HW6

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problem solved

A1

1

2

3

4

5

A1

$$f(x) = \frac{1}{(x-0.3)^2+0.01} + \frac{1}{(x-0.9)^2+0.04} - 6$$

cubic spline with a not-a-knot condition :

① all intervals are equal $h_i = h = 0.1$

we can use simplified form : (now we just use the internal points $11-2=9$)

$$\underbrace{\begin{bmatrix} \boxed{} & & & & \\ 1 & 4 & 1 & & \\ & 1 & 4 & 1 & \\ & & \ddots & \ddots & \ddots \\ & & 1 & 4 & 1 \\ & & & & \boxed{} \end{bmatrix}}_{7\times 9} \underbrace{\begin{Bmatrix} S_1 \\ S_2 \\ S_3 \\ \vdots \\ S_9 \end{Bmatrix}}_{9\times 1} = \frac{6}{(0.1)^2} \underbrace{\begin{Bmatrix} y_2 - 2y_1 + y_0 \\ y_3 - 2y_2 + y_1 \\ y_4 - 4y_3 + y_2 \\ \vdots \\ y_{10} - 2y_9 + y_8 \end{Bmatrix}}_{9\times 1} \quad \text{I}$$

7 eqs

here we have 9 unknown & 7 eqs for the other two eqs

we can use not-a-knot condition & write S_0, S_{10} in terms of S_1, S_2 & S_8, S_9

$$\left.\begin{aligned} 6hS_1 &= 6(f_1^{[1]} - f_0^{[1]}) \rightarrow \underline{6S_1} = \frac{6}{h^2}(f_1^{[1]} - f_0^{[1]}) \\ 6hS_9 &= 6(f_{10}^{[1]} - f_9^{[1]}) \rightarrow \underline{6S_9} = \frac{6}{h^2}(f_{10}^{[1]} - f_9^{[1]}) \end{aligned}\right\} \star\star$$

plugging $\star\star$ into matrix I :

$$\underbrace{\begin{bmatrix} 6 & 0 & 0 & \cdots & & \\ 1 & 4 & 1 & & & \\ & 1 & 4 & 1 & & 0 \\ & & \ddots & \ddots & \ddots & \\ 0 & & & & & \\ & & & 1 & 4 & 1 \\ & & \cdots & & 0 & 6 \end{bmatrix}}_{9\times 9} \underbrace{\begin{Bmatrix} S_1 \\ S_2 \\ S_3 \\ \vdots \\ S_9 \end{Bmatrix}}_{9\times 1} = \frac{6}{(0.1)^2} \underbrace{\begin{Bmatrix} y_2 - 2y_1 + y_0 \\ y_3 - 2y_2 + y_1 \\ y_4 - 4y_3 + y_2 \\ \vdots \\ y_{10} - 2y_9 + y_8 \end{Bmatrix}}_{9\times 1} \quad \text{II}$$

9 eqs

Then we can calculate S_0 & S_{10} by :

we have $g_0'''(x_1) = g_1'''(x_1) \rightarrow a_1 = a_0$

$g_9'''(x_{10}) = g_8'''(x_{10}) \rightarrow a_9 = a_8$

$(h = h_i) \longrightarrow$ $\boxed{S_0 = 2S_1 - S_2}$ $\boxed{S_{10} = 2S_9 - S_8}$

now we have all S_i for $i = 0, \dots, 10$

$\rightarrow$ for a cubic spline as : $\boxed{g_i(x) = d_i + c_i(x-x_i) + b_i(x-x_i)^2 + a_i(x-x_i)^3}$

we have : $\boxed{d_i = y_i}$

$\boxed{b_i = S_i/2}$

$\boxed{a_i = (S_{i+1} - S_i)/6h}$

$\boxed{c_i = \frac{y_{i+1} - y_i}{h} - \frac{2hS_i + hS_{i+1}}{6}}$

for $i = 0, \dots 9$

This gives us 10 cubic-splines for 11 nodes

After Solving II $\rightarrow$

	S values
S1	2012.1
S2	7179.31
S3	-18611.1
S4	7466.26
S5	1108.5
S6	783.726
S7	555.393
S8	-140.898
S9	-956

