Robotics and Mechatronics

Homework Four

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***Abstract*—This assignment explores serial manipulator dynamics in robotics, covering inverse dynamics (IDP), forward dynamics (FDP), inverse kinematics (IKP), and forward kinematics (FKP). By applying Newtonian and Lagrangian methods, we derive essential equations and concentrate on them.**

***Index Terms*—Dynamics, Newtonian, Lagrangian, kinematics, velocity, acceleration, force**

I. Introduction

Robotic manipulators are integral to modern industries, from manufacturing to healthcare. In this assignment, we delve into the intricacies of IDP, FDP, IKP, and FKP. These concepts form the foundation for efficient robot control and design, enabling precise movements and task execution.

II. Problem 1: IKP

FKP Equations: Having θ_1, θ_2 and θ_3, we want to find x, y and ϕ of end-effector.

$$x = L_1\cos(\theta_1) + L_2\cos(\theta_1+\theta_2) + L_3\cos(\theta_1+\theta_2+\theta_3) \tag{1}$$

$$y = L_1\sin(\theta_1) + L_2\sin(\theta_1+\theta_2) + L_3\sin(\theta_1+\theta_2+\theta_3) \tag{2}$$

$$\phi = \theta_1+\theta_2+\theta_3 \tag{3}$$

we're given:

$$\begin{aligned} x =& L_1\cos\left(\frac{\pi}{4}+\frac{\pi}{9}\sin\left(\frac{\pi}{5}t\right)\right) + L_2\cos\left(\frac{5\pi}{12}+\frac{\pi}{9}\sin\left(\frac{\pi}{5}t\right)+\frac{\pi}{18}\cos\left(\frac{\pi}{10}t\right)\right) + \\ & L_3\cos\left(\frac{11\pi}{36}+\frac{\pi}{9}\sin\left(\frac{\pi}{5}t\right)+\frac{\pi}{18}\cos\left(\frac{\pi}{10}t\right)-\frac{\pi}{36}\sin\left(\frac{\pi}{15}t\right)\right) \\ y =& L_1\sin\left(\frac{\pi}{4}+\frac{\pi}{9}\sin\left(\frac{\pi}{5}t\right)\right) + L_2\sin\left(\frac{5\pi}{12}+\frac{\pi}{9}\sin\left(\frac{\pi}{5}t\right)+\frac{\pi}{18}\cos\left(\frac{\pi}{10}t\right)\right) + \\ & L_3\sin\left(\frac{11\pi}{36}+\frac{\pi}{9}\sin\left(\frac{\pi}{5}t\right)+\frac{\pi}{18}\cos\left(\frac{\pi}{10}t\right)-\frac{\pi}{36}\sin\left(\frac{\pi}{15}t\right)\right) \\ \phi =& \frac{\pi}{4}+\frac{\pi}{9}\sin\left(\frac{\pi}{5}t\right)+\frac{\pi}{6}+\frac{\pi}{18}\cos\left(\frac{\pi}{10}t\right)-\frac{\pi}{9}-\frac{\pi}{36}\sin\left(\frac{\pi}{15}t\right) \end{aligned}$$

comparing the given equations and the derived ones gives:

$$\theta_1 = \frac{\pi}{4}+\frac{\pi}{9}\sin\left(\frac{\pi}{5}t\right) \tag{4}$$

$$\theta_2 = \frac{\pi}{6}+\frac{\pi}{18}\cos\left(\frac{\pi}{10}t\right) \tag{5}$$

$$\theta_3 = -\frac{\pi}{9}-\frac{\pi}{36}\sin\left(\frac{\pi}{15}t\right) \tag{6}$$

IKP Equations: Assuming desired value(s) for t and substituting in the given equations for x, y and ϕ, we can find the task-space parameters. Having x, y and ϕ of end-effector, we want to find θ_1, θ_2 and θ_3 using IKP equations.

$$\begin{aligned} &x = L_1\cos\theta_1 + L_2\cos(\theta_1+\theta_2) + L_3\cos\phi \\ &y = L_1\sin\theta_1 + L_2\sin(\theta_1+\theta_2) + L_3\sin\phi \\ &\rightarrow (x - L_1\cos\theta_1 - L_3\cos\phi)^2 + (y - L_1\sin\theta_1 - L_3\sin\phi)^2 = L_2^2 \end{aligned}$$

$$A\cos\theta_1 + B\sin\theta_1 = C \tag{7}$$

$$A = 2L_1(x - L_3\cos\phi) \tag{8}$$

$$B = 2L_1(y - L_3\sin\phi) \tag{9}$$

$$C = x^2 + y^2 + L_1^2 - L_2^2 + L_3^2 - 2L_3(x\cos\phi + y\sin\phi) \tag{10}$$

$$Answers \longrightarrow \begin{cases} \theta_1^+ \longrightarrow \theta_2^+ \longrightarrow \theta_3^+ \\ \\ \theta_1^- \longrightarrow \theta_2^- \longrightarrow \theta_3^- \end{cases}$$

We use MATLAB to iteratively calculate the output of both methods for a desired range of $t = [0, 10](s)$ with a step size of $\Delta t = 0.1(s)$ resulting in 101 samples. The MATLAB code is attached to this report, while here are its results:

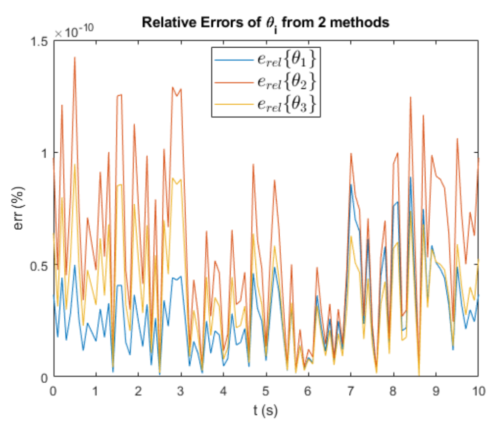

Fig. 1. The differences between joint space angles computed directly and from IKP equations.

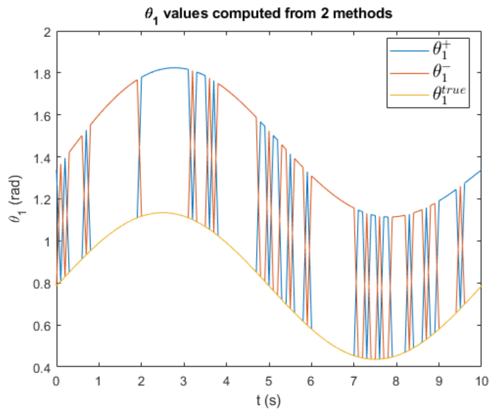

Fig. 2. θ_1 angles computed directly or from IKP

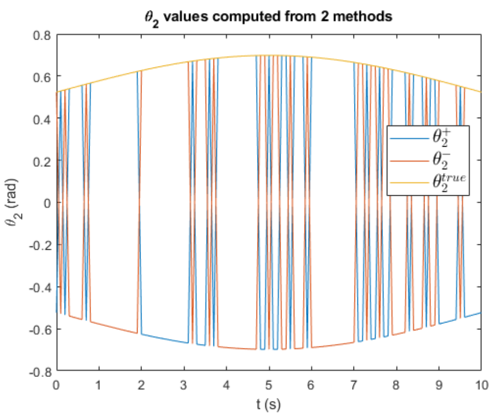

Fig. 3. θ_2 angles computed directly or from IKP

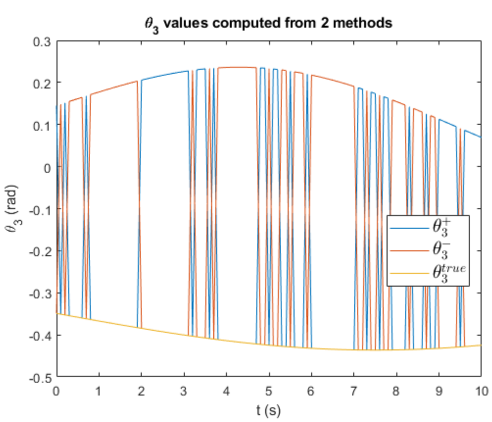

Fig. 4. θ_3 angles computed directly or from IKP

It's worth mentioning that solving via IKP gives two values for each of θ_i, while one of them is the desired one. By eliminating the answer which is further from the actual one computed directly, the following result will be achieved:

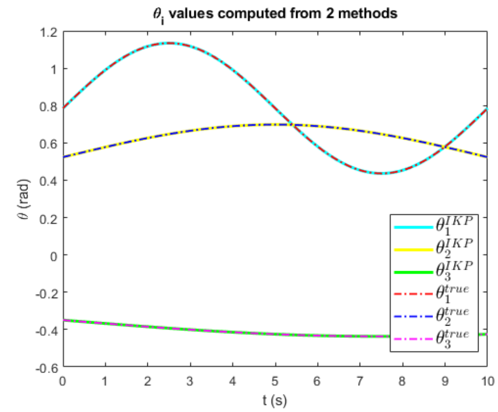

Fig. 5. summary of final θ_i angles computed directly or from IKP

As it can be seen, the duality of answers for each angle found in Figures 2 to 4 has vanished in Figure 5. It's also seen that both methods predict the same angles and have a quite nice conformity. It must also be noted that Figure 1 shows the relative error between the results of both direct and IKP methods, where the answer chose as the result of IKP method is the one shown in Figure 5, meaning the closest answer to the expected one. Thus, it's seen that the errors are all quite low, in an order of 10^{-10}.

