Solutions to Selected Exercises for Chapter 0

Solutions to Exercises for § 0.1

0.1.1 (a) ($\Rightarrow$) Suppose that $x \in (A \cap B)^c$. It follows that $x \notin (A \cap B)$. This can be for one of only three reasons.

(a) $x \in A$ but $x \notin B$.
(b) $x \in B$ but $x \notin A$.
(c) $x \notin A$ and $x \notin B$.

If (a) holds, then $x \in B^c$ so that $x \in A^c \cup B^c$ as well. Similarly, if (b) holds, then $x \in A^c$ so that, again, $x \in A^c \cup B^c$ as well. Finally, if (c) holds, then $x \in A^c$ and $x \in B^c$. Again, this implies that $x \in A^c \cup B^c$. In other words, each of (a), (b), and (c) leads us to conclude that $x \in A^c \cup B^c$. Thus we have shown that $(A \cap B)^c \subseteq A^c \cup B^c$.

($\Leftarrow$) Suppose that $x \in A^c \cup B^c$. Then, by the definition of $\cup$, there are three possibilities.

(a) $x \in A^c$ but $x \notin B^c$.
(b) $x \in B^c$ but $x \notin A^c$.
(a) $x \in A^c$ and $x \in B^c$.

Now if (a) holds, then $x \notin A$ so that $x \notin A \cap B$. But then $x \in (A \cap B)^c$. Similarly, if (b) holds, then $x \notin B$ so that $x \notin A \cap B$. But, again, we have $x \in (A \cap B)^c$. Finally, if (c) holds, then $x \notin A$ and $x \notin B$, so a fortiori $x \notin A \cap B$, which implies that $x \in (A \cap B)^c$. In other words, each of the three possibilities (a), (b), and (c) leads us to conclude that $x \in (A \cap B)^c$. Thus we have shown that $A^c \cup B^c \subseteq (A \cap B)^c$.

By the Principle of Extensionality, ($\Rightarrow$) and ($\Leftarrow$) together may be taken to show that $(A \cap B)^c = A^c \cup B^c$. Q.E.D.

0.1.4.[hwk] For any finite sets A and B, where we write $A = \{a_1, a_2, \ldots, a_{card(A)}\}$ and $B = \{b_1, b_2, \ldots, b_{card(B)}\}$, we have

$$
\begin{aligned}
card(A \times B) &= card(\{\langle a,b\rangle \mid a \in A \text{ and } b \in B\}) \\
&= card\Big(\bigcup_{a \in A} \{\langle a,b\rangle \mid b \in B\}\Big) \\
&= \{\langle a_1,b_1\rangle, \langle a_1,b_2\rangle, \ldots, \langle a_1, b_{card(B)}\rangle\} \cup \\
&\quad\ \{\langle a_2,b_1\rangle, \langle a_2,b_2\rangle, \ldots, \langle a_2, b_{card(B)}\rangle\} \cup \\
&\quad\ \ldots \qquad\qquad\qquad\qquad\qquad\qquad \cup \\
&\quad\ \{\langle a_{card(A)},b_1\rangle, \langle a_{card(A)},b_2\rangle, \ldots, \langle a_{card(A)}, b_{card(B)}\rangle\} \\
&= \underbrace{card(B) + card(B) + \ldots + card(B)}_{card(A)\text{ times}} \\
&= card(A) \cdot card(B)
\end{aligned}
$$

This is the union of a family of $card(A)$ mutually disjoint sets of ordered pairs, each of cardinality $card(B)$.

0.1.5. We have $i_1 = 3$, $i_2 = 5$, $i_3 = 1$, $i_4 = 2$, and $i_5 = 4$.

Solutions to Exercises for § 0.2

0.2.1. (a) $b, bc, abb, cabc, cccb$
(b) $\varepsilon, a, b, ab, ba$
(c) $aaaa, abab, baba, aaab, bbba$
(d) $\alpha\alpha\alpha\alpha, \alpha\beta\alpha\beta, \beta\alpha\beta\alpha, \alpha\alpha\alpha\beta, \beta\beta\beta\alpha$

(e) *aab,aaab,aaaaab,aaaaaaab*
(f) the languages of (a) and (b) only

0.2.2. (a) ∞
(b) 81
(c) ∞
(d) 26^2=676

0.2.3. (a) *aba* is a member of Σ_1^*, Σ_2^*, and Σ_3^*, and has length 3 in each case.
(b) *bAb* is a member of Σ_3^* only and has length 2.
(c) *cba* is a member of Σ_1^* only and has length 3.
(d) *cab* is a member of Σ_1^* of length 3 and a member of Σ_2^* of length 2.
(e) *caab* is a member of Σ_1^* of length 4 and a member of Σ_2^* of length 3.
(f) *baAb* is a member of Σ_3^* of length 3.

0.2.4. *add, dad, cab, cad, dab*

0.2.5. All are in Σ^* except the third word, *baaaaabaaaab*.

0.2.6. w has n+1 prefixes, n+1 suffixes, and n proper prefixes.

0.2.7. There are just two: *abab* and *baba*.

0.2.8. (a) *abbab*
(b) *ba*
(c) *babba*
(d) *abbab*

0.2.9.[hwk] All are palindromes.

Solutions to Exercises for § 0.3

0.3.1 (b)
$$\begin{aligned} \log_2 xy &= \log_2(2^{\log_2 x} \cdot 2^{\log_2 y}) && \text{by Remark 0.3.4} \\ &= \log_2(2^{\log_2 x + \log_2 y}) \\ &= \log_2 x + \log_2 y && \text{by Remark 0.3.3} \end{aligned}$$

0.3.2 (a) $g(100)=4$ and $g(109)=11$.
(b) g will attain its maximal value of 13 when n is 13 or 27 or 41 and so on.
(c) $h(100)=6$ and $h(109)=12$.
(d) h will attain its maximal value of 24 when n is 167. $h(83)$ will also be maximal.

0.3.3 (a) Function $f(n)$ is surjective but not injective and hence not bijective.
(b) Function $g(n)$ is injective but not surjective and hence not bijective.
(c) Function $h(w)$ is neither injective nor surjective and hence is not bijective.
(d) Function $f(n)$ is both injective and surjective and hence bijective.