$\rightarrow$ $\boxed{S_0 = 2S_1 - S_2}$ $\boxed{S_{10} = 2S_9 - S_8}$

$S_0 = -3155.11$ & $S_{10} = -1771.1$

now : for $i = 0 : 9$

$\boxed{d_i = y_i}$

$\boxed{b_i = S_i/2}$

$\boxed{a_i = (S_{i+1} - S_i)/6h}$

$\boxed{c_i = \frac{y_{i+1} - y_i}{h} - \frac{2hS_i + hS_{i+1}}{6}}$

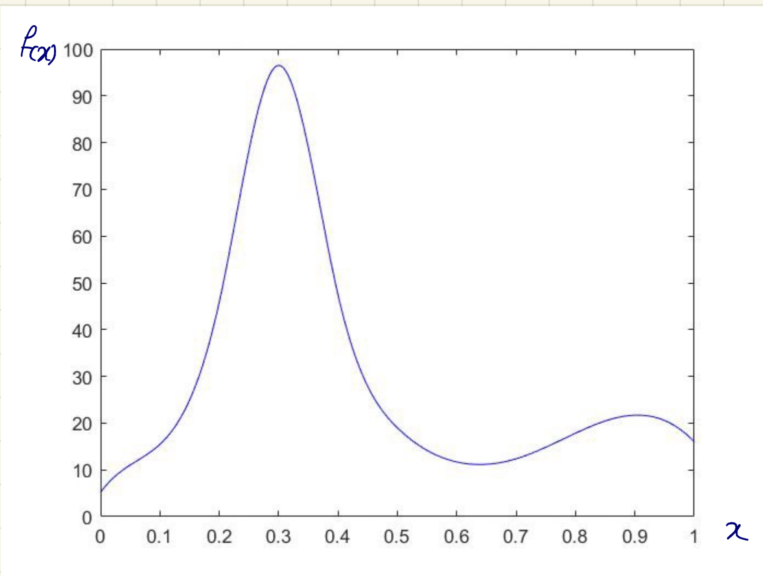
we have:

	a	b	c	d
i=0	8612.02	-1577.56	174.585	5.176
i=1	8612.02	1006.05	117.435	15.471
i=2	-42984.1	3589.66	577.005	45.887
i=3	43462.3	-9305.57	5.41376	96.5
i=4	-10596.3	3733.13	-551.83	47.448
i=5	-541.296	554.252	-123.092	19
i=6	-380.555	391.863	-28.4807	11.692
i=7	-1160.49	277.697	38.4752	12.382
i=8	-1358.5	-70.4491	59.1999	17.846
i=9	-1358.5	-478	4.35503	21.703

we can plug these coeficient into the equation below to get the cubic polynomials

$$\boxed{g_i(x) = d_i + c_i(x-x_i) + b_i(x-x_i)^2 + a_i(x-x_i)^3}$$

when we plot it:



1

x	0	1	2	2.5	3	4
y	1.4	0.6	1	0.65	0.6	1
	0	1	2	3	4	5

→ applying smoothness condition we have →

$$h_{i-1} S_{i-1} + 2(h_{i-1} + h_i) S_i + h_i S_{i+1} = 6\left(\frac{y_{i+1} - y_i}{h_i} - \frac{y_i - y_{i-1}}{h_{i-1}}\right)$$

implementing this in a matrix form :

$$h = \begin{array}{c} \begin{Bmatrix} 1 \\ 1 \\ 0.5 \\ 0.5 \\ 1 \end{Bmatrix} \end{array} \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array}$$

two terms

$$\begin{bmatrix} \triangle & \square & & \\ h_1 & 2(h_1+h_2) & h_2 & 0 \\ 0 & h_2 & 2(h_2+h_3) & h_3 \\ & & \bigcirc & \times \end{bmatrix} \underbrace{\begin{Bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \end{Bmatrix}}_{4\times1} = \frac{6}{} \underbrace{\begin{Bmatrix} \frac{y_2 - y_1}{h_1} - \frac{y_1 - y_0}{h_0} \\ \frac{y_3 - y_2}{h_2} - \frac{y_2 - y_1}{h_1} \\ \frac{y_4 - y_3}{h_3} - \frac{y_3 - y_2}{h_2} \\ \frac{y_5 - y_4}{h_4} - \frac{y_4 - y_3}{h_3} \end{Bmatrix}}_{4\times1} \quad \text{I}$$

