Numerical method HW8

Mohammad

problem solved

A1

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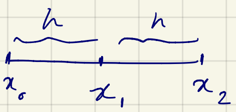
26

31

A1 Simpson 1/3 rule



$$f(x) = f(x_1) + (x-x_1) f'(x_1) + (x-x_1)^2 \frac{1}{2} f''(x_1) + (x-x_1)^3 \frac{1}{6} f^3(x_1) + (x-x_1)^4 \frac{1}{24} f^4(x_1)$$

$\rightarrow \int_{x_0}^{x_2} f(x)\,dx$:

$$\left[f(x_1)\, x + \tfrac{1}{2}(x-x_1)^2 f'(x_1) + \tfrac{1}{6}(x-x_1)^3 f''(x_1) + \tfrac{1}{24}(x-x_1)^4 f^3(x_1) + \ldots \right]_{x_0}^{x_2}$$

$$f(x_1)(x_2-x_0) + \tfrac{1}{2}\left[(x_2-x_1)^2-(x_0-x_1)^2\right] f'(x_1) + \tfrac{1}{6}\left[(x_2-x_1)^3-(x_0-x_1)^3\right] f''(x_1)$$
$$+ \tfrac{1}{24}\left[(x_2-x_1)^4-(x_0-x_1)^4\right] f^3(x) + \frac{1}{120}\left[(x_2-x_1)^5-(x_0-x_1)^5\right] f^4(x_1)$$

$\boxed{x_2 - x_1 = h}$ $\boxed{x_0 - x_1 = -h}$ $\Rightarrow$

$$\int_{x_0}^{x_2} f(x)\,dx = 2h f(x_1) + \frac{2}{6} h^3 f''(x_1) + \frac{2}{120} h^5 f^4(x_1)$$

$$\boxed{\int_{x_0}^{x_2} f(x)\,dx = 2h f(x_1) + \frac{h^3}{3} f''(x_1) + \frac{h^5}{60} f^4(x_1)}$$

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b)

$$f(x_2) = f(x_1) + (x_2-x_1) f'(x_1) + (x_2-x_1)^2 \frac{1}{2} f''(x_1) + \frac{1}{6}(x_2-x_1)^3 f^3(x_1) + \frac{1}{24}(x_2-x_1)^4 f^4(x_1)$$

$\rightarrow \boxed{x_2 - x_1 = h} \rightarrow$

$$f(x_2) = f(x_1) + h f'(x_1) + \frac{h^2}{2} f''(x_1) + \frac{h^3}{6} f^3(x_1) + \frac{h^4}{24} f^4(x_1)$$

$\boxed{x_0 - x_1 = -h} \rightarrow$

$$f(x_0) = f(x_1) - h f'(x_1) + \frac{h^2}{2} f''(x_1) - \frac{h^3}{6} f^3(x_1) + \frac{h^4}{24} f^4(x_1)$$

---

c) Error $= I_{\text{Simpson}} - I_{\text{exact}}$

$$I_{\text{Simpson}} = \frac{h}{3}(f_0 + 4f_1 + f_2) \Rightarrow$$

$$\frac{h}{3}\left[f(x_1) - h f'(x_1) + \frac{h^2}{2} f''(x_1) - \frac{h^3}{6} f^3(x_1) + \frac{h^4}{24} f^4(x_1) + 4f(x_1) + f(x_1) + h f'(x_1) + \frac{h^2}{2} f''(x_1) + \frac{h^3}{6} f^3(x_1) + \frac{h^4}{24} f^4(x_1) \right]$$

$$\frac{h}{3}\left[6 f(x_1) + h^2 f''(x_1) + \frac{h^4}{12} f^4(x_1) \right] \Rightarrow \boxed{I_{\text{simp}} = 2h f(x_1) + \frac{h^3}{3} f''(x_1) + \frac{h^5}{36} f^4(x_1)}$$

from part a) $\boxed{I_{exact} = 2h f(x_1) + \frac{h^3}{3} f''(x_1) + \frac{h^5}{60} f^{4}(x_1)}$

$$error = I_{simpson} - I_{exact} = \left(\frac{1}{36} - \frac{1}{60}\right) h^5 f^{4}(x_1) \Rightarrow \boxed{error = \frac{1}{90} h^5 f^{4}(x_1)}$$ 10

