U M B B
Faculty of Science
Departement of Mathematics

First-Year C S and C E
Module : Analysis 1
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Series n° 2 -Numerical Sequences-

Generalities

Exercise 1. *Comment if the following statements are true or false? And give a valid reason for saying so.*

- *Every bounded sequence is convergent.*
- *Every sequence $(u_n)_n$ satisfies $u_{n+1} \leq u_n$ is decreasing.*
- *The sum of two divergent sequences is a divergent sequence.*
- *The product of two divergent sequences is a divergent sequence.*
- *If the sequence $(u_n)_{n\in\mathbb{N}}$ is increasing and $u_n \leq n$, then $(u_n)_n$ is convergent.*
- *If $\lim\limits_{n\to+\infty} u_n^2 = \ell^2$, then $\lim\limits_{n\to+\infty} u_n = \ell$.*
- *If $\lim\limits_{n\to+\infty} |u_n| = |\ell|$, then $\lim\limits_{n\to+\infty} u_n = \ell$.*

Convergent Sequences

Exercise 2. *I) Consider the real sequence $(u_n)_{n\in\mathbb{N}^*}$ defined as:* $u_n = \dfrac{n+1}{2n}$.

$(u_n)_n$ *converges to* $\ell \Longleftrightarrow (\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N};\ n \geq n_0 \Rightarrow |u_n - \ell| < \varepsilon)$.

Find the absolute value of this inequality $0.49 < u_n < 0.51$, *then deduce* ℓ, ε *and* n_0.

▶ *Use the previous definition to prove each of the following:*

$$\lim_{n\to+\infty} \frac{2\ln(n+1)}{\ln(n)} = 2; \qquad \lim_{n\to+\infty} \frac{5n+3}{7n+1} = \frac{5}{7}.$$

II) Study the convergence of the following sequences:

1) $u_n = \dfrac{\sin(n)}{n}$, 2) $u_n = \dfrac{1}{n^3}\displaystyle\sum_{k=1}^{n} k,\ n \in \mathbb{N}^*$, 3) $u_n = \displaystyle\sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k}}$,

4) $u_n = \dfrac{3^n - 7^n}{3^n + 7^n}$, 5) $u_n = \dfrac{1}{n^2}\displaystyle\sum_{k=1}^{n} [kx],\ n \in \mathbb{N}^*$, 6) $u_n = \sqrt[3]{1+n} - \sqrt[3]{n}$,

7) $u_n = \left(\dfrac{n^2 - n + 3}{n^2 + 3n - 1}\right)^n$, 8) $u_n = \dfrac{5 \times 7 \times 9 \cdots (2n+5)}{4 \times 7 \times 10 \cdots (3n+4)}$.

Adjacent Sequences

Exercise 3. *I) Let $a, b \in \mathbb{R}$, $0 < a \leq b$. Let the numerical sequences $(u_n)_n$ and $(v_n)_n$ be given by*

$$\begin{cases} u_0 = a, \\ u_{n+1} = \dfrac{2u_n + v_n}{3}, \ \forall n \in \mathbb{N}. \end{cases} \qquad \begin{cases} v_0 = b, \\ v_{n+1} = \dfrac{2v_n + u_n}{3}, \ \forall n \in \mathbb{N}. \end{cases}$$

1. *Determine u_1 and v_1;*
2. *Prove that: $\forall n \in \mathbb{N}$, $u_n \leq u_{n+1} \leq v_{n+1} \leq v_n$;*
3. *Express $(v_n - u_n)_n$ in terms of $(b-a)$. Deduce that $(u_n)_n$ and $(v_n)_n$ are adjacent;*
4. *Express $(v_n + u_n)_n$ in terms of $(a+b)$. Set the limit of each sequence $(u_n)_n$ and $(v_n)_n$.*

II) Consider the sequence of general term: $v_n = 1 - \dfrac{1}{2} + \dfrac{1}{3} + \cdots + \dfrac{(-1)^{n+1}}{n}$, $\forall n \in \mathbb{N}^*$.

