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Nonlinear Electrodynamics: Lagrangians and Equations of Motion

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After a brief discussion of well-known classical fields we formulate two principles: When the field equations are hyperbolic, particles move along rays like disturbances of the field; the waves associated with stable particles are exceptional. This means that these waves will not transform into shock waves. Both principles are applied to nonlinear electrodynamics. The starting point of the theory is a Lagrangian which is an arbitrary nonlinear function of the two electromagnetic invariants. We obtain the laws of propagation of photons and of charged particles, along with an anisotropic propagation of the wavefronts. The general "exceptional" Lagrangian is found. It reduces to the Lagrangian of Born and Infeld when some constant (probably simply connected with the Planck constant) vanishes. A nonsymmetric tensor is introduced in analogy to the Born–Infeld theory, and finally, electromagnetic waves are compared with those of Einstein–Schrödinger theory.

1. WAVE DYNAMICS

A. Waves, Rays, and Exceptional Waves

Quite generally we have seen that, as far as wave propagation is concerned, the field equations can always be written as a quasilinear system of partial differential equations that is a system of equations which are linear with respect to the highest derivatives of the dependent field variables.[1] We assume that across the hypersurface S,

$$\varphi(x^\alpha) = 0, \quad \alpha = 0, 1, \cdots, n,$$

these derivatives are discontinuous. Precisely, if q is the highest order of the derivatives of the field component u, we assume that, after a change of variables,

$$x^\alpha \to \varphi(x^\alpha),\ \xi^i(x^\alpha), \quad i = 1, 2, \cdots, n,$$

the jump

$$\left[\frac{\partial^q u}{\partial \varphi^q}\right] = \left(\frac{\partial^q u}{\partial \varphi^q}\right)_{\varphi=+0} - \left(\frac{\partial^q u}{\partial \varphi^q}\right)_{\varphi=-0} = \delta^q u$$

is finite, while

$$\delta^r u = 0, \quad 0 \le r < q.$$

S is called the wave surface, and since Hadamard[2] it is well known that

$$[\partial_{\alpha_1\alpha_2\cdots\alpha_q} u] = \varphi_{\alpha_1}\varphi_{\alpha_2}\cdots\varphi_{\alpha_q}\delta^q u,$$

where a subscript α denotes partial differentiation with respect to x^α.

To allow for the above-mentioned discontinuities, φ must be a solution of some characteristic equation of the form

$$\psi = G^{\alpha\beta\cdots\nu}\varphi_\alpha\varphi_\beta\cdots\varphi_\nu, \quad \psi = 0. \tag{1.1}$$

When the field is nonlinear, the completely symmetric tensor G may depend on the field and all its continuous derivatives.

The discontinuities—or disturbances—propagate themselves along the rays:

$$\frac{dx^\alpha}{d\sigma} = \frac{\partial \psi}{\partial \varphi_\alpha}, \quad \frac{d\varphi_\alpha}{d\sigma} = -\frac{\partial \psi}{\partial x^\alpha}. \tag{1.2}$$

Thus in general relativity the ray velocity $\partial\psi/\partial\varphi_\alpha$ must be a timelike (or possibly null) vector, i.e.,

$$\mathcal{N} = \left(g^{\alpha\beta}\frac{\partial\psi}{\partial\varphi_\alpha}\frac{\partial\psi}{\partial\varphi_\beta}\right)^{\frac{1}{2}}$$

must exist (be real). We normalize (if $\mathcal{N} \neq 0$) the ray velocity

$$u^\alpha = \frac{1}{\mathcal{N}}\frac{\partial\psi}{\partial\varphi_\alpha}, \quad u_\alpha u^\alpha = 1.$$

Equation (1.1) shows that

$$u^\alpha\varphi_\alpha = 0, \tag{1.3}$$

which now implies

$$g^{\alpha\beta}\varphi_\alpha\varphi_\beta \le 0.$$

Now what about the growth of the disturbances of the field (such as $\delta^q u$)? Though we cannot here enter into details, the following results generally hold[3]: It is possible (because of the nonlinearity of the field) by a suitable choice of the (initial) disturbance to produce an accelerated wave (e.g., in fluid mechanics, push the piston into the cylinder). Consequently, the disturbances will cease to be finite after some critical time, thus tending to a shock. From this moment on, the field equations are no longer valid and must be replaced by others (e.g., Rankine–Hugoniot conditions). However, the form of the field equations might be such that this phenomenon does not appear on some wavefront; then we say with Lax[4] that this wave is exceptional. If this is true for all the waves, the system of field equations is called completely exceptional.[4,5]

The condition for (1.1) to be exceptional is[1,3]

$$\delta\psi = 0, \quad \varphi_\alpha \varphi_\beta \cdots \varphi_\nu \delta G^{\alpha\beta\cdots\nu} = 0, \tag{1.4}$$

or, equivalently,

$$\varphi_\alpha \delta u^\alpha = 0. \tag{1.5}$$

For instance, gravitational waves,

$$g^{\alpha\beta}\varphi_\alpha\varphi_\beta = 0,$$

are exceptional, for $g^{\alpha\beta}$ only depends on the metric tensor $g_{\alpha\beta}$ whose first-order derivatives are continuous[6] (hence $\delta g^{\alpha\beta} = 0$).

Alfvén waves of magnetohydrodynamics are exceptional too[3,5]; Alfvén shocks cannot be produced directly from a continuous initial state.[7]

B. Wave Dynamics: Two Principles

In this section we want to show briefly how particles, sources of hyperbolic fields, may be considered as disturbances of the field (in the sense of Sec. 1A) and therefore move along the rays of the associated waves. Let us distinguish several cases.

(1) The field equations contain the four-dimensional velocity u^α. They give the trajectories and, in order to be able to identify them with a family of rays, u^α must be a ray velocity. Hence, by virtue of (1.3), the characteristic equation

$$u^\alpha \varphi_\alpha = 0 \tag{1.6}$$

must exist. Let us examine the equations derived from the conservation laws

$$\nabla_\alpha T^{\alpha\beta} = 0$$

of two classical energy tensors.

i. Incoherent Matter

$$T^{\alpha\beta} = \rho u^\alpha u^\beta,$$

$$\nabla_\alpha(\rho u^\alpha) = 0, \tag{1.7}$$

$$u^\alpha \nabla_\alpha u^\beta = 0. \tag{1.8}$$

The discontinuities are of first order and a convenient way to obtain them is to make the replacement

$$\partial_\alpha, \nabla_\alpha \to \varphi_\alpha \delta. \tag{1.9}$$

(The Christoffel symbols are continuous; see the end of the previous section.) From the very definition of δ it is obvious that it operates as does the operator of differentiation, i.e., $f(u_1, u_2, \cdots)$ being a continuously differentiable function of the continuous field variables $u_1, u_2, \cdots$:

$$\delta f = \frac{\partial f}{\partial u_1}\delta u_1 + \frac{\partial f}{\partial u_2}\delta u_2 + \cdots.$$

From

$$\varphi_\alpha \delta(\rho u^\alpha) = 0, \quad u^\alpha \varphi_\alpha \delta u^\beta = 0,$$

we immediately get (1.6) and (1.5); the wave is exceptional. Equation (1.8) describes the time track of a test particle (geodesic).

ii. Perfect Fluids

The entropy satisfies the adiabatic condition

$$u^\alpha \partial_\alpha S = 0;$$

otherwise the motion of the fluid would not be determined.[8] On the other hand, an equation similar to (1.7) holds and it is easy to see that we obtain again (1.6) and (1.5) for the entropy wave,

$$\delta S \neq 0.$$

Similar results are valid for magnetohydrodynamics.[9] Thus we have seen on these examples that particles move in the same way as the disturbances of the wave (1.6) and that this wave is exceptional. This, of course, does not give us more information about the particle paths; they are what they are, already given by the field equations. But let us go further.

