

Neural Networks

Variational Autoencoders

(based on slides from Dan Schwarts and Tom Manzini)

Recap

- Neural networks are universal approximators
- They can model
 - Boolean functions
 - Classification functions
 - Regressions
- They can be
 - Feature extractors
 - Classifiers
 - Predictors

A new problem

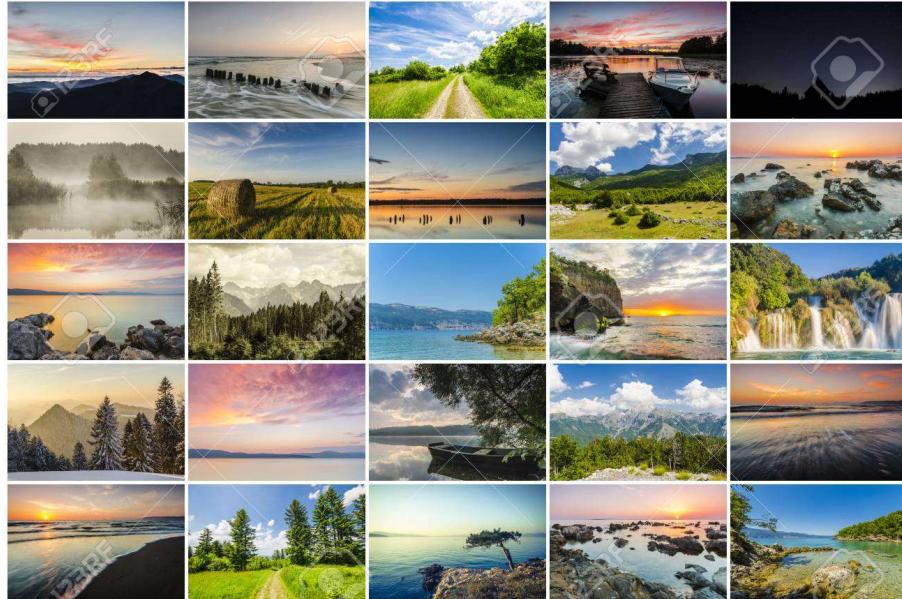
- All of the previous cases considered neural networks that are functions
 - They can *operate on*, or *process* a given input data
 - They can learn to perform these tasks from data
- Can networks also *generate* data?
 - And learn to do so from examples
 - Topic for next series of lectures

A new problem



- From a large collection of images of faces, can a network learn to *generate* new portrait
 - Generate samples from the distribution of “face” images
 - How do we even characterize this distribution?

A new problem



- From a large collection of landscapes, can a network learn to *generate* new landscape pictures
 - Generate samples from the distribution of “landscape” images
 - How do we even characterize this distribution?

Neural nets as generative models

- We've seen how neural nets can perform classification or regression
 - MLPs, CNNs, RNNs..
- Next step: NNs as generic generative models
 - Model the distribution of *any* data
 - Such that we can draw samples from it

But first...



The story of generative models

- What are generative models
- How to estimate them
 - *Expectation maximization*



What is a generative model

- A model for the probability distribution of a data x
 - E.g. a multinomial, Gaussian etc.
- Computational equivalent: a model that can be used to “generate” data with a distribution similar to the given data x
 - Typical setting: a box that takes in random seeds and outputs random samples like x



- Question: how do we generate the random seeds...

It's turtles all the way down (kinda)...



Some “simple” generative models

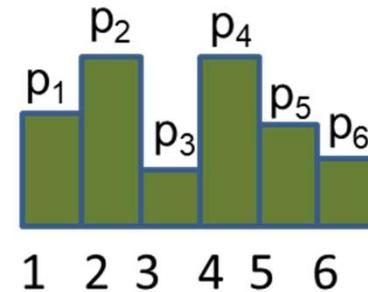
- The category PMF

$$P(x = v) \equiv P(v)$$

- For discrete data
 - v belongs to a discrete set
- Can be expressed as a table of probabilities if the set of possible vs is finite
- Else, requires a parametric form, e.g. Poisson

$$P(x = k) = \frac{\lambda^k e^{-\lambda}}{k!} \text{ for } k \geq 0$$

- λ is the Poisson parameter

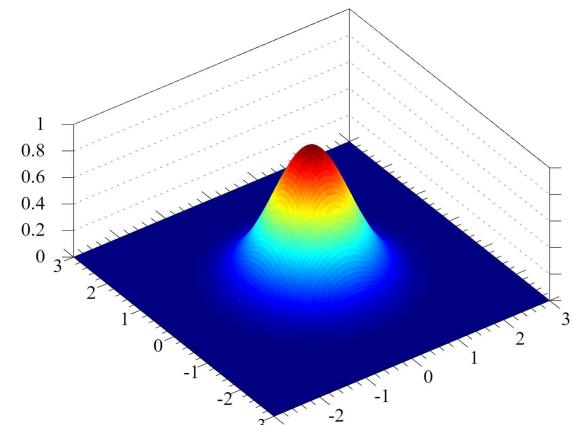
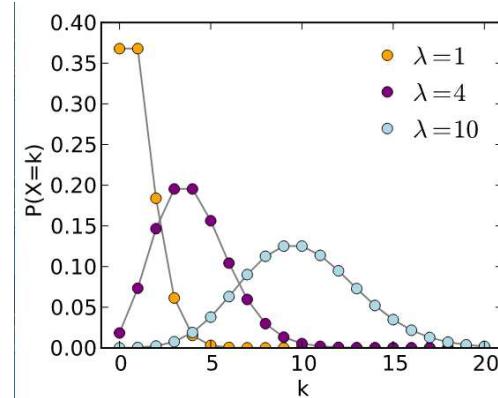


- The Gaussian PDF

$$P(x = v)$$

$$= \frac{1}{\sqrt{2\pi|\Sigma|^D}} \exp(-0.5(x - \mu)^T \Sigma^{-1}(x - \mu))$$

- For continuous-valued data
- μ is the mean of the distribution
- Σ is the Covariance matrix



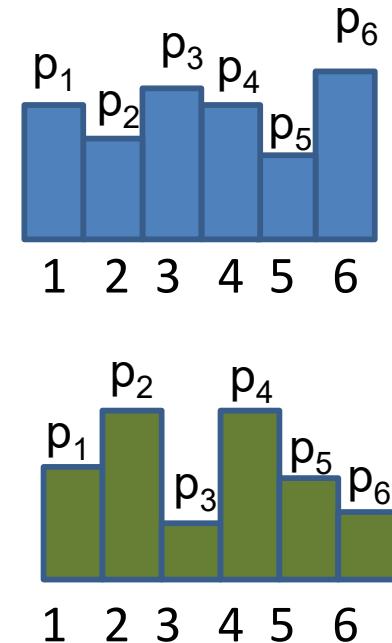
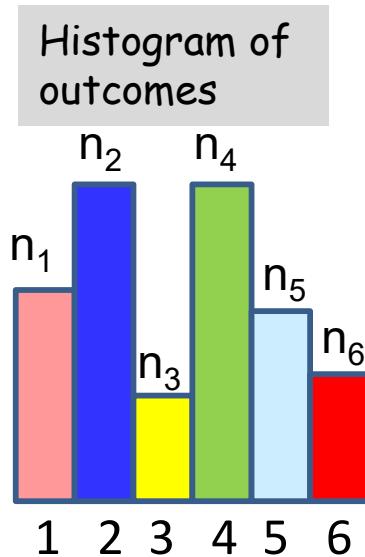
Learning a generative model for data

- You are given some set of observed data $X = \{x\}$.
- You choose a model $P(x; \theta)$ for the distribution of x
 - θ are the parameters of the model
- Estimate the θ such that $P(x; \theta)$ best “fits” the observations $X = \{x\}$
 - Hoping it will also represent data outside the training set.

An example: Multinomials

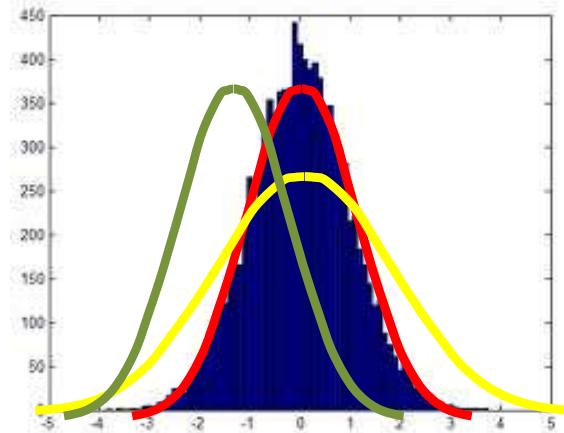
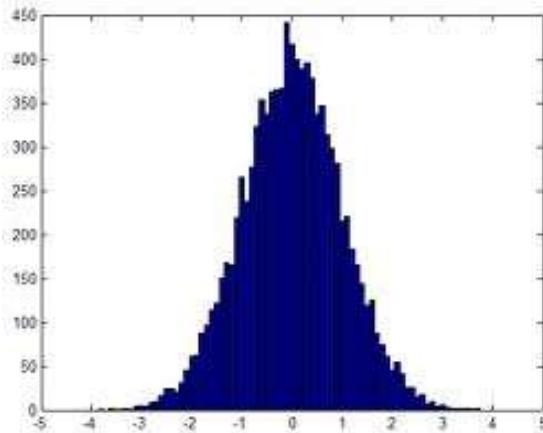


6 3 1 5 4 1 2 4 ...



- A dice roller rolls dice and you plot the histogram of outcomes
 - Shown to right
- The distribution is a multinomial
 - Parameters to be learned: $p_1, p_2, p_3, p_4, p_5, p_6$
- Which of the two probability distributions shown to the right is more likely to be the distribution for the dice?
 - Why?

An example



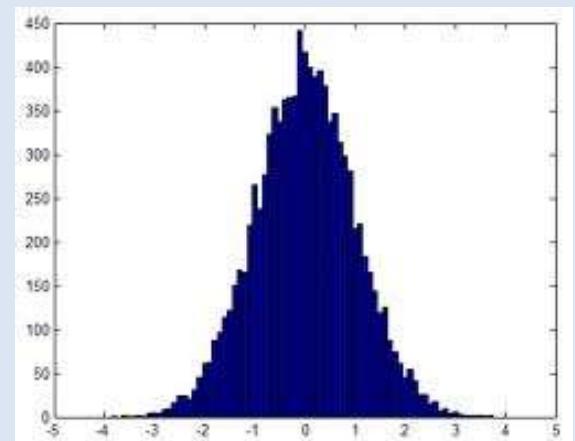
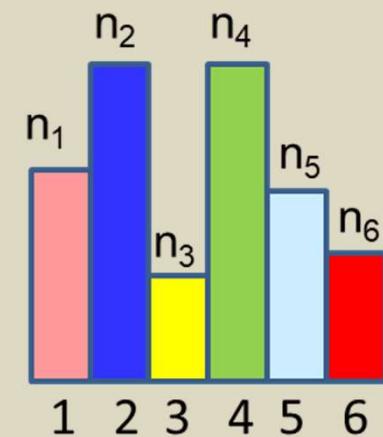
- The left figure shows the histogram of a collection of observations
- We decide to model the distribution as Gaussian
 - Parameters: Mean μ and variance σ^2
- Which of the three Gaussians shown in the right figure is most likely to be the actual PDF of the RV?
 - Why?

Defining “Best Fit”: Maximum likelihood

- The data are generated by draws from the distribution
 - I.e. the generating process draws from the distribution
- Assumption: The world is a boring place
 - The data you have observed are very typical of the process
- Consequent assumption: The distribution has a high probability of generating the observed data
 - Not necessarily true
- Select the distribution that has the *highest* probability of generating the data
 - Should assign lower probability to less frequent observations and vice versa

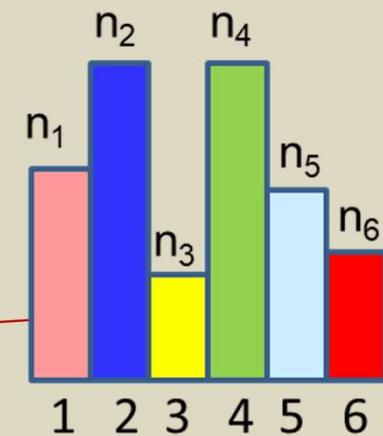
Maximum likelihood

- The maximum likelihood principle:
 - $\underset{\theta}{\operatorname{argmax}} P(X; \theta) = \underset{\theta}{\operatorname{argmax}} \log(P(X; \theta))$

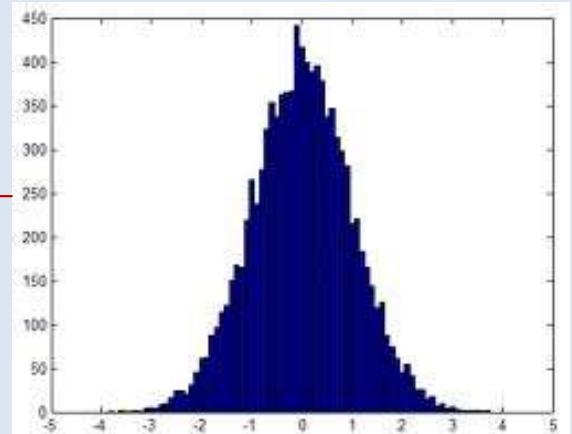


Maximum likelihood

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 - $\underset{\theta}{\operatorname{argmax}} P(X; \theta) = \underset{\theta}{\operatorname{argmax}} \log(P(X; \theta))$
- For the histogram
 - $\underset{\{p_1, p_2, p_3, p_4, p_5, p_6\}}{\operatorname{argmax}} \log(\prod_{x \in X} P(x))$



- For the Gaussian
 - $\underset{\mu, \sigma^2}{\operatorname{argmax}} \log(\prod_{x \in X} P(x))$

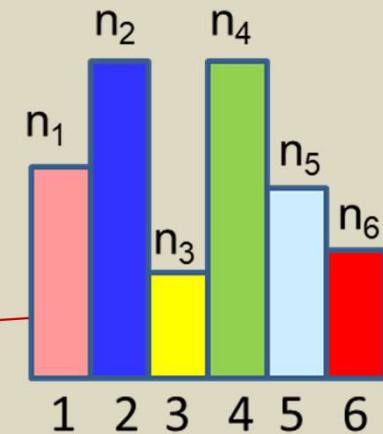


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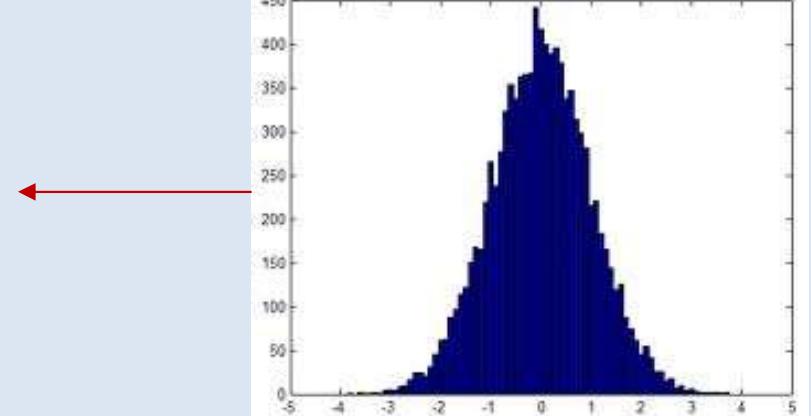
- $\underset{\{p_1, p_2, p_3, p_4, p_5, p_6\}}{\operatorname{argmax}} \log(\prod_{x \in X} P(x))$



Can be grouped by value (every instance of i has the same probability)

- For the Gaussian

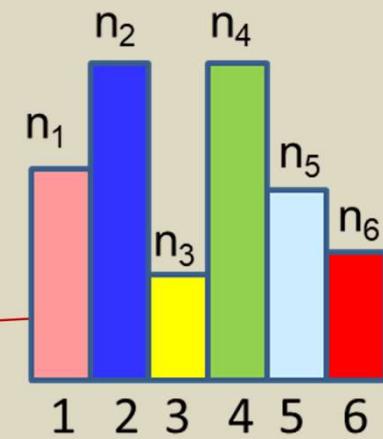
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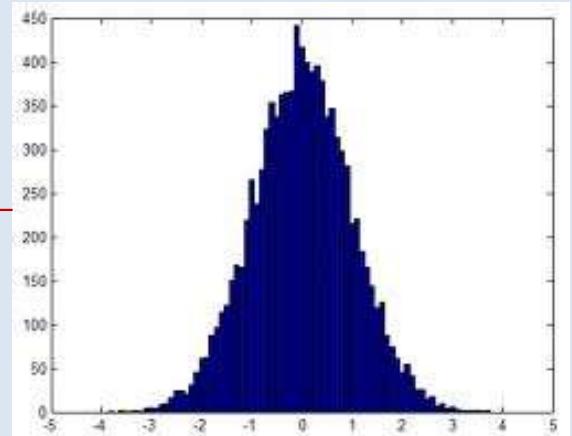
This probability is a Gaussian

Maximum likelihood

- The maximum likelihood principle:
 - $\underset{\theta}{\operatorname{argmax}} P(X; \theta) = \underset{\theta}{\operatorname{argmax}} \log(P(X; \theta))$
- For the histogram
 - $\underset{\{p_1, p_2, p_3, p_4, p_5, p_6\}}{\operatorname{argmax}} \log(\prod_i p_i^{n_i})$



- For the Gaussian
 - $\underset{\mu, \sigma^2}{\operatorname{argmax}} \log(\prod_{x \in X} \text{Gaussian}(x; \mu, \sigma^2))$

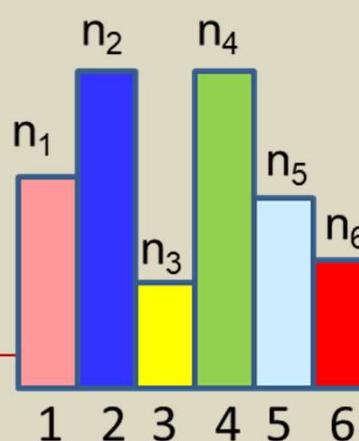


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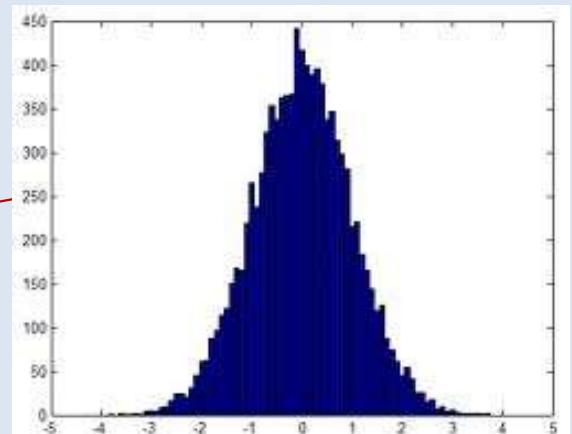
- $\operatorname{argmax}_{\theta} P(X; \theta) = \operatorname{argmax}_{\theta} \log(P(X; \theta))$

- For the histogram

- $\operatorname{argmax}_{\{p_1, p_2, p_3, p_4, p_5, p_6\}} \sum_i n_i \log(p_i)$ 
 $\Rightarrow p_i = \frac{n_i}{N}$ (N is the total number of observations)

- For the Gaussian

- $\operatorname{argmax}_{\mu, \sigma^2} \sum_{x \in X} \left(-0.5 \log(2\pi\sigma^2) - \frac{(x-\mu)^2}{2\sigma^2} \right)$
 $\Rightarrow \mu = \frac{1}{N} \sum_{x \in X} x ; \quad \sigma^2 = \frac{1}{N} \sum_{x \in X} (x - \mu)^2$



Poll 1

Maximum-likelihood estimation of probability distributions is based on the theory that the world is a terribly boring place

- True
- False

Maximum-likelihood estimation estimates the values of the parameters of a probability distribution such that they maximize the probability of the training data

- True
- False

Poll 1

Maximum-likelihood estimation of probability distributions is based on the theory that the world is a terribly boring place

- **True**
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- False

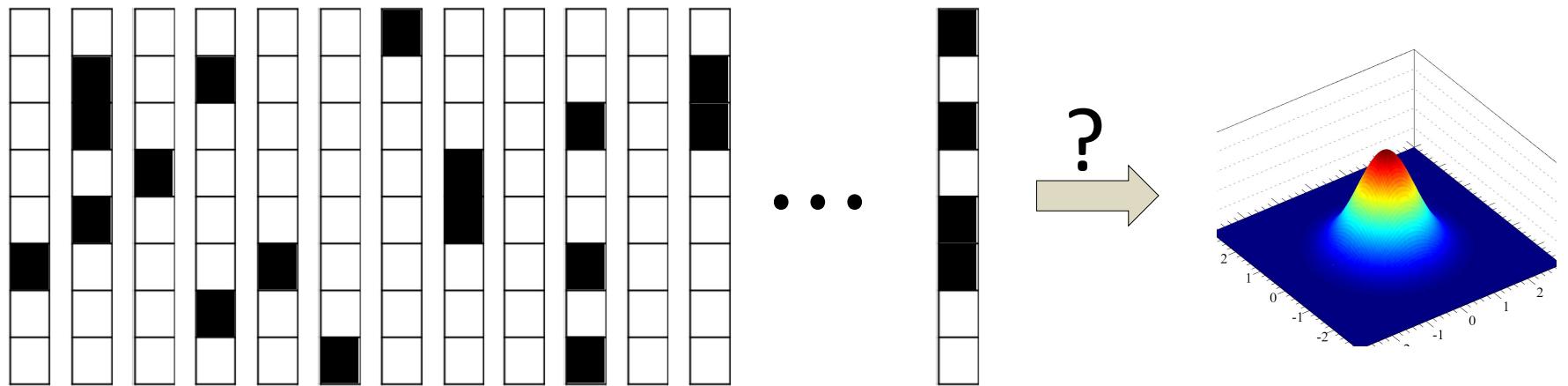
Maximum Likelihood Estimation

- Sometimes the data provided may be incomplete
 - May be insufficient to write out the complete log probability
 - Insufficient to estimate your model parameters directly
- This could be because the data themselves have missing components
 - E.g. Data vectors have some missing components
- Or because of the structure of the model
 - Mixture models, multi-stage Generative models

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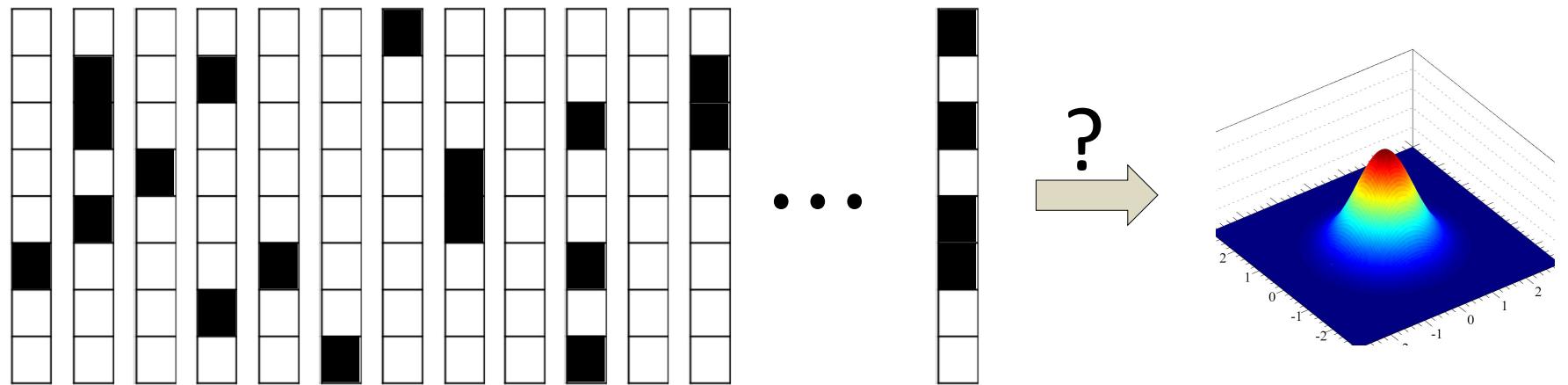
Examples of incomplete data: missing data



Blacked-out components are missing from data

- Objective: Estimate a Gaussian distribution from a collection of vectors
- Problem: Several of the vector components are missing
- Must estimate the mean and covariance of the Gaussian with these incomplete data
 - What would be a good way of doing this?

Maximum likelihood estimation with incomplete data



Blacked-out components are missing from data

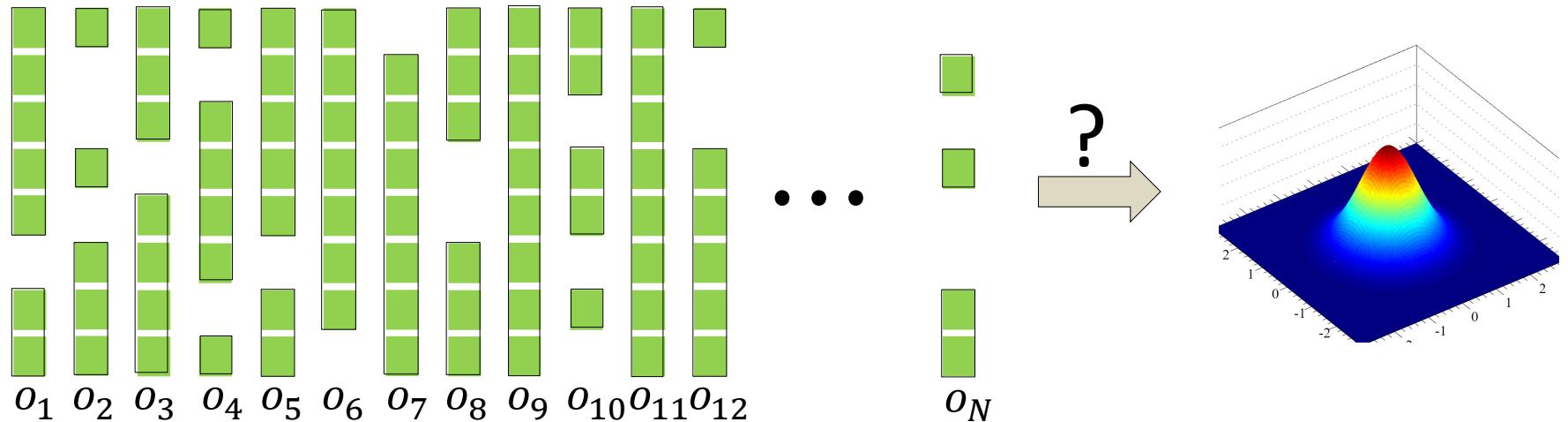
- Original problem: Estimate the Gaussian given a collection $X = \{x\}$ of *complete* vectors

$$\operatorname{argmax}_{\mu, \sigma^2} \log(P(X)) \text{ where } X \text{ is the entire data}$$

$$= \operatorname{argmax}_{\mu, \sigma^2} \sum_{x \in X} \log P(x) \text{ where } P() \text{ is a Gaussian}$$

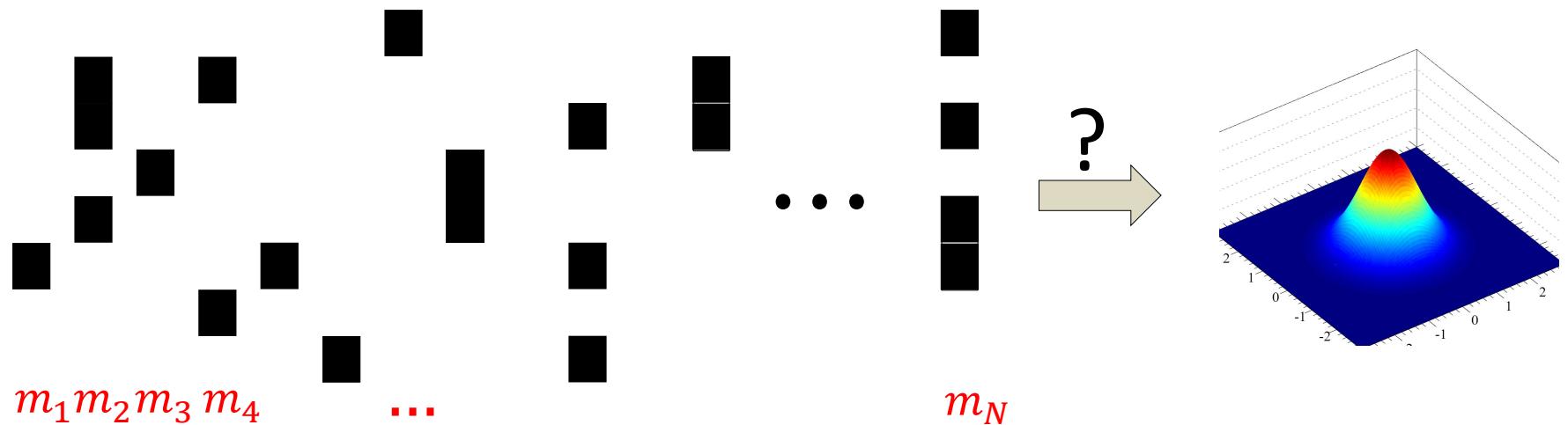
- Unfortunately, many components of each vector are missing in our data

Maximum likelihood estimation with incomplete data



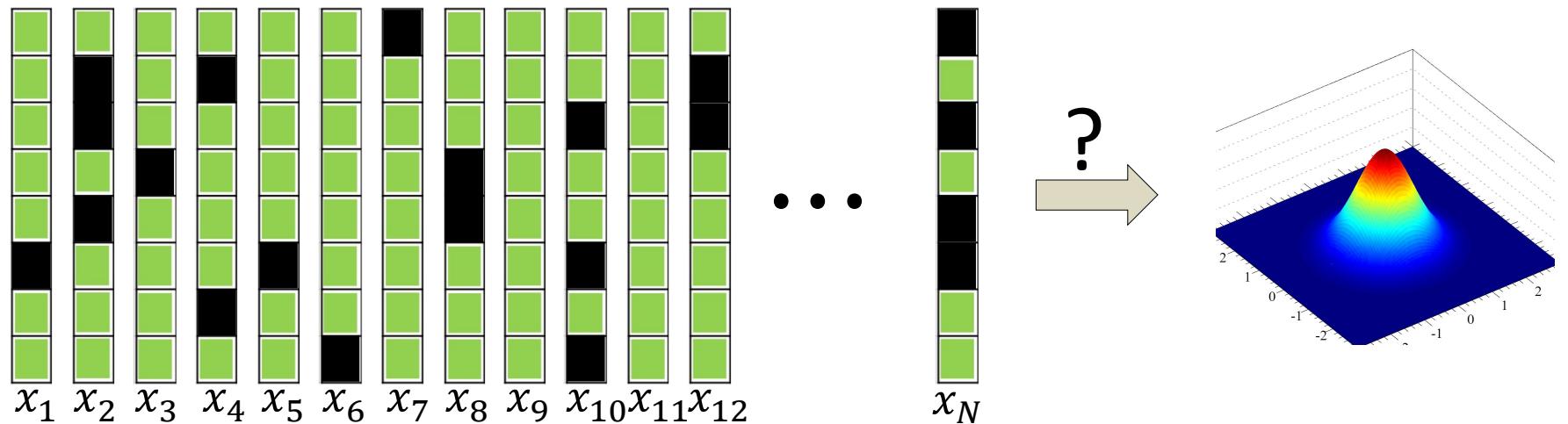
- These are the actual data we have: A set $O = \{o_1, \dots, o_N\}$ of *incomplete* vectors
 - Comprising only the *observed* components of the data

Maximum likelihood estimation with incomplete data



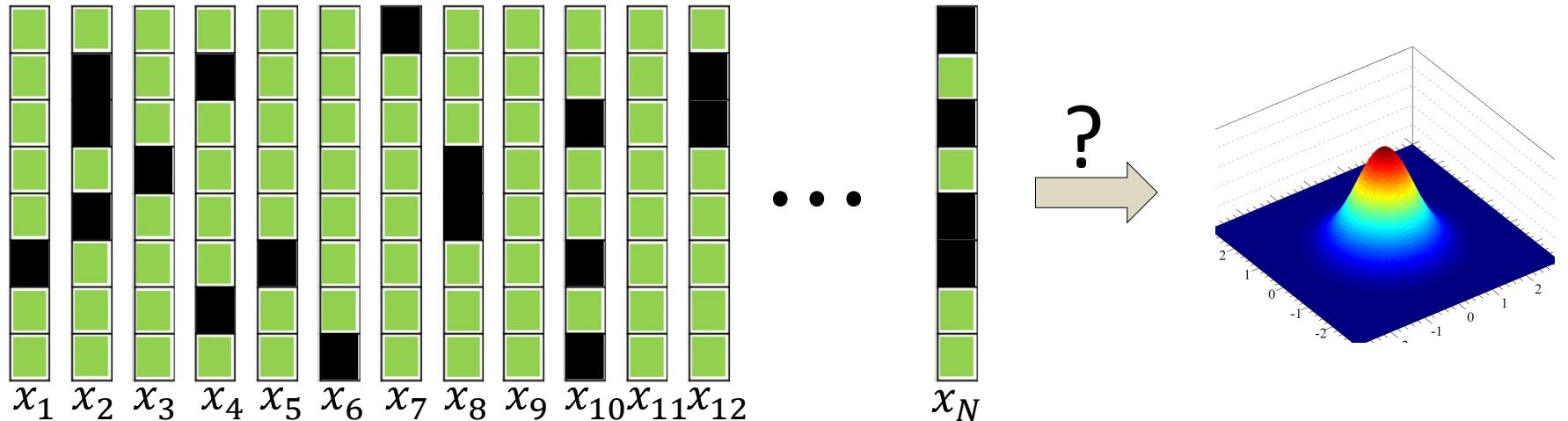
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- We are *missing* the data $M = \{m_1, \dots, m_N\}$
 - Comprising the *missing* components of the data

Maximum likelihood estimation with incomplete data



- These are the actual data we have: A set $O = \{o_1, \dots, o_N\}$ of *incomplete* vectors
 - Comprising only the *observed* components of the data
- We are *missing* the data $M = \{m_1, \dots, m_N\}$
 - Comprising the *missing* components of the data
- The *complete* data includes both the observed and missing components
$$X = \{x_1, \dots, x_N\}, \quad x_i = (o_i, m_i)$$
 - Keep in mind that at the complete data are *not* available (the missing components are missing)

Maximum likelihood estimation with incomplete data



- Maximum likelihood estimation: Maximize the likelihood of the *observed* data
 - That is all we really have

$$\underset{\mu, \sigma^2}{\operatorname{argmax}} \log(P(O)) = \underset{\mu, \sigma^2}{\operatorname{argmax}} \sum_{o \in O} \log P(o)$$

- Unfortunately, the Gaussian is defined on the *complete* vector :
 - $P(x) = \text{Gaussian}(x; \mu, \sigma^2)$
 - In order to compute $P(o)$ we must *derive* it from $P(x)$

The log likelihood of incomplete data

- The probability of any vector x with observed and missing parts o and m

$$P(x) = P(o, m)$$

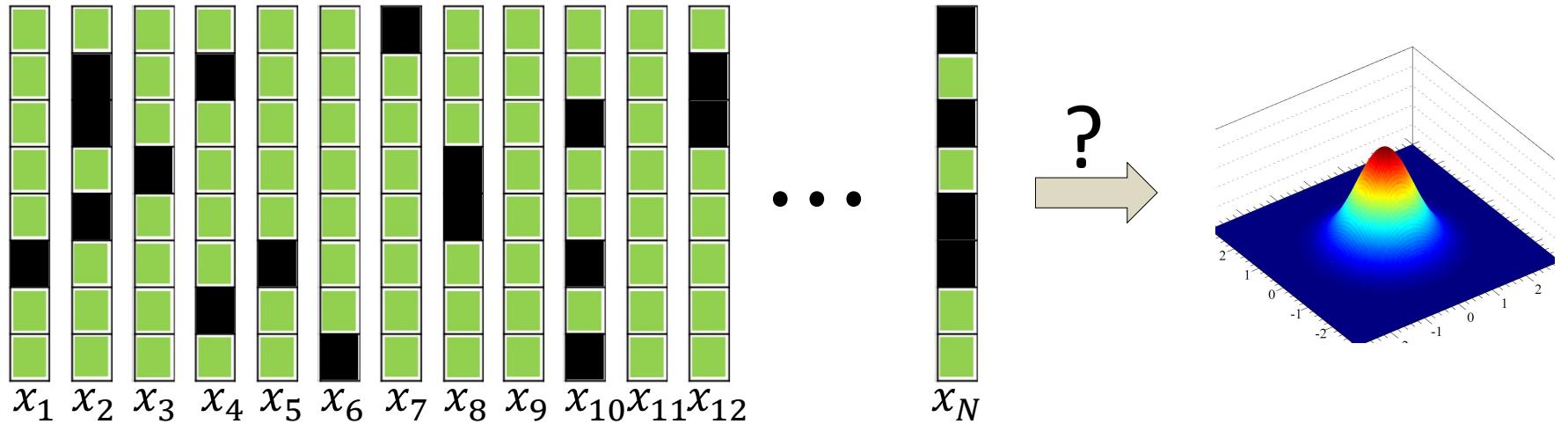
- Compute the probability of the observed components by marginalizing out the missing components

$$P(o) = \int_{-\infty}^{\infty} P(x) dm = \int_{-\infty}^{\infty} P(o, m) dm$$

- The log probability of the *entire observed training data*:

$$\sum_{o \in O} \log \int_{-\infty}^{\infty} P(o, m) dm$$

Maximum likelihood estimation with incomplete data



- Maximum likelihood estimation: Maximize the likelihood of the *observed* data

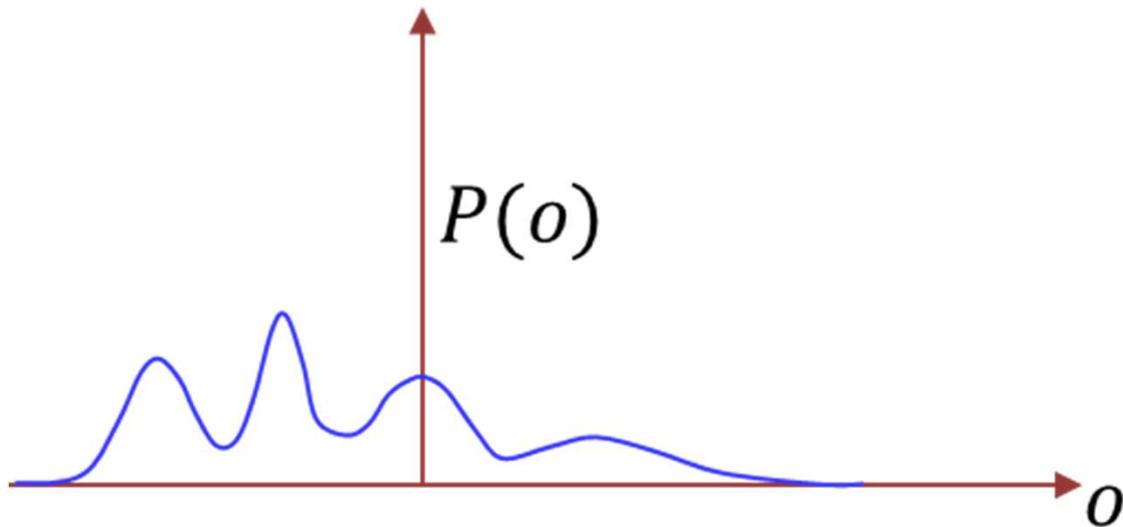
$$\underset{\mu, \sigma^2}{\operatorname{argmax}} \log(P(O)) = \underset{\mu, \sigma^2}{\operatorname{argmax}} \sum_{o \in O} \log \int_{-\infty}^{\infty} P(o, m) dm$$