III. Problem 2: A Walk in the Park

$$\dot{\vec{P}} = \left]\dot{\vec{P}}\right[+ \vec{\omega} \times \vec{P} \tag{11}$$

$$\ddot{\vec{P}} = \left]\ddot{\vec{P}}\right[+ \vec{\omega} \times \dot{\vec{P}} \tag{12}$$

$$\vec{\omega}(t) = t\hat{\imath} - t^2\hat{\jmath} + \frac{1}{t+1}\hat{k}$$

$$\vec{P}(t) = 2t^2\hat{\imath}' + t\hat{\jmath}'$$

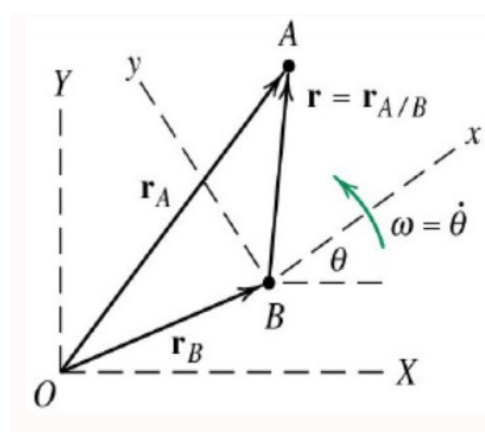

Fig. 6. Motion Relative to Rotating Axex [4]

$$
\begin{aligned}
]\dot{\vec{P}}[&= 4t\hat{\imath}' + \hat{\jmath}' \\
\dot{\vec{P}} &= (4t\hat{\imath}' + \hat{\jmath}') + (t\hat{\imath} - t^2\hat{\jmath} + \frac{1}{t+1}\hat{k}) \times (2t^2\hat{\imath}' + t\hat{\jmath}') \\
&= \begin{bmatrix} 4t \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} t \\ -t^2 \\ \frac{1}{t+1} \end{bmatrix} \times \begin{bmatrix} 2t^2 \\ t \\ 0 \end{bmatrix} = \begin{pmatrix} 4\,t - \frac{t}{t+1} \\ \frac{2\,t^2}{t+1} + 1 \\ 2\,t^4 + t^2 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
]\ddot{\vec{P}}[&= \frac{d(\dot{\vec{P}})}{dt} = (4 - \frac{1}{(t+1)^2})\hat{\imath}' + (\frac{2\,t\,(t+2)}{(t+1)^2})\hat{\jmath}' + (8t^3 + 2t)\hat{k}' \\
\ddot{\vec{P}} &= \begin{bmatrix} 4 - \frac{1}{(t+1)^2} \\ \frac{2\,t\,(t+2)}{(t+1)^2} \\ 8t^3 + 2t \end{bmatrix} + \begin{bmatrix} t \\ -t^2 \\ \frac{1}{t+1} \end{bmatrix} \times \begin{bmatrix} 4\,t - \frac{t}{t+1} \\ \frac{2\,t^2}{t+1} + 1 \\ 2\,t^4 + t^2 \end{bmatrix} \\
&= \begin{pmatrix} \frac{4\,t^2+8\,t+3}{(t+1)^2} - t^2\,\left(2\,t^4 + t^2\right) - \frac{\frac{2\,t^2}{t+1}+1}{t+1} \\ \frac{4\,t-\frac{t}{t+1}}{t+1} - t\,\left(2\,t^4 + t^2\right) + \frac{2\,t\,(t+2)}{(t+1)^2} \\ 2\,t + t^2\,\left(4\,t - \frac{t}{t+1}\right) + t\,\left(\frac{2\,t^2}{t+1} + 1\right) + 8\,t^3 \end{pmatrix}
\end{aligned}
$$

Deriving the dynamic equations according to Figure 6 gives [5]:

$$
\vec{V}_A = \vec{V}_B + \vec{\omega} \times \vec{r}_{B/A} + \vec{V}_{rel} \tag{13}
$$

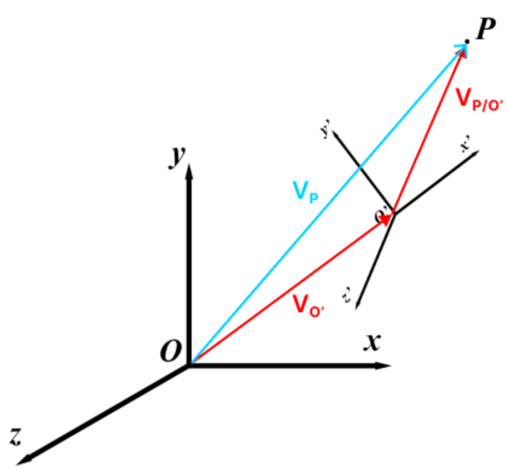

Fig. 7. Dynamics of problem under review

In this problem, with a glance of Figure 7 and comparing it with Figure 6, we can modify equation 13 as:

$$
\mathbf{V}_P = \mathbf{V}_{O'} + \mathbf{V}_{P/O'} \tag{14}
$$

$$
\mathbf{a}_P = \mathbf{a}_{O'} + \mathbf{a}_{P/O'} \tag{15}
$$

where,

$$
\begin{aligned}
\vec{O'}(t) &= (1+t)\hat{\imath} + t\hat{\jmath} + t\hat{k} \longrightarrow \\
\mathbf{V}_{O'} &= \dot{\vec{O'}}(t) = \hat{\imath} + \hat{\jmath} + \hat{k} \\
\mathbf{a}_{O'} &= \dot{\vec{V}}_{O'}(t) = \vec{0}
\end{aligned}
$$

