Home work 10

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question solved

16

22

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16 I wrote the following matlab code ↓

```
%mid_point
h=0.01;
a=0;
b=5;
y(1)=1
x=linspace(a,b,(b-a)/h);
y(2)=y(1)+h*f(x(1),y(1));
for i=1:length(x)-2
y(i+2)=y(i)+2*h*f(x(i+1),y(i+1));
end
yy=ff(x);
plot(x,y);
hold on
plot(x,yy)
xlabel('x');
ylabel('y');
legend("mid-point", "exact");


function yprime=f(x,y)
%yprime=-y.^2;
%yprime=(y./4).*(1-y./20);
%yprime=-y+2.*cos(x);
yprime=y-2.*sin(x);
end
function y=ff(x)
%y=1./(1+x);
%y=20./(1+19.*exp(-x./4));
%y=cos(x)+sin(x);
y=cos(x)+sin(x);
end
```

we will see mid point method is not stable

$$y(n+1) = y(n-1) + 2h\, f[x_n, y_n]$$

→ general solution → $y_n = \beta_0 r_0^n + \beta_1 r_1^n$

$$r_0 = h\lambda + \sqrt{1+h^2\lambda^2}$$

$$r_1 = h\lambda - \sqrt{1+h^2\lambda^2}$$

$$\beta_0 = \frac{y_1 - r_1 y_0}{r_0 - r_1}$$

$$\beta_1 = \frac{y_0 r_0 - y_1}{r_0 - r_1}$$

→

$n \to \infty \to h \to 0$

$\beta_1 r_1^n \to$ ↑ increase

$\beta_0 r_0^n \to 0$ → Source of error in our case

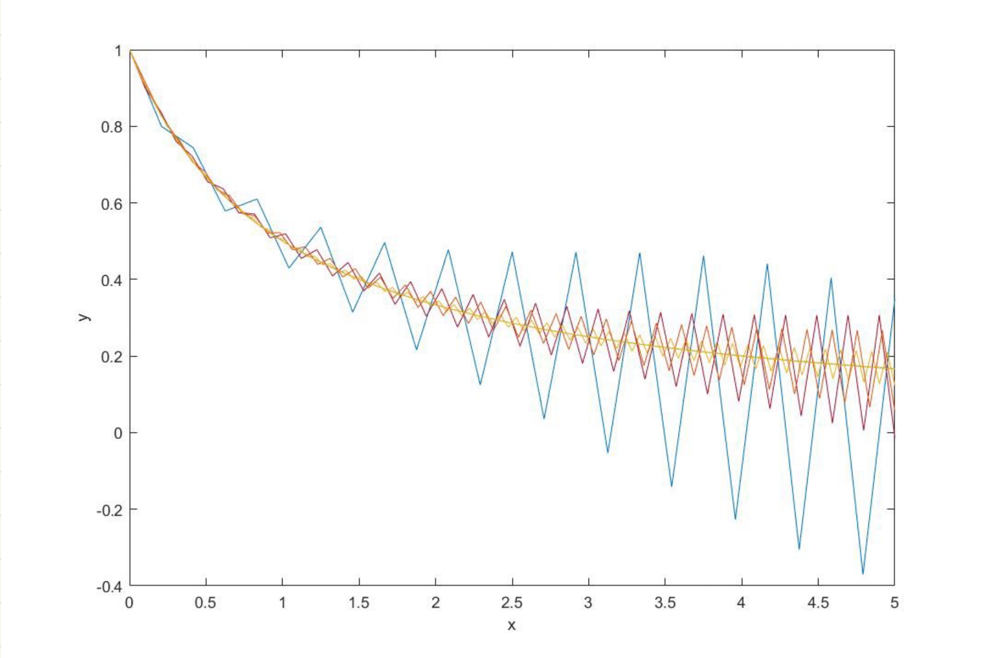
So I expect for big h & small number of n

