Navier-Stokes non-dimensionalisation

Valid for incompressible flow with the Boussinesq approximation

1 Derivation of the Boussinesq Approximation

The Boussinesq approximation can be derived from the Navier-Stokes equations by assuming an incompressible fluid,

$$\rho = \rho_0 \tag{1}$$

that has a non-vanishing expansivity β (also referred to as the volumetric coefficient of thermal expansion),

$$\beta \equiv \frac{1}{v}\left(\frac{\partial v}{\partial T}\right)_P \neq 0 \tag{2}$$

where v is the specific volume. Further, we assume that the temperature can be expressed as a baseline value T_0 plus a small deviation T',

$$T = T_o + T' \tag{3}$$

Finally, the pressure is assumed to be equal to the hydrostatic pressure P_0, plus a small deviation P',

$$\begin{aligned} P &= P_0 + P' \\ &= \underbrace{-\rho_0 g z + \text{constant}}_{P_0} + P' \end{aligned} \tag{4}$$

The "constant" simply indicates the pressure at $z = 0$, such as an atmospheric pressure. Note that in the following derivation, $g = |g|$, that is, g is a positive number.

The following sections derive the Boussinesq approximation using just the above assumptions. Many derivation steps are omitted for brevity, but can be followed in detail in [1]. For clarity, in the derivation that follows, any equations that are explicitly restricted to the Boussinesq approximation will be indicated; otherwise, parts of the derivation are quite general and simply represent different (equivalent) forms of conservation equations transformed between one another using thermodynamic relations.

1.1 Conservation of Energy

To derive the Boussinesq approximation, we begin from the conservation of energy equation. This conservation equation is expressed in terms of total specific energy E, or

$$E \equiv \underbrace{e}_{\text{internal}} + \underbrace{\frac{1}{2}u_i u_i}_{\text{kinetic}} \tag{5}$$

The conservation of energy equation is derived by balancing the power supplied by contact and body forces with the heat addition. Power is equal to force times velocity. We equate the material derivative of E with product of the Cauchy equation forces[1] and velocity,

$$\frac{D}{Dt}\int \rho E dV = \underbrace{\int \rho g_i u_i dV + \int \sigma_{ij} u_i n_j dS}_{\text{product of Cauchy forces with velocity}} + \underbrace{\int \dot{q} dV - \int q_i n_i dS}_{\text{heat addition}} \tag{6}$$

By (i) inserting the Newtonian constitutive relation for σ_{ij}, (ii) neglecting volume viscous heating terms and volume dissipative effects, and (iii) inserting the definitions for specific enthalpy h and specific total enthalpy H,

$$h \equiv e + Pv \tag{7}$$

$$H \equiv E + Pv \tag{8}$$

then Eq. (6) becomes

$$\frac{\partial(\rho E)}{\partial t} + \nabla \cdot (\rho H \vec{u}) - \rho \vec{g} \cdot \vec{u} = -\nabla \cdot \vec{q} + \dot{q} \tag{9}$$

Conservation of internal energy can be obtained by subtracting the kinetic energy contributions from Eq. (9). In other words, subtract the Cauchy force balance, multiplied by velocity, from Eq. (9) to obtain just the internal energy component,

$$\rho \frac{De}{Dt} = \sigma_{ij} \frac{\partial u_i}{\partial x_j} - \nabla \cdot \vec{q} + \dot{q} \tag{10}$$

Finally, we need to convert this balance of internal energy into the entropy form. Assuming a "simple" system[2], the first law of thermodynamics is

$$de = \delta q + \delta w \tag{11}$$

Representing the heat differential as $\delta q = T ds$ and the work term as pressure-volume work gives

$$de = T ds - P dv \tag{12}$$

Inserting Eq. (12) into the internal energy balance in Eq. (10) gives an equation for entropy conservation,

$$\rho \left(T \frac{Ds}{Dt} - P \frac{Dv}{Dt} \right) = \sigma_{ij} \frac{\partial u_i}{\partial x_j} - \nabla \cdot \vec{q} + \dot{q} \tag{13}$$

[1] The "Cauchy equation" is the fundamental expression of momentum conservation derived by balancing linear momentum for a continuum. The Cauchy equation is the "starting point" for deriving the Navier-Stokes equations, which make the further assumption that the stress tensor constitutive relationship for a Newtonian fluid is $\sigma_{ij} = -P\delta_{ij} + 2\mu e_{ij} + \lambda \nabla \cdot \vec{u} \delta_{ij}$.