so,

$$
\mathbf{V}_P = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 4\,t - \frac{t}{t+1} \\ \frac{2\,t^2}{t+1} + 1 \\ 2\,t^4 + t^2 \end{bmatrix} = \begin{bmatrix} \frac{(2\,t+1)^2}{t+1} \\ \frac{2\,t^2}{t+1} + 2 \\ 2\,t^4 + t^2 + 1 \end{bmatrix} \tag{16}
$$

$$
\mathbf{a}_P = \vec{0} + \ddot{\vec{P}} = \begin{bmatrix} -\frac{2\,t^8+4\,t^7+3\,t^6+2\,t^5+t^4-2\,t^2-7\,t-2}{(t+1)^2} \\ -\frac{t\,\left(2\,t^6+4\,t^5+3\,t^4+2\,t^3+t^2-6\,t-7\right)}{(t+1)^2} \\ \frac{t\,\left(12\,t^3+13\,t^2+3\,t+3\right)}{t+1} \end{bmatrix} \tag{17}
$$

IV. Problem 3: Isaac vs Joseph

Assuming massless rods in pendulums, their *Inertia* is equal to $I = m_p l^2$ where m_p is the mass of the pendulum and l is the length of its connecting rod. Now we can derive the **Newton** equilibrium after sketching FBD for each element of the system. Note that it's assumed that the free length[1] of the spring is equal to l. The positive direction of x-axis is also assumed to be horizontally rightwards and the y-axis, vertically upwards. The carts don't have vertical displacement either.

A. Newtonian Approach

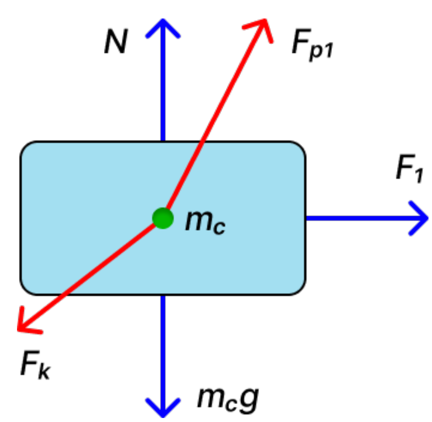

Fig. 8. Free Body Diagram of upper cart

[1]Initial length of the spring where it doesn't apply any force; in this case, the length of the spring when it has fully vertical orientation, resulting in $x_1 = x_2 = 0$.

$$\sum F_x = ma_x \rightarrow F_1 + F_{p_1} \sin\theta_1 - F_k \cos\alpha = m_c \ddot{x}_1$$
$$\sum F_y = ma_y \rightarrow N_1 + F_{p_1} |\cos\theta_1| - F_k \sin\alpha - m_c g = 0$$
$$\alpha = \arctan\left(\frac{l}{x_1 + x_2}\right) \quad \text{(the angle spring makes with x-axis)}$$
$$F_k = k\Delta l = k\left(\sqrt{(x_1+x_2)^2 + l^2} - l\right) \tag{18}$$

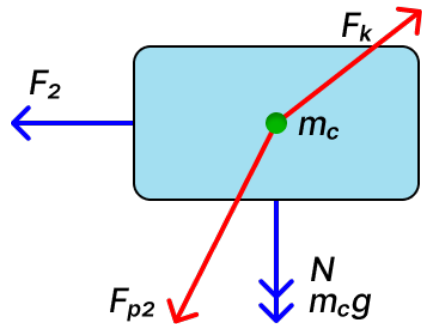

Fig. 9. Free Body Diagram of lower cart

$$\sum F_x = ma_x \rightarrow -F_2 - F_{p_2} \sin\theta_2 + F_k \cos\alpha = m_c \ddot{x}_2$$
$$\sum F_y = ma_y \rightarrow -N_2 - F_{p_2} |\cos\theta_2| + F_k \sin\alpha - m_c g = 0 \tag{19}$$

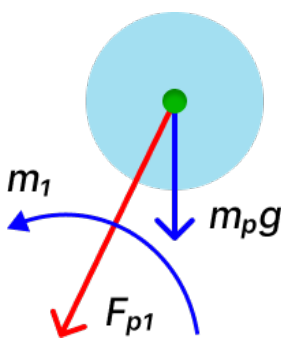

Fig. 10. Free Body Diagram of upper pendulum

Writing the equilibrium of momentum for pendulums is a little bit tricky. In the following equations, a second term is added to the right hand side of the equation which might seem confusing. That's because these equations are NOT written about the centre of mass; rather, they're about the point where the connecting rod is attached to the carts, at their center point. Thus, a *Torque of CoM* also appears in the RHS.

$$\sum M_A = I_A \ddot{\theta}_1 + m_p \bar{a} d \Longrightarrow$$
$$M_1 - m_p g l \sin\theta_1 = m_p l^2 \ddot{\theta}_1 + m_p \left(l^2 \ddot{\theta}_1 - \ddot{x}_1 l \cos\theta_1\right) \tag{20}$$
$$\sum F_x = ma_x \Longrightarrow$$
$$-F_{p_1} \sin\theta_1 = m_p \left(\ddot{x}_1 + l\dot{\theta}_1^2 \sin\theta_1 - l\ddot{\theta}_1 \cos\theta_1\right) \tag{21}$$

