HW#4 EGM6341

<u>From the textbook by Atkinson</u>:

pp 496-503:
#6a, (just expand the determinant using cofactors and minor)
Solution:

$$f_n(x) = \det \left. \begin{bmatrix} x & 1 & 0 & & \ldots & 0 \\ 1 & x & 1 & 0 & \ldots & 0 \\ 0 & 1 & x & 1 & & \\ & & & \ldots & & \\ & & 0 & 1 & x & 1 \\ 0 & \ldots & 0 & 0 & 1 & x \end{bmatrix} \right\} n \times n$$

Expand the determinant using first row, we obtain

$$f_n(x) = x \det \underbrace{\begin{bmatrix} x & 1 & 0 & & \ldots & 0 \\ 1 & x & 1 & 0 & \ldots & 0 \\ 0 & 1 & x & 1 & & \\ & & & \ldots & & \\ & & 0 & 1 & x & 1 \\ 0 & \ldots & 0 & 0 & 1 & x \end{bmatrix}}_{(n-1) \times (n-1)} - 1 * \left. \begin{bmatrix} x & 1 & 0 & & \ldots & 0 \\ 1 & x & 1 & 0 & \ldots & 0 \\ 0 & 1 & x & 1 & & \\ & & & \ldots & & \\ & & 0 & 1 & x & 1 \\ 0 & \ldots & 0 & 0 & 1 & x \end{bmatrix} \right\} (n-2) \times (n-2)$$

$$= x f_{n-1}(x) - f_{n-2}(x)$$

Which is equivalent to $f_{n+1}(x) = x f_n(x) - f_{n-1}(x)$

#21

Probe the following: for $x \in \mathbf{C}^n$

(a) $\|x\|_\infty \le \|x\|_1 \le n\|x\|_\infty$

(b) $\|x\|_\infty \le \|x\|_2 \le \sqrt{n}\,\|x\|_\infty$

(c) $\|x\|_2 \le \|x\|_1 \le \sqrt{n}\,\|x\|_2$

Solution.

(a) $\|x\|_\infty \le \|x\|_1 \le n\|x\|_\infty$

(1) $\|x\|_1 - \|x\|_\infty = \sum_{j=1}^{n} |x_j| - \underset{1\le k\le n}{\mathrm{Max}} |x_k| = \sum_{\substack{j=1 \\ j\ne k}}^{n} |x_j| \ge 0$

where k denotes index for maximum component of x vector

$\therefore \|x\|_1 \ge \|x\|_\infty$

(2) $n\|x\|_\infty - \|x\|_1$

$$n\|x\|_\infty - \|x\|_1 = n\left(\underset{1\le j\le n}{\mathrm{Max}} |x_j|\right) - \sum_{j=1}^{n} |x_j| = \sum_{k=1}^{n} 1\times\left(\underset{1\le j\le n}{\mathrm{Max}} |x_j|\right) - \sum_{k=1}^{n} |x_k|$$

$$= \sum_{k=1}^{n}\left(\underset{1\le j\le n}{\mathrm{Max}} |x_j|\right) - \sum_{k=1}^{n} |x_k| = \sum_{k=1}^{n}\left[\left(\underset{1\le j\le n}{\mathrm{Max}} |x_j|\right) - |x_k|\right] \ge 0$$

$\therefore n\|x\|_\infty \ge \|x\|_1$

From (1) and (2) $\|x\|_\infty \le \|x\|_1 \le n\|x\|_\infty$

(b) $\|x\|_\infty \le \|x\|_2 \le \sqrt{n}\,\|x\|_\infty$

(1) $\|x\|_2 - \|x\|_\infty = \left[\sum_{j=1}^{n} |x_j|^2\right]^{1/2} - \left[\left(\underset{1\le j\le n}{\mathrm{Max}} |x_j|\right)^2\right]^{1/2}$