- This requires the maximization of the log of an integral!
 - No closed form
 - Challenging on a good day, impossible on a bad one

Maximum Likelihood Estimation

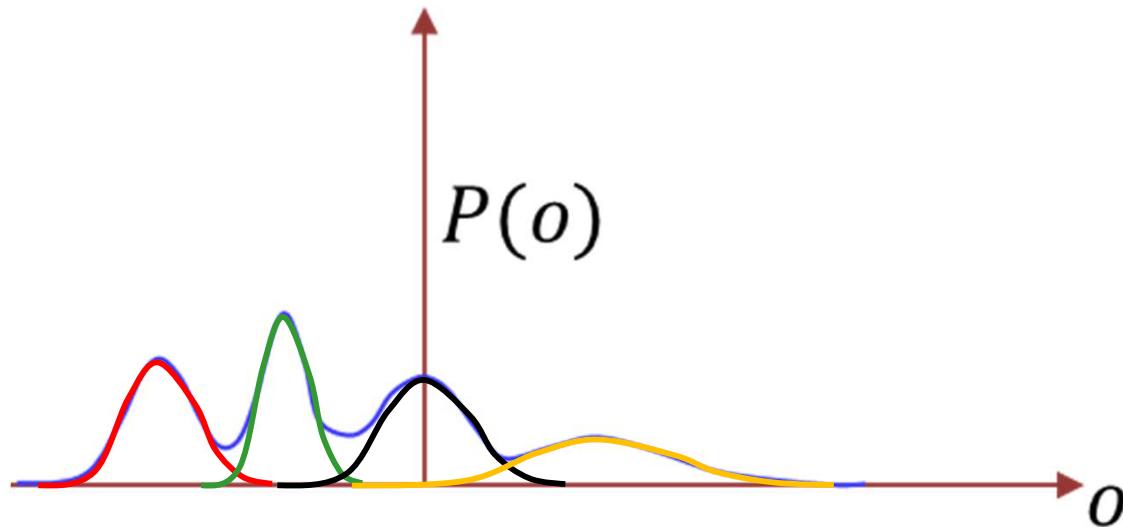
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The Gaussian Mixture



- Often, when trying to model a complicated distribution

The Gaussian Mixture

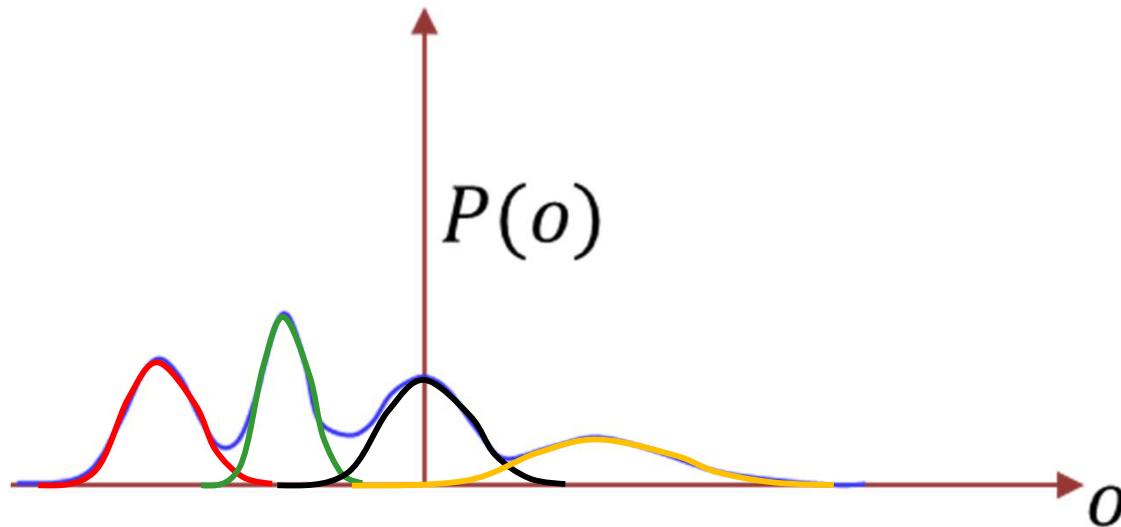


- Often, when trying to model a complicated distribution, we model it as a *mixture of Gaussians* (GMM)
 - A weighted sum of Gaussians

$$P(o) = \sum_k P(k)N(o; \mu_k, \sigma_k^2)$$

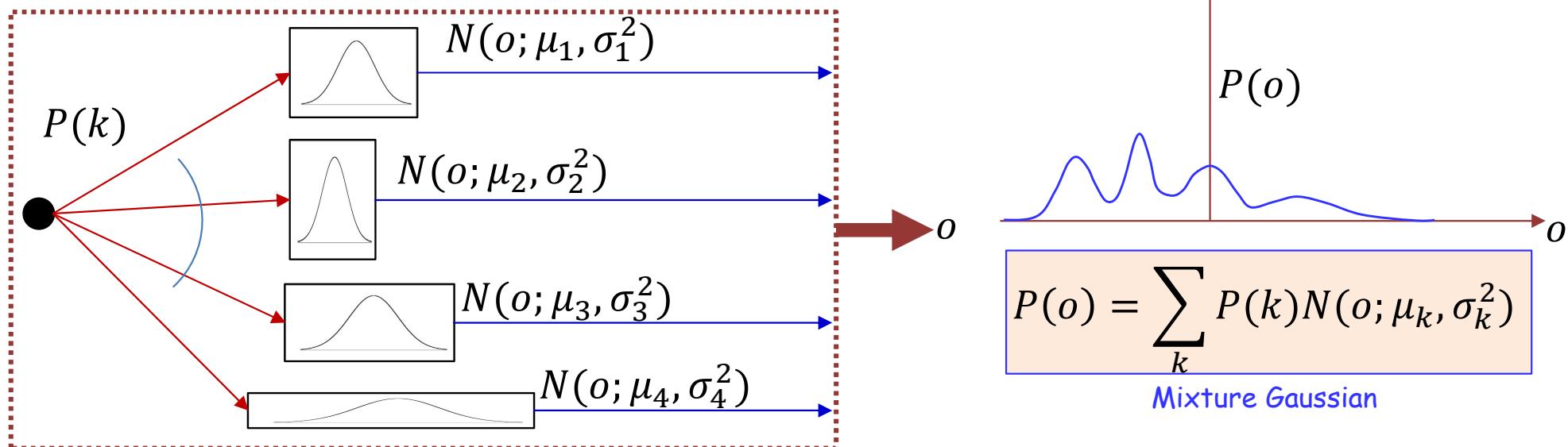
- The weights sum to 1.0

The Gaussian Mixture



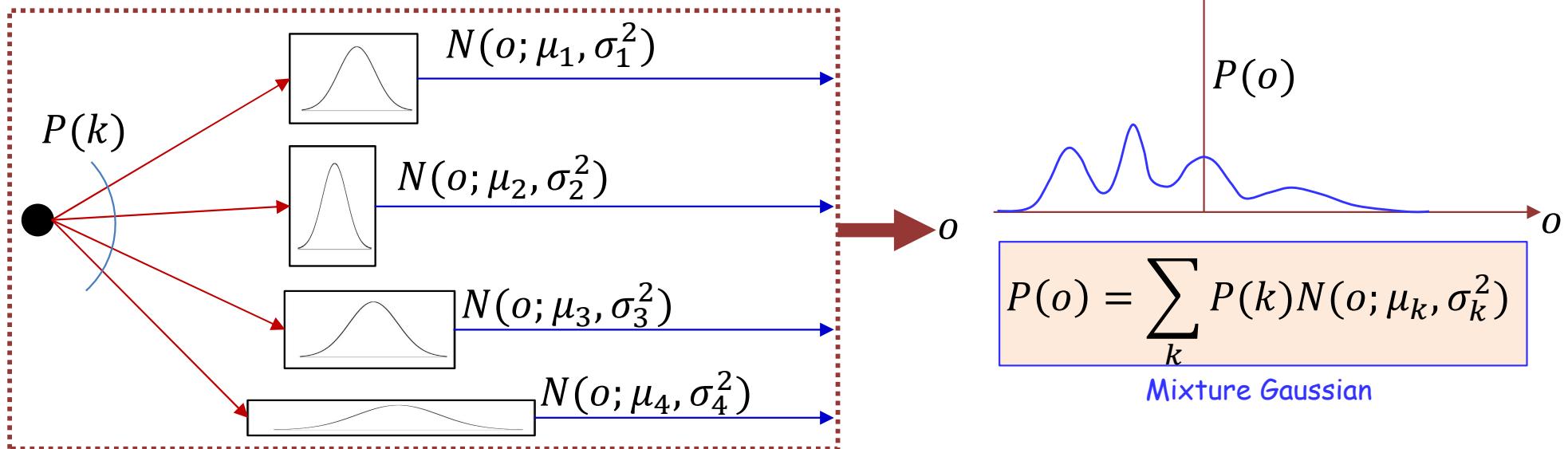
- Often, when trying to model a complicated distribution, we model it as a *mixture of Gaussians* (GMM)
 - A weighted sum of Gaussians
 - The weights sum to 1.0
- Problem: Given a number of samples from the original (complicated) distribution, how to determine the parameters of the parameters of the GMM to best fit them

Examples of incomplete data: missing information in Gaussian mixtures



- The generative model characterizes the data as the outcome of a two-level process
 - In the first step the process chooses a Gaussian from a collection
 - In the second, it draws the vector o from the chosen Gaussian
 - The overall model is a *mixture Gaussian*
- Objective: Learn the parameters of all the Gaussians from training data
 - Learn the means and variances of the individual Gaussians
 - And also the probability with which each Gaussian is selected for the draw

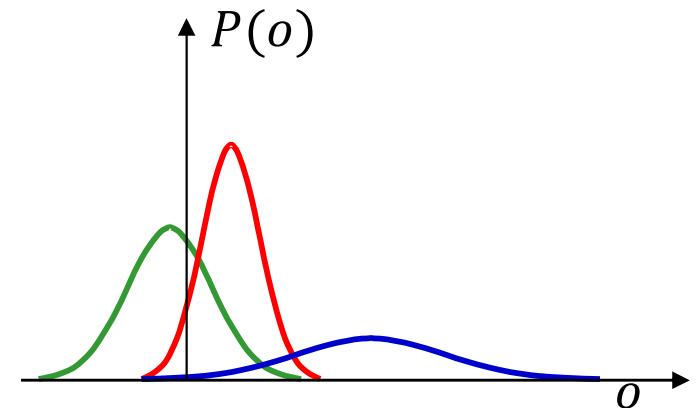
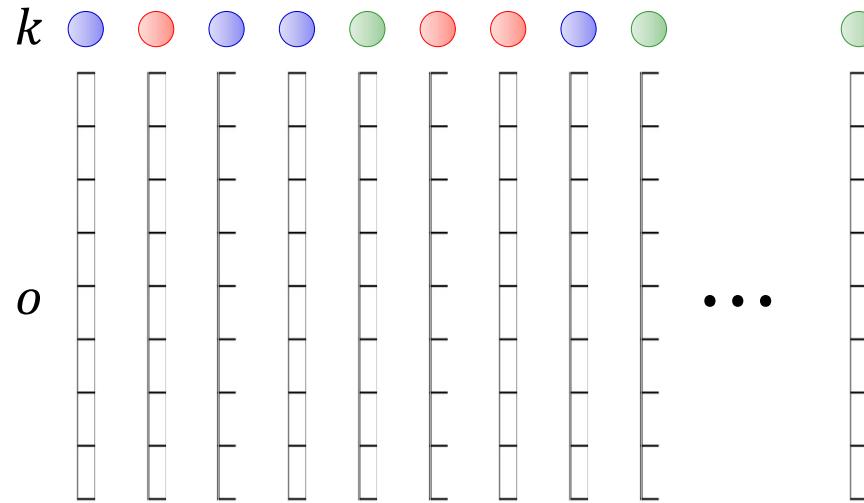
The Gaussian Mixture generative model



- Note, the process actually draws *two* variables for each observation, k and o .
- The probability of a particular draw is actually the joint probability of both variables
$$P(k, o) = P(k)P(o|k) = P(k)N(o; \mu_k, \sigma_k^2)$$
- To compute the probability of obtaining any observation o , we are *marginalizing out* the Gaussian index variable

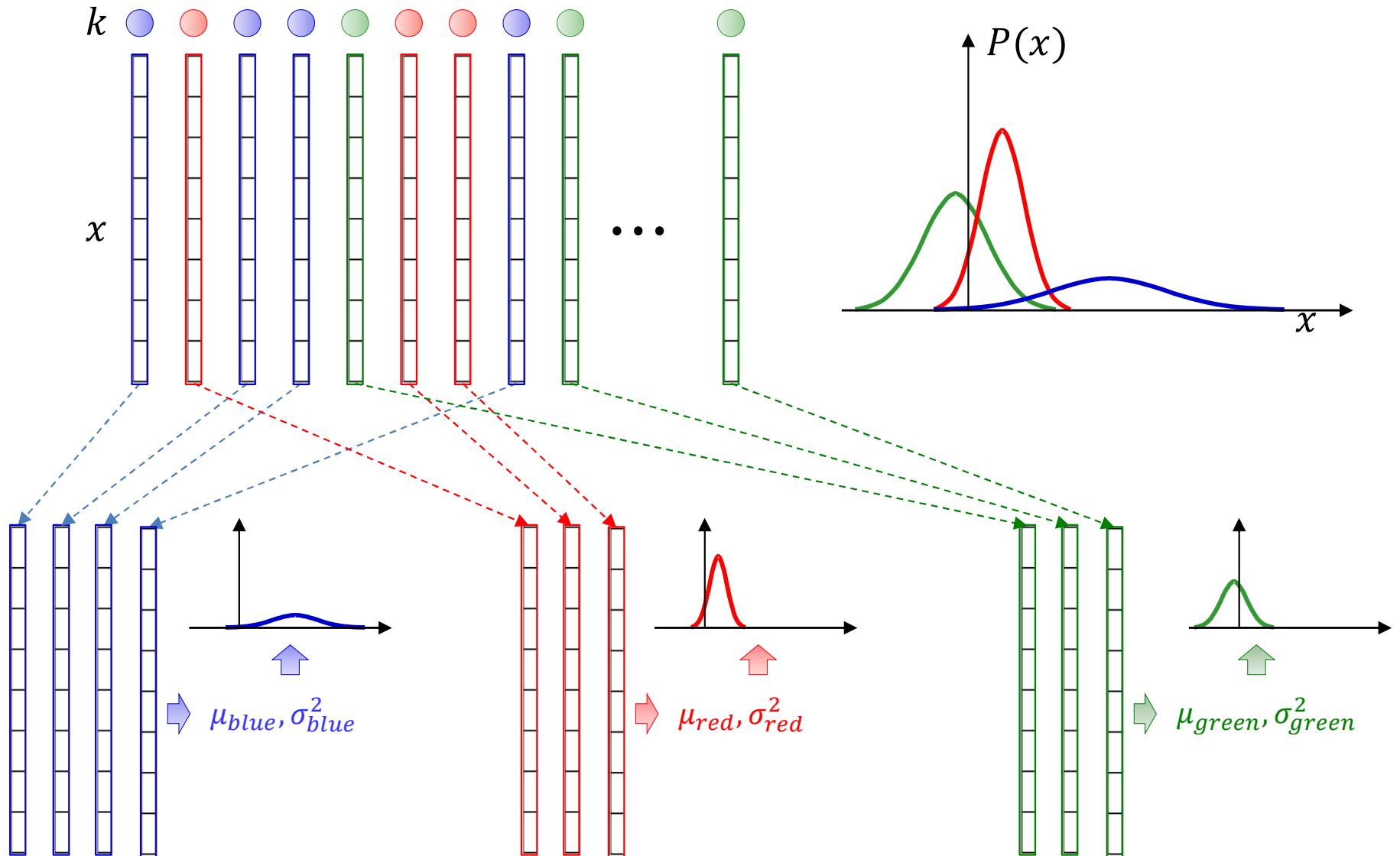
$$P(o) = \sum_k P(k, o) = \sum_k P(k)N(o; \mu_k, \sigma_k^2)$$

The *complete* data needed to precisely learn the model

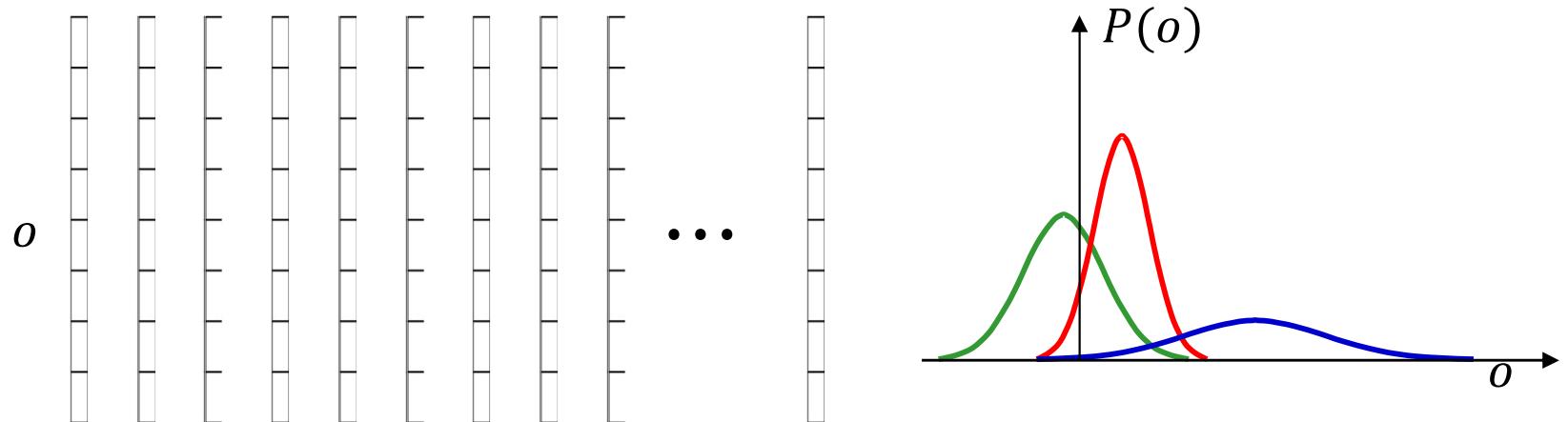


- Ideal data: Each training instance includes both the data vector o and the Gaussian k it was drawn from
 - In order to estimate the parameters of any Gaussian, you only need to segregate the training instances from that Gaussian, and compute the mean and variance from them

Learning a GMM with “complete” data



The GMM problem of incomplete data: missing information



- Problem : We are not given the actual Gaussian for each observation
 - Our data are incomplete
- What we want : $(o_1, k_1), (o_2, k_2), (o_3, k_3) \dots$
- What we have: $o_1, o_2, o_3 \dots$

ML estimation with only *observed data*

- The maximum likelihood estimation problem:
 - Given *observed data* $O = \{o_1, o_2, o_3 \dots\}$,
 - estimate $\{(\mu_k, \sigma_k^2), \forall k\}$ – the parameters of all the Gaussians

$$\operatorname{argmax}_{\{(\mu_k, \sigma_k^2), \forall k\}} \log(P(O)) = \operatorname{argmax}_{\{(\mu_k, \sigma_k^2), \forall k\}} \sum_{o \in O} \log P(o)$$

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$$\operatorname{argmax}_{\{(\mu_k, \sigma_k^2), \forall k\}} \sum_{o \in O} \log \sum_k P(k)N(o; \mu_k, \sigma_k^2)$$

- This includes the log of a sum, which defies direct optimization

The general form of the problem

- The “presence” of missing data or variables requires them to be marginalized out of your probability
 - By summation or integration
- This results in a maximum likelihood estimate of the form

$$\hat{\theta} = \operatorname{argmax}_{\theta} \sum_{o} \log \sum_{h} P(h, o; \theta)$$

- The inner summation may also be an integral in some problems
- Explicitly introducing θ in the RHS to show that the probability is computed by a model with parameter θ which must be estimated
- The log of a sum (or integral) makes estimation challenging
 - No closed form solution
 - Need efficient iterative algorithms

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By summation or integration

Can we get an approximation to this that is more tractable?
(i.e without a summation or integral within the log)

$$\hat{\theta} = \operatorname{argmax}_{\theta} \sum_{o} \log \sum_{h} P(h, o)$$

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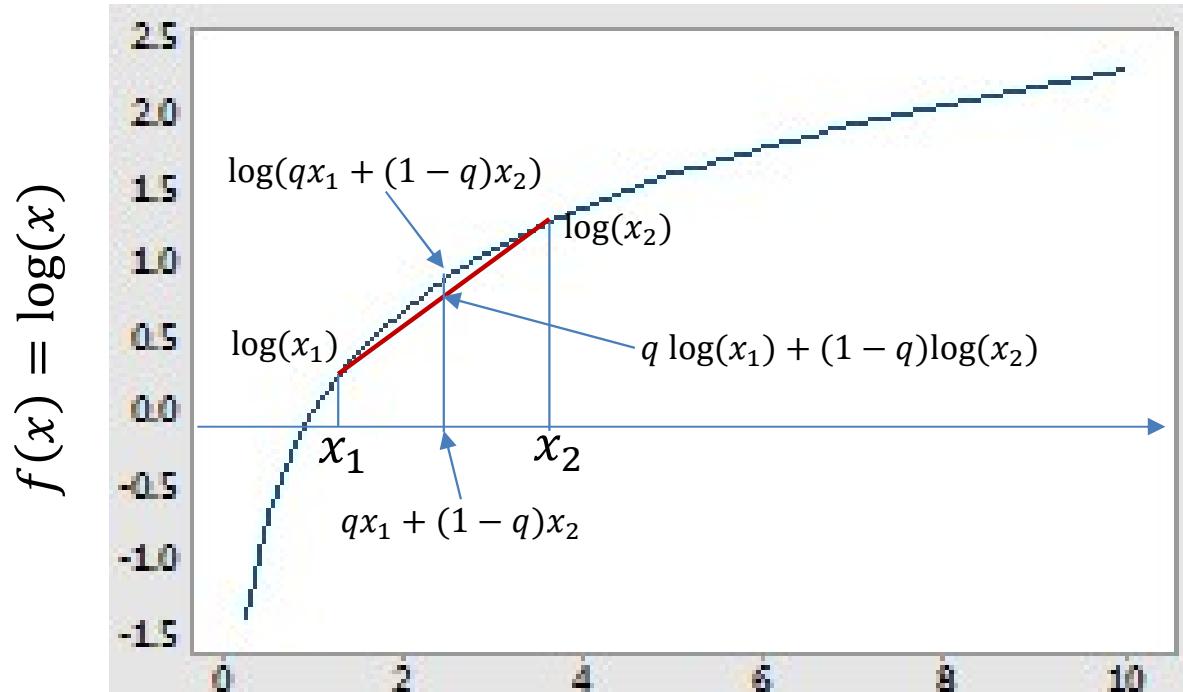
The variational lower bound

- We can rewrite

$$\log P(o) = \log \sum_h P(h, o) = \log \sum_h Q(h) \frac{P(h, o)}{Q(h)}$$

- Where $Q(h)$ is some function such that $Q(h) \geq 0$ and $\sum_h Q(h) = 1$
 - I.e. a probability distribution

The logarithm is a concave function



- For any x_1 and x_2 , for any $0 \leq q \leq 1$,
$$\log(qx_1 + (1 - q)x_2) \geq q \log(x_1) + (1 - q)\log(x_2)$$
- More generally for any set of $\{x_i\}$, and any weights $\{q_i\}$ s.t. $q_i \geq 0$ and $\sum_i q_i = 1$

$$\log\left(\sum_i q_i x_i\right) \geq \sum_i q_i \log(x_i)$$

The variational lower bound

- By the concavity of the log function

$$\log \sum_h Q(h) \frac{P(h, o)}{Q(h)} \geq \sum_h Q(h) \log \frac{P(h, o)}{Q(h)}$$

- For any $Q(h) \geq 0$ and $\sum_h Q(h) = 1$
- Note, the LHS is exactly equal to $\log P(o)$
- This is the *variational lower bound* on $\log P(o)$
 - Also called the Evidence Lower BOund, or ELBO

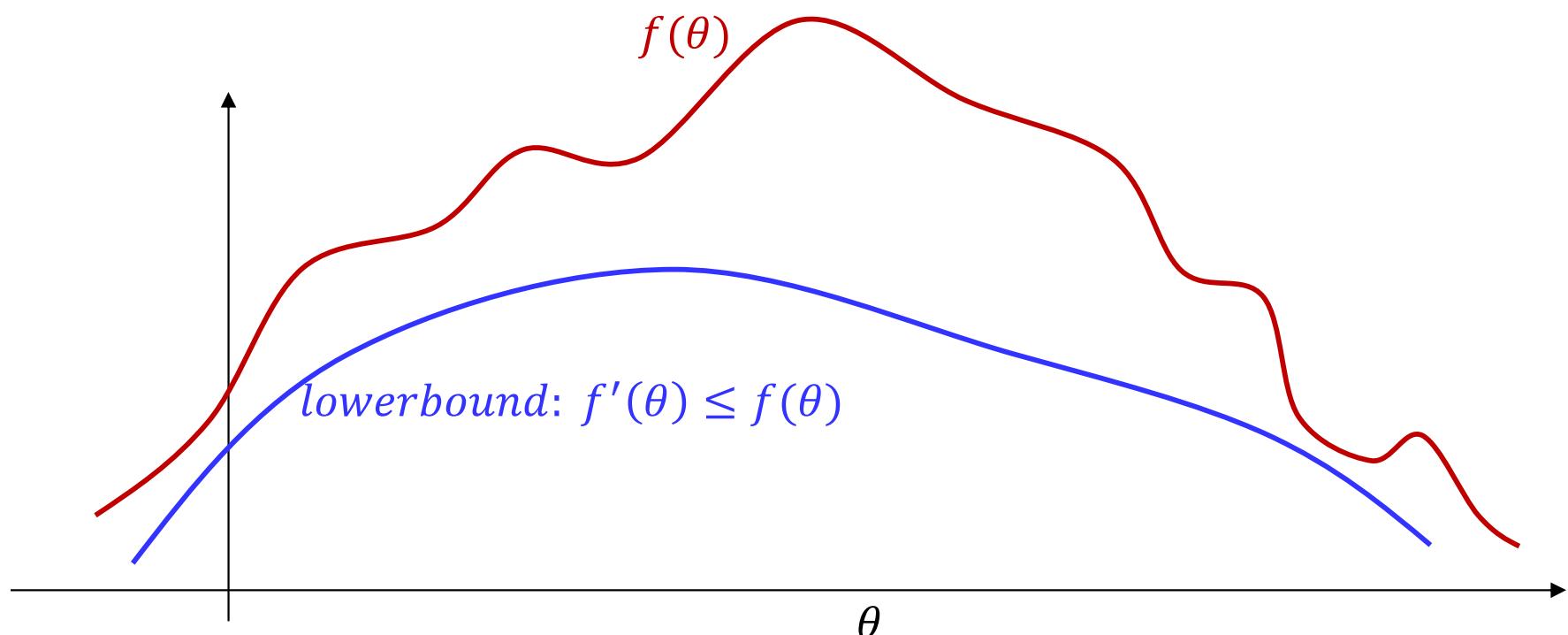
Or more explicitly

- By the concavity of the log function

$$\log P(o; \theta) \geq \sum_h Q(h) \log \frac{P(h, o; \theta)}{Q(h)}$$

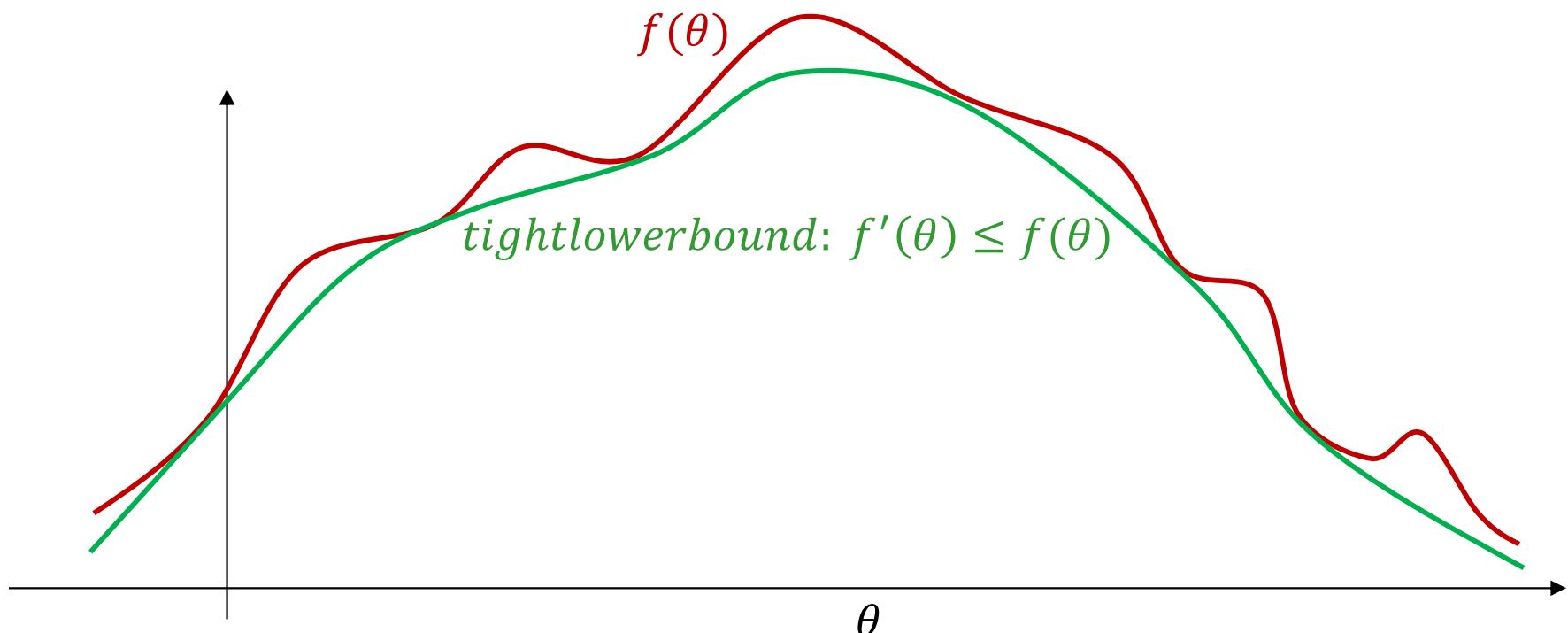
- Explicitly showing that the probability is computed by a model with parameter θ
 - We must maximize $P(o; \theta)$ w.r.t θ
- This is the *variational lower bound* or ELBO on $\log P(o; \theta)$

The (variational) lower bound



- The lower bound is always at or below the original function

The (variational) lower bound



- The lower bound is always at or below the original function
- If it is a tight lower bound, the max of the lower bound can be expected to be near the max of the function
 - To make the lower bound tight, we need to choose $Q(h)$ properly

The two-step process

- By the concavity of the log function

$$\log P(o; \theta) \geq \sum_h Q(h) \log \frac{P(h, o; \theta)}{Q(h)}$$

- **Step 1:** Determine a $Q(h)$ that maximizes the RHS, using the current estimate of θ
 - Makes the bound tight
- **Step 2:** Fix $Q(h)$ and maximize the RHS with respect to θ to get the next estimate

The two-step process

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Maximizing w.r.t. Q(h)

$$\sum_h Q(h) \log \frac{P(h, o; \theta)}{Q(h)}$$

- Take the derivative w.r.t. $Q(h)$ for all h and equate to 0
 - With the constraint that $Q(h) \geq 0, \sum_h Q(h) = 1$
- If $Q(h)$ is specifically modeled by a neural net or some other restricted function, then we cannot simply take the derivative and equate to 0
 - We may need gradient descent, with backpropagation
- Note: The optimized $Q(h)$ depends on $P(h, o; \theta)$ and is a function of θ

Choosing a good $Q(h)$

- For any $P(h, o; \theta)$, the optimal $Q(h) = P(h|o; \theta)$:

$$\begin{aligned}\sum_h Q(h) \log \frac{P(h, o; \theta)}{Q(h)} &= \sum_h P(h|o; \theta) \log \frac{P(h, o; \theta)}{P(h|o; \theta)} \\ &= \sum_h P(h|o; \theta) \log P(o; \theta) \\ &= \log P(o; \theta) \sum_h P(h|o; \theta) = \log P(o; \theta)\end{aligned}$$

- At this value of $Q(h)$ the variational lower bound achieves its maximum possible value

Choosing a good $Q(h)$

- Let $Q(h) = P(h|o; \theta')$

$$\log P(o; \theta) \geq \sum_h P(h|o; \theta') \log \frac{P(h, o; \theta)}{P(h|o; \theta')}$$

- Let

$$J(\theta, \theta') = \sum_h P(h|o; \theta') \log \frac{P(h, o; \theta)}{P(h|o; \theta')}$$

- We get

$$\log P(o; \theta) \geq J(\theta, \theta')$$

- And

$$\log P(o; \theta) = J(\theta, \theta)$$

Choosing a good $Q(h)$

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$$= \sum_h P(h|o; \theta) \log P(o; \theta)$$

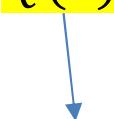
$$\log P(o; \theta) \sum_h P(h|o; \theta) = \log P(o; \theta)$$

Expectation Maximization

- We have

$$J(\theta, \theta') = \sum_h P(h|o; \theta') \log \frac{P(h, o; \theta)}{P(h|o; \theta')}$$

$Q(h)$



- where

$$\log P(o; \theta) \geq J(\theta, \theta')$$

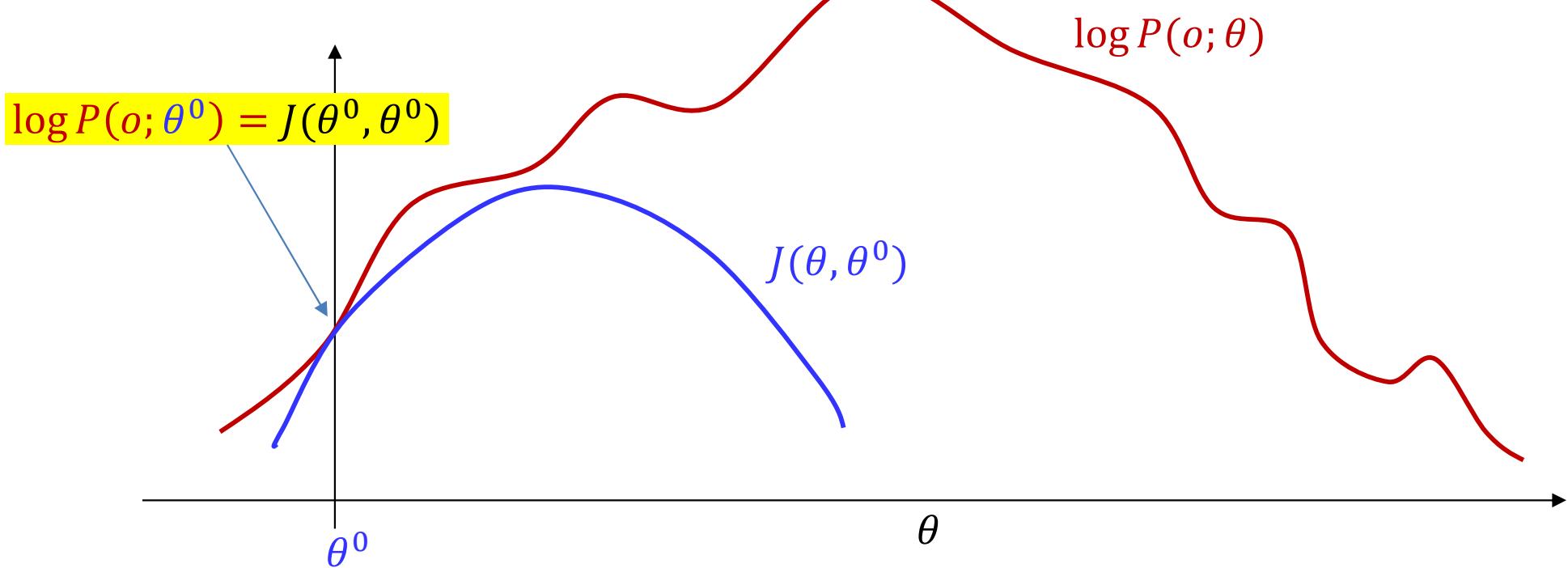
- And

$$\log P(o; \theta) = J(\theta, \theta)$$

- This gives us the following iterative algorithm that guarantees non-decreasing $P(o; \theta)$ with iterations:

