Gravity from Thermodynamics

Deriving Field Equations from Thermodynamics Laws

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Preliminary Math
Vectors and Dual Vectors

- Manifold: a space that resembles Euclidean (flat) space near each of its points.
- How we define variation?

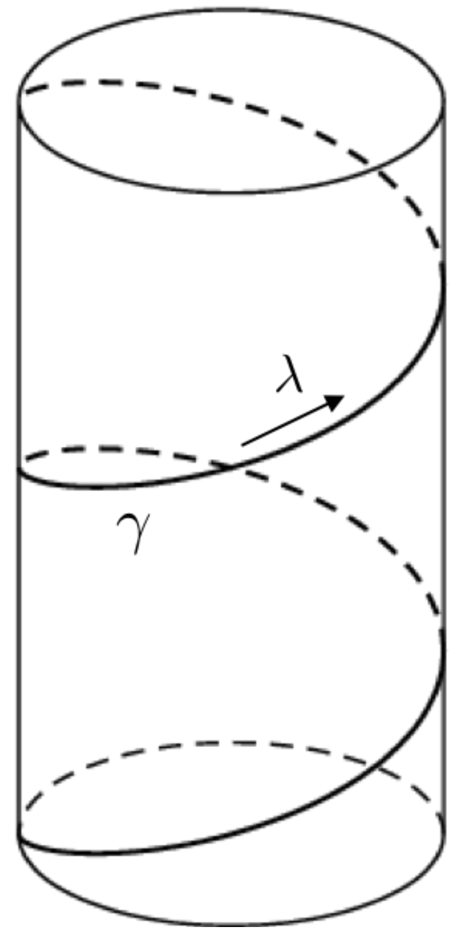
$$\frac{df}{d\lambda} = \sum_{\alpha} \frac{\partial f}{\partial x^\alpha}\frac{dx^\alpha}{d\lambda} \equiv \overbrace{\frac{\partial f}{\partial x^\alpha}}^{\text{gradient of } f} \underbrace{\frac{dx^\alpha}{d\lambda}}_{\text{vector tangent to } \gamma}$$

Caption

- The gradient is referred to be "Dual Vector"
- They transform under coordinate transformations:

$$f_{,\alpha'} = \frac{\partial f}{\partial x^{\alpha'}} = \frac{\partial x^\alpha}{\partial x^{\alpha'}} f_{,\alpha}$$
$$u^{\alpha'} = \frac{dx^{\alpha'}}{d\lambda} = \frac{\partial x^{\alpha'}}{\partial x^\alpha} u^\alpha$$