0.3.4. hwk

(a) 0, 1, 8, 27, 64, 125, ...
(b) 0, 3, 6, 9, 12, 15, ...
(c) 1, 2, 4, 8, 16, 32, ...
(d) 1, 2, 3, ...

0.3.5 (a) 3
(b) 4
(c) 0
(d) 3
(e) 3
(f) $p(n)$ is not total since it is undefined for $n=1$.

Solutions to Exercises for § 0.4

0.4.1 (a)

```
function fib(n:integer):integer;        {Pascal}
        begin
                if (n=0 or n=1)
                        then fib:=1
                else fib:=fib(n-2)+fib(n-1)
        end;
```

(b)

```
function fib(n:integer):integer;        {Pascal}
        var val1, val2, temp,i:integer;
        begin
                if (n=0 or n=1)
                        then fib:=1
                else begin
                        i:=1;
                        val1:=1;
                        val2:=1;
                        while n<>i do begin
                                temp:=val2;
                                val2:=val1+val2;
                                val1:=temp;
                                i:=i+1
                        end; {while}
                        fib:=val2
                end
        end; {fib}
```

Solutions to Exercises for § 0.5

0.5.1 (a) strictly monotone increasing
(b) not monotone increasing
(c) strictly monotone increasing
(d) monotone increasing but not strictly
(e) not monotone increasing
(f) monotone increasing but not strictly

0.5.2 (a) The addition function is strictly monotone in both its first and second arguments.

(b) The multiplication function is monotone, but not strictly, in both its first and second arguments since, for $m<k$, $m \cdot 0 = k \cdot 0$ and $0 \cdot m = 0 \cdot k$.

(c) Subtraction over the integers is strictly monotone in its first argument but is not monotone in its second argument.

(d) $j(n,m) = n \text{ div } m$ is monotone in its first argument but not in its second.

(e) $k(n,m) = n \bmod m$ is monotone neither in its first argument nor in its second.

(f) $f(n,m) = n^m$ is monotone, but not strictly, in both its first and second arguments. (Note that $5^0 = 6^0 = 1$ and that $1^0 = 1^1 = 1$.)

0.5.3. Only (c) is O(1).

0.5.4. A function f is O(0) if and only if f is constantly 0 for sufficiently large arguments n. Equivalently, f is O(0) if and only if f vanishes for all but a finite number of arguments.

0.5.5 (a) That $f(n)$ is $O(g(n))$ means that there exist natural number n_1 and constant C_1 such that for $n \geq n_1$ we have

$$f(n) \leq C_1 \cdot g(n)$$

Similarly, that $g(n)$ is $O(h(n))$ means that there exist natural number n_2 and constant C_2 such that for $n \geq n_2$ we have

$$g(n) \leq C_2 \cdot h(n)$$

Then for $n \geq max(n_1,n_2)$, we have

$$f(n) \leq C_1 \cdot g(n) \leq C_1 \cdot C_2 \cdot h(n)$$

Letting $C=C_1 \cdot C_2$, we have $f(n) \leq C \cdot h(n)$ for sufficiently large n. That is, $f(n)$ is $O(h(n))$.

(b) By hypothesis, we have, for some constant C, that $f(n) \leq C \cdot g(n)$ for sufficiently large n. But then $c \cdot f(n) \leq c \cdot C \cdot g(n)$ for sufficiently large n and so, setting $C_1 = c \cdot C$, we have $c \cdot f(n) \leq C_1 \cdot g(n)$ for sufficiently large n. By Definition 0.1, this means that function $c \cdot f(n)$ is $O(g(n))$.

(c) By hypothesis, we have, for some constants C_f, n_f, C_g, and n_g, that $f(n) \leq C_f \cdot h(n)$ for $n \geq n_f$ and $g(n) \leq C_g \cdot h(n)$ for $n \geq n_g$. So, letting $n_0 = max(n_f, n_g)$ and $C = C_f + C_g$, we can write

$$\begin{aligned} f(n) + g(n) &\leq C_f \cdot h(n) + C_g \cdot h(n) \text{ for } n \geq n_0 \\ &= [C_f + C_g] \cdot h(n) \text{ for } n \geq n_0 \\ &= C \cdot h(n) \text{ for } n \geq n_0 \end{aligned}$$

which is to say that function $f(n) + g(n)$ is $O(h(n)$.

0.5.6 (a) For cases $n=1$, $n=2$, $n=3$, we have 2^{n-1} taking values 1, 2, and 4, respectively. For $n \geq 4$,

Everything else cancels.

$$n = 2 \cdot \frac{3}{2} \cdot \frac{4}{3} \cdot \ldots \cdot \frac{n-1}{n-2} \cdot \frac{n}{n-1}$$

There are $n-1$ factors on the right here and each is ≤ 2. So $n \leq 2^{n-1}$.

(b) By (a), we have that $n \le 2^{n-1}$ for all $n>0$. It follows that $\log_2 n \le \log_2 2^{n-1} = n-1$ for all $n>0$. Now let x be an arbitrary real number > 0 and let n_x be the least natural number that strictly exceeds x. Then n_x is a natural number > 0 and $n_x-1 \le x < n_x$. So by the preceding remark, we have that $\log_2 n_x \le n_x-1$. Putting this all together, we have

$$\log_2 x < \log_2 n_x \le n_x-1 \le x$$

0.5.7 (a) For $n>0$ we have

$$n! = n\cdot(n-1)\cdot\ldots\cdot 3\cdot 2\cdot 1$$

But each of the n factors here is $\le n$. So we see that the right-hand side here is $< n^n$. That is, for $n>0$, $n! < n^n$ and hence $n!$ is $O(n^n)$.

(b) By (d) of the text, we have that $\log_2 n < \sqrt[3]{n}$ for sufficiently large n. It follows immediately that $n\cdot\log_2 n < n\cdot\sqrt[3]{n}$ for sufficiently large n.

0.5.8 [hwk]

(a) $k=2$
(b) $k=4$
(c) $k=8$

(d) $k=1$. To see this, consider that $f(n) = \lfloor \sqrt[3]{n+2} \rfloor \le \sqrt[3]{n+2} = (n+2)^{\frac{1}{3}} \le n^{\frac{1}{3}} + 2^{\frac{1}{3}}$ and that $n^{\frac{1}{3}} + 2^{\frac{1}{3}} \le n$ for all $n \ge 4$, certainly.

