Fundamentals of theory of computation 2

1st lecture

lecturer: Tichler Krisztián
ktichler@inf.elte.hu

Brief syllabus

- Two models of mathematical logic. Propositional logic and first order logic.
- Asympthotical properties of functions.
- Turing machines (TM) as a model of algorithms
- Multitape TM, nondeterministic TM, counting TM
- Cardinality of infinite sets.
- Algorithmic language classes RE,R and their properties
- Undecidable problems, reduction of a language (problem) to another language
- Algorithmic vs. Chomsky language classes.
- Basic concepts of complexity theory, time complexity classes, NP-completeness, polynomial time reduction
- NP-complete problems (SAT, graph problems, ...)
- Offline TM, space complexity.

Two models of mathematical logic

Basic concept: atomic **proposition** (or **statement**): either **true** or **false** independent of context. **true** and **false** are called **truth values**. Compound propositions consist of atomic propositions and linguistic connectors corresponding to Boolean operators.

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Internal structure of atomic propositions are not considered.
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First order logic has more expressive power but a less simple model.

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Formal syntax

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disjunction	$\vee$		

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Note: sometimes further Boolean operators are used as well, such as equivalence ($\leftrightarrow$), exclusive or ($\oplus$), nor ($\downarrow$), nand ($\uparrow$).

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The tree representation of formulas

Definition

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Definition

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(Formulas, that are atoms have no principal operator).

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The string representation of formulas

Input: a formula F given by its tree representation

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Algorithm $\textsc{Inorder}(F)$

---

if F is a leaf **then**
    write its label
    return
let F_1 and F_2 be the left and right subtrees of F
write a left parenthesis '('
if the label of the root is not $\neg$ **then**
    $\textsc{Inorder}(F_1)$
write the label of the root of F
$\textsc{Inorder}(F_2)$
write a right parenthesis ')'

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The need of parenthesis

Example:

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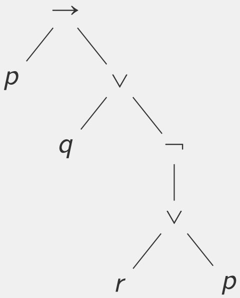

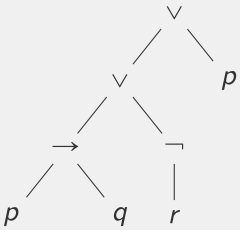

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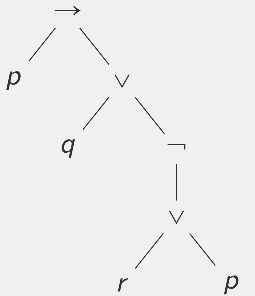
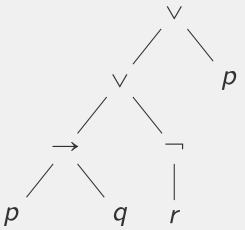


String representation:

$(p \rightarrow (q \vee (\neg(r \vee p))))$



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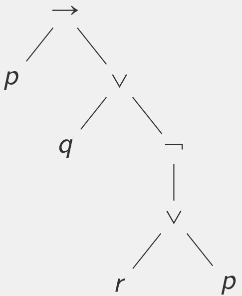

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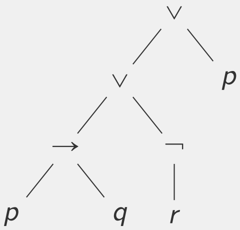

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Without parenthesis there would be an ambiguity as both string representation were $p \rightarrow q \vee \neg r \vee p$.

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Analogy: arithmetic expression, e.g., $3 + 8 \cdot 7$.

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Let A be a formula and let P_A be the set of atoms appearing in A. An **interpretation** for A is a total function $I_A : P_A \rightarrow \{T, F\}$ that assigns one of the truth values T or F to every atom in P_A.

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$A = p \rightarrow q \vee \neg(r \vee p)$. $P_A = \{p, q, r\}$. One possibility is the following: $I(p) = T,\ I(q) = F,\ I(r) = F$.

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$A = p \to q \vee \neg(r \vee p)$. $P_A = \{p, q, r\}$. One possibility is the following: $I(p) = T,\ I(q) = F,\ I(r) = F$.
There are 8 possibilities for I in this case.

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$v_I(\neg A) = F$	if $v_I(A) = T$
$v_I(A_1 \wedge A_2) = T$	if $v_I(A_1) = T$ and $v_I(A_2) = T$
$v_I(A_1 \wedge A_2) = F$	in the other three cases

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$v(q) = F$, $v(\neg(r \vee p)) = F$, so $v(q \vee \neg(r \vee p)) = F$

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$v(p) = T$, $v(q \vee \neg(r \vee p)) = F$, so $v(A) = F$.

Semantics of propositional logic

Truth table

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Definition

A **truth table** for a formula A is a table with $n+1$ columns and 2^n rows, where $n = |P_A|$. There is a column for each atom in P_A , plus a column for the formula A. The first n columns specify the interpretation I that maps atoms in P_A to $\{T, F\}$. The last column shows $v_I(A)$, the truth value of A for the interpretation I.