$x = \lambda, y = \cos(2\lambda), z = \sin(2\lambda)$



Preliminary Maths
Lie Derivatives

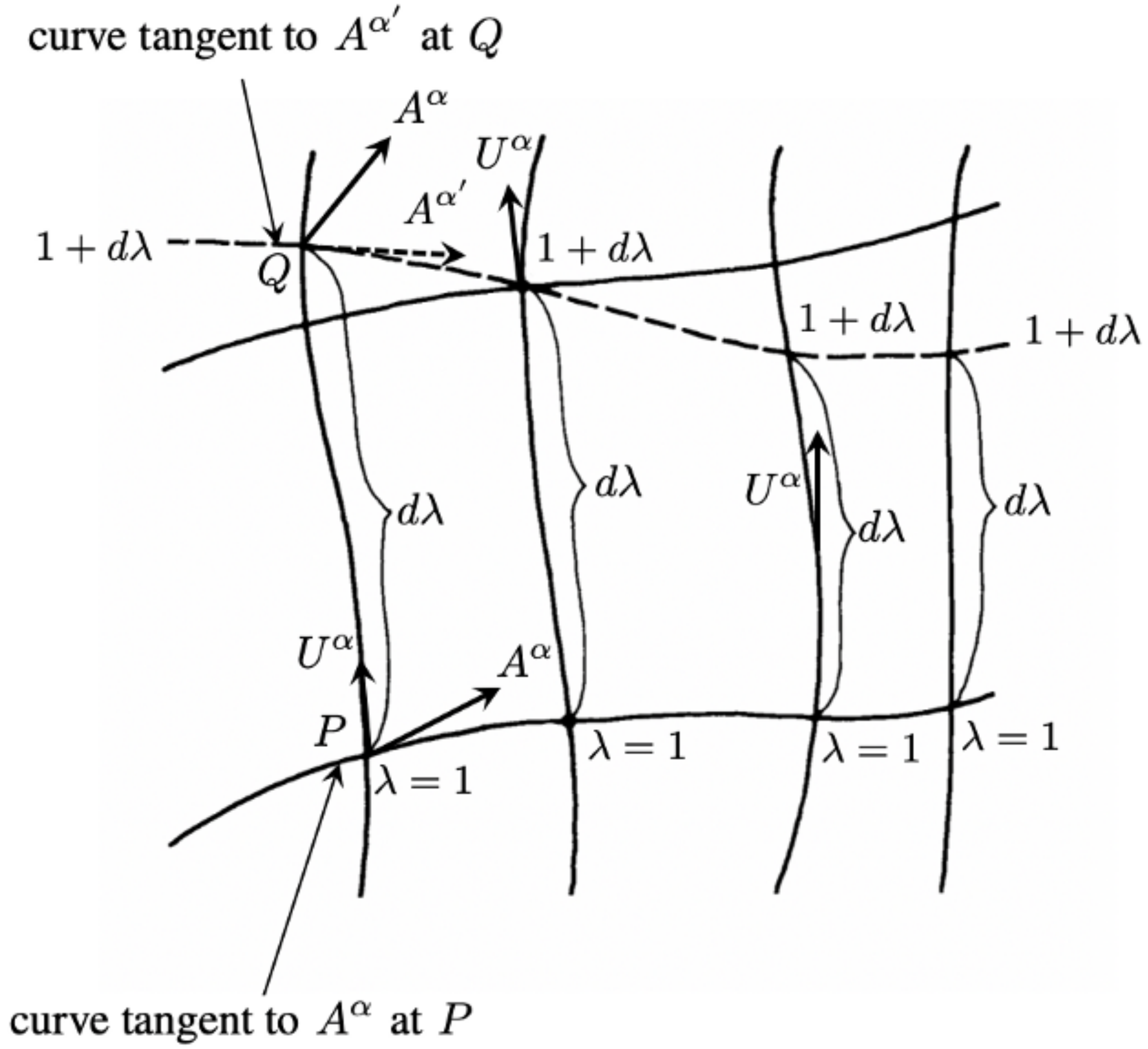
- Lie Derivative : Concept from differential geometry