1. $R(n) = [I(n/2) - I(n/4)] / [I(n) - I(n/2)]$

a) : $I = \int_0^1 \exp(-x^2)\, dx$

applying the trapezoidal rule for differen n with the matlab code below

```
% trapizoidal integration:
format long
b=1;
a=0;
n=[2,4,8,16,32,64,128,256,512];
for i=1:length(n)
h=(b-a)/n(i);
nn=n(i);
x=linspace(a,b,n(i)+1);
y=f(x);
exact(i)=trapz(x,y);
trpz(i) = (y(1)+2*sum(y(2:nn))+y(nn+1))*(h/2);
x=[];
y=[];
end
for j=1:length(n)-2
R(j)=(trpz(j+1)-trpz(j))/(trpz(j+2)-trpz(j+1));
end
table(n',trpz',[nan(),nan(),R]','VariableNames',["n","I(n)","R(n)"])
function y=f(x)
y=exp(-x.^2);
end
```

n	I(n)	R(n)
2	0.731370251828563	NaN
4	0.742984097800381	NaN
8	0.745865614845695	4.03046235339984
16	0.746584596788222	4.00777387424953
32	0.746764254652294	4.00195085385311
64	0.746809163637828	4.00048814144145
128	0.746820390541618	4.00012206161753
256	0.746823197246152	4.0000305169821
512	0.746823898920947	4.00000763007098

$R(n) = [I(n/2) - I(n/4)] / [I(n) - I(n/2)]$

It's trapezoidal rule & we expect the order of convergence to be $P \sim (h_i / h_{i+1})^2 = 4$

because $P \sim h^2 \rightarrow \left(\frac{h}{h/2}\right)^2 = 4$

$\rightarrow P \sim 4$

b) $I = \int_0^1 x^{2.5}\, dx$

here it has weak singularity at $f^3(x)$ but in trapezoidal the error is proportional to $f'(x)$ → so it doesn't affect the accuracy

n	I(n)	R(n)
2	0.338388347648318	NaN
4	0.298791496231346	NaN
8	0.288974739670143	4.03359818185403
16	0.286528567896037	4.01311006247428
32	0.285917779698734	4.00494276887683
64	0.285765152250462	4.00182407699343
128	0.285727001721098	4.00066397020729
256	0.285717464659795	4.00023950280965
512	0.285715080445649	4.00008586398653

$P \sim 4$

c)

$$I = \int_{-4}^{4} \frac{1}{1+x^2}\, dx$$

n	I(n)	R(n)
2	4.23529411764706	NaN
4	2.91764705882353	NaN
8	2.65882352941176	5.0909090909091
16	2.65050680499416	31.1208495575227
32	2.65134716346583	-9.89663899154082
64	2.65156325136377	3.88896592399237
128	2.6516173061521	3.99757180834173
256	2.65163082190308	3.99939214754328
512	2.65163420096925	3.99984796583638

$P \sim 4$

d) $$I = \int_{0}^{2\pi} \frac{1}{2+\cos x}\, dx$$

n	I(n)	R(n)
2	4.18879020478639	NaN
4	3.66519142918809	NaN
8	3.62779151664536	14.0000000000003
16	3.62759873359101	193.99999999932
32	3.62759872846844	37634.0079189321
64	3.62759872846844	5767507.5
128	3.62759872846844	-1
256	3.62759872846844	-1
512	3.62759872846844	-1

it's trapezoid but
The order of convergence is not 4 when we make h half.

The reason for this is the fact that the function f & all of it's derivatives $(f', f'', f^3, \dots)$ are also periodic in the interval $[0, 2\pi]$ so it pushes error back to zero because $T.E \approx \frac{-h^2}{12}[f'(b) - f'(a)]$

& for all m $f^{m}(2\pi) = f^{m}(0) \rightarrow$ So the error will be zero

e) $I = \int_0^{\pi} e^{x} \cos(4x)\, dx$

n	I(n)	R(n)
2	26.5163358570775	NaN
4	3.24905049448466	NaN
8	1.62452524724233	14.3225138557185
16	1.37572251765279	6.52937067821721
32	1.32031187842362	4.4901616918823
64	1.30684788549749	4.11546853397757
128	1.30350565849719	4.02844957116734
256	1.30267157945597	4.00708666103559
512	1.30246315192811	4.00177006267977

$p \sim 2$

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2 Simpson's Rule

I applied Simpson rule with the Matlab code below:

```
% simpson integration:
format long
b=1;
a=0;
n=[2,4,8,16,32,64,128,256,512];
for i=1:length(n)
h=(b-a)/n(i);
nn=n(i);
x=linspace(a,b,n(i)+1);
y=f(x);
simps(i) = (y(1)+4*sum(y(2:2:nn))+2*sum(y(3:2:nn-1))+y(nn+1))*(h/3);
x=[];
y=[];
end
for j=1:length(n)-2
R(j)=(simps(j+1)-simps(j))/(simps(j+2)-simps(j+1));
end

t=table(n',simps',[nan(),nan(),R]','VariableNames',["n","I(n)","R(n)"])

function y=f(x)
y=exp(-x.^2);
%y=x.^(2.5);
%y = 1./(1+x.^2);
%y=1./(2+cos(x));
%y=exp(x).*cos(4*x);
end
```

note :

here we make h half in each Time & Since The error in Simpson has the order of h^4 we expect to have

