$$

Morever, $u_1 - v_1 = \dfrac{2a+b-a-2b}{3} = \dfrac{a-b}{3} \leq 0$ *(because $a \leq b$), i.e.,* $u_1 \leq v_1$.
Therefore, $\quad u_0 \leq u_1 \leq v_1 \leq v_0$. *So $P(0)$ is true.*
Step 02. *Assume that $P(n)$ is true $\forall n \in \mathbb{N}$ and show that $P(n+1)$ is true.*
We have

$$u_{n+2} = \frac{2u_{n+1} + v_{n+1}}{3} \geq \frac{2u_{n+1} + u_{n+1}}{3} = u_{n+1},$$
$$v_{n+2} = \frac{2v_{n+1} + u_{n+1}}{3} \leq \frac{2v_{n+1} + v_{n+1}}{3} = v_{n+1},$$

and

$$u_{n+2} - v_{n+2} = \frac{2u_{n+1} + v_{n+1}}{3} - \frac{u_{n+1} + 2v_{n+1}}{3} = \frac{u_{n+1} - v_{n+1}}{3} \leq 0 \text{ (because } u_{n+1} \leq v_{n+1}).$$

Consequently, $u_{n+1} \leq u_{n+2} \leq v_{n+2} \leq v_{n+1}$, *i.e., $P(n+1)$ is true, so then $\forall n \in \mathbb{N}$,*

$$u_n \leq u_{n+1} \leq v_{n+1} \leq v_n.$$

3. *Express $(v_n - u_n)_n$ in terms of $(b-a)$. Deduce that $(u_n)_n$ and $(v_n)_n$ are adjacent;*
• *As* $\quad v_{n+1} - u_{n+1} = \dfrac{2v_n + u_n - 2u_n - v_n}{3} = \dfrac{v_n - u_n}{3}$, *the sequence $(v_n - u_n)_n$ is a geometric sequence of reason $\dfrac{1}{3}$, and so*

$$v_n - u_n = \left(\frac{1}{3}\right)^n (v_0 - u_0) = \left(\frac{1}{3}\right)^n (b-a).$$

Given that $0 < \dfrac{1}{3} < 1$, *the sequence $(v_n - u_n)_n$ converges to 0. That is* $\lim\limits_{n\to+\infty} \left(\dfrac{1}{3}\right)^n (b-a) = 0$.
According to the previous question, we have

$$\begin{cases} 1) \quad u_n \leq u_{n+1} \Leftrightarrow (u_n)_n \text{ is increasing.} \\ 2) \quad v_{n+1} \leq u_n \Leftrightarrow (v_n)_n \text{ is decreasing.} \\ 3) \quad \lim\limits_{n\to+\infty} v_n - u_n = 0. \end{cases}$$

Hence, we deduce that $(u_n)_n$ and $(v_n)_n$ are adjacent. They therefore converge to the same limit ℓ.
4. *Express $(v_n + u_n)_n$ in terms of $(a+b)$, and set the limit of each sequence $(u_n)_n$ and $(v_n)_n$.*

$$v_{n+1} + u_{n+1} = \frac{2v_n + u_n + 2u_n + v_n}{3} = v_n + u_n.$$

Thus, the sequence $(v_n + u_n)_n$ is constant. Then

$$\forall n \in \mathbb{N}, \quad v_n + u_n = v_0 + u_0 = a + b.$$

5. *Determine the limit of $(u_n)_n$ and $(v_n)_n$.*
As $(u_n)_n$ and $(v_n)_n$ are adjacent, so

$$\lim_{n\to+\infty} u_n = \lim_{n\to+\infty} v_n = \ell.$$

Thus $$\lim_{n\to}(u_n + v_n) = 2\ell.$$
In the other side, we have $v_n + u_n = a + b$, i.e., $\lim\limits_{n\to\infty} v_n + u_n = a + b$. *Therefore*

$$2\ell = a + b \Rightarrow \ell = \frac{a+b}{2}.$$

II) *Consider the sequence of general term:* $v_n = 1 - \frac{1}{2} + \frac{1}{3} + \cdots + \frac{(-1)^{n+1}}{n}, \;\; \forall\, n \in \mathbb{N}^*$.