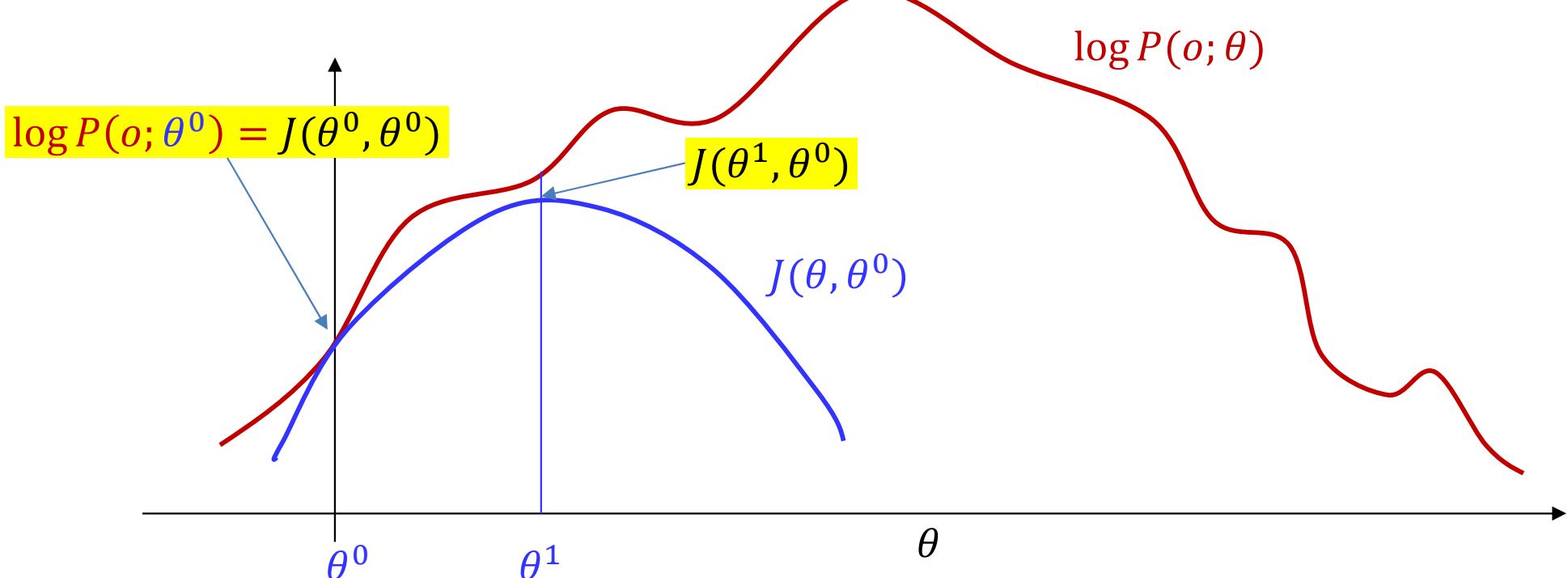
$$\theta^{k+1} \leftarrow \operatorname{argmax}_{\theta} J(\theta, \theta^k)$$

$$\theta^{k+1} \leftarrow \operatorname{argmax}_{\theta} J(\theta, \theta')$$



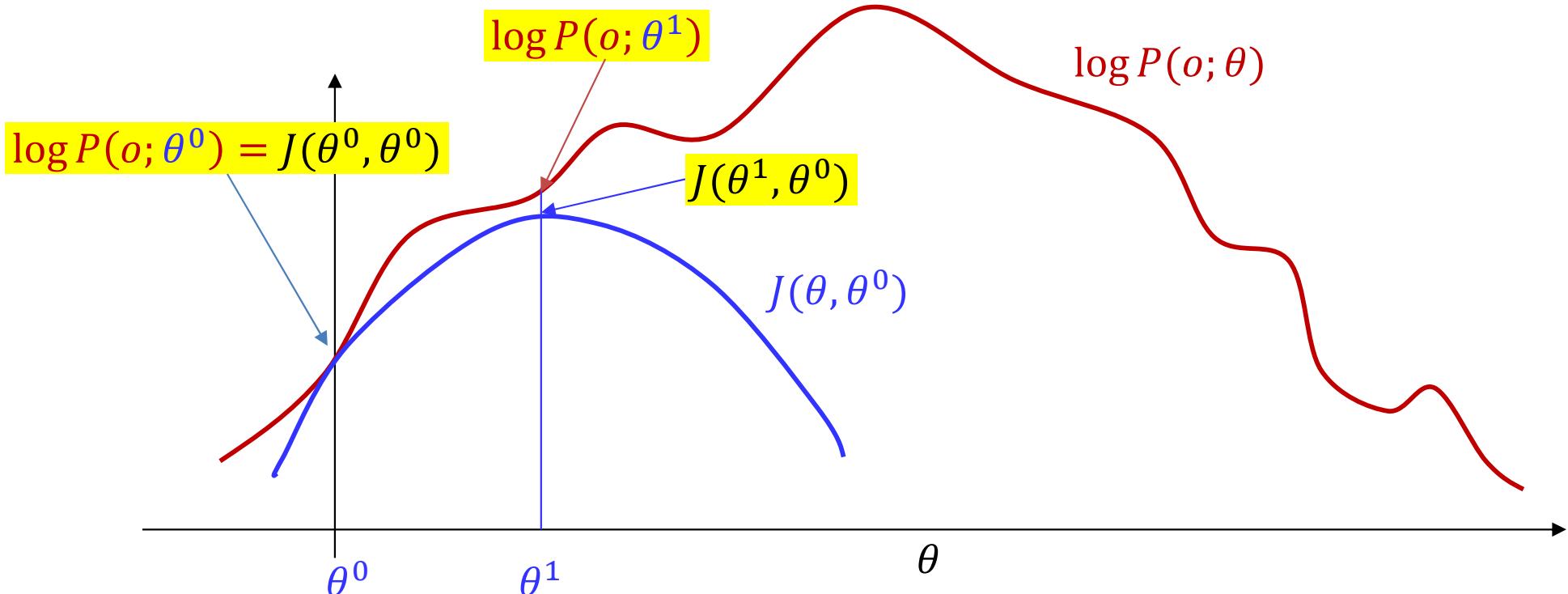
- Initialize θ^0
- Construct $J(\theta, \theta^0)$
 - It touches $\log P(o; \theta)$ at θ^0 because $\log P(o; \theta^0) = J(\theta^0, \theta^0)$

$$\theta^{k+1} \leftarrow \operatorname{argmax}_{\theta} J(\theta, \theta')$$



- Find $\theta^1 = \operatorname{argmax}_{\theta} J(\theta, \theta^0)$
 - $J(\theta^1, \theta^0) \geq J(\theta^0, \theta^0)$ (since you're maximizing $J(\theta, \theta^0)$ w.r.t θ)

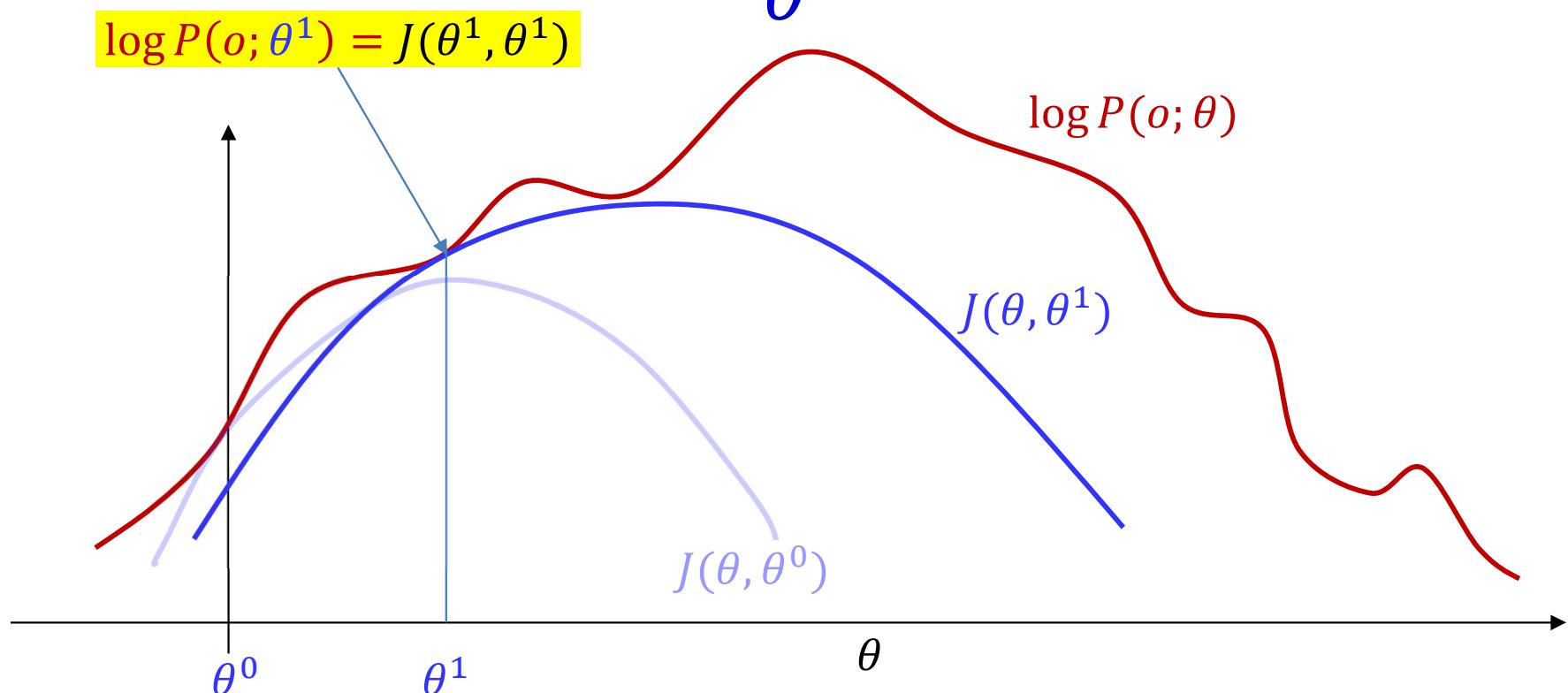
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 - $J(\theta^1, \theta^0) \geq J(\theta^0, \theta^0)$ (since you're maximizing $J(\theta, \theta^0)$ w.r.t θ)
- $\log P(o; \theta^1) \geq J(\theta^1, \theta^0)$
 - since $J(\theta, \theta^0)$ is a lower bound on $\log P(o; \theta)$
- So the iteration increases $\log P(o; \theta)$

$$\theta^{k+1} \leftarrow \operatorname{argmax}_{\theta} J(\theta, \theta')$$

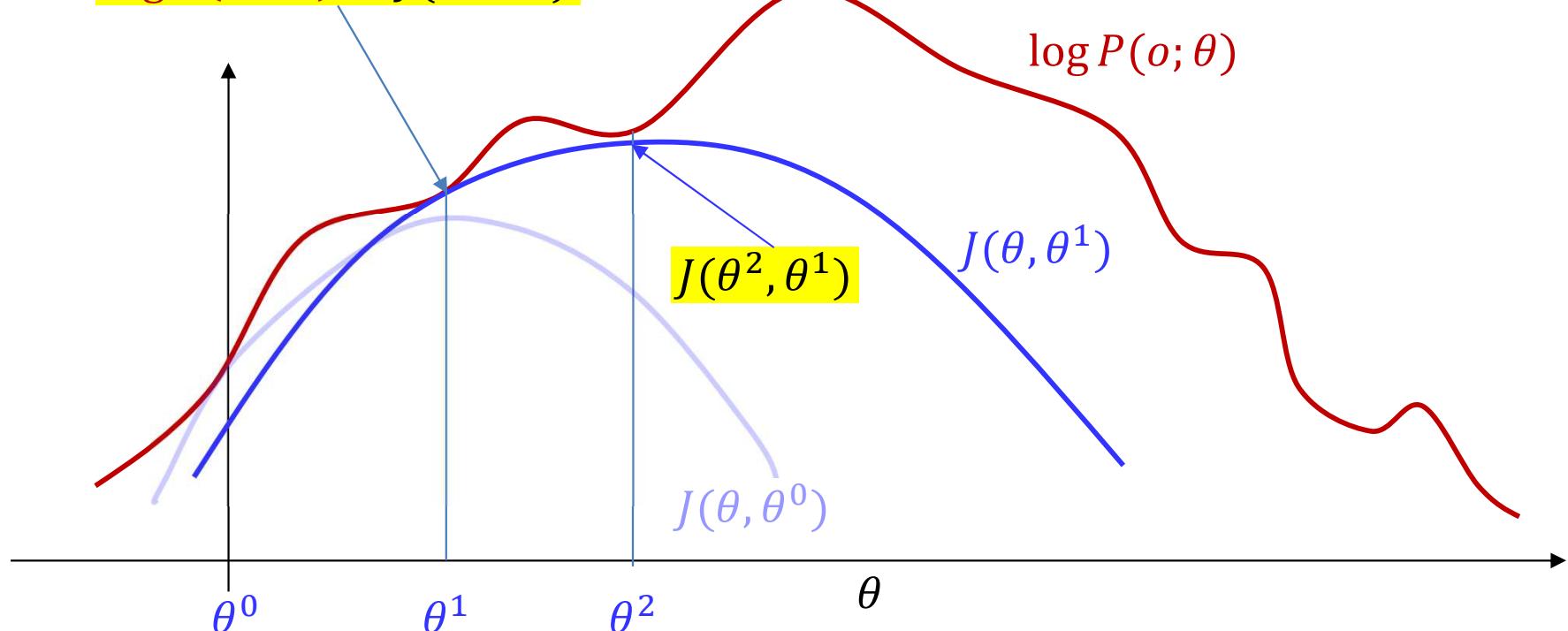
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- Construct $J(\theta, \theta^1)$
 - It touches $\log P(o; \theta)$ at θ^1 because $\log P(o; \theta^1) = J(\theta^1, \theta^1)$

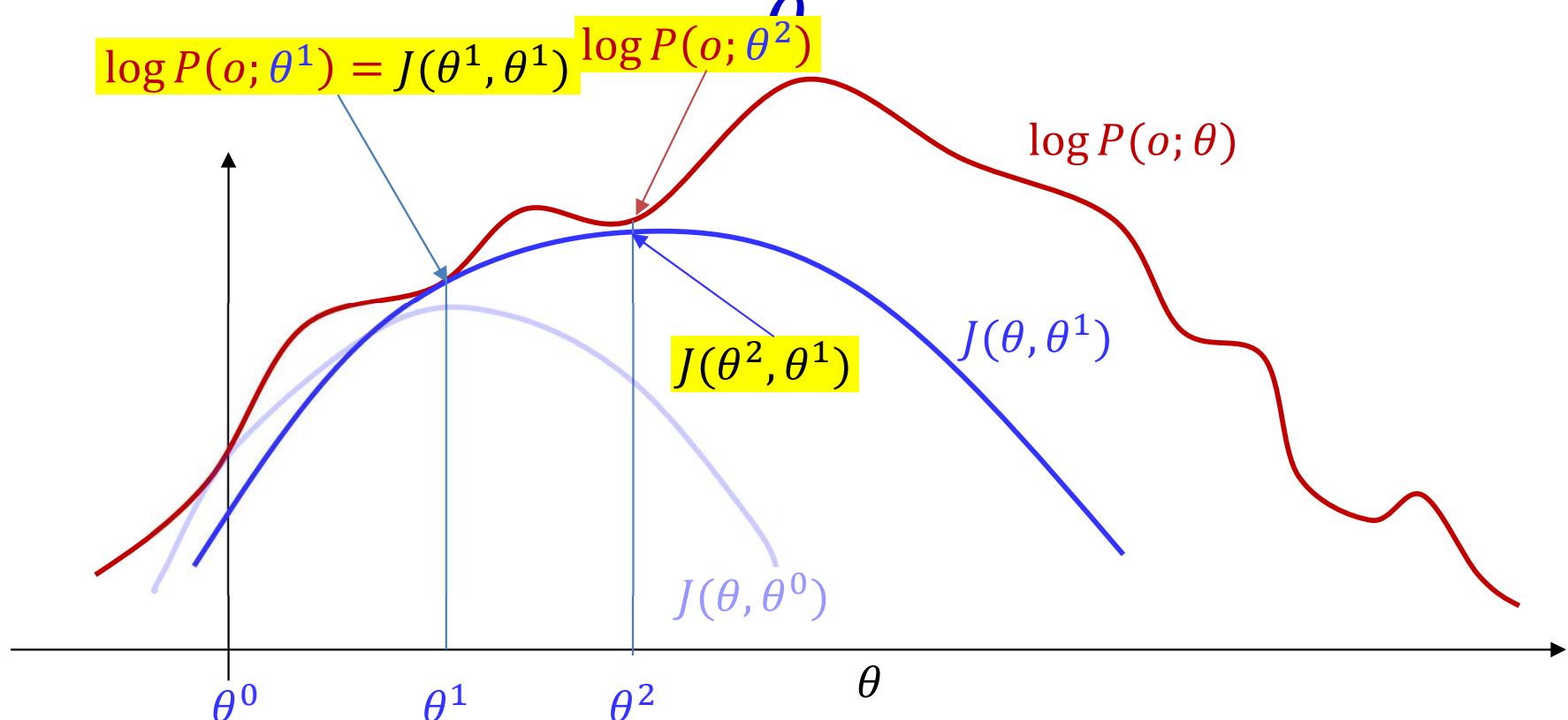
$$\theta^{k+1} \leftarrow \operatorname{argmax}_{\theta} J(\theta, \theta')$$

$$\log P(o; \theta^1) = J(\theta^1, \theta^1)$$



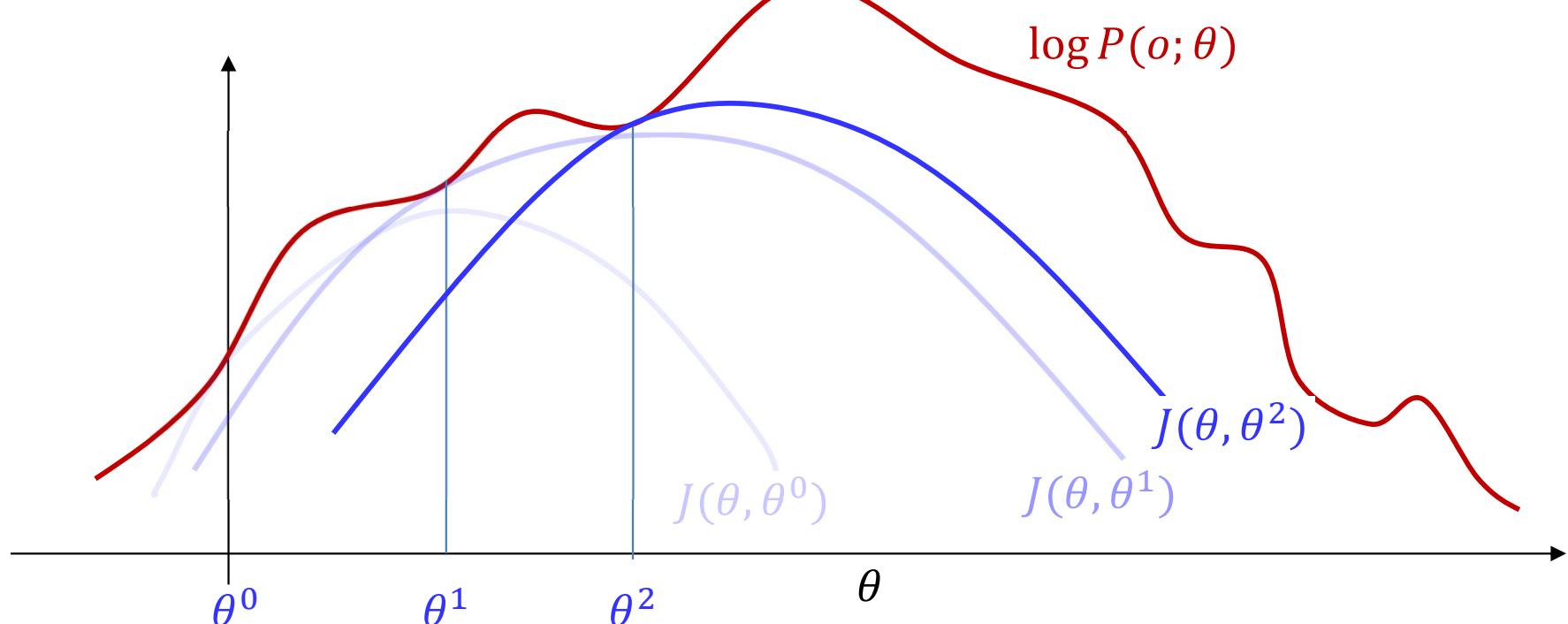
- Find $\theta^2 = \operatorname{argmax}_{\theta} J(\theta, \theta^1)$
 - $J(\theta^2, \theta^1) \geq J(\theta^1, \theta^1)$ (since you're maximizing $J(\theta, \theta^1)$ w.r.t θ)

$$\theta^{k+1} \leftarrow \operatorname{argmax}_{\theta} J(\theta, \theta')$$



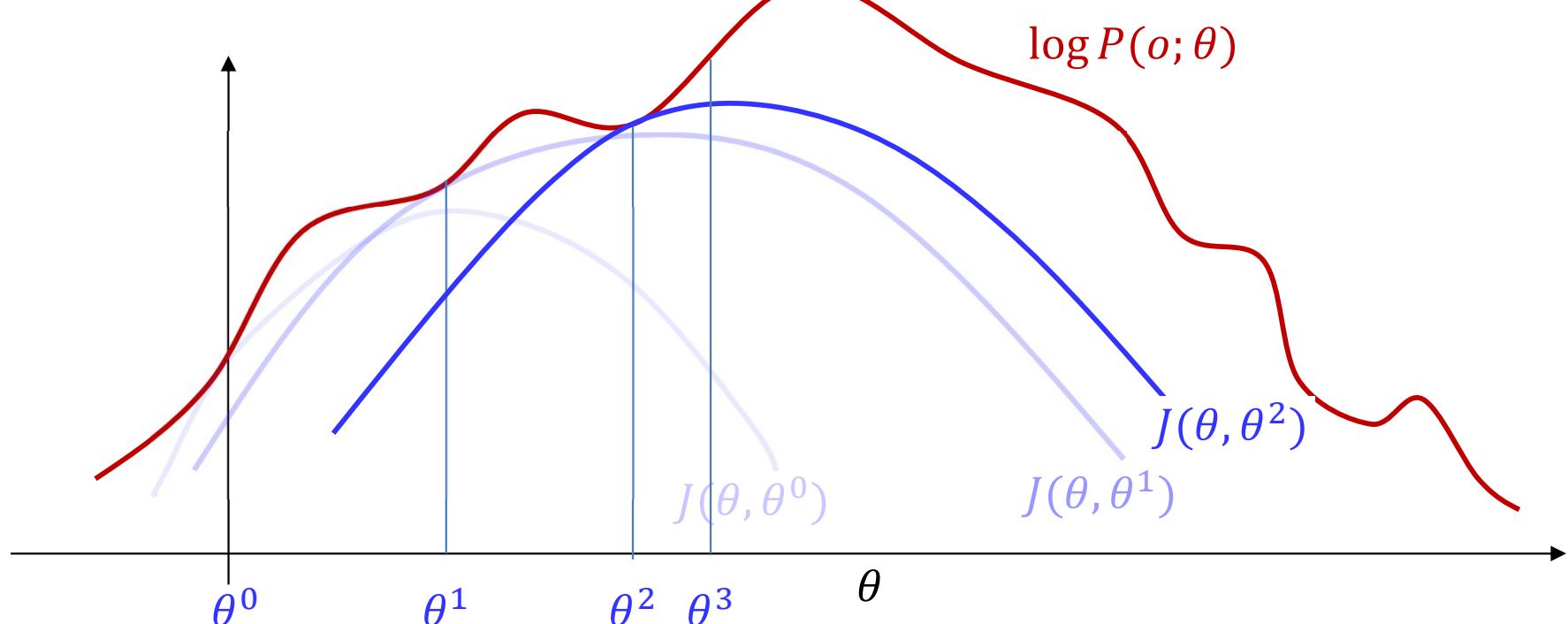
- Find $\theta^2 = \operatorname{argmax}_{\theta} J(\theta, \theta^1)$
 - $J(\theta^2, \theta^1) \geq J(\theta^1, \theta^1)$ (since you're maximizing $J(\theta, \theta^1)$ w.r.t θ)
- $\log P(o; \theta^2) \geq J(\theta^2, \theta^1)$
 - Since $J(\theta, \theta^1)$ is a lower bound on $\log P(o; \theta)$
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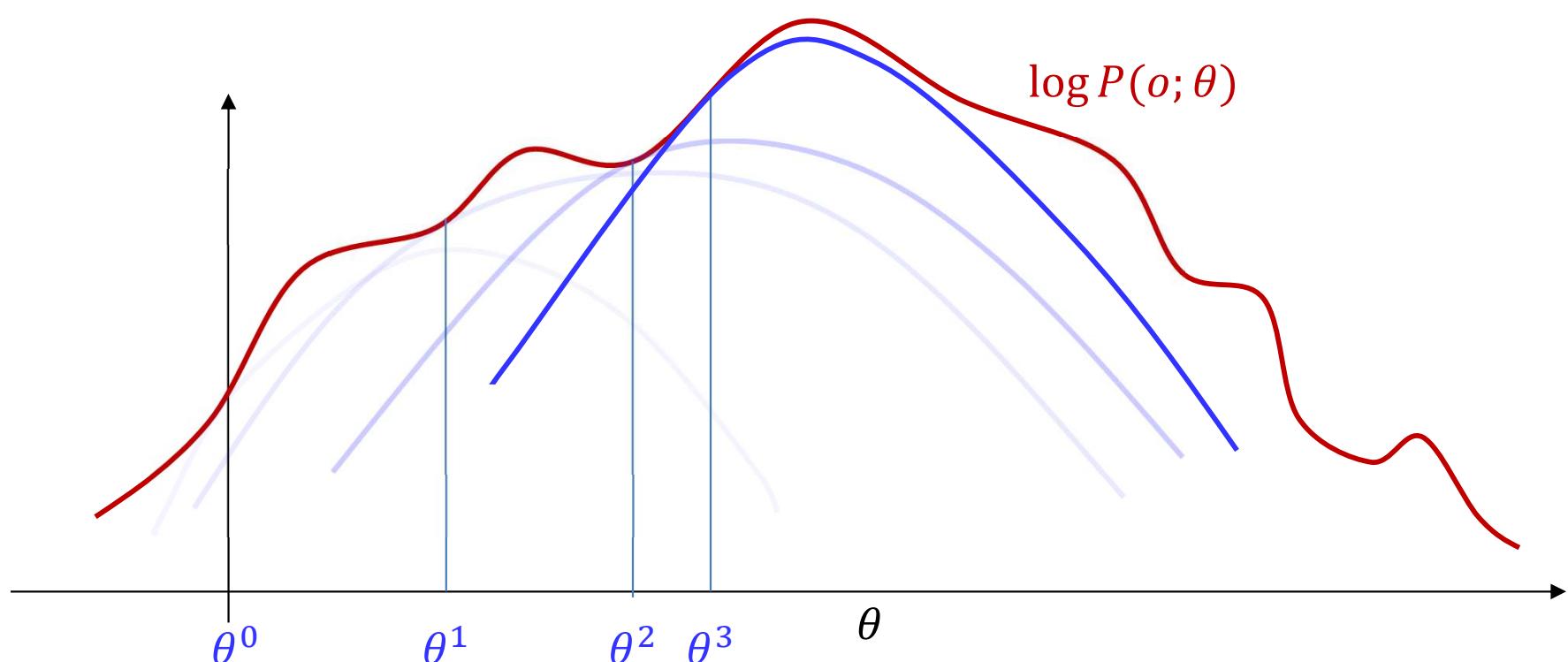
- Repeat the steps:
 - Compose $J(\theta, \theta^k)$ to “touch” $\log P(o; \theta)$ at the current estimate θ^k
 - Set $\theta^{k+1} \leftarrow \operatorname{argmax}_{\theta} J(\theta, \theta^k)$
- Each step is guaranteed to increase (or at least not decrease) $\log P(o; \theta)$
 - Stop when $\log P(o; \theta)$ stops increasing

$$\theta^{k+1} \leftarrow \operatorname{argmax}_{\theta} J(\theta, \theta')$$



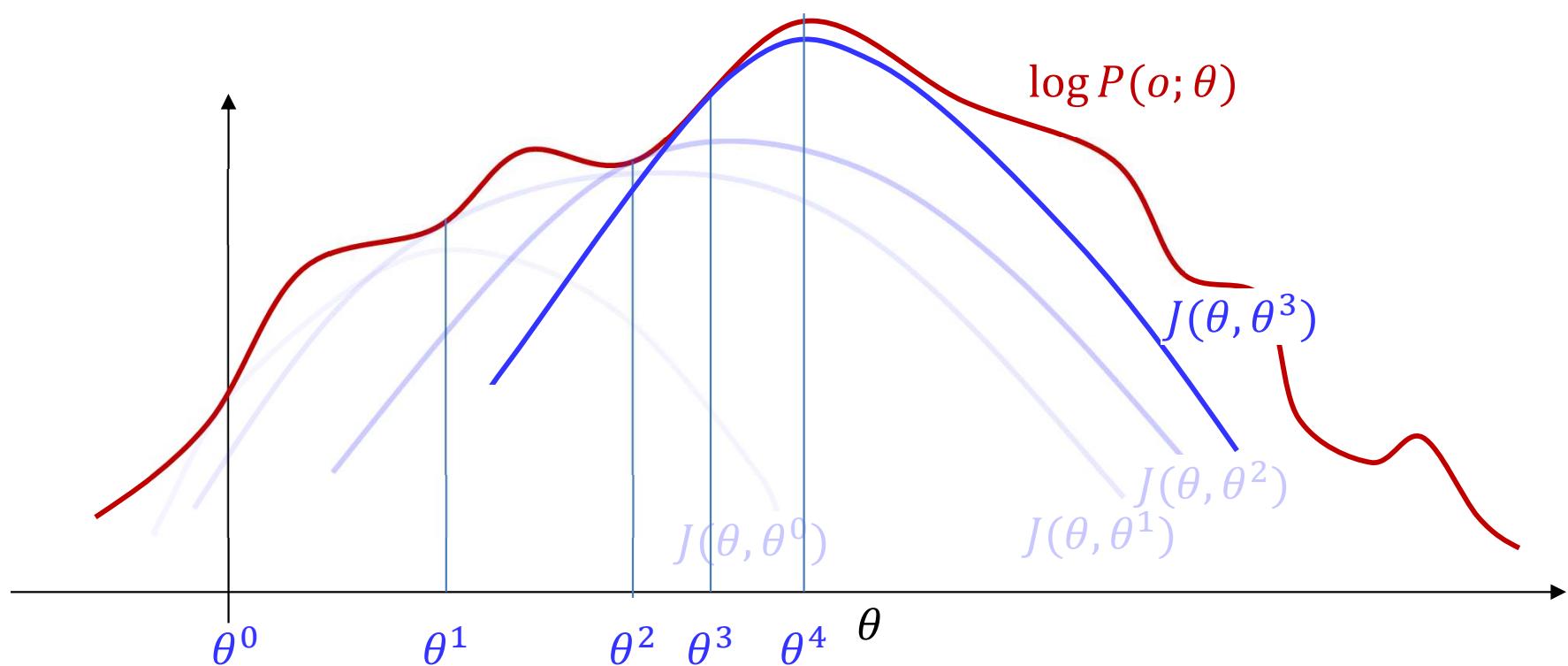
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Expectation Maximization

- Initialize θ^0
- $k = 0$
- Iterate (over k) until $\log P(O; \theta)$ converges:
 - Construct ELBO function

$$J(\theta, \theta^k) = \sum_{o \in O} \sum_h P(h|o; \theta^k) \log \frac{P(h, o; \theta)}{P(h|o; \theta^k)}$$

- Maximization step
$$\theta^{k+1} \leftarrow \operatorname{argmax}_{\theta} J(\theta, \theta^k)$$
- Let's simplify a bit

Expectation Maximization

- Initialize θ^0
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$$J(\theta, \theta^k) = \sum_{o \in O} \sum_h P(h|o; \theta^k) \log P(h, o; \theta) - \sum_{o \in O} \sum_h P(h|o; \theta^k) \log P(h|o; \theta^k)$$

- Maximization step

$$\theta^{k+1} \leftarrow \operatorname{argmax}_{\theta} J(\theta, \theta^k)$$

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Not a function of θ

Can be ignored for maximization

$$\theta^{k+1} \leftarrow \operatorname{argmax}_{\theta} J(\theta, \theta^k)$$

Expectation Maximization

- Initialize θ^0
- $k = 0$
- Iterate (over k) until $\log P(O; \theta)$ converges:
 - Expectation Step:
Compute $P(h|o; \theta^k)$ for all $o \in O$ for all h
 - Maximization step

$$\theta^{k+1} \leftarrow \operatorname{argmax}_{\theta} \sum_{o \in O} \sum_h P(h|o; \theta^k) \log P(h, o; \theta)$$

The two-step process

- By the concavity of the log function

$$\log P(o; \theta) \geq \sum_h Q(h) \log \frac{P(h, o; \theta)}{Q(h)}$$

- **Step 1:** Determine a $Q(h)$ that maximizes the RHS, using the current estimate of θ
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$$\theta \leftarrow \arg \max_{\theta} \sum_h Q(h) (\log P(h, o; \theta) - \log Q(h))$$

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Training by maximizing a variational lower bound

- **Step 1:** Determine a $Q(h)$ that maximizes the RHS, using the current estimate of θ
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Special case: Expectation Maximization

- Initialize θ^0
- $k = 0$
- Iterate (over k) until $\log P(O; \theta)$ converges:
 - Expectation Step: $Q(h) = P(h|o; \theta^k)$
Compute $P(h|o; \theta^k)$ for all $o \in O$ for all h
 - Maximization step

$$\theta^{k+1} \leftarrow \operatorname{argmax}_{\theta} \sum_{o \in O} \sum_h P(h|o; \theta^k) \log P(h, o; \theta)$$

Poll 2

Mark all that are correct about the EM algorithm

- It is an iterative algorithm that can be used to estimate probability distributions when the data are incomplete and have missing components or variables
- It iteratively maximizes an “ELBO” function with respect to model parameters
- It provides a closed form formula to estimate the parameters of the distribution

Mark all that are true of the ELBO (Empirical Lower Bound) function

- It is a lower bound on the actual log probability of the training data as computed by the model
- It is a function of the model parameters
- There are some settings of the model parameters where the ELBO can be greater than the log probability of the training data

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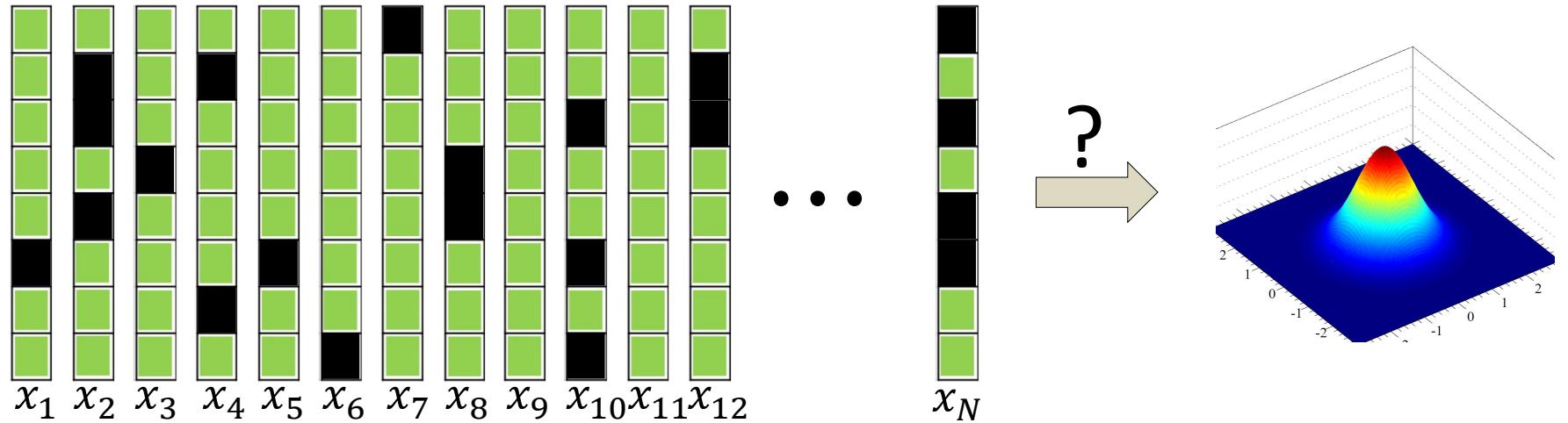
That's so much math, but what does it really do?

- What does EM practically do when we have missing data?
 - What is the intuition behind how it resolves the problem?

Recap: Maximum Likelihood Estimation

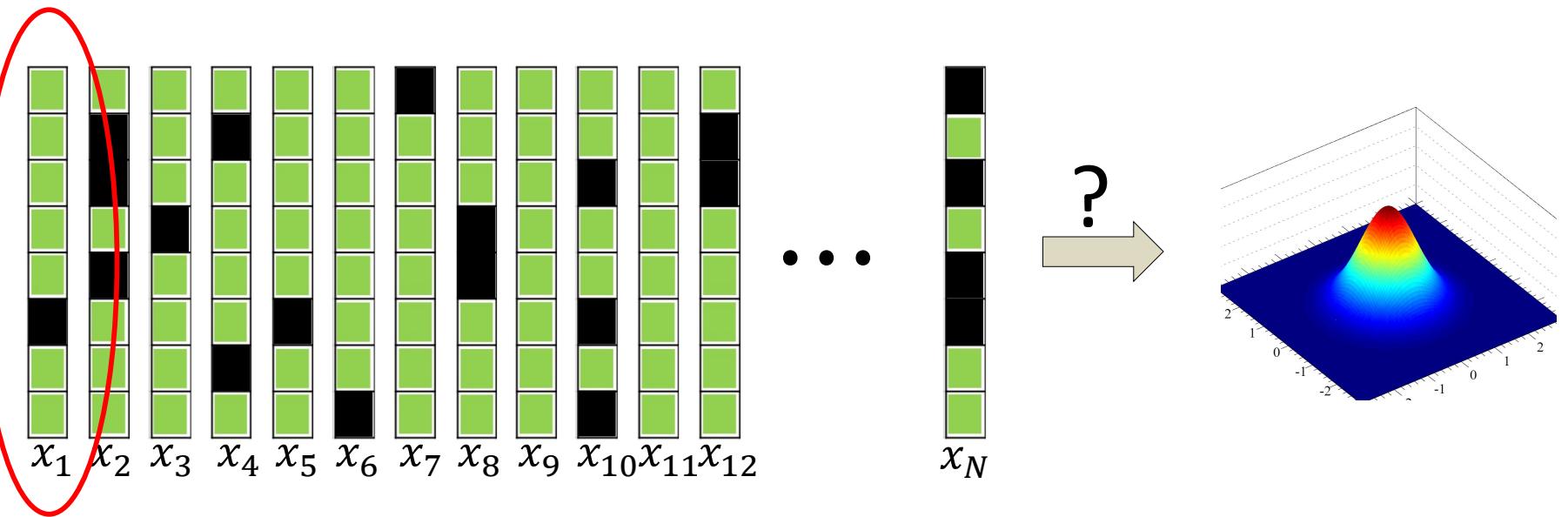
- Sometimes the data provided may be incomplete
 - Insufficient to estimate your model parameters directly
- This could be because the data themselves have missing components
 - E.g. Data vectors have some missing components
- Or because of the structure of the network
 - Mixture models, multi-stage Generative models

Recall this: Gaussian estimation with incomplete vectors



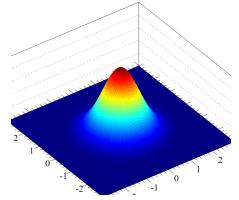
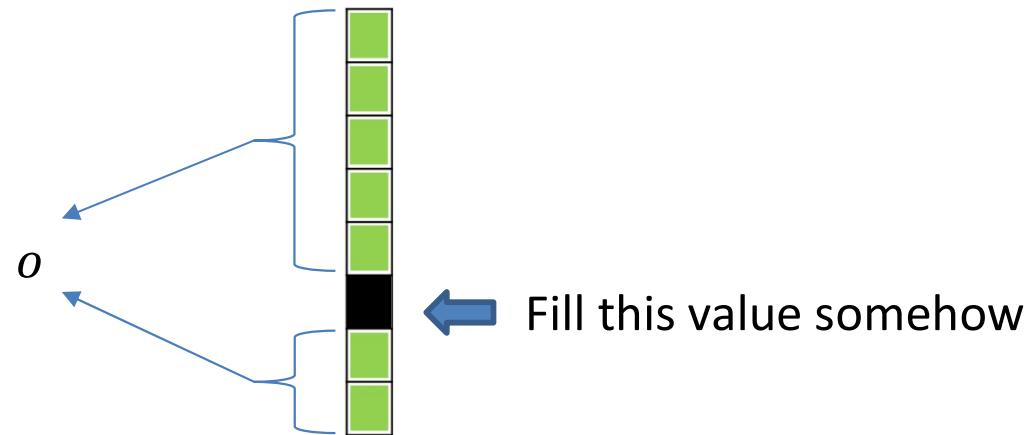
- These are the actual data we have: A set $O = \{o_1, \dots, o_N\}$ of *incomplete* vectors
 - Comprising only the *observed* components of the data
- We are *missing* the data $M = \{m_1, \dots, m_N\}$
 - Comprising the *missing* components of the data
- The *complete* data includes both the observed and missing components
$$X = \{x_1, \dots, x_N\}, \quad x_i = (o_i, m_i)$$
 - Keep in mind that at the complete data are *not* available (the missing components are missing)

Let's look at a single vector



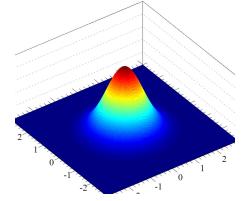
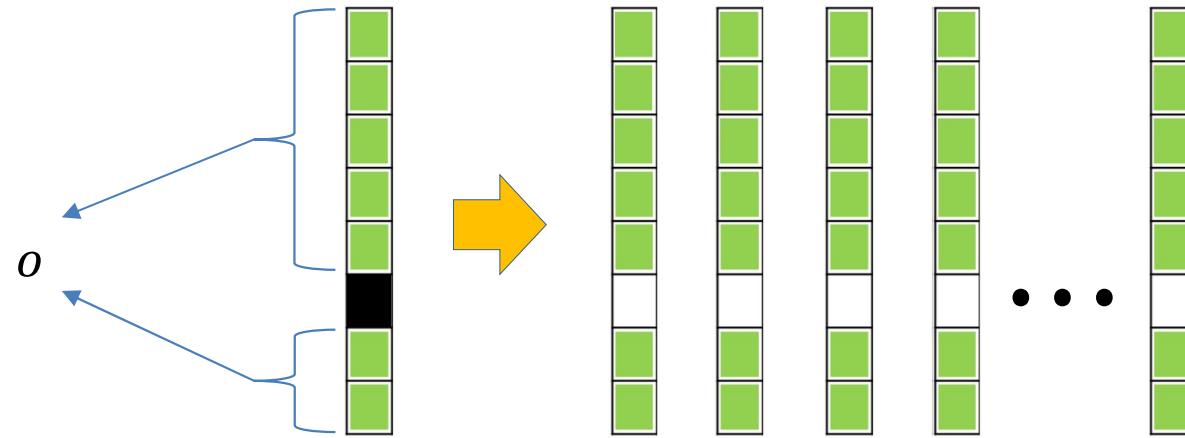
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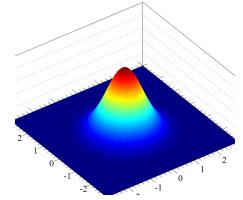
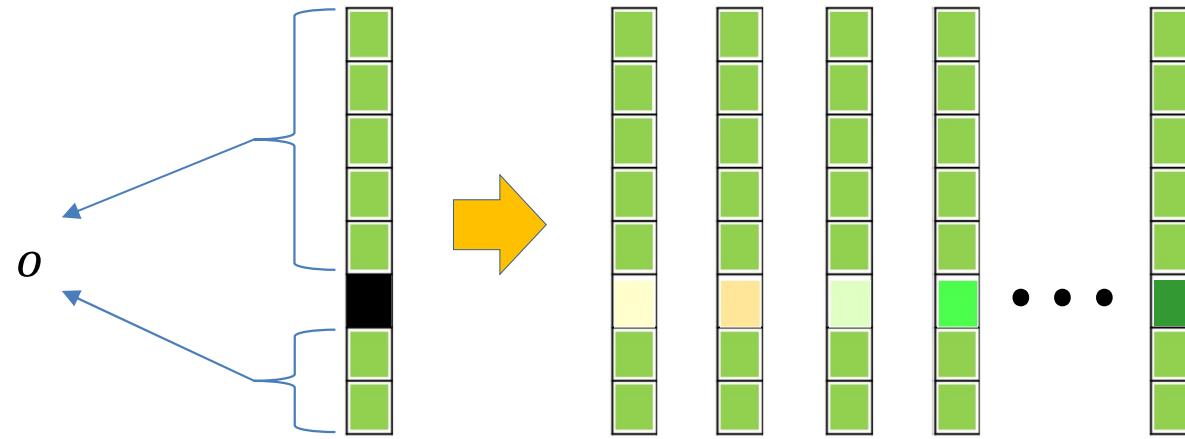
- We will try to complete the vector by filling in the missing value with *plausible* values that match the observed components
- Plausible: Values that “go with” the observed values, according to the distribution of the data

Lets look at a single vector



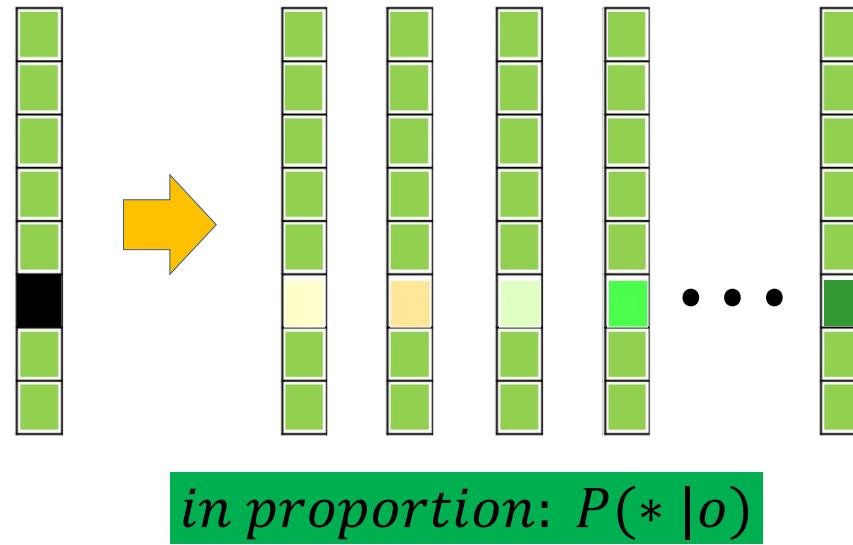
- Question: If we have a very large number of vectors from the Gaussian, all with the same observed components o , what would their missing components be?