consider

$$\sum_{j=1}^{n} |x_j|^2 - \left(\underset{1\le k\le n}{\mathrm{Max}} |x_k|\right)^2 = \sum_{\substack{j=1 \\ j\ne k}}^{n} |x_j|^2 \ge 0$$

where k denotes index for maximum component of x vector

$\sqrt{a} - \sqrt{b} \ge 0$ if $a \ge b$, for positive a and b

$$\therefore \left[\sum_{j=1}^{n} |x_j|^2\right]^{1/2} - \left[\left(\underset{1\le j\le n}{\mathrm{Max}} |x_j|\right)^2\right]^{1/2} \ge 0$$

$\therefore \|x\|_2 \geq \|x\|_\infty$

(2) $\sqrt{n}\,\|x\|_\infty - \|x\|_2 = \left[n\left(\underset{1\leq j\leq n}{\text{Max}} |x_j| \right)^2 \right]^{1/2} - \left[\sum_{j=1}^{n} |x_j|^2 \right]^{1/2}$

consider $n\left(\underset{1\leq j\leq n}{\text{Max}} |x_j| \right)^2 - \sum_{j=1}^{n} |x_j|^2$

$$n\left(\underset{1\leq k\leq n}{\text{Max}} |x_k| \right)^2 - \sum_{j=1}^{n} |x_j|^2 = \sum_{j=1}^{n}\left(\underset{1\leq k\leq n}{\text{Max}} |x_k| \right)^2 - \sum_{j=1}^{n} |x_j|^2 = \sum_{j=1}^{n}\left[\left(\underset{1\leq k\leq n}{\text{Max}} |x_k| \right)^2 - |x_j|^2 \right]$$

$\left(\underset{1\leq k\leq n}{\text{Max}} |x_k| \right)^2 - |x_j|^2 \geq 0$

$\therefore \sum_{j=1}^{n}\left[\left(\underset{1\leq k\leq n}{\text{Max}} |x_k| \right)^2 - |x_j|^2 \right] \geq 0 \quad \Rightarrow \quad n\left(\underset{1\leq j\leq n}{\text{Max}} |x_j| \right)^2 - \sum_{j=1}^{n} |x_j|^2 \geq 0$

$\sqrt{n}\,\|x\|_\infty - \|x\|_2 = \left[n\left(\underset{1\leq j\leq n}{\text{Max}} |x_j| \right)^2 \right]^{1/2} - \left[\sum_{j=1}^{n} |x_j|^2 \right]^{1/2} \geq 0$

$\therefore \sqrt{n}\,\|x\|_\infty \geq \|x\|_2$

From (1) and (2) $\|x\|_\infty \leq \|x\|_2 \leq \sqrt{n}\,\|x\|_\infty$

(c) $\|x\|_2 \leq \|x\|_1 \leq \sqrt{n}\,\|x\|_2$

(1) $\|x\|_1 - \|x\|_2 = \left[\left(\sum_{j=1}^{n} |x_j| \right)^2 \right]^{1/2} - \left[\sum_{j=1}^{n} \left(|x_j| \right)^2 \right]^{1/2}$

consider $\left(\sum_{j=1}^{n} |x_j| \right)^2 - \sum_{j=1}^{n} \left(|x_j| \right)^2 = \sum_{j=1}^{n}\sum_{i=1}^{n} |x_i||x_j| - \sum_{j=1}^{n} \left(|x_j| \right)^2$

$= \sum_{j=1}^{n}\left(\sum_{i=1}^{n} |x_i||x_j| - \left(|x_j| \right)^2 \right) = \sum_{j=1}^{n}\left(\sum_{\substack{i=1 \\ i\neq j}}^{n} |x_i||x_j| \right) \geq 0$