$\beta_0 r_0^n \to$ causes some error & instability

→ & for really small h & large interval

$\beta_1 r_1^n \longrightarrow$ cause instability

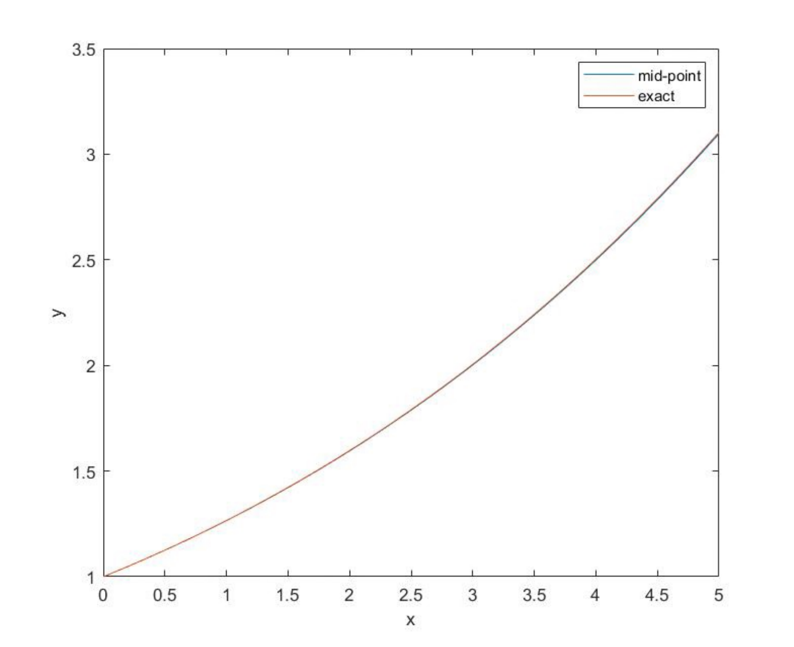
a) $y' = -y^2$ & $y(0) = 1$ & $Y(x) = \frac{1}{x+1} = \frac{1/x}{1+1/x}$



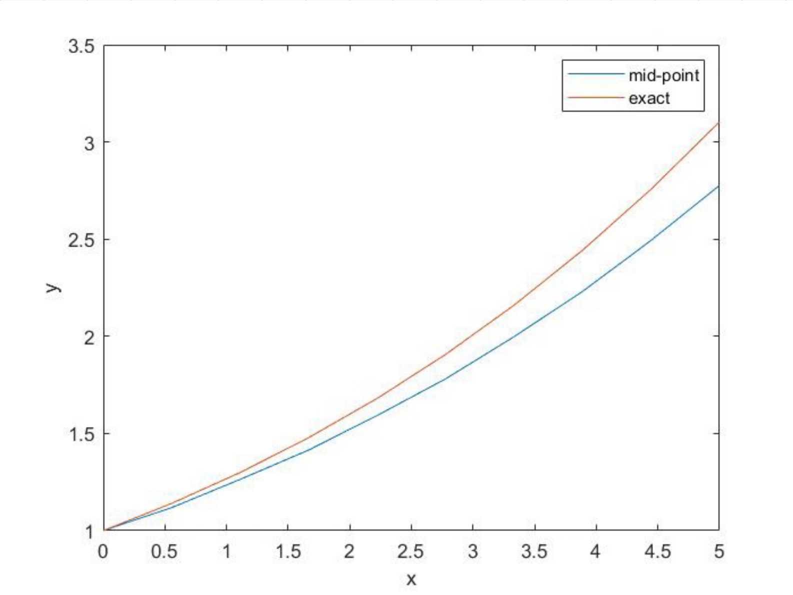
here using midpoint rule we can see the estimated y values oscillate around the exact value & as the step size becomes smaller the oscillations become smaller

(b) $y' = (y/4)(1 - y/20) \quad y(0) = 1 \;, \quad Y(x) = \dfrac{20}{1 + 19e^{-x/4}}$