$\left(\frac{h}{n/2}\right)^4 = 16$ order of convergence

as for Simpson $\frac{1}{3}$ rule for n number of intervals

$$T.E = \frac{-h^4}{90}\left[f'''(b) - f'''(a)\right]$$

a) $\int_0^1 exp(-x^2)\,dx$

$$\left(\frac{h}{n/2}\right)^4 = 16 \rightarrow \frac{TE_n}{TE_{n/2}} = 16$$

we can see R(n) converges to 16

n	I(n)	R(n)
2	0.74718042890951	NaN
4	0.746855379790987	NaN
8	0.746826120527467	11.1092720530826
16	0.74682425743573	15.7046821420663
32	0.746824140606985	15.9472031548105
64	0.746824133299672	15.9879221046643
128	0.746824132842881	15.9970496377366
256	0.74682413281433	15.9992922749084
512	0.746824132812546	16.0006222000996

$P \sim 4$

$\int_0^1 x^{2.5}\,dx$:

here due to weak singularity

The $R(n)$ doesn't converge to 16

→ $\boxed{P \sim 3.5}$

→ we can do Richardson extrapolation to mitigate that

n	I(n)	R(n)
2	0.284517796864425	NaN
4	0.285592545759022	NaN
8	0.285702487483075	9.77562343914868
16	0.285713177304669	10.284710842913
32	0.285714183632966	10.6225986308251
64	0.285714276434372	10.8438906725777
128	0.285714284877977	10.9907328067267
256	0.285714285639361	11.0898063347152
512	0.2857142857076	11.1575633089019

$f(x) = x^{2.5}$ $f'(x) = 2.5\,x^{1.5}$ $f''(x) = (2.5)(1.5)\,x^{+0.5}$ $\boxed{f^{3}(x) = +(2.5)(1.5)(0.5)\,x^{-0.5}}$

the reason for this $T.E = \frac{-h^4}{90}\left[f^3(b) - f^3(a)\right]$ → we have weak singularity

we can $I_h = I_{extrap} - error_h$; $I_h \sim I_{extrap} - R(n)\, error_{h/2}$

$I_{h/2} = I_{extrap} - error_{h/2}$; $I_{h/2} \sim I_{extrap} - error_{h/2}$

$R(n)\, I_{h/2} - I_h = (R(n) - 1)\, I_{extrap}$ → $\boxed{I_{extrap} \sim \frac{R(n)\, I_{h/2} - I_h}{(R(n) - 1)}}$

with applying this extrapolation we can mitigate the effect of the weak singularity

n	I(n)	R(n)	Iextrap	R of extrap
2	0.284517796864425	NaN	NaN	NaN
4	0.285592545759022	NaN	NaN	NaN
8	0.285702487483075	9.77562343914868	0.285715015564392	NaN
16	0.285713177304669	10.284710842913	0.285714328640642	NaN
32	0.285714183632966	10.6225986308251	0.285714288212647	16.9912892449563
64	0.285714276434372	10.8438906725777	0.285714285861682	17.1963341518533
128	0.285714284877977	10.9907328067267	0.285714285723121	16.9670104439218
256	0.285714285639361	11.0898063347152	0.285714285714822	16.696832669989
512	0.2857142857076	11.1575633089019	0.285714285714319	16.4859947066608
1024	0.285714285713691	11.2043932005651	0.285714285714288	16.3387387387387

$P \sim 4$

c $\int_{-4}^{4} \frac{1}{x^2+1}\,dx$

n	I(n)	R(n)
2	5.49019607843137	NaN
4	2.47843137254902	NaN
8	2.57254901960784	-32.0000000000001
16	2.64773456352162	1.25180509655895
32	2.65162728295638	19.3144009409791
64	2.65163528066308	486.729456859926
128	2.65163532441487	182.797216762095
256	2.6516353271534	15.9763993660054
512	2.65163532732465	15.9917585552317