need two more equations } we will get them from → not - a - knot condition :

$$\underbrace{\frac{(h_0 + h_1)(h_0 + 2h_1)}{h_1}}_{\triangle} S_1 + \underbrace{\frac{h_1^2 - h_0^2}{h_1}}_{\square} S_2 = 6\left(\frac{y_2 - y_1}{h_1} - \frac{(y_1 - y_0)}{h_0}\right)$$ } for the first interval

$$\underbrace{\frac{h_3^2 - h_4^2}{h_3}}_{\bigcirc} S_3 + \underbrace{\frac{(h_4 + h_3)(h_4 + 2h_3)}{h_3}}_{\times} S_4 = 6\left(\frac{y_5 - y_4}{h_4} - \frac{y_4 - y_3}{h_3}\right)$$ } for the last interval

plugging the coeficient of S_1, S_2, S_3, S_4 for the first & last equation into the matrix

$$\begin{bmatrix} 6 & 0 & 0 & 0 \\ 1 & 3 & 0.5 & 0 \\ 0 & 0.5 & 2 & 0.5 \\ 0 & 0 & -1.5 & 6 \end{bmatrix} \begin{Bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \end{Bmatrix} = \begin{Bmatrix} 7.2 \\ -6.6 \\ 3.6 \\ 3 \end{Bmatrix} \rightarrow \text{Solving for } S \rightarrow \begin{Bmatrix} 1.2 \\ -2.9796 \\ 2.2776 \\ 1.0694 \end{Bmatrix} \begin{array}{c} S_1 \\ S_2 \\ S_3 \\ S_4 \end{array}$$

now
$$\begin{cases} S_0 = \left(\frac{h_0}{h_1}+1\right) S_1 - \left(\frac{h_0}{h_1}\right) S_2 \rightarrow \boxed{S_0 = 5.3796} \\ S_5 = \left(1+\frac{h_4}{h_3}\right) S_4 - \left(\frac{h_4}{h_3}\right) S_3 \rightarrow \boxed{S_5 = -1.3469} \end{cases}$$

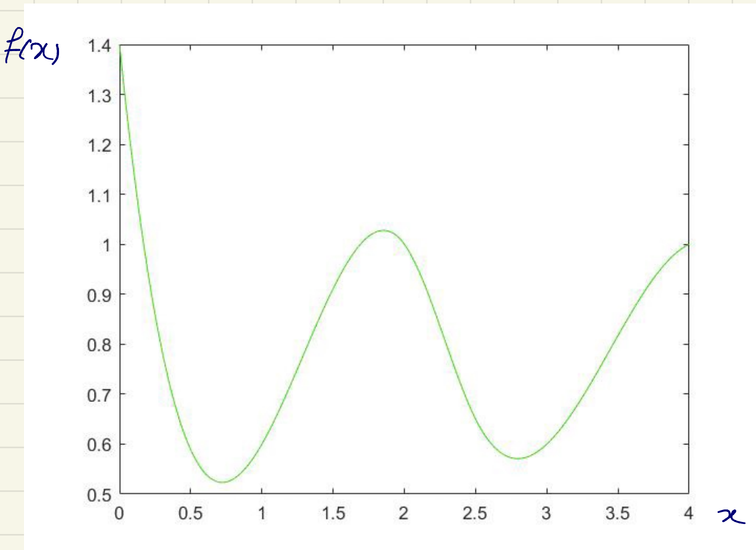
now : for $i = 0:4$

$$h = \begin{Bmatrix} 1 \\ 1 \\ 0.5 \\ 0.5 \\ 1 \end{Bmatrix} \qquad \begin{cases} \boxed{d_i = y_i} \\ \boxed{b_i = S_i/2} \\ \boxed{a_i = (S_{i+1} - S_i)/6h_i} \\ \boxed{c_i = \dfrac{y_{i+1} - y_i}{h_i} - \dfrac{2h_iS_i + h_iS_{i+1}}{6}} \end{cases}$$

	a	b	c	d
i=0	-0.6966	2.6898	-2.7932	1.4
i=1	-0.6966	0.6	0.4966	0.6
i=2	1.75238	-1.4898	-0.3932	1
i=3	-0.40272	1.13878	-0.56871	0.65
i=4	-0.40272	0.53469	0.26803	0.6

$$\boxed{g_i(x) = d_i + c_i(x-x_i) + b_i(x-x_i)^2 + a_i(x-x_i)^3}$$

we can plug the values into the function & plot it :