Let's look at a single vector



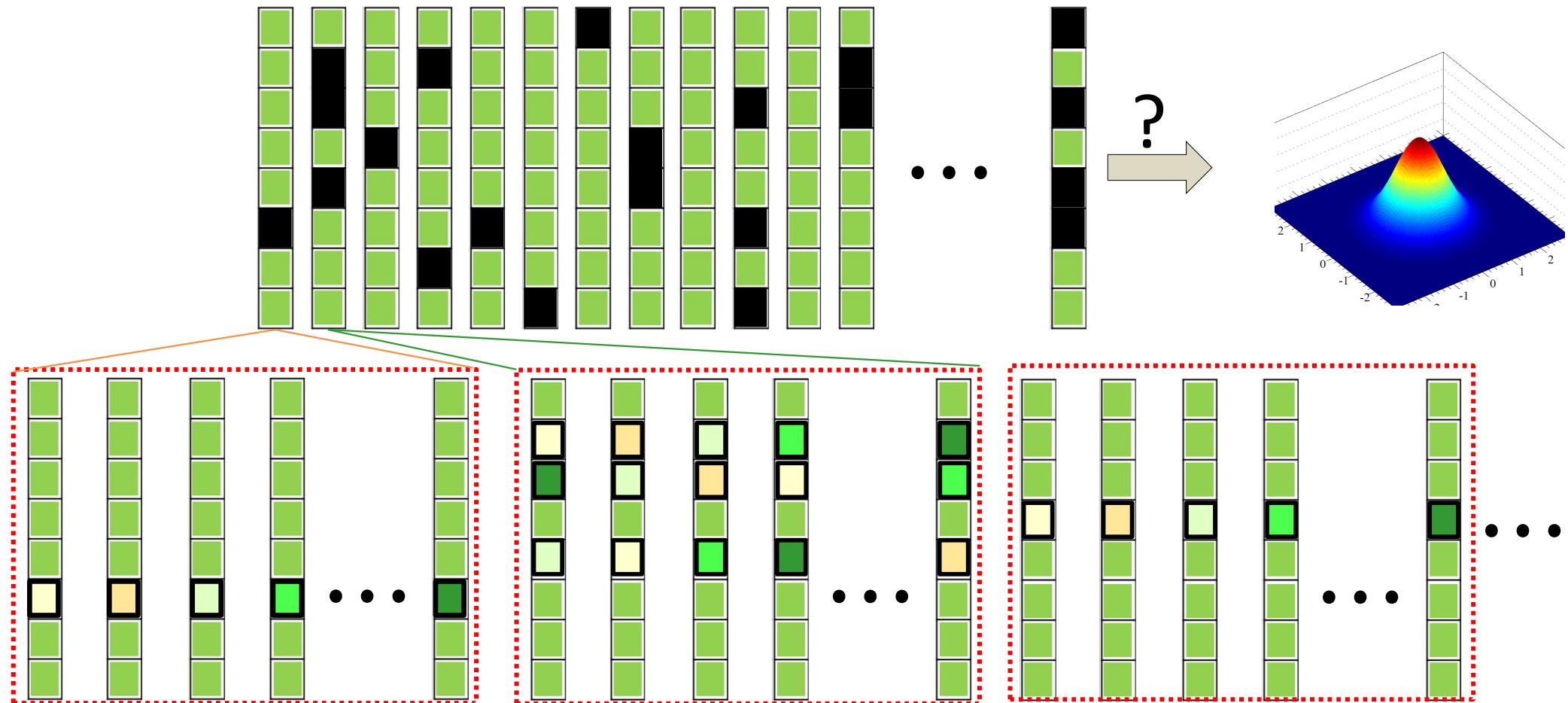
- Question: If we have a very large number of vectors from the Gaussian, all with the same observed components o , what would their missing components be?
- We would see every possible value, but in proportion to their probability: $P(m|o)$ (conditioned on the observations)

Completing incomplete vectors



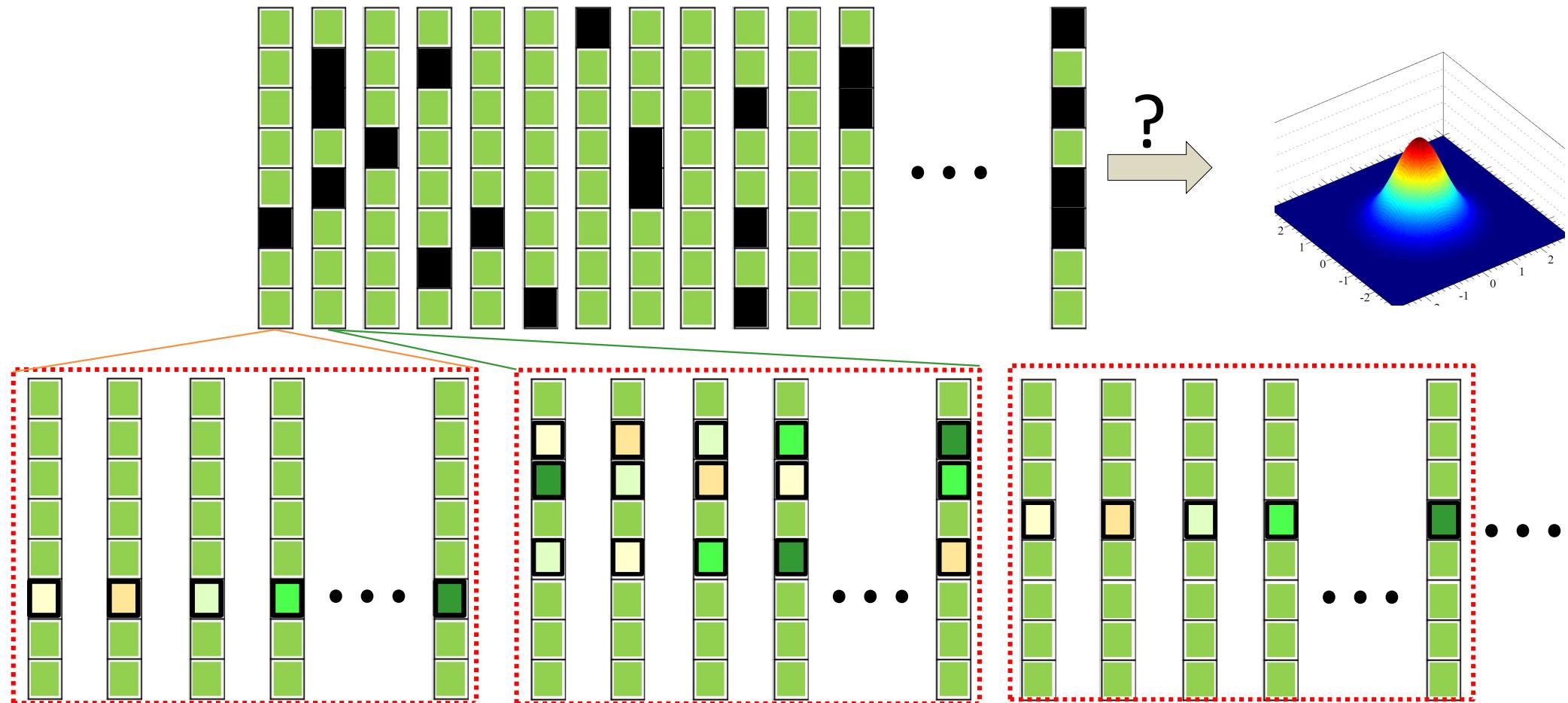
- Complete vector by filling up the missing components with *every possible value*
 - I.e. make many complete “clones” of the incomplete vector
- But assign a *proportion* to each value
 - Proportion is $P(m|o)$
 - Which can be computed if we know $P(x) = P(o, m)$

Gaussian estimation with incomplete vectors



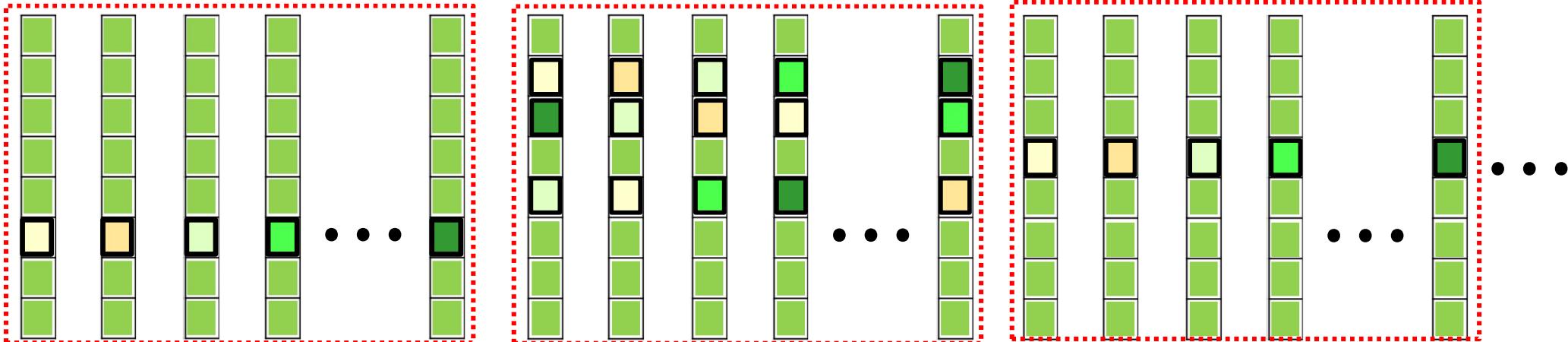
- “Expand” every incomplete vector out into all possibilities
 - In appropriate proportions $P(m|o)$
 - For already complete observations, there is no expansion
- Estimate the statistics from the expanded data

Gaussian estimation with incomplete vectors



- “Expand” every incomplete vector out into all possibilities
 - In appropriate proportions $P(m|o)$ From a previous estimate of the model
 - For already complete observations, there is no expansion
- Estimate the statistics from the expanded data

Estimating the Gaussian Parameters



- Compute the statistics from the (proportionately) expanded set
- Let $x_i(m)$ be the “completed” version of the observation o_i , when the missing components are filled with value m
 - There will be one such vector for every value of m

$$\mu^{k+1} = \frac{1}{N} \sum_{x_i(m)} x_i(m)$$

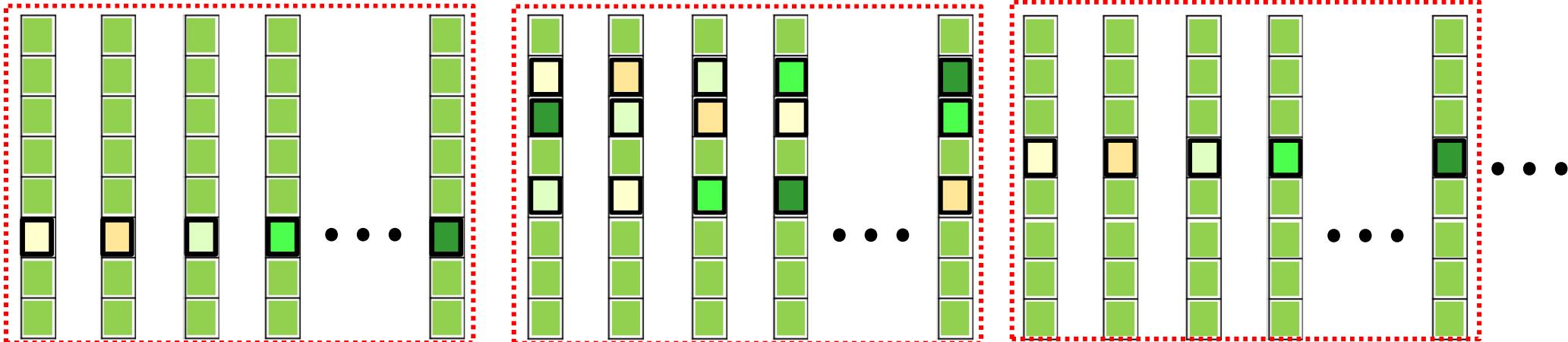
- We have several $x_i(m)$ for each o_i . Group the sum by o_i .
- Recall that for each o_i , the number of $x_i(m)$ for each m is proportion to $P(m|o; \theta^k)$.

$$\mu^{k+1} = \frac{1}{N_o N_{m|o}} \sum_{o \in O} \sum_m P(m|o; \theta^k) x_i(m) = \frac{1}{N_o} \sum_{o \in O} \frac{1}{N_{m|o}} \sum_m P(m|o; \theta^k) x_i(m)$$

- In the limit, if we consider *every* value of m

$$\mu^{k+1} = \frac{1}{N_o} \sum_{o \in O} \int_{-\infty}^{\infty} P(m|o; \theta^k) x_i(m) dm$$

Estimating the Gaussian Parameters

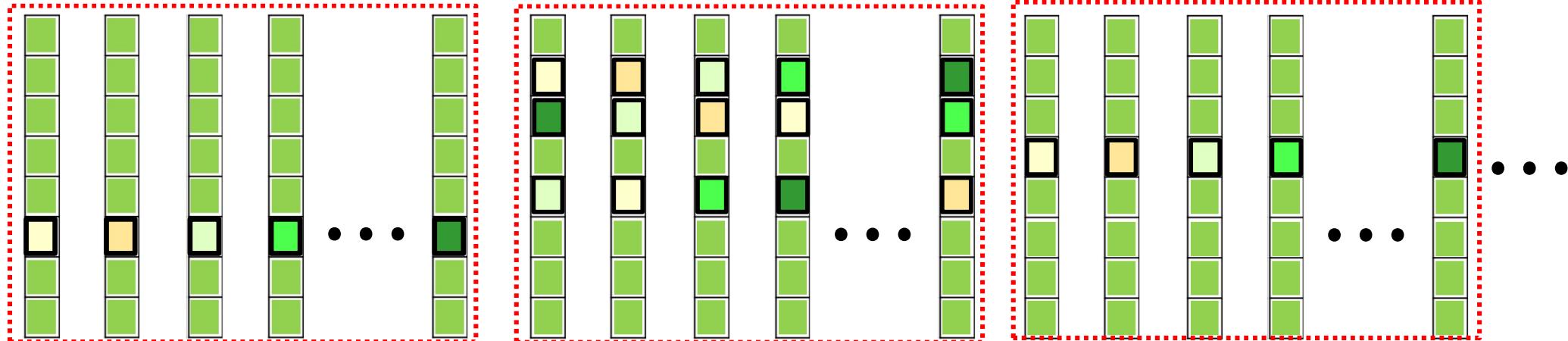


- Compute the statistics from the (proportionately) expanded set
- Let $x_i(m)$ be the “completed” version of the observation o_i , when the missing components are filled with value m
 - There will be one such vector for every value of m
- Estimate the statistics from the expanded data

$$\mu^{k+1} = \frac{1}{N} \sum_{o \in O} \int_{-\infty}^{\infty} P(m|o; \theta^k) x_i(m) dm$$

$$\Sigma^{k+1} = \frac{1}{N} \sum_{o \in O} \int_{-\infty}^{\infty} P(m|o; \theta^k) (x_i(m) - \mu^{k+1})(x_i(m) - \mu^{k+1})^T dm$$

EM for computing the Gaussian Parameters



- Initial $\theta^0 = (\mu^0, \Sigma^0)$
- Until $P(O; \theta)$ converges:

$$\mu^{k+1} = \frac{1}{N} \sum_{o \in O} \int_{-\infty}^{\infty} P(m|o; \theta^k) x_i(m) dm$$

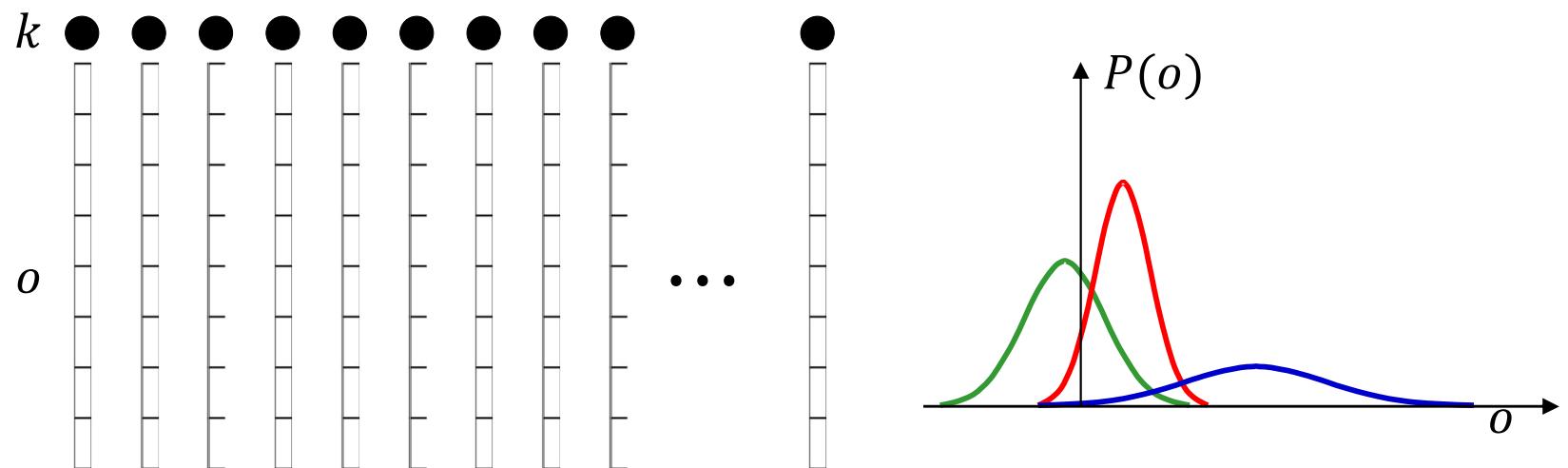
$$\Sigma^{k+1} = \frac{1}{N} \sum_{o \in O} \int_{-\infty}^{\infty} P(m|o; \theta^k) (x_i(m) - \mu^{k+1})(x_i(m) - \mu^{k+1})^T dm$$

Where $x_i(m) = (m, o_i)$ and the parameters of $P(m|o; \theta^k)$ are derived from the $P(x; \theta^k) = Gaussian(x; \mu^k, \Sigma^k)$

Recap: Maximum Likelihood Estimation

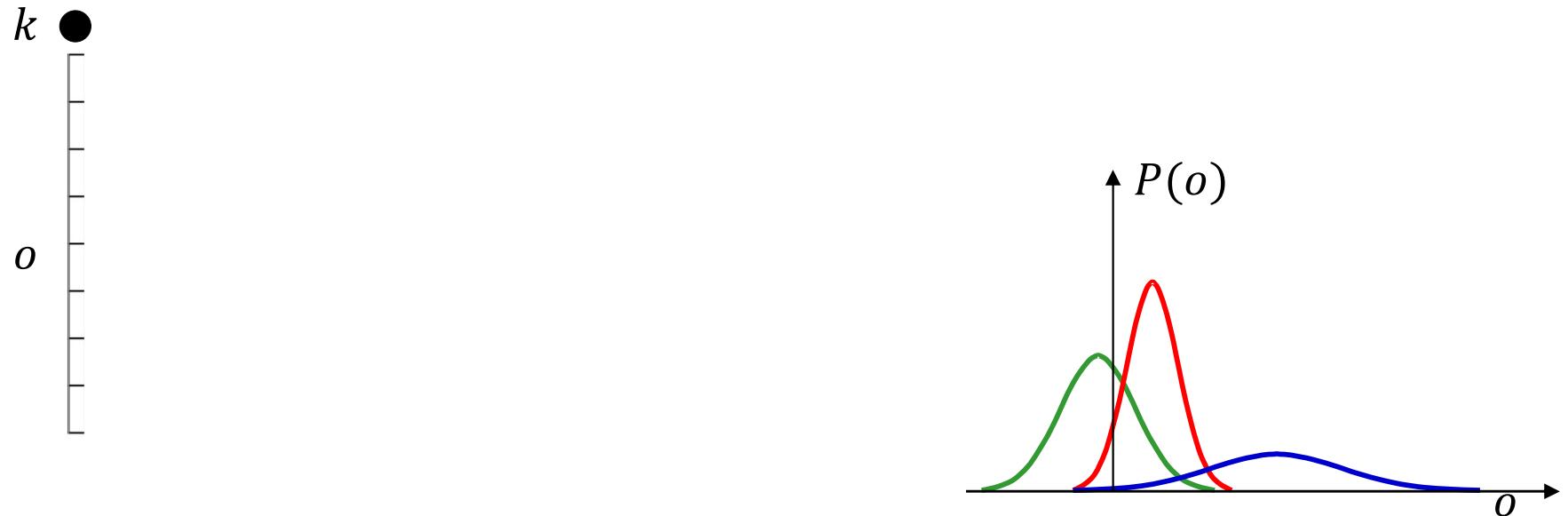
- Sometimes the data provided may be incomplete
 - Insufficient to estimate your model parameters directly
- This could be because the data themselves have missing components
 - E.g. Data vectors have some missing components
- Or because of the structure of the network
 - Mixture models, multi-stage Generative models

The GMM problem of incomplete data: missing information



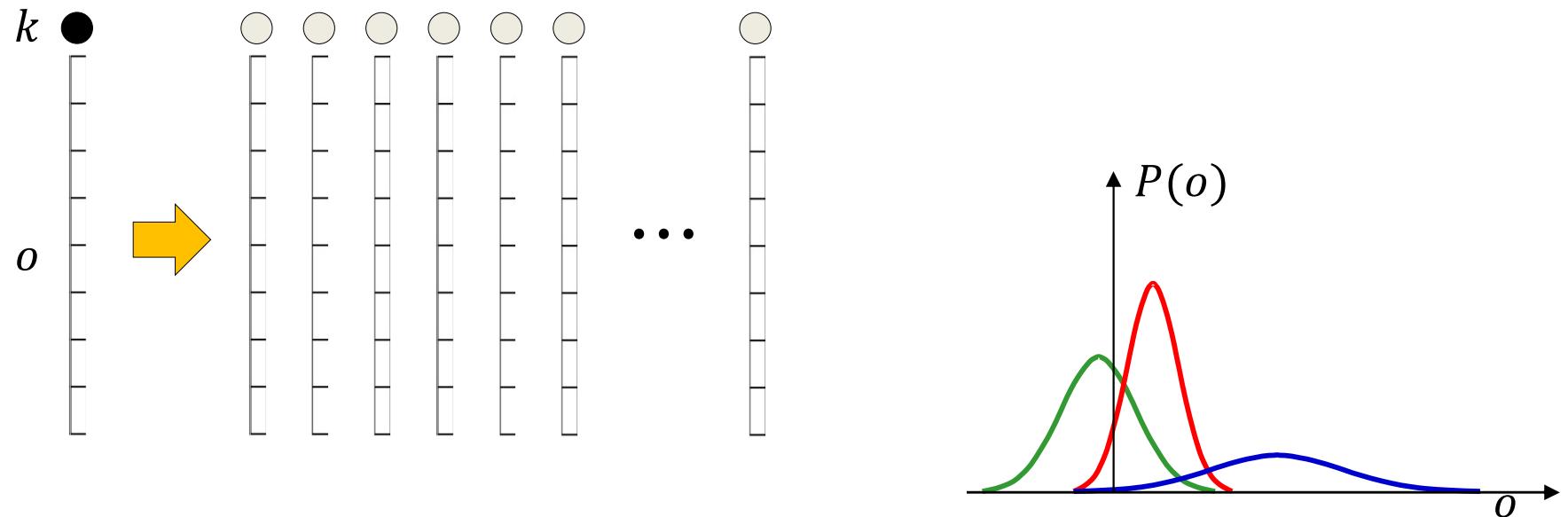
- Problem : We are not given the actual Gaussian for each observation
 - Our data are incomplete
- What we want : $(o_1, k_1), (o_2, k_2), (o_3, k_3) \dots$
- What we have: $o_1, o_2, o_3 \dots$

Consider a single vector



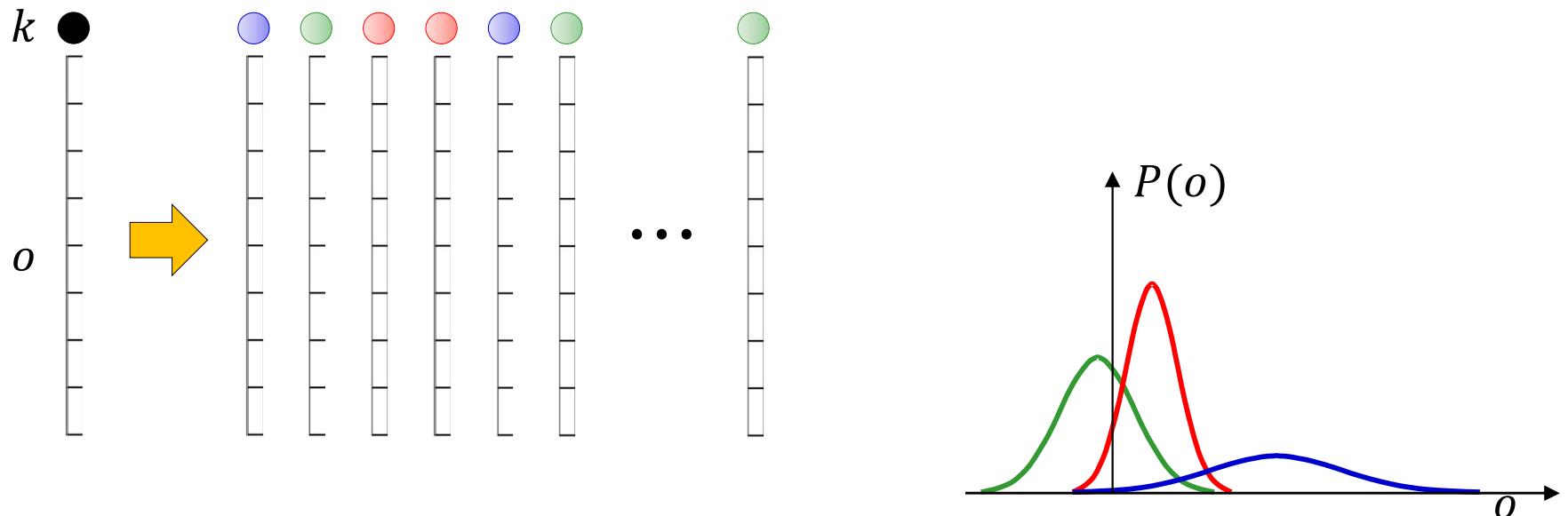
- *Every* Gaussian is capable of generating this vector
 - With different probabilities

Consider a single vector



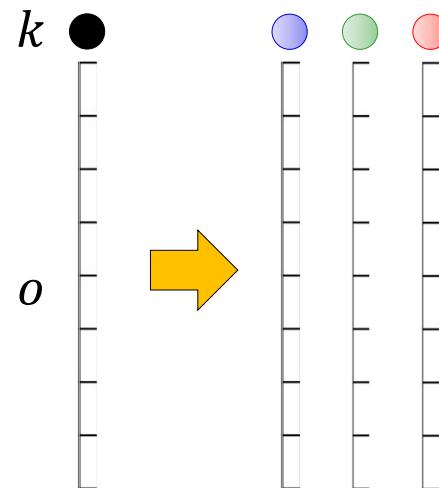
- Every Gaussian is capable of generating this vector
 - With different probabilities
- If we saw a large number of these vectors, how many of these would have come from each Gaussian?

Consider a single vector



- Every Gaussian is capable of generating this vector
 - With different probabilities
- If we saw a large number of these vectors, how many of these would have come from each Gaussian
- All of them, but in proportion to $P(k|o)$

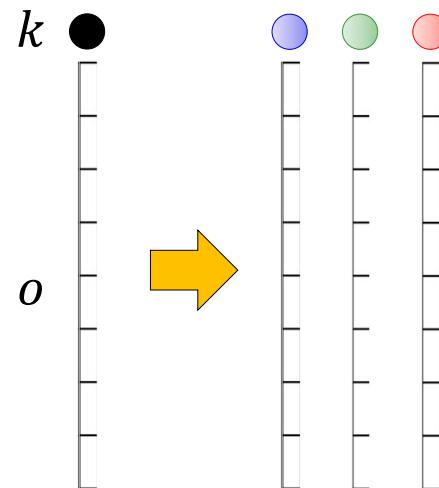
Completing incomplete vectors



in proportion: $P()|o)$*

- Complete the data by attributing to *every Gaussian*
 - I.e. make many complete “clones” of the data
- But assign a *proportion* to each completed vector
 - Proportion is $P(k|o)$
 - Which can be computed if we know $P(k)$ and $P(o|k)$
- Then estimate the parameters using the complete data

Completing incomplete vectors



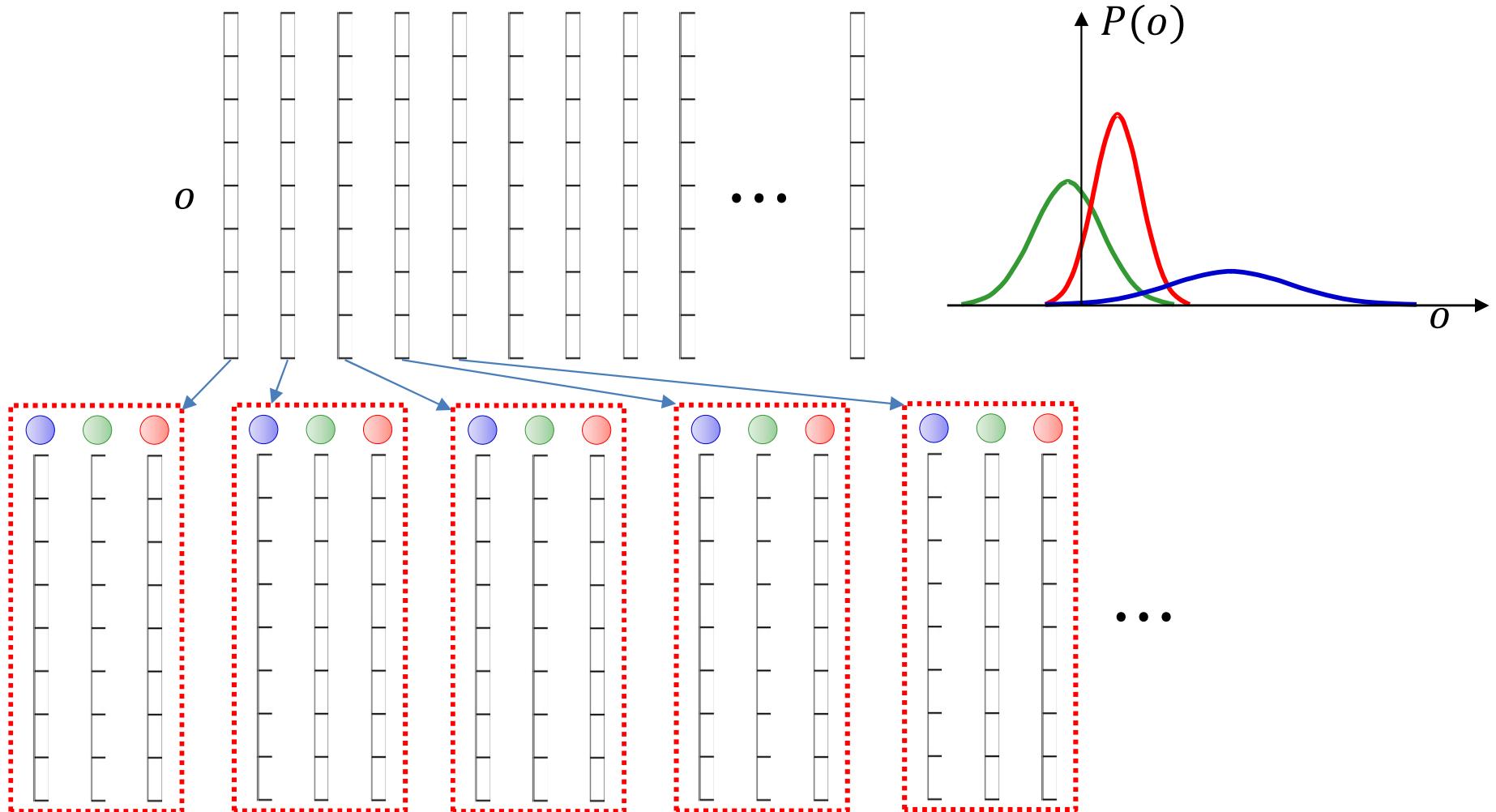
in proportion: $P()|o)$*

- Complete the data by attributing to *every Gaussian*
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- Then estimate the parameters using the complete data

