University of Illinois Urbana-Champaign
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CP02

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$$0 \le r \le R$$

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1.1.1.3 Boundary Conditions

$$\frac{dT_{ss}(r)}{dr}|_{r=0} = 0$$

$$-k\frac{dT_{ss}(r)}{dr}|_{r=R} = h(T_{ss}(R) - T_{b0})$$

1.1.2 Data

Table 1 shows problem (a) parameters.

Table 1 Problem (a) Parameters

Parameter	Value	Unit
R	0.5	cm
k	2	W m^{-1} K^{-1}
h	45	kW m^{-2} K^{-1}
T_{b0}	300	°C

1.1.3 Solution

The ODE is

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT_{ss}(r)}{dr}\right)+\frac{\dot{q}_0}{k}=0$$

Rearranging and integrating

$$\int d\left(r\frac{dT_{ss}(r)}{dr}\right)=\int\frac{\dot{q}_0}{k}rdr$$

$$r\frac{dT_{ss}(r)}{dr}=\frac{\dot{q}_0}{2k}r^2+c_1$$

Applying the boundary condition at $r=0$

$$\frac{dT_{ss}(r)}{dr}|_{r=0}=0$$

Then

$$c_1=0$$

And the ODE becomes

$$r\frac{dT_{ss}(r)}{dr}=-\frac{\dot{q}_0}{2k}r^2$$

Rearranging and integrating

$$\int dT_{ss}(r)=\int-\frac{\dot{q}_0}{2k}rdr$$

$$T_{ss}(r)=-\frac{\dot{q}_0}{4k}r^2+c_2$$

Applying the boundary condition at $r=R$

$$-k\frac{dT_{ss}(r)}{dr}|_{r=R}=h(T_{ss}(R)-T_{b0})$$

$$-k\left(-\frac{\dot{q}_0}{2k}R\right)=h\left(-\frac{\dot{q}_0}{4k}R^2+c_2-T_{b0}\right)$$

Then

$$c_2 = T_{b0} + \frac{\dot{q}_0 R}{2h} + \frac{\dot{q}_0 R^2}{4k}$$

And the solution becomes

$$T_{ss}(r) = \frac{\dot{q}_0}{4k}(R^2 - r^2) + \frac{\dot{q}_0 R}{2h} + T_{b0}$$

Substituting the numerical values of the constants yields

$$T_{ss}(r) = \frac{\dot{q}_0}{8}(2.5 \times 10^{-5} - r^2) + 5.56 \times 10^{-8}\, \dot{q}_0 + 300$$

Where
$r\ in\ m$
$\dot{q}_0\ in\ W/m^3$
$T_{ss}\ in\ ℃$

Or

$$T_{ss}(r) = \frac{\dot{q}_0}{8}(25 - 100r^2) + 5.56 \times 10^{-2}\, \dot{q}_0 + 300$$

Where
$r\ in\ cm$
$\dot{q}_0\ in\ W/cm^3$
$T_{ss}\ in\ ℃$

The maximum temperature is at $r = 0$

$$T_{ss_{max}} = \dot{q}_0 \left(\frac{R^2}{4k} + \frac{R}{2h}\right) + T_{b0}$$

$$\dot{q}_0 = \frac{T_{ss_{max}} - T_{b0}}{\frac{R^2}{4k} + \frac{R}{2h}}$$

$$\dot{q}_0 = 1.26 \times 10^8\ W/m^3 = 1.26 \times 10^5\ kW/m^3$$

$$T_{ss}(r) = 700 - 1.57 \times 10^7\ r^2$$

Where
$r\ in\ m$
$T_{ss}\ in\ ℃$

Figure 1 shows the steady state temperature distribution.

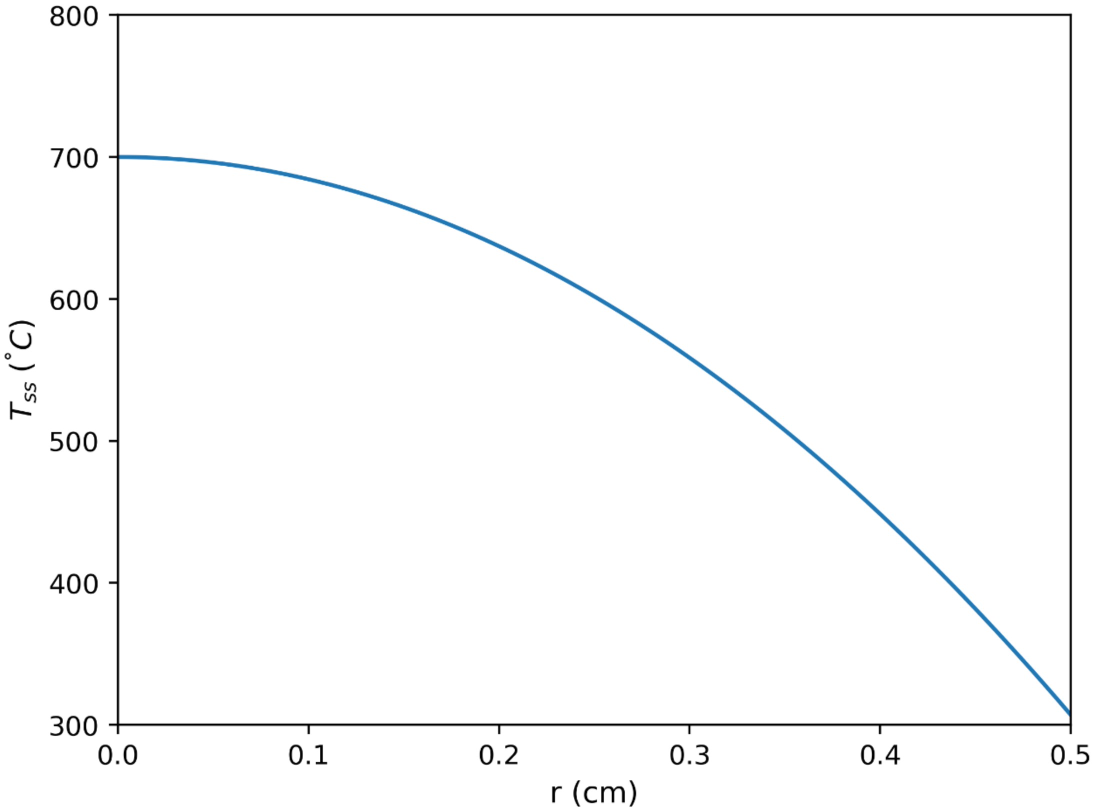

Figure 1 Steady State Temperature Distribution

1.2 b

1.2.1 Formulation

1.2.1.1 Domain

$$0 \le r \le R$$

1.2.1.2 PDE

$$\frac{1}{\alpha}\frac{\partial T(r,t)}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T(r,t)}{\partial r}\right) + \frac{\dot{q}(t)}{k}$$

1.2.1.3 Boundary Conditions

$$\frac{\partial T(r,t)}{\partial r}|_{r=0} = 0$$

$$-k\frac{\partial T(r,t)}{\partial r}|_{r=R} = h(T(R,t) - T_b(t))$$

Initial Condition

$$T(r,0) = \frac{\dot{q}_0}{4k}(R^2 - r^2) + \frac{\dot{q}_0 R}{2h} + T_{b0}$$

1.2.2 Data

Table 2 shows problem (b) parameters.

Table 2 Problem (b) Parameters

Parameter	Value	Unit
R	0.5	cm
k	2	W m^{-1} K^{-1}
h	45	kW m^{-2} K^{-1}
T_{b0}	300	°C
$\dot{q}_0$	1.26×10^5	kW m^{-3}
ρ	10.75	g cm^{-3}
c_p	84	J mol^{-1} K^{-1}

1.2.3 Solution

The eigenvalue problem formulation is

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\psi(r)}{dr}\right) + \lambda^2\psi(r) = 0$$

$$\lim_{r\to 0}\psi(r) \neq \pm\infty$$

$$\frac{d\psi(r)}{dr}|_{r=R} + \frac{h}{k}\psi(R) = 0$$

The solution is

$$\psi(r) = c_1 J_0(\lambda r) + c_2 Y_0(\lambda r)$$

Applying the boundary condition at $r = 0$

$$\lim_{r\to 0}\psi(r) \neq \pm\infty$$

$$c_2 = 0$$

$$\psi(r) = c_1 J_0(\lambda r)$$

$$\frac{d\psi(r)}{dr} = -c_1\lambda\, J_1(\lambda r)$$

Applying the boundary condition at $r = R$

$$\frac{d\psi(r)}{dr}|_{r=R} + \frac{h}{k}\psi(R) = 0$$

$$-c_1\lambda\, J_1(\lambda R) + \frac{h}{k}c_1 J_0(\lambda R) = 0$$

Then the eigenvalues equation becomes

$$\frac{\lambda_n\, J_1(\lambda_n R)}{J_0(\lambda_n R)} = \frac{h}{k} \qquad where \qquad n = 1, 2, 3, \dots, \infty$$

Operate on the original PDE by

$$\int_0^R r\ \psi_n(r) dr$$

$$\int_0^R \frac{1}{\alpha}\frac{\partial T(r,t)}{\partial t}\ r\,\psi_n(r)dr = \int_0^R \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T(r,t)}{\partial r}\right)\ r\ \ \psi_n(r)dr + \int_0^R \frac{\dot{q}(t)}{k}\ r\ \ \psi_n(r)dr$$

