MISR UNIVERSITY FOR SCIENCE AND TECHNOLOGY
COLLEGE OF ENGINEERING
MECHATRONICS DEPARTMENT

MTE 506 DIGITAL CONTROL

LAB 6 – SPRING 2019

Lab 6

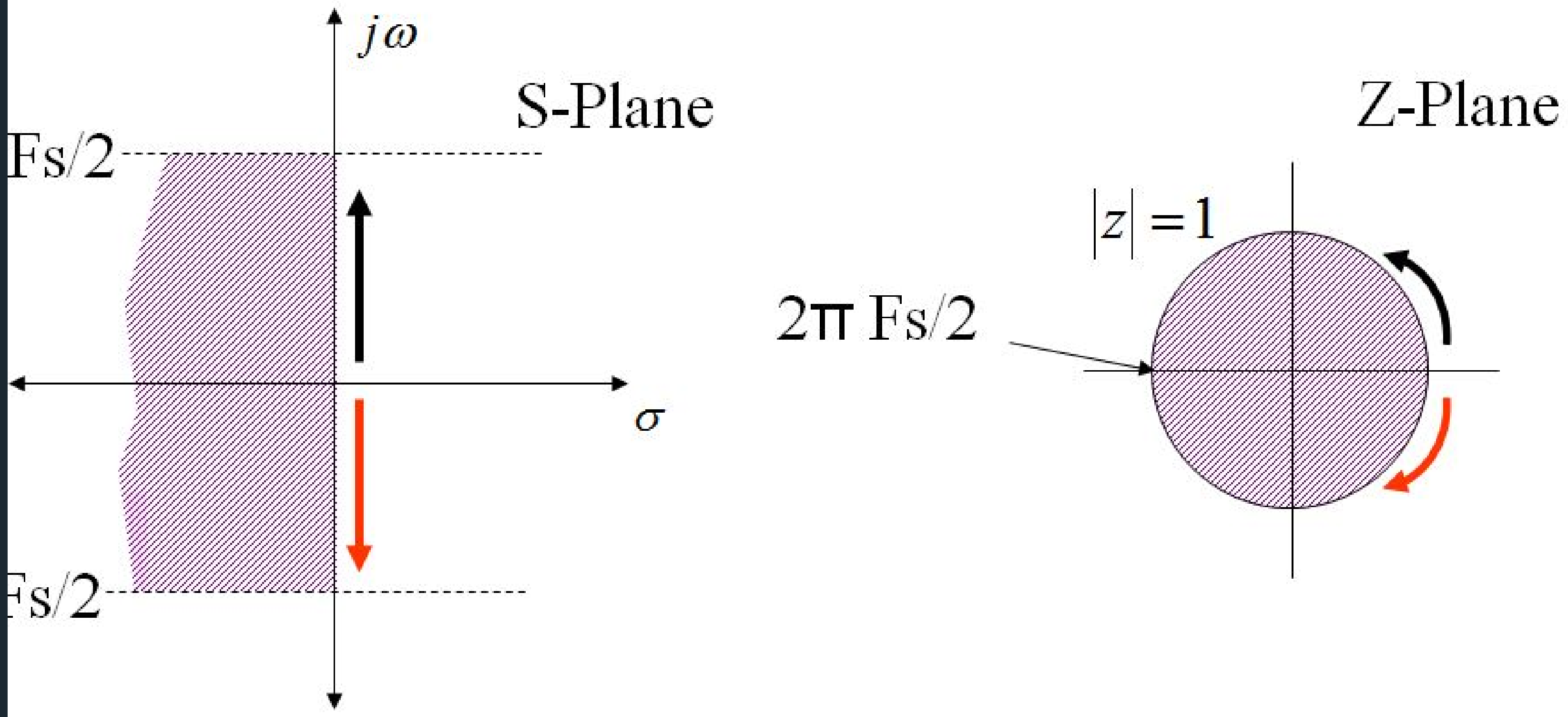
Goals of The Lab

- Z- Transform and unit delay
- Conversion to Z-Transform

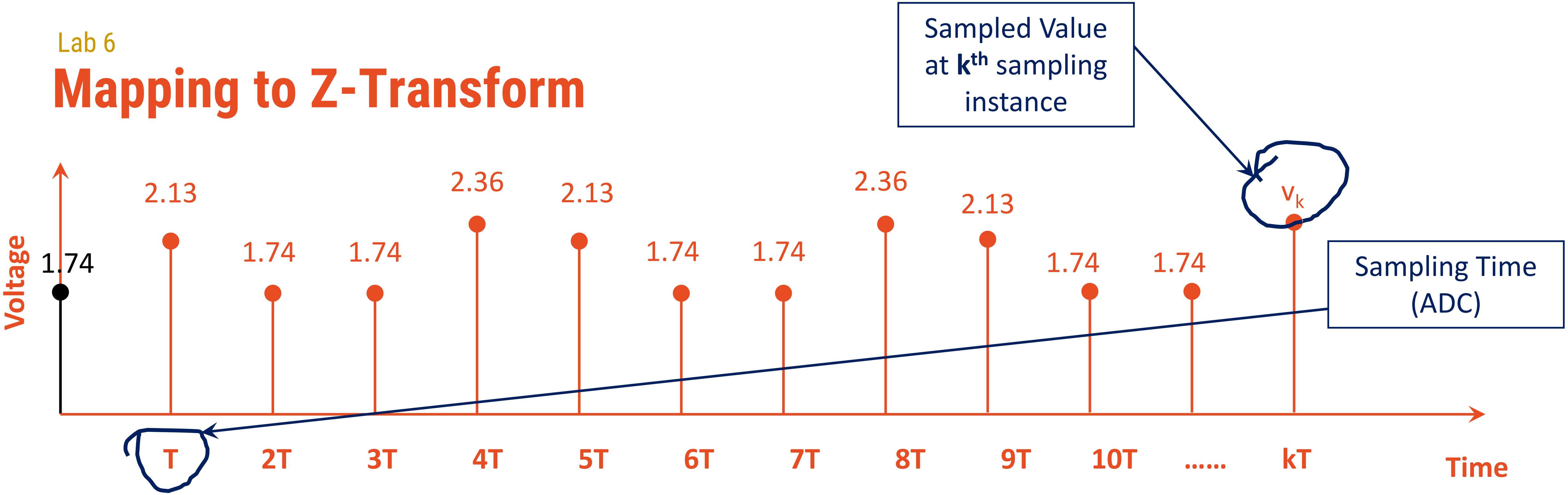
Z - Transform

Mapping from s to z

$$z = e^{sT} = e^{\frac{s}{F_s}}$$



Mapping to Z-Transform



$$v(t) = 1.74\,\delta(0) + 2.13\,\delta(t-T) + 1.74\,\delta(t-2T) + 1.74\,\delta(t-3T) + 2.36\,\delta(t-4T) + 2.13\,\delta(t-5T) + \cdots + \boldsymbol{v_k\,\delta(t-kT)}$$

$$V(s) = 1.74 \quad + 2.13\,e^{-sT} \quad + 1.74e^{-2sT} \quad + 1.74e^{-3sT} \quad + 2.36\,e^{-4sT} \quad + 2.13\,e^{-5sT} \quad + \cdots + \boldsymbol{v_k}\,e^{-ksT}$$