From previous estimate
of model

$$P(k) \text{ and } P(o|k)$$

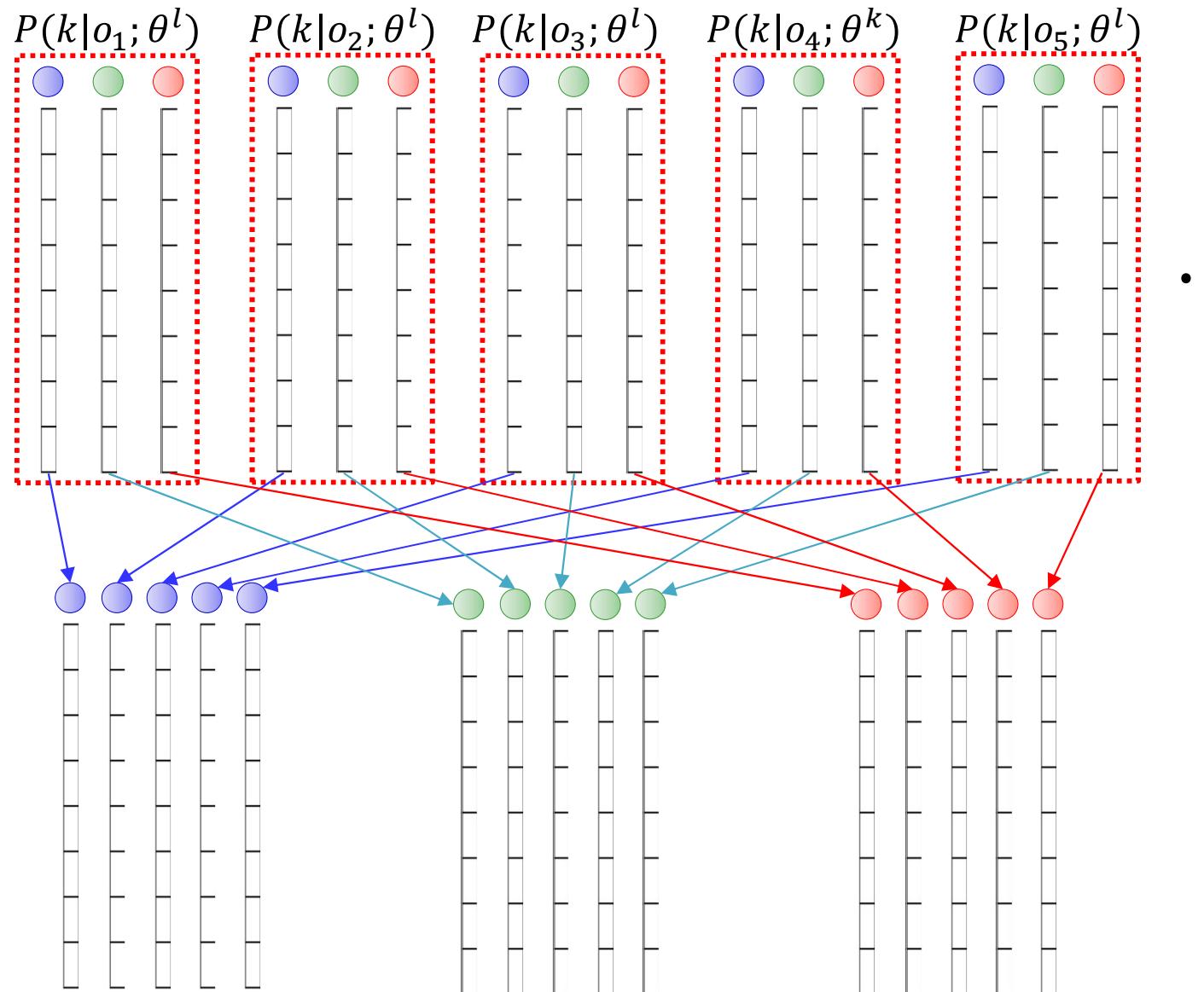
EM for GMMs



- “Complete” each vector in every possible way:
 - assign each vector to every Gaussian
 - In proportion $P(k|o; \theta^l)$ (computed from current model estimate)
- Compute statistics from “completed” data

EM for GMMs

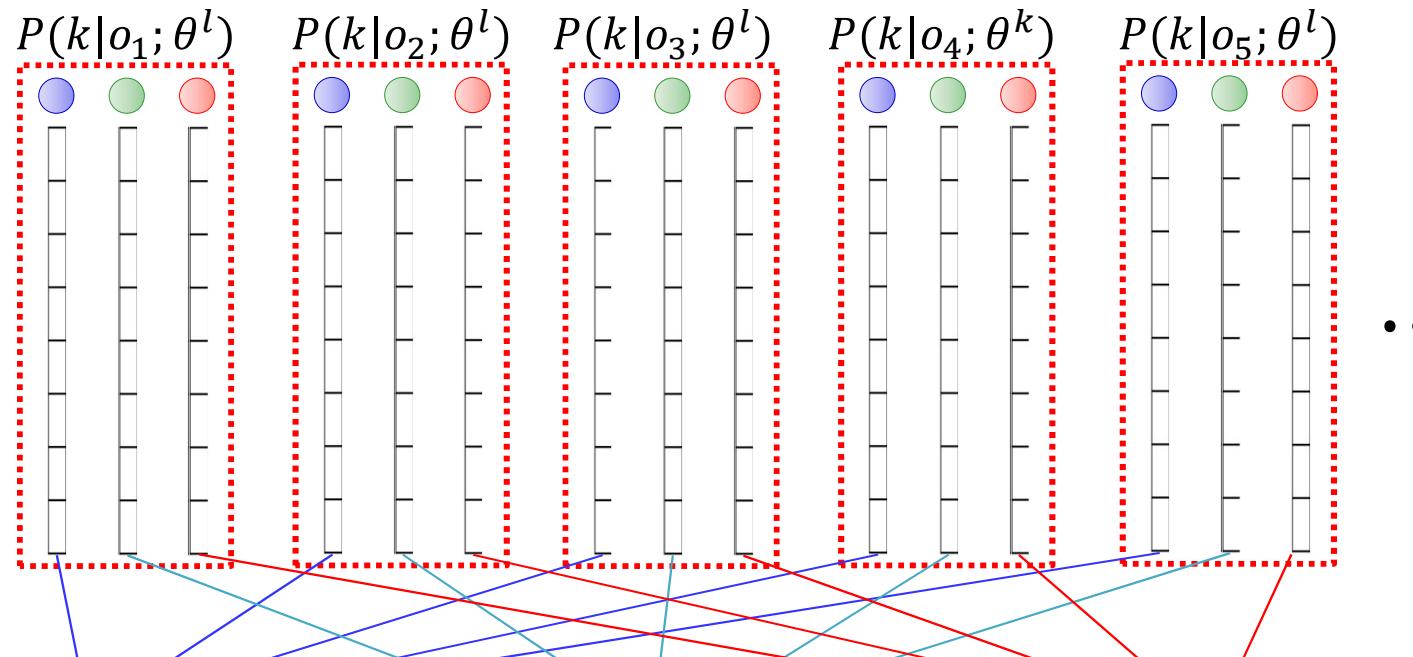
In proportion to



- Now you can segregate the vectors by Gaussian
 - The number of segregated complete vectors from each observation will be in proportion to $P(k|o; \theta^l)$

EM for GMMs

In proportion to



$$\mu_k^{l+1} = \frac{1}{\sum_o P(k|o; \theta^l)} \sum_o P(k|o; \theta^l) o$$

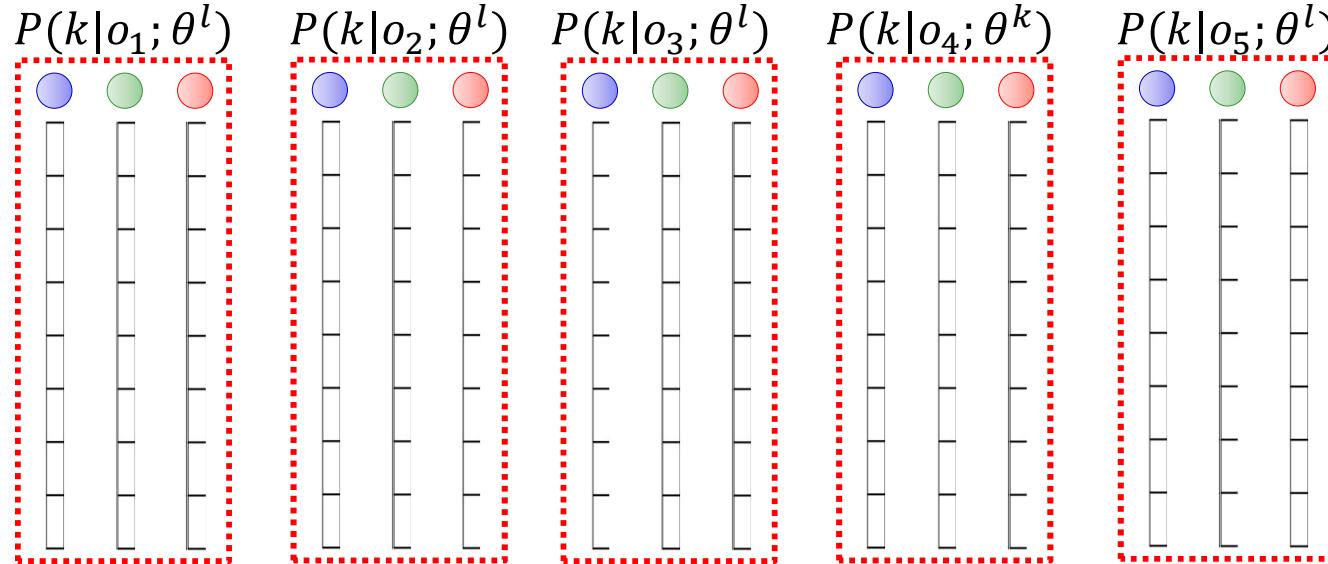
$$\Sigma_k^{l+1} = \frac{1}{\sum_o P(k|o; \theta^l)} \sum_o P(k|o; \theta^l) (o - \mu_k^{l+1})(o - \mu_k^{l+1})^T$$

- Now you can segregate the vectors by Gaussian

- The number of segregated complete vectors from each observation will be in proportion to $P(k|o; \theta^l)$

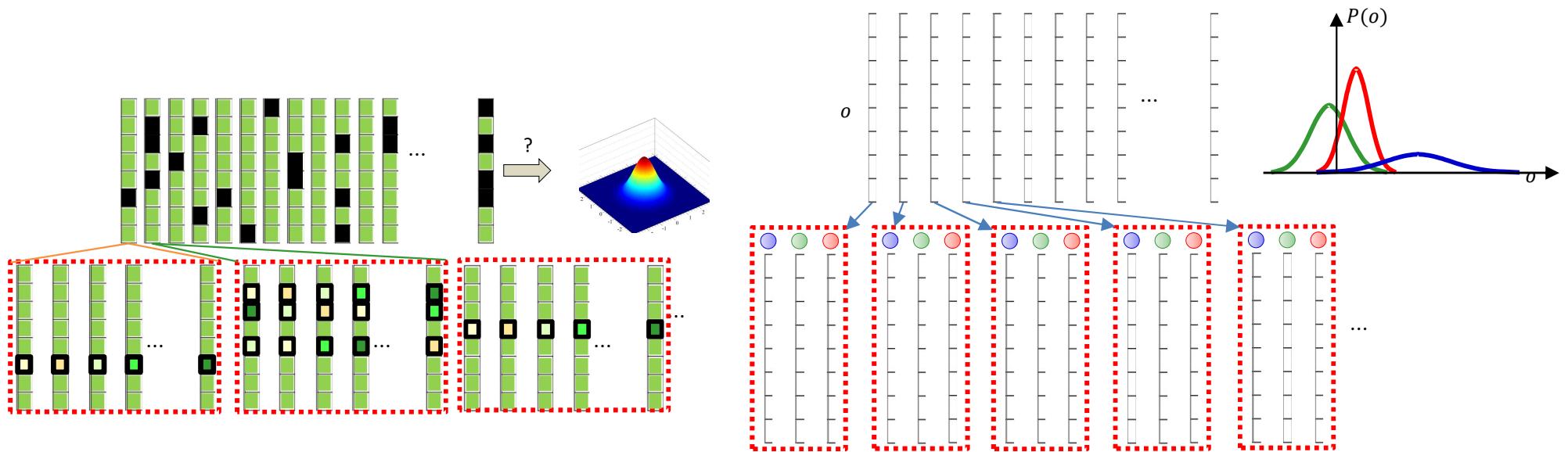
EM for GMMs

In proportion to



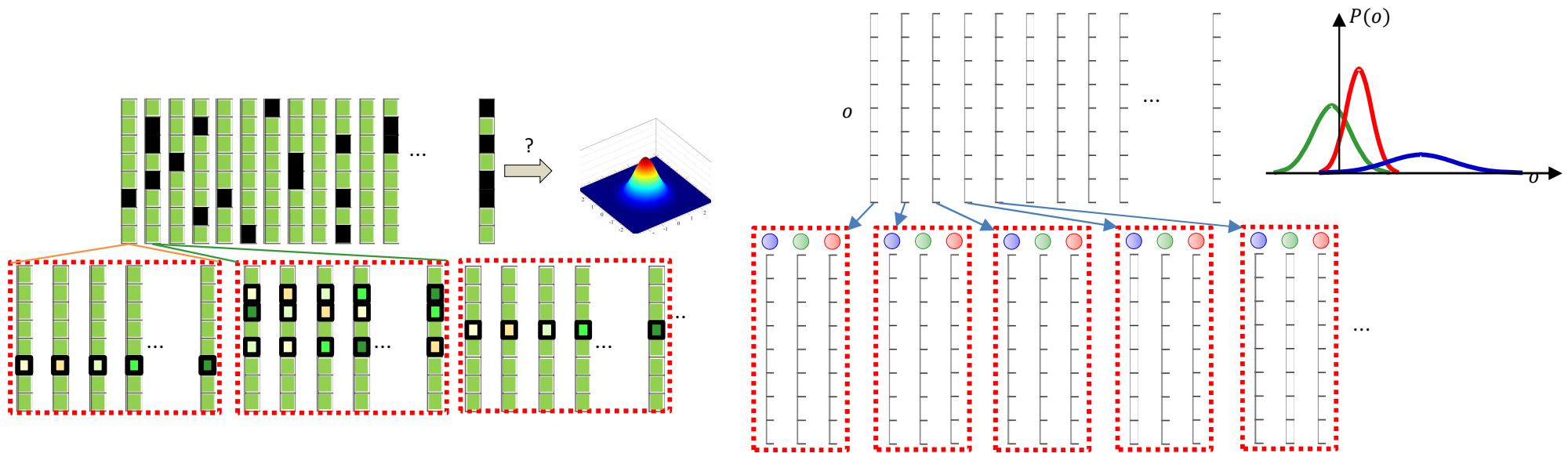
- Initialize μ_k^0 and Σ_k^0 for all k
- Iterate (over l):
 - Compute $P(k|o; \theta^l)$ for all o
 - Compute the proportions by which o is assigned to all Gaussians
 - Update:
 - $\mu_k^{l+1} = \frac{1}{\sum_o P(k|o; \theta^l)} \sum_o P(k|o; \theta^l) o$
 - $\Sigma_k^{l+1} = \frac{1}{\sum_o P(k|o; \theta^l)} \sum_o P(k|o; \theta^l) (o - \mu_k^{l+1})(o - \mu_k^{l+1})^T$

General EM principle



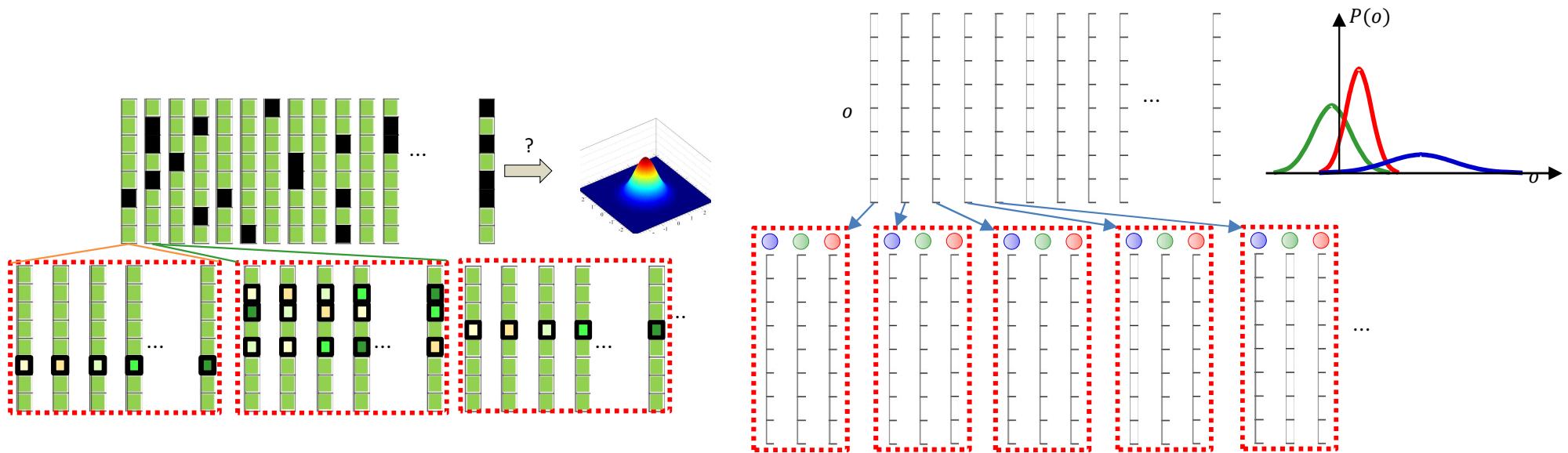
- “Complete” the data by considering *every* possible value for missing data/variables
 - In proportion to their posterior probability, given the observation, $P(m|o)$ (or $P(k|o)$)
- Reestimate parameters from the “completed” data

General EM principle



- “Complete” the data by considering *every* possible value for missing data/variables
 - In proportion to their posterior probability, given the observation, $P(m|o)$ (or $P(k|o)$)
- Reestimate parameters from the “completed” data

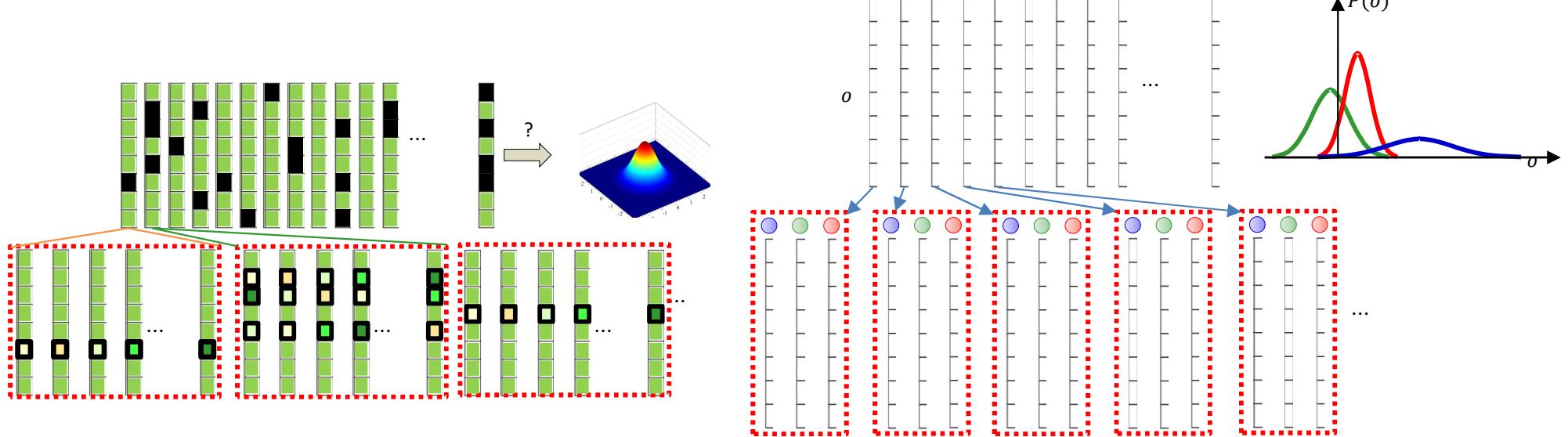
General EM principle



- “Complete” the data by considering *every* possible value for missing data/variables
 - In proportion to their posterior probability, given the observation, $P(m|o)$ (or $P(k|o)$)

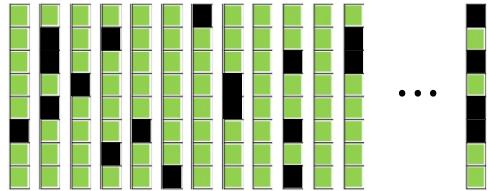
Sufficient to “complete” the data by *sampling* missing values from the posterior $P(m|o)$ (or $P(k|o)$) instead

Alternate EM principle

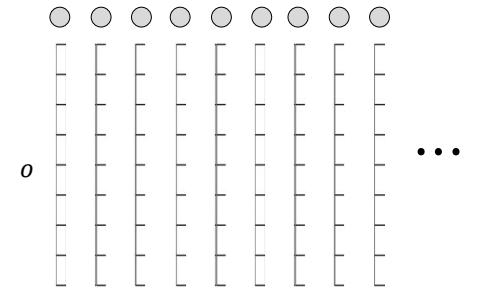


- “Complete” the data by *sampling* possible value for missing data/variables from $P(m|o)$ (or $P(k|o)$)
- Reestimate parameters from the “completed” data

Overall EM principle: Remember this



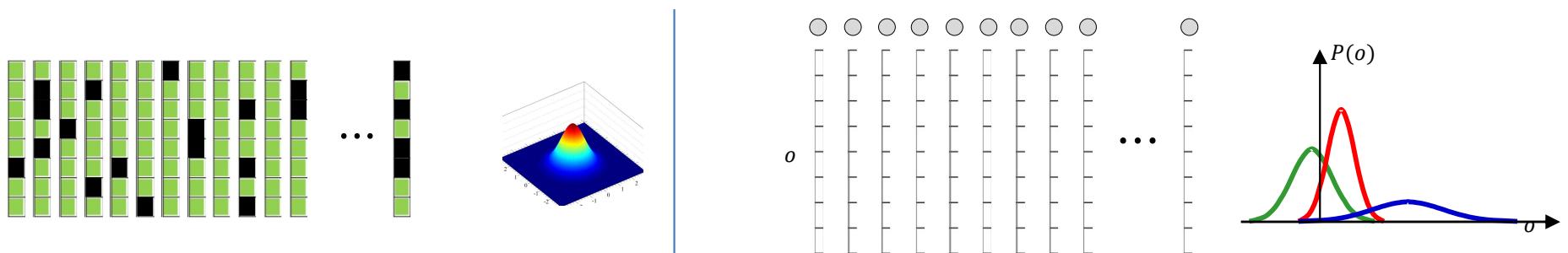
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...

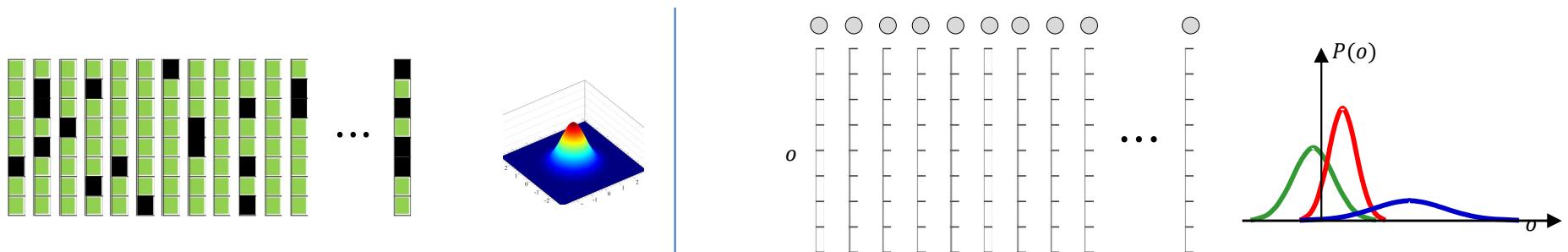
- Initially, some data/information are missing

Overall EM principle: Remember this



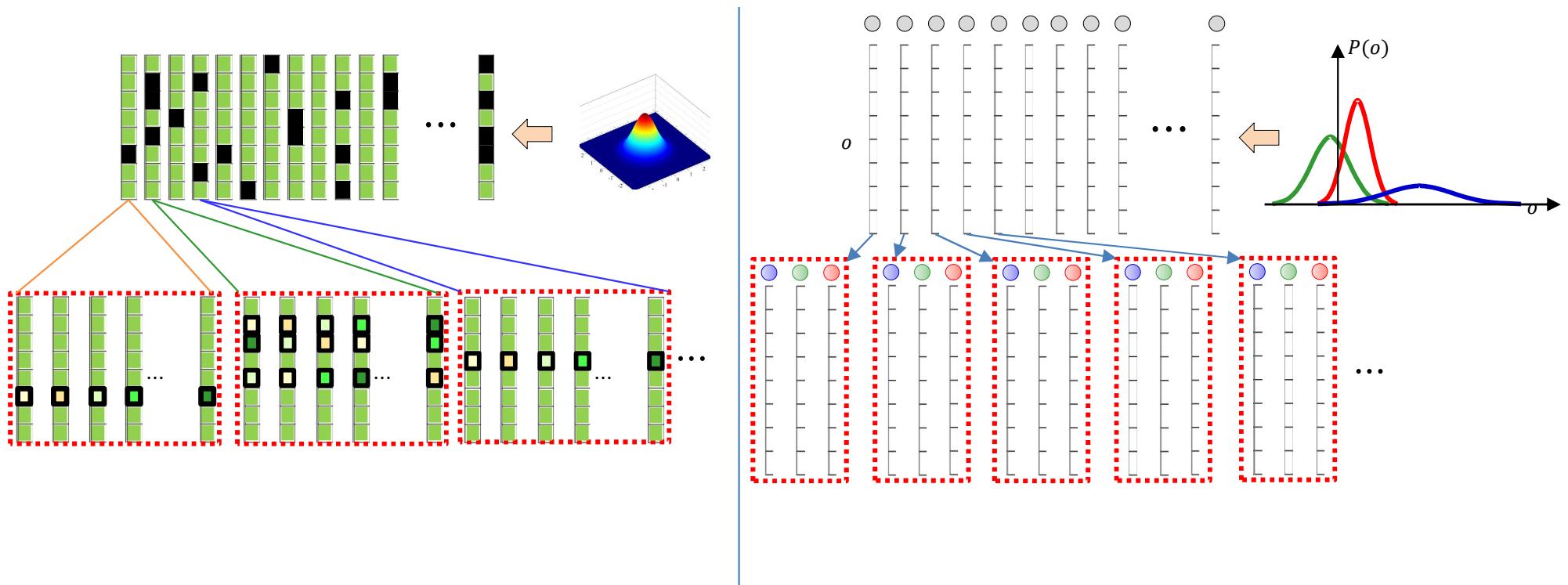
- Initially, some data/information are missing
- *Initialize model parameters*

Overall EM principle: Remember this



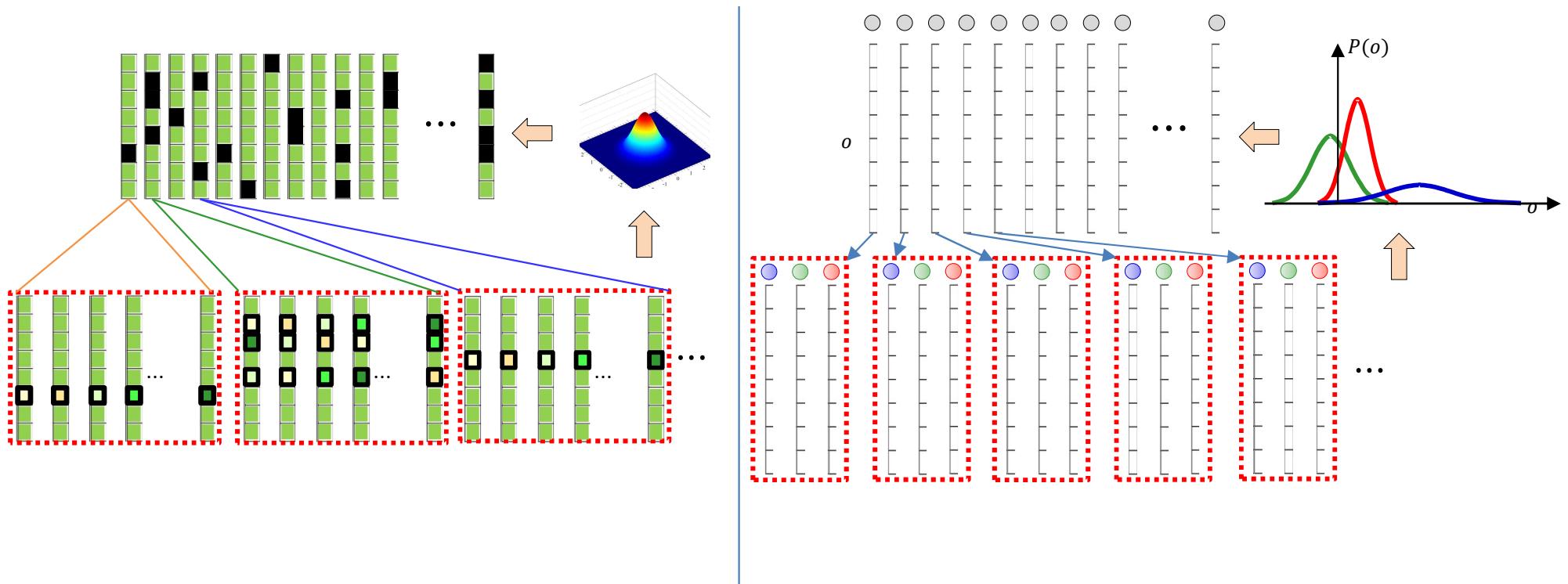
- Initially, some data/information are missing
- Initialize model parameters
- **Iterate:**

Overall EM principle: Remember this



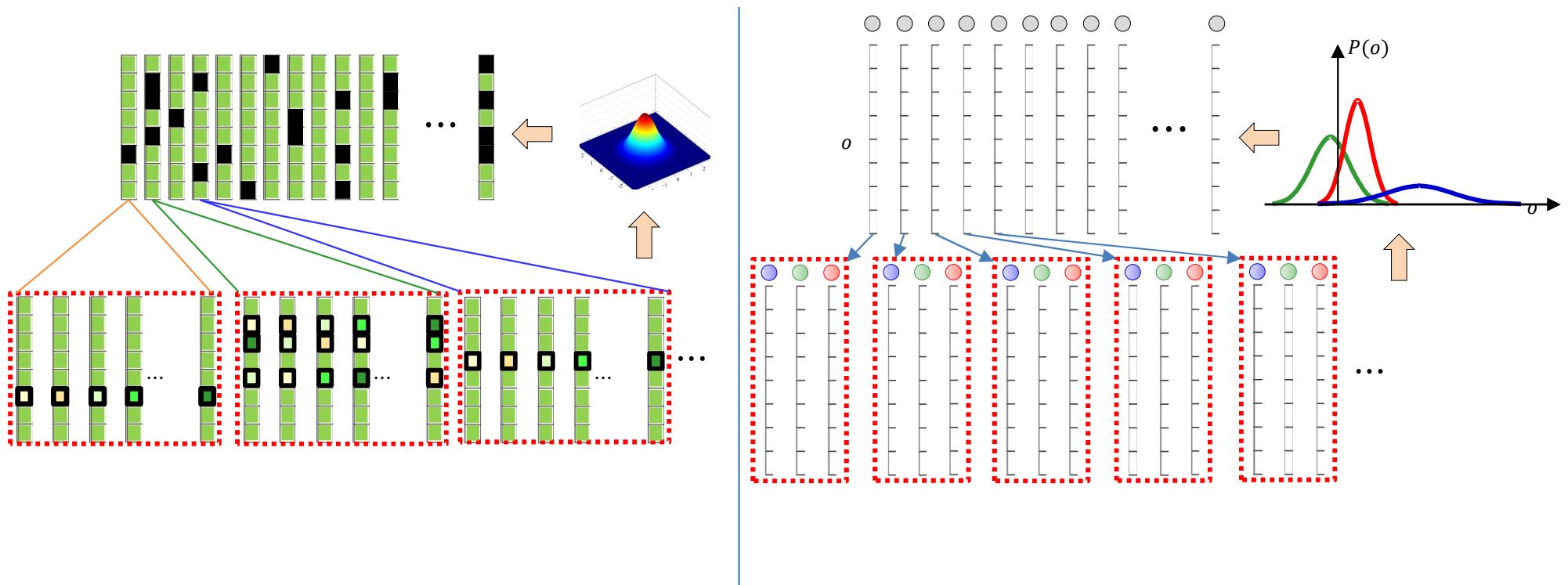
- Initially, some data/information are missing
- Initialize model parameters
- Iterate
 - **Complete the data according to the posterior probabilities $P(m|o)$ computed by the current model**
 - By implicitly considering every possible value, with its posterior-based proportionality
 - Or by explicit completion through sampling the posterior probability distribution $P(m|o)$

Overall EM principle: Remember this



- Initially, some data/information are missing
- Initialize model parameters
- Iterate
 - Complete the data according to the posterior probabilities $P(m|o)$ computed by the current model
 - By considering every possible value, with its posterior-based proportionality
 - Or by sampling the posterior probability distribution $P(m|o)$
 - **Reestimate the model**

Overall EM principle: Remember this



- Initially, some data/information are missing
- Initialize model parameters
- Iterate
 - Complete the data according to the posterior probabilities $P(m|o)$ computed by the current model
 - By implicitly considering every possible value, with its posterior-based proportionality
 - Or by explicit completion through sampling the posterior probability distribution $P(m|o)$
 - Reestimate the model

Poll 3

Select all that are true of EM estimation

- In each iteration we “complete” the data, by filling in the missing components/variables, and estimate parameters from the entire completed data
- A data instance can be completed by filling in the missing terms with every possible value, in proportion to their a-posteriori probability, given the observed components of the data
- A data instance can be completed by randomly drawing samples of the missing components from their a-posteriori probability distribution, given the observed components of the data
- “Data completion” must be performed only once during the entire training (with EM)

Poll 3 (@1207)

Select all that are true of EM estimation

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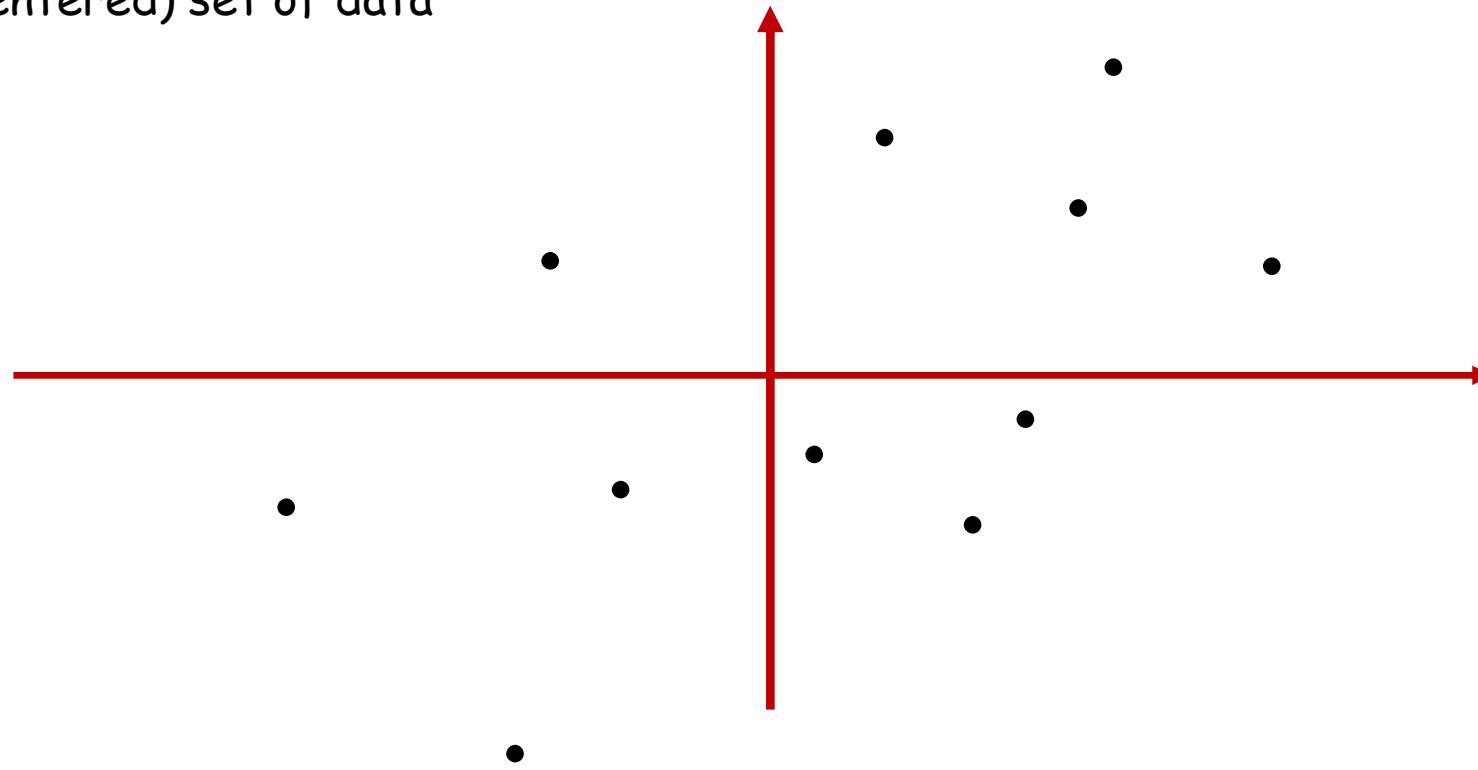
**And now
for something
completely different...**



Principal Component Analysis

Principal Component Analysis

Given a (centered) set of data

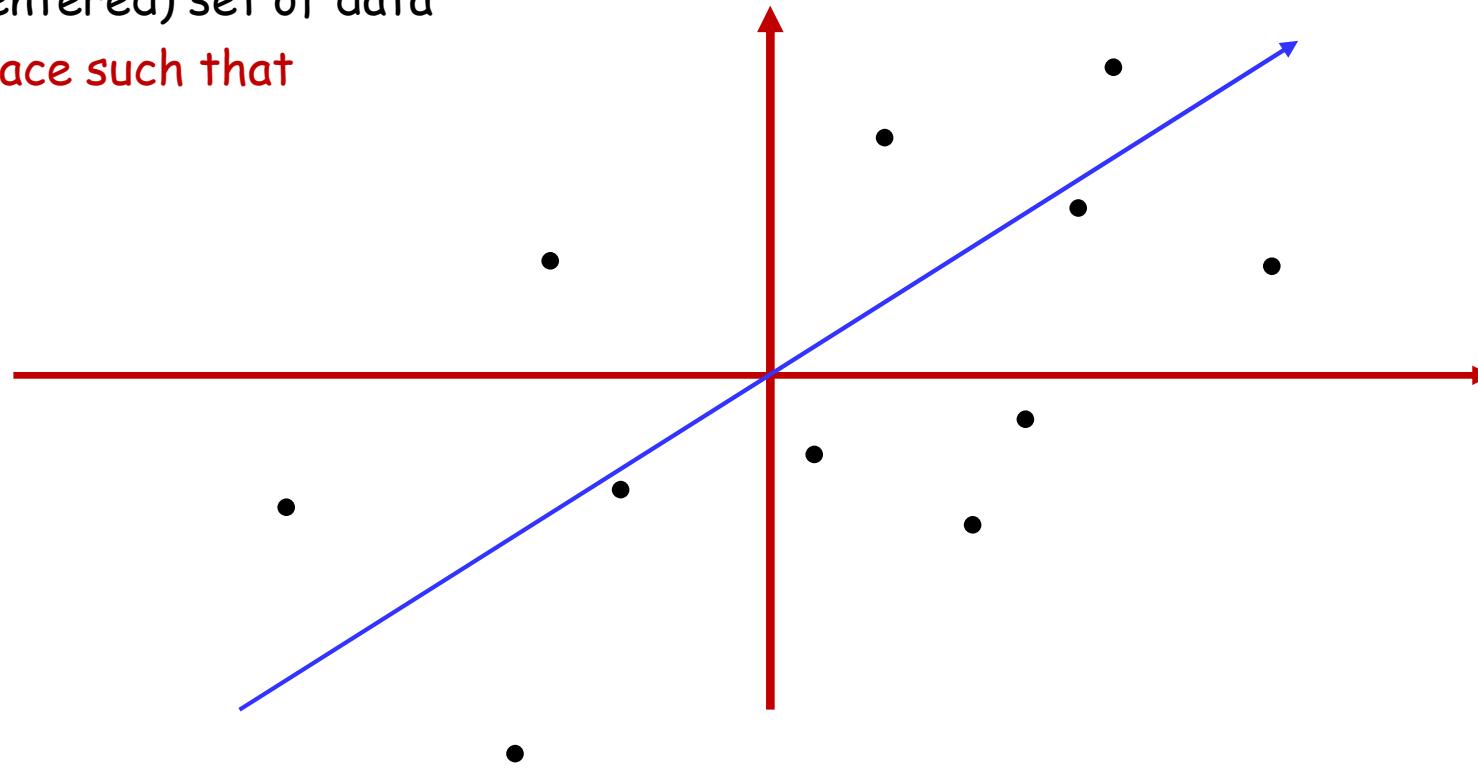


- Find the principal subspace such that when all vectors are approximated as lying on that subspace, the approximation error is minimal
 - Assuming “centered” (zero-mean) data