Define the following integral transform

$$\bar{T}_n(t) = \int_0^R T(r,t)\,r\ \ \psi_n(r)dr$$

$$\bar{q}_n(t) = \int_0^R \dot{q}(t)\,r\,\psi_n(r)dr = \dot{q}(t)\ I_{1n}$$

Where

$$I_{1n} = \int_0^R \ r\,\psi_n(r)dr$$

The PDE becomes

$$\frac{1}{\alpha}\frac{d\bar{T}_n(t)}{dt} = \left[r\ \ \psi_n(r)\frac{\partial T(r,t)}{\partial r} - r\,T(r,t)\ \frac{d\,\psi_n(r)}{dr}\right]_0^R - \lambda_n^2\bar{T}(t) + \frac{\bar{q}_n(t)}{k}$$

$$\frac{1}{\alpha}\frac{d\bar{T}_n(t)}{dt} + \lambda_n^2\bar{T}(t) = R\ \ \psi_n(R)\frac{\partial T(r,t)}{\partial r}|_{r=R} - R\,T(R,t)\ \frac{d\,\psi_n(r)}{dr}|_{r=R} + \frac{\bar{q}_n(t)}{k}$$

Apply the boundary conditions

$$\frac{1}{\alpha}\frac{d\bar{T}_n(t)}{dt} + \lambda_n^2\bar{T}_n(t) = R\ \ \psi_n(R)\ \left(-\frac{h}{k}(T(R,t) - T_b(t))\right) - R\,T(R,t)\ \left(-\frac{h}{k}\ \psi_n(R)\right) + \frac{\bar{q}_n(t)}{k}$$

Which simplifies to

$$\frac{1}{\alpha}\frac{d\bar{T}_n(t)}{dt} + \lambda_n^2\bar{T}_n(t) = \frac{hR\,\psi_n(R)}{k}T_b(t) + \frac{\bar{q}_n(t)}{k}$$

Substituting the eigenfunction and multiplying by α

$$\frac{d\bar{T}_n(t)}{dt} + \alpha\lambda_n^2\bar{T}_n(t) = \frac{\alpha hR\,J_0(\lambda_n R)}{k}T_b(t) + \frac{\alpha\bar{q}_n(t)}{k}$$

An integrating factor for this ODE is

$$\mu = e^{\alpha\lambda_n^2 t}$$

multiplying by the integrating factor and integrating yields

$$\bar{T}_n(t) = c_n e^{-\alpha\lambda_n^2 t} + e^{-\alpha\lambda_n^2 t}\int\left(\frac{\alpha hR\, J_0(\lambda_n R)}{k}T_b(t) + \frac{\alpha I_{1n}\dot{q}(t)}{k}\right)e^{\alpha\lambda_n^2 t}dt$$

The temperature expansion is

$$T(r,t) = \sum_{n=1}^{\infty} d_n \psi_n(r)$$

Operate by $\int_0^R r\,\psi_m(r)dr$ and apply orthogonality

$$d_n = \frac{\int_0^R T(r,t)\, r\, \psi_n(r)dr}{\int_0^R\, r\, \psi_n^2(r)dr} = \frac{\bar{T}_n(t)}{\int_0^R\, r\, J_0^2(\lambda_n r)dr} = \frac{\bar{T}_n(t)}{I_{2n}}$$

Then the expansion becomes

$$T(r,t) = \sum_{n=1}^{\infty}\frac{\psi_n(r)\bar{T}_n(t)}{I_{2n}}$$

$$T(r,t) = \sum_{n=1}^{\infty}\frac{J_0(\lambda_n r)}{I_{2n}}\left(c_n e^{-\alpha\lambda_n^2 t} + e^{-\alpha\lambda_n^2 t}\int\left(\frac{\alpha hR\, J_0(\lambda_n R)}{k}T_b(t) + \frac{\alpha I_{1n}\dot{q}(t)}{k}\right)e^{\alpha\lambda_n^2 t}dt\right)$$

Apply the initial condition

$$T(r,0) = \frac{\dot{q}_0}{4k}(R^2 - r^2) + \frac{\dot{q}_0 R}{2h} + T_{b0}$$

$$\frac{\dot{q}_0}{4k}(R^2 - r^2) + \frac{\dot{q}_0 R}{2h} + T_{b0} = \sum_{n=1}^{\infty}\frac{J_0(\lambda_n r)}{I_{2n}}\left(c_n + \left[\int\left(\frac{\alpha hR\, J_0(\lambda_n R)}{k}T_b(t) + \frac{\alpha I_{1n}\dot{q}(t)}{k}\right)e^{\alpha\lambda_n^2 t}dt\right]_0\right)$$

Operate by $\int_0^R r\, J_0(\lambda_m r)dr$ and apply orthogonality and let

$$I_{3n} = \int_0^R \left(\frac{\dot{q}_0}{4k}(R^2 - r^2) + \frac{\dot{q}_0 R}{2h} + T_{b0}\right) r\, J_0(\lambda_n r)dr$$

Then

$$I_{3n} = c_n + \left[\int \left(\frac{\alpha hR\, J_0(\lambda_n R)}{k} T_b(t) + \frac{\alpha I_{1n}\dot{q}(t)}{k}\right) e^{\alpha\lambda_n^2 t} dt\right]_0$$

Then c_n could be found from

$$c_n = I_{3n} - \left[\int \left(\frac{\alpha hR\, J_0(\lambda_n R)}{k} T_b(t) + \frac{\alpha \bar{q}_n(t)}{k}\right) e^{\alpha\lambda_n^2 t} dt\right]_0$$

Then

$$\bar{T}_n(t) = e^{-\alpha\lambda_n^2 t}\left(I_{3n} + \int_0^t \left(\frac{\alpha hR\, J_0(\lambda_n R)}{k} T_b(t) + \frac{\alpha I_{1n}}{k}\dot{q}(t)\right) e^{\alpha\lambda_n^2 t} dt\right)$$

And the solution becomes

$$T(r,t) = \sum_{n=1}^{\infty} \frac{J_0(\lambda_n r) e^{-\alpha\lambda_n^2 t}\left(I_{3n} + \int_0^t \left(\frac{\alpha hR\, J_0(\lambda_n R)}{k} T_b(t) + \frac{\alpha I_{1n}}{k}\dot{q}(t)\right) e^{\alpha\lambda_n^2 t} dt\right)}{I_{2n}}$$

Where

$$I_{1n} = \int_0^R r\ J_0(\lambda_n r) dr$$

$$I_{2n} = \int_0^R r\, J_0^2(\lambda_n r) dr$$

$$I_{3n} = \int_0^R \left(\frac{\dot{q}_0}{4k}(R^2 - r^2) + \frac{\dot{q}_0 R}{2h} + T_{b0}\right) r\, J_0(\lambda_n r) dr$$

1.3 c

1.3.1 Formulation

1.3.1.1 Domain

$$0 \leq r \leq R$$

1.3.1.2 PDE

$$\frac{1}{\alpha}\frac{\partial T_{split}(r,t)}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_{split}(r,t)}{\partial r}\right) + \frac{\dot{q}_0}{2k}$$

1.3.1.3 Boundary Conditions

$$\frac{\partial T_{split}(r,t)}{\partial r}|_{r=0} = 0$$

$$-k\frac{\partial T_{split}(r,t)}{\partial r}|_{r=R} = h(T_{split}(R,t) - \frac{T_{b0}}{2})$$

Initial Condition

$$T(r,0) = \frac{\dot{q}_0}{4k}(R^2 - r^2) + \frac{\dot{q}_0 R}{2h} + T_{b0}$$

1.3.2 Solution

Let

$$T_{split}(r,t) = T_{tr}(r,t) + T_{ss}(r)$$

First we solve for the steady state

$$T_{ss}(r) = \frac{\dot{q}_0}{8k}(R^2 - r^2) + \frac{\dot{q}_0 R}{4h} + \frac{T_{b0}}{2} = \frac{T_{split}(r,0)}{2}$$

Then we solve for the transient

$$\frac{1}{\alpha}\frac{\partial T_{tr}(r,t)}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_{tr}(r,t)}{\partial r}\right)$$

Let

$$T_{tr}(r,t) = \Phi(r)\Gamma(t)$$

Then

$$T_{tr}(r,t) = \sum_{n=1}^{\infty} c_n J_0(\lambda_n r) e^{-\alpha \lambda_n{}^2 t}$$

And

$$T_{split}(r,t) = T_{ss}(r) + \sum_{n=1}^{\infty} c_n J_0(\lambda_n r) e^{-\alpha \lambda_n{}^2 t}$$

Apply the initial condition

$$T_{split}(r,0) = \frac{T_{split}(r,0)}{2} + \sum_{n=1}^{\infty} c_n J_0(\lambda_n r)$$

$$\frac{T_{split}(r,0)}{2} = \sum_{n=1}^{\infty} c_n J_0(\lambda_n r)$$

Operate by $\int_0^R r\, J_0(\lambda_n r) dr$ and apply orthogonality

$$c_n = \frac{\int_0^R r\, J_0(\lambda_n r) T_{split}(r,0) dr}{2\int_0^R r\, J_0{}^2(\lambda_n r) dr}$$