$P \sim 4$

$\left(\frac{h}{h/2}\right)^P = 2^4 = 16$

d)

$$\int_0^{2\pi} \frac{1}{2+\cos x}\,dx$$

n	I(n)	R(n)
2	4.88692190558412	NaN
4	3.49065850398866	NaN
8	3.61532487913111	-11.1999999999999
16	3.6275344725729	10.2105263157901
32	3.62759872676091	190.020196647123
64	3.62759872846843	37630.0372790443
128	3.62759872846844	1922501
256	3.62759872846843	-1
512	3.62759872846844	-0.666666666666667

here again since 2π is period of f & all of its derivatives the Integral converge to its exact value really fast with increasing number of intervals

as $T.E \sim -\frac{1}{90} h^4 [f'''(b) - f'''(a)]$ for n number of intervals

e) $I = \int_0^{\pi} e^x \cos(4x)\,dx$

n	I(n)	R(n)
2	22.7150773714852	NaN
4	-4.50671129304627	NaN
8	1.08301683149489	-4.8699664917542
16	1.29278827445628	26.6467544181878
32	1.30184166534723	23.1704833568
64	1.30235988785545	17.4700842733717
128	1.30239158283042	16.350305015931
256	1.3023935531089	16.0865457813213
512	1.3023936760855	16.0215726944268

#18 one & two point Gaussian quadrature

~~two~~ two point:

$$I = \int_0^1 x f(x)\,dx = \alpha f(x_1) + \beta f(x_2)$$

if $f(x)$ is polynomial of degree 3 or less eq * is exact
So for any combination of monomials up to degree 3 should be exact
as well So:

$\int_0^1 x f(x)\,dx \rightarrow f(x) = x^0 \rightarrow \int_0^1 x\,dx = \alpha + \beta \rightarrow$

$f(x) = x \rightarrow \int_0^1 x^2\,dx = \alpha x_1 + \beta x_2 \rightarrow$

$f(x) = x^2 \rightarrow \int_0^1 x^3\,dx = \alpha x_1^2 + \beta x_2^2 \rightarrow$

$f(x) = x^3 \rightarrow \int_0^1 x^4\,dx = \alpha x_1^3 + \beta x_2^3 \rightarrow$

$$\left\{\begin{aligned} \tfrac{1}{2} &= \alpha - \beta \\ \tfrac{1}{3} &= \alpha x_1 + \beta x_2 \\ \tfrac{1}{4} &= \alpha x_1^2 + \beta x_2^2 \\ \tfrac{1}{5} &= \alpha x_1^3 + \beta x_2^3 \end{aligned}\right\}$$

$$\left.\begin{aligned} \alpha + \beta - \tfrac{1}{2} &= 0 \\ \alpha x_1 + \beta x_2 - \tfrac{1}{3} &= 0 \\ \alpha x_1^2 + \beta x_2^2 - \tfrac{1}{4} &= 0 \\ \alpha x_1^3 + \beta x_2^3 - \tfrac{1}{5} &= 0 \end{aligned}\right\}$$

Solving non linear system of equation with Jacobian

$$\left.\begin{aligned} \alpha &= 0.0452 \\ \beta &= 0.4548 \\ x_1 &= 0.3436 \\ x_2 &= 0.9064 \end{aligned}\right\} \longrightarrow$$

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see solution

$$\boxed{\int_0^1 x f(x) = \alpha f(x_1) + \beta (f(x_2))}$$

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one point:

$$\int_0^1 x f(x) = \alpha f(x_1)$$

if $f(x)$ is polynomial of degree 1 or less eq * is exact
So for any combination of monomials up to degree 3 should be exact
as well So:

$f(x) = x^0 \rightarrow \int_0^1 x\,dx = \alpha \rightarrow \tfrac{1}{2} = \alpha$

$f(x) = x \rightarrow \int_0^1 x^2\,dx = \alpha x_1 \rightarrow \tfrac{1}{3} = \alpha x_1$

$$\boxed{x_1 = \tfrac{2}{3} \quad \& \quad \alpha = \tfrac{1}{2}}$$

$$\boxed{\int_0^1 x f(x) = \tfrac{1}{2} f\left(\tfrac{2}{3}\right)}$$

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26 predict the order of convergence

given the I values we can calculate $\boxed{R = [\, I(n/2) - I(n/4) \,] / [\, I(n) - I(n/2) \,]}$

& evaluate our data based on that

$R \simeq 11 \rightarrow \left(\frac{h}{h/2}\right)^p \sim 2^{3.5} \rightarrow \boxed{P \sim 3.5}$ Since Simpson's rule is used

it means there's a weak singularity → if we use Richardson

extrapolation we can make the order of convergence closer to 4

n	I(n)	R(n) $\boxed{P \sim 3.5}$	Iextrap	R of extrap $\boxed{P \sim 4}$
2	0.28451779686	NaN	NaN	NaN
4	0.28559254576	NaN	NaN	NaN
8	0.28570248748	9.77562384870744	0.28571501556027	NaN
16	0.28571317731	10.2847023759767	0.285714328647928	NaN
32	0.28571418363	10.6226945703731	0.285714288207776	16.9858990648322
64	0.28571427643	10.8439655151282	0.285714285857095	17.2035888918903