#2

x	0.1	0.2	0.4	0.6	0.9	1.3	1.5	1.7	1.8
y	0.75	1.25	1.45	1.25	0.85	0.55	0.35	0.28	0.18

using regression $\rightarrow$ $y = \alpha x e^{\beta x}$ $\rightarrow$ $(z = y/x)$ $\rightarrow$ $z = \alpha e^{\beta_1 x)}$

$\rightarrow \underbrace{\ln(z)} = \underbrace{\ln(\alpha)} + \underline{\beta} x$

$$\underbrace{\begin{bmatrix} 1 & x \end{bmatrix}}_{A} \underbrace{\begin{Bmatrix} \ln\alpha \\ \beta \end{Bmatrix}}_{X} = \underbrace{\begin{Bmatrix} \ln(z) \end{Bmatrix}}_{b} \rightarrow Ax = b \rightarrow$$

{ it doesn't have any exact solution }

$\rightarrow$ but we can make the error very small : $X_{LS} = \text{argmin} \|Ax - b\|_2^2$

$$\|Ax - b\|^2 = (Ax - b)^T(Ax - b) = x^TA^TAx - x^TA^Tb - b^TAx + b^Tb$$

b^TAx is scalar $\rightarrow$ $b^TAx = x^TA^Tb$ $\rightarrow$

$$\|Ax - b\|^2 = x^TA^TAx - 2b^TAx + b^Tb \rightarrow \frac{d\|Ax - b\|^2}{dx} = 0 \leftarrow$$

{ for finding the minimum of summation }

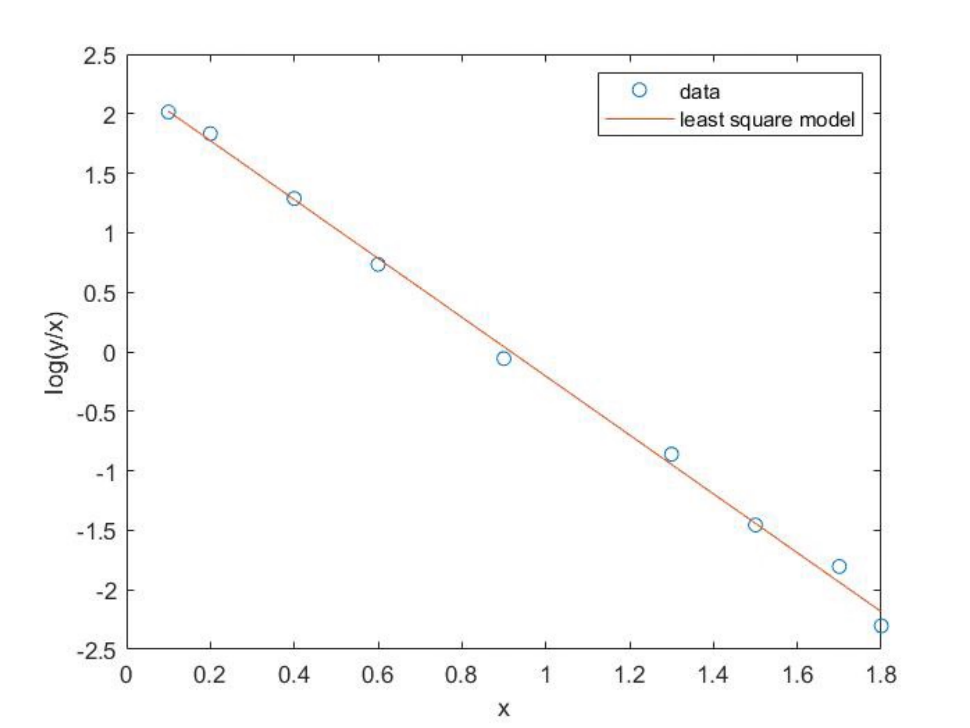
$$2x^TA^TA - 2b^TA = 0 \rightarrow A^TAx = A^Tb \rightarrow \boxed{x_{LS} = (A^TA)^{-1}A^Tb}$$ ✩