[2] All systems have at least two modes of energy transfer - work and heat. One thermodynamic property is needed to fix the state of a system for each mode of energy transfer. The work in "simple" systems is exclusively pressure-volume work (i.e. not subjected to electromagnetic fields or other forms of work). Therefore, simple systems can be "fixed" with just two thermodynamic properties.

For a simple system, any thermodynamic property can be expressed in terms of two other thermodynamic properties. Taking entropy as a function of pressure and temperature,

$$
\begin{aligned}
\frac{Ds}{Dt} &= \left(\frac{\partial s}{\partial T}\right)_P \frac{DT}{Dt} + \left(\frac{\partial s}{\partial P}\right)_T \frac{DP}{Dt} \\
&= \frac{C_p}{T}\frac{DT}{Dt} - \beta v \frac{DP}{Dt} \\
&= \frac{C_{p,0}}{T_0}\frac{DT'}{Dt} - \beta_0 v_0 \frac{D\left(P_0 + P'\right)}{Dt} \quad \text{(Boussinesq only)}
\end{aligned} \tag{14}
$$

where the second line is shown in terms of the thermodynamic definitions for C_p and β. A "0" subscript indicates evaluating that parameter at P_0 and T_0, which we select for evaluating the terms outside derivates because the Boussinesq approximation assumes that P' and T' are small. Alternatively, expressing entropy in terms of e and v gives (using a different set of thermodynamic identities)

$$
\begin{aligned}
\frac{Ds}{Dt} &= \frac{1}{T}\frac{De}{Dt} + \frac{P}{T}\frac{Dv}{Dt} \\
&= \frac{1}{T_0}\frac{De}{Dt} + \frac{P_0}{T_0}\frac{Dv}{Dt} \quad \text{(Boussinesq only)}
\end{aligned} \tag{15}
$$

Therefore, combining Eqs. (14) and (15) to expand the term $T\frac{Ds}{Dt}$ in Eq. (13) gives

$$
\begin{aligned}
T\frac{Ds}{Dt} &= \left(T_0 + T'\right)\frac{Ds}{Dt} && \text{(Boussinesq only)} \\
&= \underbrace{\frac{De}{Dt}}_{\text{the } T_0 Ds/Dt \text{ term}} + \underbrace{T'\left[\frac{C_{p,0}}{T_0}\frac{DT'}{Dt} - \beta_0 v_0 \frac{D\left(P_0 + P'\right)}{Dt}\right]}_{\text{the } T' Ds/Dt \text{ term}} && \text{(Boussinesq only)}
\end{aligned} \tag{16}
$$

where we have also cancelled terms that are zero for incompressible flow (a part of the Boussinesq approximation). Next, take the total derivative of the $P_0 = -\rho_0 g z$ assumption to get

$$
\frac{DP_0}{Dt} = -\rho_0 g u_z \quad \text{(Boussinesq only)} \tag{17}
$$

and insert into Eq. (16) to give

$$
T\frac{Ds}{Dt} = \frac{De}{Dt} + T'\left\{\frac{C_{p,0}}{T_0}\frac{DT'}{Dt} - \beta_0 v_0 \left[-\rho_0 g u_z + \frac{DP'}{Dt}\right]\right\} \quad \text{(Boussinesq only)} \tag{18}
$$

To first order in the primed variables (e.g. include T', but assume $DT'/Dt \approx 0$), the above becomes

$$
T\frac{Ds}{Dt} = \frac{De}{Dt} + T'\beta_0 g u_z \quad \text{(Boussinesq only)} \tag{19}
$$

Finally, inserting (19) into Eq. (13) gives the Boussinesq energy equation,

$$\rho_0 \frac{De}{Dt} = \sigma_{ij} \frac{\partial u_i}{\partial x_j} - \nabla \cdot \vec{q} + \dot{q} - \rho_0 T' \beta_0 g u_z \quad \text{(Boussinesq only)} \tag{20}$$

where the Dv/Dt term in Eq. (13) was again set to zero due to the incompressibility assumption. At this point, we have all the derivation in place to make the argument for what the additional buoyancy force should be in the momentum equation. As seen in Eq. (20), this form of the internal energy conservation equation is the same as the classical form except for (i) the use of incompressible assumptions and (ii) the appearance of a term $-\rho_0 T' \beta_0 g u_z$. We had embarked on the derivation of the conservation of energy equation by first taking the product of the Cauchy momentum equation with velocity; so to conclude which *force* we need to add to the momentum conservation equation, we simply need to divide this extra Boussinesq term by velocity, giving

