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# HybridBF Algorithm

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## Abstract

Efficient and accurate root-finding algorithms are critical in numerical analysis across science and engineering. Here we compare three methods - bisection, false position, and a hybrid technique - for solving nonlinear equations. The hybrid approach combines aspects of false position (an open method) and bisection (a bracketing method), leveraging the speed of the former and reliability of the latter. Extensive testing on 25 diverse sample equations shows the hybrid method requires significantly fewer iterations and less computation time to identify roots with similar or better accuracy versus the other two techniques. It reduces iterations and CPU time, while achieving function values closer to the desired zero. By integrating strengths of open and bracketing root-finders, the new hybrid method delivers faster, more efficient, and numerically stable performance in locating roots for a wide variety of equation types.

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# 1 Equations

We have used the same 25 equations with each method and run each method (Bisection, False Position, and Hybrid) 500 times for each problem and then we have calculated the average time. We have also calculated the number of iterations each method have taken for each problem.

We have also used the same tolerance for each method which is  $10^{-14}$

These are the equations that we have used with each method:

## 1.1 Our Equations

In these equations we have tried to use different types of functions and intervals to test our methods.

Table 1: Our Equations

No	Equation	Equation Code	Interval
P1	$f(x) = x^3 + 4x^2 - 10 = 0$	<code>x**3 + 4*x**2 - 10</code>	[0, 4]
P2	$f(x) = x^2 - 4$	<code>x**2 - 4</code>	[0, 4]
P3	$f(x) = e^x - 2$	<code>math.exp(x) - 2</code>	[0, 2]
P4	$f(x) = \sin(x)$	<code>math.sin(x)</code>	[2, 6]
P5	$f(x) = x^3 - 6x^2 + 11x - 6$	<code>x**3 - 6*x**2 + 11*x - 6</code>	[1, 2.5]
P6	$f(x) = x^2 + 3x + 2$	<code>x**2 + 3*x + 2</code>	[-2.5, -1.5]
P7	$f(x) = \cos(x) - x$	<code>math.cos(x) - x</code>	[0, 1]
P8	$f(x) = 2^x - 8$	<code>2**x - 8</code>	[2, 4]
P9	$f(x) = \tan(x)$	<code>math.tan(x)</code>	[-1, 1]
P10	$f(x) = x^4 - 8x^3 + 18x^2 - 9x + 1$	<code>x**4 - 8*x**3 + 18*x**2 - 9*x + 1</code>	[2, 4]

## 1.2 Equations From Paper

We got these equations from [this paper](#) and we have used the same intervals too.

Table 2: Equations From Paper

No	Equation	Equation Code	Interval	Reference
P11	$f(x) = x^2 - 3$	<code>x**2 - 3</code>	[1, 2]	Harder [18]
P12	$f(x) = x^2 - 5$	<code>x**2 - 5</code>	[2, 7]	Srivastava[9]
P13	$f(x) = x^2 - 10$	<code>x**2 - 10</code>	[3, 4]	Harder [18]
P14	$f(x) = x^2 - x - 2$	<code>x**2 - x - 2</code>	[1, 4]	Moazzam [10]
P15	$f(x) = x^2 + 2x - 7$	<code>x**2 + 2*x - 7</code>	[1, 3]	Nayak[11]
P16	$f(x) = x^3 - 2$	<code>x**3 - 2</code>	[0, 2]	Harder [18]
P17	$f(x) = xe^x - 7$	<code>x * math.exp(x) - 7</code>	[0, 2]	Callhoun [19]

No	Equation	Equation Code	Interval	Reference
P18	$f(x) = x - \cos(x)$	<code>x - math.cos(x)</code>	[0, 1]	Ehiwario [6]
P19	$f(x) = x \sin(x) - 1$	<code>x * math.sin(x) - 1</code>	[0, 2]	Mathews [20]
P20	$f(x) = x \cos(x) + 1$	<code>x * math.cos(x) + 1</code>	[-2, 4]	Esfandiari [21]
P21	$f(x) = x^{10} - 1$	<code>x**10 - 1</code>	[0, 1.3]	Chapra [17]
P22	$f(x) = x^2 + e^{x/2} - 5$	<code>x**2 + (2.71828**(x/2)) - 5</code>	[1, 2]	Esfandiari [21]
P23	$f(x) = \sin(x) \sinh(x) + 1$	<code>math.sin(x) * math.sinh(x) + 1</code>	[3, 4]	Esfandiari [21]
P24	$f(x) = e^x - 3x - 2$	<code>(2.71828**x) - 3*x - 2</code>	[2, 3]	Hoffman [22]
P25	$f(x) = \sin(x) - x^2$	<code>math.sin(x) - x**2</code>	[0.5, 1]	Chapra[17]

## 2 Results

These are the results we got with each method. We have run each method 500 times on each equation and took the average time to get the highest accuracy possible.

