U M B B
Faculty of Science
Departement of Mathematics

First-Year C S and C E
Module : Analysis 1
Academic year : 2023/2024

Series n° 4 Differentiation on $\mathbb{R}$

Exercise 1. *Indicate, justifying your answers, whether the following propositions are true or false:*

1. *Any function differentiable at a point is continuous at this same point.*
2. *Any continuous function is differentiable.*
3. *Any function differentiable on right-hand and left-hand at* $\mathbf{a}$ *is differentiable at* $\mathbf{a}$.
4. *If the derivative of a function vanishes at* $\mathbf{a}$, *this function admits an extremum at* $\mathbf{a}$.
5. *If* f *is differentiable on an interval* I *containing a point* a, $\lim\limits_{x\to a} f'(x) = f'(a)$.
6. *If* f *is a function differentiable on* $\mathbb{R}$ *and even then* f' *is odd.*

Exercise 2. *Calculate the following limits by using the definition of a function's derivative:*

1) $\lim\limits_{x\to 0} \frac{\sin(x)}{x}$.

2) $\lim\limits_{x\to a} \frac{xf(a)-af(x)}{x-a}$ ($f : \mathbb{R} \to \mathbb{R}$ *differentiable at* a).

Exercise 3. *Study the differentiability of the following functions at the indicated point* x_0*:*

1) $f(x) = |x-n|\,[x]$, $x_0 = n \in \mathbb{Z}$,

2) $g(x) = \begin{cases} \dfrac{\sin^2(\pi x)}{\pi - x}, & x \neq \pi, \\ 0, & x = \pi, \end{cases} \quad x_0 = \pi.$

Exercise 4. *Let*

$$f_n(x) = \begin{cases} x^n \sin\left(\frac{1}{x}\right), & x \neq 0, \\ 0, & x = 0, \end{cases} \qquad n \in \mathbb{N}.$$

Study according to the values of n, the differentiability of the function f_n *and calculate its derivative.*

Exercise 5. *Calculate the n-th derivative of the following functions:*

1) $f(x) = \dfrac{1}{1-x^2}$,

2) $g(x) = x^2(1+x)^n$, $n \geq 2$.

Exercise 6.
1. State the theorem of finite increments.
2. Can the theorem of finite increments be applied in the following cases?

a) $f_1(x) = |1-x|$ *on* $[0,3]$,

b) $f_2(x) = \sqrt[3]{x - \dfrac{1}{2}}$ *on* $[-1,1]$,

3. Show the following inequalities:

$$\forall x > 0,\ \frac{x}{1-x^2} \leq \arctan x \leq x,$$

$$\forall x > 0,\ \frac{x}{x+1} \leq \ln(x+1) - \ln(x) \leq \frac{1}{x},$$

Exercise 7.
1. State Rolle's theorem.
2. Can Rolle's theorem be applied in the following cases?

a) $f(x) = \begin{cases} 3 - x^2, & x \leq 0, \\ \dfrac{1}{x}, & x > 0, \end{cases}$ *on* $[0, 2]$,

b) $g(x) = \sqrt[3]{(x-2)^2}$ *on* $[0, 4]$,

Exercise 8.

1. State the Mac-Laurin-Lagrange formula at order n.

2. Let the following function:

$$f(x) = e^x$$

i) Calculate the n-th derivative of f.

ii) Give the Mac-Laurin-Lagrange formula of f at order n.

3. Applications:

i) Show that:

$$1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!} < e < 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!} + \frac{e}{(n+1)!}.$$

ii) Let the sequence $(u_n)_n$ of general term

$$u_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!}.$$

Calculate $\lim\limits_{n \to +\infty} u_n$.

Exercise 9.

1. Determine the definition set of the following functions:

a) $f(x) = \arcsin(\dfrac{x}{x+1})$ *b)* $g(x) = \arccos(2x+1) - \arcsin(3x^2)$.

2. Solve the following equations:

a) $\arcsin x = \arccos 5x$. *b)* $\arcsin x = \arcsin \dfrac{2}{5} + \arcsin \dfrac{3}{5}$.

3. Show that:

$$\forall x \in]-1, 1[, \ \arcsin x + \arccos x = \frac{\pi}{2}.$$