$$\text{extra Boussinesq momentum term} = \rho_0 \beta_0 T' g \hat{z} \tag{21}$$

1.2 Conservation of Momentum

For the conservation of momentum equation, we simply take the Navier-Stokes conservation equation and (i) assume incompressibility, (ii) substitute $P = P_0 + P'$, and (iii) add Eq. (21) to give

$$\begin{aligned} \rho \frac{Du_i}{Dt} &= -\frac{\partial P}{\partial x_j}\delta_{ij} + \rho g_i + \frac{\partial \tau_{ij}}{\partial x_j} \\ \rho_0 \frac{Du_i}{Dt} &= -\frac{\partial P'}{\partial x_j}\delta_{ij} + \frac{\partial \tau_{ij}}{\partial x_j} + \rho_0 \beta_0 T' g \hat{z} \qquad \text{(Boussinesq only)} \end{aligned} \tag{22}$$

2 Momentum Equation

$$\rho\left(\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla u\right) = -\nabla p + \mu \nabla^2 \vec{u} + \rho \vec{f} \tag{23}$$

$$= -\nabla p + \mu \nabla^2 \vec{u} + \rho_0 \vec{g} - \rho_0 \beta (T - T_0) \vec{g} \tag{24}$$

$$= -\nabla (p - \rho_0 g z) + \mu \nabla^2 \vec{u} - \rho_0 \beta (T - T_0) \vec{g} \tag{25}$$

$$= -\nabla \overline{p} + \mu \nabla^2 \vec{u} - \rho_0 \beta (T - T_0) \vec{g} \tag{26}$$

$$= -\nabla \overline{p} + \mu \nabla^2 \vec{u} + \rho_0 \beta (T - T_0) g \hat{z} \tag{27}$$

$$\text{where } \overline{p} = p - \rho_0 g z. \tag{28}$$

It is understood in this notation that the Boussinesq term is only applied to the z-component of momentum. Taking $\tilde{u} = u/U$, $\tilde{x} = x/L$, $\tilde{t} = tU/L$, $\tilde{p} = \overline{p}/(\rho U^2)$, and $\theta = \frac{T - T_0}{\Delta T}$ we get:

$$\rho\left[\left(\frac{U}{L/U}\right)\frac{\partial\tilde{u}}{\partial\tilde{t}}+\left(\frac{U^2}{L}\right)\tilde{u}\cdot\tilde{\nabla}\tilde{u}\right]=-\frac{1}{L}\nabla\overline{p}+\frac{U}{L^2}\mu\nabla^2\tilde{u}+\rho_0\beta(T-T_0)g\hat{z} \quad (29)$$

$$\frac{\rho U^2}{L}\left(\frac{\partial\tilde{u}}{\partial\tilde{t}}+\tilde{u}\cdot\tilde{\nabla}\tilde{u}\right)=-\frac{1}{L}\nabla\overline{p}+\frac{U}{L^2}\mu\nabla^2\tilde{u}+\rho_0\beta(T-T_0)g\hat{z} \quad (30)$$

$$\left(\frac{\partial\tilde{u}}{\partial\tilde{t}}+\tilde{u}\cdot\tilde{\nabla}\tilde{u}\right)=-\frac{1}{\rho U^2}\nabla\overline{p}+\frac{\mu}{\rho UL}\nabla^2\tilde{u}+\frac{\rho_0}{\rho}\beta\frac{gL}{U^2}(T-T_0)\hat{z} \quad (31)$$

$$\left(\frac{\partial\tilde{u}}{\partial\tilde{t}}+\tilde{u}\cdot\tilde{\nabla}\tilde{u}\right)=-\nabla\tilde{p}+\frac{1}{Re}\nabla^2\tilde{u}+\frac{\rho_0}{\rho}\frac{g\beta\Delta TL}{U^2}\theta\hat{z} \quad (32)$$

Taking $\rho\approx\rho_0$ (Boussinesq assumption $\Delta\rho << \rho$) and $Ri=\frac{g\beta\Delta TL}{U^2}$, we get

$$\left(\frac{\partial\tilde{u}}{\partial\tilde{t}}+\tilde{\nabla}\cdot\tilde{u}\right)=-\nabla\tilde{p}+\frac{1}{Re}\nabla^2\tilde{u}+Ri\,\theta\hat{z} \quad (33)$$

References

[1] A.J. Novak, S. Schunert, R.W. Carlsen, P. Balestra, D. Andrš, J. Kelly, R.N. Slaybaugh, and R.C. Martineau. Pronghorn Theory Manual. Technical Report INL/EXT-18-44453-Rev001, Idaho National Laboratory, 2020.