Mapping to Z-Transform

$$Let\ \boldsymbol{z} = \boldsymbol{e^{sT}}$$

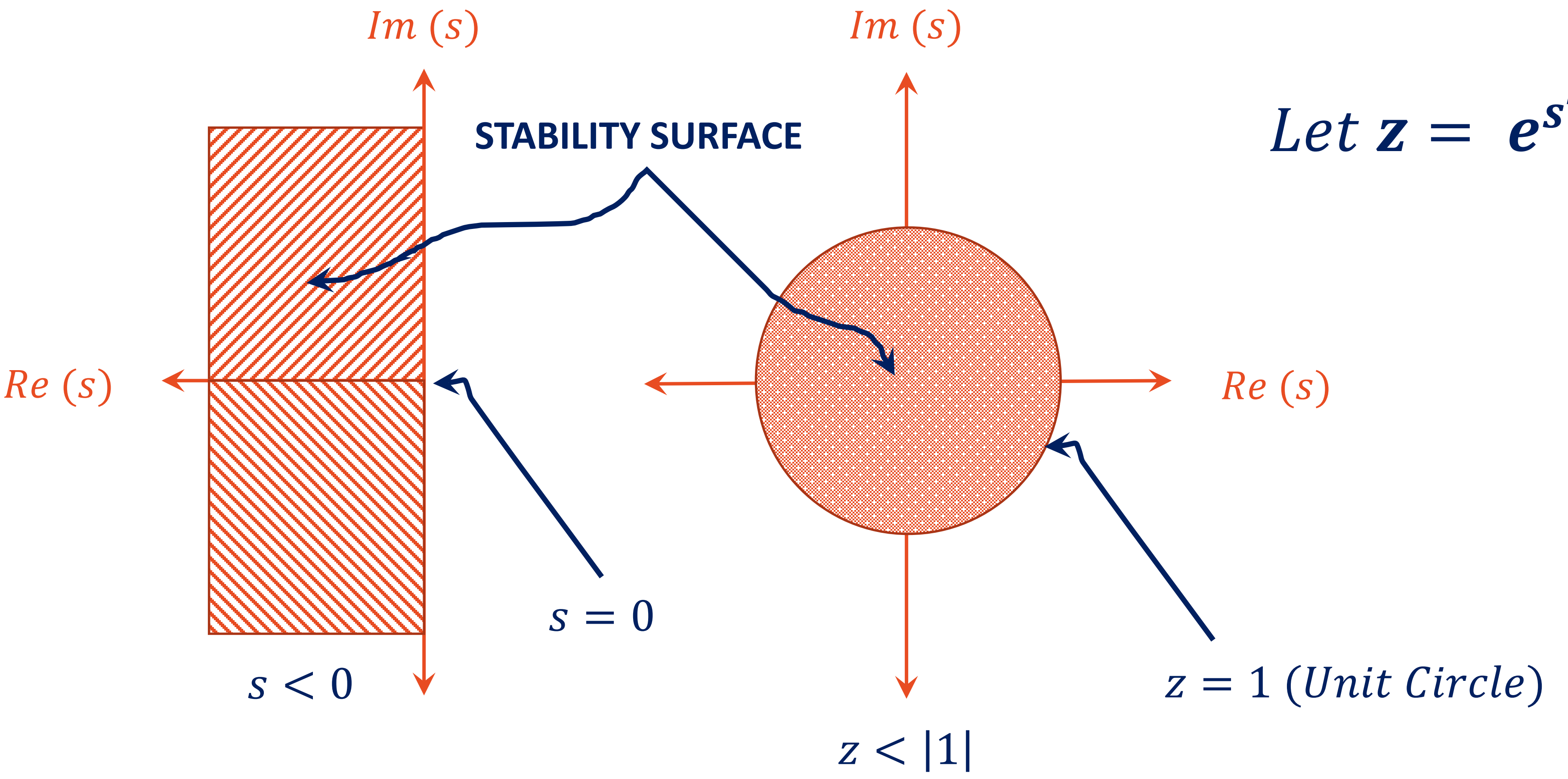
$$v(t) = 1.74\,\delta(0) + 2.13\,\delta(t-T) + 1.74\,\delta(t-2T) + 1.74\,\delta(t-3T) + 2.36\,\delta(t-4T) + 2.13\,\delta(t-5T) + \cdots + \boldsymbol{v_k\,\delta(t-kT)}$$

$$V(s) = 1.74 + 2.13\,e^{-sT} + 1.74e^{-2sT} + 1.74e^{-3sT} + 2.36\,e^{-4sT} + 2.13\,e^{-5sT} + \cdots + \boldsymbol{v_k\,e^{-ksT}}$$

$$V(z) = 1.74 + 2.13\,z^{-T} + 1.74\,z^{-2T} + 1.74z^{-3T} + 2.36\,z^{-4T} + 2.13\,z^{-5T} + \cdots + \boldsymbol{v_k\,z^{-kT}}$$

$$V(z) = \underbrace{\sum_{k=0}^{N} v_k\,z^{-kT}}_{\text{Sequence}} \qquad V(z) = \sum_{k=0}^{N} v_k\,z^{-k}\ , T = 1$$

Mapping to Z-Transform



$$Let\ z = e^{sT}\ or\ s = \frac{1}{T}\ln z$$

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Z – Transform Closed Form

Transforms

$f(kT), k \ge 0$	$F(z)$
$\begin{cases} 1, k=0 \\ 0, k \ne 0 \end{cases}$	1
$\begin{cases} 1, k=n \\ 0, k \ne n \end{cases}$	z^{-n}
1	$\frac{z}{z-1}$
kT	$\frac{Tz}{(z-1)^2}$
$\frac{1}{2}(kT)^2$	$\frac{T^2 z(z+1)}{2(z-1)^3}$
e^{-akT}	$\frac{z}{z-e^{-aT}}$
$(kT)e^{-akT}$	$\frac{Te^{-aT}z}{(z-e^{-aT})^2}$