Semantics of propositional logic

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p	q	r	$r \vee p$	$\neg(r \vee p)$	$q \vee \neg(r \vee p)$	$p \to q \vee \neg(r \vee p)$
T	T	T	T	F	T	T
T	T	F	T	F	T	T
T	F	T	T	F	F	F
T	F	F	T	F	F	F
F	T	T	T	F	T	T
F	T	F	F	T	T	T
F	F	T	T	F	F	T
F	F	F	F	T	T	T

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- A is **falsifiable**, denoted $\not\models A$, iff it is not valid, that is, if $v_I(A) = F$ for some interpretation I.

Semantics of propositional logic

Semantic properties of formulas, examples

Let A_1, A_2 be formulas.

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If $v_I(A_1) = v_I(A_2)$ for all interpretations I , then A_1 is **logically equivalent** to A_2 , denoted $A_1 \equiv A_2$.

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- $p \vee q$ and $q \vee p$ are logically equivalent formulas.

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- $A \vee \bot \equiv A$ and $A \wedge \top \equiv A$ (identity laws),
- $A \vee \neg A \equiv \top$ and $A \wedge \neg A \equiv \bot$,
- $\neg\neg A \equiv A$ (double negation law),
- $A \vee A \equiv A$ and $A \wedge A \equiv A$ (idempotent laws),
- $A \rightarrow B \equiv \neg A \vee B$,
- $A \rightarrow B \equiv \neg B \rightarrow \neg A$ (law of transposition),

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Substitution

Definition

Let A be a subformula of B and let A' be any formula. $B\{A \leftarrow A'\}$, the **substitution** of A for A' in B, is the formula obtained by replacing all occurrences of the subtree for A in B by A'.

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Theorem

Let A be a subformula of B and let A' be a formula such that $A \equiv A'$. Then $B \equiv B\{A \leftarrow A'\}$.

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Proof:
Let $B' = B\{A \leftarrow A'\}$ and I be an arbitrary interpretation. Since $A \equiv A'$ we have $v(A) = v(A')$. We have to show that $v(B) = v(B')$.

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In B_1 , the depth of A is less than d. By the inductive hypothesis, $v(B_1) = v(B_1') = v(B_1\{A \leftarrow A'\})$, and similarly $v(B_2) = v(B_2') = v(B_2\{A \leftarrow A'\})$.

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By the definition of v, $v(B)$ depends only on $v(B_1)$ and $v(B_2)$, so $v(B) = v(B')$ proving the theorem.

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$\equiv \neg B \vee \top \;\equiv \top$.

So $A \to (B \to A)$ is a valid formula.

Semantic properties of a set of formulas

Satisfiability

Definition

A set of formulas $U = \{A_1, \ldots\}$ is (simultaneously) **satisfiable** iff there exists an interpretation I such that $v_I(A_i) = T$ for all i. The satisfying interpretation is a **model** of U, denoted $I \models U$. U is **unsatisfiable** iff for every interpretation I, there exists an i such that $v_I(A_i) = F$.

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Which of the three is satisfiable?
$I \models U_1$, where $I(p) = I(q) = I(r) = T$.

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A set of formulas $U = \{A_1, \ldots\}$ is (simultaneously) **satisfiable** iff there exists an interpretation I such that $v_I(A_i) = T$ for all i. The satisfying interpretation is a **model** of U, denoted $I \models U$. U is **unsatisfiable** iff for every interpretation I, there exists an i such that $v_I(A_i) = F$.

Example:

$U_1 = \{p, \neg p \vee q, q \wedge r\}$,
$U_2 = \{p, \neg p \vee q, \neg p\}$.
$U_3 = \{p, \neg p \vee q, \neg q\}$.
Which of the three is satisfiable?
$I \models U_1$, where $I(p) = I(q) = I(r) = T$.
p and $\neg p$ can not be T simultaneously, so for all interpretations I:
$I \not\models U_2$.

Semantic properties of a set of formulas

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One can check that for all interpretations I: $I \not\models U_3$.

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Logical consequence

Proposition

If U is satisfiable, then U' is satisfiable, too, for all $U' \subseteq U$.

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Let U be a set of formulas and A a formula. A is a **logical consequence** of U, denoted $U \models A$, iff every model of U is a model for A.

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Note, that A is not valid, since it is not true in the interpretation I where $I(p) = F, I(q) = T, I(r) = T$.

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Definition

A **literal** is either an atom or a negation of an atom.

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For every formula A in propositional logic there is a logically equivalent formula in CNF.

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Proof Let $P_A = \{p_1, \ldots, p_n\}$ and let $A^F = \{I \mid v_I(A) = F\}$ be the set of those interpretation evaluating A for false.

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For every $I \in A^F$

$$B_I = \bigvee_{x:I(x)=T} \neg x \vee \bigvee_{x:I(x)=F} x$$

is a disjunction of literals with the property $B_I^F = \{I\}$.