(e) $k=2$

0.5.9 (a) $n!$
(b) n^4
(c) n^n
(d) 0
(e) n^2
(f) $\lfloor \log_2 n \rfloor$
(g) n^4
(h) 1

0.5.10 (a) Polynomial function $p(n)$ is given as

$$a_k n^k + a_{k-1}n^{k-1} + a_{k-2}n^{k-2} + \ldots + a_1 n^1 + a_0 n^0$$

where the coefficients are taken to be integers. In general, some of the coefficients other than a_k may be strictly negative and even exceed a_k in magnitude. But plainly, letting $a^+ = |a_k| + |a_{k-1}| + \ldots + |a_1| + |a_0|$, we can nonetheless write

$$\begin{aligned} p(n) &\le |a_k|n^k + |a_{k-1}|n^k + \ldots + |a_1|n^k + |a_0|n^k \\ &= a^+\cdot n^k \end{aligned}$$

for all $n \ge 1$. Consequently, $p(n)$ is seen to be $O(n^k)$.

We must also see that $p(n)$ is $\Omega(n^k)$ even though n^k may in fact exceed $p(n)$ for almost all n—the (possibly) negative coefficients again. To this end, we set

$$N = 2\cdot(|a_{k-1}| + \ldots + |a_1| + |a_0|)$$

from which it follows that

$$\begin{aligned}
N^k &= (2\cdot(|a_{k-1}|+\ldots+|a_1|+|a_0|))^k \\
&= 2^k\cdot(|a_{k-1}|+\ldots+|a_1|+|a_0|)^k \\
&= (2^{k+1}-2^k)(|a_{k-1}|+\ldots+|a_1|+|a_0|)^k \text{ by } 2^{k+1} = 2^k+2^k \text{ for } k\geq 1 \\
&\leq (a_k\cdot 2^{k+1}-2^k)(|a_{k-1}|+\ldots+|a_1|+|a_0|)^k \text{ by } a_k\geq 1 \\
&= 2(a_k2^k-2^{k-1})(|a_{k-1}|+\ldots+|a_1|+|a_0|)^k \\
&= 2(a_k2^k(|a_{k-1}|+\ldots+|a_1|+|a_0|)^k - 2^{k-1}(|a_{k-1}|+\ldots+|a_1|+|a_0|)^k) \\
&= 2(a_k2^k(|a_{k-1}|+\ldots+|a_1|+|a_0|)^k - (|a_{k-1}|+\ldots+|a_1|+|a_0|)2^{k-1}(|a_{k-1}|+\ldots+|a_1|+|a_0|)^{k-1}) \\
&= 2\cdot\{a_k[2\cdot(|a_{k-1}|+\ldots+|a_1|+|a_0|)]^k - |a_{k-1}|[2\cdot(|a_{k-1}|+\ldots+|a_1|+|a_0|)]^{k-1} - \ldots \\
&\qquad - |a_1|[2\cdot(|a_{k-1}|+\ldots+|a_1|+|a_0|)]^{k-1} - |a_0|[2\cdot(|a_{k-1}|+\ldots+|a_1|+|a_0|)]^{k-1}\} \\
&\leq 2\cdot\{a_k[2\cdot(|a_{k-1}|+\ldots+|a_1|+|a_0|)]^k + a_{k-1}[2\cdot(|a_{k-1}|+\ldots+|a_1|+|a_0|)]^{k-1} + \ldots \\
&\qquad + a_1[2\cdot(|a_{k-1}|+\ldots+|a_1|+|a_0|)]^1 + a_0[2\cdot(|a_{k-1}|+\ldots+|a_1|+|a_0|)]^0\} \\
&= 2\cdot p(2(|a_{k-1}|+\ldots+|a_1|+|a_0|)) \\
&= 2\cdot p(N)
\end{aligned}$$

Further, since the k^{th}-degree term of $p(n)$ dominates, we have that $n^k \leq 2\cdot p(n)$ for $n\geq N$. Thus n^k is $O(p(n))$, from which it follows that $p(n)$ is $\Omega(n^k)$.

(b) We have $n^k\leq n^{k+1}$ for $n\geq 0$, from which it follows that n^k is $O(n^{k+1})$. By transitivity and (a), $p(n)$ is $O(n^{k+1})$ as well. We also must show that $p(n)$ is not $\Omega(n^{k+1})$. Suppose, for the sake of deriving a contradiction, that $p(n)$ is indeed $\Omega(n^{k+1})$. By definition, we have n^{k+1} is $O(p(n))$. By (a) and transitivity again, n^{k+1} is $O(n^k)$. That is, for some constants C and n_0,

$$(*) \qquad n^{k+1} \leq C\cdot n^k$$

for $n\geq n_0$. But letting N be the larger of $C+1$ and n_0, we have $N\geq n_0$ and yet $N^{k+1}=N\cdot N^k > C\cdot N^k$ contradicting $(*)$.

(c) By (a), $p(n)$ is $O(n^k)$. By Theorem 0.1(c), $g(n)=n^k$ is $O(2^n)$ so that, by transitivity, $p(n)$ is $O(2^n)$ as well. This is half of what was to be proved. As for the other half, suppose for the sake of proving a contradiction that $h(n)=2^n$ were $O(p(n))$. Then, by transitivity, $h(n)=2^n$ is $O(n^k)$ by (a) again. By Theorem 0.1(c), $i(n)=n^{k+1}$ is $O(2^n)$ so that, by transitivity, $i(n)=n^{k+1}$ is $O(n^k)$, contradicting (b).

