HW9

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problem solved

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A1

A2

14 4 point gauss – Legendre

Use $n = 2, 4, 8, 16, 32, \ldots$

$$I = \int_{-1}^{1} f(t)\,dt = \sum_{i=1}^{4} w_i f(t_i)$$ → we need change of variable for each interval separately

$$I = \int_a^b f(x)\,dx \rightarrow \frac{b-a}{dx} = \frac{2}{dt} \rightarrow dx = \frac{(b-a)}{2}\,dt \qquad \boxed{x = \frac{b-a}{2}t + \frac{a+b}{2}} \quad \Bigg\} \; I = \int_{-1}^{1} \frac{b-a}{2} f\left(\frac{b-a}{2}t + \frac{a+b}{2}\right)dt$$

$$\rightarrow I = \frac{b-a}{2} \int_{-1}^{1} f\left(\frac{b-a}{2}t + \frac{a+b}{2}\right) dt$$ for each interval

$$\rightarrow I = \frac{b-a}{2} \times \sum_{i=1}^{4} w_i f(t_i)$$ for each interval

I wrote this code to implement it

- global interval
- different # of intervals
- t_i & w_i
- h for each n
- local interval for each 4 node Gauss – quadrature
- change of variable for each local interval & calculating $\sum w_i f(t_i)$
- $I = \frac{b-a}{2} \times \sum w_i f(t_i)$
- adding them all together to get the integral value for the global interval
- – comparing with the exact result to get the error

```
%composite 4 point Gauss Quadrature integration:
format long
b=1;
a=0;
n=[2,4,8,16,32,64,128,256,512];
%%%%%%%
t=[-0.861136311594052575224
-0.339981043584856264802 7
0.339981043584856264802 7
0.861136311594052575224];
%%%%%%%%%%%%%%
w=[0.347854845137453857373 1
 0.652145154862546142626 9
 0.652145154862546142626 9
 0.347854845137453857373 1];
for i=1:length(n)
GQ=[]
h=(b-a)/n(i);
nn=n(i);
for j=1:nn
 x1=(j-1)*h+a;
 x2=(j)*h+a;
 gg=0
 for k=1:4
 gg=w(k)*f(((x2-x1)/2)*t(k)+((x1+x2)/2))+gg;
 end
 GQ(j)=gg*((x2-x1)/2);
end
I(i)=sum(GQ)

end
syms x
expr = exp(-x.^2);
F = double(int(expr,[a b]));
err=F-I;
t=table(n',I',err','VariableNames',["n","I(n)","error"])
function y=f(x)
y=exp(-x.^2);
%y=x.^(2.5);
%y = 1./(1+x.^2);
%y=1./(2+cos(x));
%y=exp(x).*cos(4*x);
end
```

(a) $\int_0^1 exp(-x^2)\,dx$

n	I(n)	error
2	0.746824133277504	-4.65076976929879e-10
4	0.746824133281397	-1.54276591501912e-12
8	0.746824132812433	-5.88418203051333e-15
16	0.746824132812427	0
32	0.746824132812427	1.11022302462516e-16
64	0.746824132812427	0
128	0.746824132812427	0
256	0.746824132812427	0
512	0.746824132812427	1.11022302462516e-16

(b) $\int_0^1 x^{2.5}\,dx$

weak singularity

n	I(n)	error
2	0.285714783081152	-4.97366866414328e-07
4	0.285714329677606	-4.39633204285528e-08
8	0.285714289600139	-3.88585369437422e-09
16	0.28571428605775	-3.43464257035464e-10
32	0.285714285744644	-3.03581604299552e-11
64	0.285714285716969	-2.68329802821654e-12
128	0.285714285714523	-2.37199149211165e-13
256	0.285714285714307	-2.09832151654155e-14
512	0.285714285714288	-1.83186799063151e-15

(c) $\int_{-4}^{4} \frac{1}{x^2+1}\,dx$

n	I(n)	error
2	2.6554264455908	-0.00379111825473233
4	2.65081776509432	0.000817562241747005
8	2.65164513212222	-9.80478615764113e-06
16	2.6516353507247	-2.33886310319065e-08
32	2.65163532733615	-8.26005930321116e-14
64	2.65163532733606	4.44089209850063e-16
128	2.65163532733606	4.44089209850063e-16
256	2.65163532733606	4.44089209850063e-16
512	2.65163532733606	4.44089209850063e-16

(d) $\int_0^{2\pi} \frac{1}{2+\cos x}\,dx$

n	I(n)	error
2	3.6253693784055	0.00222935006293712
4	3.62761515990057	-1.64314321340342e-05
8	3.62759893368501	-2.05216577953138e-07
16	3.62759872847391	-5.46940270851337e-12
32	3.62759872846844	0
64	3.62759872846844	0
128	3.62759872846844	4.44089209850063e-16
256	3.62759872846844	0
512	3.62759872846843	8.88178419700125e-16