► *Prove that the two subsequences* $(v_{2n})_n$ *and* $(v_{2n+1})_n$ *are adjacent, then deduce the nature of* $(v_n)_n$.

For every $n \in \mathbb{N}^*$, *we have* $v_{2n} = 1 - \frac{1}{2} + \cdots + \frac{(-1)^{2n-1}}{2n} = 1 - \frac{1}{2} + \cdots + \frac{1}{2n-1} - \frac{1}{2n}$. *Then*

$$\begin{aligned} v_{2(n+1)} - v_{2n} &= \left(1 - \frac{1}{2} + \cdots + \frac{1}{2n+1} - \frac{1}{2n+2}\right) - \left(1 - \frac{1}{2} + \cdots + \frac{1}{2n-1} - \frac{1}{2n}\right) \\ &= \frac{1}{2n+1} - \frac{1}{2n+2} = \frac{1}{(2n+1)(2n+2)} > 0. \end{aligned}$$

That is, $(v_{2n})_n$ *is increasing. Furthermore*

$$\begin{aligned} v_{2(n+1)+1} - v_{2n+1} &= \left(1 - \frac{1}{2} + \cdots - \frac{1}{2n+2} + \frac{1}{2n+3}\right) - \left(1 - \frac{1}{2} + \cdots - \frac{1}{2n} + \frac{1}{2n+1}\right) \\ &= \frac{-1}{2n+2} + \frac{1}{2n+3} = \frac{-1}{(2n+2)(2n+3)} < 0. \end{aligned}$$

That is, $(v_{2n+1})_n$ *is decreasing.*

Now, $\lim\limits_{n\to+\infty} v_{2n} - v_{2n+1} = -\frac{1}{2n+1} \underset{n\to+\infty}{\longrightarrow} 0$. *Under the above-obtained results, we deduce that,* $(v_{2n})_n$ *and* $(v_{2n+1})_n$ *are adjacent sequences.*

Conclusion. *As* $(v_{2n})_n$ *and* $(v_{2n+1})_n$ *are adjacent, these two sequences are convergent and have the same limit* ℓ. *Consequently,* $(v_n)_n$ *is converges to* ℓ.

Divergent Sequences

Exercise 4. ***I)*** *Consider the real sequence* $(u_n)_{n\in\mathbb{N}^*}$ *defined as:* $u_n = n\sqrt{n}$.

$$(u_n)_n \text{ diverges to } \pm\infty \iff \begin{pmatrix} \forall\, A > 0,\, \exists\, n_0 \in \mathbb{N}, \forall\, n \in \mathbb{N};\; n \geq n_0 \Rightarrow u_n > A \\ \forall\, A > 0,\, \exists\, n_0 \in \mathbb{N}, \forall\, n \in \mathbb{N};\; n \geq n_0 \Rightarrow u_n < -A \end{pmatrix}.$$

Explain the definition that corresponds to the limit of $(u_n)_n$, *such that* $u_n \in (10^6; +\infty)$, *then deduce the values of* A *and* n_0.

We have $u_n \in (10^6; +\infty)$ *i.e,* $u_n > 10^6$, *then* u_n *is bounded below.*

On the other hand, $u_n = n\sqrt{n}$ *is increasing.*

So, $(u_n)_n$ *is bounded below and increasing, then,* $(u_n)_n$ *is divergent to* $+\infty$.

Therfore, we use the first definition, by it we conclude:

$$u_n > 10^6 \Rightarrow A = 10^6,$$

and $n\sqrt{n} > 10^6 \Leftrightarrow (\sqrt{n})^3 > 10^6 \Leftrightarrow n > 10^4$, *it's enough to take* $n_0 = [10^4] + 1 = 10001$.

