University of Tehran

College of Engineering

School of Electrical and Computer Engineering (ECE)

School of Mechanical Engineering (ME)

Mechatronics & Robotics

Homework 4:
Dynamics of Serial Robotic Manipulators

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Deadline: May 17, 2024 (Ordibehesht 28), 23:59

List of Problems

Problem 1: IKP (25 points)

Given the position of point P represented by its coordinates (x,y) and the orientation ϕ of the last link of a serial robot, on which point P is located, accomplish the following tasks:

a) Utilize the closed-form solution of the Inverse Kinematic Problem (IKP) to determine the angle between the links as a function of time.

b) Employ MATLAB to numerically solve the IKP for a desired time step and validate the obtained results by comparing them with the solution derived in section a. For section b, you may assume $L_1 = 17$ cm, $L_2 = 19$ cm, $L_3 = 23$ cm.

$$x = L_1 \cos\left(\frac{\pi}{4} + \frac{\pi}{9}\sin\left(\frac{\pi}{5}t\right)\right) + L_2 \cos\left(\frac{5\pi}{12} + \frac{\pi}{9}\sin\left(\frac{\pi}{5}t\right) + \frac{\pi}{18}\cos\left(\frac{\pi}{10}t\right)\right)$$
$$+ L_3 \cos\left(\frac{11\pi}{36} + \frac{\pi}{9}\sin\left(\frac{\pi}{5}t\right) + \frac{\pi}{18}\cos\left(\frac{\pi}{10}t\right) - \frac{\pi}{36}\sin\left(\frac{\pi}{15}t\right)\right)$$

$$y = L_1 \sin\left(\frac{\pi}{4} + \frac{\pi}{9}\sin\left(\frac{\pi}{5}t\right)\right) + L_2 \sin\left(\frac{5\pi}{12} + \frac{\pi}{9}\sin\left(\frac{\pi}{5}t\right) + \frac{\pi}{18}\cos\left(\frac{\pi}{10}t\right)\right)$$
$$+ L_3 \sin\left(\frac{11\pi}{36} + \frac{\pi}{9}\sin\left(\frac{\pi}{5}t\right) + \frac{\pi}{18}\cos\left(\frac{\pi}{10}t\right) - \frac{\pi}{36}\sin\left(\frac{\pi}{15}t\right)\right)$$

$$\phi = \frac{\pi}{4} + \frac{\pi}{9}\sin\left(\frac{\pi}{5}t\right) + \frac{\pi}{6} + \frac{\pi}{18}\cos\left(\frac{\pi}{10}t\right) - \frac{\pi}{9} - \frac{\pi}{36}\sin\left(\frac{\pi}{15}t\right)$$

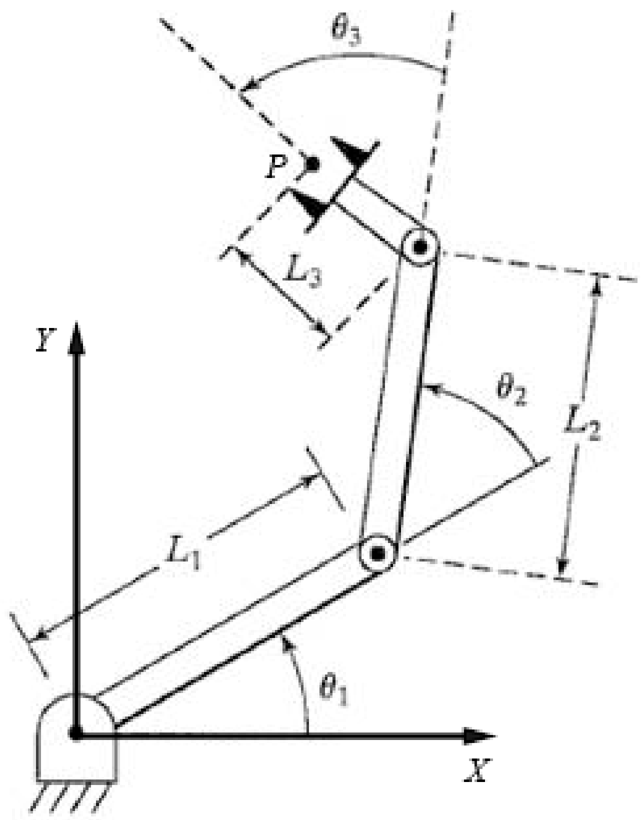

Figure 1: Point P is located at the end of the last link

Answer:

a)