0.5.11. For all real $x>0$ we have that

$$\begin{aligned}
\log_{10}x &= \log_{10} 2^{\log_2 x} && \text{by Remark 0.3.4} \\
&= \log_2 x \cdot \log_{10}2 \\
&= \log_{10}2 \cdot \log_2 x
\end{aligned}$$

From this it follows that, for all natural numbers $n\geq 1$, we have $\lceil \log_{10}n \rceil \leq \lceil \log_{10}2 \rceil \cdot \lceil \log_2 n \rceil$. But $\lceil \log_{10}2 \rceil = 1$ is a natural number constant so that, by Definition 0.5, **function $f(n)=\lceil \log_{10}n \rceil$ is $O(\lceil \log_2 n \rceil)$.** Similarly, we have $\log_2 x = (1/\log_{10}2)\cdot \log_{10}x$ for all real $x>0$ so that, for all natural numbers $n\geq 1$, $\lceil \log_2 n \rceil \leq \lceil 1/\log_{10}2 \rceil \cdot \lceil \log_{10}n \rceil$. But, once again, $\lceil 1/\log_{10}2 \rceil = 4$ is a natural number constant so that **function $f(n)=\lceil \log_2 n \rceil$ is $O(\lceil \log_{10}n \rceil)$.** Putting the two results together, we have that **function $f(n)=\lceil \log_{10}n \rceil$ is $\Theta(\lceil \log_2 n \rceil)$.**

0.5.12. That (i) implies (ii) is trivial. By (i) and letting h be f, we have that $g(n)$ is $O(f(n))$ since obviously $f(n)$ is $O(f(n))$.

As for (ii) implying (i), this is also easy. Assume $g(n)$ is $O(f(n))$ and that $f(n)$ is $O(h(n))$ for some given h. Then, by transitivity, we have that $g(n)$ is $O(h(n))$.

0.5.13. Suppose that function $f(n)$ is $O(g(n))$ for some polynomial $g(n)$. By Definition 0.5, it follows that $f(n) \le C \cdot g(n)$ for $n \ge n_0$, where C and n_0 are natural number constants. But, of course, $C \cdot g(n)$ is itself a polynomial in n.

Solutions to Exercises for § 0.6

0.6.1. **Base case.** $n=1$. This is trivial since $4=1\cdot(3\cdot 1+1)$.

Inductive case. $n=k+1$. We assume as induction hypothesis that

$$4 + 10 + 16 + \ldots + (6k-2) = k(3k+1)$$

All we are doing here is displaying one more term of the series.

We then have

$$\underline{4 + 10 + 16 + \ldots + (6(k+1)-2)} = 4 + 10 + 16 + \ldots + \mathbf{(6k-2)} + (6(k+1)-2)$$

$$= k(3k+1) + (6(k+1)-2) \text{ by induction hypothesis}$$

$$\begin{aligned} &= 3k^2+k+6k+6-2 \\ &= 3k^2+7k+4 \\ &= (k+1)(3k+4) \\ &= (k+1)(3k+3+1) \\ &= \underline{(k+1)(3(k+1)+1)} \end{aligned}$$

These are straightforward arithmetic transformations. Our goal is to produce the right-hand side of our proposition with $n=k+1$.

Q.E.D.

0.6.2. **Base case** $n=0$. We have $\underline{|\sin 0\cdot x|} = |\sin 0| = 0 = \underline{0\cdot|\sin x|}$

Inductive case $n=k+1$. As induction hypothesis, we have

$$|\sin kx| \le k|\sin x|$$

We can write

$$\begin{aligned} |\sin (k+1)\cdot x| &= |\sin (kx+x)| \\ &\le |\sin kx+\sin x| \text{ by (i)} \\ &\le |\sin kx|+|\sin x| \text{ by (ii)} \\ &\le k|\sin x|+|\sin x| \text{ by induction hypothesis and (iii)} \\ &\le (k+1)\cdot|\sin x| \end{aligned}$$

Q.E.D.

0.6.3. hwk **Base case.** $n=0$. This is trivial. Clearly, $0=0\cdot(0+1)/2$.

Inductive case. $n=k+1$. As induction hypothesis, we assume that $0+1+2+\ldots+k = k\cdot(k+1)/2$. Then we can write

$$\begin{aligned} \underline{0+1+2+\ldots+k} + (k+1) &= \underline{k\cdot(k+1)/2} + (k+1) \text{ by induction hypothesis} \\ &= k\cdot(k+1)/2 + (2k+2)/2 \\ &= [k\cdot(k+1)+(2k+2)]/2 \\ &= (k^2+3k+2)/2 \\ &= (k+1)(k+2)/2 \\ &= (k+1)((k+1)+1)/2 \end{aligned}$$

Q.E.D.

0.6.4. **Base case.** $n=0$. This is trivial since $0^2=0\cdot(0+1)(2\cdot0+1)/6$.

Inductive case. $n=k+1$. As induction hypothesis, we have that $0^2+1^2+2^2+\ldots+k^2 = k(k+1)(2k+1)/6$. Then we can write

$$\begin{aligned}\underline{0^2+1^2+2^2+\ldots+k^2} + (k+1)^2 &= \underline{k(k+1)(2k+1)/6} + (k+1)^2 \text{ by induction hypothesis}\\ &= (k^2+k)(2k+1)/6 + (k^2+2k+1)\\ &= (2k^3+3k^2+k)/6 + (6k^2+12k+6)/6\\ &= (2k^3+9k^2+13k+6)/6\\ &= (k^2+3k+2)(2k+3)/6\\ &= (k+1)(k+2)(2k+3)/6\\ &= (k+1)((k+1)+1)(2(k+1)+1)/6\end{aligned}$$

Q.E.D.

0.6.5. **Base case.** $n=0$. This is trivial since $0^3=[0\cdot(0+1)/2]^2$.

Inductive case. $n=k+1$. As induction hypothesis we have that $0^3+1^3+2^3+\ldots+k^3 = [k(k+1)/2]^2$. Then we can write

$$\begin{aligned}\underline{0^3+1^3+2^3+\ldots+k^3} + (k+1)^3 &= \underline{[k(k+1)/2]^2} + (k+1)^3\\ &= [(k^2+k)/2]^2 + (k^3+3k^2+3k+1)\\ &= (k^4+2k^3+k^2)/4 + (4k^3+12k^2+12k+4)/4\\ &= (k^4+6k^3+13k^2+12k+4)/4\\ &= [(k^2+3k+2)/2]^2\\ &= [(k+1)(k+2)/2]^2\\ &= [(k+1)((k+1)+1)/2]^2\end{aligned}$$

Q.E.D.