$$x^{\alpha'} = x^\alpha + dx^\alpha = x^\alpha + u^\alpha d\lambda$$

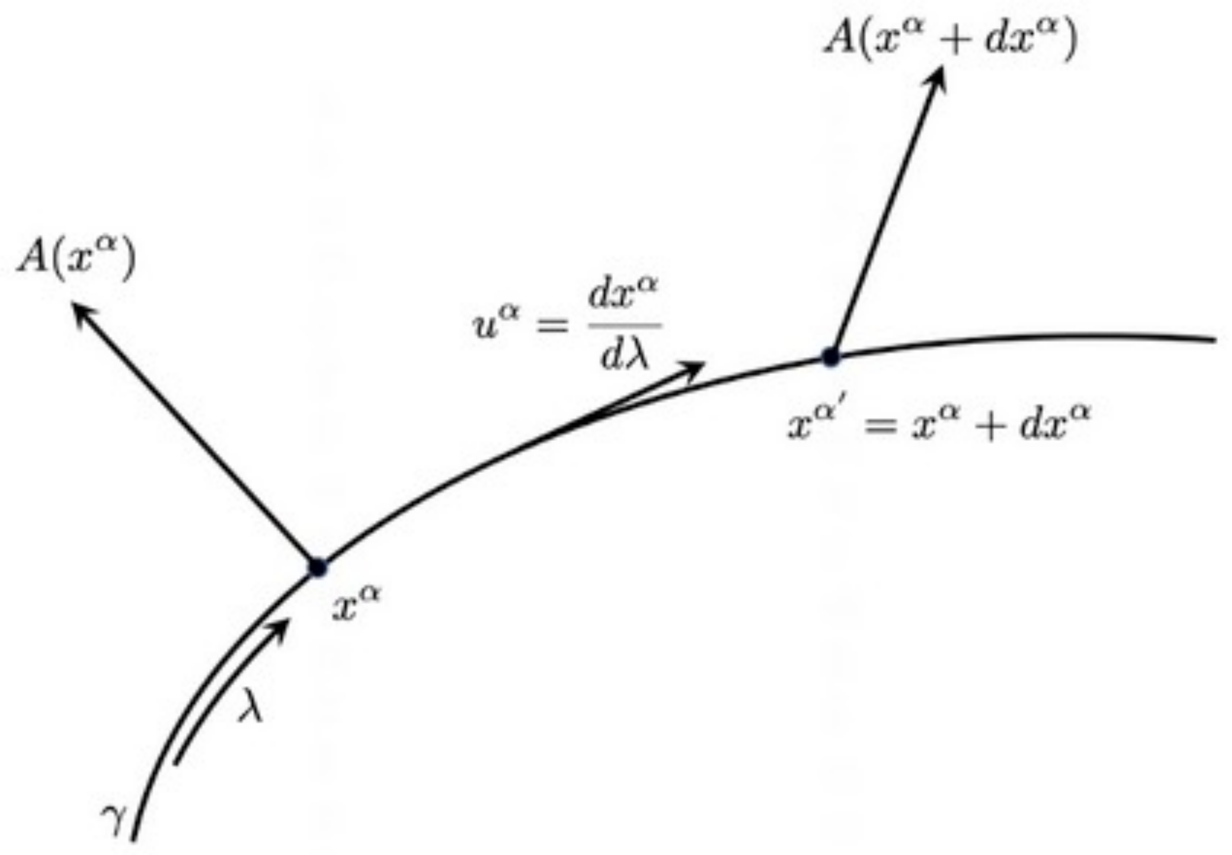
$$A^{\alpha'}(x+dx) = \frac{\partial x^{\alpha'}}{\partial x^\beta} A^\beta(x) = \left(\delta^\alpha_\beta + \frac{\partial u^\alpha}{\partial x^\beta} d\lambda\right) A^\beta(x) = A^\alpha(x) + \frac{\partial u^\alpha}{\partial x^\beta} A^\beta(x) d\lambda$$

$$A^\alpha(x+dx) = A^\alpha(x) + u^\beta \frac{\partial A^\alpha}{\partial x^\beta} A^\beta(x) d\lambda$$

$$\mathcal{L}_u A^\alpha(x) = \frac{A^\alpha(x+dx) - A^{\alpha'}(x+dx)}{d\lambda} \Rightarrow \mathcal{L}_u A^\alpha(x) = \frac{\partial A^\alpha}{\partial x^\beta} u^\beta - \frac{\partial u^\alpha}{\partial x^\beta} A^\beta \equiv A^\alpha_{\ ,\beta} u^\beta - u^\alpha_{\ ,\beta} A^\beta$$



$$\mathcal{L}_u A^\alpha(P) = \frac{A^\alpha(Q) - A^{\alpha'}(Q)}{d\lambda}$$



Preliminary Maths
Killing Vectors and Symmetries

- Killing vector is a vector field such that the lie derivative of the metric along its direction is zero.