1.4 d

Let

$$\dot{q}(t) = \frac{\dot{q}_0}{2}(1 + e^{-\eta t})$$

$$T_b(t) = \frac{T_{b0}}{2}(1 + e^{-mt})$$

Then

$$T(r,t) = \sum_{n=1}^{\infty} \frac{J_0(\lambda_n r) e^{-\alpha\lambda_n^2 t}\left(I_{3n} + \int_0^t \left(\frac{\alpha hR\, J_0(\lambda_n R) T_{b0}}{2k}(1 + e^{-mt}) + \frac{\alpha \dot{q}_0 \int_0^R r\, J_0(\lambda_n r) dr}{2k}(1 + e^{-\eta t})\right) e^{\alpha\lambda_n^2 t} dt\right)}{I_{2n}}$$

The solution becomes

$$\begin{aligned} T(r,t) = \sum_{n=1}^{\infty} \frac{J_0(\lambda_n r)}{\int_0^R r\, J_0^2(\lambda_n r) dr}\Bigg(& \int_0^R \left(\frac{\dot{q}_0}{4k}(R^2 - r^2) + \frac{\dot{q}_0 R}{2h} + T_{b0}\right) r\, J_0(\lambda_n r) dr\, e^{-\alpha\lambda_n^2 t} \\ & + \frac{\alpha hR\, J_0(\lambda_n R) T_{b0}}{2k}\left(\frac{m - \alpha\lambda_n^2 + (2\alpha\lambda_n^2 - m) e^{-\alpha\lambda_n^2 t} - \alpha\lambda_n^2 e^{-mt}}{\alpha\lambda_n^2 (m - \alpha\lambda_n^2)}\right) \\ & + \frac{\alpha \dot{q}_0 I_{1n}}{2k}\left(\frac{\eta - \alpha\lambda_n^2 + (2\alpha\lambda_n^2 - \eta) e^{-\alpha\lambda_n^2 t} - \alpha\lambda_n^2 e^{-\eta t}}{\alpha\lambda_n^2 (\eta - \alpha\lambda_n^2)}\right)\Bigg) \end{aligned}$$

And

$$\begin{aligned} T_{split}(r,t) = {} & \frac{\dot{q}_0}{8k}(R^2 - r^2) + \frac{\dot{q}_0 R}{4h} + \frac{T_{b0}}{2} \\ & + \sum_{n=1}^{\infty} \frac{\int_0^R r\, J_0(\lambda_n r)\left(\frac{\dot{q}_0}{8k}(R^2 - r^2) + \frac{\dot{q}_0 R}{4h} + \frac{T_{b0}}{2}\right) dr}{\int_0^R r\, {J_0}^2(\lambda_n r) dr} J_0(\lambda_n r) e^{-\alpha{\lambda_n}^2 t} \end{aligned}$$

1.5 e

Figure 2 shows the heat generation and Figure 3 shows the fluid temperature for 3 different time constants ($m = 0.015, 0.020, 0.025\ s^{-1}$)

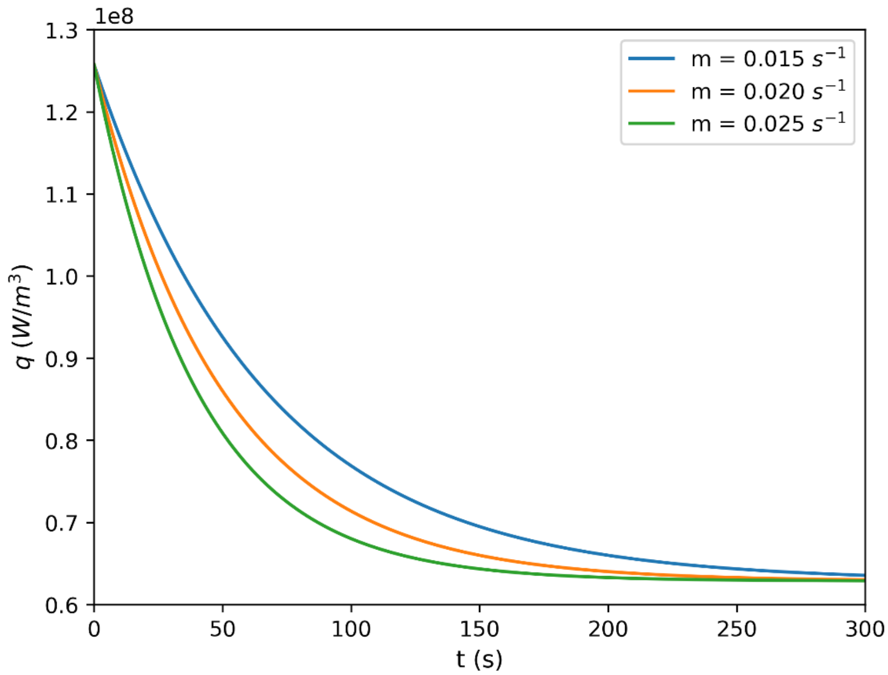

Figure 2 Heat generation for three different time constants

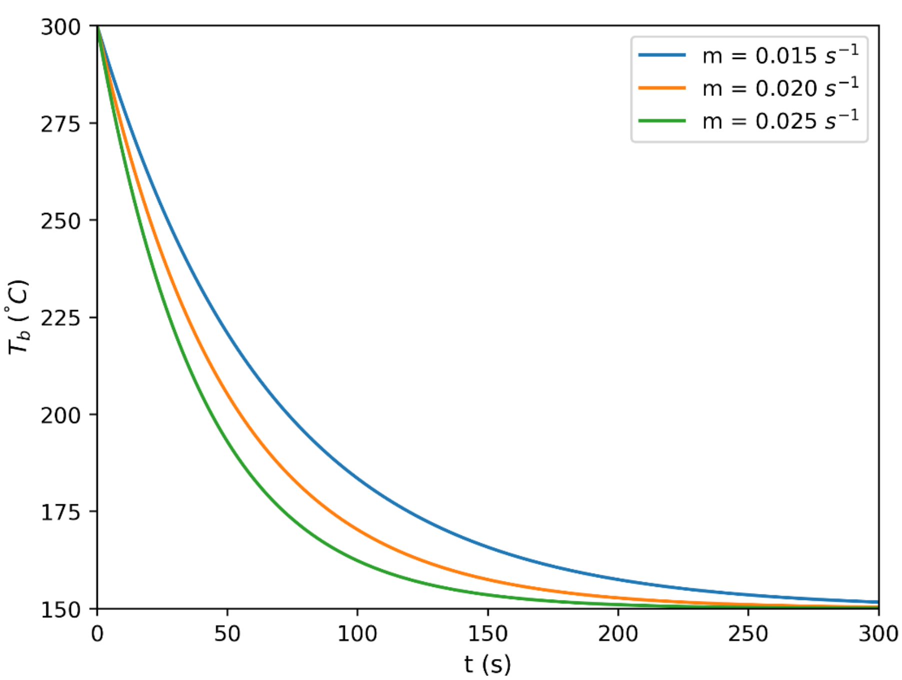

Figure 3 Fluid temperature for 3 different time constants

1.6 f

Figure 4 shows the initial temperature distribution T_i vs the steady state temperature distribution T_{ss}. The distribution is reduced by a factor of 2 because the both the heat generation and the fluid temperature have been reduced by a factor of 2.

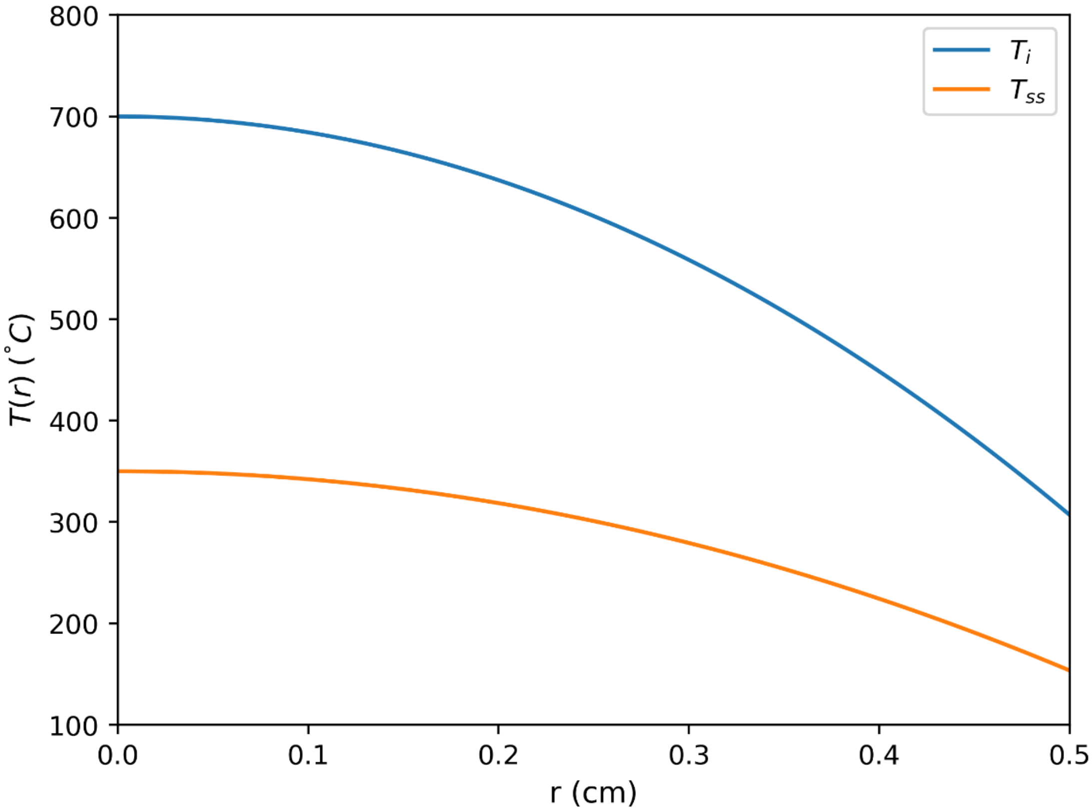

Figure 4 T_i vs T_{ss}

1.7 g

Figure 5 shows the temperature distributions by the two methods at six timestamps. It is clear from the figure that T_{split} approaches the steady state faster because q and T_b were assumed to be at their steady state values. It is also clear from the figure that both methods reach the same steady state.