$$A = \begin{bmatrix} 1 & 0.1 \\ 1 & 0.2 \\ 1 & 0.4 \\ 1 & 0.6 \\ 1 & 0.9 \\ 1 & 1.3 \\ 1 & 1.5 \\ 1 & 1.7 \\ 1 & 1.8 \end{bmatrix} \qquad b = \ln(z) = \begin{Bmatrix} 2.0149 \\ 1.83258 \\ 1.28785 \\ 0.73397 \\ -0.05716 \\ -0.8602 \\ -1.45529 \\ -1.80359 \\ -2.30259 \end{Bmatrix}$$

with the A & b values & solving for least square parameters using equation ✩ we have :

$$\begin{Bmatrix} \ln\alpha \\ \beta \end{Bmatrix} = \begin{Bmatrix} 2.2682 \\ -2.4733 \end{Bmatrix} \rightarrow \begin{Bmatrix} \alpha \\ \beta \end{Bmatrix} = \begin{Bmatrix} 9.662 \\ -2.4733 \end{Bmatrix}$$

plotting it :


data
least square model
log(y/x)
x
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8

3

x	y	z
0	0	5
2	1	10
2.5	2	9
1	3	0
4	6	3
7	2	27

$$\varepsilon_2 = \sum_{i=1}^{n} (z_i - a_0 - a_1 x_i - a_2 y_i)^2$$

$\rightarrow$ we can write this in a matrix form like this

$$\underbrace{\begin{bmatrix} 1 & X & Y \end{bmatrix}}_{A} \underbrace{\begin{Bmatrix} a_0 \\ a_1 \\ a_2 \end{Bmatrix}}_{x} = \underbrace{\begin{Bmatrix} Z \end{Bmatrix}}_{b}$$

in which $X = \begin{Bmatrix} x_1 \\ \vdots \\ x_6 \end{Bmatrix}_{6\times 1}$, $Y = \begin{Bmatrix} y_1 \\ \vdots \\ y_6 \end{Bmatrix}_{6\times 1}$

$Z = \begin{Bmatrix} z_1 \\ \vdots \\ z_6 \end{Bmatrix}_{6\times 1}$ & $1 = \begin{Bmatrix} 1 \\ \vdots \\ 1 \end{Bmatrix}_{6\times 1}$

now $Ax = b \rightarrow$ & we are trying to solve for $\{x\}$, but there might not be any exact solution. However we can find $\{x\}$ such that $\rightarrow x_{LS} = \operatorname{Arg\,min}\{\|Ax - b\|^2\} \rightarrow$ in which x_{LS} is the solution of least square problem by which we minimized the error of above summation

Same as previous question:

$$\|Ax - b\|^2 = (Ax - b)^T (Ax - b) = x^T A^T A x - x^T A^T b - b^T A x + b^T b$$

$b^T A x$ is scalar $\rightarrow b^T A x = x^T A^T b \rightarrow$

$$\|Ax - b\|^2 = x^T A^T A x - 2b^T A x + b^T b \rightarrow \frac{d\|Ax - b\|^2}{dx} = 0$$

: for finding the minimum of summation

$$2x^T A^T A - 2b^T A = 0 \rightarrow A^T A x = A^T b \rightarrow \boxed{x_{LS} = (A^T A)^{-1} A^T b}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 2.5 & 2 \\ 1 & 1 & 3 \\ 1 & 4 & 6 \\ 1 & 7 & 2 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} 5 \\ 10 \\ 9 \\ 0 \\ 3 \\ 27 \end{Bmatrix} \longrightarrow \text{least square solution} \quad \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} 5 \\ 4 \\ -3 \end{Bmatrix}$$