$h = 0.01$ $\longrightarrow$



$h = 0.5$ $\longrightarrow$

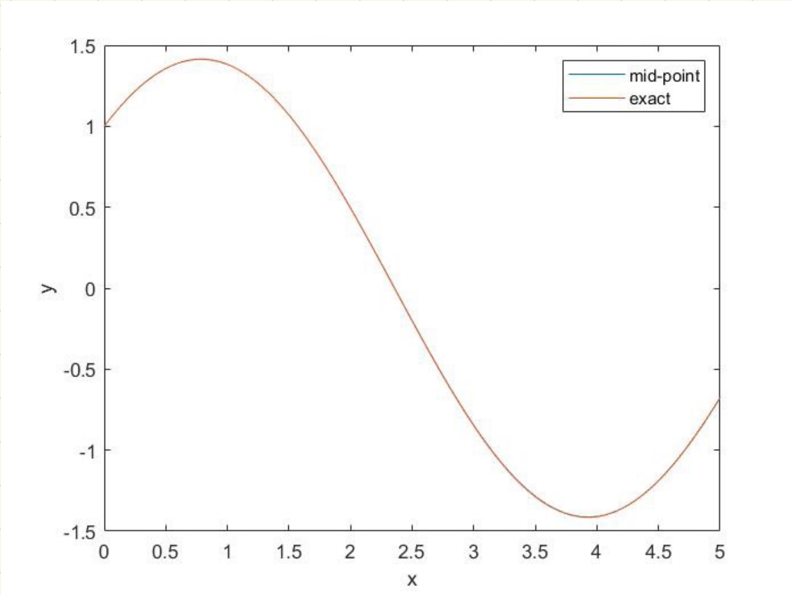


big h

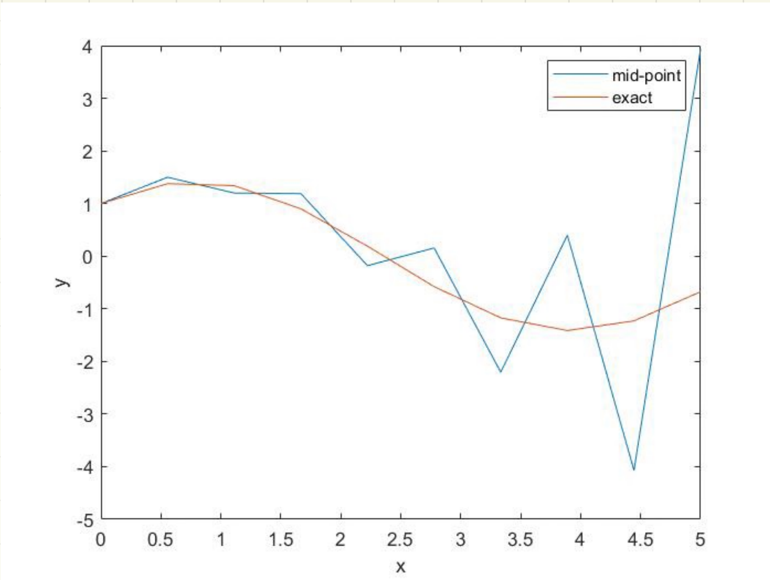
not stable

c) $y = -y + 2\cos(x)$ $\quad y(0) = 1 \quad$ $Y(x) = \cos x + \sin x$

$h = 0.01$

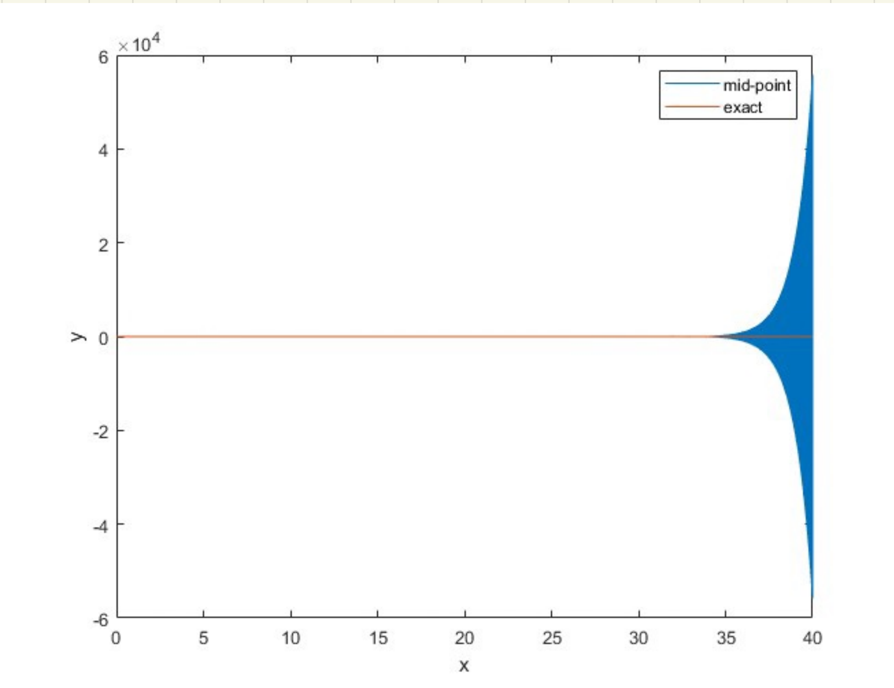


$h = 0.5$



big h
not stable

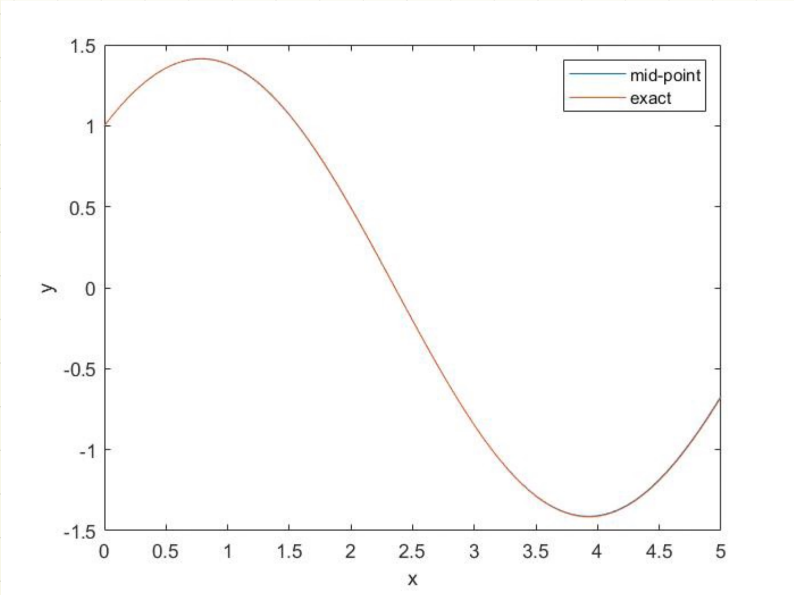
$h = 0.000001$



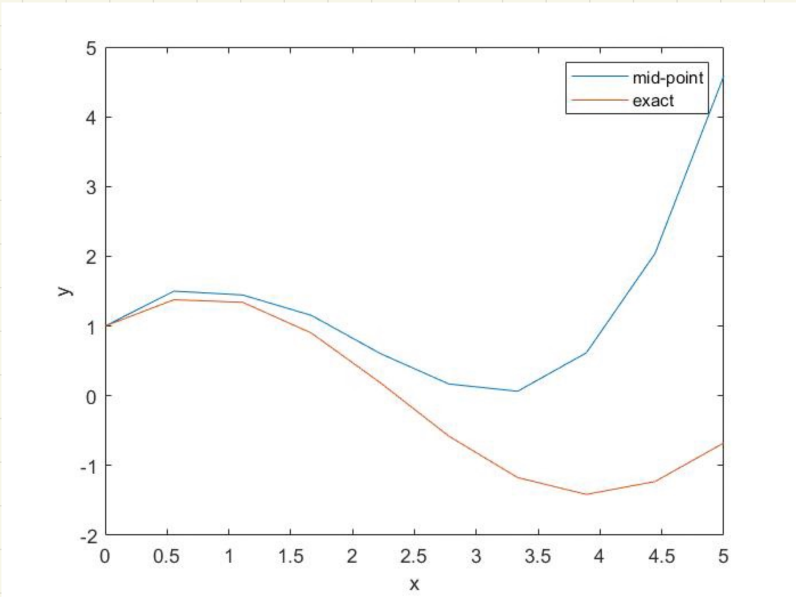
small
h
not stable

d) $y' = y - 2\sin x$, $y(0) = 1$, $Y(x) = \cos x + \sin x$

$h = 0.01$ $\longrightarrow$

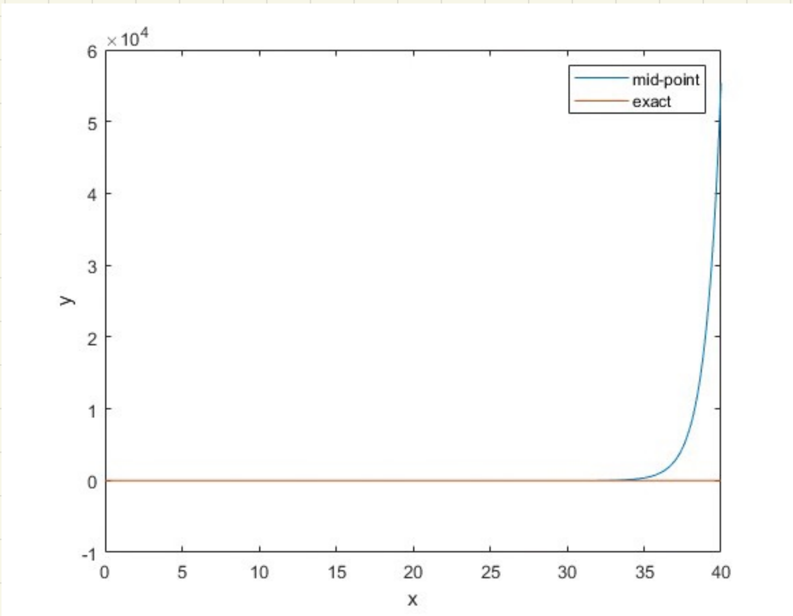


$h = 0.5$ $\rightarrow$



big h
not stable

$h = 0.000001$ $\longrightarrow$



small
h
not stable

22 wrote this Matlab code

```
format long
%fourth-order Adams-Moulton
h=[0.5,0.25];
a=0;
b=5;
y(1)=1;
y2(1)=1;
x=a:h(1):b;
y(2)=ff(x(2));