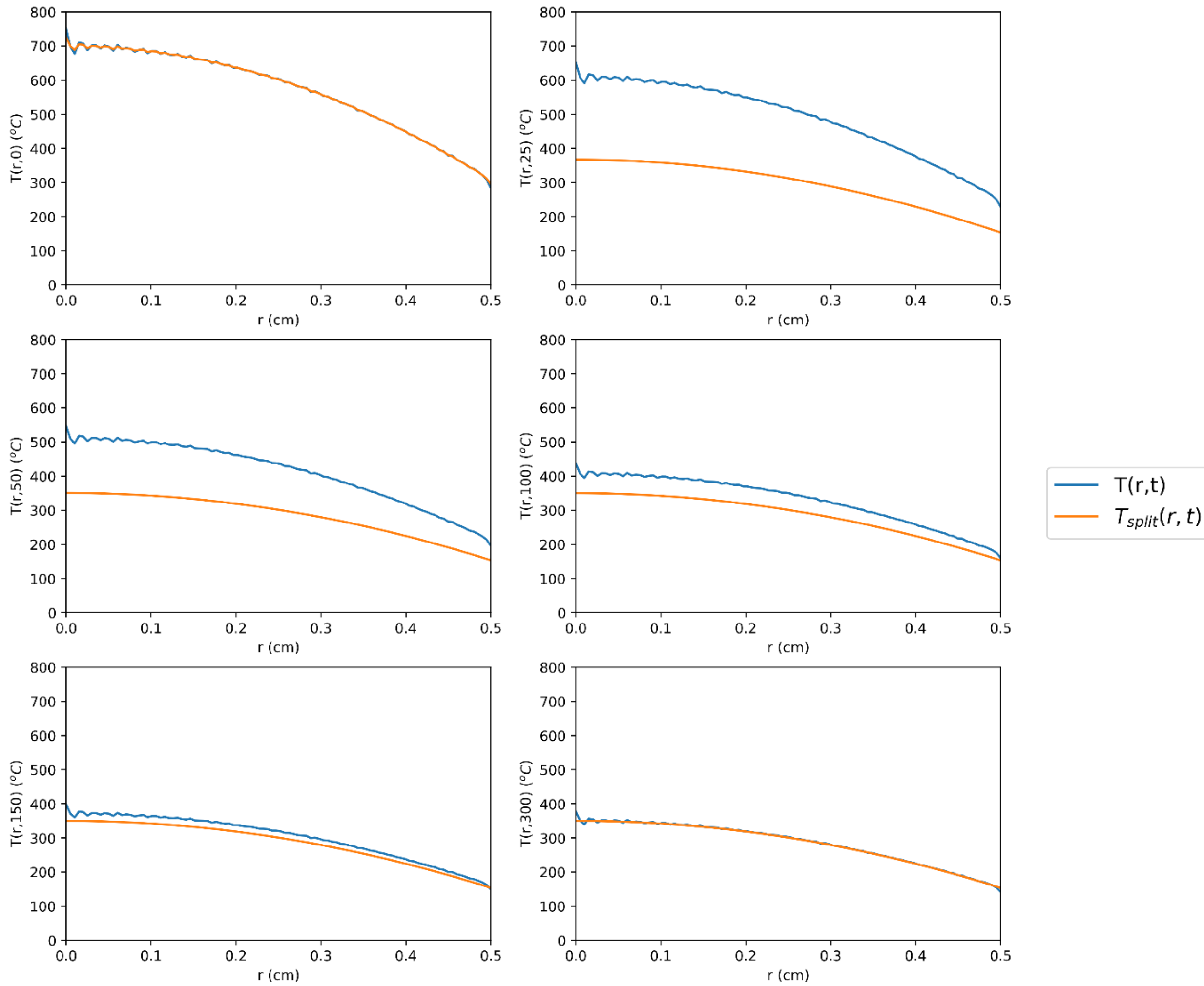

Figure 5 Temperature distributions for the two methods at six different timestamps

1.8 h

1.8.1 Eigenvalue Evaluation

To evaluate the analytical temperature, we need to solve the eigenvalues equation to obtain the eigenvalues. Since the eigenvalue equation is transcendental, it will be solved numerically. We first plot the eigenvalue equation (see Figure 6) to estimate good initial guesses for the numerical solver to decrease the number of iterations required. After that, the eigenvalues equation is solved numerically to obtain the eigenvalues. Table 2 shows the first 10 eigenvalues.

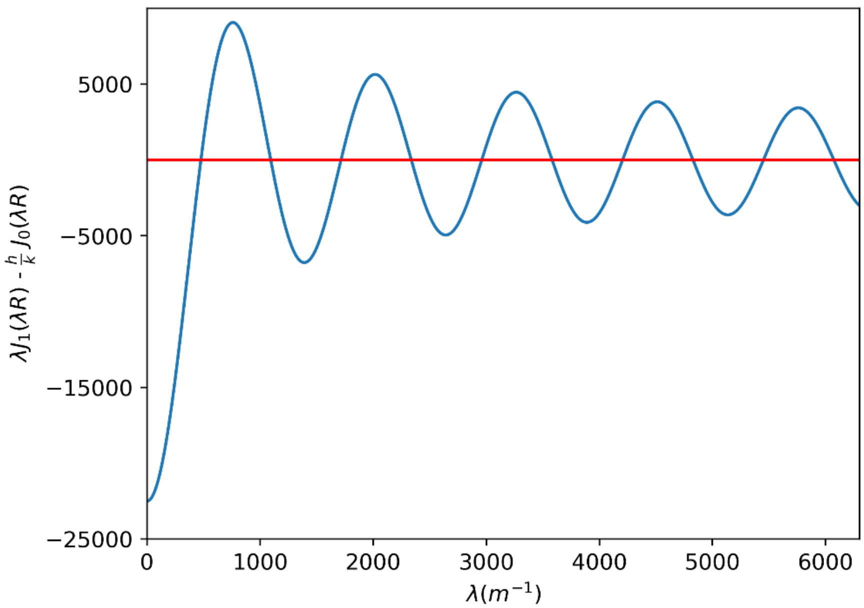

Figure 6 Eigenvalue equation

Table 2 First 10 eigenvalues and corresponding c value

n	λ_n
1	4.76709447722924E+02
2	1.09425332812053E+03
3	1.71545892441610E+03
4	2.33751165072225E+03
5	2.95990818489364E+03
6	3.58249504818516E+03
7	4.20521301209909E+03
8	4.82803663383521E+03
9	5.45095482217612E+03
10	6.07396325551783E+03

1.8.2 Convergence

To estimate the number of terms required to achieve good convergence, the following procedure was applied: 6 timestamps (0s, 25s, 50s, 100s, 150s, 300s) were chosen to capture the time evolution of the temperature distribution. The domain was divided into 100 equally sized elements. For each timestamp the temperature distribution was evaluated for increasingly number of terms in the series. Figure 7 shows the temperature distribution for different number of terms for the six timestamps. At least 50 terms are required to achieve reasonable convergence for all timestamps. To achieve better smoothness we take 200 terms in our final solution. Figure 8 shows the converged analytical temperature distribution at the 6 timestamps.

