$$= 256\left[\frac{1}{3}u^{2}\right]_{0} = \frac{256}{3}.$$

Problem 2. *Let C denote the line segment $y = 2x + 1, -1 \leq x \leq 0$ and $G(x, y) = 3x^2 + 6y^2$. Evaluate the following line integrals :*

(a) $\int_C G(x,y)\ dx.$ (b) $\int_C G(x,y)\ dy.$ (c) $\int_C G(x,y)\ ds.$

Solution. Here C is defined by an explicit function $y = f(x) = 2x + 1$, $-1 \leq x \leq 0$. Then

$$dy = f'(x)\,dx = 2dx \text{ and } ds = \sqrt{1+[f'(x)]^2}\ dx = \sqrt{1+2^2}\,dx = \sqrt{5}dx.$$

(a) $$\int_C G(x,y)\ dx = \int_a^b G(x,\ f(x))\ dx = \int_{-1}^{0} 3x^2 + 6(2x+1)^2\ dx$$

$$= \int_{-1}^{0} 27x^2 + 24x + 6\ dx = \left[9x^3 + 12x^2 + 6x\right]_{-1}^{0}$$

$$= \left[0 - (-9 + 12 - 6)\right] = 3.$$

(b) $$\int_C G(x,y)\ dy = \int_a^b G(x,\ f(x))\, f'(x)\ dx$$

$$= \int_{-1}^{0} \left[3x^2 + 6(2x+1)^2\right]\ 2\,dx = 2\int_{-1}^{0} 27x^2 + 24x + 6\ dx$$

$$= 2\left[9x^3 + 12x^2 + 6x\right]_{-1}^{0} = 2\left[0 - (-9 + 12 - 6)\right] = 6.$$

(c) $$\int_C G(x,y)\ ds = \int_a^b G(x,\ f(x))\sqrt{1+[f'(x)]^2}\ dx.$$

$$= \int_{-1}^{0} \left[3x^2 + 6(2x+1)^2\right]\ \sqrt{5}\,dx$$

$$= \sqrt{5}\int_{-1}^{0} 27x^2 + 24x + 6\ dx = \sqrt{5}\left[9x^3 + 12x^2 + 6x\right]_{-1}^{0}$$

$$= \sqrt{5}\,[0-(-9+12-6)] = 3\sqrt{5}.$$

Problem 3. *Evaluate* $\int_C xy\,dx + x^2dy$, *where* C *is given by* $y = x^3$, $-1 \le x \le 2$.

Solution. Here C is defined by an explicit function $y = f(x) = x^3$, $-1 \le x \le 2$. Then $dy = 3x^2\,dx$.

$$\int_C xy\,dx + x^2dy = \int_{-1}^{2} x(x^3)\,dx + x^2(3x^2dx)$$

$$= \int_{-1}^{2} [x^4 + 3x^4]\,dx = 4\int_{-1}^{2} x^4\,dx$$

$$= 4\left[\frac{x^5}{5}\right]_{-1}^{2} = 4\left[\frac{32}{5} - \frac{-1}{5}\right] = \frac{132}{5}.$$

Problem 4. *Evaluate* $\oint_C x^2y^3\,dx - xy^2dy$, *where* C *is the square with vertices* $(-1, -1)$, $(1, -1)$, $(1, 1)$ *and* $(-1, 1)$, *in the counter clockwise direction.*

Solution. Let C_1, C_2, C_3 and C_4 be the sides of the square C, joining $(-1,-1)$ to $(1,-1)$, $(1,-1)$ to $(1, 1)$, $(1, 1)$ to $(-1, 1)$ and $(-1, 1)$ to $(-1, -1)$, respectively. (refer figure 3.2).

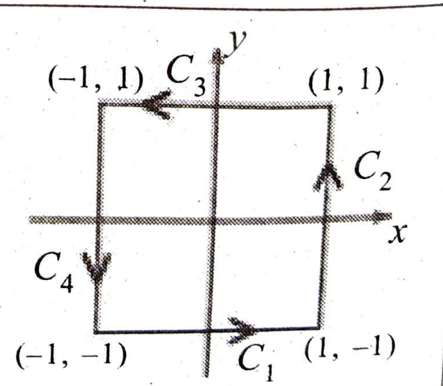

FIGURE 3.2

Then C is composed of smooth curves C_1, C_2, C_3 and C_4. The parametric representations of C_1, C_2, C_3 and C_4 are

C_1: $x = t$, $y = -1$, $-1 \le t \le 1$,

C_2: $x = 1$, $y = t$, $-1 \le t \le 1$,

C_3: $x = -t$, $y = 1$, $-1 \le t \le 1$,

C_4: $x = -1$, $y = -t$, $-1 \le t \le 1$.