$\left(\sum_{j=1}^{n} |x_j| \right)^2 - \sum_{j=1}^{n} \left(|x_j| \right)^2 \geq 0 \quad \Rightarrow \quad \left[\left(\sum_{j=1}^{n} |x_j| \right)^2 \right]^{1/2} - \left[\sum_{j=1}^{n} \left(|x_j| \right)^2 \right]^{1/2} \geq 0$

$$\|x\|_1 - \|x\|_2 = \left[\left(\sum_{j=1}^{n}|x_j|\right)^2\right]^{1/2} - \left[\sum_{j=1}^{n}\left(|x_j|\right)^2\right]^{1/2} \geq 0$$

$$\therefore \|x\|_1 \geq \|x\|_2$$

(2) $\sqrt{n}\,\|x\|_2 - \|x\|_1$

Use Hölder's inequality:

$$\|x\|_1 = |x_1| + |x_2| + \cdots + |x_n| = 1\cdot|x_1| + 1\cdot|x_2| + \cdots + 1\cdot|x_n| = |1\cdot x_1| + |1\cdot x_2| + \cdots + |1\cdot x_n|$$

$$\leq \left(1^2 + 1^2 + \cdots 1^2\right)^{1/2}\left(x_1^2 + x_2^2 + \cdots x_n^2\right)^{1/2} = \sqrt{n}\,\|x\|_2$$

$$\sqrt{n}\,\|x\|_2 \geq \|x\|_1$$

From (1) and (2) $\|x\|_2 \leq \|x\|_1 \leq \sqrt{n}\,\|x\|_2$

Problem 23

Show

$$\lim_{p\to\infty}\left[\sum_{j=1}^{n}|x_j|^p\right]^{1/p}=\operatorname*{Max}_{1\le i\le n}|x_i|$$

Solution.

Let B= $\displaystyle\operatorname*{Max}_{1\le i\le n}|x_i|$.

Note that there may be several (k≥1) elements of x_i having the same maximum magnitude of B.

Since $\displaystyle |x_j|\le \operatorname*{Max}_{1\le i\le n}|x_i|=B \quad\Rightarrow\quad |x_j|^p\le B^p$ for 1≤j≤n

For $\displaystyle\sum_{j=1}^{n}|x_j|^p=B^p\sum_{j=1}^{n}(|x_j|/B)^p$

Inside the summation of $\displaystyle\sum_{j=1}^{n}(|x_j|/B)^p$, there are k terms of 1 and the rest of the terms are less than 1 due to the definition of B.

As p→∞, $\displaystyle\lim_{p\to\infty}\sum_{j=1}^{n}(|x_j|/B)^p$ = k≥1 since $\displaystyle\lim_{p\to\infty}a^p=0$ for a < 1.

Thus, $\displaystyle\lim_{p\to\infty}\sum_{j=1}^{n}|x_j|^p=k\lim_{p\to\infty}B^p$,

Hence, $$\lim_{p\to\infty}\left(\sum_{j=1}^{n}|x_j|^p\right)^{1/p}=\lim_{p\to\infty}\left(kB^p\right)^{1/p}=B\lim_{p\to\infty}(k)^{1/p}=B$$

(because $\displaystyle\lim_{p\to\infty}b^{1/p}=1$ for b ≥ 1.

Finally, $$\lim_{p\to\infty}\left(\sum_{j=1}^{n}|x_j|^p\right)^{1/p}=\operatorname*{Max}_{1\le i\le n}|x_i|$$

pp 574-583:
#2

$$6x_1 + 2x_2 + 2x_3 = -2$$

$$2x_1 + \frac{2}{3}x_2 + \frac{1}{3}x_3 = 1$$

$$x_1 + 2x_2 - x_3 = 0$$

Soln: First, with only 4 significant digits, the above system of equations becomes

$6.000x_1 + 2.000x_2 + 2.000x_3 = -2.000$ -----Row 1
$2.000x_1 + 0.6667x_2 + 0.3333x_3 = 1.000$ -----Row 2
$1.000x_1 + 2.000x_2 - 1.000x_3 = 0.000$ -----Row 3