► *Use the previous definitions to prove each of the following:*

- $\lim\limits_{n\to+\infty} 3^{2n+1} = +\infty$. *Using the definition, we have*

$$\begin{aligned} \forall\, A > 0,\, \exists\, n_0 \in \mathbb{N}, \forall\, n \in \mathbb{N};\; n \geq n_0 &\Rightarrow 3^{2n+1} > A \\ &\Rightarrow (2n+1)\ln(3) > \ln(A) \\ &\Rightarrow n > \frac{2\ln(A)}{2\ln(3)} - \frac{1}{2}. \end{aligned}$$

Then, it's enough to take $n_0 = \left[\frac{\ln(A)}{2\ln(3)} - \frac{1}{2}\right] + 1$.

- $\lim\limits_{n\to+\infty} \frac{1}{2}(e^{\frac{1}{n}} - n) = -\infty$. *Using the definition, we have*

$$\begin{aligned}
\forall A > 0, \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N};\ n \geq n_0 &\Rightarrow \frac{1}{2}(e^{\frac{1}{n}} - n) < -A \\
&\Rightarrow e^{\frac{1}{n}} - n < -2A \\
&\Rightarrow -n < -(2A + e^{\frac{1}{n}}) \\
(0 < \frac{1}{n} \leq 1 \Leftrightarrow 1 < e^{\frac{1}{n}} \leq e) &\Rightarrow -n < -(2A + e) \\
&\Rightarrow n > 2A + e
\end{aligned}$$

Then, it's enough to take $n_0 = [2A + e] + 1$.

II) *Choose the suitable way (subsequences, comparison test) to show that the below sequences are divergent:*

1) $u_n = \cos\left(\frac{n\pi}{4}\right)$, *we choose the subsequences way:*
If $n = 4k,\ k \in \mathbb{N}$, *we find* $u_{4k} = \cos\left(\dfrac{4k\pi}{4}\right) = cos(k\pi) = (-1)^k$,
since, the subsequence $(-1)^k$ *is divergent, then* $(u_n)_n$ *is divergent.*

2) $u_n = \left(1 + \frac{1}{n}\right)^{n^2}$, *we choose the comparison test:*
Since $\left(1 + \frac{1}{n}\right)^n$ *is a monotonically increasing sequence, therefore* $\left(1 + \frac{1}{n}\right)^n \geq 2$.

$$u_n = \left(1 + \frac{1}{n}\right)^{n^2} = \left(\left(1 + \frac{1}{n}\right)^n\right)^n \geq 2^n \underset{n\to+\infty}{\longrightarrow} +\infty.$$

By comparison test, we deduce $(u_n)_n$ *is divergent to* $+\infty$.

3) $u_n = \dfrac{n + (-1)^n n}{n - (-1)^n \frac{n}{2}}$, *we choose the subsequences way:*
If $n = 2k,\ k \in \mathbb{N}$, *we get*

$$u_{2k} = \frac{2k + 2k}{2k - \frac{2k}{2}} = 4 \text{ is a constant sequence converges to } 4.$$

If $n = 2k + 1,\ k \in \mathbb{N}$, *we get*

$$u_{2k+1} = \frac{2k + 1 - 2k - 1}{2k + 1 - \frac{2k+1}{2}} = 0 \text{ is a constant sequence converges to } 0.$$

Then, $\lim\limits_{n\to+\infty} u_{2k} \neq \lim\limits_{n\to+\infty} u_{2k+1}$.
Hence, $(u_n)_n$ *is divergent.*

4) $\dfrac{n! + 3^n}{n - 2^n}$, *we choose the comparison test:*
The order of growth of the denominator is 2^n : $\dfrac{n! + 3^n}{n - 2^n} = \dfrac{\frac{n!}{2^n} + \left(\frac{3}{2}\right)^n}{\frac{n}{2^n} - 1}$.
We have, $n < 2^n \Rightarrow \frac{n}{2^n} - 1 < 0 \Rightarrow \dfrac{1}{\frac{n}{2^n} - 1} < 0$, *and* $\frac{n}{2^n} \underset{n\to+\infty}{\longrightarrow} 0$. $(*)$
On the other hand, we have

$$\frac{n!}{2^n} + \left(\frac{3}{2}\right)^n > \frac{n}{2^n} + \left(\frac{3}{2}\right)^n$$

$$(*) \Downarrow$$

$$u_n < \frac{\frac{n}{2^n} + \left(\frac{3}{2}\right)^n}{\frac{n}{2^n} - 1} \underset{n\to+\infty}{\longrightarrow} -\infty.$$

By comparison test, we deduce $(u_n)_n$ *is divergent to* $-\infty$.