On C_1 $x = t$, $y = -1$ and $-1 \le t \le 1$. Then $dx = dt$, $dy = 0$ and so

$$\int_{C_1} x^2y^3dx - xy^2dy = \int_{-1}^{1} t^2(-1)^3dt - t(-1)^2 \cdot 0$$

$$= -\left[t^3/3\right]_{-1}^{1} = -2/3.$$

On C_2 $x = 1$, $y = t$ and $-1 \le t \le 1$. Then $dx = 0$, $dy = dt$ and so

$$\int_{C_2} x^2y^3dx - xy^2dy = \int_{-1}^{1} 1\cdot t^3 \cdot 0 - 1 \cdot t^2 \cdot dt$$

$$= -\left[t^3/3\right]_{-1}^{1} = -2/3.$$

On C_3 $x = -t$, $y = 1$ and $-1 \le t \le 1$. Then $dx = -dt$, $dy = 0$ and so

$$\int_{C_3} x^2y^3dx - xy^2dy = \int_{-1}^{1} (-t)^2(-1)^3(-dt) - (-t)\cdot 1 \cdot 0$$

$$= -\left[t^3/3\right]_{-1}^{1} = -2/3.$$

On C_4 $x = -1$, $y = -t$ and $-1 \le t \le 1$. Then $dx = 0$, $dy = -dt$ and so

$$\int_{C_4} x^2y^3dx - xy^2dy = \int_{-1}^{1} (-1)^2(-t)^3 \cdot 0 - (-1)\cdot(-t)^2 \cdot (-dt)$$

$$= -\left[t^3/3\right]_{-1}^{1} = -2/3.$$

Since C is composed of smooth curves C_1, C_2, C_3 and C_4, the integral over C is equal to the sum of the integrals over C_1, C_2, C_3 and C_4.

$$\therefore \oint_{C} x^2y^3\,dx - xy^2dy = -\frac{2}{3} + \left(-\frac{2}{3}\right) + \left(-\frac{2}{3}\right) + \left(-\frac{2}{3}\right) = -\frac{8}{3}.$$

Remark. It is important to be aware that a line integral is independent of the parametrization of the curve C provided C is given the same orientation by all sets of parametric equations defining the curve. Also it can be shown that

$$\int_{-C} P\,dx + Q\,dy = -\int_{C} P\,dx + Q\,dy\ .$$

1.2. Line Integrals in Space

Line integrals of a function $G(x, y, z)$ of three variables along a curve C in space are defined in a way similar to line integrals in plane. Let C be a smooth curve in 3-space defined by the parametric equations $x = f(t)$, $y = g(t)$, $z = h(t)$, $a \le t \le b$. Then line integrals of G along C with respect to x, y, z and s are evaluated by using

$$\int_C G(x,y,z)\ dx = \int_a^b G(f(t),g(t),h(t))\,f'(t)\ dt$$

$$\int_C G(x,y,z)\ dy = \int_a^b G(f(t),g(t),h(t))\,g'(t)\ dt$$

$$\int_C G(x,y,z)\ dz = \int_a^b G(f(t),g(t),h(t))\,h'(t)\ dt$$

$$\int_C G(x,y,z)\ ds = \int_a^b G(f(t),g(t),h(t))\sqrt{[f'(t)]^2+[g'(t)]^2+[h'(t)]^2}\ dt.$$

Problem 5. *Let C denote the curve defined by* $x = (1/3)t^3$, $y = t^2$, $z = 2t$, $0 \le t \le 1$. *Evaluate the following line integrals :*

(a) $\displaystyle\int_C 4xyz\ dx.$ (b) $\displaystyle\int_C 4xyz\ dy.$

(c) $\displaystyle\int_C 4xyz\ dz.$ (c) $\displaystyle\int_C 4xyz\ ds.$

Solution. Here $G\ (x, y) = 4xyz$, $f\ (t) = (1/3)t^3$, $g\ (t) = t^2$, $h\ (t) = 2t$, $a = 0$ and $b = 1$. Then $dx = f'(t)\ dt = t^2\ dt$, $dy = g'(t)\ dt = 2t\ dt$, $dz = h'(t)\ dt = 2dt$ and

$$ds = \sqrt{[f'(t)]^2+[g'(t)]^2+[h'(t)]^2}\ dt = \sqrt{t^4+4t^2+4}\,dt = (t^2+2)\,dt.$$