(e) $\int_0^{\pi} e^{x} \cos(4x)\, dx$

n	I(n)	error
2	1.29515117409637	0.00724251018476307
4	1.30244219026869	-4.85059875603433e-05
8	1.30239383656061	-1.52279481024920e-07
16	1.30239368484315	-5.62011770455229e-10
32	1.30239368428329	-2.16049400592055e-12
64	1.30239368428114	-5.32907051820075e-15
128	1.30239368428113	3.10862446895044e-15
256	1.30239368428113	2.66453525910038e-15
512	1.30239368428113	3.5527136788005e-15

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A1

$x = \xi^q \rightarrow dx = q\,\xi^{q-1}\,d\xi$ $\quad p = \sqrt{2}-1 < 1$ $\quad$ $\checkmark$: integrable singularity

$$\int_0^1 \frac{e^x}{x^{\sqrt{2}-1}}\,dx = \int \frac{e^{\xi^q}}{(\xi^q)^{(\sqrt{2}-1)}}\,q\,\xi^{q-1}\,d\xi \rightarrow q\int e^{\xi^q}\,\xi^{[q-1-q(\sqrt{2}-1)]}\,d\xi$$

$\rightarrow$ eliminating singularity $\rightarrow$ $q-1-q(\sqrt{2}-1)=0 \rightarrow q(1+1-\sqrt{2})=1$

$$q = \frac{1}{1+1-\sqrt{2}} \rightarrow \boxed{q = \frac{1}{2-\sqrt{2}}} \rightarrow \int_0^1 \frac{e^x}{x^{\sqrt{2}-1}}\,dx \equiv \boxed{\frac{1}{2-\sqrt{2}}\int_0^1 e^{\xi^{\left(\frac{1}{2-\sqrt{2}}\right)}}\,d\xi}$$

Solving it with Matlab int function

```
format long
f = @(x) (1/(2-sqrt(2)))*exp(x.^(1/(2-sqrt(2))));
I = integral(f,0,1)
```

$\Big\}$ $I =$ 2.588371477311493

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A2 $\int_0^\infty \frac{\sin(x)}{x}\,dx$

a) → using Matlab function $I = \text{integral}(f, 0, \text{inf})$

```
format long
f = @(x) sin(x)./x;
I = integral(f,0,inf)
```

$I = 3.449670307502231$

but the exact value is $I_{exact} = \pi/2 = 1.570796326794897$

the result is wrong : because of Singularity & unbounded interval

→ we are dealing with a Impooper Integral

b) → In order to use Gauss-Laguerre : $\int_0^\infty f(x)\, e^{-x}\, dx$

→ So : $\int_0^\infty \underbrace{\frac{\sin(x)}{x} e^{x}}_{f(x)} e^{-x} dx = \sum_{i=1}^{256} w_i f(t_i) = 1.503712893933262$

the error is still large → $\boxed{2 \text{ reasons}}$

The reason for this is the existance of singularity

in other words close to singularity we cannot capture the growth of the function accurately even with 256 terms as well as the fact That

$\frac{\sin(x)}{x}$ does not decay enough so it cant cancel out the growth of e^x

The last term

```
x=5.934978037627280E+02
x =
 5.934978037627280e+02
>> y=sin(x)*exp(x)/x
y =
 2.481273370162274e+254
>> y*1.300383017776430E-257
ans =
 0.003226605753020
```

→ The last term is still a big number

$\boxed{\text{we could do change of variables}}$ & dividing into two subintervals :

c

$$I = \int_0^{\infty} \frac{\sin x}{x}\,dx = \underbrace{\int_0^{b} \frac{\sin(x)}{x}\,dx}_{I_1} + \underbrace{\int_b^{\infty} \frac{\sin(x)}{x}\,dx}_{I_2}$$

$b = 10000$

$$I_2 \simeq \left(\frac{1}{b} - \frac{2}{b^3}\right)\cos(b) + \left(\frac{1}{b^2} - \frac{6}{b^4}\right)\sin b$$

$\rightarrow$

```
b=10000
f = @(x) sin(x)./x;
I1 = integral(f,0,b)
I2 = ((1/b)-(2/b.^3))*cos(b)+((1/b.^2)-(6/b.^4))*sin(b)
I1+I2
1.570796326794892
```

use b=1000

report I1 and I2 values as well

$$\boxed{I = 1.570796326794892}$$

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$\text{error} = \frac{\pi}{2} - I = 4.662936703425657e\text{-}15$

error is close to machine epsilon & acceptable