Principal Component Analysis

Given a (centered) set of data

find subspace such that



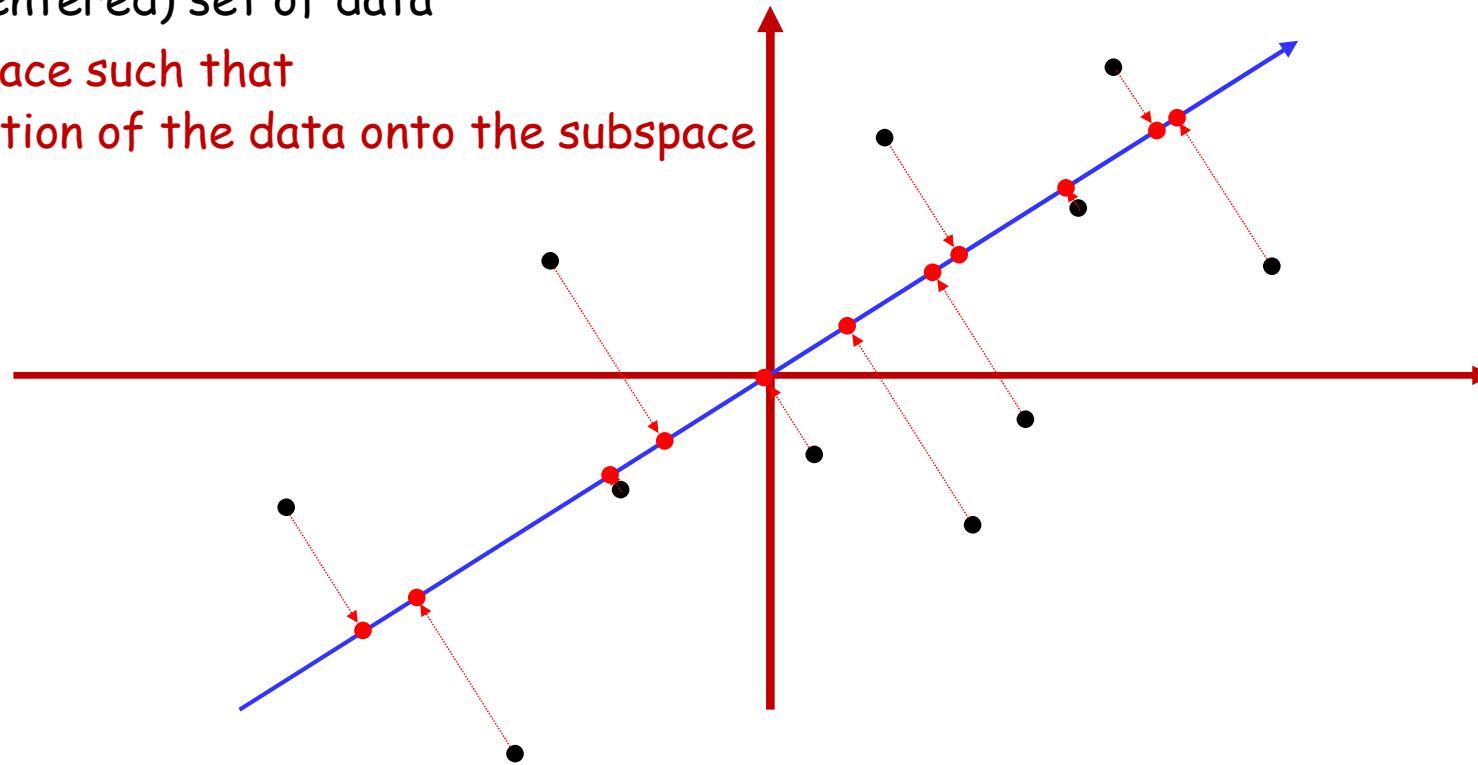
- Find the principal subspace such that when all vectors are approximated as lying on that subspace, the approximation error is minimal
 - Assuming “centered” (zero-mean) data

Principal Component Analysis

Given a (centered) set of data

find subspace such that

the projection of the data onto the subspace



- Find the principal subspace such that when all vectors are approximated as lying on that subspace, the approximation error is minimal
 - Assuming “centered” (zero-mean) data

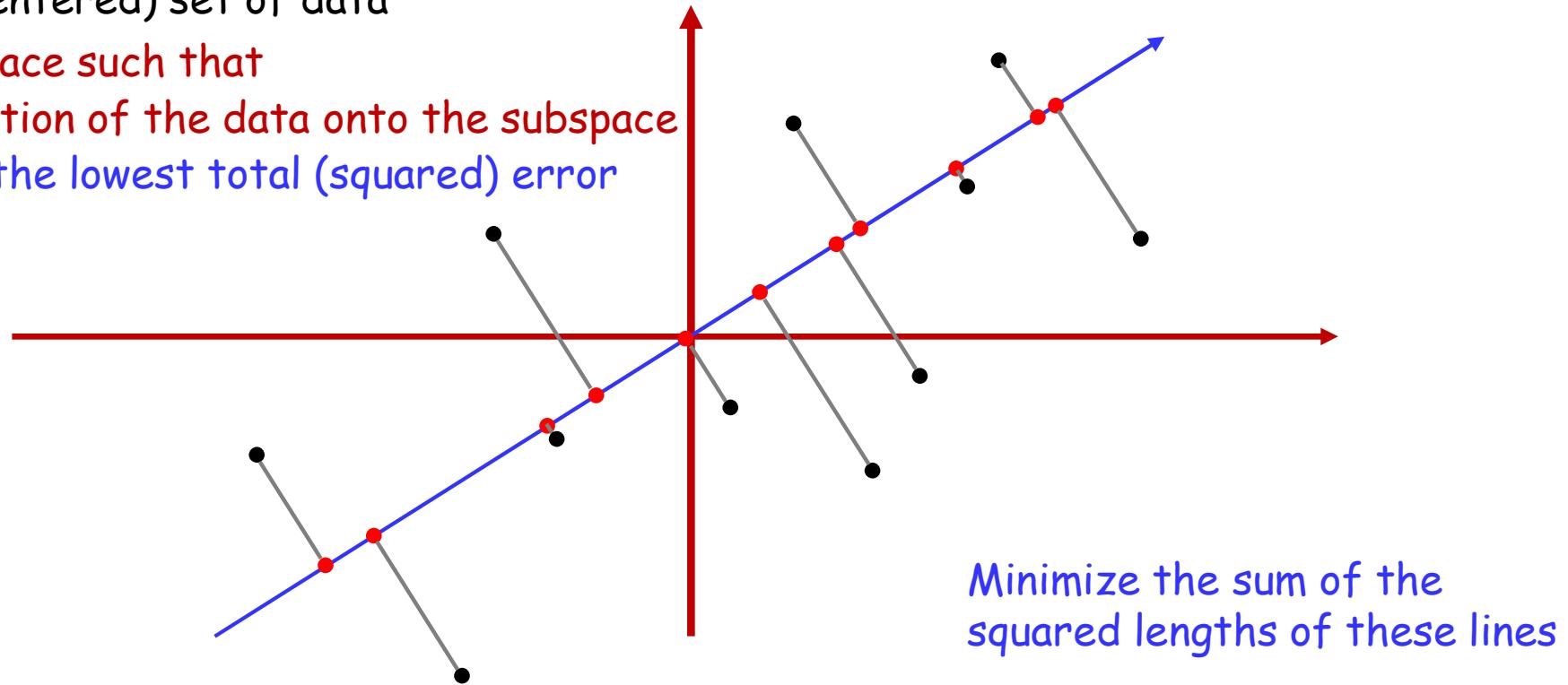
Principal Component Analysis

Given a (centered) set of data

find subspace such that

the projection of the data onto the subspace

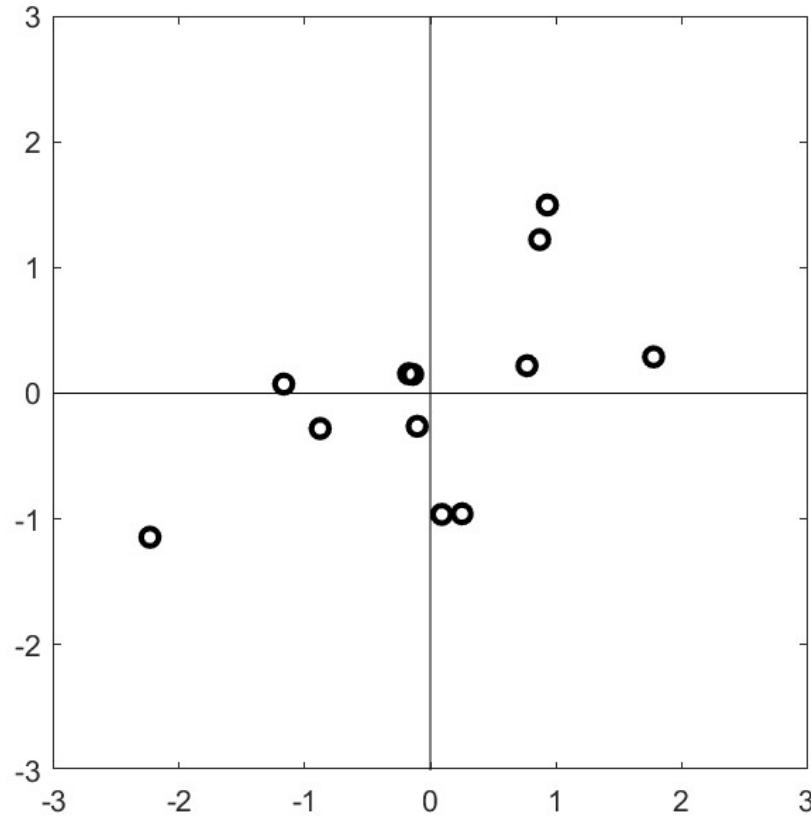
results in the lowest total (squared) error



- Find the principal subspace such that when all vectors are approximated as lying on that subspace, the approximation error is minimal
 - Assuming “centered” (zero-mean) data

Principal Component Analysis

Animation:
Original centered data

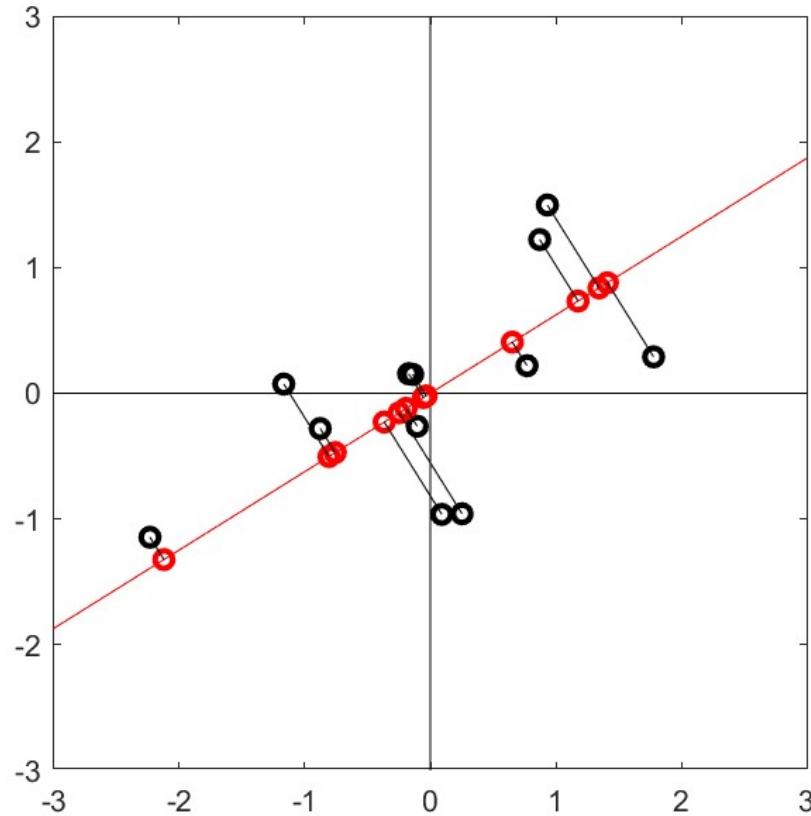


- Find the principal subspace such that when all vectors are approximated as lying on that subspace, the approximation error is minimal
 - Assuming “centered” (zero-mean) data

Principal Component Analysis

Animation:
Original centered data

Principal axis we're
searching for

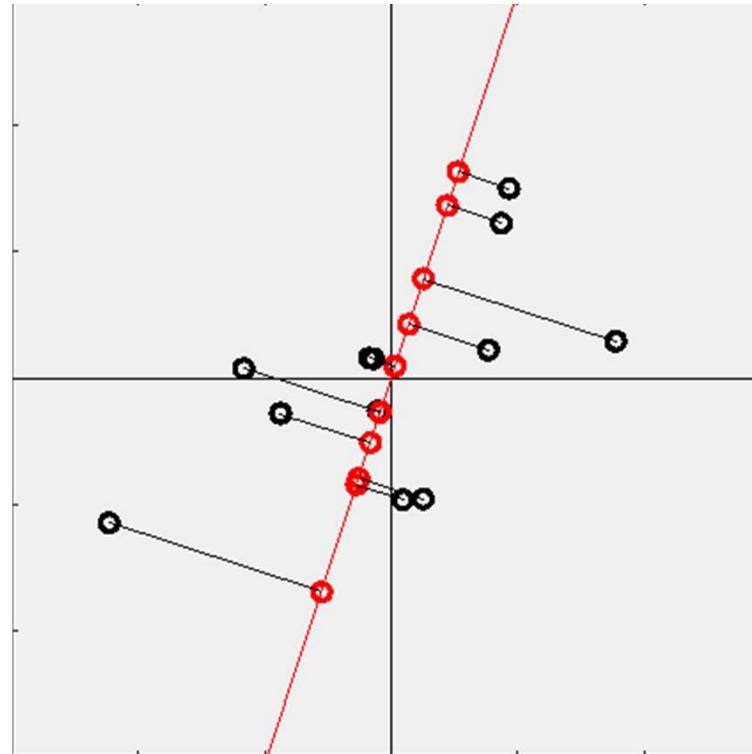


- Find the principal subspace such that when all vectors are approximated as lying on that subspace, the approximation error is minimal
 - Assuming “centered” (zero-mean) data

Principal Component Analysis

Animation:
Original centered data

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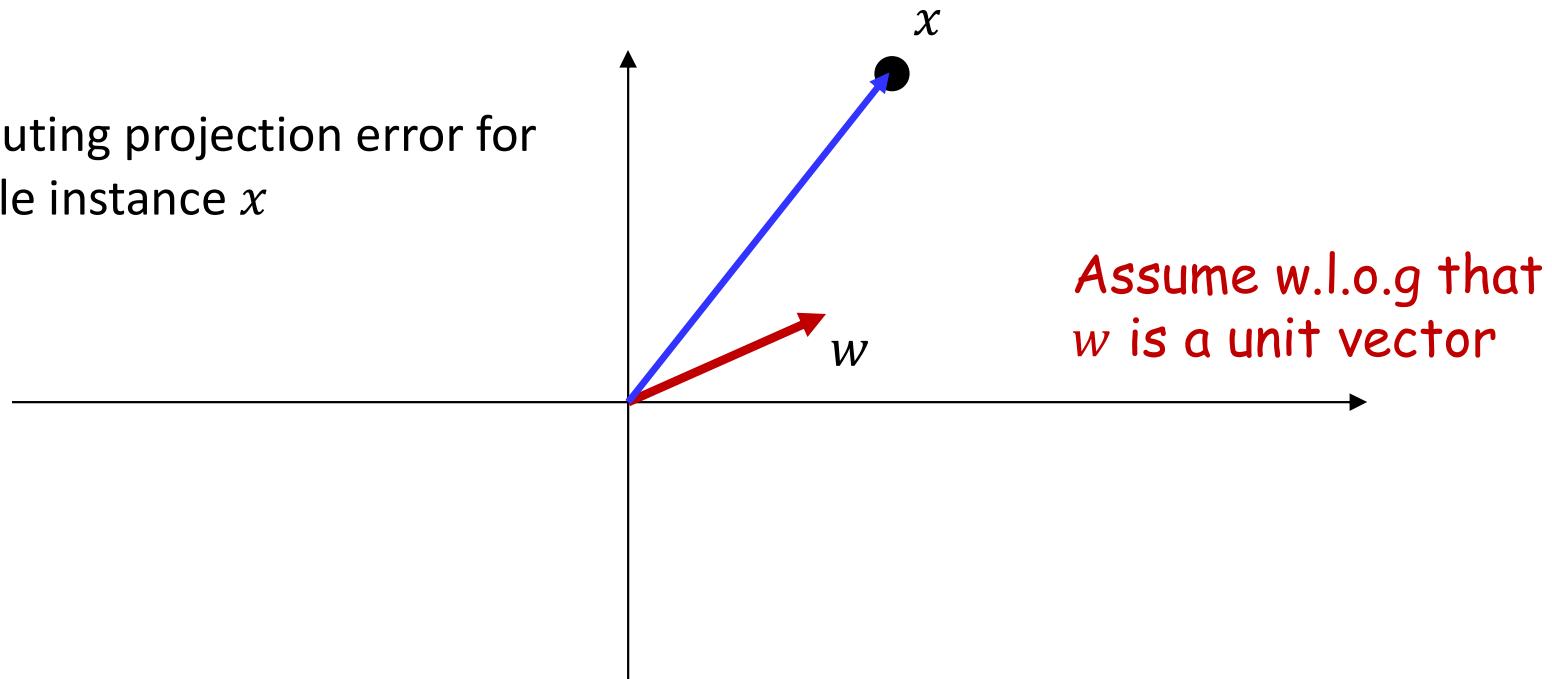


Search through all
subspaces to find the
one with minimum
projection error

- Find the principal subspace such that when all vectors are approximated as lying on that subspace, the approximation error is minimal
 - Assuming “centered” (zero-mean) data

Can be done in closed form

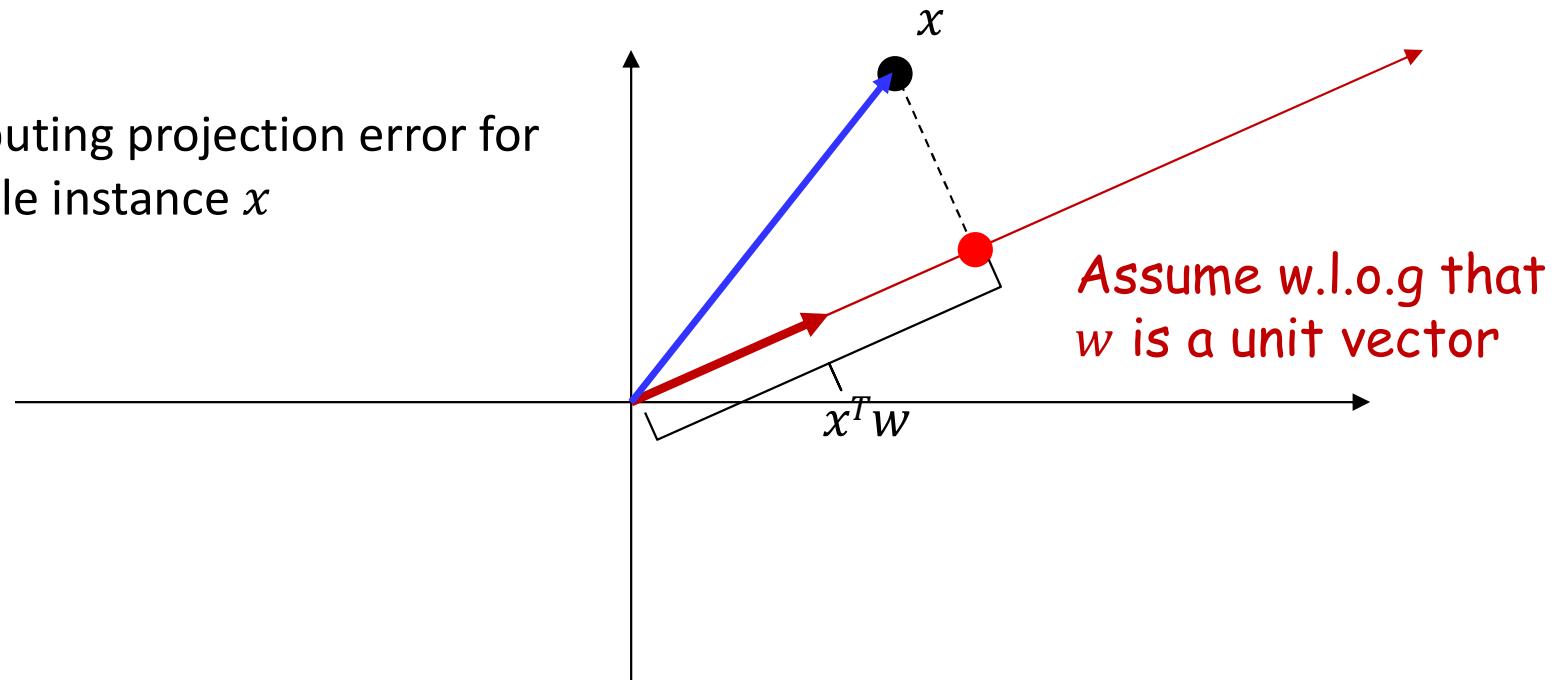
Computing projection error for
a single instance x



- Since we're minimizing quadratic L_2 error, we can find a closed form solution

Can be done in closed form

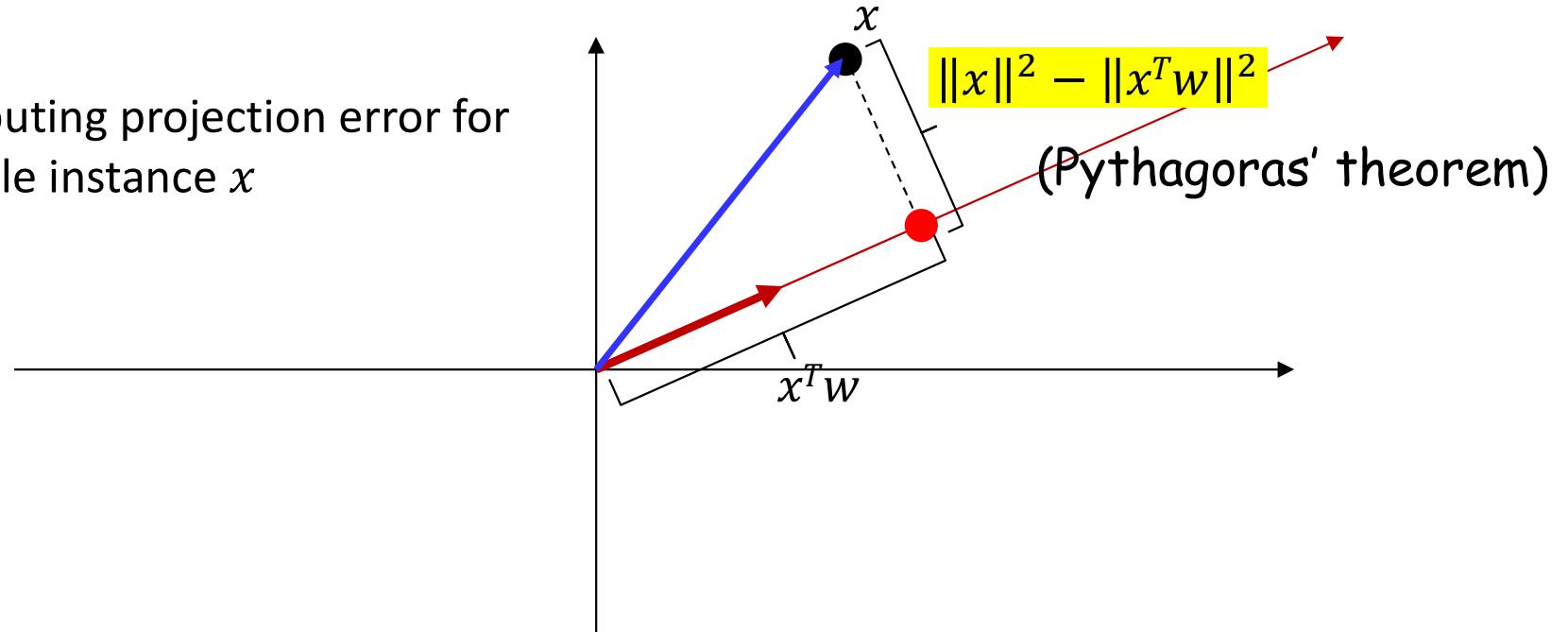
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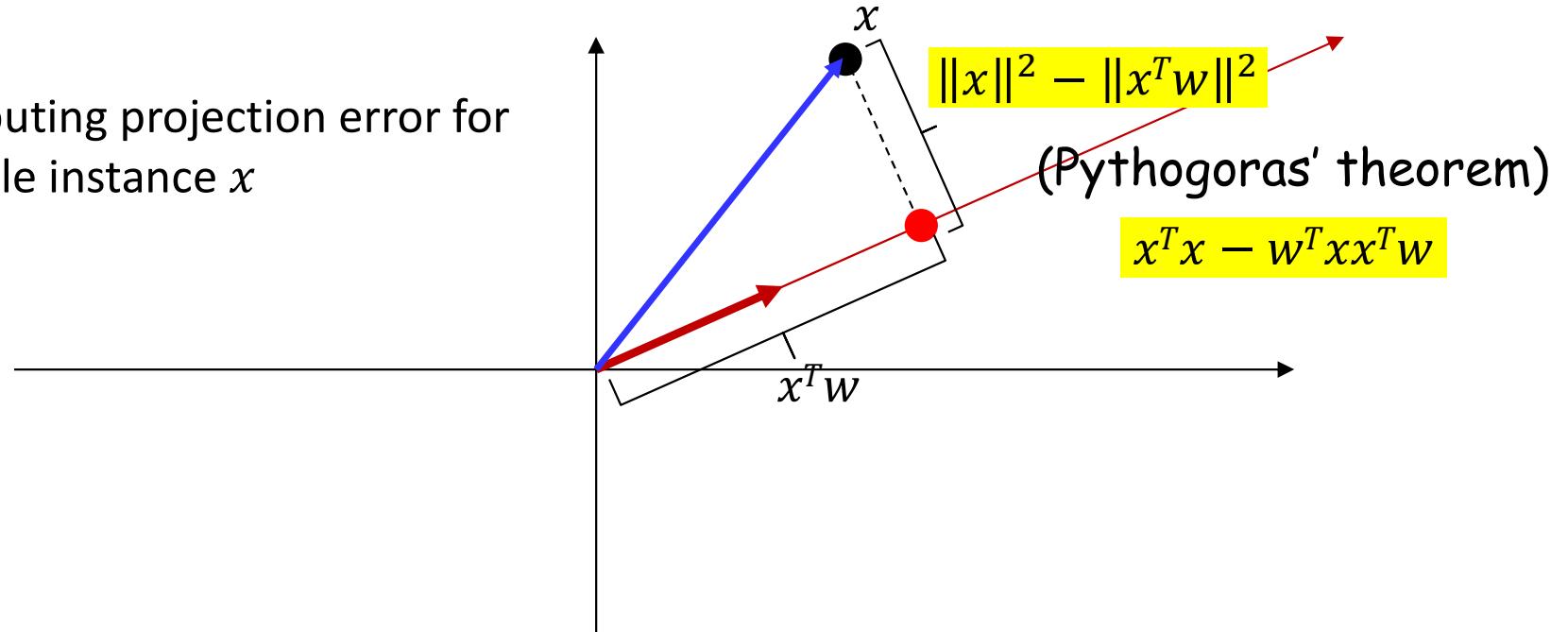
Computing projection error for
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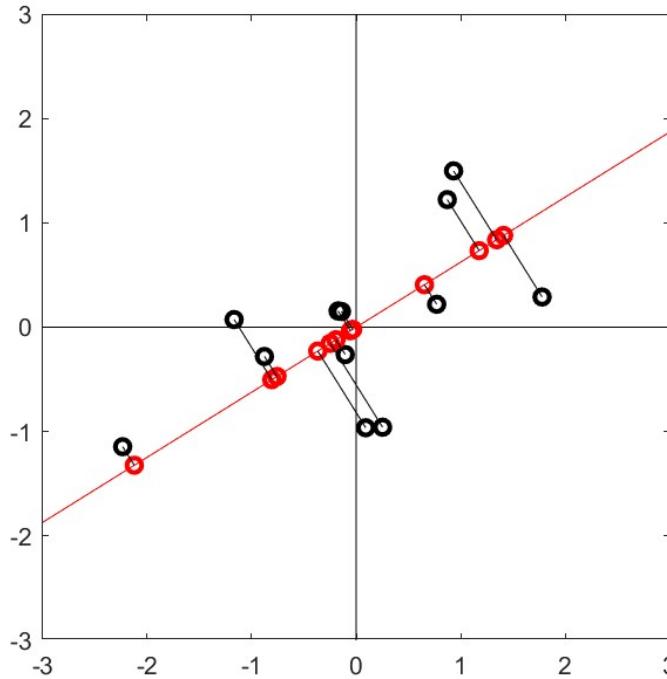
Can be done in closed form

Computing projection error for
a single instance x



- Since we're minimizing quadratic L_2 error, we can find a closed form solution

Can be done in closed form



- Since we're minimizing quadratic L_2 error, we can find a closed form solution
- Total projection error for all data:

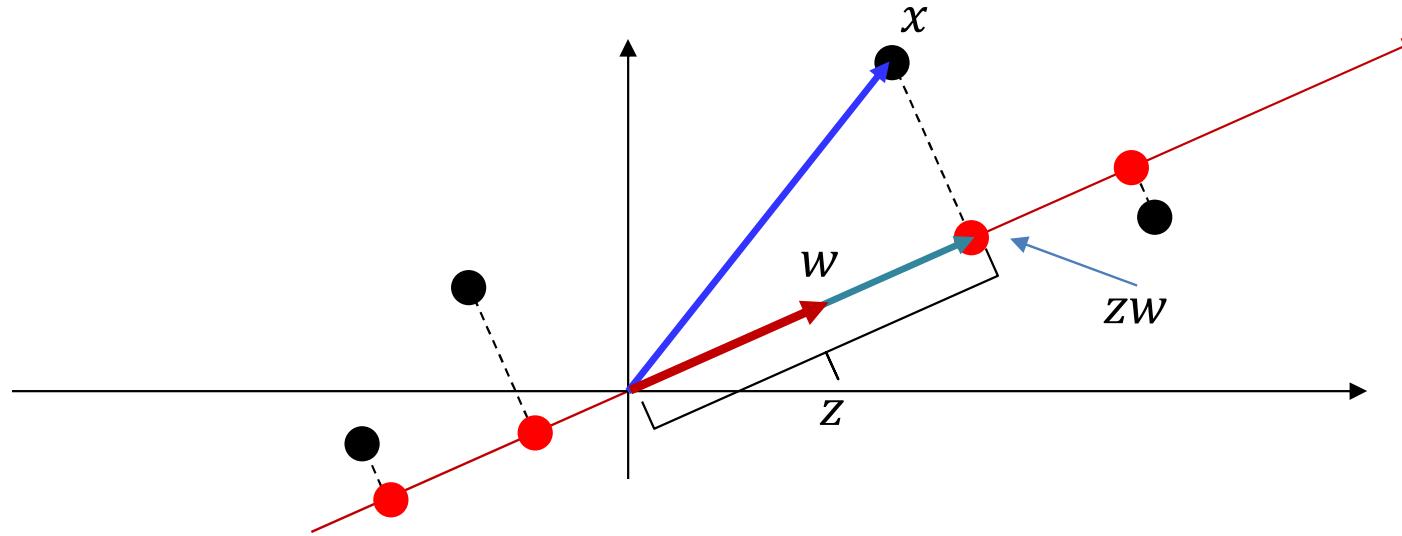
$$L = \sum_x x^T x - w^T x x^T w$$

- Minimizing this w.r.t w (subject to w = unit vector) gives you the Eigenvalue equation

$$\left(\sum_x x^T x \right) w = \lambda w$$

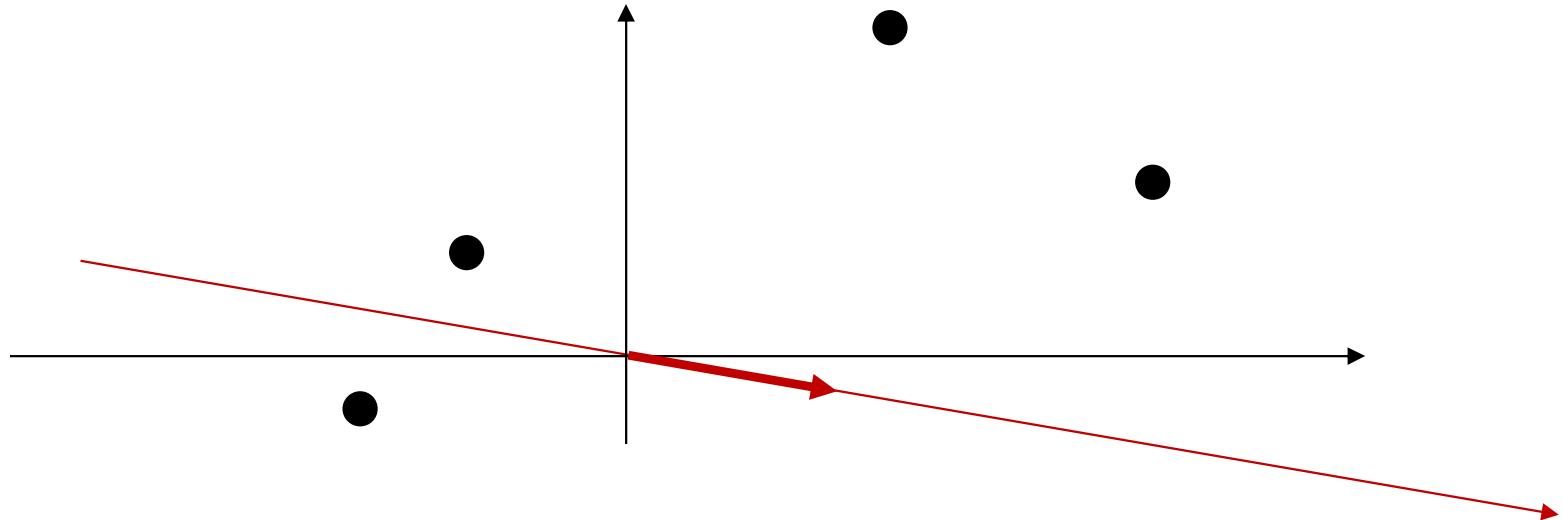
- This can be solved to find the principal subspace

There's also an iterative solution



- Objective: find a vector (subspace) w and a *position* z on w such that $zw \approx x$ most closely (in an L_2 sense) for the entire (training) data
- Let $X = [x_1 x_2 \dots x_N]$ be the entire training set (arranged as a matrix)
 - Objective: find vector bases (for the subspace) W and the set of *position vectors* $Z = [z_1 z_2 \dots z_N]$ for all vectors in X such that $WZ \approx X$
- Initialize W
- Iterate until convergence:
 - Given W , find the best position vectors Z : $Z \leftarrow W^+X$
 - Given position vectors Z , find the best subspace: $W \leftarrow XZ^+$
 - Guaranteed to find the principal subspace

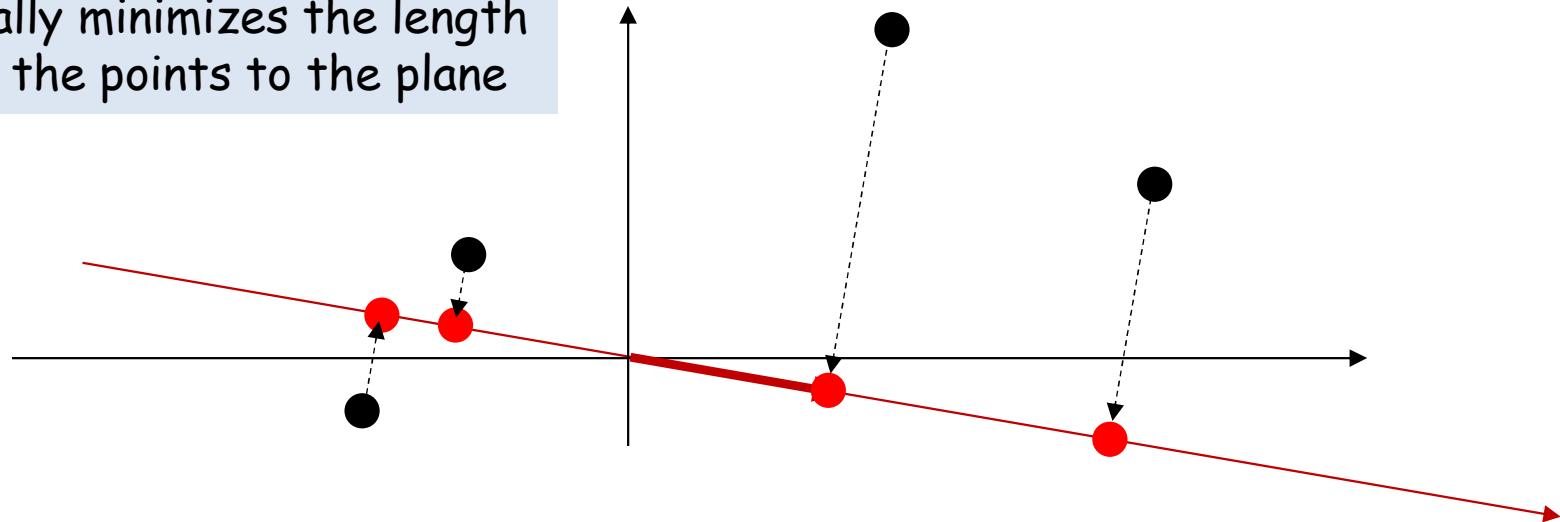
The iterative algorithm



- Initialize a subspace (the basis w)

The iterative algorithm

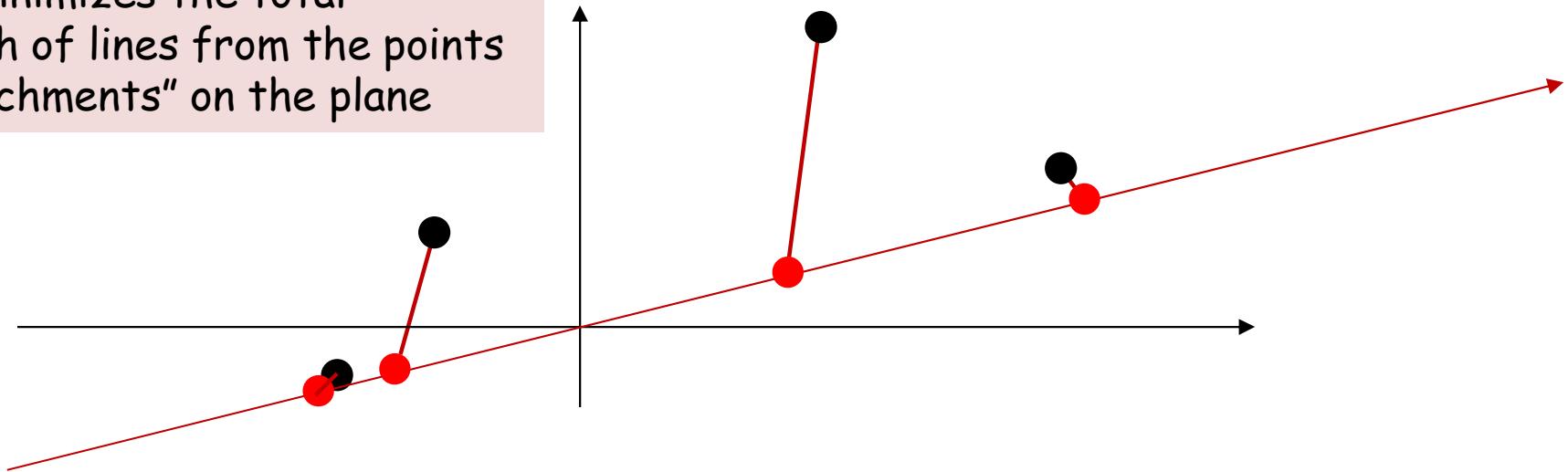
This individually minimizes the length of lines from the points to the plane