Solutions to Exercises for § 0.7

0.7.1 (a) *a c d e c d e c b*
(b) *a b d a*
(c) *y x y x y*

0.7.3 (a) 23 = 7+7+3+5+1
(b) Path $v_1\ v_3\ v_6$ has cost 5, which is minimal. However, it is not unique since path $v_1\ v_3\ v_2\ v_4\ v_6$ also has cost 5, as does $v_1\ v_3\ v_5\ v_6$.

0.7.4 hwk (a) Path $w\ x\ v\ w$ is a Hamiltonian circuit. So are $w\ v\ x\ w$ and $v\ x\ w\ v$. There are three others, all of length 3.
(b) There are no Hamiltonian circuits within the undirected graph of Figure 0.7.4(b). Why?
(c) Path $v_1\ v_4\ v_5\ v_6\ v_3\ v_2\ v_1$ is a Hamiltonian circuit, and there are several others.

0.7.5. hwk (a) The Hamilton circuit $v_1\ v_3\ v_5\ v_6\ v_4\ v_2\ v_1$ has cost 11 but is not unique since $v_1\ v_2\ v_4\ v_6\ v_5\ v_3\ v_1$ has cost 11 also.
(b) Assign weight 1 to each and every edge within undirected graph $G=(V,E)$. A shortest Hamiltonian circuit starting at vertex $v\in V$ is then a least-cost Hamiltonian circuit starting at vertex $v\in V$ within the induced weighted graph.

Solutions to Exercises for § 0.8

0.8.1 (a) Satisfiable. Let p be true.
(b) Satisfiable. Let p be false or let q be true.
(c) Satisfiable. Let p be true and let r be false.
(d) Unsatisfiable. Note that each possible assignment of truth values to p and q makes one of the conjuncts here false.

0.8.7[hwk] (a)

p	q	$p{\downarrow}q$
T	T	F
T	F	T
F	T	T
F	F	T

(b) $\neg p \Leftrightarrow_{\text{def.}} p{\downarrow}p$

$p\&q \Leftrightarrow_{\text{def.}} (p{\downarrow}q){\downarrow}(p{\downarrow}q)$ (Note that $p\&q \Leftrightarrow \neg\neg(p\&q) \Leftrightarrow \neg(p{\downarrow}q)$.)

$p\vee q \Leftrightarrow_{\text{def.}} [(p{\downarrow}p){\downarrow}(q{\downarrow}q)]{\downarrow}[(p{\downarrow}p){\downarrow}(q{\downarrow}q)] \downarrow [(p{\downarrow}p){\downarrow}(q{\downarrow}q)]{\downarrow}[(p{\downarrow}p){\downarrow}(q{\downarrow}q)]$
(Note that $p\vee q \Leftrightarrow \neg\neg(p\vee q) \Leftrightarrow \neg(\neg p\&\neg q)$ by DeMorgan.)

Note that $p{\rightarrow}q \Leftrightarrow \neg p\vee q$ and use the fact that $p{\leftrightarrow}q \Leftrightarrow (p{\rightarrow}q)\&(q{\rightarrow}p)$.

Solutions to Exercises for § 0.10

0.10.1. First, by clause (i), each of p, q, r, and s are wffs. Next, by (ii), $(q \rightarrow r)$ is a wff. Also, by (ii) again, $(p \rightarrow (q \rightarrow r))$ is a wff. Finally, since both $(p \rightarrow (q \rightarrow r))$ and s are wffs, so is $((p \rightarrow (q \rightarrow r)) \;\&\; s)$, letting S_1 be $(p \rightarrow (q \rightarrow r))$ and S_2 be s in (ii).

0.10.2. Suppose that $(p \;\&\; (q \rightarrow r)$ is a wff. By clause (iii), $(p \;\&\; (q \rightarrow r)$ must be seen to be a wff by clauses (i) and (ii). But $(p \;\&\; (q \rightarrow r)$ is not a sentence letter. So (i) does not imply that $(p \;\&\; (q \rightarrow r)$ is a wff. As for (ii), $(p \;\&\; (q \rightarrow r)$ is of the form $(S_1 \;\&\; S_2)$ only if, in (ii), we let S_1 be p and S_2 be $(q \rightarrow r$. It follows that $(q \rightarrow r$ must be a wff. But, by (iii) again, $(q \rightarrow r$ is a wff only if (i) and (ii) say so. But neither (i) nor (ii) imply that $(q \rightarrow r$ is a wff. Thus our assumption that $(p \;\&\; (q \rightarrow r)$ is a wff has produced a contradiction, namely, that $(q \rightarrow r$ both is and is not a wff.

Solutions to Exercises for § 0.11

0.11.1 (a) Let S be a countable set of finite sets. Countability means that the elements of S can be enumerated as

$$S_1, S_2, \ldots, S_n \ldots \qquad \text{(i)}$$

Moreover, since each of the S_i here is finite, an enumeration of the sumset of S is obtained from (i) by replacing each S_i with the finite list $s_{i1}, s_{i2}, \ldots, s_{ik_i}$ of its elements to obtain

$$s_{11}, s_{12}, \ldots, s_{1k_1}, s_{21}, s_{22}, \ldots, s_{2k_2}, \ldots, s_{n1}, s_{n2}, \ldots, s_{nk_n}, \ldots \qquad \text{(ii)}$$

Since the sumset of S is enumerated by (ii), it follows that the sumset of S is countable.