(2) From the Einstein equations

$$S_{\alpha\beta} = \chi T_{\alpha\beta},$$

we get

$$g^{\alpha\beta}\varphi_\alpha\varphi_\beta = 0, \quad \delta^2 g_{\alpha\beta} \neq 0,$$

no matter what the continuous energy–momentum tensor is. This is the equation of a null surface propagating with the speed of light. The wave is exceptional (Sec. 1A) and it is well known that the rays are null geodesics and the trajectories of particles with vanishing mass.

(3) The field equations do not give (directly) the laws of motion of particles. This is the case of nonlinear electrodynamics which we investigate in Sec. 2.

We propose now the following principles[10]:

Ray Principle: When the field equations are hyperbolic, particles move along rays like disturbances of the field.

Principle of Exception: The waves associated with stable particles are exceptional.

Generally other waves—that is, waves which are not associated with particles (e.g., fast or slow waves in magnetohydrodynamics) or which are associated with unstable particles—are not exceptional. In the latter case the critical time might be connected with the lifetime of the particle: if the instability lies in the nature of the particle, it must also lie in the nature of the disturbance that represents it.

However, if all the waves of the field can represent stable particles, the system of field equations must be completely exceptional.

As yet we have only checked the above principles on the examples of (1) and (2). We are now going to see what comes out of them when applied to nonlinear electrodynamics.

2. NONLINEAR ELECTRODYNAMICS

A. Field Equations

It is well known that from the electromagnetic tensor $F_{\alpha\beta}$ and its dual,

$$F^{*\gamma\delta} = \tfrac{1}{2}\eta^{\alpha\beta\gamma\delta}F_{\alpha\beta}, \quad \eta^{\alpha\beta\gamma\delta} = -\frac{1}{(-g)^{\frac{1}{2}}}\epsilon^{\alpha\beta\gamma\delta}, \quad (2.1)$$

one can construct the two invariants

$$Q = \tfrac{1}{4}F_{\alpha\beta}F^{\alpha\beta}, \quad R = \tfrac{1}{4}F^*{}_{\alpha\beta}F^{\alpha\beta}. \quad (2.2)$$

[In a pseudo-Cartesian frame at some point, $Q = \frac{1}{2}(\mathbf{H}^2 - \mathbf{E}^2)$, $R = \mathbf{E}\cdot\mathbf{H}$, where $\mathbf{E}$ and $\mathbf{H}$ are the ordinary electric and magnetic 3-vectors.]

Let us introduce the 4-potential ϕ_α through

$$F_{\alpha\beta} = \partial_\alpha\phi_\beta - \partial_\beta\phi_\alpha \quad (2.3)$$

and the Lagrangian $L(Q, R)$ arbitrary function of the invariants (2.2).

Then the Euler equations for the variational principle applied to

$$\int (-g)^{\frac{1}{2}} L\, dx$$

read as

$$\nabla_\alpha L^{\alpha\beta} = 0, \quad L^{\alpha\beta} = L_Q F^{\alpha\beta} + L_R F^{*\alpha\beta}, \quad L_Q = \partial L/\partial Q, \quad (2.4)$$

which, together with

$$\nabla_\alpha F^{*\alpha\beta} = 0, \quad (2.5)$$

locally satisfied by (2.3), and the Einstein equations (cf. Ref. 11),

$$\begin{aligned} S_{\alpha\beta} &= \chi T_{\alpha\beta}, \\ T^{\alpha\beta} &= Lg^{\alpha\beta} - L^{\alpha\rho}F^{\beta}{}_{\rho} \\ &= L_Q\tau^{\alpha\beta} + (L - QL_Q - RL_R)g^{\alpha\beta}, \end{aligned} \quad (2.6)$$

form the system of the field equations. We recall that

$$\tau^{\alpha\beta} = Qg^{\alpha\beta} - F^{\alpha\rho}F^{\beta}{}_{\rho} \quad (2.7)$$

is the Maxwellian energy tensor.

Remark: Equation (2.4) shows that L needs to be defined only up to multiplicative and additive constants [more precisely, because of (2.5), to an additive linear function of R]. The first constant can be absorbed by χ and the second one can cancel a "cosmological" constant of the Einstein tensor, so we shall not bother here ensuring the conditions

$$L_Q(0, 0) = 1, \quad L(0, 0) = 0.$$

The following relations are useful too:

$$F^*{}_{\alpha\rho}F^{\beta\rho} = Rg^\beta_\alpha, \quad F^*{}_{\alpha\rho}F^{*\beta\rho} = F_{\alpha\rho}F^{\beta\rho} - 2Qg^\beta_\alpha, \quad (2.8)$$

$$\tau_{\alpha\rho}\tau^{\beta\rho} = P^2 g^\beta_\alpha, \quad P = (Q^2 + R^2)^{\frac{1}{2}}. \quad (2.9)$$

We shall suppose, of course, that L is nonlinear, and more precisely that

$$L_Q(L_{QQ} + L_{RR}) \neq 0.$$

B. Wave Surface

We assume that, across the wavefront $\varphi(x^\alpha) = 0$, ϕ_α has a discontinuity of the second order,

$$\pi_\alpha = \delta^2\phi_\alpha,$$

so that, from (2.3),

$$\delta F_{\alpha\beta} = \varphi_\alpha\pi_\beta - \varphi_\beta\pi_\alpha, \quad \varphi_\alpha = \partial_\alpha\varphi. \quad (2.10)$$

Equations (2.4) and (2.5) give [see Eq. (1.9)]

$$\varphi_\alpha(F^{\alpha\beta}\delta L_Q + F^{*\alpha\beta}\delta L_R + L_Q\delta F^{\alpha\beta}) = 0, \qquad \varphi_\alpha\delta F^{*\alpha\beta} = 0, \quad (2.11)$$

or by introducing the vectors

$$U^\beta = F^{\alpha\beta}\varphi_\alpha, \quad V^\beta = F^{*\alpha\beta}\varphi_\alpha$$

and taking account of (2.10),

$$U_\alpha\delta L_Q + V_\alpha\delta L_R + L_Q\{\mathcal{G}\pi_\alpha - (\pi^\beta\varphi_\beta)\varphi_\alpha\} = 0. \quad (2.12)$$

If $\mathcal{G} = g^{\alpha\beta}\varphi_\alpha\varphi_\beta \neq 0$, this equation shows that

$$\pi_\alpha = aU_\alpha + bV_\alpha + c\varphi_\alpha, \quad (2.13)$$

with

$$a\mathcal{G}L_Q + \delta L_Q = 0, \quad b\mathcal{G}L_Q + \delta L_R = 0. \quad (2.14)$$

The coefficient c is undetermined, but the last term of (2.13) has no effect on the disturbance (2.10) of the electromagnetic tensor; the Lorentz condition ($\nabla_\alpha\phi^\alpha = 0$) will make it vanish (provided $\mathcal{G} \neq 0$).