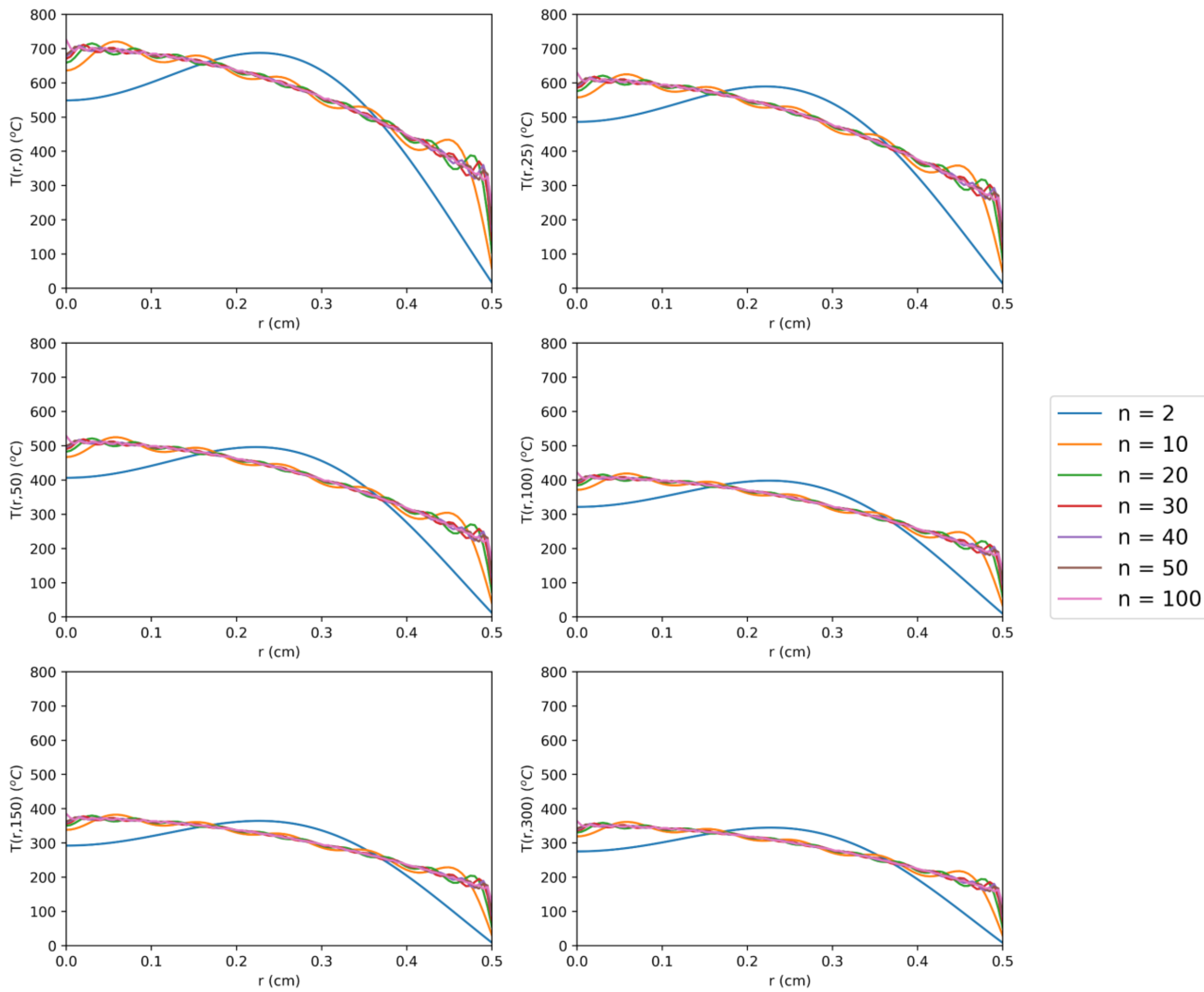

Figure 7 Analytical temperature distribution at six different timestamps for an increasing number of terms in the series

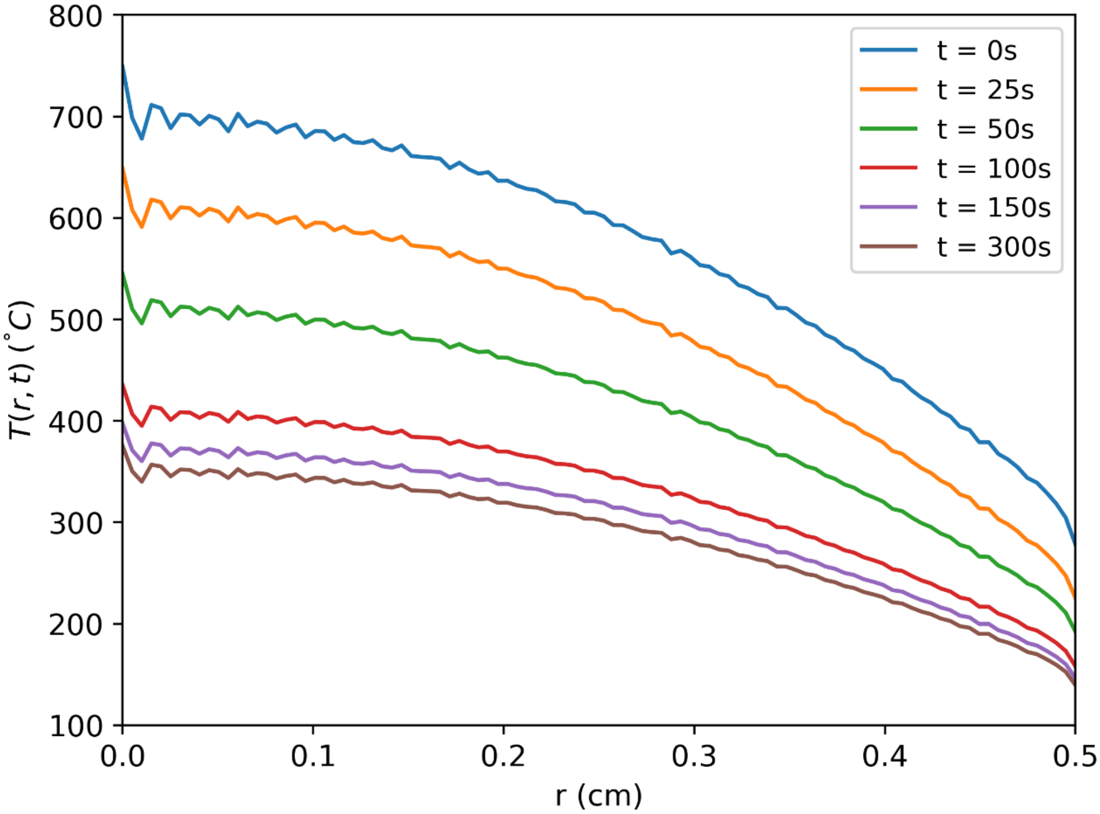

Figure 8 Analytical Temperature Distribution

1.9 i

Figures 9, 10, 11 and 12 show the temperature distributions for the two methods at six timestamps for different values of m and η. The difference between the cases is very small.

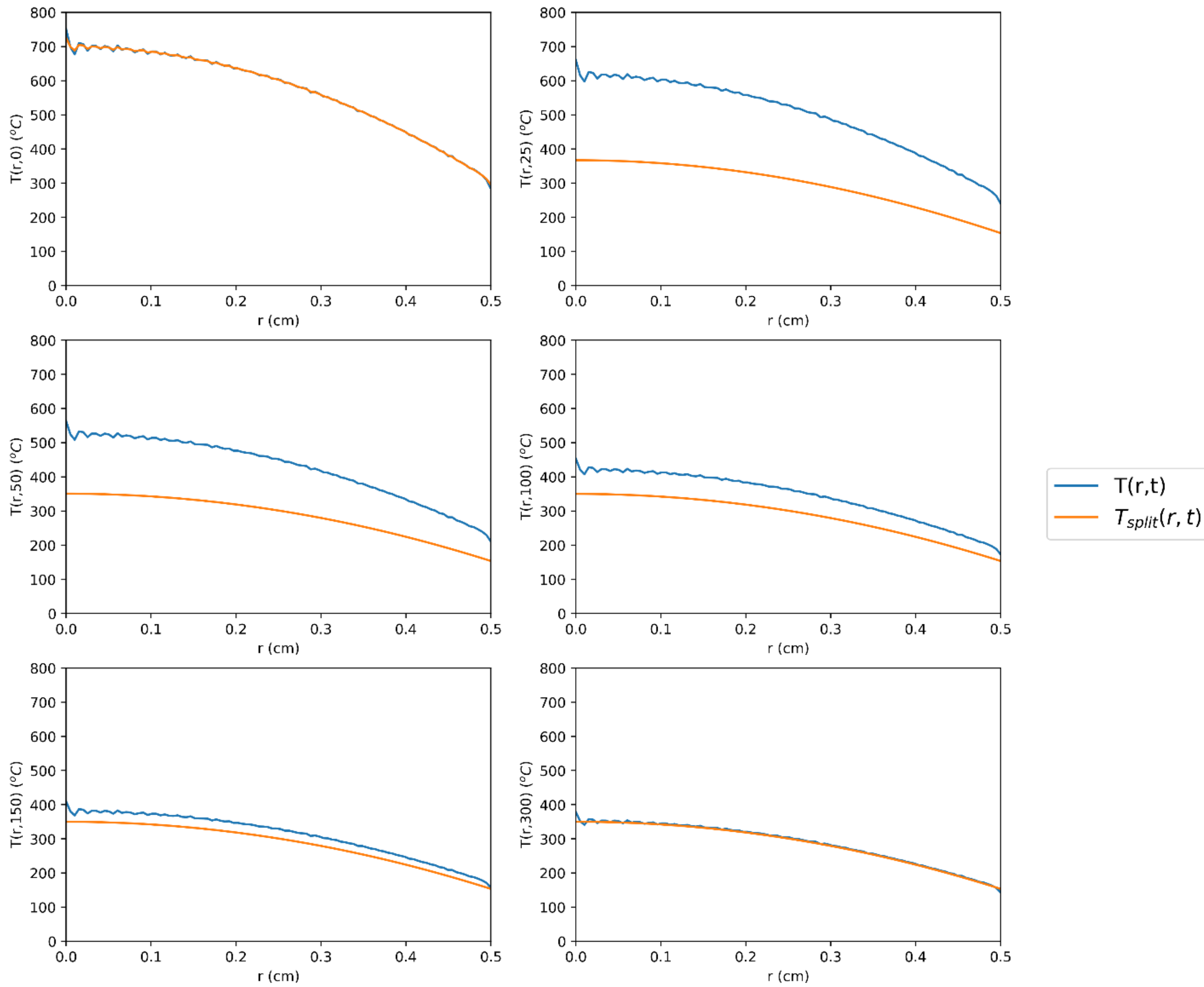

Figure 9 (m = 0.015, η = 0.020)