### 2.1 False Position

These are the results we got with False Position method:

Table 3: False Position

Problem	False Position Algorithm					
	Iter	Avg CPU Time	Approximate Root	Function Value	Lower Bound	Upper Bound
P1	80	0.000229008	1.3652300134140964	-7.11E-15	1.3652300134140964	4
P2	33	4.399728775024414e-05	1.9999999999999978	-8.88E-15	1.9999999999999978	4
P3	51	5.6000232696533204e-05	0.6931471805599422	-6.22E-15	0.6931471805599422	2
P4	8	6.000041961669922e-06	3.141592653589793	1.2246467991473532e-16	3.141592653589793	3.1415926535899232
P5	2	0	1	0	1	2.5
P6	31	4.800844192504883e-05	-2	-5.33E-15	-2.5	-2
P7	12	1.101541519165039e-05	0.7390851332151551	9.2148511043888e-15	0.7390851332151551	1
P8	30	4.401159286499024e-05	2.9999999999999987	-7.11E-15	2.9999999999999987	4
P9	2	1.991748809814453e-06	0	0	0	1
P10	13	4.0007591247558594e-05	3.1117486563092474	0	3.1117486563092474	3.1117486563092482
P11	14	1.7997264862060548e-05	1.732050807568876	-4.00E-15	1.732050807568876	2
P12	50	6.600427627563476e-05	2.2360679774997876	-9.77E-15	2.2360679774997876	7
P13	17	2.2464752197265626e-05	3.162277660168379	-1.78E-15	3.162277660168379	4
P14	38	5.301380157470703e-05	1.9999999999999971	-8.66E-15	1.9999999999999971	4
P15	21	3.1998634338378904e-05	1.8284271247461896	-2.66E-15	1.8284271247461896	3
P16	41	5.600643157958984e-05	1.2599210498948719	-6.22E-15	1.2599210498948719	2
P17	30	3.40123176574707e-05	1.5243452049841437	-7.99E-15	1.5243452049841437	2
P18	12	1.2005805969238282e-05	0.7390851332151551	-9.21E-15	0.7390851332151551	1
P19	7	7.99846649169922e-06	1.1141571408719306	8.881784197001252e-16	1.0997501702946164	1.1141571408719306
P20	13	1.1332988739013672e-05	2.0739328090912146	7.771561172376096e-16	2.0739328090912146	2.5157197710146586
P21	139	0.000183961	0.9999999999999991	-8.88E-15	0.9999999999999991	1.3
P22	16	3.3281803131103514e-05	1.6490135532979475	-1.78E-15	1.6490135532979475	2
P23	45	7.994651794433594e-05	3.2215883990939416	6.328271240363392e-15	3.2215883990939416	4
P24	45	6.818151473999023e-05	2.1253934262332246	-9.77E-15	2.1253934262332246	3

Problem	False Position Algorithm					
	Iter	Avg CPU Time	Approximate Root	Function Value	Lower Bound	Upper Bound
<i>P25</i>	17	2.703714370727539e-05	0.8767262153950554	7.882583474838611e-15	0.8767262153950554	1

## 2.2 Bisection Method

These are the results we got with Bisection method:

Table 4: Bisection

Problem	Bisection Algorithm					
	Iter	Avg CPU Time	Approximate Root	Function Value	Lower Bound	Upper Bound
<i>P1</i>	50	7.303380966186524e-05	1.3652300134140951	-2.84E-14	1.3652300134140916	1.3652300134140987
<i>P2</i>	1	0	2	0	0	4
<i>P3</i>	49	4.200363159179687e-05	0.6931471805599436	-3.33E-15	0.6931471805599401	0.6931471805599472
<i>P4</i>	50	3.406333923339844e-05	3.141592653589793	1.2246467991473532e-16	3.1415926535897896	3.1415926535897967
<i>P5</i>	48	7.496118545532227e-05	2.0000000000000018	0	1.999999999999964	2.000000000000007
<i>P6</i>	1	1.9011497497558594e-06	-2	0	-2.5	-1.5
<i>P7</i>	48	3.201484680175781e-05	0.7390851332151591	2.55351295663786e-15	0.7390851332151556	0.7390851332151627
<i>P8</i>	1	1.9893646240234374e-06	3	0	2	4
<i>P9</i>	1	1.991748809814453e-06	0	0	-1	1
<i>P10</i>	49	9.199857711791992e-05	3.111748656309249	1.0658141036401503e-14	3.1117486563092456	3.1117486563092527
<i>P11</i>	48	4.000377655029297e-05	1.7320508075688785	4.440892098500626e-15	1.732050807568875	1.732050807568882
<i>P12</i>	50	3.901958465576172e-05	2.2360679774997854	-1.95E-14	2.236067977499781	2.236067977
<i>P13</i>	48	3.7988662719726566e-05	3.1622776601683817	1.5987211554602254e-14	3.162277660168378	3.1622776601683853
<i>P14</i>	50	4.400014877319336e-05	1.9999999999999991	-2.66E-15	1.999999999999964	2.0000000000000018
<i>P15</i>	49	5.607509613037109e-05	1.828427124746188	-1.15E-14	1.8284271247461845	1.8284271247461916
<i>P16</i>	49	3.8086414337158205e-05	1.2599210498948743	5.329070518200751e-15	1.2599210498948707	1.2599210498948779
<i>P17</i>	49	3.905820846557617e-05	1.5243452049841473	3.375077994860476e-14	1.5243452049841437	1.5243452049841508
<i>P18</i>	48	2.9998779296875e-05	0.7390851332151591	-2.55E-15	0.7390851332151556	0.7390851332151627
<i>P19</i>	49	0.000136974	1.114157140871928	-3.00E-15	1.1141571408719244	1.1141571408719315
<i>P20</i>	51	5.606412887573242e-05	2.0739328090912155	-1.33E-15	2.073932809091213	2.073932809091218
<i>P21</i>	48	4.004716873168945e-05	1.0000000000000001	1.1102230246251565e-14	0.999999999999966	1.0000000000000058
<i>P22</i>	44	5.988311767578125e-05	1.649013553297948	0	1.6490135532978911	1.6490135532980048
<i>P23</i>	48	6.889772415161133e-05	3.2215883990939425	-5.55E-15	3.221588399093939	3.221588399093946
<i>P24</i>	48	4.5994281768798825e-05	2.1253934262332272	5.329070518200751e-15	2.1253934262332237	2.125393426233231
<i>P25</i>	47	6.799602508544923e-05	0.8767262153950632	-8.88E-16	0.8767262153950597	0.8767262153950668

## 2.3 Hybrid Method

These are the results we got with hybrid method:

Table 5: Hybrid

Problem	Hybrid Algorithm					
	Iter	Avg CPU Time	Approximate Root	Function Value	Lower Bound	Upper Bound
<i>P1</i>	10	3.6006927490234375e-05	1.3652300134140964	-7.11E-15	1.365230013413779	1.3675001980274413
<i>P2</i>	1	1.9969940185546874e-06	2	0	0	4
<i>P3</i>	10	1.399993896484375e-05	0.6931471805599453	0	0.6931471805599334	0.695162706
<i>P4</i>	6	1.006174087524414e-05	3.141592653589793	1.2246467991473532e-16	3.1415903579556947	3.141592653604888
<i>P5</i>	1	3.940105438232422e-06	1	0	1	2.5
<i>P6</i>	1	1.9888877868652345e-06	-2	0	-2.5	-1.5