4

$f(x) = \sin[(\pi x/2)] \qquad -1 \leq x \leq 1$

$\rightarrow P_n(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n \rightarrow$ expanding around $\underline{x=0}$

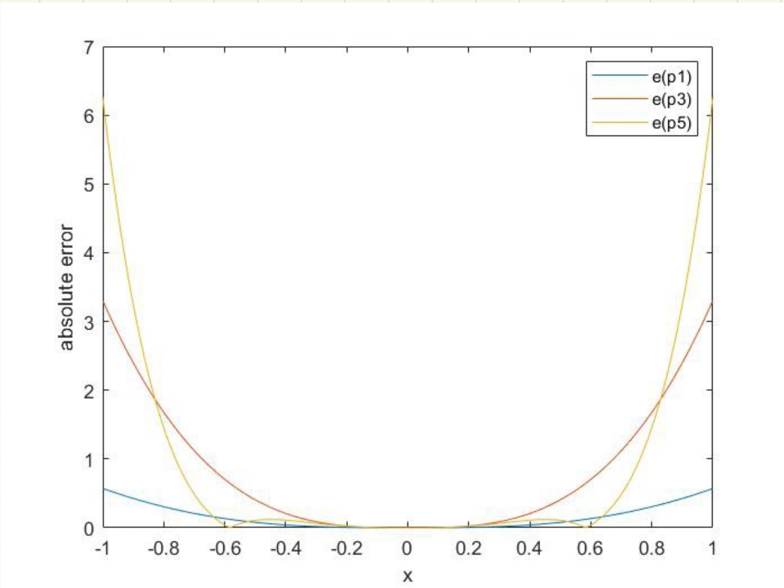
$\rightarrow P_1(x) = \sin(0) + \frac{\pi/2 \cos(0)}{1!} x \rightarrow \boxed{P_1 = \frac{\pi}{2} x}$

$P_3(x) = \sin(0) + \pi/2 \cos(0) x + \sin(0)(\sim) - (\frac{\pi}{2})^3 \frac{x^3}{3!} \rightarrow \boxed{P_3(x) = \frac{\pi}{2} x - (\frac{\pi}{2})^3 \frac{x^3}{3!}}$

$$\boxed{P_5(x) = \frac{\pi}{2} x - (\frac{\pi}{2})^3 \frac{x^3}{3!} + (\frac{\pi}{2})^5 \frac{x^5}{5!}}$$

Plotting $\underline{\text{error} = \left(\sin(\frac{\pi}{2} x) - P_n(x)\right)}$ for $n = 1, 3, 5$

in $\boxed{x \in [-1, 1]}$



the general trend is that as we go away from $\underline{x=0}$ around which we expanded our taylor expansion we have larger errors & this error is even larger for higher order expansion because the term $(\frac{\pi}{2})^n \rightarrow$ for n order polynomial amplify the error even more at the ends of the interval

5

$$R(x) = \frac{a+bx}{1+cx}$$

$$\boxed{R(0) = a}$$

$$R^{(j)}(0) = f^{j}(0) \qquad j = 0, 1, 2$$

$$\rightarrow R'(x) = \frac{b(1+cx) - c(a+bx)}{(1+cx)^2} \rightarrow \boxed{R'(0) = \frac{b-ca}{1}}$$

$$R^2(x) = \frac{-2c(1+cx)\left[b(1+cx) - c(a+bx)\right]}{(1+cx)^3} \rightarrow \boxed{R^2(0) = \frac{-2c(b-ca)}{1}}$$

$$\left.\begin{aligned} f(0) &= a \\ b - ca &= f'(0) \\ -2c(b-ca) &= f''(0) \end{aligned}\right\} \quad 3 \text{ unknown } \& \text{ 3 equations} \rightarrow -2c(f'(0)) = f''(0)$$

$$\rightarrow \boxed{c = \frac{-f''(0)}{2f'(0)}} \quad \& \quad \boxed{a = f(0)}$$

$$b = f'(0) - ca \longrightarrow \boxed{b = f'(0) + \frac{f''(0) f(0)}{2f'(0)}}$$

→ there is always a solution for it → as we said f should be 3 times continuously differentiable → so we $\boxed{\text{know } f'(0) \neq 0}$ if that's the case there is always a solution