According to page 73 of the lecture notes, for a 3-link serial robot, the forward kinematics equations are

$$x = L_1 \cos(\theta_1) + L_2 \cos(\theta_1 + \theta_2) + L_3 \cos(\theta_1 + \theta_2 + \theta_3)$$

$$y = L_1 \sin(\theta_1) + L_2 \sin(\theta_1 + \theta_2) + L_3 \sin(\theta_1 + \theta_2 + \theta_3)$$

$$\phi = \theta_1 + \theta_2 + \theta_3$$

A direct comparison between these and the given equations, yields

$$\theta_1 = \frac{\pi}{4} + \frac{\pi}{9} \sin(\frac{\pi}{5} t)$$

$$\theta_2 = \frac{\pi}{6} + \frac{\pi}{18} \cos(\frac{\pi}{10} t)$$

$$\theta_3 = -\frac{\pi}{9} - \frac{\pi}{36} \sin(\frac{\pi}{15} t)$$

b)

Problem 2: A Walk in the Park (15 points)

Imagine a non-rotating frame $\{xyz\}$ with unit vectors **i**, **j**, and **k**, and a rotating frame $\{x'y'z'\}$ with unit vectors **i'**, **j'**, and **k'**, where the angular velocity of the rotating frame $\omega(t)$ is defined as

$$\omega(t) = t\mathbf{i} - t^2\mathbf{j} + \frac{1}{t+1}\mathbf{k}$$

and the position of the origin of the rotating frame $O'(t)$ is defined as

$$O'(t) = (1+t)\mathbf{i} + t\mathbf{j} + t\mathbf{k}.$$

A point P, where is the standing point of Parsa is defined in the rotating frame with its position as

$$P(t) = 2t^2\mathbf{i'} + t\mathbf{j'}$$

Determine the velocity and acceleration of Parsa in the non-rotating frame as functions of time. Assume that at $t = 0$, the axes of the rotating and the non-rotating frame are parallel.

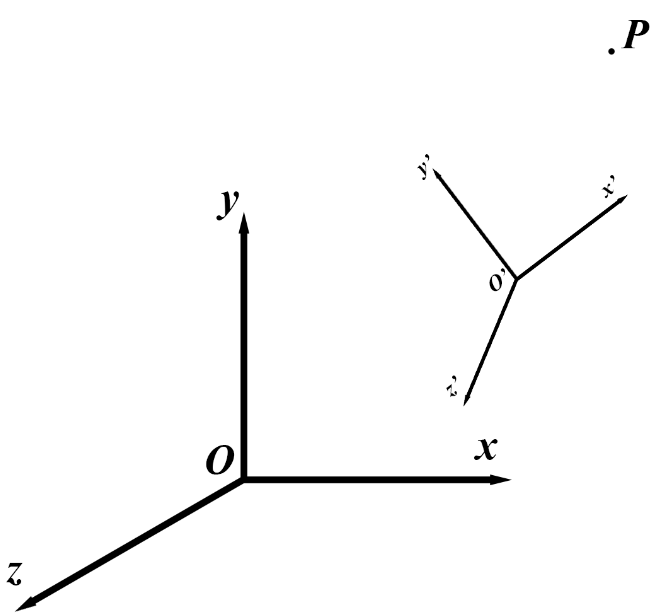

Figure 2: The frames and point P

Answer:

We start by enumerating the non-rotating frame as frame 1 and the rotating one as frame 2. According to page 203 of the lecture notes:

$$\dot{\vec{P}}_1 = \dot{\vec{P}}_2 + \vec{\omega} \times \vec{P}_2$$

Therefore

$$[\frac{d}{dt}\mathbf{i'}]_1 = [\frac{d}{dt}\mathbf{i'}]_2 + \vec{\omega} \times [\mathbf{i'}]_2$$

$$[\frac{d}{dt}\mathbf{i'}]_1 = \vec{0} + \begin{bmatrix} 1+t \\ t \\ t \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$[\frac{d}{dt}\mathbf{i'}]_1 = \frac{1}{t+1}\mathbf{j} + t^2\mathbf{k}$$