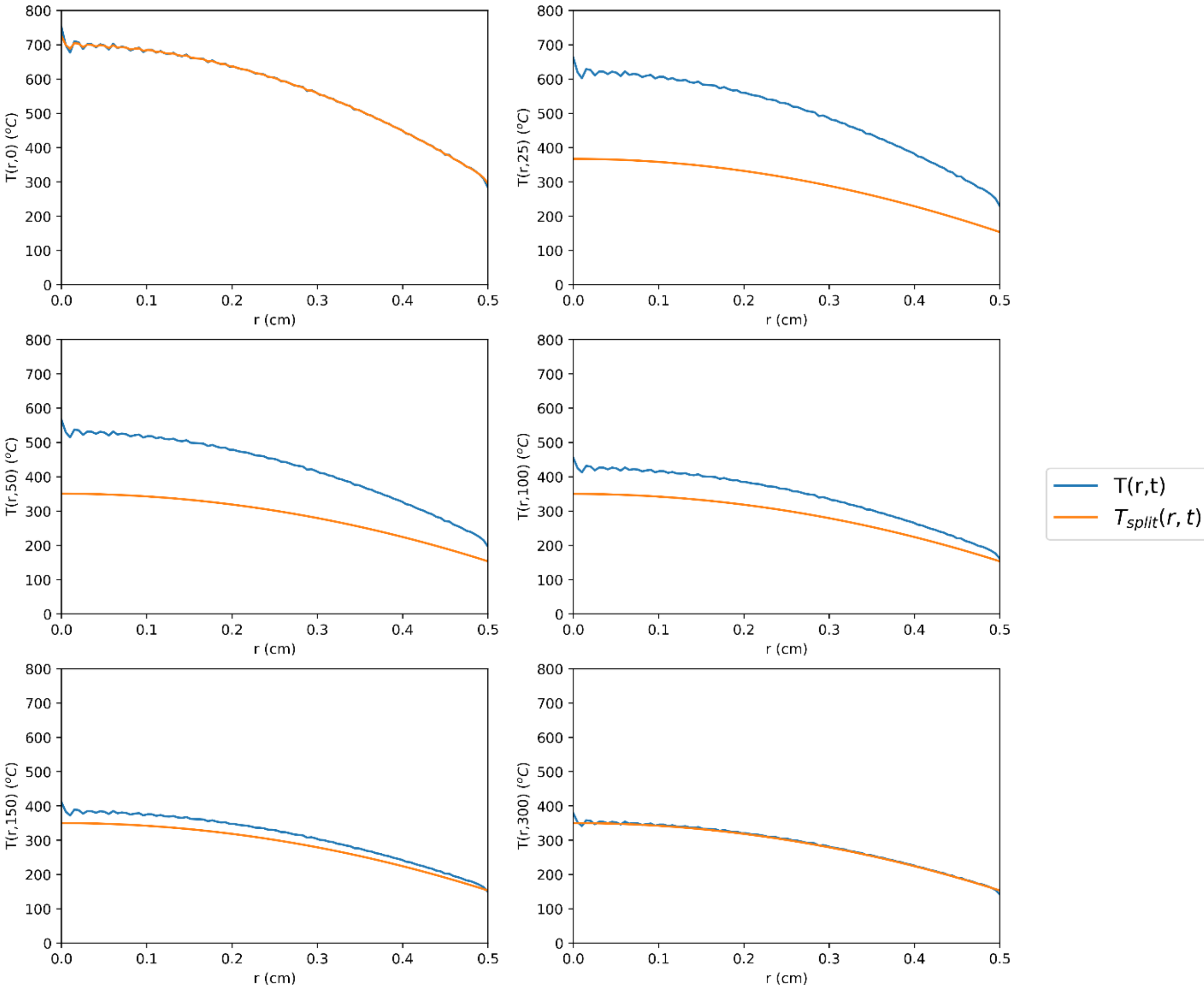

Figure 10 (m = 0.020, η = 0.015)

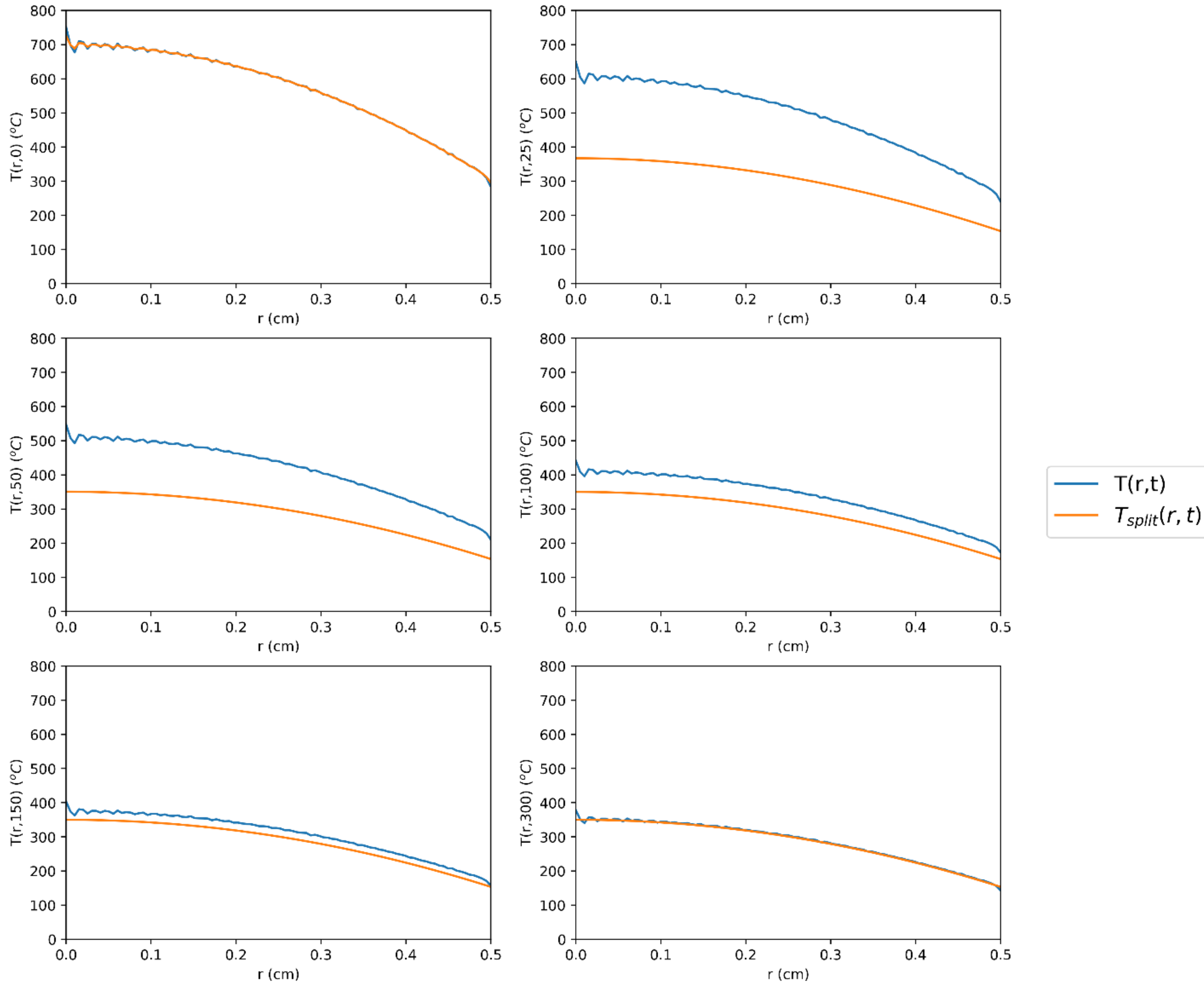

Figure 11 (m = 0.015, η = 0.025)