It is easy to plug in $x_1 = 2.6,\ x_2 = -3.8,\ x_3 = -5.0$ and verify that all three equations are satisfied.

a) Without pivoting, the $2.000x_1$ term in line 2 is eliminated as follows:

Row 2 – Row 1 *(0.3333) =>

$$+(0.6667 - 2.000 * 0.3333)x_2 + (0.3333 - 2.000 * 0.3333)x_3$$
$$= 1.000 + 2.000 * 0.3333$$

That is,

$$0.0001x_2 - 0.3333x_3 = 1.667$$

For the 3rd equation, without pivoting, the $1.000x_1$ term is eliminated as follows:

Row 3 – Row1 *(0.1667) =>

$$+(2.000 - 2.000 * 0.1667)x_2 + (-1.000 - 2.000 * 0.1667)x_3$$
$$= 0.000 + 2.000 * 0.1667$$

That is,

$$1.667x_2 - 1.333x_3 = 0.3334$$

Thus, the system becomes:

$6.000x_1 + 2.000x_2 + 2.000x_3 = -2.000$ -----Row 1
$0.0001x_2 - 0.3333x_3 = 1.667$ -----Row 2
$1.667x_2 - 1333x_3 = 0.3334$ -----Row 3

In matrix form, the augmented matrix is

$$\begin{bmatrix} 6.000 & 2.000 & 2.000 & -2.000 \\ 0 & 0.0001 & -0.3333 & 1.667 \\ 0 & 1.667 & -1.333 & 0.3334 \end{bmatrix}$$

Now, to eliminate the $1.667x_2$ term in the 3[rd] equation, we use

Row 3 – 16670* Row 2 =>

$$[-1.333 - 16670 * (-0.3333)]x_3 = 0.3334 - 16670 * 1.667$$

$\Rightarrow$ $5556x_3 = -27780$

$x_3 = -5.0$

Now, using $0.0001x_2 - 0.3333x_3 = 1.667$ we obtain

$$x_2 = [1.667 + 0.3333 * (-5.002)]/0.0001 = 0.000$$

Using $6.000x_1 + 2.000x_2 + 2.000x_3 = -2.000$, we obtain

x_1 = [-2.000 -2.0000*(-5.0000)]/6.0000 =1.333

Clearly, x1 and x2 are completely wrong.

b) Now use partial pivoting.

In matrix form, after the reduction of the first column in part a), the augmented matrix is

$$\begin{bmatrix} 6.000 & 2.000 & 2.000 & -2.000 \\ 0 & 0.0001 & -0.3333 & 1.667 \\ 0 & 1.667 & -1.333 & 0.3334 \end{bmatrix}$$

Now switch Row (2) with Row (3) in the above so that the augmented matrix becomes

$$\begin{bmatrix} 6.000 & 2.000 & 2.000 & -2.000 \\ 0 & 1.6667 & -1.333 & 0.3334 \\ 0 & 0.0001 & -0.3333 & 1.667 \end{bmatrix}$$

In the form of system of equations, we have

$6.000x_1 + 2.000x_2 + 2.000x_3 = -2.000$ -----Row 1

$1.667x_2 - 1.333x_3 = 0.3334$ -----Row 2

$0.0001x_2 - 0.3333x_3 = 1.667$ -----Row 3

- Row 3 – Row 2 * (0.0001/1.667) =>

$$(-0.3333 + 1.333 * 0.00005999)x_3 = 1.667 - 0.3334 * 0.00005999$$

$\Rightarrow$ $-0.3332x_3 = 1.667$ => $x_3 = -5.003$

$\Rightarrow$ Row 2:

$$x_2 = [0.3334 + 1.333 * (-5.003)]/1.667 = 1.667$$

= [0.3334-6.669]/1.667 = -6.336/1.667 = -3.801 =>

$x_2 = -3.801$

$\Rightarrow$ Row 1:

$$x_1 = [-2.000 - 2.000 * (-3.801) - 2.000 * (-5.003)]/6.000$$

= [-2+7.602+10.01]/6.000

= 15.61/6.0000 = 2.602

Hence

$$x_1 = 2.602\,,\quad x_2 = -3.801\,,\quad x_3 = -5.003$$

which are much closer to the exact solution.