Similarly,

$$[\frac{d}{dt}\mathbf{j'}]_1 = [\frac{d}{dt}\mathbf{j'}]_2 + \vec{\omega} \times [\mathbf{j'}]_2$$

$$[\frac{d}{dt}\mathbf{j'}]_1 = -\frac{1}{t+1}\mathbf{i} + t\mathbf{k}$$

and

$$[\frac{d}{dt}\mathbf{k'}]_1 = [\frac{d}{dt}\mathbf{k'}]_2 + \vec{\omega} \times [\mathbf{k'}]_2$$

$$[\frac{d}{dt}\mathbf{k'}]_1 = -t^2\mathbf{i} - t\mathbf{j}$$

Integration from the recent equations yields

$$[\mathbf{i'}]_1 = \int_0^t [\frac{d}{dt}\mathbf{i'}]_1\, dt \ = \mathbf{i} + \ln{(t+1)}\mathbf{j} + \frac{t^3}{3}\mathbf{k}$$

$$[\mathbf{j'}]_1 = \int_0^t [\frac{d}{dt}\mathbf{j'}]_1\, dt \ = -\ln{(t+1)}\mathbf{i} + \mathbf{j} + \frac{t^2}{2}\mathbf{k}$$

$$[\mathbf{k'}]_1 = \int_0^t [\frac{d}{dt}\mathbf{k'}]_1\, dt \ = -\frac{t^3}{3}\mathbf{i} - \frac{t^2}{2}\mathbf{j} + \mathbf{k}$$

Putting the recently derived equations into the given position of point P leads to

$$\vec{P}_1 = \vec{O'} + \vec{P}_2$$

$$\vec{P_1} = (1+t)\mathbf{i} + t\mathbf{j} + t\mathbf{k} + 2t^2\mathbf{i'} + t\mathbf{j'}$$

$$\vec{P_1} = (t - t\ln(t+1) + \ln(t+1)\sqrt{t^2+1} + 2t^2 + 1)\mathbf{i} + (2t + 2t^2\ln(t+1) - \sqrt{t^2+1})\mathbf{j}$$
$$+(t - \frac{t^2\sqrt{t^2+1}}{2} + \frac{t^3}{2} + \frac{2t^5}{3})\mathbf{k}$$

Therefore the velocity and acceleration of point P is determined by direct derivation of P_1.

$$\vec{v} = \frac{d}{dt}\vec{P_1}$$
$$\vec{a} = \frac{d}{dt}\vec{v}$$

Problem 3: Isaac vs Joseph (30 points)

Use the Newtonian and Lagrangian methods to determine the dynamic equations of motion for the system shown. Assume the force $F_1(t)$ and $F_2(t)$ are applied on the upper and lower cart and the moment $M_1(t)$ and $M_2(t)$ are applied on the upper and lower pendulum respectively.

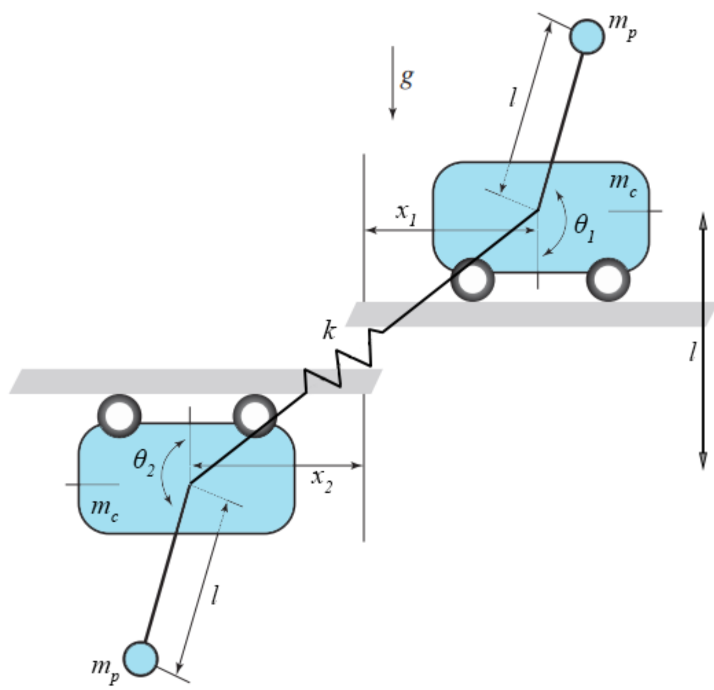

Figure 3: The dynamic system

Answer:

$$T = \frac{1}{2}m_c(\dot{x_1}^2+\dot{x_2}^2)+\frac{1}{2}m_p((\dot{x_1}+l\dot{\theta}_1\cos\theta_1)^2+(l\dot{\theta}_1\sin\theta_1)^2+(\dot{x_2}-l\dot{\theta}_2\cos\theta_2)^2+(l\dot{\theta}_2\sin\theta_2)^2)$$

$$V = m_p gl(\cos\theta_2 - \cos\theta_1)$$

$$L = T - V$$

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{x_1}}) - \frac{\partial L}{\partial x_1} = F_1$$

$$m_c\ddot{x}_1 + \frac{m_p}{2}(2\ddot{x}_1 + 2l\cos\theta_1\ddot{\theta}_1 - 2l\sin\theta_1\dot{\theta}_1^{\ 2}) = F_1$$