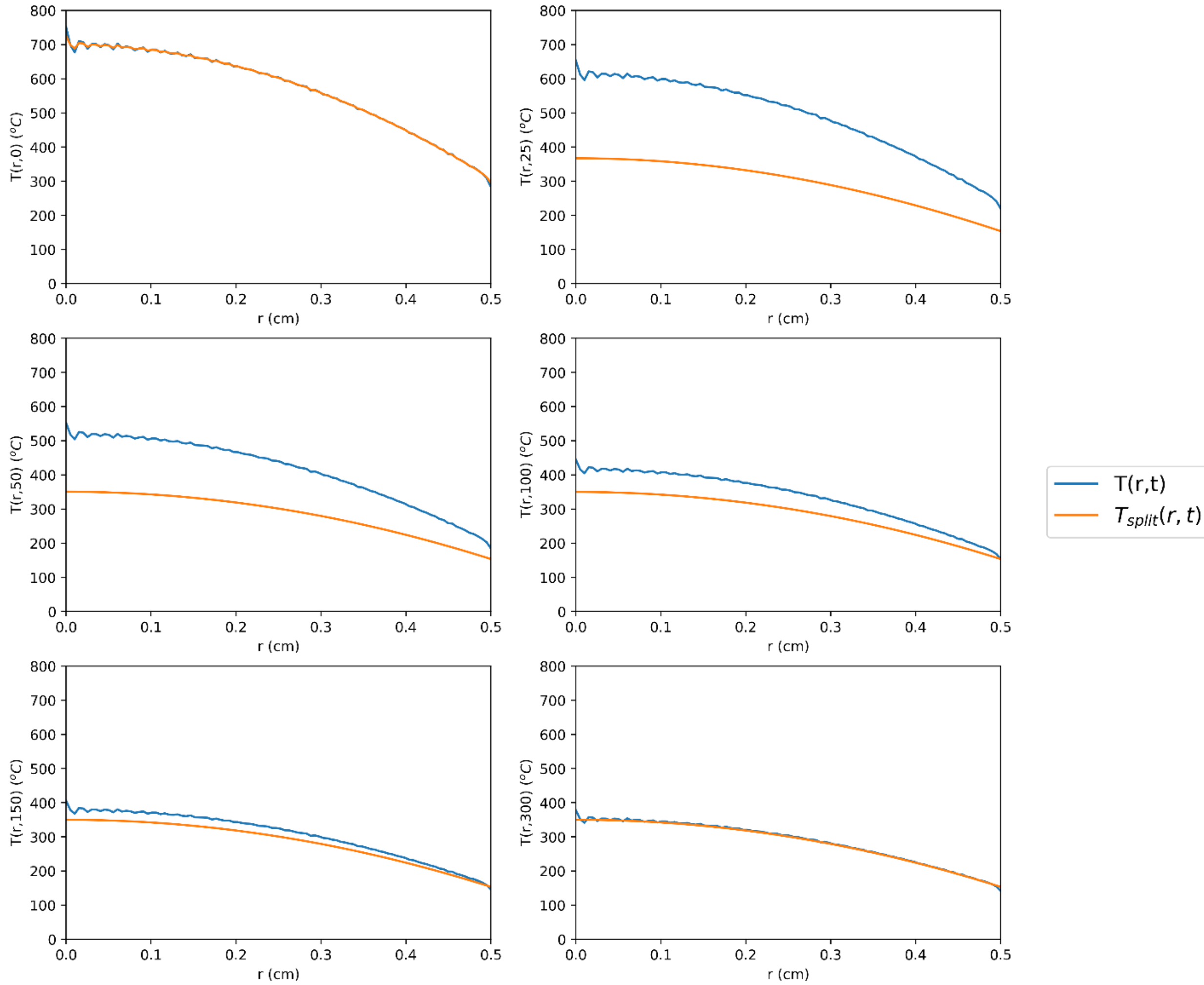

Figure 12 (m = 0.025, η = 0.015)

2. NUMERICAL SOLUTION

2.1 Finite Difference Formulation

First the domain was divided into *I+1* equally spaced nodes as shown in Figure 13. The mesh size is defined as

$$\Delta r = r_i - r_{i-1} = \frac{R}{I}$$

The time axis was divided into equally spaced $J + 1$ time instances between 0 and 300 s as shown in Figure 14. The times step is defined as

$$\Delta t = t_j - t_{j-1} = \frac{300}{J}$$

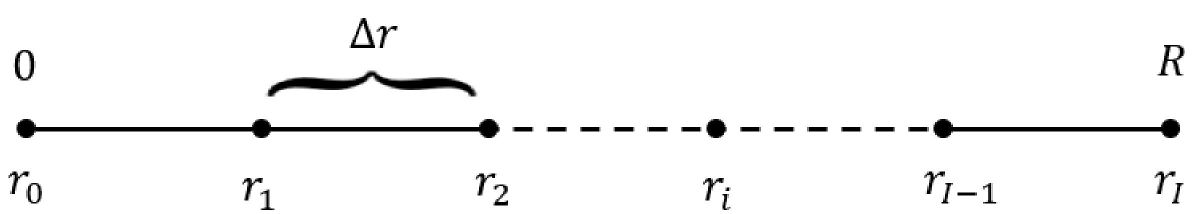

Figure 13 Mesh

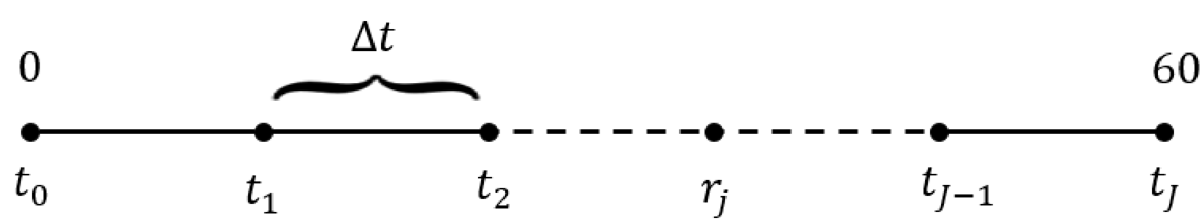

Figure 14 Time Axis Discretization

The central finite difference second order approximation for the first space derivative of the temperature is defined as

$$\frac{\partial T(r,t)}{\partial r}\Big|_i^j = \frac{T_{i+1}^j - T_{i-1}^j}{2\Delta r}$$

Then the second space derivative of the temperature becomes

$$\frac{\partial^2 T(r,t)}{\partial r^2}\Big|_i^j = \frac{T_{i+1}^j - 2T_i^j + T_{i-1}^j}{\Delta r^2}$$

The forward first order finite difference approximation for the time derivative of the temperature is defined as

$$\frac{\partial T(r,t)}{\partial t}|_i^j = \frac{T_i^{j+1} - T_i^j}{\Delta t}$$

Substituting these definitions into the PDE

$$\frac{1}{\alpha}\frac{\partial T(r,t)}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T(r,t)}{\partial r}\right) + \frac{\dot{q}(t)}{k}$$

yields

$$\frac{1}{\alpha}\frac{T_i^{j+1} - T_i^j}{\Delta t} = \frac{T_{i+1}^j - 2T_i^j + T_{i-1}^j}{\Delta r^2} + \frac{T_{i+1}^j - T_{i-1}^j}{2r_i\Delta r} + \frac{\dot{q}^j}{k}$$

Rearranging

$$T_i^{j+1} = T_i^j + \frac{\alpha\Delta t}{\Delta r^2}\left(T_{i+1}^j - 2T_i^j + T_{i-1}^j\right) + \frac{\alpha\Delta t}{2r_i\Delta r}\left(T_{i+1}^j - T_{i-1}^j\right) + \frac{\alpha\Delta t}{k}\dot{q}^j$$

This equation can be used to compute the temperature at any interior node at any time instance.

The first boundary condition could be written as

$$\frac{\partial T(r,t)}{\partial r}|_{r=0} = 0$$

If the temperature at a fictitious node outside the domain with distance Δr from the left of node r_0 is T_{-1}^j then the boundary condition could be written as.

$$\frac{\partial T(r,t)}{\partial r}|_0^j = \frac{T_1^j - T_{-1}^j}{2\Delta r} = 0$$

Which gives

$$T_{-1}^j = T_1^j$$

Substituting into the finite difference equation of node 0 yields

$$T_0^{j+1} = T_0^j + \frac{2\alpha\Delta t}{\Delta r^2}\left(T_1^j - T_0^j\right) + \frac{\alpha\Delta t}{k}\dot{q}^j$$

The second boundary condition is

$$\frac{\partial T(r,t)}{\partial r}|_{r=R} + \frac{h}{k}(T(R,t) - T_b(t)) = 0$$

Discretizing

$$\frac{T_{I+1}^j - T_{I-1}^j}{2\Delta r} + \frac{h}{k}(T_I^j - T_b{}^j) = 0$$

Rearranging

$$T_{I+1}^j = T_{I-1}^j - \frac{2\Delta r h}{k}(T_I^j - T_b{}^j)$$

The discretized equation of T_I^j is

$$T_I^{j+1} = T_I^j + \frac{\alpha\Delta t}{\Delta r^2}\left(T_{I+1}^j - 2T_I^j + T_{I-1}^j\right) + \frac{\alpha\Delta t}{2r_I\Delta r}\left(T_{I+1}^j - T_{I-1}^j\right) + \frac{\alpha\Delta t}{k}\dot{q}^j$$

Eliminating T_{I+1}^{j+1}

$$T_I^{j+1} = T_I^j + \frac{2\alpha\Delta t}{\Delta r^2}\left(T_{I-1}^j - T_I^j - \frac{\Delta r h}{k}(T_I^j - T_b{}^j)\right) - \frac{\alpha\Delta t\, h}{r_I k}\left(T_I^j - T_b{}^j\right) + \frac{\alpha\Delta t}{k}\dot{q}^j$$

Simplifying

$$T_I^{j+1} = T_I^j + \frac{2\alpha\Delta t}{\Delta r^2}\left(T_{I-1}^j - T_I^j\right) - \frac{\alpha\Delta t\, h}{k}\left(\frac{2}{\Delta r} + \frac{1}{r_I}\right)(T_I^j - T_b{}^j) + \frac{\alpha\Delta t}{k}\dot{q}^j$$

To find the values of the temperatures, the finite difference equation is applied at each time instance for all nodes.

2.2 Mesh Refinement Study

A mesh refinement study was conducted by evaluating the numerical temperature at the seven timestamps for decreasingly mesh size. We started with a mesh size of 2^{-2} cm and then reduced the mesh size by a factor of 2 each iteration. Figure 15 shows the results of this study. It is obvious that solution accuracy become independent of the mesh size starting from mesh size 2^{-5} cm. To achieve better smoothness in the solution without an unnecessary increase in the computational time we chose the mesh size to be 2^{-6} cm in the time step study and in the final evaluation of the numerical temperature.