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$B = \bigwedge_{I \in A^F} B_I$ has the property

$$B^F = \bigcup_{I \in A^F} B_I^F = \bigcup_{I \in A^F} \{I\} = A^F,$$

i.e., $B \equiv A$ and B is in CNF proving the theorem.

Conjunctive normal form (CNF)

Example: Let $A = (p \to q) \to r$. Then the truth table for A is the following

p	q	r	A
T	T	T	T
T	T	F	F
T	F	T	T
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According the previous proof

$$(\neg p \vee \neg q \vee r) \wedge (p \vee \neg q \vee r) \wedge (p \vee q \vee r)$$

is a CNF equivalent with A.

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Heuristic algorithm (can be made formal):
1st step: Eliminate implications by the law $A \to B \equiv \neg A \vee B$

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3rd step: Now, the formula is in the form of literals connected by $\wedge$'s and $\vee$'s. Use distributive laws,
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&\equiv (\neg p \wedge q) \vee (\neg p \vee q) \\
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- The formula that is the empty set of clauses is denoted by $\varnothing$.

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Corollary

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A clause is called **trivial** if it contains a pair of clashing literals.

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Proposition

Let S be a set of clauses and let $C \in S$ be a trivial clause. Then $S - \{C\}$ is logically equivalent to S.

Removing trivial clauses

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Let S be a set of clauses and let $C \in S$ be a trivial clause. Then $S - \{C\}$ is logically equivalent to S.

True, because of $A \vee \top \equiv \top$ and $A \wedge \top \equiv A$.

Removing trivial clauses

Corollary

Every formula in propositional logic can be transformed into an logically equivalent formula in clausal form.

multiple occurrences of literals and clauses $\Rightarrow$ single occurence
equivalence due to idempotent laws ($A \vee A \equiv A, \ A \wedge A \equiv A$)

Example: CNF:
$(p \vee r) \wedge (\neg q \vee \neg p \vee q) \wedge (p \vee \neg p \vee q \vee p \vee \neg p) \wedge (r \vee p)$

Clausal form: $\{\{p, r\}, \{\neg q, \neg p, q\}, \{p, \neg p, q\}\}$

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$\mathrm{cf}(A) = \{pr, \bar{q}\bar{p}q, p\bar{p}q\}$ with the shorter notation.

Logical consequence in propositional logic

Theorem

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- $\{A_1, \dots A_n\} \models B$

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In order to decide logical consequence it is enough to decide whether a set of clauses is unsatisfiable.

Resolution rule

Definition and example

If ℓ is a literal, let ℓ^c denote its complementary pair.

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Let C_1, C_2 be clauses such that $\ell \in C_1, \ell^c \in C_2$. The clauses C_1, C_2 are said to be **clashing clauses** and to **clash on the complementary pair of literals** ℓ, ℓ^c . C, the **resolvent** of C_1 and C_2 , is the clause:

$$\mathrm{Res}(C_1, C_2) = (C_1 - \{l\}) \cup (C_2 - \{l^c\}).$$

C_1 and C_2 are the parent clauses of C.

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Recall that a clause is a set so duplicated literals are removed when taking the union.

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Resolution is only performed if the pair of clauses clash on exactly one pair of complementary literals due to the following.

Proposition

If two clauses clash on more than one literal, their resolvent is a trivial clause.

Remark: It is not strictly incorrect to perform resolution on such clauses, but since trivial clauses contribute nothing to the satisfiability or unsatisfiability of a set of clauses, we agree to delete them from any set of clauses and not to perform resolution on clauses with two clashing pairs of literals.

Resolution procedure

Lemma

Let I be an interpretation. If $I \models \{C_1, C_2\}$ then $I \models \mathrm{Res}(C_1, C_2)$. If $I \models \mathrm{Res}(C_1, C_2)$, then I can be extended to $\hat{I}$, such that $\hat{I} \models \{C_1, C_2\}$.

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Corollary

Let S be a set of clauses and let $C_1, C_2 \in S$ be a pair of clashing clauses. Then S is satisfiable if and only if $S \cup \{\mathrm{Res}(C_1, C_2)\}$ is satisfiable.

If a set of clauses S contains $\square$ then S is unsatisfiable.

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Algorithm

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Algorithm RESOLUTION PROCEDURE(S)

while there is an unmarked pair of $\binom{S}{2}$ **do**
    choose an unmarked pair $\{C_1, C_2\}$ of $\binom{S}{2}$ and mark it
    if $\{C_1, C_2\}$ is a clashing pair of clauses **then**
        $C \leftarrow \mathrm{Res}(C_1, C_2)$
        if $C = \square$ **then**
            return 'S is unsatisfiable'
        else
            if C is not the trivial clause **then**
                $S \leftarrow S \cup \{C\}$
return 'S is satisfiable'

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(5)	$\bar{p}\bar{q}$	Res((3),(4))
(6)	$\bar{p}$	Res((5),(2))
(7)	$\square$	Res((6),(1))

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Theorem

The resolution procedure always halts and it is sound and complete.

(without proof)