From (2.2) and (2.10), we have

$$\delta Q = U^\alpha\pi_\alpha, \quad \delta R = V^\alpha\pi_\alpha; \quad (2.15)$$

inserting (2.13) and taking account of (2.7) and (2.8), we obtain

$$\delta Q = a(Q\mathcal{G} - \mathcal{T}) + bR\mathcal{G}, \quad \delta R = aR\mathcal{G} - b(Q\mathcal{G} + \mathcal{T}); \quad \mathcal{T} = \tau^{\alpha\beta}\varphi_\alpha\varphi_\beta. \tag{2.16}$$

Now Eqs. (2.14) give

$$a\{\mathcal{G}(L_Q + QL_{QQ} + RL_{QR}) - \mathcal{T}L_{QQ}\} + b\{\mathcal{G}(RL_{QQ} - QL_{QR}) - \mathcal{T}L_{QR}\} = 0,$$

$$a\{\mathcal{G}(QL_{QR} + RL_{RR}) - \mathcal{T}L_{QR}\} + b\{\mathcal{G}(L_Q + RL_{QR} - QL_{RR}) - \mathcal{T}L_{RR}\} = 0, \tag{2.17}$$

wherefrom we get the characteristic equations,[12]

$$\mathcal{T} + \mu\mathcal{G} = 0; \tag{2.18}$$

$\mu(Q, R)$ being a solution of

$$\varpi\mu^2 + \mu + \omega - \varpi P^2 = 0; \tag{2.19}$$

$$\varpi = \frac{L_{QQ}L_{RR} - L_{QR}^2}{L_Q(L_{QQ} + L_{RR})},$$

$$\omega = \frac{L_Q + Q(L_{QQ} - L_{RR}) + 2RL_{QR}}{L_{QQ} + L_{RR}}.$$

The roots of (2.19) always exist since the discriminant can be written[13]

$$\Delta = \left(\frac{L_{QQ} - L_{RR}}{L_{QQ} + L_{RR}} - 2Q\varpi\right)^2 + 4\left(\frac{L_{QR}}{L_{QQ} + L_{RR}} - R\varpi\right)^2. \tag{2.20}$$

(We disregard the case $\varpi = 0$ that does not seem of physical interest.)

Let $G^{\alpha\beta}$ be a normal tensor,[6]

$$G^{\alpha\beta} = s_0 V^\alpha_{(0)} V^\beta_{(0)} - \sum_{i=1}^{3} s_i V^\alpha_{(i)} V^\beta_{(i)}, \quad V_{(\mu)\alpha} V^\alpha_{(\nu)} = \eta_{\mu\nu},$$

and let the characteristic equation be

$$G^{\alpha\beta}\varphi_\alpha\varphi_\beta = 0.$$

Then the following inequalities[10,14] must hold according to the definition of the ray velocity (see Sec. 1A):

$$0 < \frac{s_i}{s_0} \leq 1, \quad i = 1, 2, 3. \tag{2.21}$$

If one equality is satisfied, light velocity is reached in two directions:

$$s_1 = s_0 \rightarrow \mathcal{G} = 0, \quad l_\alpha l^\alpha = 0:$$

$$\frac{dx^\alpha}{d\sigma} = l^\alpha \propto \varphi^\alpha \propto V^\alpha_{(0)} \pm V^\alpha_{(1)}. \tag{2.22}$$

[When $\mathcal{N} = 0$ (Sec. 1A), we note l^α instead of u^α in order to avoid confusion.]

Here

$$G^{\alpha\beta} = \tau^{\alpha\beta} + \mu g^{\alpha\beta}$$

and the eigenvalues are

$$s_0 = \mu + P; \quad s_i: \mu + P, \quad \mu - P, \quad \mu - P.$$

We must have (2.21)

$$\mu > P. \tag{2.23}$$

This inequality must be true for both values of μ. Hence,[13]

$$\varpi < 0, \quad \varpi P + \tfrac{1}{2} > 0, \quad \omega + P < 0. \tag{2.24}$$

Propagation with the speed of light [(2.22) is satisfied] will be studied in Sec. 2.D.

C. Double Root: The Born–Infeld Lagrangian

The discriminant (2.20) is positive and vanishes identically only for a solution of the system of partial differential equations:

$$p(r - t) - 2x(rt - s^2) = 0,$$
$$ps - y(rt - s^2) = 0. \tag{2.25}$$

Here we have replaced $L(Q, R)$ by $z(x, y)$ and used p, q, r, s, t to denote the partial derivatives. Since we assume that $\varpi \neq 0$, the system is equivalent to

$$p(y_q - x_p) - 2x = 0,$$
$$py_p + y = 0. \tag{2.26}$$

By integration this equation gives

$$y = -F'(q)/p.$$

Then

$$x_q = y_p = F'/p^2, \quad x = F(q)/p^2 + G(p).$$

Inserting these expressions into (2.26) results in

$$F''(q) + pG'(p) + 2G(p) = 0,$$

or (since $\varpi \neq 0$)

$$F''(q) = \text{const} = k, \quad F(q) = \tfrac{1}{2}kq^2 + a$$

(the coefficient of q in F can be taken equal to zero by changing q into q + const; cf. Remark, Sec. 2.A).

$$G(p) = (b/p^2) - \tfrac{1}{2}k.$$

p and q are easily obtained and so is z,

$$z = \text{const}[-y^2 + k(2x + k)]^{\frac{1}{2}}.$$

By (2.19) and (2.25) we have

$$\mu = -\frac{1}{2\varpi} = \frac{p}{rt - s^2}[\tfrac{1}{2}(r-t) - r]$$
$$= x - py_q = x + F'' = x + k.$$

Therefore,[13]

$$L = [-R^2 + k(2Q + k)]^{\frac{1}{2}}, \qquad (2.27)$$

$$\mu = Q + k, \qquad (2.28)$$

and according to (2.23) k must be positive. This is the Lagrangian of Born and Infeld.[11] [Note that it can also be written $L = (\mu^2 - P^2)^{\frac{1}{2}}$.] We shall see the fundamental role it plays in the theory.

The spherically symmetric solution obtained with this Lagrangian is well known[11,15]:

$$F_{01} = k^{\frac{1}{2}} \Big/ \left[1 + \left(\frac{r}{r_0}\right)^4\right]^{\frac{1}{2}}, \quad x^0 = t,\, x^1 = r. \quad (2.29)$$

It removes the singularity at the origin that appeared in the Maxwellian field:

$$r = 0: \quad F_{01} = k^{\frac{1}{2}}.$$

This finite value is called, after Born and Infeld, the "absolute field."