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31 I did 8 levels of extrapolations & calculated my error based on $\underline{int(\,)}$ function in matlab

a) $I = \int_0^1 \exp(-x^2)\,dx$

n	I(n)	I1	I2	I3	I4	I5	I6	I7	I8
2	0.731370251828563	NaN	NaN	NaN	NaN	NaN	NaN	NaN	NaN
4	0.742984097800381	0.746855379790987	NaN	NaN	NaN	NaN	NaN	NaN	NaN
8	0.745865614845695	0.746826120527467	0.746824169909899	NaN	NaN	NaN	NaN	NaN	NaN
16	0.746584596788222	0.74682425743573	0.746824133229614	0.746824132647388	NaN	NaN	NaN	NaN	NaN
32	0.746764254652294	0.746824140606985	0.746824132818402	0.746824132811875	0.74682413281252	NaN	NaN	NaN	NaN
64	0.746809163637828	0.746824133299672	0.746824132812518	0.746824132812425	0.746824132812427	0.746824132812427	NaN	NaN	NaN
128	0.746820390541618	0.746824132842881	0.746824132812428	0.746824132812427	0.746824132812427	0.746824132812427	0.746824132812427	NaN	NaN
256	0.746823197246152	0.74682413281433	0.746824132812427	0.746824132812427	0.746824132812427	0.746824132812427	0.746824132812427	0.746824132812427	NaN
512	0.746823898920947	0.746824132812546	0.746824132812427	0.746824132812427	0.746824132812427	0.746824132812427	0.746824132812427	0.746824132812427	0.746824132812427

n	erI(n)	er(I1)	er(I2)	er(I3)	er(I4)	er(I5)	er(I6)	er(I7)	er(I8)
2	0.0154538809838639	NaN	NaN	NaN	NaN	NaN	NaN	NaN	NaN
4	0.00384003501204577	-3.12469785602731e-05	NaN	NaN	NaN	NaN	NaN	NaN	NaN
8	0.000958517966731742	-1.98771503967521e-06	-3.7097471539127e-08	NaN	NaN	NaN	NaN	NaN	NaN
16	0.000239536024205456	-1.24623303343618e-07	-4.17187506762673e-10	1.6503920452493e-10	NaN	NaN	NaN	NaN	NaN
32	5.98781601329756e-05	-7.79455788801897e-09	-5.9748872516252e-12	5.52335954751015e-13	-9.27036225562006e-14	NaN	NaN	NaN	NaN
64	1.49691745990888e-05	-4.87245466196384e-10	-9.13713549266504e-14	1.99840144432528e-15	-2.22044604925031e-16	-1.11022302462516e-16	NaN	NaN	NaN
128	3.74227080912615e-06	-3.04541947215853e-11	-1.4432899320127e-15	0	0	0	0	NaN	NaN
256	9.35566274673505e-07	-1.90347737571983e-12	-1.11022302462516e-16	-1.11022302462516e-16	-1.11022302462516e-16	-1.11022302462516e-16	-1.11022302462516e-16	-1.11022302462516e-16	NaN
512	2.33891479517467e-07	-1.18904885937354e-13	1.11022302462516e-16	1.11022302462516e-16	1.11022302462516e-16	1.11022302462516e-16	1.11022302462516e-16	1.11022302462516e-16	1.11022302462516e-16

→ smaller error with increasing number of extrapolation levels

decreasing error with higher number of intervals

As we can see after two levels of extrapolation for 512 numbers of intervals we get close to machine epsilon for showing error