Cauchy Sequences

Exercise 5. ***I)*** *Consider the real sequence* $(u_n)_{n\in\mathbb{N}^*}$*, such that:*

$$|u_{n+1}-u_n| \le k|u_n - u_{n-1}|,\ \ n\in\mathbb{N}^*,\ k\in(0;1).$$

1) *Show that:* $\forall\, n\in\mathbb{N}^*,\ |u_{n+1}-u_n| \le k^{n-1}|u_2-u_1|$.
According to the above inequality, we have

$$\begin{aligned}
|u_{n+1}-u_n| &\le k|u_n-u_{n-1}| \\
&\le k\left(k|u_{n-1}-u_{n-2}|\right) \\
&\le kk\left(k|u_{n-2}-u_{n-3}|\right) \\
&\vdots \\
&\le \overbrace{kk\cdots k}^{(n-1)\ \textit{times}}|u_2-u_1| = k^{n-1}|u_2-u_1|.
\end{aligned}$$

2) *Prove that:* $\forall\, p,q\in\mathbb{N}^*,\ q<p$, *we have* $|u_p-u_q| \le \dfrac{k^{q-1}}{1-k}|u_2-u_1|$.
According to the previous question, we have

$$\begin{aligned}
|u_p-u_q| &= |u_p-u_{p-1}+u_{p-1}-u_{p-2}+u_{p-2}\cdots+u_{q-1}-u_q| \\
&\le |u_p-u_{p-1}|+|u_{p-1}-u_{p-2}|+\cdots+|u_{q-1}-u_q| \\
&\le k^{p-2}|u_2-u_1|+k^{p-3}|u_2-u_1|+\cdots+k^{q-1}|u_2-u_1| \\
&= \left(k^{p-2}+k^{p-3}+\cdots+k^{q-1}\right)|u_2-u_1| \\
&= k^{q-1}\left(k^{p-(q+1)}+k^{p-(q+2)}+\cdots+1\right)|u_2-u_1| \\
\sum_{i=0}^{p-q-1}k^i\ \ \underset{\textit{of reason } k}{\textit{is a geometric sequence}}\qquad &= k^{q-1}\left(\sum_{i=0}^{p-q-1}k^i\right)|u_2-u_1| \\
&= k^{q-1}\left(\frac{1-k^{p-q}}{1-k}\right)|u_2-u_1| \\
&= \frac{k^{q-1}}{1-k}\left(1-k^{p-q}\right)|u_2-u_1|,\quad 1-k^{p-q}<1 \\
&\le \frac{k^{q-1}}{1-k}|u_2-u_1|,
\end{aligned}$$

3) *Deduce the nature of* $(u_n)_n$*. Applying the definition, we have*

$$\forall\,\varepsilon>0,\ \exists\, n_0\in\mathbb{N},\forall\, p,q\in\mathbb{N};\ p\ge n_0, q\ge n_0 \Rightarrow |u_p-u_q|<\varepsilon$$

$$\begin{aligned}
|u_p-u_q|<\varepsilon &\Rightarrow \frac{k^{q-1}}{1-k}|u_2-u_1|<\varepsilon \\
&\Rightarrow (q-1)\ln(k) < \ln\left(\frac{(1-k)\varepsilon}{|u_2-u_1|}\right) \\
&\Rightarrow q > \ln\left(\frac{(1-k)\varepsilon}{|u_2-u_1|}\right)\times\frac{1}{\ln(k)}+1.
\end{aligned}$$

Then, it's enough to take $n_0 = \left[\ln\left(\dfrac{(1-k)\varepsilon}{|u_2-u_1|}\right)\times\dfrac{1}{\ln(k)}+1\right]+1.$
Consequently, $(u_n)_n$ *is a Cauchy sequence, so it is convergent.*

II) *In view of the Cauchy criterion, determine the nature of the sequences* $(u_n)_n$ *defined by:*