D. Propagation with the Speed of Light

At the end of Sec. 2B we have seen that, owing to (2.22), the speed of light is reached in two directions that we are going to determine. Equations (2.7), (2.8), and (2.18) give

$$U_\alpha U^\alpha = (\mu + Q)\mathcal{G},$$
$$U_\alpha V^\alpha = R\mathcal{G}, \quad V_\alpha V^\alpha = U_\alpha U^\alpha - 2Q\mathcal{G}. \quad (2.30)$$

Thus when $\mathcal{G} = 0$, U_α, V_α, φ_α are null vectors. Since on the other hand

$$U^\alpha \varphi_\alpha = V^\alpha \varphi_\alpha = 0,$$

it is necessary that

$$U_\alpha = u\varphi_\alpha, \quad V_\alpha = v\varphi_\alpha. \qquad (2.31)$$

Multiplying (2.8) by φ_β results in

$$uv = -R, \quad v^2 - u^2 = 2Q$$

or

$$u = \pm(P - Q)^{\frac{1}{2}}, \quad v = \mp(P + Q)^{\frac{1}{2}}\,\mathrm{sgn}\, R.$$

The equations [cf. (2.22) and (2.31)]

$$\frac{dx^\alpha}{d\sigma} = l^\alpha, \quad [F^{\alpha\beta} \pm (P - Q)^{\frac{1}{2}} g^{\alpha\beta}]l_\beta = 0, \quad l_\alpha l^\alpha = 0$$

give the trajectories of photons.[10]

It is worthwhile to note that for a null field ($P \equiv 0$) these equations—together with (2.4)–(2.6)—coincide with those of the Maxwellian case[16]:

$$P = 0, \quad F^{\alpha\beta} l_\beta = 0, \quad l_\alpha l^\alpha = 0,$$

implying[17,18]

$$F_{\alpha\beta} = l_\alpha p_\beta - l_\beta p_\alpha, \quad p_\alpha p^\alpha = -1,$$
$$p_\alpha l^\alpha = 0, \quad \tau_{\alpha\beta} = l_\alpha l_\beta, \quad l^\alpha \nabla_\alpha l^\beta = 0.$$

The null field still describes a fluid of photons[6] that still follow null geodesics.

In Sec. 2F we shall show that the results of this paragraph are compatible with Eq. (2.12) by calculating the associated disturbances.

E. Rays[19]

Let us define the tensors

$$G^{\alpha\beta} = (\tau^{\alpha\beta} + \mu g^{\alpha\beta})/(\mu^2 - P^2)^{\frac{1}{2}},$$
$$H_{\alpha\beta} = -(\tau_{\alpha\beta} - \mu g_{\alpha\beta})/(\mu^2 - P^2)^{\frac{1}{2}},$$

such that, by virtue of (2.9),

$$H_{\alpha\beta} G^{\beta\gamma} = g_\alpha^\gamma.$$

Let us introduce the relative spacelike 4-vectors,

$$e_\alpha = F_{\rho\alpha} u^\rho, \quad h_\alpha = F^*{}_{\rho\alpha} u^\rho, \quad S^\alpha = \eta^{\alpha\beta\gamma\delta} e_\beta h_\gamma u_\delta$$
$$(u_\alpha u^\alpha = 1),$$

which coincide in the rest frame ($u^i = 0$, $u^0 = 1$) with the electric, magnetic, and Poynting vectors, respectively. When expressed with these quantities, we have

$$F_{\alpha\beta} = u_\alpha e_\beta - u_\beta e_\alpha + \eta_{\alpha\beta\gamma\delta} h^\gamma u^\delta,$$
$$F^{*\alpha\beta} = u^\alpha h^\beta - u^\beta h^\alpha - \eta^{\alpha\beta\gamma\delta} e_\gamma u_\delta,$$
$$\tau_{\alpha\beta} = (e^2 + h^2)(u_\alpha u_\beta - \tfrac{1}{2} g_{\alpha\beta}) \qquad (2.32)$$
$$- (e_\alpha e_\beta + h_\alpha h_\beta + u_\alpha S_\beta + u_\beta S_\alpha),$$
$$e^2 = -e_\alpha e^\alpha, \cdots.$$

From Sec. 1A and Eq. (2.18), the rays are given by

$$\frac{dx^\alpha}{d\tau} = \frac{1}{(-\mathcal{G})^{\frac{1}{2}}} G^{\alpha\beta}\varphi_\beta = u^\alpha, \qquad (2.33)$$

$$\frac{d\varphi_\alpha}{d\tau} = -\frac{1}{2(-\mathcal{G})^{\frac{1}{2}}} \partial_\alpha G^{\beta\gamma}\varphi_\beta\varphi_\gamma. \qquad (2.34)$$

Equation (2.33) yields

$$\frac{\varphi_\alpha}{(-\mathcal{G})^{\frac{1}{2}}} = H_{\alpha\beta} u^\beta = \frac{1}{(\mu^2 - P^2)^{\frac{1}{2}}}\{(\zeta - e^2)u_\alpha + S_\alpha\},$$

$$\zeta = \mu - Q, \qquad (2.35)$$

while

$$u^\alpha \varphi_\alpha = 0 \rightarrow H_{\alpha\beta} u^\alpha u^\beta = 0,$$

i.e.,

$$e^2 = \zeta, \tag{2.36}$$

or equivalently,

$$\tfrac{1}{2}(e^2 + h^2) = \mu, \quad S^2 = \mu^2 - P^2$$

[because $Q = \frac{1}{2}(h^2 - e^2)$, $R = e_\alpha h^\alpha$, $S^2 = e^2 h^2 - R^2$].
Thus,

$$\frac{\varphi_\alpha}{(-\mathcal{G})^{\frac{1}{2}}} = \frac{S_\alpha}{(\mu^2 - P^2)^{\frac{1}{2}}}, \tag{2.37}$$

the Poynting vector is orthogonal to the wavefront, ζ (or rather $\zeta^{\frac{1}{2}}$) is the absolute field of the charged particle which moves along the rays according to the first principle of Sec. 1B. μ is the absolute density of energy.

According to (2.33) and (2.34), particle paths are null geodesics for the metric

$$d\bar{s}^2 \propto H_{\alpha\beta}\, dx^\alpha\, dx^\beta.$$

Consequently,

$$u^\alpha \nabla_\alpha u^\beta = (g^\beta_\alpha - u_\alpha u^\beta)\gamma^\alpha,$$

where

$$\gamma^\lambda = -\tfrac{1}{2} G^{\lambda\rho}(\nabla_\mu H_{\nu\rho} + \nabla_\nu H_{\mu\rho} - \nabla_\rho H_{\mu\nu}) u^\mu u^\nu.$$

[As for the uncharged particles, they move along geodesics according to (1.8).]