- Initialize a subspace (the basis w)
- Iterate until convergence:
 - Find the best position vectors Z on the W subspace for each training instance
 - Find the location on W that is *closest* to each instance, i.e. the perpendicular projection

The iterative algorithm

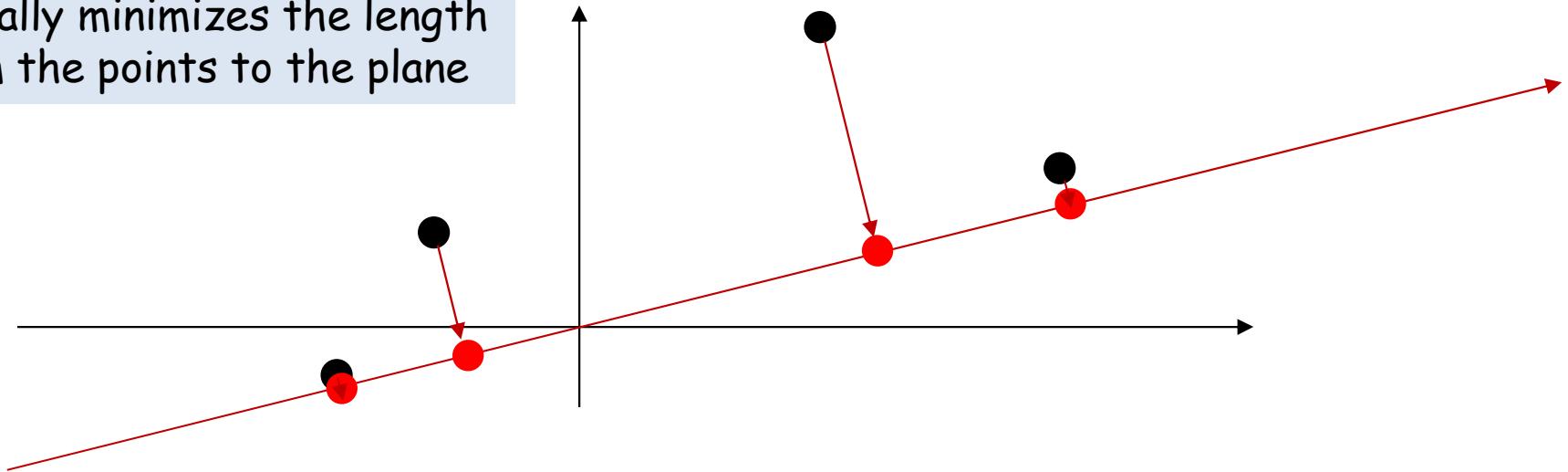
This jointly minimizes the total squared length of lines from the points to their "attachments" on the plane



- Initialize a subspace (the basis w)
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 - Find the best position vectors Z on the W subspace for each training instance
 - Find the location on W that is *closest* to each instance, i.e. the perpendicular projection
 - Let W rotate and stretch/shrink, keeping the arrangement of Z locations fixed
 - Minimize the total square length of the lines attaching the projection on the place to the instance

The iterative algorithm

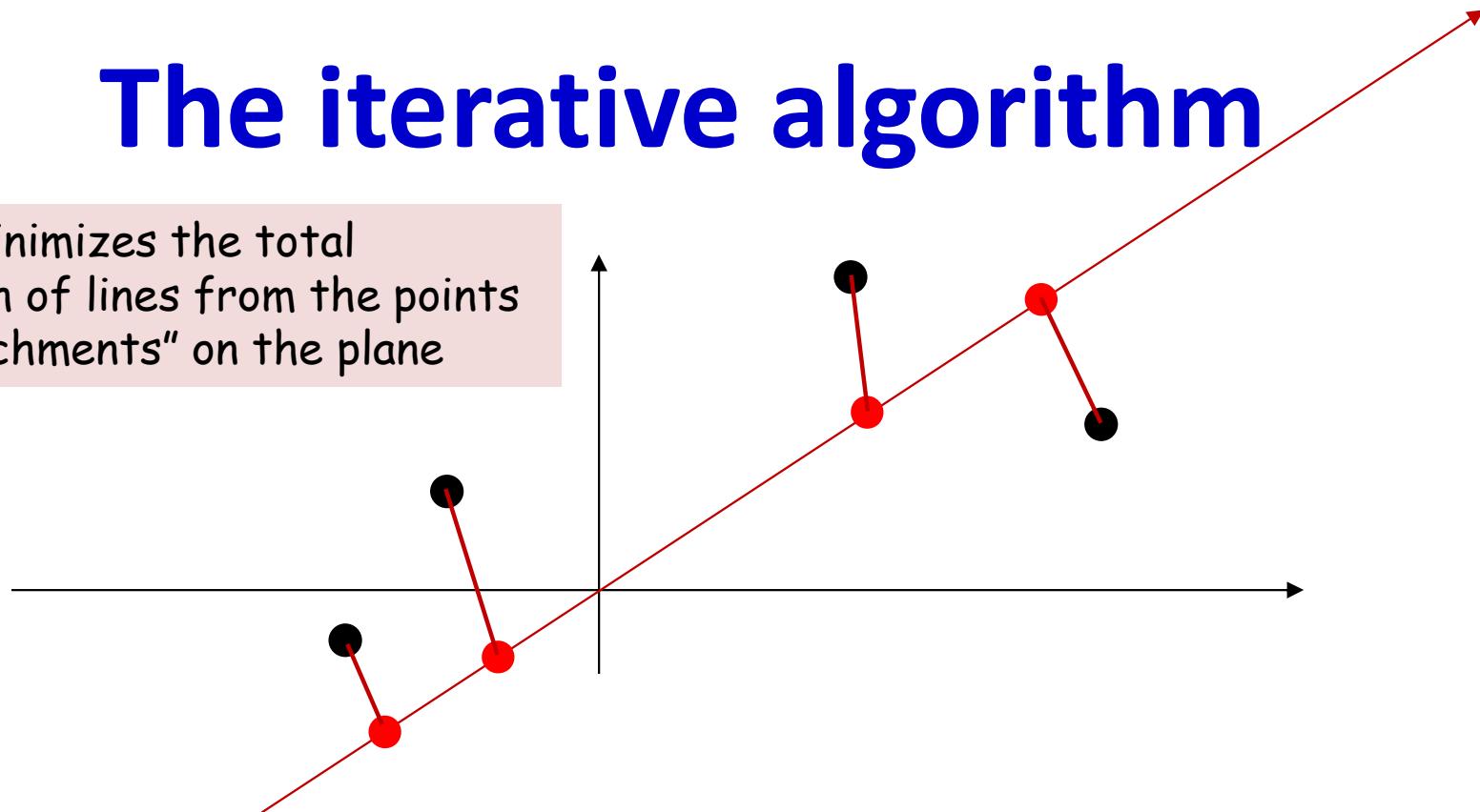
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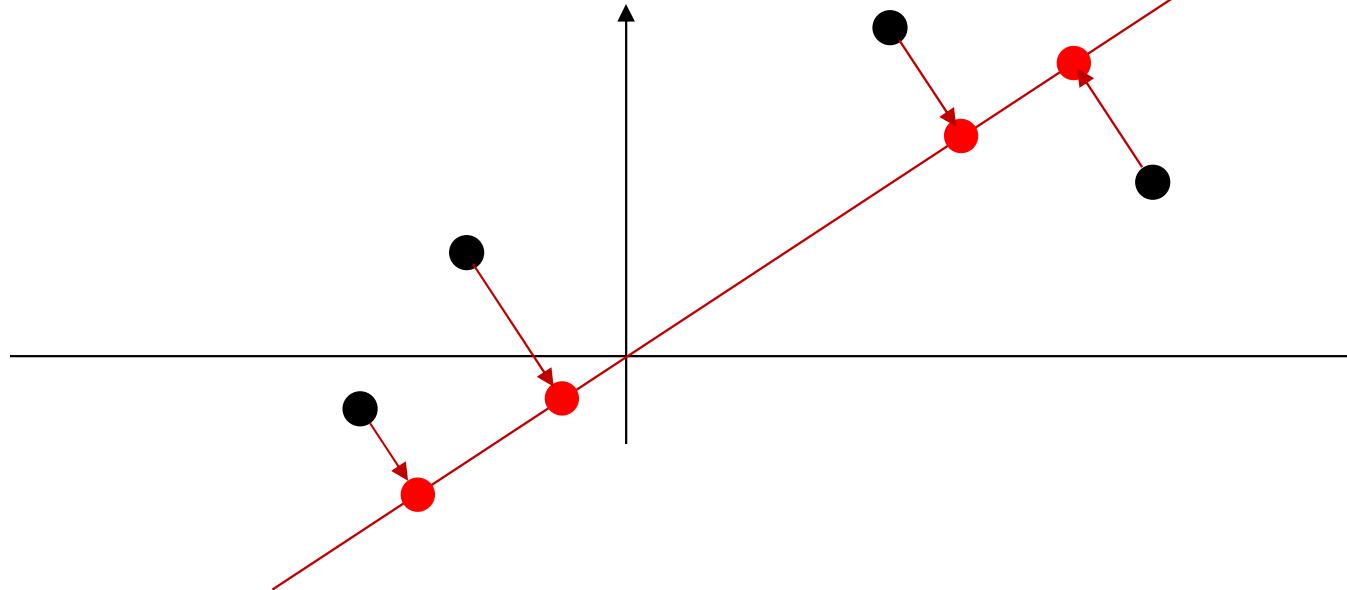
The iterative algorithm

This jointly minimizes the total squared length of lines from the points to their "attachments" on the plane



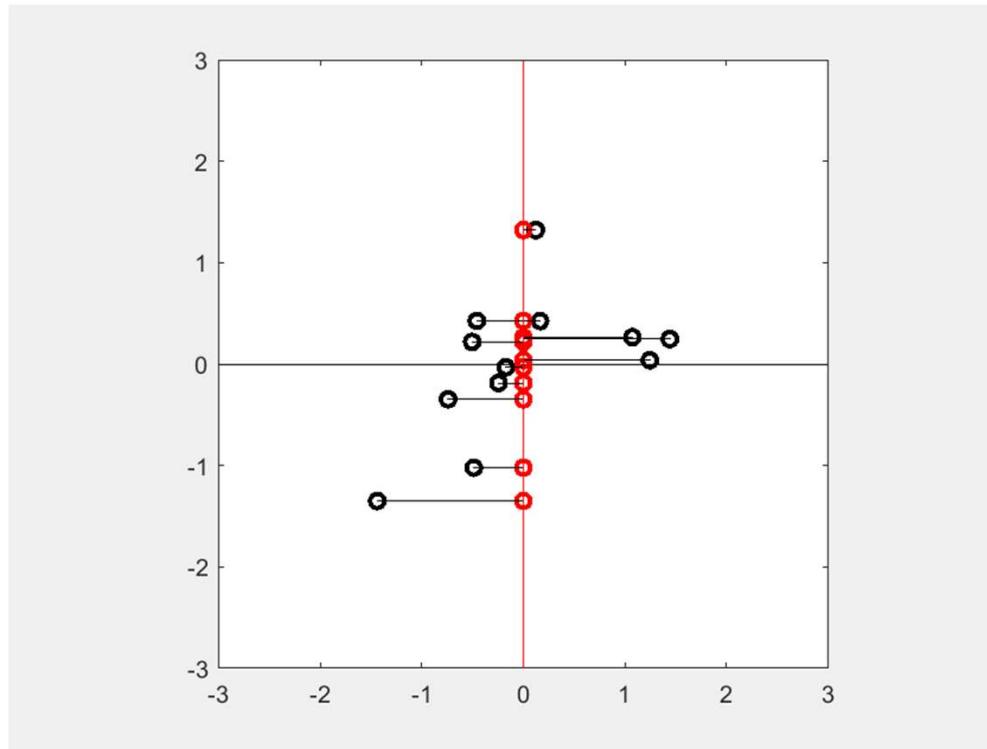
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The iterative algorithm



- Initialize a subspace (the basis w)
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A failed attempt at animation



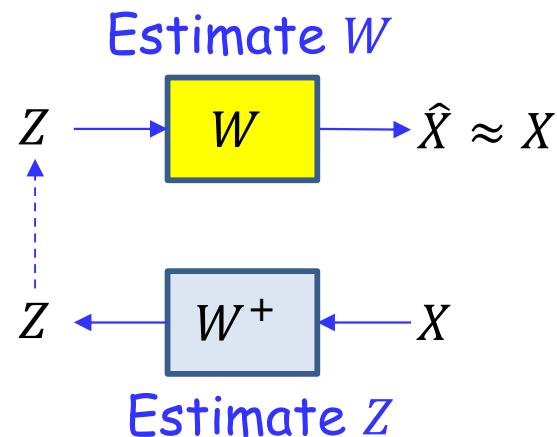
- Someone with animated-gif generation skills, help me...

A cartoon view of Iterative PCA

Iterate :

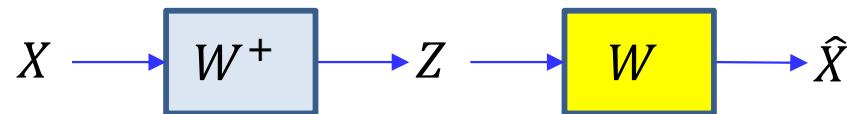
$$Z \leftarrow W^+ X$$

$$W \leftarrow X Z^+$$



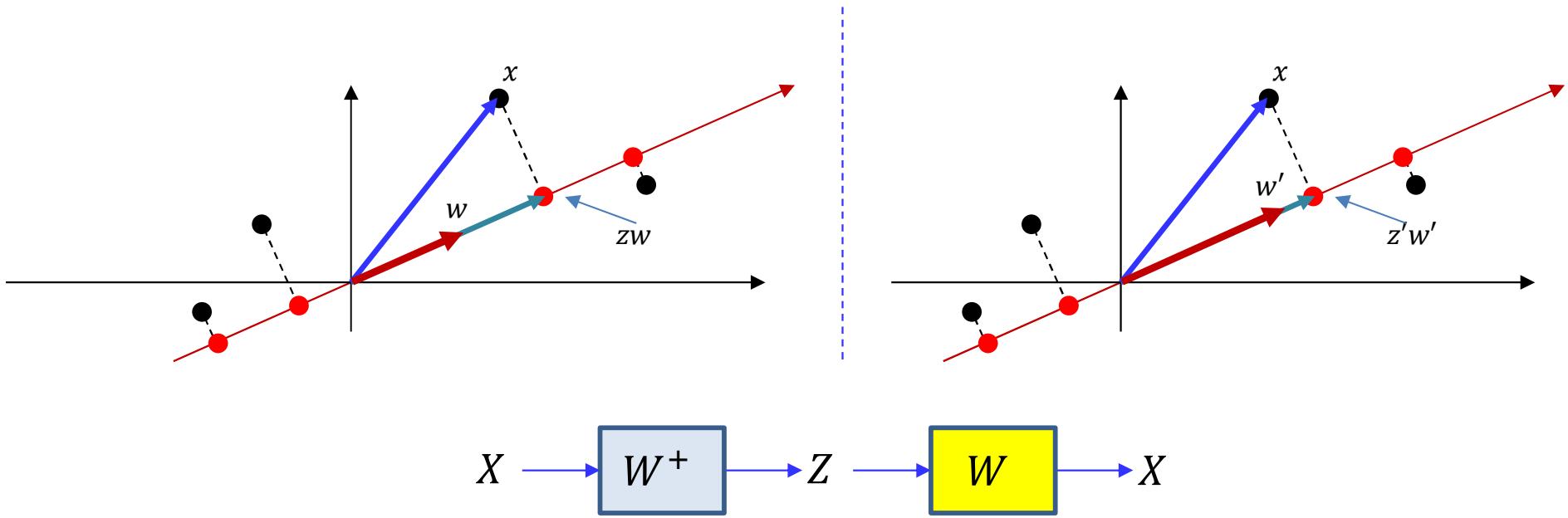
- Note that the real problem in estimating Z is computing W^+
 - If you know W^+ , Z is obtained by a direct matrix multiply

Drawing this differently



- Look familiar?
- An autoencoder with linear activations
- Backprop actually works by simultaneously updating Z (implicitly) and W in tiny increments

A minor issue: Scaling invariance

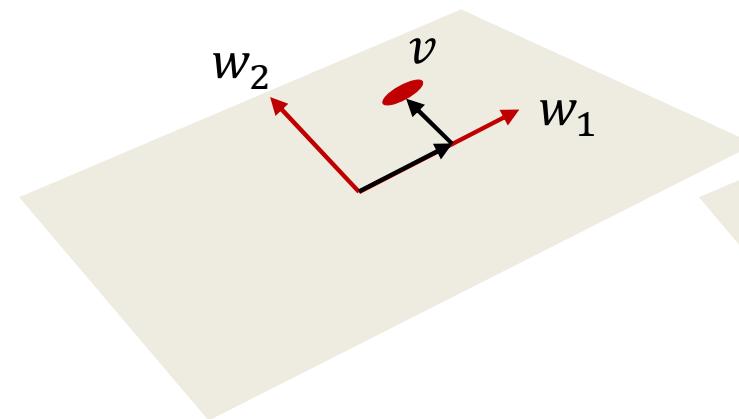


- The estimation is scale invariant
- We can increase the length of w , and compensate for it by reducing z
- The solution is not unique!

Rotation/scaling invariance

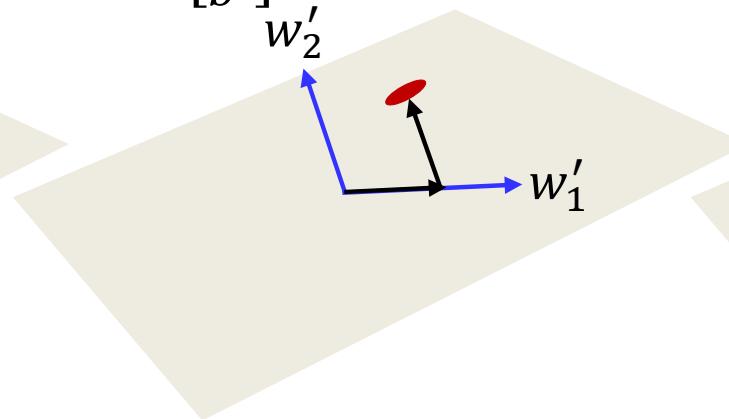
$$v = aw_1 + bw_2$$

$$z = \begin{bmatrix} a \\ b \end{bmatrix}$$



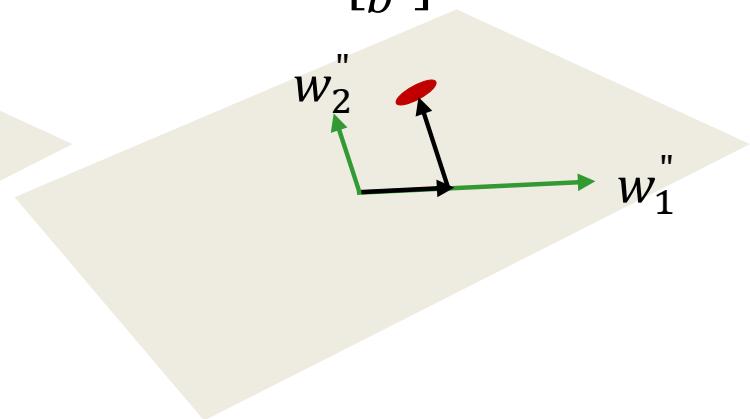
$$v = a'w'_1 + b'w'_2$$

$$z = \begin{bmatrix} a' \\ b' \end{bmatrix}$$



$$v = a''w_1'' + b''w_2''$$

$$z = \begin{bmatrix} a'' \\ b'' \end{bmatrix}$$



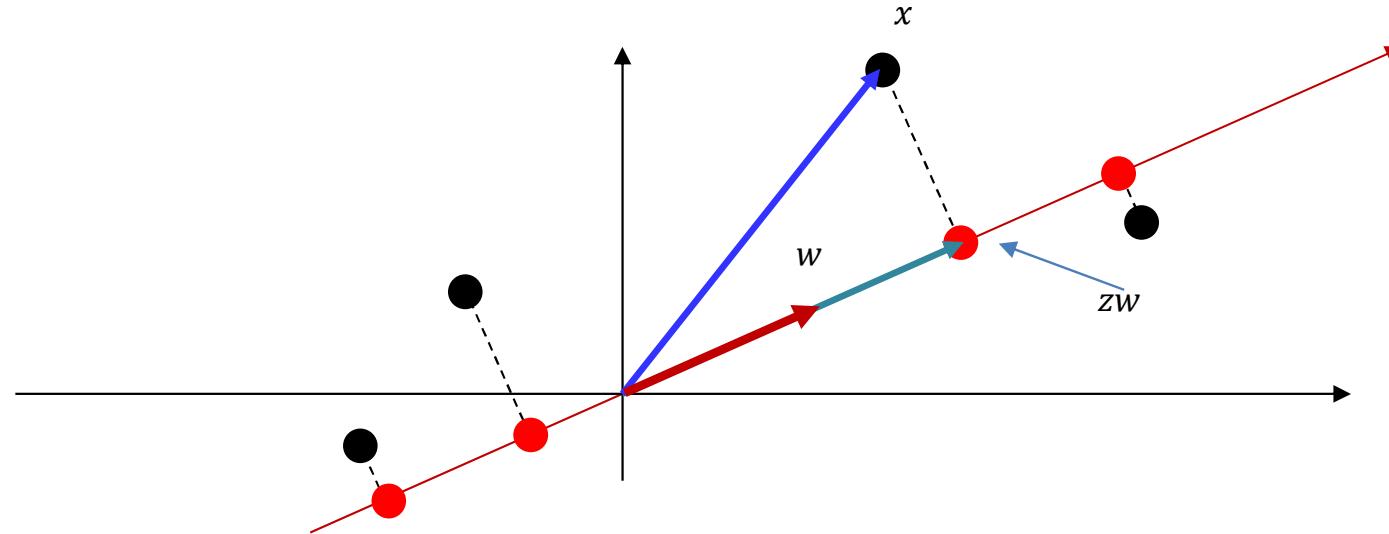
- We can rotate and scale the vectors in W without changing the actual subspace they compose
- The representation of any point in the hyperspace in terms of these vectors will also change
 - The zs in the two cases will be related through a linear transform
- The subspace is invariant to transformations of z

Transformation invariance



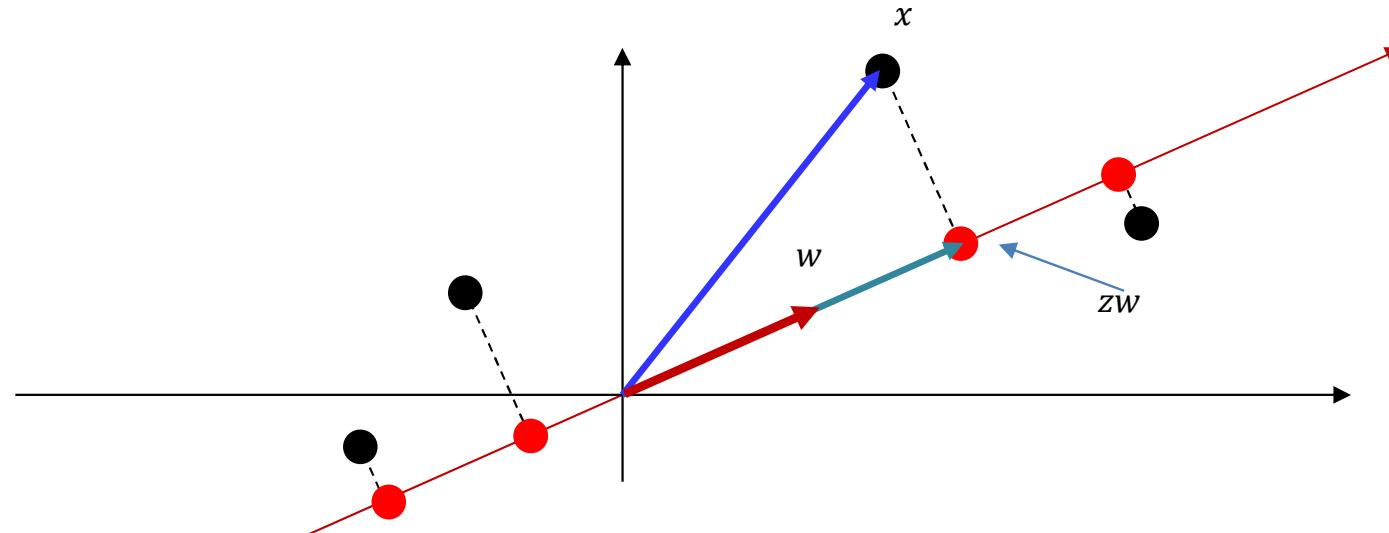
- We can modify W to $W' = WB$, and Z to $Z' = B^{-1}Z$ such that $WZ = W'Z'$ $(WB)(B^{-1}Z)$
 - A different set of bases for the same subspace
- We can modify Z to $Z' = BZ$, and W to $W' = WB^{-1}$ such that $WZ = W'Z'$ $(WB^{-1})(BZ)$
 - A different set of bases for the same subspace
- The representation is invariant to invertible transforms of either W or Z
 - Although we will always find the same *subspace*, the *bases* and the *representations* in terms of these bases are not unique
 - I.e. there is no guarantee of which of the infinite possible solutions we will actually find

Resolving this issue



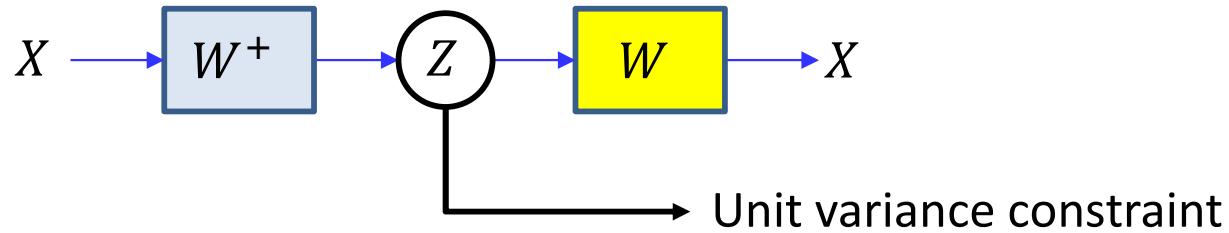
- A unique solution can be found by either
 - Requiring the vectors in W to be unit length and orthogonal
 - Standard “closed” form PCA
 - Constraining the variance of Z to be unity
- While the W s estimated with the two solutions will be different, the resulting discovered principal subspace will be the same

Resolving this issue



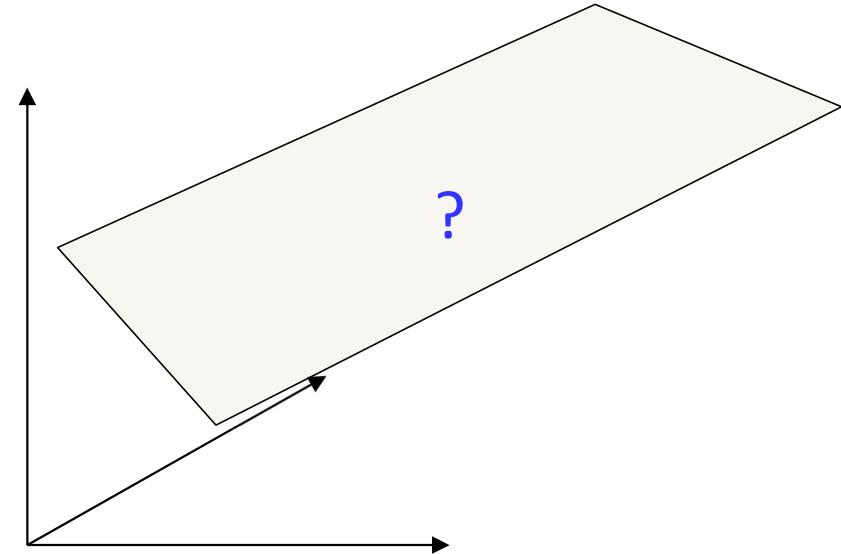
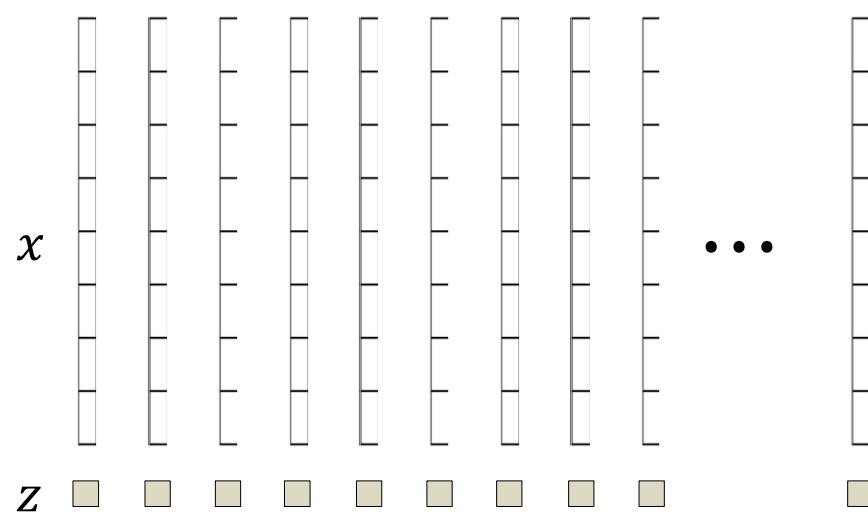
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Constraining the linear AE



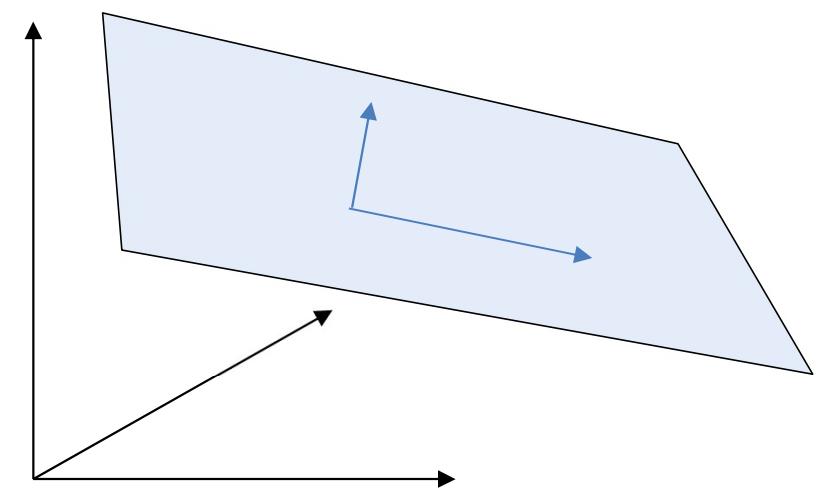
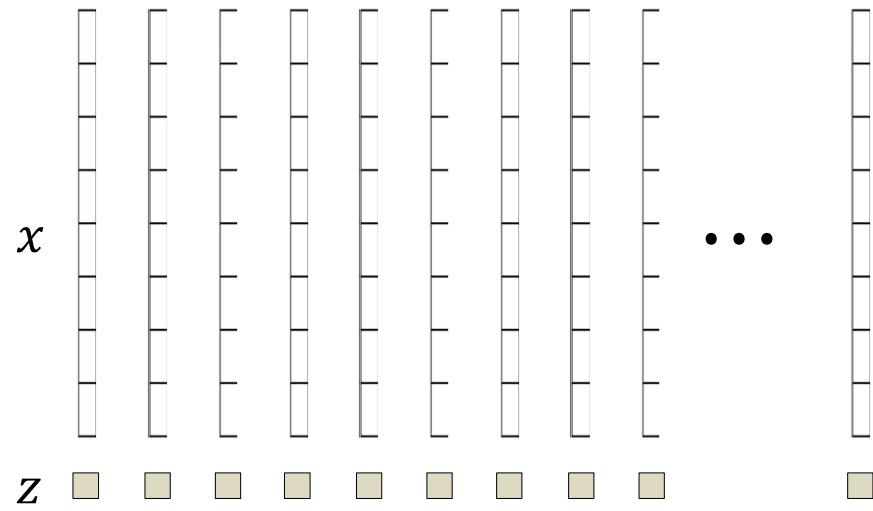
- The linear AE can be constrained to give you a unique(ish) solution
- Impose a unity constraint on the variance of Z
 - How?

So what are we doing in the iterative solution?



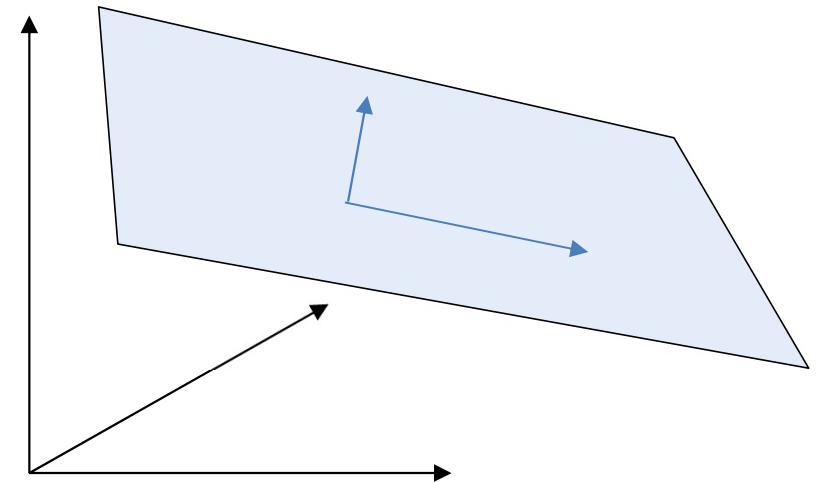
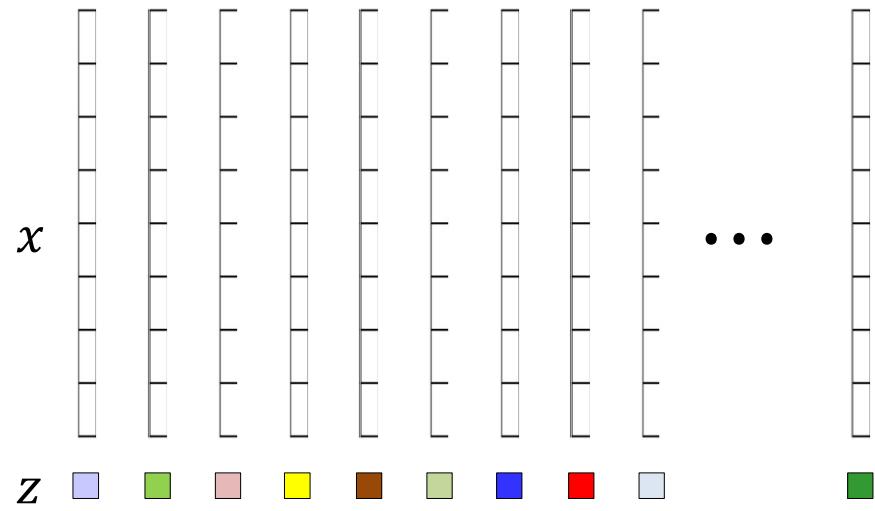
- For every training vector x , we are missing the information z about where the vector lies on the principal subspace hyperplane
- If we had z , we could uniquely identify the plane

Iterative solution



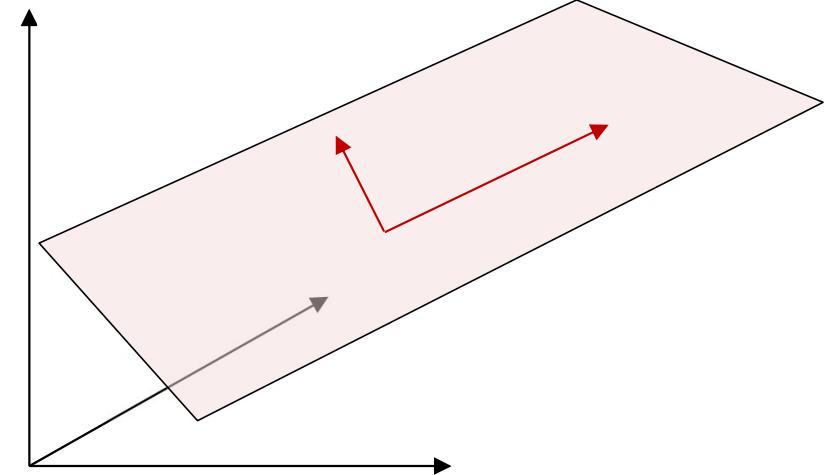
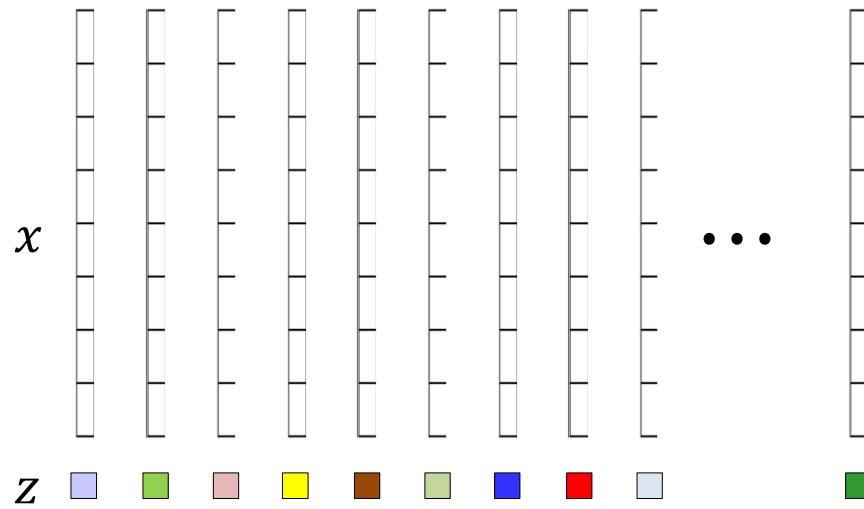
- Initialize the plane
 - Or rather, the bases for the plane