Problem	Hybrid Algorithm					
	Iter	Avg CPU Time	Approximate Root	Function Value	Lower Bound	Upper Bound
P7	8	1.1938095092773438e-05	0.7390851332151606	1.1102230246251565e-16	0.739085133	0.7422270732175922
P8	1	2.0036697387695312e-06	3	0	2	4
P9	1	2.0928382873535157e-06	0	0	-1	1
P10	8	2.4066925048828126e-05	3.1117486563092474	0	3.1085379927858856	3.1117486563092536
P11	8	1.7096519470214843e-05	1.7320508075688772	-4.44E-16	1.7320508075688001	1.7350578402209837
P12	10	1.4061450958251953e-05	2.236067977499789	-3.55E-15	2.236067977499364	2.243929153983615
P13	8	1.393747329711914e-05	3.1622776601683795	1.7763568394002505e-15	3.16227766	3.1672187190124017
P14	2	2.0089149475097657e-06	2	0	1.5	2.5
P15	5	8.056163787841797e-06	1.828427125	0	1.8284271247430004	1.8284271247493797
P16	9	1.2000083923339844e-05	1.2599210498948723	-4.00E-15	1.259921049893984	1.2611286403176987
P17	11	1.3935565948486329e-05	1.5243452049841444	0	1.5243452049841386	1.526033337108763
P18	8	1.0064601898193359e-05	0.7390851332151606	-1.11E-16	0.739085133	0.7422270732175922
P19	6	8.002758026123047e-06	1.1141571408719302	2.220446049250313e-16	1.1132427327642702	1.1141571408719768
P20	10	1.5938282012939452e-05	2.073932809091215	-2.22E-16	2.0739328090911866	2.078935003337393
P21	12	1.6058921813964842e-05	0.9999999999999999	-1.11E-15	0.9999999999999305	1.000343363282986
P22	8	2.393531799316406e-05	1.6490135532979473	-3.55E-15	1.6490135532974015	1.6531560376633945
P23	9	1.7997264862060548e-05	3.221588399093942	3.3306690738754696e-16	3.2215883990939242	3.2224168881395068
P24	9	1.2019157409667969e-05	2.125393426233225	-7.11E-15	2.1253934262325003	2.1275213330097245
P25	7	1.1998653411865234e-05	0.8767262153950581	4.773959005888173e-15	0.8767262153886713	0.8772684454348731

As we see from the table above the hybrid method tend to be faster and take much less iterations than both Bisection and False Position methods.

## 2.4 Conclusion

Our conclusions based on the tables and the plots are:

*Side Note: The equation  $f(x) = x^3 - 6x^2 + 11x - 6$  in P5 has two roots. In the false position and hybrid methods, the equation has a root at 1. On the other hand, the bisection method identifies another root at approximately 2.00000000000000018 within the interval  $[1, 2.5]$ .*

### 2.4.1 Iterations

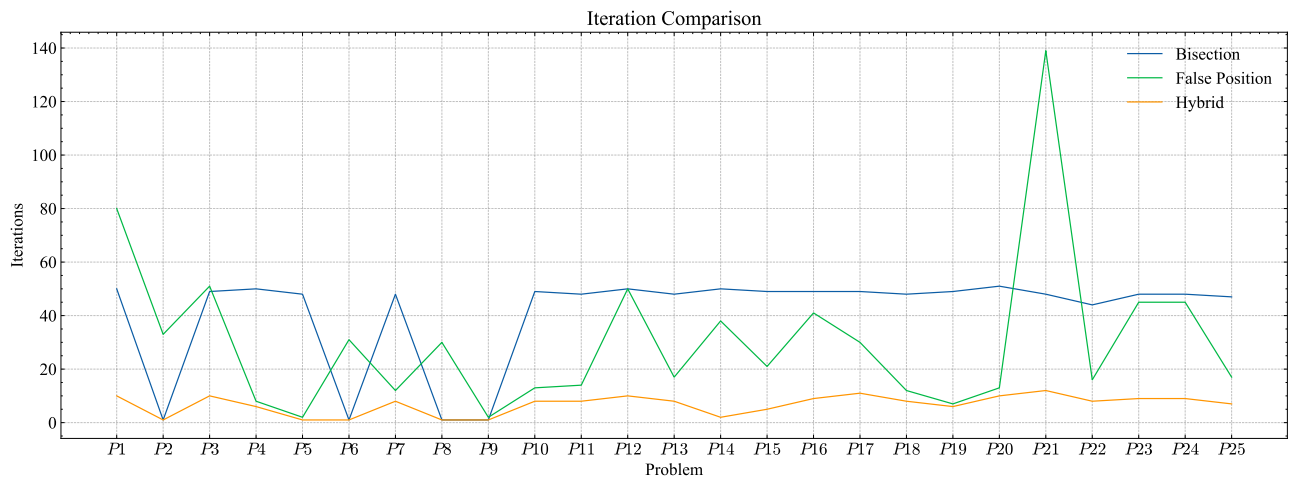


Figure 1: Iterations Comparison

The hybrid method demonstrates superior performance compared to both the bisection and false position methods in terms of the number of iterations required.

As we see here in P21 the false position method have much more number of iterations than both hybrid and bisection methods which will lead to more CPU time as we will see in the next section.