Now it is easy to show that the Born–Infeld Lagrangian is well suited to describe a particle in the spherically symmetric approximation. We have (2.29)

$$e^2 = -g^{00} g^{11} (F_{01})^2 = F^2_{01}, \quad u^i = 0.$$

On the other hand, (2.28) and (2.35),

$$\zeta = k, \tag{2.38}$$

so that at the origin $r = 0$, where the particle is at rest, the equality (2.36) is true.

To end this section, we shall say a few words about the ray velocity diagram. This diagram gives us the shape of the wavefront resulting from a point disturbance (that is, a disturbance initially localized within a small sphere) and propagating into a constant state (specified by $g_{\alpha\beta} = \eta_{\alpha\beta}$, $\mathbf{E}$, $\mathbf{H}$). A proper choice of the reference frame in the three-dimensional physical space (the z axis lies in the direction of the Poynting vector $\mathbf{S} = \mathbf{E} \times \mathbf{H}$) allows one to write the equation of the wavefront at time t[20]:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{1}{c^2}(z - Vt)^2 = t^2, \quad \mathbf{V} = \frac{\mathbf{S}}{\mu + w}, \tag{2.39}$$

with

$$V = |\mathbf{V}|, \quad w = \tfrac{1}{2}(E^2 + H^2),$$

$$a = \left(\frac{\mu + P}{\mu + w}\right)^{\frac{1}{2}}, \quad b = \left(\frac{\mu - P}{\mu + w}\right)^{\frac{1}{2}}, \quad c = ab.$$

It is immediately seen that

$$1 > a \geq b > c, \quad V^2 = (1 - a^2)(1 - b^2).$$

The propagation is anisotropic, and besides there is a draught effect of speed $\mathbf{V}$. In the limit of an infinite absolute energy density $\mu \rightarrow \infty$, the ellipsoid (2.39) becomes the usual sphere of the linear theory of Maxwell.

F. Disturbances

i. Disturbances Associated with Photons[12,16]

Inserting (2.31) into (2.15) and (2.12) results in

$$\delta Q = u \pi^\alpha \varphi_\alpha, \quad \delta R = v \pi^\alpha \varphi_\alpha, \quad (\omega - P)\pi^\alpha \varphi_\alpha = 0,$$

or [see (2.24)]

$$\pi^\alpha \varphi_\alpha = 0, \tag{2.40}$$

$$\delta Q = \delta R = 0. \tag{2.41}$$

Furthermore we have from Eqs. (2.10) and (2.11)

$$\varphi_\alpha \delta F^{\alpha\beta} = 0, \quad \varphi_\alpha \delta F^{*\alpha\beta} = 0,$$

which means that the electric and magnetic disturbances are transverse, orthogonal to each other, and of equal strength, i.e., the usual results of the linear theory. But here Eqs. (2.41), $\delta Q = \delta R = 0$, replace the null field conditions (fluid of photons) $Q = R = 0$.

It is even possible to show that (2.41) characterizes photons. Let us assume that (2.41) is satisfied. Then, by (2.15),

$$U^\alpha \pi_\alpha = V^\alpha \pi_\alpha = 0, \tag{2.42}$$

and, by (2.12),

$$\mathcal{G} \pi_\alpha = (\pi^\beta \varphi_\beta)\varphi_\alpha,$$

which means

$$\mathcal{G} = 0, \quad \pi^\alpha \varphi_\alpha = 0 \tag{2.43}$$

for a wave (i.e., a disturbance) to exist: $\delta F_{\alpha\beta} \neq 0$ (2.10). Then, taking account of (2.30),

$$U_\alpha V^\alpha = 0, \quad V_\alpha V^\alpha = U_\alpha U^\alpha. \tag{2.44}$$

π_α is certainly a spacelike vector; otherwise it would be colinear to φ_α, which is just what we wanted to avoid when writing (2.43). U_α cannot be timelike for $U^\alpha \varphi_\alpha = 0$; if U_α were spacelike, then according to (2.42)–(2.44) φ_α would be orthogonal to the three linearly independent spacelike vectors π_α, U_α, V_α and would be timelike (and not null). Therefore, U_α is a null vector and we are led back to the conclusions of Sec. 2D: light rays are eigendirections of the electromagnetic tensor.

ii. Disturbances Associated with Charged Particles[12]

The first equation (2.17) and (2.18) yield

$$a = -\pi\{R\eta_Q + (\mu - Q)\eta_R\}, \quad \eta = \log|L_Q|,$$
$$b = \pi\{1 + (\mu + Q)\eta_Q + R\eta_R\}; \tag{2.45}$$

$\pi(x^\alpha)$ is a function of the coordinates whose growth in the course of time has been briefly discussed in Sec. 1A. Equations (2.16) give

$$\delta Q = \mathcal{G}\{a(2Q + \zeta) + bR\}, \quad \delta R = \mathcal{G}(aR + b\zeta), \tag{2.46}$$

or

$$\delta Q = \pi\mathcal{G}(R - f^2\eta_R), \quad \delta R = \pi\mathcal{G}(\zeta + f^2\eta_Q), \tag{2.47}$$

with the definition

$$f(Q, R; \zeta) = [-R^2 + \zeta(2Q + \zeta)]^{\frac{1}{2}}\ [= (\mu^2 - P^2)^{\frac{1}{2}}].$$

The expressions (2.45) and (2.47) are not valid for the Born–Infeld Lagrangian: the coefficients of a and b in (2.17) all vanish; a and b are arbitrary. It had to be so because of the double root in Sec. 2C.

Making use of the relation

$$-\eta^{\alpha\beta\gamma\delta}S_\alpha = \epsilon^{\beta\gamma\delta}_{\lambda\mu\nu}e^\lambda h^\mu u^\nu = (e^\beta h^\gamma - e^\gamma h^\beta)u^\delta + (e^\gamma h^\delta - e^\delta h^\gamma)u^\beta + (e^\delta h^\beta - e^\beta h^\delta)u^\gamma$$

and of Eqs. (2.32) and (2.37), we can write (2.13) as

$$\pi_\alpha = -\frac{1}{f(-\mathcal{G})^{\frac{1}{2}}}(e_\alpha\delta Q + h_\alpha\delta R + c\mathcal{G}S_\alpha),$$

while (2.10) takes the form

$$\delta F_{\alpha\beta} = \frac{1}{f^2}\{(e_\alpha S_\beta - e_\beta S_\alpha)\delta Q + (h_\alpha S_\beta - h_\beta S_\alpha)\delta R\}; \tag{2.48}$$

the disturbance of the dual tensor is easily obtained[14]:

$$\delta F^{*\gamma\delta} = \frac{1}{f^2}(-\eta^{\alpha\beta\gamma\delta}S_\alpha)(e_\beta\delta Q + h_\beta\delta R),$$

i.e.,

$$\delta F^{*\gamma\delta} = \mathcal{G}\{a(u^\gamma h^\delta - u^\delta h^\gamma) - b(u^\gamma e^\delta - u^\delta e^\gamma)\}. \tag{2.49}$$