Iterative solution



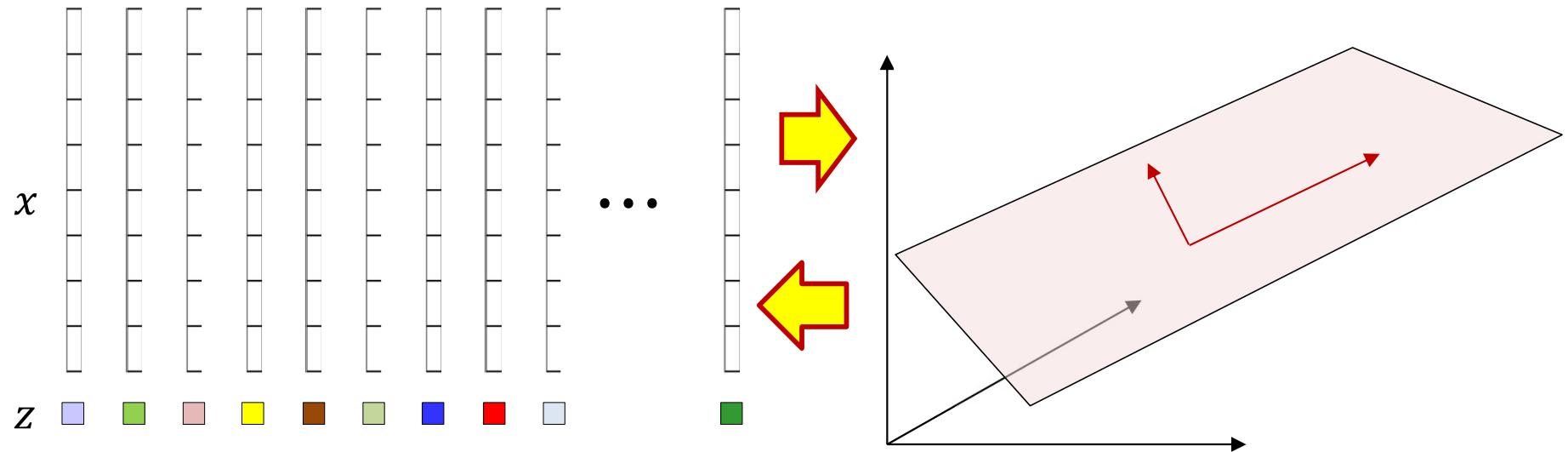
- Initialize the plane
 - Or rather, the bases for the plane
- “Complete” the data by computing the appropriate zs for the plane

Iterative solution



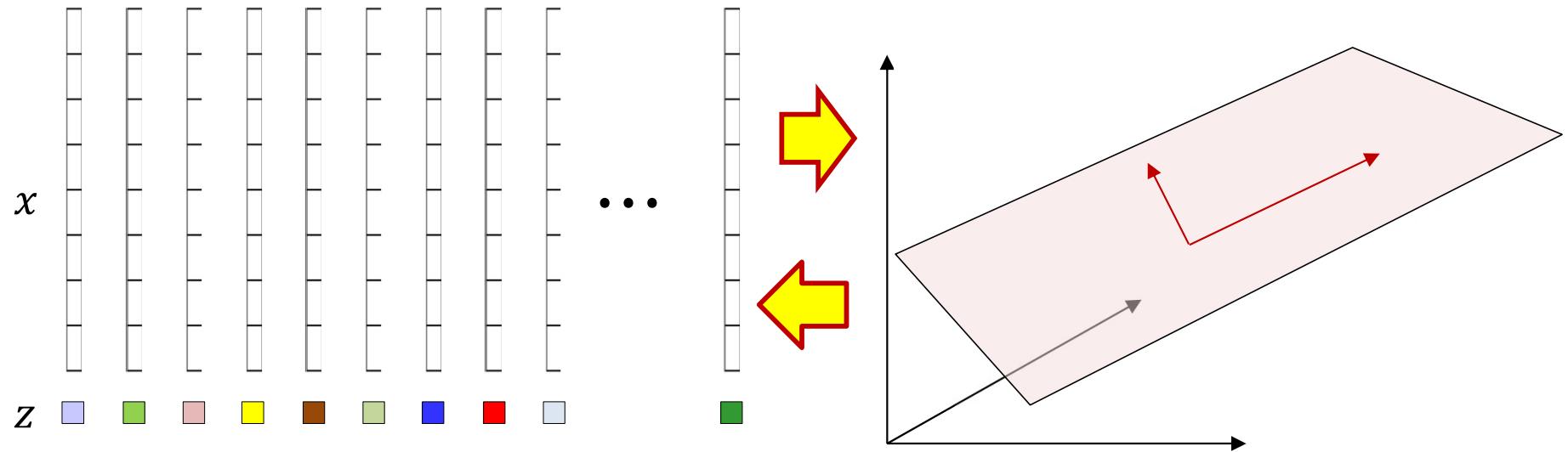
- Initialize the plane
 - Or rather, the bases for the plane
- “Complete” the data by computing the appropriate zs for the plane
- Reestimate the plane using the zs

Iterative solution



- Initialize the plane
 - Or rather, the bases for the plane
- “Complete” the data by computing the appropriate zs for the plane
- Reestimate the plane using the zs
- Iterate

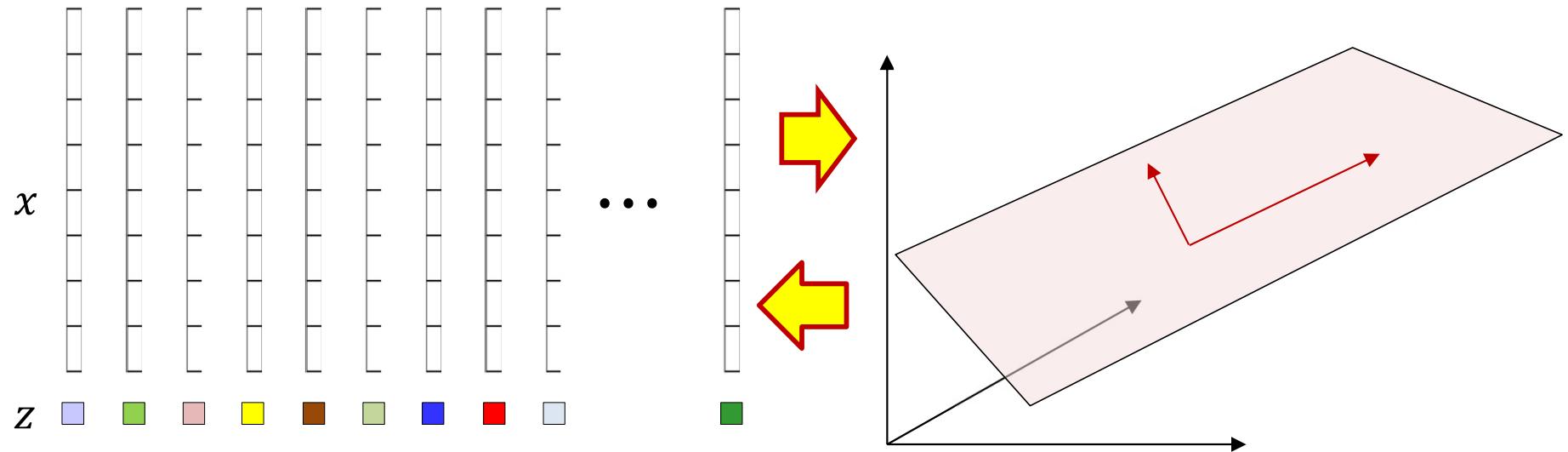
Iterative solution



- Initialize the plane
 - Or rather, the bases for the plane
- “Complete” the \dots
- Reestimate
- Iterate

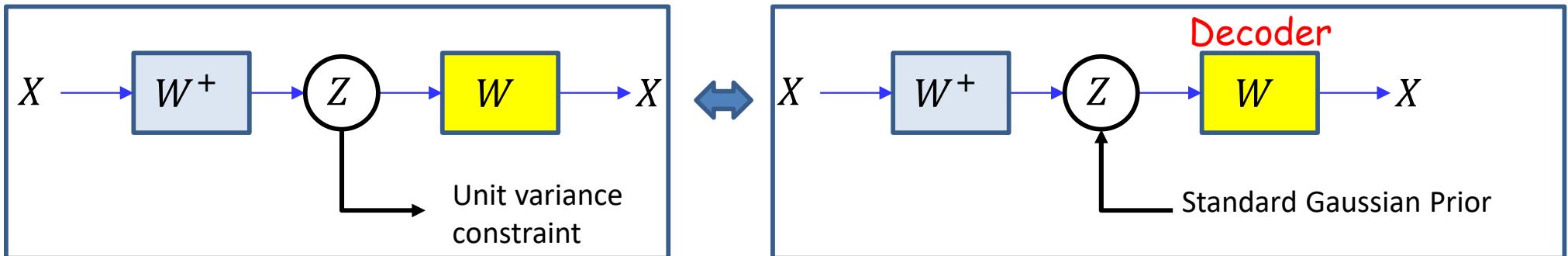
Look Familiar?

Iterative solution



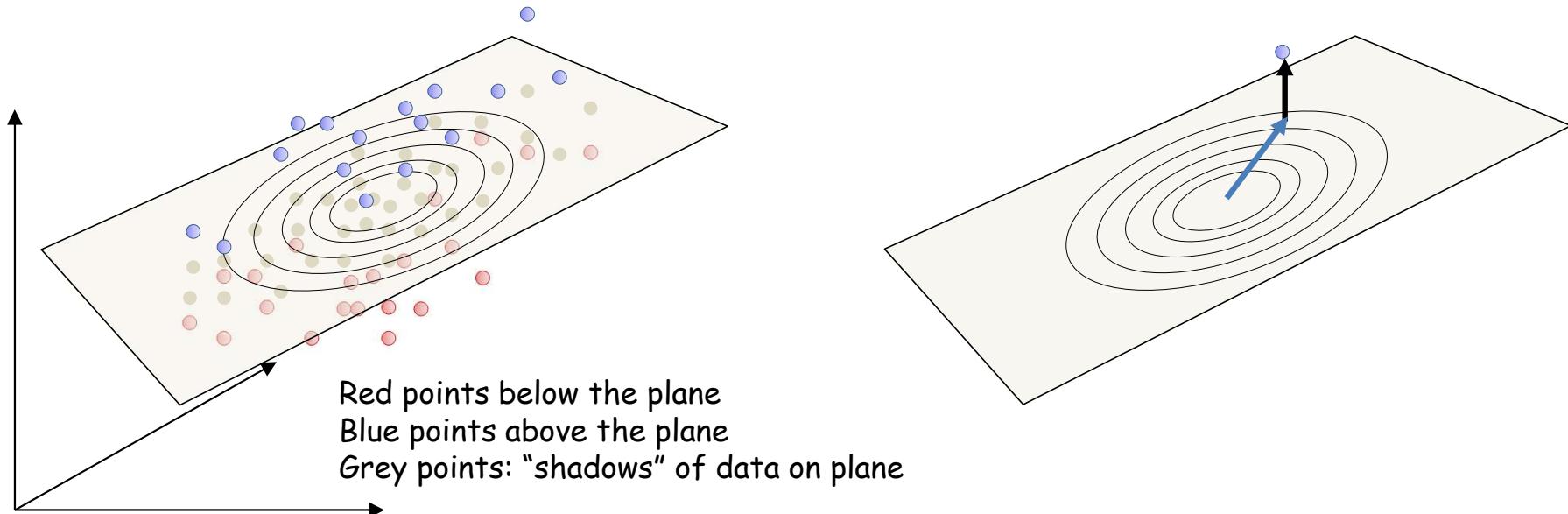
- This looks like EM
 - In fact it is
- But what is the generative model?
- And what distribution is this encoding?

Constraining the linear AE



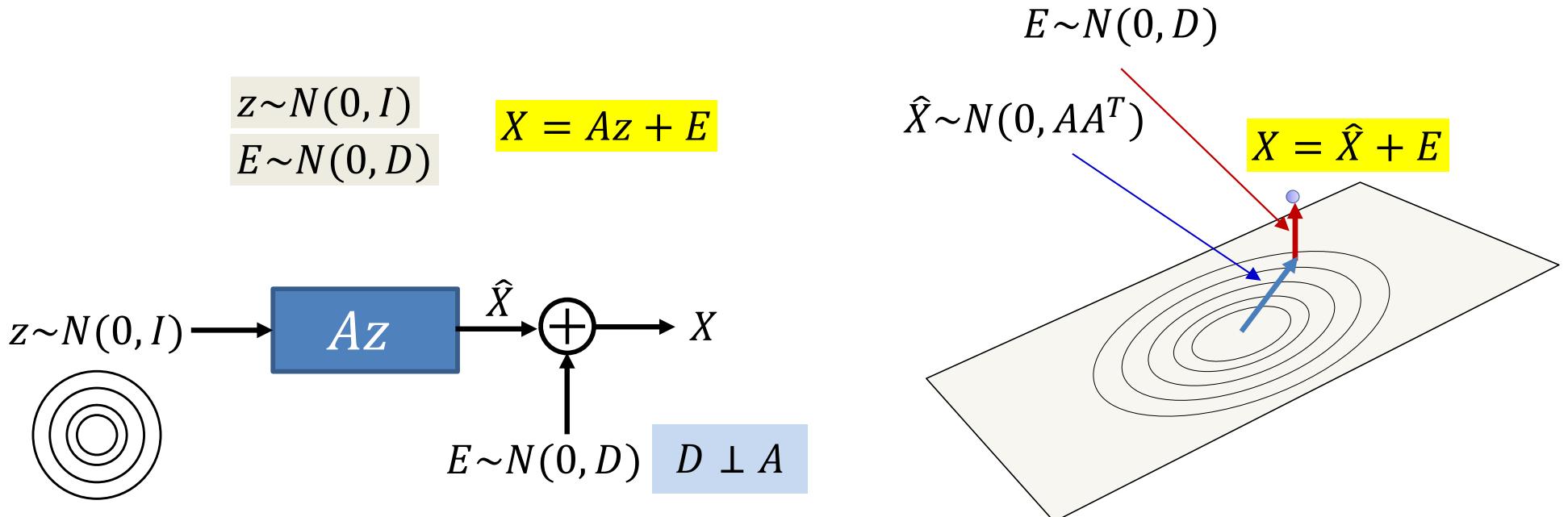
- Imposing the constraint that z must have unit variance is the same as assuming that z is drawn from a standard Gaussian
 - 0 mean, unit variance!
- The decoder of the AE with the unit-variance constraint on z is in fact a Generative model

The *generative* story behind PCA (linear AEs)



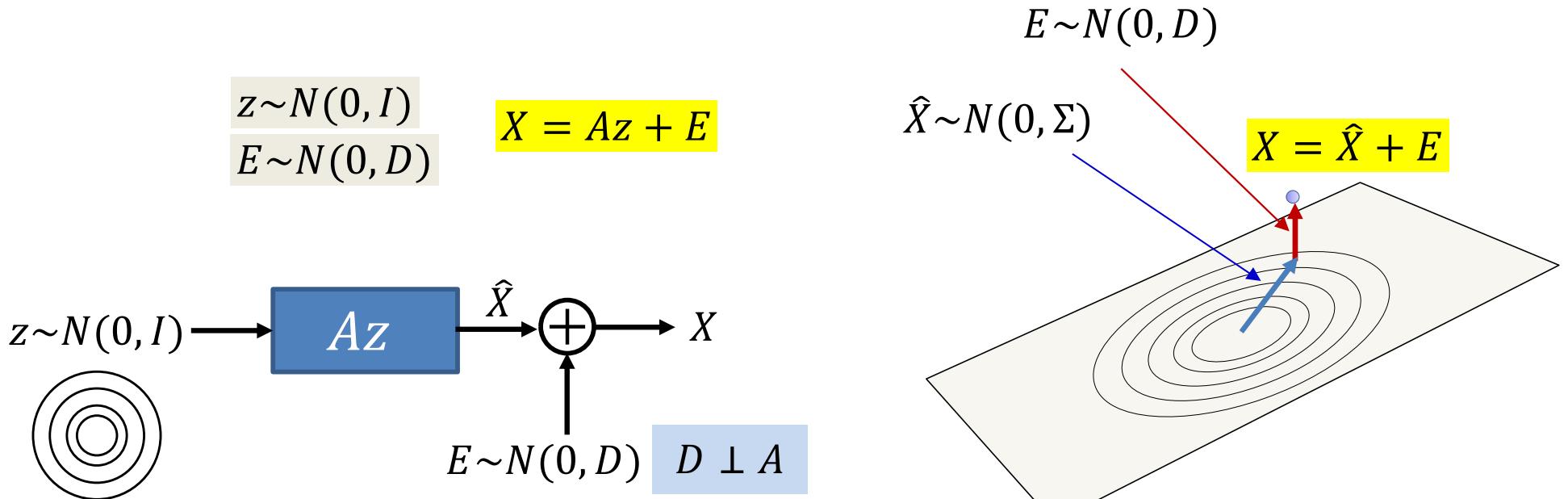
- Linear AEs actually have a generative story
- In order to generate any point
 - We first take a Gaussian step on the principal plane
 - Then we take an orthogonal *Gaussian* step from where we land to generate a point
 - PCA / Linear AEs find the plane and the characteristics of the Gaussian steps from the data

The *generative* story behind PCA (linear AEs)



- **Generative story for PCA:**
 - z is drawn from a K -dim isotropic Gaussian
 - K is the dimensionality of the principal subspace
 - A is “basis” matrix
 - Matrix of principal Eigen vectors scaled by Eigen values
 - E is a 0-mean Gaussian noise that is orthogonal to the principal subspace
 - **The covariance of the Gaussian is low-rank and orthogonal to the principal subspace!**

The *generative* story behind PCA (linear AEs)



PCA implicitly obtains maximum likelihood estimate of A and D , from training data X

- **Generative story for PCA:**

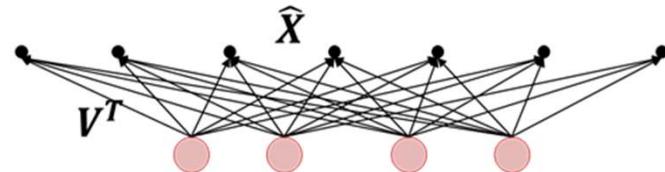
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The *generative* (PCA) story of linear AEs

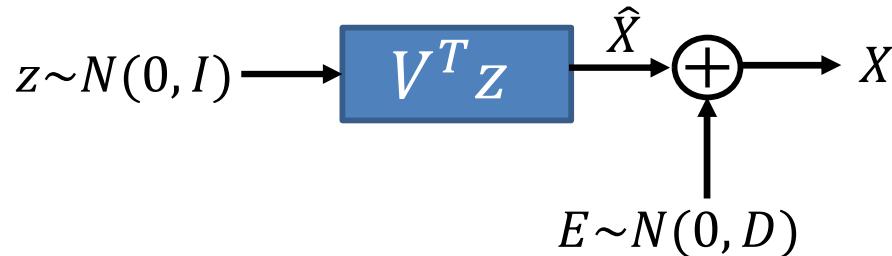
Changed notation

$$V^T = A$$

$$\hat{X} = Az$$



Note: the generative model
is the decoder

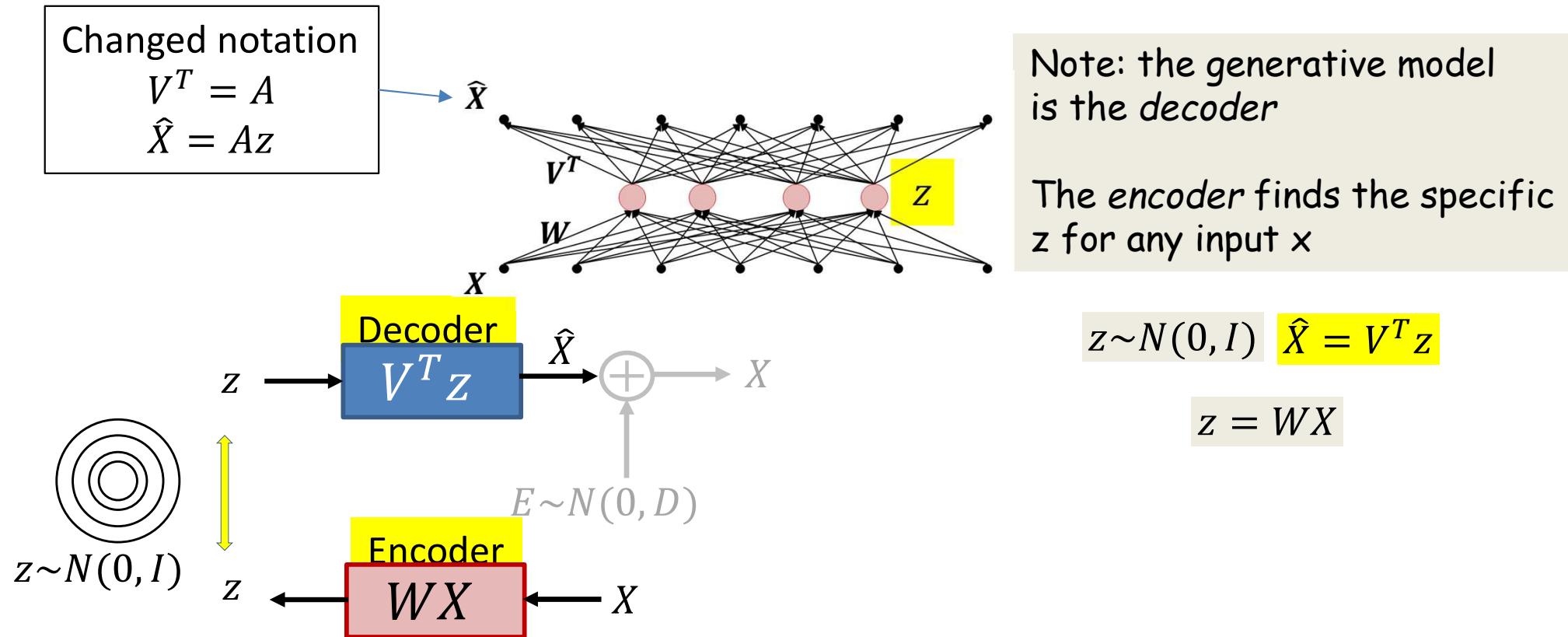


$$z \sim N(0, I) \quad \hat{X} = V^T z$$

$$E \sim N(0, D)$$

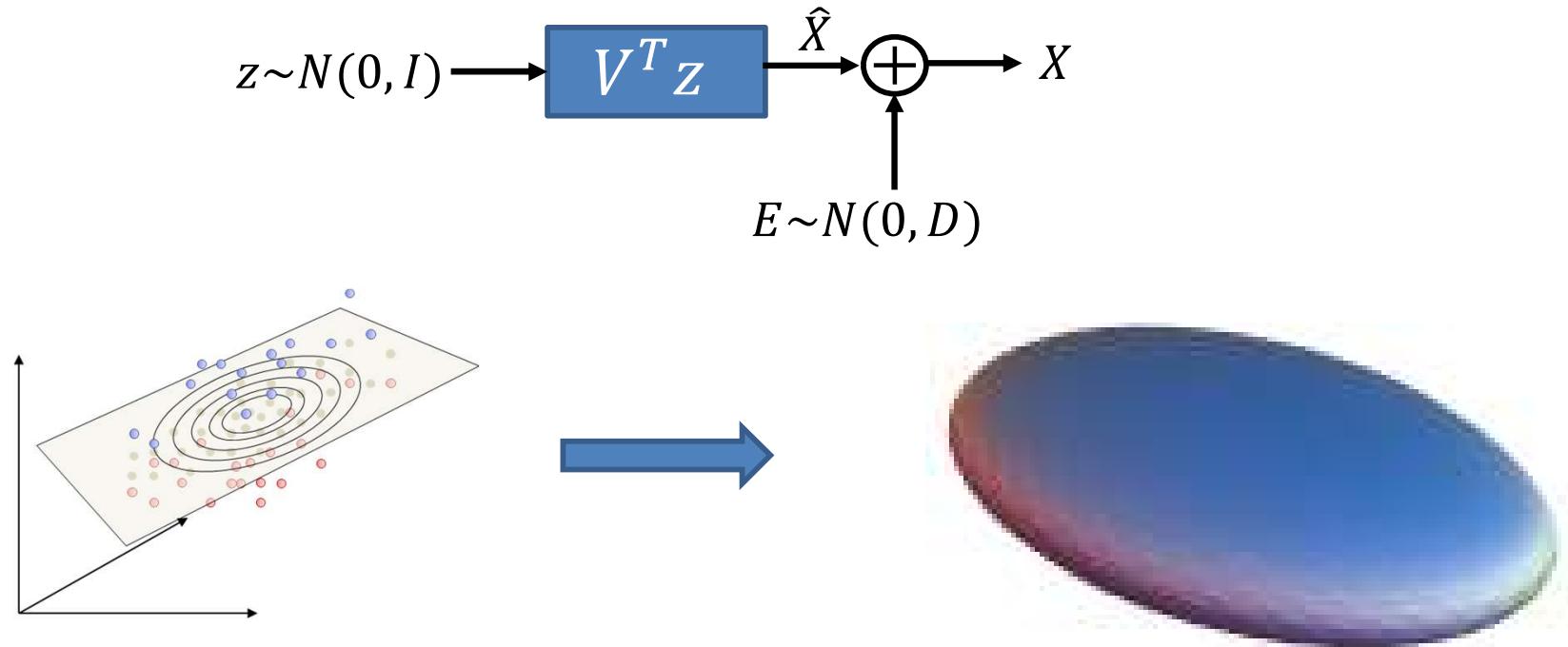
- The decoder weights are just the PCA basis matrix

The *generative* (PCA) story of linear AEs



- The decoder weights are just the PCA basis matrix
- The encoder only projects the data into latent Gaussian position variable z
- Encoder: transforms input X into Gaussian z
- Decoder: transforms Gaussian z into principal subspace reconstruction \hat{X}

The distribution modelled by PCA



- If z is Gaussian, \hat{X} is Gaussian
- \hat{X} and E are Gaussian $\Rightarrow X$ is Gaussian
- PCA model: The observed data are Gaussian
 - Gaussian data lying very close to a principal subspace
 - Comprising “clean” Gaussian data on the subspace plus orthogonal noise

Poll 4

Select all that are true about PCA

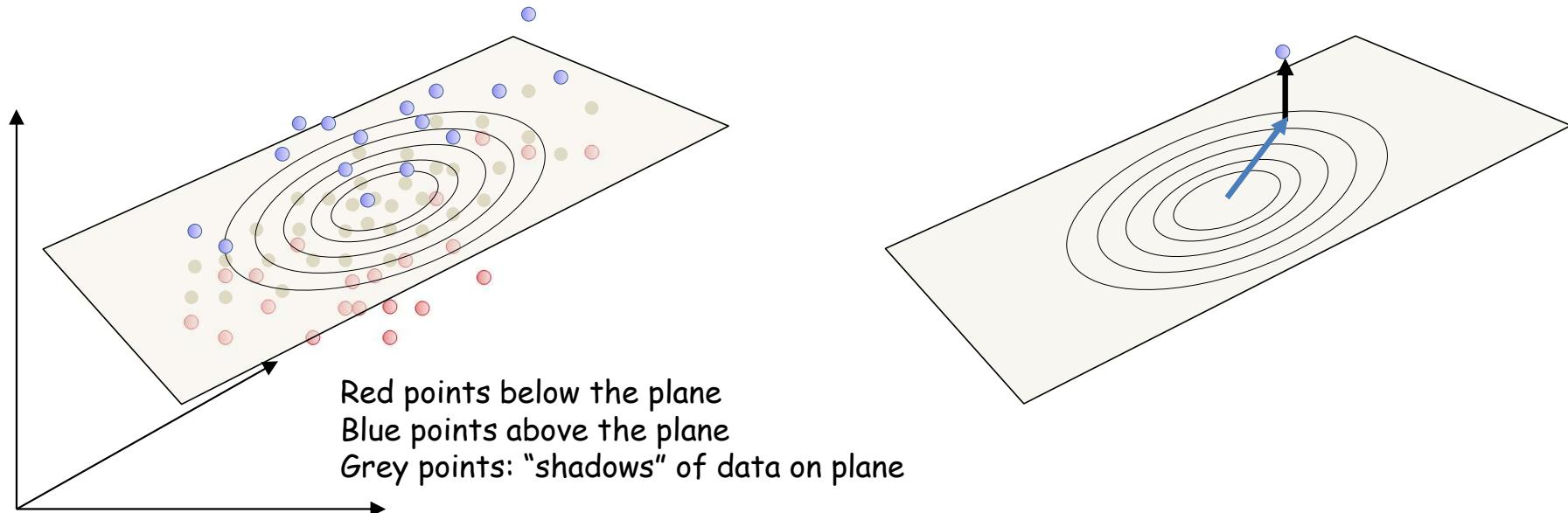
- PCA finds the principal subspace, such that approximating all training data by their projections onto this subspace results in the lowest error
- An optimal autoencoder with linear activations reconstructs all data as their projections on the principal subspace
- The bases of this subspace can be uniquely estimated without constraints
- One way to uniquely estimate the subspace is to require the bases of the subspace (the decoder weights of the AE) to be orthonormal
- Another way to estimate the subspace uniquely is to require the distribution of the latent variable Z to be standard Gaussian
- The decoder weights estimated using both above solutions will be the same

Poll 4

Select all that are true about PCA

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Can we do better?



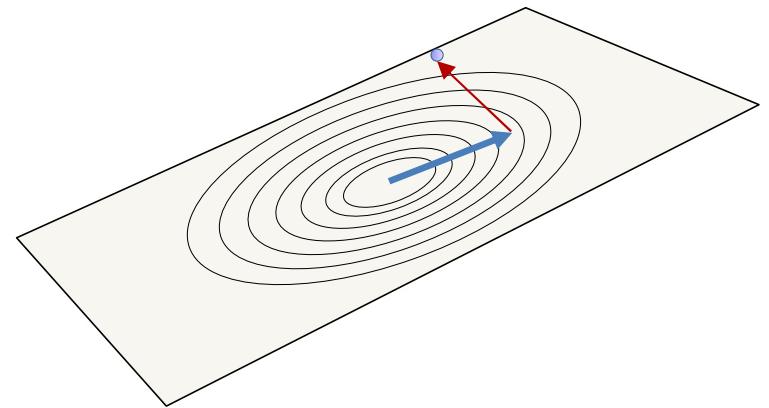
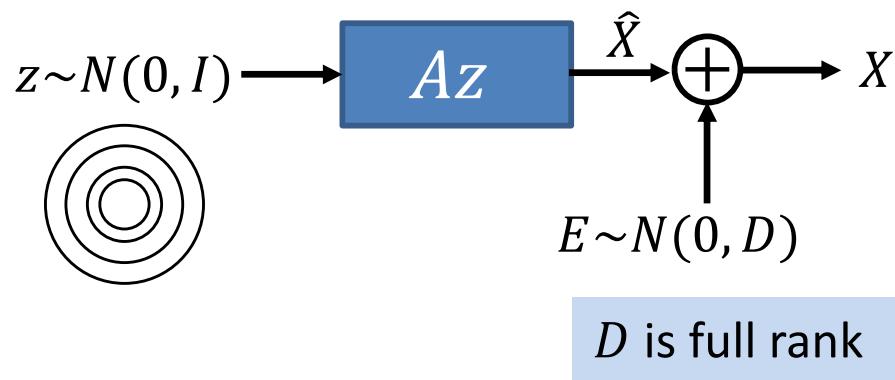
- PCA assumes the noise is always orthogonal to the data
 - Not always true
 - Noise in images can look like images, random noise can sound like speech, etc.
- Let us generalize the model to permit non-orthogonal noise

The Linear Gaussian Model

$$z \sim N(0, I)$$

$$E \sim N(0, D)$$

$$X = Az + E$$



- Update the model: The noise added to the output of the encoder can lie in *any* direction
 - Uncorrelated, but not just orthogonal to the principal subspace
- Generative model: to generate any point
 - Take a Gaussian step on the hyperplane
 - Add *full-rank* Gaussian uncorrelated noise that is independent of the position on the hyperplane
 - Uncorrelated: diagonal covariance matrix
 - Direction of noise is unconstrained
 - Need not be orthogonal to the plane

The linear Gaussian model

$$z \sim N(0, I)$$

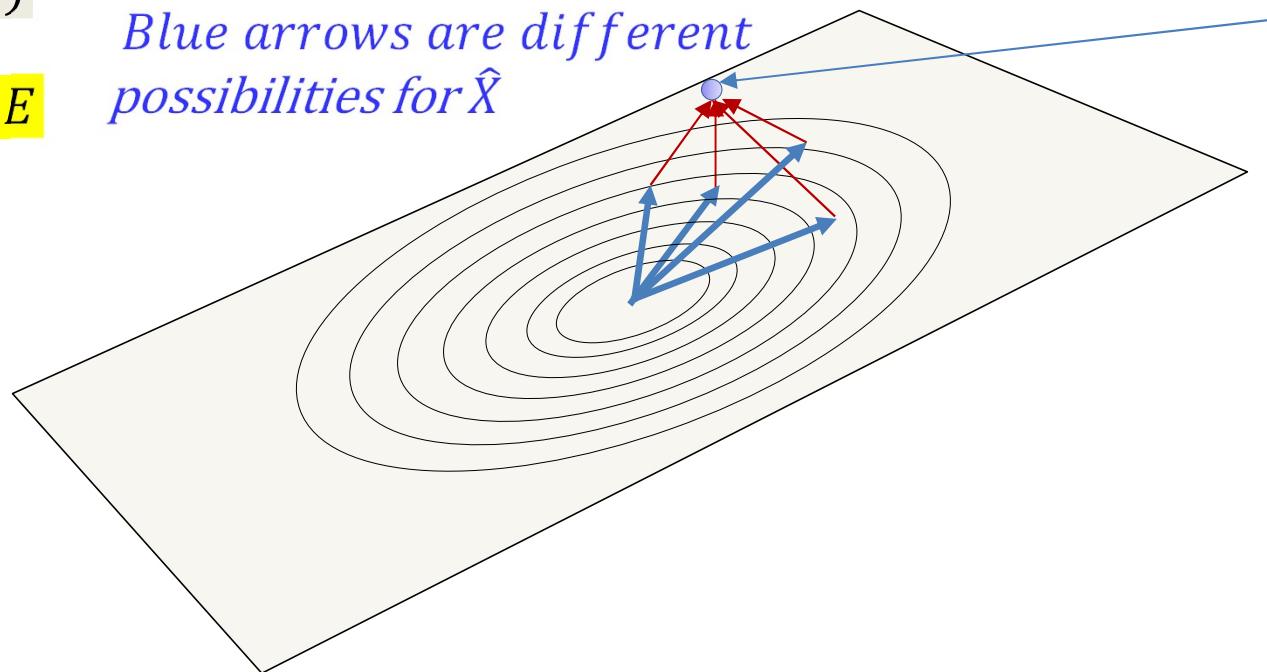
$$E \sim N(0, D)$$

$$X = Az + E$$

Red arrows are different possibilities for E

Blue arrows are different possibilities for \hat{X}

$$X = \hat{X} + E$$



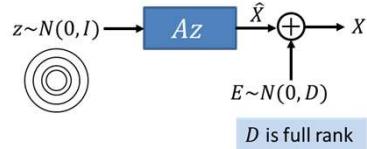
- The way to produce any data instance is no longer unique
 - though different corrections may have different probabilities

The linear Gaussian model

$$z \sim N(0, I)$$

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$$X = Az + E$$



Red arrows are different possibilities for E

Blue arrows are different possibilities for \hat{X}

$$X = \hat{X} + E$$

$$P(X) = N(X; 0, AA^T + D)$$

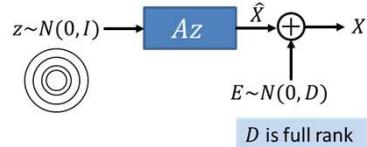
- The way to produce any data instance is no longer unique
 - though different corrections may have different probabilities
- This is still a parametric model for a Gaussian distribution
 - Parameters are A and D (assuming 0 mean)

The linear Gaussian model

$$z \sim N(0, I)$$

$$E \sim N(0, D)$$

$$X = Az + E$$



Red arrows are different possibilities for E

Blue arrows are different possibilities for \hat{X}

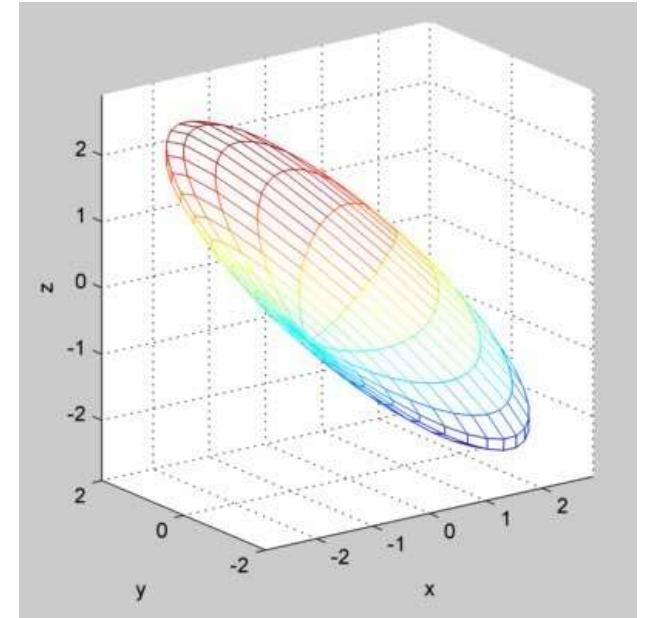
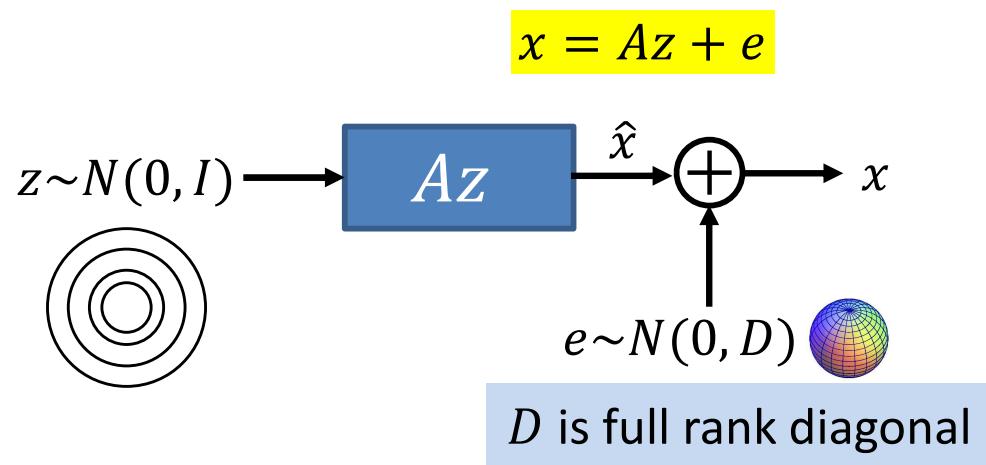
$$X = \hat{X} + E$$

$$P(X) = N(X; 0, AA^T + D)$$

Also known as Factor Analysis:

- The way to p
– though diff
• This is in fact
– Parameters are A and D (assuming 0 mean)
- A is the loading matrix
• z are the factors
• D is diagonal
- X has a multivariate normal distribution

The probability distribution modelled by the LGM



- The noise added to the output of the encoder can lie in *any* direction
- The probability density of x is Gaussian lying mostly close to a hyperplane
 - With uncorrelated Gaussian noise

Story for the day

- EM: An iterative technique to estimate probability models for data with missing components or information
 - By iteratively “completing” the data and reestimating parameters
- PCA: Is actually a generative model for Gaussian data
 - Data lie close to a linear manifold, with orthogonal noise
- Factor Analysis: Also a generative model for Gaussian data
 - Data lie close to a linear manifold
 - Like PCA, but without directional constraints on the noise
- Will continue with FA and Variational AutoEncoders in the next class