$f(t), t \ge 0$	$F(s)$	$f(kT), k \ge 0$
$1-e^{-at}$	$\frac{a}{s(s+a)}$	$1-e^{-akT}$
$e^{-at}-e^{-bt}$	$\frac{b-a}{(s+a)(s+b)}$	$e^{-akT}-e^{-bkT}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$	$\sin(\omega kT)$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$	$\cos(\omega kT)$
$e^{-at}\sin(\omega t)$	$\frac{\omega}{(s+a)^2+\omega^2}$	$e^{-akT}\sin(\omega kT)$
$e^{-at}\cos(\omega t)$	$\frac{s+a}{(s+a)^2+\omega^2}$	$e^{-akT}\cos(\omega kT)$
—	—	a^k
—	—	$k \cdot a^{k-1}$

Z-Transform Infinite Power Series

$$U(z) = u_0 + u_1\, z^{-1} + u_2\, z^{-2} + u_3\, z^{-3} + \cdots + u_k\, z^{-k}$$
$$= \sum_{k=0}^{N} u_k\, z^{-k}\ (sequence)$$

Z-Transform Infinite Power Series

$$U(z) = u_0 + u_1\ z^{-1} + u_2\ z^{-2} + u_3\ z^{-3} + \cdots + u_k\ z^{-k}$$

$$= \sum_{k=0}^{N} u_k\ z^{-k}\ (sequence)$$

and $$\sum_{k=0}^{\infty} a^k = \frac{1}{1-a}$$

$(power\ infinite\ series)$

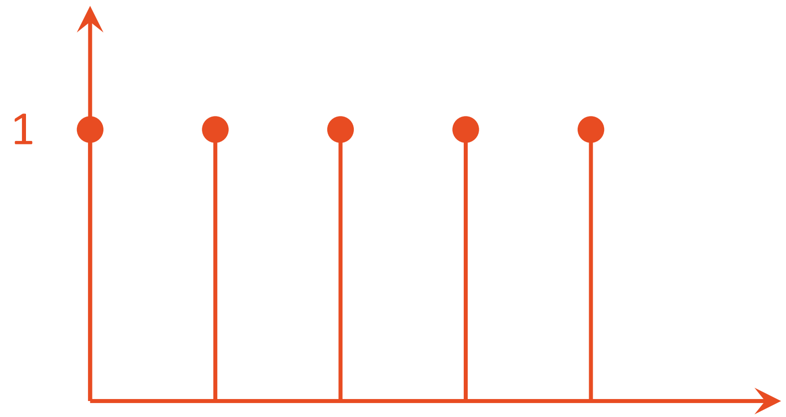
Sampled Unit Step 1(k)

$$\{u(k)\}_{k=0}^{\infty} = \{1,1,1,1, \ldots\}$$

$$U(z) = 1 + 1\ z^{-1} + 1\ z^{-2} + 1\ z^{-3} + \cdots + 1\ z^{-k}$$

$$= \sum_{k=0}^{N} 1\ z^{-k}$$

$$\boldsymbol{U(z) = \frac{1}{1-z^{-1}} = \frac{z}{z-1}}$$



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Z-Transform Infinite Power Series