G. Exceptional Waves

We apply now the second principle. According to (2.7), (2.10), and (2.15), we have

$$\varphi_\alpha\varphi_\beta\delta(\tau^{\alpha\beta} + \mu g^{\alpha\beta}) = \mathcal{G}\delta(\mu + Q) - 2U_\rho(\varphi^\alpha\pi^\rho - \varphi^\rho\pi^\alpha)\varphi_\alpha = \mathcal{G}\delta\zeta.$$

Thus the wave will be exceptional [cf. (1.4)] if [12]

$$\delta\zeta = 0. \tag{2.50}$$

From (2.38) it is already obvious that the Born–Infeld waves are exceptional. When the absolute field is not constant, this gives

$$\zeta_Q\delta Q + \zeta_R\delta R = 0,$$

or, by (2.47),

$$f^2J = R\zeta_Q + \zeta\zeta_R, \quad J = \zeta_Q\eta_R - \zeta_R\eta_Q. \tag{2.51}$$

If we assume the Jacobian $J \neq 0$,

$$\zeta_Q = JR_\eta, \quad \zeta_R = -JQ_\eta,$$

and (2.51) become

$$f + f_\eta = 0,$$

whose integration is easy,[13]

$$fL_Q = Z(\zeta), \tag{2.52}$$

Z being an arbitrary function of ζ. When $J = 0$ this equation still holds:

$$J = 0 \quad \text{and} \quad \varpi \neq 0 \rightarrow \zeta = \zeta(\eta),$$
$$RL_{QQ} + \zeta L_{QR} = 0.$$

With the change of variables of Sec. 2C, this last equation is equivalent to

$$yy_q - \zeta(p)x_q = 0, \quad \tfrac{1}{2}y^2 - \zeta(p)x = \text{funct}\,(p),$$

which is analogous to (2.52) if ζ is not constant.

Now Eq. (2.19) can be written

$$(2x + \zeta)r + 2ys + \zeta t + p + \frac{f^2}{p}(rt - s^2) = 0, \tag{2.53}$$

which is an equation of Monge–Ampère since, by virtue of (2.52), ζ can be considered as a function of x, y, p:

$$pf(x, y; \zeta) = Z(\zeta). \tag{2.54}$$

It is then easily seen that, along the characteristic curves of (2.53) (see, for instance, Ref. 21),

$$\begin{cases} f^2\dfrac{dp}{p} + \zeta\,dx - y\,dy = 0, \\ f^2\dfrac{dq}{p} - y\,dx + (2x + \zeta)\,dy = 0, \end{cases}$$

the quantities

$$\zeta, \quad q + y\frac{p}{\zeta}$$

are constant. As a result z is obtained by eliminating

ζ between the equations

$$z = F(\zeta)f + yG(\zeta) + H(\zeta) \quad (F = Z/\zeta),$$
$$\frac{\partial z}{\partial \zeta} = 0: \quad F'f^2 + (yG' + H')f + (x + \zeta)F = 0,$$

where the prime denotes differentiation with respect to ζ.

Therefore the Lagrangian is of the form[22]

$$L = F(\zeta)f(Q, R; \zeta) + RG(\zeta) + H(\zeta), \quad (2.55)$$

where the absolute field $\zeta(Q, R)$ is obtained by solving

$$F'f^2 + (RG' + H')f + (Q + \zeta)F = 0, \quad (2.56)$$

F, G, H being arbitrary functions of ζ.

Moreover,

$$L_Q = \frac{F\zeta}{f}, \quad L_R = -R\frac{F}{f} + G. \quad (2.57)$$

H. Exceptional Lagrangians[22]

To each value μ_1, μ_2 of μ solution of (2.19) corresponds a wave (2.18). If both families of waves can describe trajectories of stable particles,they must both be exceptional. Then the field equations will be completely exceptional (Sec. 1A) and we say that the Lagrangian is exceptional. It has already been made clear that the Born–Infeld Lagrangian is exceptional ($\zeta_1 = \zeta_2 = \text{const} = k$).

In the general case when neither ζ_1 nor ζ_2 is constant, (2.50) must be true for each value; the equations

$$\begin{cases} \delta_1\zeta_1 = 0, \\ \delta_2\zeta_2 = 0 \end{cases}$$

must be satisfied simultaneously, where the subscript 1 in δ_1 means that we must put $\zeta = \zeta_1$ in the expressions (2.47) (and also, of course, $\pi = \pi_1$, $\mathcal{G} = \mathcal{G}_1$).

Consequently, the Eqs. (2.55)–(2.57) must be valid no matter which value of ζ is used. In other words, the following relations hold:

$$F_1f_1 + RG_1 + H_1 = F_2f_2 + RG_2 + H_2, \quad (2.58)$$

$$\frac{F_1\zeta_1}{f_1} = \frac{F_2\zeta_2}{f_2}, \quad -R\frac{F_1}{f_1} + G_1 = -R\frac{F_2}{f_2} + G_2. \quad (2.59)$$

We eliminate Q and R between these equations to obtain the identity

$$\zeta_1\zeta_2(G_1 - G_2)^2 - (\zeta_1 - \zeta_2)(F_1^2\zeta_1 - F_2^2\zeta_2) + (H_1 - H_2)^2 \equiv 0. \quad (2.60)$$

(ζ_1, ζ_2 cannot be linked by any relation, for Q and R would then be functions of each other.)

This identity can also be written in the form

$$\psi_{12} + H_1^2 - F_1^2\zeta_1^2 + H_2^2 - F_2^2\zeta_2^2 \equiv 0 \quad (2.61)$$

with

$$\psi_{12} = \zeta_1\theta_2 + \zeta_2\theta_1 - 2\phi_1\phi_2 - 2H_1H_2,$$
$$\theta = \zeta(G^2 + F^2), \quad \phi = G\zeta. \quad (2.62)$$

According to the Remark of Sec. 2A, we shall make use of the possibility to change G and H into G + const and H + const, respectively, thus avoiding unnecessary constants of integration. We shall mention in parentheses the function to which the constant is added.

By virtue of (2.61), ψ_{12} is equal to the sum of a function of ζ_1 and of a function of ζ_2; hence,

$$\frac{\partial^2\psi_{12}}{\partial\zeta_1\partial\zeta_2} = 0,$$

i.e.,

$$\theta_1' + \theta_2' - 2\phi_1'\phi_2' - 2H_1'H_2' = 0. \quad (2.63)$$

It is only a matter of simple calculations to see that if both (ϕ')'s were constant, the same would be true of the absolute fields. Hence we assume $\phi_2'' \neq 0$. Differentiating (2.63) with respect to ζ_2, we get

$$(\theta_2''/\phi_2'') - 2\phi_1' - 2H_1'(H_2''/\phi_2'') = 0. \quad (2.64)$$

The assumption $H_1' = \text{const}$ would lead to conclusions analogous to the above ones. Thus we are left with

$$\frac{H_2''}{\phi_2''} = \text{const} = \gamma \neq 0,$$
$$H_2 = \gamma\phi_2. \quad (G_2, H_2)$$

Equations (2.64) and (2.63) give

$$\phi_1 + \gamma H_1 = 0, \quad (H_1)$$
$$\theta_1 = -a\zeta_1 + b, \quad \theta_2 = a\zeta_2 + c.$$