▶ *Prove that the two subsequences $(v_{2n})_n$ and $(v_{2n+1})_n$ are adjacent, then deduce the nature of $(v_n$*

Divergent Sequences

Exercise 4. *I) Consider the real sequence $(u_n)_{n\in\mathbb{N}^*}$ defined as:* $u_n = n\sqrt{n}$.

$$(u_n)_n \text{ diverges to } \pm\infty \iff \begin{pmatrix} 1)\ \forall A > 0,\ \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N};\ n \geq n_0 \Rightarrow u_n > A. \\ 2)\ \forall A > 0,\ \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N};\ n \geq n_0 \Rightarrow u_n < -A. \end{pmatrix}$$

Explain the definition that corresponds to the limit of $(u_n)_n$, such that $u_n \in (10^6; +\infty)$, then deduce the values of A and n_0.

▶ *Use the previous definitions to prove each of the following:*

$$\lim_{n\to+\infty} 3^{2n+1} = +\infty; \qquad \lim_{n\to+\infty} \frac{1}{2}(e^{\frac{1}{n}} - n) = -\infty.$$

II) Choose the suitable way (subsequences, comparison test) to show that the below sequences are divergent:

$$1)\ u_n = \cos\left(\frac{n\pi}{4}\right), \quad 2)\ u_n = \left(1 + \frac{1}{n}\right)^{n^2}, \quad 3)\ u_n = \frac{n + (-1)^n n}{n - (-1)^n \frac{n}{2}}, \quad 4)\ \frac{n! + 3^n}{n - 2^n}.$$

Cauchy Sequences

Exercise 5. *I) Consider the real sequence $(u_n)_{n\in\mathbb{N}^*}$, such that:*

$$|u_{n+1} - u_n| \leq k|u_n - u_{n-1}|, \quad n \in \mathbb{N}^*,\ k \in (0; 1).$$

1. *Show that:* $\forall n \in \mathbb{N}^*,\ |u_{n+1} - u_n| \leq k^{n-1}|u_2 - u_1|$;
2. *Prove that:* $\forall p, q \in \mathbb{N}^*,\ q < p$, *we have* $|u_p - u_q| \leq \dfrac{k^{q-1}}{1-k}|u_2 - u_1|$;
3. *Deduce the nature of $(u_n)_n$.*

II) In view of the Cauchy criterion, determine the nature of the sequences $(u_n)_n$ defined by:

$$1)\ u_n = \sum_{k=1}^{n} \frac{1}{\sqrt{k}}, \qquad 2)\ u_n = \sum_{k=1}^{n} \frac{\sin k}{2^k}.$$

Recurrence Sequences

Exercise 6. *Let the sequence $(u_n)_n$ be recursively defined by:*

$$\begin{cases} u_0 \in \mathbb{R}, \\ u_{n+1} = \dfrac{u_n}{(u_n)^2 + 1}, \quad n \in \mathbb{N}. \end{cases}$$

1. *We set $u_0 = \alpha$.*

   a) *Determine α, such that the sequence $(u_n)_n$ is zero.*

2. *For $u_n > 0$.*

   (a) *Prove that:* $\forall n \in \mathbb{N},\ u_n > 0$.

   (b) *Show that $(u_n)_n$ is strictly decreasing.*

   (c) *Deduce that $(u_n)_n$ is convergent and compute its limit.*

3. *For $u_n < 0$.*

   (a) *Prove that:* $\forall n \in \mathbb{N},\ u_n < 0$.

   (b) *Show that $(u_n)_n$ is strictly increasing.*

   (c) *Deduce that $(u_n)_n$ is convergent and compute its limit.*

4. *Determine* sup, inf, max, *and* min *if they exist in the set*

$$E = \{|u_n|,\quad n \in \mathbb{N}\}.$$