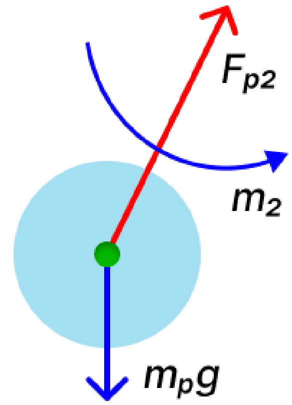

Fig. 11. Free Body Diagram of lower pendulum

$$\sum M_B = I_B \ddot{\theta}_2 + m_p \bar{a} d \Longrightarrow$$
$$M_2 + m_p g l \sin\theta_2 = m_p l^2 \ddot{\theta}_2 + m_p \left(l^2 \ddot{\theta}_2 - \ddot{x}_2 l \cos\theta_2\right) \tag{22}$$
$$\sum F_x = ma_x \Longrightarrow$$
$$F_{p_2} \sin\theta_2 = m_p \left(\ddot{x}_2 + l\dot{\theta}_2^2 \sin\theta_2 - l\ddot{\theta}_2 \cos\theta_2\right) \tag{23}$$

Combining equations 18 to 23 gives

$$\begin{aligned} M_1 &= 2m_p l^2 \ddot{\theta}_1 - m_p l \ddot{x}_1 \cos(\theta_1) + m_p g l \sin(\theta_1) \\ M_2 &= 2m_p l^2 \ddot{\theta}_2 - m_p l \ddot{x}_2 \cos(\theta_2) - m_p g l \sin(\theta_2) \\ F_1 &= +k\left(\sqrt{(x_1+x_2)^2 + l^2} - l\right) \cos\left(\tan^{-1}\left(\frac{l}{x_1+x_2}\right)\right) \\ &\quad - m_p\left(\ddot{x}_1 + l\dot{\theta}_1^2 \sin\theta_1 - l\ddot{\theta}_1 \cos\theta_1\right) + m_c \ddot{x}_1 \\ F_2 &= -k\left(\sqrt{(x_1+x_2)^2 + l^2} - l\right) \cos\left(\tan^{-1}\left(\frac{l}{x_1+x_2}\right)\right) \\ &\quad + m_p\left(\ddot{x}_2 + l\dot{\theta}_2^2 \sin\theta_2 - l\ddot{\theta}_2 \cos\theta_2\right) - m_c \ddot{x}_2 \end{aligned} \tag{24}$$

B. Lagrangian Approach

$$T = \sum_1^2 \frac{1}{2} m_c \dot{x}_i + \sum_1^2 \frac{1}{2} m_p v_{p_i} = \frac{1}{2} m_c \left(\dot{x}_1^2 + \dot{x}_2^2\right) + \frac{1}{2} m_p \times$$
$$\left(\dot{x}_1^2 + \dot{x}_2^2 + l^2\left(\dot{\theta}_1^2 + \dot{\theta}_2^2\right) - 2l\left(\dot{x}_2 \dot{\theta}_2 \cos\theta_2 + \dot{x}_1 \dot{\theta}_1 \cos\theta_1\right)\right)$$
$$V = U_k + {U_g}^2 = \frac{1}{2} k \Delta l^2 + \sum_1^2 m_p g h_i =$$
$$\frac{1}{2} k\left(\sqrt{(x_1+x_2)^2 + l^2} - l\right)^2 + mgl\left(\cos\theta_2 - \cos\theta_1\right)$$
$$\mathcal{L} = T - V \qquad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = Q_i \qquad q_i = x_1, x_2, \theta_1, \theta_2$$

Now differentiating gives:

$$q_1{=}x_1{,}Q_1{=}F_1 \rightarrow \qquad m_c\ddot{x}_1 + m_p\left(\ddot{x}_1 + l\left(\dot{\theta}_1^2\sin\theta_1 - \ddot{\theta}_1\cos\theta_1\right)\right) - k\frac{(x_1+x_2)\left(\sqrt{(x_1+x_2)^2+l^2}-l\right)}{\sqrt{(x_1+x_2)^2+l^2}} = F_1 \tag{25}$$

$$q_2{=}x_2{,}Q_2{=}F_2 \rightarrow \qquad m_c\ddot{x}_2 + m_p\left(\ddot{x}_2 + l\left(\dot{\theta}_2^2\sin\theta_2 - \ddot{\theta}_2\cos\theta_2\right)\right) - k\frac{(x_1+x_2)\left(\sqrt{(x_1+x_2)^2+l^2}-l\right)}{\sqrt{(x_1+x_2)^2+l^2}} = F_2 \tag{26}$$

$$q_3{=}\theta_1{,}Q_3{=}M_1 \rightarrow \qquad m_pl\left(l\ddot{\theta}_1 - \ddot{x}_1\cos(\theta_1) + g\sin(\theta_1)\right) = M_1 \tag{27}$$

$$q_4{=}\theta_2{,}Q_4{=}M_1 \rightarrow \qquad m_pl\left(l\ddot{\theta}_2 - \ddot{x}_2\cos(\theta_2) + g\sin(\theta_2)\right) = M_2 \tag{28}$$