By (2.62) we have

$$\psi_{12} = c\zeta_1 + b\zeta_2,$$

and taking account of (2.61) we finally obtain the systems (I) and (II):

$$\text{(I)} \quad \begin{cases} G_1\zeta_1 + \gamma H_1 = 0, \\ \zeta_1(G_1^2 + F_1^2) = -a\zeta_1 + b, \\ H_1^2 - F_1^2\zeta_1^2 + c\zeta_1 + l = 0; \end{cases}$$

$$\text{(II)} \quad \begin{cases} \gamma G_2\zeta_2 - H_2 = 0, \\ \zeta_2(G_2^2 + F_2^2) = a\zeta_2 + c, \\ H_2^2 - F_2^2\zeta_2^2 + b\zeta_2 - l = 0. \end{cases}$$

Any one of these systems may be used with (2.55) and (2.56) to determine the Lagrangian. We note that, on passing from (I) to (II), that is,from the value ζ_1 to the value ζ_2 of the absolute field, we make the following changes (C):

$$\text{(C)} \quad (\zeta_1 \to \zeta_2): \gamma \to -\frac{1}{\gamma}, a \to -a, b \rightleftarrows c, l \to -l.$$

From (I) and (II) we get

$$H_1^2 = \frac{-1}{1+\gamma^2}\{a\zeta_1^2 - (b-c)\zeta_1 + l\},$$

$$H_2^2 = \frac{\gamma^2}{1+\gamma^2}\{a\zeta_2^2 - (b-c)\zeta_2 + l\}.$$

If $a \neq 0$, we can always assume $a > 0$. The discriminant of the polynomial inside brackets is necessarily positive, so we can put

$$k = \frac{1}{2}\frac{b-c}{a}, \quad l = a(k^2 - h^2).$$

The new constants h and k are C invariant, i.e., invariant by the transformations (C). Then,

$$H_1^2 = [a/(1+\gamma^2)]\{h^2 - (\zeta_1 - k)^2\};$$

the Lagrangian (2.55) reduces to the Born–Infeld Lagrangian when $h = 0$ ($\zeta_1 = k$), provided we assume $k > 0$. We have for $h = 0$

$$F_1^2(k) = \frac{1}{2}\frac{b+c}{k}, \quad G_1(k) = H_1(k) = 0,$$

which shows that the C invariant

$$\frac{1}{2}\frac{b+c}{k} \tag{2.65}$$

must be positive.

We introduce the dimensionless constant [23]

$$s = \frac{1}{2}\frac{b+c}{k}\frac{1}{a},$$

and, assuming that L has been divided by the square root of (2.65), the solution of (I) can be taken equal to

$$F_1^2 = \frac{1}{s(1+\gamma^2)}\left\{\frac{\gamma^2}{\zeta_1^2}(k^2 - h^2) + 2K\frac{\gamma}{\zeta_1} - 1\right\},$$

$$G_1 = -\frac{\gamma}{\zeta_1}H_1, \quad H_1^2 = \frac{1}{s(1+\gamma^2)}\{h^2 - (\zeta_1 - k)^2\},$$

where the C-invariant constant K is defined by

$$2K = \frac{1}{a\gamma}(b + c\gamma^2) = \frac{k}{\gamma}\{s + 1 + \gamma^2(s-1)\}.$$

The law of transformation (C) of γ suggests to one to put

$$\gamma = tg\tfrac{1}{2}\theta.$$

The absolute field depends on the new constants in the following way:

$$\zeta_1 = \zeta(Q, R; k, h, s, \theta),$$
$$\zeta_2 = \zeta(Q, R; k, h, -s, \theta + \pi);$$
$$h = 0: \quad \zeta_1 = \zeta_2 = k.$$

The constant h might well be simply connected with the Planck constant. When it vanishes, both values of ζ reduce to the constant absolute field k of Born–Infeld, thus allowing for a spherical approximation of the particle (Sec. 2E). The spin of the stable particle seems to be represented by s and θ is some constant angle.

To illustrate the role played by θ we consider the first-order approximation of the absolute field when the parameter h/k is small (which is probably the case). Let Q_0, R_0 be the values of the invariants built from a solution of the Born–Infeld field equations and let

$$f_0 = f(Q_0, R_0; k).$$

We introduce the three coplanar dimensionless vectors **A**, **B**, **C**:

$$|\mathbf{A}| = \frac{f_0^2}{k^2}, \quad |\mathbf{B}| = s\,|1 - (R_0^2/k^2)|,$$

$$|\mathbf{C}| = 2s(|R_0|/k),$$

the last ones being orthogonal to each other ($\mathbf{B}\cdot\mathbf{C} = 0$) and making with **A** the constant angles

$$(\mathbf{A}, \mathbf{B}) = \left|\theta + \frac{\pi}{2}[1 - \operatorname{sgn}(k^2 - R_0^2)]\right|,$$

$$(\mathbf{A}, \mathbf{C}) = \left|\frac{\pi}{2} + \theta \operatorname{sgn} R_0\right|;$$

then, using (2.56), we find[23]

$$\zeta_1 \cong k \pm h\frac{|\mathbf{A}| - |\mathbf{B}+\mathbf{C}|}{|\mathbf{A}+\mathbf{B}+\mathbf{C}|},$$

$$\zeta_2 \cong k \pm h\frac{|\mathbf{A}| + |\mathbf{B}+\mathbf{C}|}{|\mathbf{A}+\mathbf{B}+\mathbf{C}|}.$$

Remarks: (1) Quite generally in one-dimensional propagation when a field has only two components and when the field equations are completely exceptional,[3] the characteristics (wave surfaces in the x, t plane) belong to two families of isocline curves. This means that the slope of curves of one family is constant along each curve of the other family; in other words, curves of the same family cannot intersect. This phenomenon had already been noted for the Born–Infeld one-dimensional field equations.[24]

(2) It has also been predicted[25] that shocks might arise when the Lagrangian of Heisenberg–Euler[26] is used. This is true since this Lagrangian is not exceptional. This Lagrangian, however, as it is given, appears as the first terms of an expansion with respect to a small parameter. As we ignore the exact form of this Lagrangian, we cannot even calculate the absolute field. Nevertheless it is probable that the exact form is not exceptional since it is used in connection with the production of unstable particles.

I. Disturbance of the Ray Velocity

It is given by the formula[14,1]

$$\delta u^\alpha = [1/f(-\mathcal{G})^{\frac{1}{2}}](\delta^\alpha_\beta - u^\alpha u_\beta)\varphi_\gamma \delta(\tau^{\beta\gamma} + \mu g^{\beta\gamma})$$

and calculations show that it is equal to

$$\delta u^\alpha = (S^\alpha / f^2)\delta\zeta.$$

When the wave is exceptional,

$$\delta u^\alpha = 0,$$

from (2.48) and (2.49) we have

$$\delta e^\alpha = u_\rho \delta F^{\rho\alpha} = 0, \quad \delta h^\alpha = u_\rho \delta F^{*\rho\alpha} = \mathcal{G}(a h^\alpha - b e^\alpha),$$

$$\delta S_\alpha = \eta_{\alpha\lambda\mu\nu} e^\lambda \delta h^\mu u^\nu = a\mathcal{G} S_\alpha.$$

Thus a stable particle appears in its proper frame as a disturbance of the sole magnetic field.