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(b) $\int_0^1 x^{2.5}\,dx$

n	I(n)	I1	I2	I3	I4	I5	I6	I7	I8
2	0.338388347648318	NaN	NaN	NaN	NaN	NaN	NaN	NaN	NaN
4	0.298791496231346	0.285592545759022	NaN	NaN	NaN	NaN	NaN	NaN	NaN
8	0.288974739670143	0.285702487483075	0.285709816931345	NaN	NaN	NaN	NaN	NaN	NaN
16	0.286528567896037	0.285713177304669	0.285713889959442	0.285713954610681	NaN	NaN	NaN	NaN	NaN
32	0.285917779698734	0.285714183632966	0.285714250721519	0.285714256447901	0.285714257631577	NaN	NaN	NaN	NaN
64	0.285765152250462	0.285714276434372	0.285714282621132	0.285714283127475	0.285714283232101	0.285714283257126	NaN	NaN	NaN
128	0.285727001721098	0.285714284877977	0.285714285440884	0.285714285485642	0.28571428549489	0.285714285497101	0.285714285497648	NaN	NaN
256	0.285717464659795	0.285714285639361	0.28571428569012	0.285714285694076	0.285714285694894	0.285714285695089	0.285714285695137	0.28571428569515	NaN
512	0.285715080445649	0.285714285707601	0.28571428571215	0.2857142857125	0.285714285712572	0.285714285712589	0.285714285712593	0.285714285712594	0.285714285712595

when comparing with exact integration value attained from built in int( ) Matlab function we have

n	erI(n)	er(I1)	er(I2)	er(I3)	er(I4)	er(I5)	er(I6)	er(I7)	er(I8)
2	-0.0526740619340327	NaN	NaN	NaN	NaN	NaN	NaN	NaN	NaN
4	-0.0130772105170602	0.000121739955263933	NaN	NaN	NaN	NaN	NaN	NaN	NaN
8	-0.00326045395585711	1.17982312106135e-05	4.46878294035891e-06	NaN	NaN	NaN	NaN	NaN	NaN
16	-0.000814282181751669	1.10840961681236e-06	3.95754843895979e-07	3.31103604256366e-07	NaN	NaN	NaN	NaN	NaN
32	-0.0002034939844483	1.02081319508507e-07	3.49927663512162e-08	2.92663842049734e-08	2.8082708825039e-08	NaN	NaN	NaN	NaN
64	-5.08665361766969e-05	9.27991383736781e-09	3.09315345559114e-09	2.58681037701436e-09	2.48218462450822e-09	2.45715969793281e-09	NaN	NaN	NaN
128	-1.27160068125187e-05	8.36308855500789e-10	2.73401856709654e-10	2.28643881605706e-10	2.19396167899788e-10	2.17184270567827e-10	2.16637263683594e-10	NaN	NaN
256	-3.17894550971864e-06	7.49245665510045e-11	2.41656139543522e-11	2.02095007395542e-11	1.93920990376739e-11	1.91965887630374e-11	1.9148238550315e-11	1.91361371193466e-11	NaN
512	-7.9473136355146e-07	6.68515243162915e-12	2.13590256592511e-12	1.78618231316818e-12	1.71390679426509e-12	1.69664282623216e-12	1.69236846758736e-12	1.69131375571396e-12	1.69103619995781e-12

here even after 8 levels of extrapolation we cannot reach machine epsilon ↓

because of the presence of weak singularity at $x = 0$

we have the same trend with more extrapolation → we reduce the error c $I = \int_{-4}^{4} \frac{1}{1+x^2}\, dx$

n	I(n)	I1	I2	I3	I4	I5	I6	I7	I8
2	4.23529411764706	NaN	NaN	NaN	NaN	NaN	NaN	NaN	NaN
4	2.91764705882353	2.47843137254902	NaN	NaN	NaN	NaN	NaN	NaN	NaN
8	2.65882352941176	2.57254901960784	2.57882352941176	NaN	NaN	NaN	NaN	NaN	NaN
16	2.65050680499416	2.64773456352162	2.65274693311587	2.65392032047625	NaN	NaN	NaN	NaN	NaN
32	2.65134716346583	2.65162728295638	2.65188679758537	2.65187314464044	2.65186511649991	NaN	NaN	NaN	NaN
64	2.65156325136377	2.65163528066308	2.65163581384353	2.65163182997461	2.65163088364258	2.65163065467596	NaN	NaN	NaN
128	2.6516173061521	2.65163532441487	2.65163532733166	2.65163531960925	2.65163533329409	2.6516353376437	2.65163533878728	NaN	NaN
256	2.65163082190308	2.6516353271534	2.65163532733597	2.65163532733604	2.65163532736634	2.65163532736054	2.65163532735803	2.65163532735734	NaN
512	2.65163420096925	2.65163532732465	2.65163532733606	2.65163532733606	2.65163532733606	2.65163532733603	2.65163532733603	2.65163532733603	2.65163532733603

with increasing the number of extrapolation the integral value converges to the exact result