V. Problem 4: Dynamics of Serial Robots

TABLE I
THE D-H PARAMETERS OF THE ROBOT UNDER REVIEW

i	a_i	b_i	α_i	θ_i
1	0	a	$\pi/2$	θ_1
2	c	b	0	θ_2

$$I_1 = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix} \qquad I_2 = \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix}$$

$$[Q_1] = \begin{bmatrix} \cos\theta_1 & 0 & \sin\theta_1 \\ \sin\theta_1 & 0 & -\cos\theta_1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \vec{\mathbf{a_1}} = \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix}$$

$$[Q_2] = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 \\ \sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \vec{\mathbf{a_2}} = \begin{bmatrix} c\cos\theta_2 \\ c\cos\theta_2 \\ b \end{bmatrix}$$

$$T_i = \frac{1}{2}m_i\,\dot{\vec{c}}_i^T\,\dot{\vec{c}}_i + \frac{1}{2}\vec{\omega}_1^T\,I_i\,\vec{c}_i \tag{29}$$

$$\vec{s}_i : O_{i+1} \Longrightarrow C_i \tag{30}$$

$$\vec{c}_1 = \vec{a}_1 + \vec{s}_1 \tag{31}$$

$$\vec{c}_2 = \vec{a}_1 + \vec{a}_2 + \vec{s}_2 \tag{32}$$

[2]The *gravitational potential energy* of carts is excluded in this equation. That's because the carts move only horizontally and don't have height alternation. Since we are finally going to use the differentiate of the *Lagrangian* with respect to the system's degrees of freedom, these constant terms of "potential energies of carts" will be omitted. Therefore, we've ignored them from the very beginning.

$$[\vec{s}_1]_2 = \frac{b}{2}\hat{k}_2 \qquad [\vec{a}_1]_1 = Q_1[\vec{a}_1]_2 \rightarrow [\vec{a}_1]_2 = Q_1^T[\vec{a}_1]_1$$

$$[\vec{a}_1]_2 = \begin{bmatrix} \cos\theta_1 & \sin\theta_1 & 0 \\ 0 & 0 & 1 \\ \sin\theta_1 & -\cos\theta_1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix} = \begin{bmatrix} 0 \\ a \\ 0 \end{bmatrix}$$

$$\vec{c}_1 = \begin{bmatrix} 0 \\ a \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ b/2 \end{bmatrix} = \begin{bmatrix} 0 \\ a \\ b/2 \end{bmatrix}$$

$$[\vec{s}_2]_2 = -\frac{c}{2}\hat{\imath}_2 \qquad [\vec{a}_2]_2 = \begin{bmatrix} c\cos\theta_2 \\ c\cos\theta_2 \\ b \end{bmatrix}$$

$$\vec{c}_2 = \begin{bmatrix} 0 \\ a \\ 0 \end{bmatrix} + \begin{bmatrix} c\cos\theta_2 \\ c\cos\theta_2 \\ b \end{bmatrix} + \begin{bmatrix} -c/2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} c\cos\theta_2 - c/2 \\ a + c\cos\theta_2 \\ b \end{bmatrix}$$

The following approach is used to find the expression for **Kinetic Energy** of each link T_i:

$$T_i = \frac{1}{2}t_i^T M_i t_i \tag{33}$$

$$\text{where,}\quad t_i = [\omega^T, 0]C_i^T \tag{34}$$

$$\text{inertia matrix:}\quad M_i = \begin{bmatrix} I & 0 \\ 0 & m_1 \end{bmatrix} \tag{35}$$

$$[\boldsymbol{\omega}_1]_2 = \mathbf{Q}_1^T\dot{\theta}_1\,[\mathbf{e}_1]_1 = \begin{bmatrix} 0 & \dot{\theta}_1 & 0 \end{bmatrix}^T$$

$$[\boldsymbol{\omega}_2]_3 = \mathbf{Q}_2^T\left([\boldsymbol{\omega}_1]_2 + \dot{\theta}_2\,[\mathbf{e}_2]_2\right) = \begin{bmatrix} \sin\theta_2\dot{\theta}_1 & \cos\theta_2\dot{\theta}_1 & \dot{\theta}_2 \end{bmatrix}^T$$

Note that since the inertia matrices were given in link-fixed coordinates in this problem (meaning: $[I_i]_{i+1}$), we need $[t_i]_{i+1}$. Thus, the equations above have been derived using the $[Q_i]$ matrices achieved earlier. Moreover,

$$[\mathbf{c}_1]_2 = \mathbf{Q}_1^T [\mathbf{c}_1]_1 = \begin{bmatrix} 0 & a & \frac{b}{2} \end{bmatrix}^T$$

$$[\mathbf{c}_2]_3 = \mathbf{Q}_2^T \left[\overrightarrow{O_1O_2}\right]_2 + \left[\overrightarrow{O_2C_2}\right]_3 = \begin{bmatrix} a\sin\theta_2 \\ a\cos\theta_2 \\ 0 \end{bmatrix} + \begin{bmatrix} c/2 \\ 0 \\ b \end{bmatrix} = \begin{bmatrix} c/2 + a\sin\theta_2 \\ a\cos\theta_2 \\ b \end{bmatrix}$$