On the other hand, we obtain[10], by solving Eqs. (2.46),

$$a\mathcal{G} = \frac{\delta f}{f} - \frac{\mu}{f^2}\delta\zeta, \quad b\mathcal{G} = \frac{f}{\zeta}\delta\left(\frac{R}{f}\right) + \frac{R}{\zeta}\frac{\mu}{f^2}\delta\zeta.$$

Inserting these expressions into the above relations, we immediately see that in the exceptional case ($\delta\zeta = 0$), the tetrad of vectors

$$V^\alpha_{(\gamma)}: u^\alpha, \quad \frac{e^\alpha}{\zeta^{\frac{1}{2}}}, \quad \frac{Re^\alpha + \zeta h^\alpha}{f(\zeta)^{\frac{1}{2}}}, \quad \frac{S^\alpha}{f}; \qquad (2.66)$$

$$V_{(\mu)\alpha} V^\alpha_{(\nu)} = \eta_{\mu\nu},$$

is not disturbed by the stable particle wave:

$$\delta V^\alpha_{(\mu)} = 0. \qquad (2.67)$$

J. Introduction of a Nonsymmetric Tensor

In their famous paper,[11] Born and Infeld introduce the nonsymmetric tensor

$$a_{\alpha\beta} = g_{\alpha\beta} + F_{\alpha\beta}, \qquad (2.68)$$

i.e., identify the symmetric and antisymmetric part with the gravitational and electromagnetic field tensors, respectively, and use the Lagrangian density,

$$\mathfrak{L} = A(-|a_{\alpha\beta}|)^{\frac{1}{2}} \pm B(|F_{\alpha\beta}|)^{\frac{1}{2}} + C(-|g_{\alpha\beta}|)^{\frac{1}{2}}. \qquad (2.69)$$

If a positive constant k is introduced for the sake of dimension, we rewrite (2.68) as

$$a_{\alpha\beta} = k^{\frac{1}{2}} g_{\alpha\beta} + F_{\alpha\beta}, \qquad (2.70)$$

while (2.69) gives the Born–Infeld Lagrangian,

$$L = Af(Q, R; k) + BR + C,$$
$$\mathfrak{L} = (-g)^{\frac{1}{2}} L, \quad g = |g_{\alpha\beta}|. \qquad (2.71)$$

Now an obvious generalization of (2.70) and (2.71) consists of replacing the constant k by the function $\zeta(Q, R)$ of Q and R,

$$a_{\alpha\beta} = \zeta^{\frac{1}{2}} g_{\alpha\beta} + F_{\alpha\beta}, \qquad (2.72)$$

$$L = F(\zeta)f(Q, R; \zeta) + G(\zeta)R + H(\zeta). \qquad (2.73)$$

This is nothing but the Lagrangian (2.55), and, from the results of Sec. 2G, we know that ζ is indeed an absolute field if it is defined by

$$\partial L / \partial \zeta = 0,$$

a result which can also be established by direct calculation.[16,27]

Let us compare waves of nonlinear electrodynamics with waves of the Einstein–Schrödinger theory. To this aim we introduce the adjoint tensor of (2.72),[23]

$$a_{\alpha\beta} \dot{a}^{\alpha\gamma} = a_{\beta\alpha} \dot{a}^{\gamma\alpha} = \delta^\gamma_\beta,$$

i.e.,

$$\dot{a}^{\alpha\beta} = \frac{\zeta^{\frac{1}{2}}}{f^2}(\tau^{\alpha\beta} + \mu g^{\alpha\beta}) + \frac{1}{f^2}(\zeta F^{\alpha\beta} - RF^{*\alpha\beta}).$$

We see at once that the quantity

$$\partial_\alpha[(-a)^{\frac{1}{2}} \dot{a}^{[\alpha\beta]}], \quad a = |a_{\alpha\beta}| = gf^2,$$

vanishes [cf. (2.4)] for the Born–Infeld field[11,28] and that the waves,

$$\dot{a}^{(\alpha\beta)} \varphi_\alpha \varphi_\beta = 0, \qquad (2.74)$$

are to be found in Einstein–Schrödinger theory too.[6]

However, in nonlinear electrodynamics we do not find the other family of waves of the Einstein–Schrödinger field. If we put

$$g'_{\alpha\beta} = a_{(\alpha\beta)} = \zeta^{\frac{1}{2}} g_{\alpha\beta},$$

and hence

$$\dot{g}'^{\alpha\beta} = g^{\alpha\beta}/\zeta^{\frac{1}{2}}, \quad g' = \zeta^2 g,$$

this family is given by[29]

$$\left(\dot{a}^{(\alpha\beta)} - 2\frac{g'}{a}\dot{g}'^{\alpha\beta}\right)\varphi_\alpha\varphi_\beta = 0,$$

i.e.,

$$\{\tau^{\alpha\beta} + (Q - \zeta)g^{\alpha\beta}\}\varphi_\alpha\varphi_\beta = 0;$$

it would correspond to the change of ζ into $-\zeta$, which would not be acceptable since, by (2.23) and (2.35), ζ must be positive.

Another difference must be stressed: in Einstein–Schrödinger theory, the nonsymmetric tensor is assumed to be continuous in the first order[6] and to admit discontinuities of the second order, which is natural because the field equations involve second-order derivatives of this tensor. On the contrary we have here

$$\delta a_{\alpha\beta} \neq 0.$$

Thus if we want to try to write the field equations of nonlinear electrodynamics in terms of the components of a nonsymmetric tensor and its first- and second-order derivatives, we had better choose this tensor in such a way that

$$\delta a_{\alpha\beta} = 0.$$

For this purpose we might think of the vectors (2.66). (A set of four vectors is used by Møller in his tetrad theory.[30])

For instance, the tensor

$$a_{\alpha\beta} = g_{\alpha\beta} + f_{\alpha\beta},$$
$$f_{\alpha\beta} = V_{(0)\alpha}V_{(1)\beta} - V_{(1)\alpha}V_{(0)\beta} + \eta_{\alpha\beta\gamma\delta}V^{\gamma}_{(2)}V^{\delta}_{(0)},$$

is analogous to (2.72) when the ratio P/ζ is small,

$$g_{\alpha\beta} + f_{\alpha\beta} \simeq g_{\alpha\beta} + (F_{\alpha\beta}/\zeta^{\frac{1}{2}}), \quad P/\zeta \ll 1.$$

Furthermore,

$$f_{\alpha\beta}V^{\beta}_{(2)} = 0 \rightarrow |f_{\alpha\beta}| = 0$$

and

$$a = |a_{\alpha\beta}| = 2\,|a_{(\alpha\beta)}| = 2g.$$

This is a singular case in Einstein–Schrödinger theory, but it also allows for the choice of another connection.[28,31]

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