1) $u_n = \sum\limits_{k=1}^{n}\dfrac{1}{\sqrt{k}} = 1+\dfrac{1}{\sqrt{2}}+\cdots+\dfrac{1}{\sqrt{n}}$*. Using the Cauchy criterion:*

$$\forall\,\varepsilon>0,\ \exists\, n_0\in\mathbb{N}^*,\forall\, p,q\in\mathbb{N}^*;\ p>q,\ p\ge n_0, q\ge n_0 \Rightarrow |u_p-u_q|<\varepsilon.$$

$$|u_p - u_q| = \left|1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{q}} + \frac{1}{\sqrt{q+1}} + \cdots + \frac{1}{\sqrt{p}} - \left(1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{q}}\right)\right|$$
$$= \left|\frac{1}{\sqrt{q+1}} + \frac{1}{\sqrt{q+2}} + \cdots + \frac{1}{\sqrt{p}}\right|.$$

If we take $p = 2n > q = n$, *we get*

$$|u_{2n} - u_n| = \left|\frac{1}{\sqrt{n+1}} + \frac{1}{\sqrt{n+2}} + \cdots + \frac{1}{\sqrt{2n}}\right|$$
$$\geq \left|\frac{n}{\sqrt{2n}}\right| = \sqrt{\frac{n}{2}} \geq \frac{1}{\sqrt{2}}.$$

Therefore, $\exists\, \varepsilon = \dfrac{1}{\sqrt{2}} > 0,\ \forall\, n_0 \in \mathbb{N}^*,\ \exists\, p = 2n\ \wedge\ \exists\, q = n,\ \exists\, p \geq n_0 \wedge q \geq n_0\ \wedge\ |u_p - u_q| \geq \varepsilon.$
Thus, $(u_n)_n$ *isn't Cauchy and deduce that* $(u_n)_n$ *is divergent.*

2) $u_n = \displaystyle\sum_{k=1}^{n} \frac{\sin k}{2^k} = \frac{\sin(1)}{2} + \frac{\sin(2)}{2^2} + \cdots + \frac{\sin(n)}{2^n}$. *Using the Cauchy criterion:*

$$\forall\, \varepsilon > 0,\ \exists\, n_0 \in \mathbb{N}^*, \forall\, p, q \in \mathbb{N}^*;\ p > q,\ p \geq n_0, q \geq n_0 \Rightarrow |u_p - u_q| < \varepsilon.$$

$$|u_p - u_q| = \left|\sum_{k=1}^{p} \frac{\sin k}{2^k} - \sum_{k=1}^{q} \frac{\sin k}{2^k}\right| = \left|\sum_{k=q+1}^{p} \frac{\sin k}{2^k}\right|$$
$$\leq \sum_{k=q+1}^{p} \frac{1}{2^k} = \frac{1}{2^{q+1}} + \frac{1}{2^{q+2}} + \cdots + \frac{1}{2^p}$$
$$= \frac{1}{2^{q+1}}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{p-q-1}}\right) \quad \textit{is a geometric sequence of reason } 1/2$$
$$= \frac{1}{2^{q+1}}\left(1 \times \frac{1 - \left(\frac{1}{2}\right)^{p-q}}{11 - \frac{1}{2}}\right) = \frac{1}{2^q}\left(1 - \left(\frac{1}{2}\right)^{p-q}\right)$$
$$\leq \frac{1}{2^q} < \varepsilon \Rightarrow 2^q > \frac{1}{\varepsilon} \Rightarrow q > \frac{\ln(1/\varepsilon)}{\ln(2)}.$$

Then, it's enough to take $n_0 = \left[\dfrac{\ln(1/\varepsilon)}{\ln(2)}\right] + 1 \in \mathbb{N}$, *therefore* $(u_n)_n$ *is convergent.*

Recurrence Sequences

Exercise 6. *Let the sequence* $(u_n)_n$ *be recursively defined by:*

$$\begin{cases} u_0 \in \mathbb{R}, \\ u_{n+1} = \dfrac{u_n}{(u_n)^2 + 1}, \quad n \in \mathbb{N}. \end{cases}$$

1. *We set* $u_0 = \alpha$. *Determine* α, *such that the sequence* $(u_n)_n$ *is zero.*

$$\Big((u_n)_n \textit{ is zero} \iff u_0 = u_1 = u_2 = \cdots = u_n = 0\Big) \Rightarrow \Big(\alpha = 0 \textit{ because } u_0 = \alpha\Big).$$

2. *For* $u_n > 0$.