Therefore,

$$[\dot{\mathbf{c}}_1]_2 = [\boldsymbol{\omega}_1 \times \mathbf{c}_1]_2 = \begin{bmatrix} \frac{b}{2}\dot{\theta}_1 & 0 & 0 \end{bmatrix}^T$$

$$[\dot{\mathbf{c}}_2]_3 = [\boldsymbol{\omega}_2 \times \mathbf{c}_2]_3 = \begin{bmatrix} \cos\theta_2 \left(b\dot{\theta}_1 - a\dot{\theta}_2\right) \\ \frac{c}{2}\dot{\theta}_2 - \sin\theta_2 \left(b\dot{\theta}_1 - a\dot{\theta}_2\right) \\ -\frac{c}{2}\cos\theta_2\dot{\theta}_1 \end{bmatrix}$$

The total **Kinetic Energy** is given by

$$T = \frac{1}{2}\mathbf{t}_1^T \mathbf{M}_1 \mathbf{t}_1 + \frac{1}{2}\mathbf{t}_2^T \mathbf{M}_2 \mathbf{t}_2$$

with

$$\mathbf{t}_1 = [\mathbf{t}_1]_2 = \begin{bmatrix} 0 \\ \dot{\theta}_1 \\ 0 \\ \frac{b}{2}\dot{\theta}_1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{t}_2 = [\mathbf{t}_2]_3 = \begin{bmatrix} \sin\theta_2\dot{\theta}_1 \\ \cos\theta_2\dot{\theta}_1 \\ \dot{\theta}_2 \\ \cos\theta_2 \left(b\dot{\theta}_1 - a\dot{\theta}_2\right) \\ \frac{c}{2}\dot{\theta}_2 - \sin\theta_2 \left(b\dot{\theta}_1 - a\dot{\theta}_2\right) \\ -\frac{c}{2}\cos\theta_2\dot{\theta}_1 \end{bmatrix}$$

$$[\mathbf{M}_1]_2 = \begin{bmatrix} I_{11} & I_{12} & I_{13} & 0 & 0 & 0 \\ I_{12} & I_{22} & I_{23} & 0 & 0 & 0 \\ I_{13} & I_{23} & I_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & m_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & m_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & m_1 \end{bmatrix}$$

$$[\mathbf{M}_2]_3 = \begin{bmatrix} J_{11} & J_{12} & J_{13} & 0 & 0 & 0 \\ J_{12} & J_{22} & J_{23} & 0 & 0 & 0 \\ J_{13} & J_{23} & J_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & m_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & m_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & m_2 \end{bmatrix}$$

Finally, performing the required calculations, we obtain

$$\begin{aligned} T =& \frac{\dot{\theta}_1^2}{2}\left(I_{22} + \frac{m_1 b^2}{4} + J_{11}\sin^2\theta_2 + J_{22}\cos^2\theta_2 + 2J_{12}\sin\theta_2\cos\theta_2 \right. \\ &\left. + m_2 b^2 + \frac{m_2 c^2}{4}\cos^2\theta_2\right) \\ &+ \frac{\dot{\theta}_2^2}{2}\left(J_{33} + m_2 a^2 + \frac{m_2 c^2}{4} + m_2 ac\sin\theta_2\right) \\ &+ \dot{\theta}_1\dot{\theta}_2\left(J_{13}\sin\theta_2 + J_{23}\cos\theta_2 - m_2 ab - \frac{1}{2}m_2 bc\sin\theta_2\right) \end{aligned}$$

We also need to find the **Potential Energy** expression of each link and then, of the entire system. Taking the ground level on z_2 axis, we would only have the potential energy of the second link. That's because the center of math of the first link always has a constant height which is invariant with respect to each degree of freedom of the system. In other words, its height wouldn't change and since the potential energy expression is used after differentiation, this constant value would be omitted.

$$V = \sum U_g = [mgh]_2 = m_2 g\frac{c}{2}\sin\theta_2 \tag{36}$$

Now we can sum up,

$$\mathcal{L} = T + V \tag{37}$$

Now that we have the **Lagrangian** expression, we can write the Lagrange Equation of the system dynamics.

$$\underset{\sim}{M}\ddot{\vec{\theta}} + \dot{\underset{\sim}{M}}\dot{\vec{\theta}} - \frac{\partial T}{\partial\vec{\theta}} + \frac{\partial V}{\partial\vec{\theta}} = \vec{\tau} \tag{38}$$

$$\underset{\sim}{\mathbf{M}}(\theta) = \begin{bmatrix} \dfrac{\partial^2 T}{\partial\dot{\vec{\theta}}_1^{\,2}} & \dfrac{\partial^2 T}{\partial\dot{\vec{\theta}}_1\partial\dot{\vec{\theta}}_2} \\ \dfrac{\partial^2 T}{\partial\dot{\vec{\theta}}_1\partial\dot{\vec{\theta}}_2} & \dfrac{\partial^2 T}{\partial\dot{\vec{\theta}}_2^{\,2}} \end{bmatrix} \tag{39}$$