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{x_2}}) - \frac{\partial L}{\partial x_2} = F_2$$

$$m_c\ddot{x}_2 + \frac{m_p}{2}(2\ddot{x}_2 - 2l\cos\theta_2\ddot{\theta}_2 + 2l\sin\theta_2\dot{\theta}_2^{\ 2}) = F_2$$

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{\theta}_1}) - \frac{\partial L}{\partial \theta_1} = M_1$$

$$\frac{m_p}{2}(2l\cos\theta_1(\ddot{x}_1 - l\sin\theta_1\dot{\theta}_1^{\ 2} + l\cos\theta_1\ddot{\theta}_1) - 2l\sin\theta_1\dot{\theta}_1(\dot{x_1} + l\cos\theta_1\dot{\theta}_1)+$$

$$+2l^2\sin{\theta_1}^2\ddot{\theta}_1 + 4l^2\sin\theta_1\cos\theta_1\dot{\theta}_1^{\ 2})+$$

$$+m_p gl\sin\theta_1 - \frac{m_p}{2}(2l^2\cos\theta_1\sin\theta_1\dot{\theta}_1^{\ 2} - 2l\sin\theta_1\dot{\theta}_1(\dot{x_1} + l\cos\theta_1\dot{\theta}_1)) = M_1$$

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{\theta}_2}) - \frac{\partial L}{\partial \theta_2} = M_2$$

$$\frac{m_p}{2}(2l\cos\theta_2(l\cos\theta_2\ddot{\theta}_2 - l\sin\theta_2\dot{\theta}_2^{\ 2} - \ddot{x}_2) - 2l\sin\theta_2(l\cos\theta_2\dot{\theta}_2 - \dot{x_2})\dot{\theta}_2+$$

$$+4l^2\sin\theta_2\cos\theta_2\dot{\theta}_2^{\ 2} + 2l^2\sin{\theta_2}^2\ddot{\theta}_2)+$$

$$-m_p gl\sin\theta_2 - \frac{mp}{2}(2l^2\cos\theta_2\sin\theta_2\dot{\theta}_2^{\ 2} + 2l\sin\theta_2\dot{\theta}_2(\dot{x_2} - l\cos\theta_2\dot{\theta}_2)) = M_2$$

Problem 4: Dynamics of Serial Robots (30 points)

A two-revolute pointing manipulator is shown in the figure. The centroidal inertia matrices of the links are denoted by I_1 and I_2. These are given, in link-fixed coordinates, by:

$$I_1 = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix}, I_2 = \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix}$$

Moreover, the mass centers of the links are denoted by C_1 and C_2, respectively, and are shown in the same figure, the masses being denoted by m_1 and m_2. Determine the dynamic equations of motion for the system.

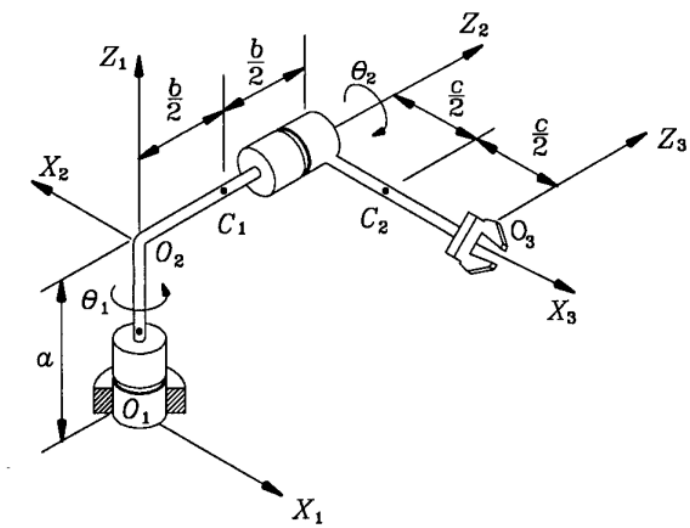

Figure 4: The serial robot

Answer:

$$T = T_1 + T_2$$

$$T_1 = \frac{1}{2}m_1\dot{\vec{C_1}} \cdot \dot{\vec{C_1}} + \frac{1}{2}\vec{\omega_1}^T I_1\vec{\omega_1}$$

$$T_1 = \frac{1}{2}m_1 \begin{bmatrix} \frac{b}{2}\dot{\theta}_1\cos\theta_1 \\ \frac{b}{2}\dot{\theta}_1\sin\theta_1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} \frac{b}{2}\dot{\theta}_1\cos\theta_1 \\ \frac{b}{2}\dot{\theta}_1\sin\theta_1 \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 0 & \dot{\theta}_1 \end{bmatrix} \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix}$$

$$T_1 = \frac{1}{8}m_1 b^2\dot{\theta}_1^{\ 2} + \frac{1}{2}I_{33}\dot{\theta}_1^{\ 2}$$

$$T_2 = \frac{1}{2}m_2\dot{\vec{C_2}} \cdot \dot{\vec{C_2}} + \frac{1}{2}\vec{\omega_2}^T I_2\vec{\omega_2}$$

$$T_2 = \frac{1}{2}m_2(\begin{bmatrix} \frac{b}{2}\dot{\theta}_1\cos\theta_1 \\ \frac{b}{2}\dot{\theta}_1\sin\theta_1 \\ 0 \end{bmatrix} + \begin{bmatrix} -\cos\theta_1 & 0 & -\sin\theta_1 \\ -\sin\theta_1 & 0 & \cos\theta_1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{c}{2}\dot{\theta}_2\cos\theta_1 \\ \frac{c}{2}\dot{\theta}_2\sin\theta_1 \\ 0 \end{bmatrix})\cdot$$

$$\cdot(\begin{bmatrix} \frac{b}{2}\dot{\theta}_1\cos\theta_1 \\ \frac{b}{2}\dot{\theta}_1\sin\theta_1 \\ 0 \end{bmatrix} + \begin{bmatrix} -\cos\theta_1 & 0 & -\sin\theta_1 \\ -\sin\theta_1 & 0 & \cos\theta_1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{c}{2}\dot{\theta}_2\cos\theta_1 \\ \frac{c}{2}\dot{\theta}_2\sin\theta_1 \\ 0 \end{bmatrix})+$$

$$+\frac{1}{2}(\begin{bmatrix} 0 & 0 & \dot{\theta}_1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & \dot{\theta}_2 \end{bmatrix} \begin{bmatrix} -\cos\theta_1 & -\sin\theta_1 & 0 \\ 0 & 0 & 1 \\ -\sin\theta_1 & \cos\theta_1 & 0 \end{bmatrix}) \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix}$$

$$(\begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} + \begin{bmatrix} -\cos\theta_1 & 0 & -\sin\theta_1 \\ -\sin\theta_1 & 0 & \cos\theta_1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{bmatrix})$$

$$V = m_2 g\frac{c}{2}\sin\theta_2$$

$$\mathbf{M} = \frac{\partial^2}{\partial\dot{\vec{\theta}}^2}T$$

$$\mathbf{M}\ddot{\vec{\theta}} + \dot{\mathbf{M}} - \frac{\partial}{\partial\vec{\theta}}T + \frac{\partial}{\partial\vec{\theta}}V = \vec{\tau}$$

Homework Guidelines and Instructions

- The deadline for submitting this homework will be the end of Friday, May 17.
- This time cannot be extended and you can use time grace if needed.
- The implementation must be in Python programming language and your codes must be executable and uploaded along with the report.
- This homework should be done individually.
- If any similarity is observed in the work report or implementation codes, this will be considered fraud for the parties.
- Using ready-made codes without mentioning the source and without changing them will constitute cheating and your practice score will be considered zero.
- If you do not follow the format of the work report, you will not be awarded the grade of the report.
- Handwritten exercise delivery is not acceptable.
- All pictures and tables used in the work report must have captions and numbers.
- A large part of your grade is related to the work report and problem-solving process.
- Please upload the report, code file and other required attachments in the following format in the system: `HW3_[Lastname]_[StudentNumber].zip`
  For example the: `HW3_Rahmati_810699209.zip`
- If you have questions or doubts, you can contact the corresponding teaching assistant via the following Telegram link.
  - `https://t.me/erfunbsarmadi` (Erfun B. Sarmadi)
- Stay happy and healthy