   (a) Prove that: "$\forall n \in \mathbb{N},\ u_n > 0$". *By induction, we have*
   Step 01. *For* $n = 0$, *we get from the data* $u_0 > 0$.
   Step 02. *Suppose that* $u_n > 0,\ \forall n \in \mathbb{N}$ *is true and we must show that* $u_{n+1} > 0,\ \forall n \in \mathbb{N}$.
   The condition $u_n > 0$ *give us* $\dfrac{1}{(u_n)^2 + 1} > 0$, *and so* $u_{n+1} = \dfrac{u_n}{(u_n)^2 + 1} > 0$.
   Therefore; by induction, $\forall n \in \mathbb{N},\ u_n > 0$.

(b) *Show that $(u_n)_n$ is strictly decreasing.*
We have $u_n > 0$ and $\dfrac{1}{(u_n)^2+1} < 1 \Longleftrightarrow \dfrac{u_n}{(u_n)^2+1} < u_n \Longleftrightarrow u_{n+1} < u_n$.
Then $(u_n)_n$ is strictly decreasing.

(c) *Deduce that $(u_n)_n$ is convergent and compute its limit.*
The sequence $(u_n)_n$ is strictly decreasing and bounded below by 0, then $(u_n)_n$ is C.V.
On the other hand, we have $u_{n+1} = f(u_n)$, where $f(x) = \dfrac{x}{x^2+1}$ is a continuous function. Then

$$f(\ell) = \ell \Longleftrightarrow \ell = \frac{\ell}{\ell^2+1} \Longleftrightarrow \ell^2 = 0 \Longleftrightarrow \ell = 0.$$

3. *For $u_n < 0$. (In the same previous way).*

4. *Determine* sup, inf, max, *and* min *if they exist in the set $E = \{|u_n|, \quad n \in \mathbb{N}\}$.*
• *Si $u_0 = 0$ then $E = \{u_n = 0, \quad \forall n \in \mathbb{N}\} = \{0\}$. Then $\sup(E) = \max(E) = \inf(E) = \min(E) = 0$.*
• *Si $u_n > 0$ then $E = \{u_n, n \in \mathbb{N}\}$.*
Here, the sequence $(u_n)_n$ is decreasing and convergent to 0. Thus

$$\inf(E) = \lim_{n\to+\infty} u_n = 0 \notin E \Longrightarrow \min(E) \text{ does not exist,}$$

and

$$\sup(E) = \max(E) = u_0 \in E \text{ (because } u_0 > u_1 > u_2 > \cdots).$$

• *Si $u_n < 0$, then $E = \{-u_n, n \in \mathbb{N}\}$.*
In this case, we have $(u_n)_n$ is increasing and convergent to 0. Thus

$$\inf\{u_n,\ n \in \mathbb{N}\} = u_0 \text{ (because } u_0 < u_1 < u_2 < \cdots),$$

and

$$\sup\{u_n,\ n \in \mathbb{N}\} = \lim_{n\to+\infty} u_n = 0.$$

As a consequence, $(-u_n)_n$ is decreasing and convergent to 0, that is

$$\sup(E) = \max(E) = -u_0 \in E \quad \text{and} \quad \inf(E) = \lim_{n\to+\infty} -u_n = 0 \notin E,\ \text{i.e.,}\ \min(E)\ \nexists.$$

Finally, we can summarize the three cases by:

1. *if $u_0 = 0$:*
$$\sup(E) = \max(E) = \inf(E) = \min(E) = 0.$$

2. *if $u_0 \neq 0$:*
$$\sup(E) = \max(E) = |u_0| \quad \text{and} \quad \inf(E) = 0,\ \text{with}\ \min(E)\ \nexists.$$