so, here

$$\underset{\sim}{M} = \begin{bmatrix} \sigma_1 & \sigma_2 \\ \sigma_2 & \sigma_3 \end{bmatrix} \qquad \dot{M} = \frac{\partial M}{\partial t} = \begin{bmatrix} \sigma_6 & \sigma_7 \\ \sigma_7 & \sigma_8 \end{bmatrix}$$

$$\frac{\partial T}{\partial\vec{\theta}} = \begin{bmatrix} \frac{\partial T}{\partial\theta_1} \\ \frac{\partial T}{\partial\theta_2} \end{bmatrix} = \begin{bmatrix} 0 \\ \sigma_4 \end{bmatrix} \qquad \frac{\partial V}{\partial\vec{\theta}} = \begin{bmatrix} \frac{\partial V}{\partial\theta_1} \\ \frac{\partial V}{\partial\theta_2} \end{bmatrix} = \begin{bmatrix} 0 \\ \sigma_5 \end{bmatrix}$$

$$\begin{aligned} \sigma_1 =& I_{22} + \frac{m_1 b^2}{4} + J_{11}\sin^2\theta_2 + J_{22}\cos^2\theta_2 + 2J_{12}\sin\theta_2\cos\theta_2 + m_2 b^2 \\ &+ \frac{m_2 c^2}{4}\cos^2\theta_2 \\ \sigma_2 =& J_{13}\sin\theta_2 + J_{23}\cos\theta_2 - m_2 ab - \frac{1}{2}m_2 bc\sin\theta_2 \\ \sigma_3 =& J_{33} + m_2 a^2 + \frac{m_2 c^2}{4} + m_2 ac\sin\theta_2 \\ \sigma_4 =& \frac{\dot{\theta}_1^2}{2}\left(2J_{11}\sin\theta_2\cos\theta_2 - 2J_{22}\sin\theta_2\cos\theta_2 + 2J_{12}\cos 2\theta_2 - \frac{m_2 c^2}{2}\right. \\ &\left. \cos\theta_2\sin\theta_2\right) + \frac{\dot{\theta}_2^2}{2}m_2 ac\cos\theta_2 + \dot{\theta}_1\dot{\theta}_2\left(J_{13}\cos\theta_2 - J_{23}\sin\theta_2\right. \\ &\left. -\frac{m_2}{2}bc\cos\theta_2\right) \qquad \sigma_5 = \frac{m_2 gc}{2}\cos\theta_2 \\ \sigma_6 =& \dot{\theta}_2\left(J_{11}\sin 2\theta_2 - J22\sin 2\theta_2 + 2J_{12}\cos 2\theta_2 - \frac{m_2 c^2}{4}\sin 2\theta_2\right) \\ \sigma_7 =& \dot{\theta}_2\left(J_{13}\cos\theta_2 - cJ_{23}\sin\theta_2 - \frac{m_2 bc}{2}\cos\theta_2\right) \\ \sigma_8 =& \dot{\theta}_2\left(m_2 ac\cos\theta_2\right) \end{aligned}$$

After differentiating M with respect to t to derive $\dot{M} = \frac{\partial M}{\partial t}$, we've fulfilled all of the required parameters in equation 38. Now, we can substitute achieved values into it and find the final **Dynamic Equation** of this system.

$$
\begin{bmatrix} \sigma_1 & \sigma_2 \\ \sigma_2 & \sigma_3 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} \sigma_6 & \sigma_7 \\ \sigma_7 & \sigma_8 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} - \begin{bmatrix} 0 \\ \sigma_4 \end{bmatrix} + \begin{bmatrix} 0 \\ \sigma_5 \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}
$$

VI. Conclusion

In this assignment, we've explored the intricate world of serial robot dynamics. By combining theoretical principles with practical methods, we gain insights into how robots move, react to external forces, and achieve precise tasks. Whether it's designing a robotic arm for assembly lines or programming a surgical robot, mastering these concepts is essential.

References

[1] J. Angeles, "Fundamentals of Robotic Mechanical Systems", Theory, Methods, and Algorithms, 4th edition, Springer.

[2] J. J. Craig, "Introduction to Robotics", 3d edition, Pearson Education, Inc.

[3] M. Kazim, J. Hong, M. Kim, K. K. Kim, (2023). "Recent advances in path integral control for trajectory optimization: An overview in theoretical and algorithmic perspectives". Annual Reviews in Control, 57, 100931. [Online]. Available: https://doi.org/10.1016/j.arcontrol.2023.100931

[4] Mechanical Dynamics NAV, "Motion Relative to Rotating Axes", http://pioneer.netserv.chula.ac.th/~anopdana/211/55rotax.pdf

[5] S. Widnall, J. Peraire, "Relative Motion using Rotating Axes", 16.07 Dynamics, Fall 2009, [Online]. Available: https://ocw.mit.edu/courses/16-07-dynamics-fall-2009/1b1825352e94d42461eb576b540b2f23_MIT16_07F09_Lec08.pdf