#10b

$$A = \begin{bmatrix} 15 & -18 & 15 & -3 \\ -18 & 24 & -18 & 4 \\ 15 & -18 & 18 & -3 \\ -3 & 4 & -3 & 1 \end{bmatrix}$$

Solution:

Let $$A = \begin{bmatrix} b_{11} & 0 & 0 & 0 \\ b_{12} & b_{22} & 0 & 0 \\ b_{13} & b_{23} & b_{33} & 0 \\ b_{14} & b_{24} & b_{34} & b_{44} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ 0 & b_{22} & b_{23} & b_{24} \\ 0 & 0 & b_{33} & b_{34} \\ 0 & 0 & 0 & b_{44} \end{bmatrix} = LU = LL^T$$

Let us work on the first column of L:

$$b_{11}^2 = a_{11} = 15 \quad => \quad b_{11} = \sqrt{15}$$

$$b_{12} * b_{11} = a_{12} = -18 \quad => \quad b_{12} = -18/\sqrt{15} = -\frac{6}{5}\sqrt{15}$$

$$b_{13} * b_{11} = a_{13} = 15 \quad => \quad b_{13} = 15/\sqrt{15} = \sqrt{15}$$

$$b_{14} * b_{11} = a_{14} = -3 \quad => \quad b_{14} = -3/\sqrt{15} = -\sqrt{15}/5$$

Now work on the 2nd column of L:

$$b_{12} * b_{12} + b_{22}^2 = a_{22} = 24 => \quad b_{22} = \sqrt{24 - 324/15} = \sqrt{12/5} = \frac{2}{5}\sqrt{15}$$

$$b_{13} * b_{12} + b_{23} * b_{22} = a_{23} = -18$$

$$=> \quad b_{23} = (-18 + \sqrt{15} * \frac{6}{5}\sqrt{15})/b_{22} = 0$$

$$b_{14} * b_{12} + b_{24} * b_{22} = a_{24} = 4$$

$$=> \quad b_{24} = (4 - \sqrt{15} * \frac{6}{5}\sqrt{15})/b_{22} = \sqrt{15}/15$$

From the 3rd column,

$$b_{13}^2 + b_{23}^2 + b_{33}^2 = a_{33} = 18 => \quad b_{33} = \sqrt{18 - 15 - 0} = \sqrt{3}$$

$$b_{14} * b_{13} + b_{24} * b_{23} + b_{34} * b_{33} = a_{34} = -3$$

$$=> \quad b_{34} = (-3 + \sqrt{15} * \frac{1}{5}\sqrt{15})/b_{33} = 0$$

The last element b_{44} is determined from:

$$b_{14}^2 + b_{24}^2 + b_{34}^2 + b_{44}^2 = a_{44} = 1$$

=> $b_{44} = \sqrt{1 - 3/5 - 1/15} = \sqrt{3}/3$

Finally,

$$L = \begin{bmatrix} \sqrt{15} & 0 & 0 & 0 \\ -\frac{6}{5}\sqrt{15} & \frac{2}{5}\sqrt{15} & 0 & 0 \\ \sqrt{15} & 0 & \sqrt{3} & 0 \\ -\frac{1}{5}\sqrt{15} & \frac{1}{15}\sqrt{15} & 0 & \frac{1}{3}\sqrt{3} \end{bmatrix}$$

#14. Using the algorithm (8.3.23-24) for solving tridiagonal systems, solve Ax=b with

$$A=\begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ 1 & 2 & -1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}, \; \underline{b}=\begin{bmatrix} 3 \\ -2 \\ 2 \\ -2 \\ 1 \end{bmatrix}$$

Check the hypotheses and conclusions of Theorem 8.2 are satisfied by this example.