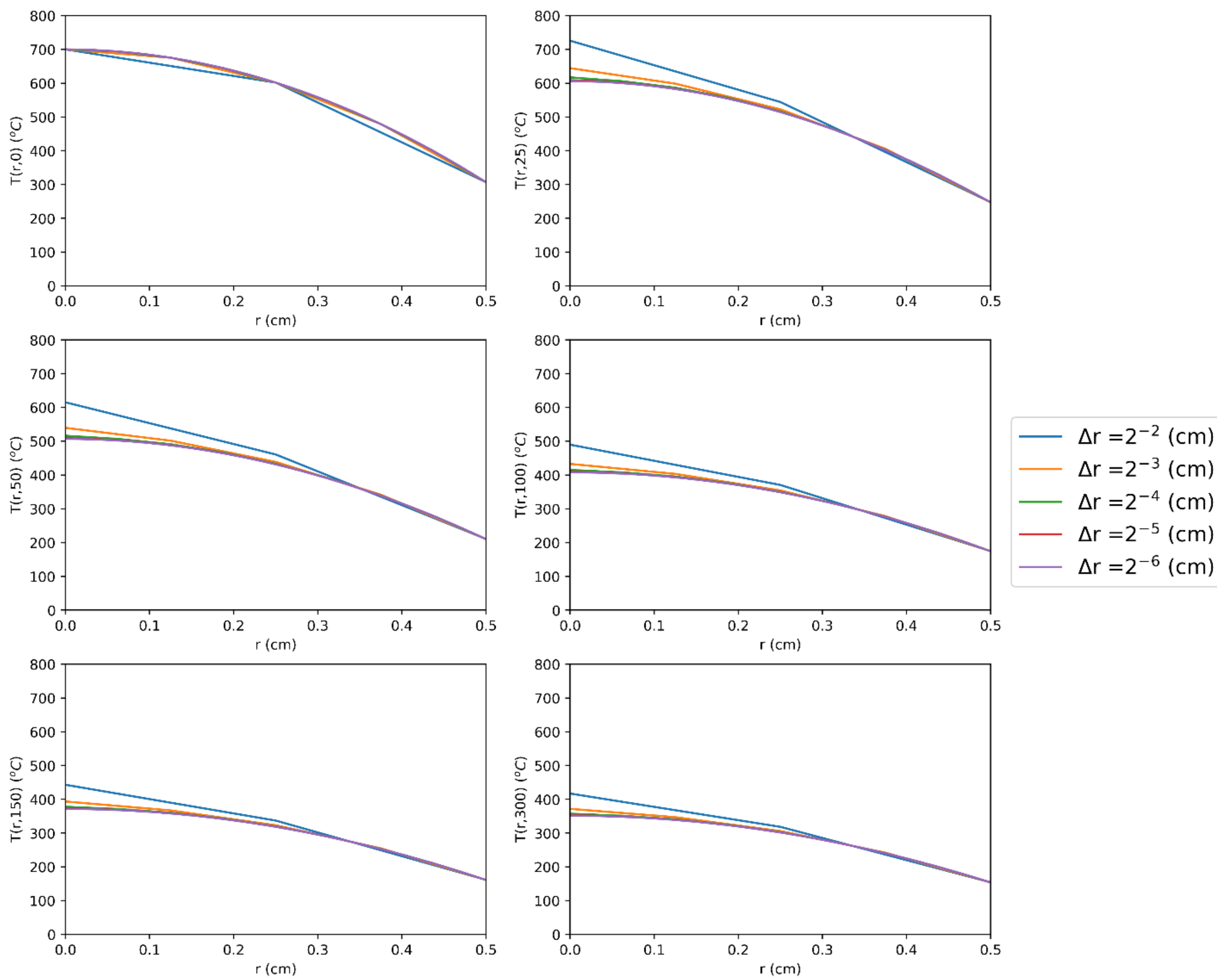

Figure 15 Temperature distribution at the six timestamps for different mesh sizes

2.3 Time Step Study

It can be shown that to achieve convergence and stability in the results of the finite difference solution the following condition must be satisfied [1]

$$\Delta t \leq \frac{1}{2}\frac{\Delta r^2}{\alpha}$$

Substituting our final mesh size $\Delta r = 2^{-6}$ cm = 1.5625×10^{-4} m into the inequality yields

$$\Delta t \leq 0.02\ s$$

And To avoid oscillation of errors, it is recommended to choose Δt less than half this value ($\Delta t \leq 0.01\ s$) [1]. To avoid oscillating errors and unnecessary computation time, we chose the time step to be 0.005s in our final evaluation. Figure 16 shows the final numerical and analytical temperature distributions. The two distributions are in agreement with each other.

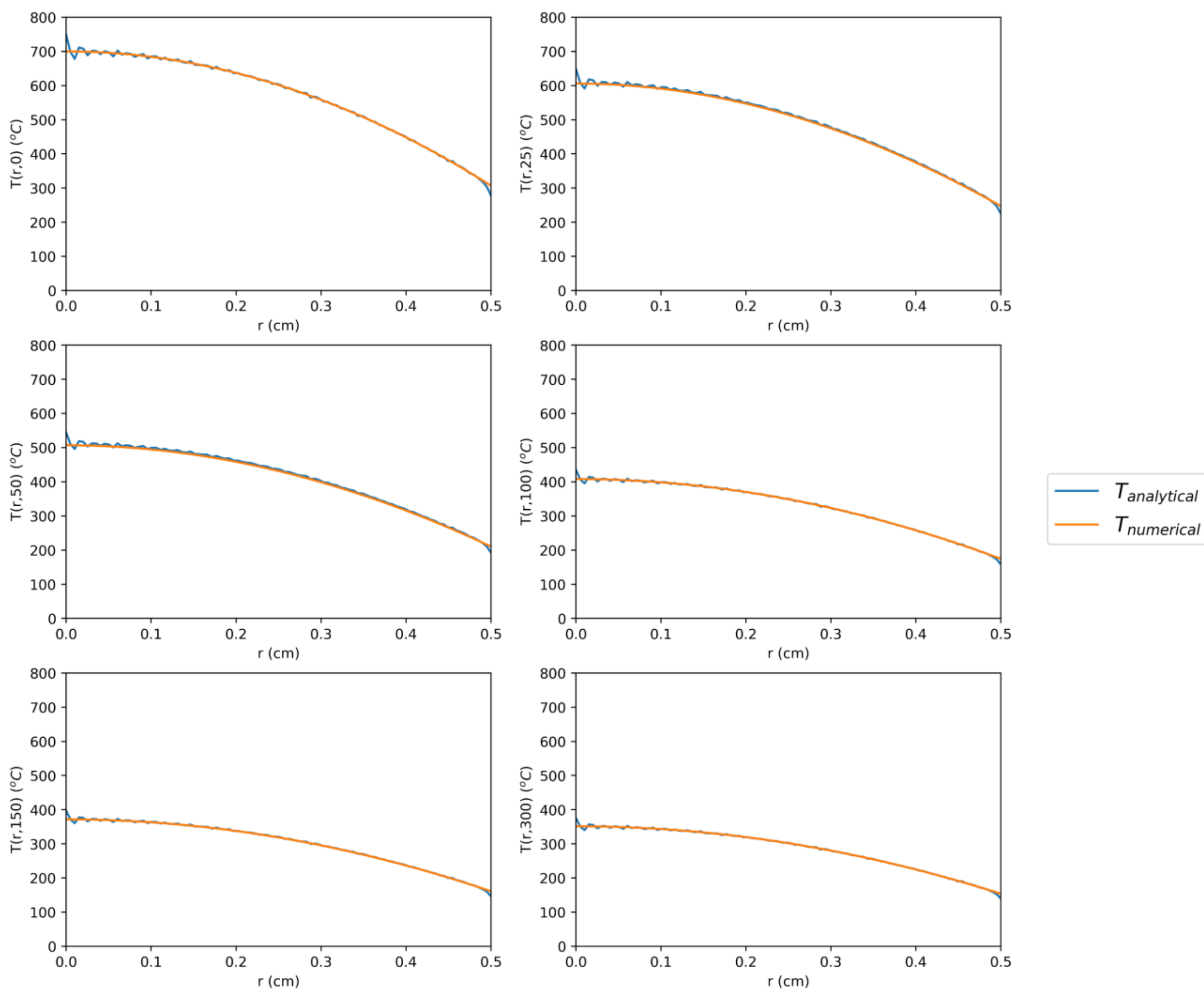

Figure 16 Analytical vs Numerical Temperature Distributions

3. CONCLUSION

The aim of this study was to solve a 1D time dependent heat conduction problem in cylindrical coordinates using the finite difference method. The problem was first solved analytically, and a study was made to assess the number of terms required from the series to achieve convergence. A suitable number of terms was found to be 50. The finite difference method was implemented in a python code. A grid refinement study was done and the mesh size chosen was 2^{-6} cm which corresponds to 32 elements in the domain. Another study was made to find a reasonable time step that achieves convergence without error oscillation and it was found to be 0.005 s. Finally the analytical and numerical solutions were compared and the difference between them was found to be negligible.

4. REFERENCES

[1] S. C. Chapra, R. P. Canale, *Numerical Methods for Engineers*, 7th ed. New York: McGraw-Hill Education, 2015.

APPENDIX

https://github.com/MohamedElkamash/501_CP02

This is the link of the Github repository of the project. The repository contains all the code files and the results including csv files and the figures in this report. I didn't put the code here because it consists of multiple files.