(b) Let S be a countable set of countable sets. Countability means that the elements of S can be enumerated as

$$S_1, S_2, \ldots, S_n \ldots \qquad \text{(i)}$$

or, perhaps more perspicuously, as

$$S_1 \qquad \text{(i)}$$

$$\begin{array}{c} S_2 \\ S_3 \\ S_4 \\ \ldots \\ \ldots \\ S_n \\ \ldots \end{array}$$

Moreover, since each of the S_i here is countable, we may enumerate each one as $s_{i1}, s_{i2}, \ldots, s_{ij}, \ldots$ (It will be convenient to assume that every such enumeration is infinite. If S_i is in fact finite, then an infinite enumeration is obtainable by repeating its last element ad infinitum.) Replacing each S_i in (i) with the list $s_{i1}, s_{i2}, \ldots, s_{ij}, \ldots$ of its elements, we obtain a two-dimensional array of the sumset of S.

$$\begin{array}{llllll} s_{11}, & s_{12}, & s_{13}, & s_{14}, & \ldots, & s_{1j}, \ldots \\ s_{21}, & s_{22}, & s_{23}, & s_{24}, & \ldots, & s_{2j}, \ldots \\ s_{31}, & s_{32}, & s_{33}, & s_{34}, & \ldots, & s_{3j}, \ldots \\ s_{41}, & s_{42}, & s_{43}, & s_{44}, & \ldots, & s_{4j}, \ldots \\ \ldots \\ \ldots \\ s_{n1}, & s_{n2}, & s_{n3}, & s_{n4}, & \ldots, & s_{nj}, \ldots \\ \ldots \end{array} \tag{ii}$$

Now (ii) is not itself an enumeration of the sumset of S. (Rather, (ii) is an enumeration of enumerations.) However, we can obtain an enumeration from (ii) by traversing the indicated path.

$$\begin{array}{llllll} s_{11}, & s_{12}, & s_{13}, & s_{14}, & \ldots, & s_{1j}, \ldots \\ s_{21}, & s_{22}, & s_{23}, & s_{24}, & \ldots, & s_{2j}, \ldots \\ s_{31}, & s_{32}, & s_{33}, & s_{34}, & \ldots, & s_{3j}, \ldots \\ s_{41}, & s_{42}, & s_{43}, & s_{44}, & \ldots, & s_{4j}, \ldots \\ \ldots \\ \ldots \\ s_{n1}, & s_{n2}, & s_{n3}, & s_{n4}, & \ldots, & s_{nj}, \ldots \\ \ldots \end{array} \tag{iii}$$

0.11.2.

$$f(n) = \begin{cases} n/2 \text{ if } n \text{ is even} \\ -(n+1)/2 \text{ if } n \text{ is odd} \end{cases}$$

0.11.4$^{\text{hwk}}$(a) Let $\{0,1,2,\ldots,n\}$ and $\{0,1,2,\ldots,m\}$ be proper initial segments of $\mathcal{N}$ with $m<n$. Suppose that there does exist a bijection f from $\{0,1,2,\ldots,n\}$ onto $\{0,1,2,\ldots,m\}$. Clearly, f induces a bijection g of subset $\{0,1,2,\ldots,m\}$ onto itself. But then $f(m+1)$, say, must be identical with $g(k)=f(k)$ for some $k\leq m$. That $f(m+1)=f(k)$ with $k\leq m$ contradicts our assumption that f is a bijective, and hence an injective, function.

(b) Let $\{0,1,2,\ldots,n\}$ be a proper initial segment of $\mathcal{N}$. Suppose that there does exist a bijection f from $\{0,1,2,\ldots,n\}$ onto a proper subset of itself. Let this proper subset be S with $card(S)=m+1<n+1$. We define function g by writing

$g(0)$ = the smallest member of S
$g(1)$ = the smallest member of $S\backslash\{g(0)\}$
$g(2)$ = the smallest member of $S\backslash\{g(0),g(1)\}$
...
$g(m)$ = the smallest member of $S\backslash\{g(0),g(1),\ldots,g(m-1)\}$, i.e., the unique member of $S\backslash\{g(0),g(1),\ldots,g(m-1)\}$

It is easy to see that $(g^{-1} \circ f)^{-1}$ is a bijection of $\{0,1,2,\ldots,n\}$ onto some $\{0,1,2,\ldots,m\}$, which contradicts Exercise 0.11.4(a).

0.11.5$^{\text{hwk}}$ ($\Leftarrow$). Our argument is indirect. Suppose that S is not finite but that, nonetheless, there does exist a bijection f from S onto a proper initial segment $\{0,1,2,\ldots,n\}$ of $\mathcal{N}$. By our definition of a finite set as one that is not infinite, there must exist a bijection g from S onto a proper subset of itself. One can then see that function $f \circ (g \circ f^{-1})$ is a bijection of $\{0,1,2,\ldots,n\}$ onto a proper subset of itself, contradicting Exercise 0.11.4(b). We have thereby shown that, if there exists a bijection from S onto a proper initial segment $\{0,1,2,\ldots,n\}$ of $\mathcal{N}$, then S must be finite.

($\Rightarrow$). Suppose that S is finite. If S happens to be empty, then the totally undefined function $\varnothing$ maps $S=\varnothing$ onto the empty initial segment $\varnothing$ of $\mathcal{N}$. That initial segment is definitely proper. So we are done.

Otherwise, suppose that S is finite and nonempty. By the Axiom of Choice, one is permitted to make a series of choices. First, we select some element x_0 of S and set $g(0)=x_0$. Next, if $S\backslash\{x_0\}$ is nonempty, we select x_1 in $S\backslash\{x_0\}$ and set $g(1)=x_1$. Further, if $S\backslash\{x_0,x_1\}$ is nonempty, we select x_2 in $S\backslash\{x_0,x_1\}$ and set $g(2)=x_2$. We continue in this way and thereby define a unary function g with $Cod(g)=S$. It is easy enough to see that, after k selections for some $k\geq 1$, set S will have been exhausted. (That is, $S\backslash\{x_0,x_1,\ldots,x_{k-1}\}=\varnothing$.) So g^{-1} is the desired 1–1 correspondence between S and a proper initial segment $\{0,1,2,\ldots,k-1\}$ of $\mathcal{N}$.

To see that our selections must eventually deplete S, consider the following. If it were possible to continue the described selection process indefinitely so as to map every natural number onto S—with or without exhausting S in the process—then function g would be defined for every natural number n. But then $[g \circ (f \circ g^{-1})]^{-1}$ would be a bijection of S onto a proper subset of itself, where $f(n)=_{\text{def}} 2n$ is the usual bijection of $\mathcal{N}$ onto the evens. This, in turn, would contradict our assumption that S is finite. Q.E.D.

0.11.6$^{\text{hwk}}$ ($\Rightarrow$). Suppose that S is countable in the sense of Definition 0.10.