Solution:

First, the general tri-diagonal system of equations is:

$$\begin{bmatrix} a_1 & c_1 & 0 & 0 & \dots & 0 \\ b_2 & a_2 & c_2 & 0 & \dots & 0 \\ 0 & b_3 & a_3 & c_3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & b_{n-1} & a_{n-1} & c_{n-1} \\ 0 & 0 & \dots & 0 & b_n & a_n \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \dots \\ x_{n-1} \\ x_n \end{bmatrix}=\begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ \dots \\ r_{n-1} \\ r_n \end{bmatrix}$$

In the present case

$$A=\begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ 1 & 2 & -1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}, \; \underline{r}=\begin{bmatrix} 3 \\ -2 \\ 2 \\ -2 \\ 1 \end{bmatrix}$$

Carrying out the reduction step using the Tridiagonal algorithm, we get

$$\begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ 0 & 5/2 & -1 & 0 & 0 \\ 0 & 0 & 12/5 & -1 & 0 \\ 0 & 0 & 0 & 29/12 & -1 \\ 0 & 0 & 0 & 0 & 70/29 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}=\begin{bmatrix} 3 \\ -7/2 \\ 17/5 \\ -99/29 \\ 70/29 \end{bmatrix}$$

⇨ $x=[1\ -1\ 1\ -1\ 1]^T$.

On the other hand we can decompose A into L*U in the form of
Furthermore, A is decomposed into LU in the form (to be consistent with the textbook):

$$\begin{bmatrix} \alpha_1 & 0 & 0 & 0 & 0 \\ b_2 & \alpha_2 & 0 & 0 & 0 \\ 0 & b_3 & \alpha_3 & 0 & 0 \\ 0 & 0 & b_4 & \alpha_4 & 0 \\ 0 & 0 & 0 & b_5 & \alpha_5 \end{bmatrix} \begin{bmatrix} 1 & \gamma_1 & 0 & 0 & 0 \\ 0 & 1 & \gamma_2 & 0 & 0 \\ 0 & 0 & 1 & \gamma_3 & 0 \\ 0 & 0 & 0 & 1 & \gamma_4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ 1 & 2 & -1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

=> $\alpha_1=2$; $\alpha_1 \gamma_1=-1$ => $\gamma_1=-1/2$

=> $b_2 \gamma_1 + \alpha_2 = a_2$ => $\alpha_2 = 2-(-1/2)=5/2$; $\alpha_2 \gamma_2=-1$ => $\gamma_2=-2/5$

=> $b_3 \gamma_2 + \alpha_3 = a_3$ => $\alpha_3 = 2-(-2/5)=12/5$; $\alpha_3 \gamma_3=-1$ => $\gamma_3=-5/12$

=> $b_4 \gamma_3 + \alpha_4 = a_4$ => $\alpha_4 = 2-(-5/12)=29/12$; $\alpha_4 \gamma_4=-1$ => $\gamma_4=-12/29$

=> $b_5 \gamma_4 + \alpha_5 = a_5$ => $\alpha_5 = 2-(-12/29)=70/29$;

* Now checking the hypothesis in Thereom 8.2 (p. 528)

1. $|\alpha_1|=2>|c_1|=1>0$ satisfied
2. $|a_i| \geq |b_i| + |c_i|$:

   i=2: $|a_2|=2 \geq |b_2| + |c_2| = 1 + 1 = 2$

   i=3: $|a_3|=2 \geq |b_3| + |c_3| = 1 + 1 = 2$

   i=4: $|a_4|=2 \geq |b_4| + |c_4| = 1 + 1 = 2$

3. $|a_n| \geq |b_n|>0$: n=5

   $|a_5|=2 \geq |b_5|=1 > 0$

All 3 conditions are satisfied.