$$\mathcal{L}_\xi g_{\alpha\beta} = 0$$

- If in a given coordinate system the metric does not depend on the coordinate σ*, the α-component of ξ is:

$$\xi^\alpha \stackrel{*}{=} \delta^\alpha_{\sigma_*}$$

- Obtain constants of motion as $\dfrac{d}{d\lambda}(u^\alpha \xi_\alpha) = 0$ $\longrightarrow$ If ***u*** is tangent to a geodesics

- Example: 3D Cartesian

$$ds^2 = dx^2 + dy^2 + dz^2 \longrightarrow$$

$$\begin{aligned}
\xi^\alpha_{(x)} &= \frac{\partial x^\alpha}{\partial x} = (1,0,0) \\
\xi^\alpha_{(y)} &= \frac{\partial x^\alpha}{\partial y} = (0,1,0) \\
\xi^\alpha_{(z)} &= \frac{\partial x^\alpha}{\partial z} = (0,0,1)
\end{aligned}$$

Preliminary Maths
Killing Horizon

- Take a null hypersurface (i.e. light cone): Every normal vector at each point is null (zero length respect to metric)
- Killing horizon **Σ**: a null hypersurface where the norm of a killing vector field vanishes. Also, since a null surface cannot have two linearly independent null tangent vectors, ***ξ*** will be normal to Σ.

- In flat spacetime (Minkowski spacetime)

$$\xi_\alpha \xi^\alpha = -x^2 + t^2 \longrightarrow x = \pm t$$

- Rindler Wedge: a sheet of hyperbolic timelike observers close to the null surface $\longrightarrow$ $a = a_\mu a^\mu + O(x)$

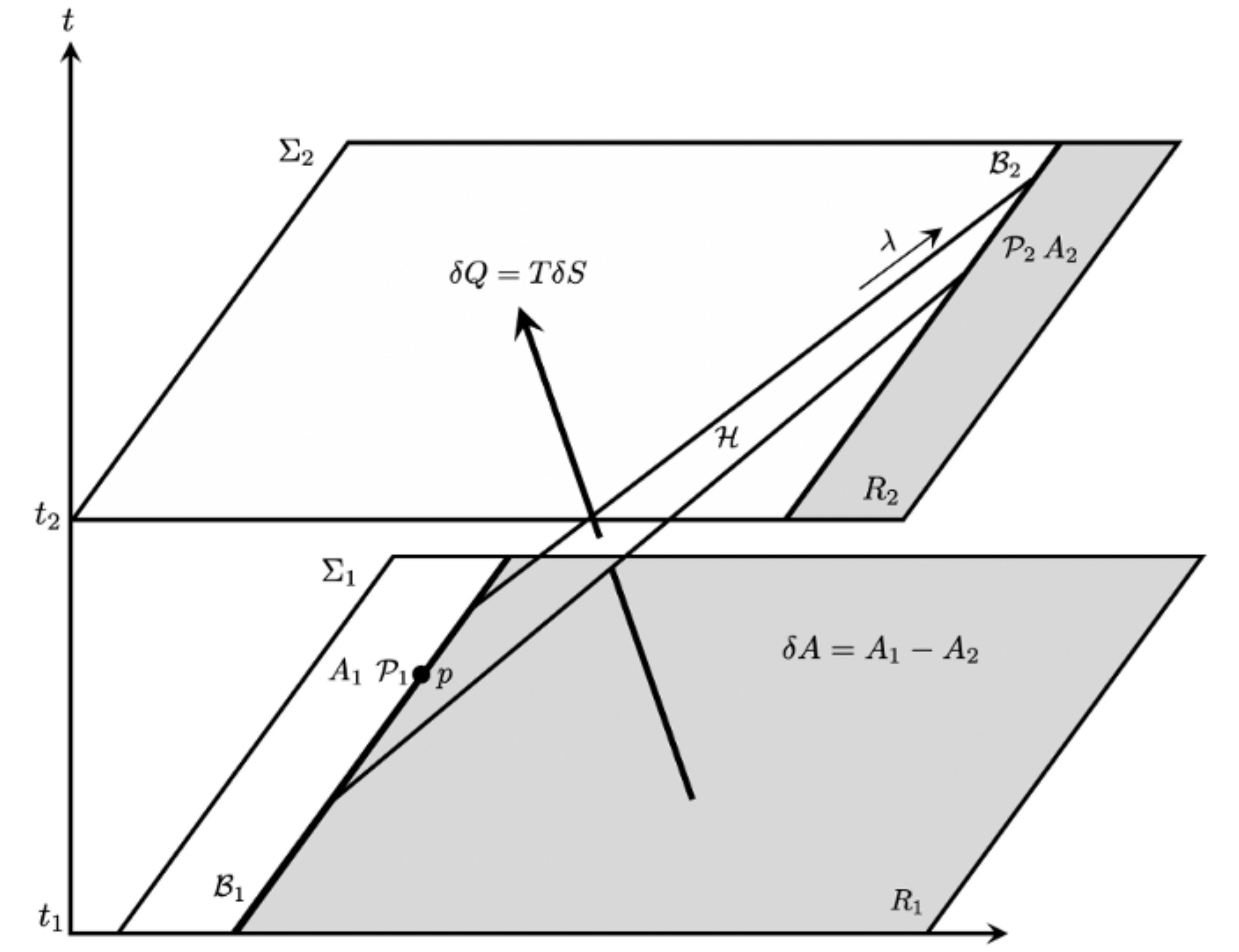
QFT $\longrightarrow$ $T = \dfrac{\hbar\kappa}{2\pi}$