## 2.4.2 CPU Time

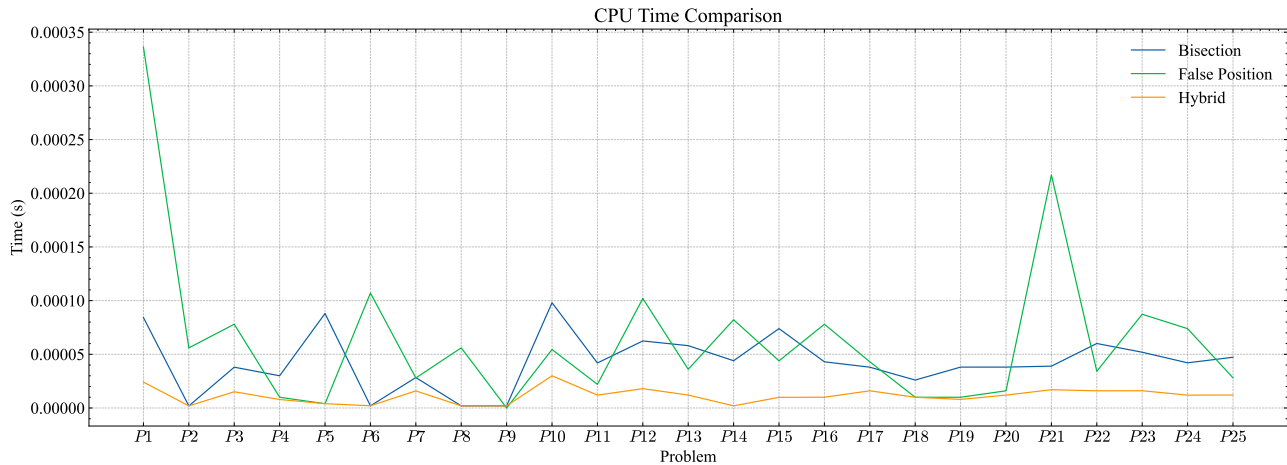


Figure 2: CPU Time Comparison

The hybrid method shows significant improvement over the bisection method in terms of CPU time, with a ratio of 21:4 .This translates to approximately 84% for the hybrid method and 16% for the bisection method.

The hybrid method shows significant improvement over the false position method in terms of CPU time, with a ratio of 19:6 .This translates to approximately 76% for the hybrid method and 24% for the false position method.

As a general trend, the hybrid method is faster than both the bisection and false position methods when it comes to finding the approximate root.

## 2.4.3 Function Value

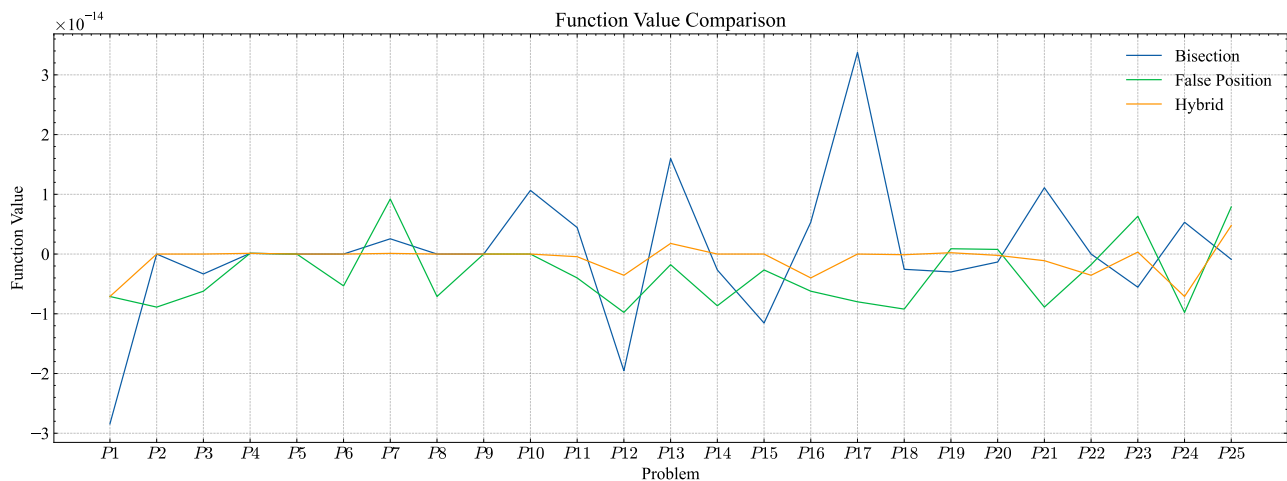


Figure 3: Function Value Comparison

The hybrid method outperforms both the bisection and false position methods in terms of function value, with smaller values that are closer to zero.