$$U(z) = u_0 + u_1\ z^{-1} + u_2\ z^{-2} + u_3\ z^{-3} + \cdots + u_k\ z^{-k}$$

$$= \sum_{k=0}^{N} u_k\ z^{-k}\ (sequence)$$

and $\displaystyle\sum_{k=0}^{\infty} a^k = \frac{1}{1-a}$

$(power\ infinite\ series)$

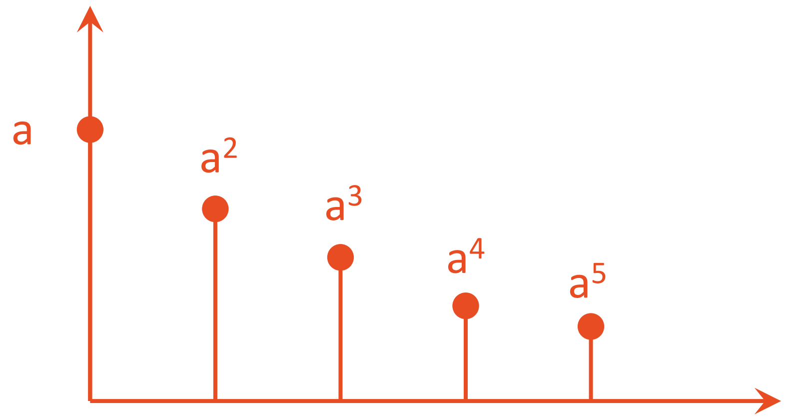
Exponential a^k

$$\{U(k)\}_{k=0}^{\infty} = \{1, a^2, a^3, a^4, \dots, a^k\}$$

$$U(z) = 1 + a\ z^{-1} + a^2\ z^{-2} + a^3\ z^{-3} + \cdots + a^k\ z^{-k}$$

$$= \sum_{k=0}^{N} a^k\ z^{-k}$$

$$\mathbf{1(z) = = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}}$$



Z-Transform Properties

Time delay

$$\mathcal{Z}\{f(k-N)\} = z^{-N}F(z)$$

Examples

$$\mathcal{Z}\{1(k-2)\} = z^{-2}F\big(1(k)\big) = z^{-2}\frac{z}{z-1} = \frac{1}{z(z-1)}$$

Z-Transform Properties

Time advance

$$\mathcal{Z}\{f(k+N)\} = z^N F(z) - z^N f(0) - z^{(N-1)} f(1) - \cdots - z f(N-1)$$

Examples

$$\mathcal{Z}\{1(k-2)\} = z^{-2} F\big(1(k)\big) = z^{-2} \frac{z}{z-1} = \frac{1}{z(z-1)}$$

Lab 6

Z-Transform Properties

Time advance

$$\mathcal{Z}\{f(k+N)\} = z^N F(z) - z^N f(0) - z^{(N-1)} f(1) - \cdots - z f(N-1)$$

Examples

$$f(k) = \{4,8,16,32,\dots\} \to f(k) = 2^{\boldsymbol{k+2}} \to f(k+2)\,, f(k) = 2^k$$

$$\mathcal{Z}\{f(k+2)\} = z^2 F(2^k) - z^2 f(0) - z f(1)$$

$$\because\ f(k) = 2k \ \to f(0) = 1, f(1) = 2$$

$$\therefore\ \mathcal{Z}\{f(k+2)\} = z^2 \frac{z}{z-2} - z^2 - 2z = \frac{(z^3) - (z^3 - 2z^2) - (2z^2 - 4z)}{z-2}$$

$$\boldsymbol{\mathcal{Z}\{f(k+2)\} = \frac{4z}{z-2}}$$

Z-Transform Properties

Multiplication by exponential

$$\mathcal{Z}\{a^{-k} f(k)\} = F(az)$$

$Examples$

$f(k) = e^{-3kT}, k = 0,1,2,3, \ldots$

$\because\ e^{-3kT} = (e^{3T})^{-k} = a^{-k} * 1(k)$

$$\therefore \mathcal{Z}\{a^{-k} * 1(k)\} = \frac{1}{1-(az)^{-1}} = \frac{z}{z-a^{-1}} = \boldsymbol{\frac{z}{z-e^{-3T}}}$$

Z-Transform Properties

Complex Differentiation

$$\mathcal{Z}\{k^m f(k)\} = \left(-z\frac{d}{dz}\right)^m F(z)$$

Examples

$f(k) = k\ , k = 0,1,2,3, \ldots$

$$\because f(k) = k\ * 1(k) = \left(-z\frac{d}{dz}\right)\frac{z}{z-1} = (-z)\frac{(z-1)(1)-(z)(1)}{(z-1)^2} = \frac{\boldsymbol{z}}{(\boldsymbol{z}-\mathbf{1})^2}$$

Solved Example

$Given\ the\ linear\ difference\ equation$

$y(k+1) - y(k) = u(k+1),$ u(k) is a $unit\ step$

$Find\ \boldsymbol{Y(z)}$

$Solution$

$$\mathcal{Z}\{y(k+1)\} = zF(z) - zf(0) = zY(z) - z(0) = zY(z)$$

$$\mathcal{Z}\{y(k)\} = Y(z)$$

$$\mathcal{Z}\{u(k+1)\} = zF(z) - zf(0) = z\frac{z}{z-1} - (z)(1) = \frac{z^2 - z^2 + z}{z-1} = \frac{z}{z-1}$$

$$\therefore zY(z) - Y(z) = \frac{z}{z-1} \rightarrow Y(z)[z-1] = \frac{z}{z-1} \rightarrow Y(z) = \frac{z}{(z-1)^2}$$

Don’t forget to pull the lab update from.

http://github.com/wbadry/mte506

Waleed El-Badry

END OF Lab 6