Deriving Equation of State

- We have point p in spacetime M and an infinitesimal neighborhood containing *p*
- codimension: M has d=4, the submanifold Σ_1 has dimension d=4–1=3).
- Let point p located on t=t_1 on adimension spacelike codimension-one hypersurface Σ_1
- Introduce a spacelike codimension-two (approximately-flat) patch P_1 which contains the point p (a submanifold with d=2) and construct a local inertial frame inside P_1.
- introduce a closed spacelike codimension-two surface B_1 (a submanifold with d=2) such that $P_1 \subset B_1$

$$k^\mu = \left(\frac{d}{d\lambda}\right)^\mu \text{ with } \lambda(p) = 0$$

- The congruence generates a null hypersurface H that emanates from P_1 with k tangent to the generators



Deriving Equation of State

- We choose an approximate boost Killing vector field ξ to be the generator of H and define it → $\xi^\mu = -\kappa\lambda k^\mu$
- The surface element for the local Rindler horizon → $d\Sigma^\mu = k^\mu dA d\lambda$
- Energy that flows across the horizon is all heat.

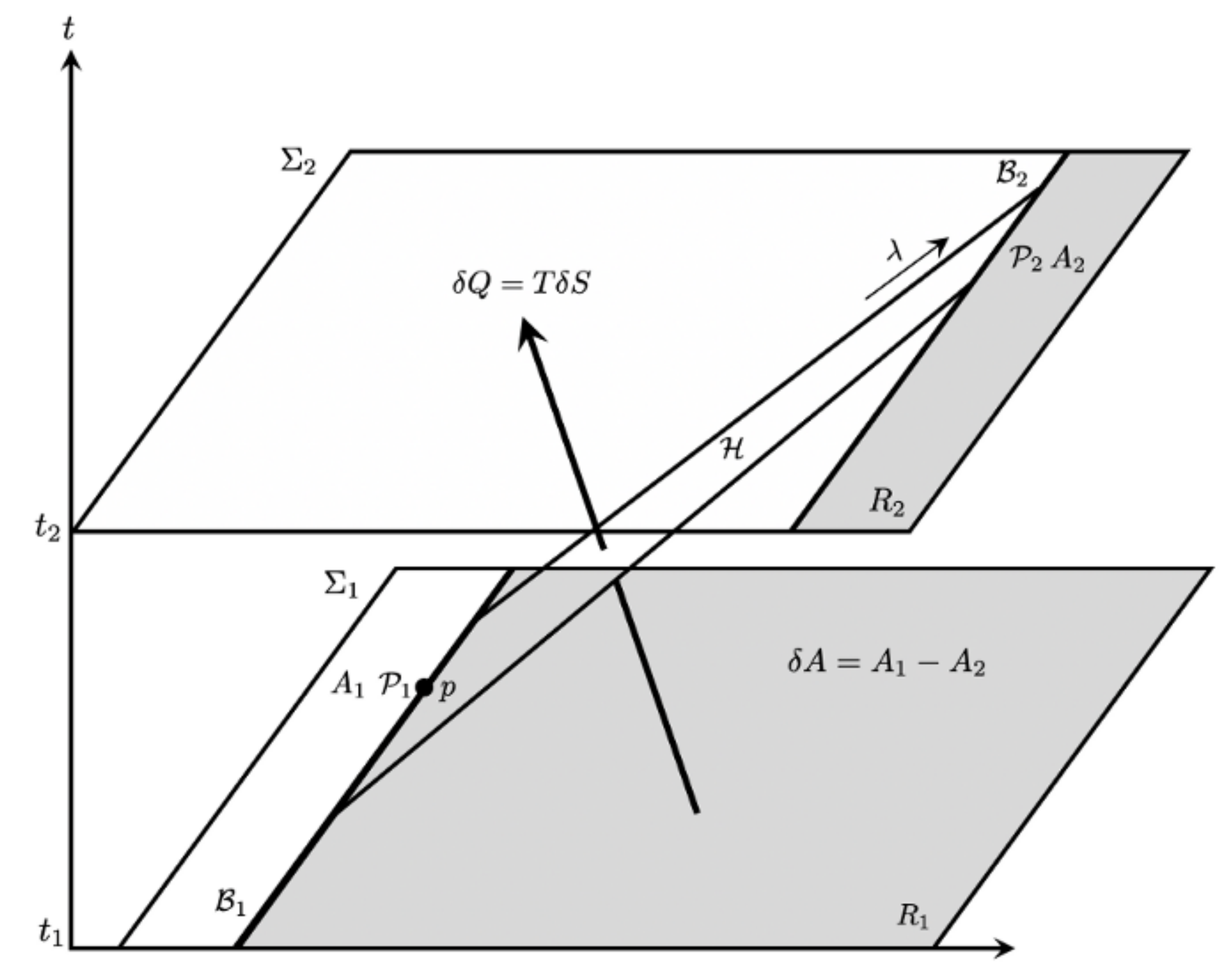
$$\delta Q = \int_{\mathcal{H}} T_{\mu\nu}\xi^\mu d\Sigma^\nu \qquad \delta A = \int_{\mathcal{H}} \theta \, d\lambda \, dA \qquad \theta = \frac{1}{\delta A}\frac{d(\delta A)}{d\lambda}$$

- Assume that

$$\delta Q = T dS \qquad S = \alpha A$$

- Raychaudhuri equation ➡ $\frac{d\theta}{d\lambda} = -\frac{\theta^2}{2} - \sigma^2 - R_{\mu\nu}k^\mu k^\nu \ \text{ where } \ \sigma^2 = \sigma^{\mu\nu}\sigma_{\mu\nu}$

$$dS = -\alpha \int_{\mathcal{H}} \lambda R_{\mu\nu}k^\mu k^\nu dA \, d\lambda \longrightarrow \frac{\hbar\alpha}{2\pi}\int_{\mathcal{H}} R_{\mu\nu}k^\mu k^\nu dA \, d\lambda = \int_{\mathcal{H}} T_{\mu\nu}k^\mu k^\nu dA \, d\lambda$$



Result

$$R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{2\pi}{\hbar\alpha}T_{\mu\nu}, \quad \text{with} \quad G = (4\hbar\alpha)^{-1}$$

References

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