n	erI(n)	er(I1)	er(I2)	er(I3)	er(I4)	er(I5)	er(I6)	er(I7)	er(I8)
2	-1.58365879031099	NaN	NaN	NaN	NaN	NaN	NaN	NaN	NaN
4	-0.266011731487464	0.173203954787046	NaN	NaN	NaN	NaN	NaN	NaN	NaN
8	-0.00718820207569948	0.0790863077282218	0.0728117979243001	NaN	NaN	NaN	NaN	NaN	NaN
16	0.00112852234190886	0.00390076381444482	-0.00111160577980707	-0.0022849931401896	NaN	NaN	NaN	NaN	NaN
32	0.000288163870238112	8.0443796814933e-06	-0.000251470249303054	-0.000237817304374222	-0.000229789163841865	NaN	NaN	NaN	NaN
64	7.2075972994124e-05	4.66729863646265e-08	-4.86507460095709e-07	3.49736145821566e-06	4.44369348118201e-06	4.67266010595324e-06	NaN	NaN	NaN
128	1.80211839682087e-05	2.9211912888627e-09	4.40492087250277e-12	7.72681518768081e-09	-5.95802651659483e-09	-1.03076369661892e-08	-1.14512186399907e-08	NaN	NaN
256	4.50543298979866e-06	1.82663661973947e-10	9.54791801177635e-14	2.70894418008538e-14	-3.02740055246886e-11	-2.4479529514565e-11	-2.19686491220727e-11	-2.12709849733983e-11	NaN
512	1.1263668118211e-06	1.14193099420845e-11	3.10862446895044e-15	1.77635683940025e-15	1.77635683940025e-15	3.15303338993544e-14	3.73034936274053e-14	3.86357612569554e-14	3.86357612569554e-14

⟶ decreasing the error

we have a bigger interval compared to previous ones

also $\boxed{f(x) = \frac{1}{x^2+1}}$

$$\left.\begin{aligned} f^{2n-1}(-4) &= -f^{2n-1}(4) \\ f^{2n}(-4) &= f^{2n}(4) \end{aligned}\right\} \text{ for } n \in N$$

$$\left.\begin{aligned} &\text{as for trapezoid error depends on } f'(4) - f'(-4) = 2f'(4) \\ &\text{\& for Simpson error depends on } f^{3}(4) - f^{3}(-4) = \underline{2f^{3}(4)} \end{aligned}\right\}$$

So we dont have cancelaton of error at all & the error adds up because the derivative values add up

→ So for these two reason we have slightly less accuracy

d) $I = \int_0^{2\pi} \frac{1}{2+\cos(x)}\,dx$

n	I(n)	I1	I2	I3	I4	I5	I6	I7	I8
2	4.18879020478639	NaN	NaN	NaN	NaN	NaN	NaN	NaN	NaN
4	3.66519142918809	3.49065850398866	NaN	NaN	NaN	NaN	NaN	NaN	NaN
8	3.62779151664536	3.61532487913111	3.62363597080728	NaN	NaN	NaN	NaN	NaN	NaN
16	3.62759873359101	3.6275344725729	3.62834844546902	3.62842324665412	NaN	NaN	NaN	NaN	NaN
32	3.62759872846844	3.62759872676091	3.62760301037344	3.62759117807034	3.62758791505629	NaN	NaN	NaN	NaN
64	3.62759872846844	3.62759872846843	3.62759872858227	3.62759866061733	3.62759868996065	3.62759870049331	NaN	NaN	NaN
128	3.62759872846844	3.62759872846844	3.62759872846844	3.62759872846663	3.62759872873271	3.62759872877061	3.62759872877751	NaN	NaN
256	3.62759872846844	3.62759872846843	3.62759872846843	3.62759872846843	3.62759872846844	3.62759872846818	3.62759872846811	3.62759872846809	NaN
512	3.62759872846844	3.62759872846844	3.62759872846844	3.62759872846844	3.62759872846844	3.62759872846844	3.62759872846844	3.62759872846844	3.62759872846844

n	erI(n)	er(I1)	er(I2)	er(I3)	er(I4)	er(I5)	er(I6)	er(I7)	er(I8)
2	-0.561191476317956	NaN	NaN	NaN	NaN	NaN	NaN	NaN	NaN
4	-0.0375927007196557	0.136940224479777	NaN	NaN	NaN	NaN	NaN	NaN	NaN
8	-0.000192788176920811	0.012273849337324	0.00396275766116005	NaN	NaN	NaN	NaN	NaN	NaN
16	-5.12257614104783e-09	6.42558955386008e-05	-0.000749717000580574	-0.00082451818568785	NaN	NaN	NaN	NaN	NaN
32	-4.44089209850063e-16	1.70752523231954e-09	-4.281905008785e-06	7.55039809563485e-06	1.08134121497017e-05	NaN	NaN	NaN	NaN
64	4.44089209850063e-16	8.88178419700125e-16	-1.13834275339286e-10	6.78511047347286e-08	3.85077831843716e-08	2.79751297682651e-08	NaN	NaN	NaN
128	-4.44089209850063e-16	-8.88178419700125e-16	-8.88178419700125e-16	1.80566672725035e-12	-2.64269939265205e-10	-3.02170288790649e-10	-3.09075431914607e-10	NaN	NaN
256	4.44089209850063e-16	8.88178419700125e-16	8.88178419700125e-16	8.88178419700125e-16	-6.21724893790088e-15	2.52242671194836e-13	3.25961480029946e-13	3.45057316053499e-13	NaN
512	-4.44089209850063e-16	-8.88178419700125e-16	-8.88178419700125e-16	-8.88178419700125e-16	-8.88178419700125e-16	-8.88178419700125e-16	-8.88178419700125e-16	-8.88178419700125e-16	-8.88178419700125e-16