- If S happens to be finite, then, by Exercise 0.11.5, there exists a bijection from S onto $\{0,1,2,\ldots,n\}$ for some n.
- If S happens to be countably infinite, then, by Definition 0.9, there exists a bijection f from $\mathcal{N}$ onto S. So f^{-1} is a bijection from S onto $\mathcal{N}$. And since $\mathcal{N}\subseteq\mathcal{N}$, we are done.

($\Leftarrow$). Suppose there exists a bijection f from S onto some $\mathcal{N}_0\subseteq\mathcal{N}$. We define function g by writing

$$\begin{aligned}
&g(0) = \text{the smallest member of } \mathcal{N}_0\\
&g(1) = \text{the smallest member of } \mathcal{N}_0\backslash\{g(0)\}\\
&g(2) = \text{the smallest member of } \mathcal{N}_0\backslash\{g(0),g(1)\}\\
&\ldots
\end{aligned}$$

so that g maps an initial segment of $\mathcal{N}$ onto $\mathcal{N}_0$. That initial segment of $\mathcal{N}$ may or may not be proper.

- If it is proper, then function $g^{-1} \circ f$ is a bijection of S onto a proper initial segment of $\mathcal{N}$, from which it follows, by Exercise 0.11.5, that S is finite. By Definition 0.10, S is countable.
- If the initial segment is not proper but, rather, all of $\mathcal{N}$, then $g^{-1} \circ f$ is a bijection from S onto $\mathcal{N}$ itself. But then $(g^{-1} \circ f)^{-1}$ is a bijection from $\mathcal{N}$ onto S. By Definitions 0.9 and 0.10, set S is countable. Q.E.D.

Solutions to Exercises for § 0.12

0.12.1. Each of (a), (b), and (f) is a CNF.

0.12.3 [hwk]

(a)

$[p \to (q \vee r)] \,\&\, [\neg(\neg r \,\&\, q)]$

$\Leftrightarrow [\neg p \vee (q \vee r)] \,\&\, [\neg(\neg r \,\&\, q)]$ by Implication

$\Leftrightarrow [\neg p \vee q \vee r] \,\&\, [\neg\neg r \vee \neg q]$ by DeMorgan

$\Leftrightarrow [\neg p \vee q \vee r] \,\&\, [r \vee \neg q]$ by Double Negation

(b)

$(p \,\&\, \neg q) \vee (r \,\&\, q \,\&\, p)$

$\Leftrightarrow [(p \,\&\, \neg q) \vee r] \,\&\, [(p \,\&\, \neg q) \vee q] \,\&\, [(p \,\&\, \neg q) \vee p)]$ by Distribution

$\Leftrightarrow [r \vee (p \,\&\, \neg q)] \,\&\, [q \vee (p \,\&\, \neg q)] \,\&\, [p \vee (p \,\&\, \neg q))]$ by Commutativity

$\Leftrightarrow [(r \vee p) \,\&\, (r \vee \neg q)] \,\&\, [(q \vee p) \,\&\, (q \vee \neg q)] \,\&\, [(p \vee p) \,\&\, (p \vee \neg q)]$ by Distribution

Note that the first use of Distribution here is the generalized version of Exercise 0.8.4.

0.12.4. Definition 0.11 is unchanged. Otherwise, we replace every occurrence of $\vee$ with & and conversely throughout Definitions 0.12 and 0.13.

Definition 0.11′ A literal is either a sentence letter or a negated sentence letter. Thus the following are literals: p, $\neg p$, q, $\neg q$, and so on.

Definition 0.12′

(i) A literal by itself is a *DNF clause*.

(ii) If each of $A_1, A_2, \ldots, A_n$ is a *DNF clause*, then so is $(A_1 \,\&\, A_2 \,\&\, \ldots \,\&\, A_n)$.

(iii) Nothing else is a *DNF clause* except what can be obtained from (i) and (ii).

Finally, here is the new, formal definition of disjunctive normal form.

Definition 0.13′

(i) A DNF clause by itself is a *DNF*.

(ii) If $A_1, A_2, \ldots, A_n$ are all *DNF*s, then so is $(A_1 \vee A_2 \vee \ldots \vee A_n)$.

(iii) Nothing else is a *DNF* except what can be obtained from (i) and (ii).

0.12.5. **Subcase (3).** Suppose that S is a conditional. That is, suppose that S is of the form

$$A \to B$$

Again, A and B are of strictly lower complexity than is S, so, by the induction hypothesis, we assume that we have equivalent CNFs A' and B'. For simplicity, let us assume that A' is $(p\vee q)\&(r\vee s)$ and that B' is $(t\vee u)\&(w\vee y)$. So

$S\,(= A \to B) \Leftrightarrow A' \to B'$

$\Leftrightarrow \neg[(p\vee q)\&(r\vee s)] \vee [(t\vee u)\&(w\vee y)]$ by Implication

$\Leftrightarrow [\neg(p\vee q)\vee \neg(r\vee s)] \vee [(t\vee u)\&(w\vee y)]$ by DeMorgan

$\Leftrightarrow [(\neg p\&\neg q)\vee (\neg r\&\neg s)] \vee [(t\vee u)\&(w\vee y)]$ by DeMorgan

$\Leftrightarrow \{[(\neg p\&\neg q)\vee (\neg r\&\neg s)]\vee(t\vee u)\}\&\{[(\neg p\&\neg q)\vee (\neg r\&\neg s)]\vee(w\vee y)\}$ by Distributivity

$\Leftrightarrow \{[((\neg p\&\neg q)\vee\neg r)\&((\neg p\&\neg q)\vee\neg s)]\vee(t\vee u)\}\&$
$\{[((\neg p\&\neg q)\vee\neg r)\& ((\neg p\&\neg q)\vee\neg s)]\vee(w\vee y)\}$ by Distributivity

$\Leftrightarrow \{[(\neg r\vee(\neg p\&\neg q))\&(\neg s\vee(\neg p\&\neg q))]\vee(t\vee u)\}\&$
$\{[(\neg r\vee(\neg p\&\neg q))\& (\neg s\vee(\neg p\&\neg q))]\vee(w\vee y)\}$ by Commutativity