(a) $$\int_C 4xyz\ dx = \int_C G(x,y,z)\ dx = \int_a^b G(f(t),g(t),h(t))\,f'(t)\ dt$$

$$= \int_0^1 4(t^3/3)\cdot t^2\cdot 2t\cdot t^2\ dt = \frac{8}{3}\int_0^1 t^8\ dt$$

$$= \frac{8}{3}\left[\frac{t^9}{9}\right]_0^1 = \frac{8}{27}.$$

(b) $$\int_C 4xyz\ dy = \int_C G(x,y,z)\ dx = \int_a^b G(f(t),g(t),h(t))\,g'(t)\ dt$$

$$= \int_0^1 4(t^3/3)\cdot t^2\cdot 2t\cdot 2t\ dt = \frac{16}{3}\int_0^1 t^7\ dt$$

$$= \frac{16}{3}\left[\frac{t^8}{8}\right]_0^1 = \frac{2}{3}.$$

(c) $$\int_C 4xyz\ dz = \int_C G(x, y, z)\ dz = \int_a^b G(f(t), g(t), h(t))\, h'(t)\ dt$$

$$= \int_0^1 4(t^3/3)\cdot t^2 \cdot 2t \cdot 2\ dt = \frac{16}{3}\int_0^1 t^6\ dt$$

$$= \frac{16}{3}\left[\frac{t^7}{7}\right]_0^1 = \frac{16}{21}.$$

(d) $$\int_C 4xyz\ ds = \int_C G(x, y, z)\ ds = \int_a^b G(f(t), g(t), h(t))\ ds$$

$$= \int_0^1 4(t^3/3)\cdot t^2 \cdot 2t \cdot (t^2 + 2)\ dt$$

$$= \frac{8}{3}\int_0^1 t^6 (t^2 + 2)\ dt = \frac{8}{3}\int_0^1 t^8 + 2t^6\ dt$$

$$= \frac{8}{3}\left[\frac{t^9}{9} + \frac{2t^7}{7}\right]_0^1 = \frac{8}{3}\left[\frac{1}{9} + \frac{2}{7}\right] = \frac{200}{189}.$$

Problem 6. *Evaluate* $\int_C y\ dx + z\, dy + x\, dz$, *where* C *consists of line segments from* (0, 0, 0) *to* (2, 3, 4) *and from* (2, 3, 4) *to* (6, 8, 5).

Solution. Let C_1 denote the line segment from (0, 0, 0) to (2, 3, 4). Then C_1 is parallel to the vector

$$(2 - 0)\ \mathbf{i} + (3 - 0)\ \mathbf{j} + (4 - 0)\ \mathbf{k} \quad \text{i.e., } 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$$

and passes through the point (0, 0, 0). Hence its parametric equation is

$$x = 0 + 2t,\ y = 0 + 3t,\ z = 0 + 4t, \quad \text{i.e., } x = 2t,\ y = 3t,\ z = 4t.$$

The above line passes through (0, 0, 0) when $t = 0$ and passes through (2, 3, 4) when $t = 1$. Hence parametrization of the line segment C_1 is

$$x = 2t,\ y = 3t,\ z = 4t,\ 0 \le t \le 1.$$

Let C_2 denote the line segment from (2, 3, 4) to (6, 8, 5). Then C_1 is parallel to the vector

$$(6-2)\,\mathbf{i}+(8-3)\,\mathbf{j}+(5-4)\,\mathbf{k} \quad \text{i.e.,}\ 4\mathbf{i}+5\mathbf{j}+\mathbf{k}$$

and passes through the point (2, 3, 4). Hence its parametric equation is

$$x = 2+4t,\ \ y = 3+5t,\ z = 4+t.$$

The above line passes through (2, 3, 4) when $t = 0$ and passes through (6, 8, 5) when $t = 1$. Hence parametrization of the line segment C_2 is

$$x = 2+4t,\ \ y = 3+5t,\ z = 4+t,\ \ 0 \le t \le 1.$$

On C_1, $x = 2t$, $y = 3t$, $z = 4t$ and $0 \le t \le 1$. Then $dx = 2dt$, $dy = 3dt$, $dz = 4dt$ and so

$$\int_{C_1} y\,dx + z\,dy + x\,dz = \int_0^1 3t \cdot 2\,dt + 4t \cdot 3\,dt + 2t \cdot 4\,dt$$

$$= \int_0^1 26t\ dt = \left[13t^2\right]_0^1 = 13.$$

On C_2, $x = 2+4t$, $y = 3+5t$, $z = 4+t$ and $0 \le t \le 1$. Then $dx = 4dt$, $dy = 5dt$, $dz = dt$ and so

$$\int_{C_1} y\,dx + z\,dy + x\,dz = \int_0^1 (3+5t)\cdot 4\,dt + (4+t)\cdot 5\,dt + (2+4t)\cdot dt$$

$$= \int_0^1 (34+29t)\ dt = \left[34t + \frac{29}{2}t^2\right]_0^1 = \frac{97}{2}.$$

Since C is composed of smooth curves C_1 and C_2, the integral over C is equal to the sum of the integrals over C_1 and C_2. Therefore

$$\int_C y\,dx + z\,dy + x\,dz = \int_{C_1} y\,dx + z\,dy + x\,dz + \int_{C_1} y\,dx + z\,dy + x\,dz$$

$$= 13 + \frac{97}{2} = \frac{123}{2}.$$