Det(A)= $\alpha_1 * \alpha_2 * \alpha_3 * \alpha_4 * \alpha_5 = 2*5/2*12/5*29/12*70/29 = 70 \neq 0$

Thus A is non-singular.

AND:

$|\gamma_1| = |-2/5| < 1$

$|\gamma_2| = |-5/12| < 1$

$|\gamma_3| = |-12/29| < 1$

$|\gamma_4| = |-29/70| < 1$

$|a_2|-|b_2| = 2-1=1 < \alpha_2 = 5/2 < |a_2|+|b_2| = 2+1=3$

$|a_3|-|b_3| = 2-1=1 < \alpha_3 = 12/5 < |a_3|+|b_3| = 2+1=3$

$|a_4|-|b_4| = 2-1=1 < \alpha_4 = 29/12 < |a_4|+|b_4| = 2+1=3$

$|a_5|-|b_5| = 2-1=1 < \alpha_5 = 70/29 < |a_5|+|b_5| = 2+1=3$

In the above,

$$L = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 1 & 5/2 & 0 & 0 & 0 \\ 0 & 1 & 12/5 & 0 & 0 \\ 0 & 0 & 1 & 29/12 & 0 \\ 0 & 0 & 0 & 1 & 70/29 \end{bmatrix}, \ U = \begin{bmatrix} 1 & -1/2 & 0 & 0 & 0 \\ 0 & 1 & -2/5 & 0 & 0 \\ 0 & 0 & 1 & -5/12 & 0 \\ 0 & 0 & 0 & 1 & -12/29 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

#18 Find the condition number $A = \begin{bmatrix} 100 & 99 \\ 99 & 98 \end{bmatrix}$

Solution: Det (A)=100*98-99*99 = (99+1)(99-1)-99*99= -1

The inverse of A is : $A^{-1} = -\begin{bmatrix} 98 & -99 \\ -99 & 100 \end{bmatrix} = \begin{bmatrix} -98 & 99 \\ 99 & -100 \end{bmatrix}$;

the conjugate transpose is: $A^* = A = \begin{bmatrix} 100 & 99 \\ 99 & 98 \end{bmatrix}$

For p=1, the column sum of A is: (100+99, 99+98). The maximum is $\|A\|_1 = 199$

The column sum of A^{-1} is: (98+99, 99+100). The maximum is $\|A^{-1}\|_1 = 199$.

Thus, the condition number is: $\text{cond}(A)_1 = \|A\|_1 \|A^{-1}\|_1 = 199*199 = 39601$

For p=2, we note that A* A is:

$$A^* A = \begin{bmatrix} 100 & 99 \\ 99 & 98 \end{bmatrix} \begin{bmatrix} 100 & 99 \\ 99 & 98 \end{bmatrix} = \begin{bmatrix} 19801 & 19602 \\ 19602 & 19405 \end{bmatrix}$$

The eigenvalues of A* A is determined from:

$$(19801-\lambda)(19405-\lambda) - 19602^2 = 0$$

=> $\lambda^2 - 39206\lambda + 1 = 0$ => $\max(\lambda) = (39206 + \sqrt{39206^2 - 4})/2 \sim 39206.0$

Thus the spectrum radius of (A*A) is 39206.0

Hence $\text{cond}(A)_2 \sim 39206.0$

For p=∞, the maximum row sum of A is: $\|A\|_\infty = 199$

The maximum row sum of A^{-1} is: $\|A^{-1}\|_\infty = 199$

Hence the condition number is: $\text{cond}(A)_\infty = \|A\|_\infty \|A^{-1}\|_\infty = 199*199 = 39601$.

The eigenvalues of A : -0.005050376, 198.0050504

And their ratio is: 198.005/0.0050504 ~ 39206 ~ $\text{cond}(A)_2$