This one is interesting because 2π is the period of function f & all of its derivative So all the error terms cancel each other out & the integral value converges to its true value really fast

$$f(0) = f(2\pi) \quad \& \quad f^{n}(0) = f^{n}(2\pi) \qquad \underline{n \in N}$$

$$\text{error} \approx h^{2n} \underbrace{\left[f^{2n-1}(b) - f^{2n-1}(a) \right]}_{\text{always zero}}$$

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e) $I = \int_0^{\pi} e^{x} \cos(4x)\,dx$

n	I(n)	I1	I2	I3	I4	I5	I6	I7	I8
2	26.5163358570775	NaN	NaN	NaN	NaN	NaN	NaN	NaN	NaN
4	3.24905049448466	-4.50671129304627	NaN	NaN	NaN	NaN	NaN	NaN	NaN
8	1.62452524724233	1.08301683149488	1.45566537313096	NaN	NaN	NaN	NaN	NaN	NaN
16	1.37572251765279	1.29278827445628	1.30677303732037	1.30440966691068	NaN	NaN	NaN	NaN	NaN
32	1.32031187842362	1.30184166534723	1.30244522473996	1.30237652930218	1.30236855621351	NaN	NaN	NaN	NaN
64	1.30684788549749	1.30235988785545	1.30239443602267	1.30239362985255	1.30239369691353	1.302393721489	NaN	NaN	NaN
128	1.30350565849719	1.30239158283042	1.30239369582875	1.30239368407964	1.30239368429229	1.30239368427996	1.30239368427087	NaN	NaN
256	1.30267157945597	1.3023935531089	1.3023936844608	1.30239368428035	1.30239368428114	1.30239368428113	1.30239368428113	1.30239368428113	NaN
512	1.30246315192811	1.3023936760855	1.30239368428393	1.30239368428113	1.30239368428113	1.30239368428113	1.30239368428113	1.30239368428113	1.30239368428113

n	erI(n)	er(I1)	er(I2)	er(I3)	er(I4)	er(I5)	er(I6)	er(I7)	er(I8)
2	-25.2139421727963	NaN	NaN	NaN	NaN	NaN	NaN	NaN	NaN
4	-1.94665681020353	5.8091049773274	NaN	NaN	NaN	NaN	NaN	NaN	NaN
8	-0.322131562961196	0.219376852786249	-0.153271688849828	NaN	NaN	NaN	NaN	NaN	NaN
16	-0.0733288333716582	0.00960540982485436	-0.00437935303923864	-0.00201598262954694	NaN	NaN	NaN	NaN	NaN
32	-0.0179181941424864	0.000552018933904108	-5.15404588259383e-05	1.7154978958267e-05	2.51280676188959e-05	NaN	NaN	NaN	NaN
64	-0.00445420121635953	3.37964256826861e-05	-7.51741532134531e-07	5.44285836401315e-08	-1.26323982385657e-08	-3.72078625687777e-08	NaN	NaN	NaN
128	-0.00111197421605347	2.1014507152195e-06	-1.15476157525052e-08	2.01494154694615e-10	-1.11610720665567e-11	1.17639231689282e-12	1.02629016396349e-11	NaN	NaN
256	-0.000277895174837745	1.31172234629773e-07	-1.79664061406015e-10	7.79598607891785e-13	-7.54951656745106e-15	3.33066907387547e-15	3.10862446895044e-15	2.44249065417534e-15	NaN
512	-6.946764698057e-05	8.19563839016269e-09	-2.80131473573419e-12	5.99520433297585e-15	2.88657986402541e-15	2.88657986402541e-15	2.88657986402541e-15	2.88657986402541e-15	2.88657986402541e-15

usual trend with doing more extra polation error decreases

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