$\Leftrightarrow \{[((\neg r\vee\neg p)\& (\neg r\vee\neg q))\&((\neg s\vee\neg p)\&(\neg s\vee\neg q))]\vee(t\vee u)\}\&$
$\{[((\neg r\vee\neg p)\&(\neg r\vee\neg q))\& ((\neg s\vee\neg p)\&(\neg s\vee\neg q))]\vee(w\vee y)\}$ by Distributivity

$\Leftrightarrow \{(t\vee u)\vee[((\neg r\vee\neg p)\& (\neg r\vee\neg q)\&(\neg s\vee\neg p)\&(\neg s\vee\neg q)]\}\&$
$\{(w\vee y)\vee[((\neg r\vee\neg p)\&(\neg r\vee\neg q))\& ((\neg s\vee\neg p)\&(\neg s\vee\neg q))]\ \}$ by Commutativity

$\Leftrightarrow (t\vee u\vee\neg r\vee\neg p)\&(t\vee u\vee\neg r\vee\neg q)\&(t\vee u\vee\neg s\vee\neg p)\&(t\vee u\vee\neg s\vee\neg q)\&$
$(w\vee y\vee\neg r\vee\neg p)\&(w\vee y\vee\neg r\vee\neg q)\&(w\vee y\vee\neg s\vee\neg p)\&(w\vee y\vee\neg s\vee\neg q)$ by Distributivity

Subcase (4). Suppose that S is a negation. That is, suppose that S is of the form

$$\neg A$$

Again, A is of strictly lower complexity than is S, so by the induction hypothesis, we assume that we have equivalent CNF A'. For simplicity, let us assume that A' is $(p \vee q) \& (r \vee s)$. So

$S\ (= \neg A) \Leftrightarrow \neg A'$
$\Leftrightarrow \neg[(p \vee q) \& (r \vee s)]$
$\Leftrightarrow \neg(p \vee q) \vee \neg(r \vee s)$ — by DeMorgan
$\Leftrightarrow (\neg p \& \neg q) \vee (\neg r \& \neg s)$ — by DeMorgan
$\Leftrightarrow ((\neg p \& \neg q) \vee \neg r) \& ((\neg p \& \neg q) \vee \neg s)$ — by Distributivity
$\Leftrightarrow (\neg r \vee (\neg p \& \neg q)) \& (\neg s \vee (\neg p \& \neg q))$ — by Commutativity
$\Leftrightarrow (\neg r \vee \neg p) \& (\neg r \vee \neg q)) \& (\neg s \vee \neg p) \& (\neg s \vee \neg q))$ — by Distributivity

Subcase (5). Suppose that S is a biconditional. That is, suppose that S is of the form

$$A \leftrightarrow B$$

Again, A and B are of strictly lower complexity than is S, so, by the induction hypothesis, we assume that we have equivalent CNFs A' and B'. So

$$S\ (= A \rightarrow B) \Leftrightarrow A' \leftrightarrow B'$$
$$\Leftrightarrow (A' \rightarrow B')\ \&\ (B' \rightarrow A')$$

By our work in Subcase (4), each of the two conjuncts here is equivalent to a CNF. But then the conjunction of those two CNFs is itself a CNF that is equivalent S. Q.E.D.

Solutions to Exercises for § 0.13

0.13.1[hwk] (a) true
(b) false
(c) false
(d) true
(e) true
(f) false
(g) true

0.13.3 (a)
10 *div* 4 = 2
10 *div* 5 = 2
0 *div* 5 = 0
5 *div* 0 is undefined.

(b)
10 *mod* 4 = 2
10 *mod* 5 = 0
0 *mod* 5 = 0
5 *mod* 0 is undefined.

(c)
4|10 is false.
5|10 is true.
5|0 is true.
0|5 is false.

Solutions to Exercises for § 0.14

0.14.1 (a) This exercise is trivial, which is our point in bringing it to the reader's attention. Namely, if, for every natural number $C \geq 1$, there exists natural number n_0 with $f(n) \leq \frac{1}{C} g(n)$ for $n \geq n_0$, then letting $C=1$ gives $f(n) \leq 1 \cdot g(n)$ for $n \geq n_0$, which is to say that $f(n)$ is $O(g(n))$.

(b) If $f(n)$ and $g(n)$ differ by a constant, then $f(n)$ will be $O(g(n))$ but $f(n)$ will not be $o(g(n))$, for example, if $g(n)=f(n)+10$, say. Similarly, for any two polynomial functions $f(n)$ and $g(n)$ of like degree and with positive leading coefficients, we have that $f(n)$ is $O(g(n))$ and, in fact, $\Theta(g(n))$. But $f(n)$ will not be $o(g(n))$ nor will $g(n)$ be $o(f(n))$.

0.14.2.[hwk] Suppose that the class of total, unary functions were countable. In that case, we may assume an enumeration

$$f_0, f_1, f_2, \ldots$$

Set $f^*(n)=f_n(n)+1$ and note that f^* is unary and total. Now derive a contradiction.

Solutions to Selected Exercises for Chapter 1

Solutions to Exercises for § 1.1

1.1.2.[hwk] (a)

i	*quot*	*b*[*i*]
—	783	—
0	391	1
1	195	1
2	97	1
3	48	1
4	24	0
5	12	0
6	6	0
7	3	0
8	1	1
9	0	1

The binary equivalent of 783_{10} is read in the right-hand column from bottom to top.

(b) The algorithm is determinate in the sense that, at any point in carrying it out, it is always clear wha the next step of the algorithm is. In addition, it is determinate in the sense that it terminates after a finit number of steps, having produced a certain definite output. Finally, this output is unambiguous in th sense that there is no question as to how it is to be interpreted.

1.1.4. Direct apprehension is suggested by the speed with which the brothers classify numbers—as either prime (composite apparently. Given our own slowness in carrying out the familiar algorithm for deciding whether a give number is prime, we want to say that the brothers simply *see* that a given natural number has one or the oth property. On the other hand, the fact that classification of a 13-digit number, although yet fast, takes the considerably longer than classification of a 10-digit number, say, suggests computation involving some algorith But what could that algorithm be, given that the brothers seem to possess no concept of division?

One attempt at an explanation would involve positing, in the case of the twins, some unusual, high specific neural representation of number that would be especially advantageous for discerning primality—a apparently not much else. Then the brothers' activity could be regarded as "symbolic" manipulation of the