y(3)=ff(x(3));
y(4)=ff(x(4));
for i=1:length(x)-4
y(i+4)=y(i+3)+(h(1)/24)*(55*f(x(i+3),y(i+3))-59*f(x(i+2),y(i+2))+37*f(x(i+1),y(i+1))-9*f(x(i),y(i)));
y(i+4)=y(i+3)+(h(1)/24)*(9*f(x(i+4),y(i+4))+19*f(x(i+3),y(i+3))-5*f(x(i+2),y(i+2))+f(x(i+1),y(i+1)));
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
x2=a:h(2):b;
y2(2)=ff(x2(2));
y2(3)=ff(x2(3));
y2(4)=ff(x2(4));
for i=1:length(x2)-4
y2(i+4)=y2(i+3)+(h(2)/24).*(55*f(x2(i+3),y2(i+3))-59*f(x2(i+2),y2(i+2))+37*f(x2(i+1),y2(i+1))-9*f(x2(i),y2(i)));
y2(i+4)=y2(i+3)+(h(2)/24).*(9*f(x2(i+4),y2(i+4))+19*f(x2(i+3),y2(i+3))-5*f(x2(i+2),y2(i+2))+f(x2(i+1),y2(i+1)));
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
yy=ff(x);
yy2=ff(x2);
er=yy-y;
er2=yy2-y2;
t=table(x',y',er','VariableNames',["x for
h=0.5","y","error"])
t2=table(x2',y2',er2','VariableNames',["x for
h=0.25","y","error"])
function yprime=f(x,y)
%yprime=-y.^2;
%yprime=(y./4).*(1-y./20);
%yprime=-y+2.*cos(x);
yprime=y-2.*sin(x);
end
function y=ff(x)
%y=1./(1+x);
%y=20./(1+19.*exp(-x./4));
%y=cos(x)+sin(x);
y=cos(x)+sin(x);
end
```

a a) $y = -y^2$ & $y(0) = 1$ & $Y(x) = \frac{1}{x+1}$

x for h=0.5	y	error
0	1	0
0.5	0.666666666666667	0
1	0.5	0
1.5	0.4	0
2	0.327937461319284	0.00539587201404962
2.5	0.280839392554231	0.00487489316005491
3	0.24612388600127	0.00387611399873
3.5	0.218982889304763	0.00323933291745965
4	0.197327638645798	0.0026723613542016
4.5	0.17958978242123	0.0022283993969518
5	0.164782358760701	0.00188430790596517

x for h=0.25	y	error
0	1	0
0.25	0.8	0
0.5	0.666666666666667	0
0.75	0.571428571428571	0
1	0.499414866890432	0.000585133109568037
1.25	0.443788205722016	0.000656238722428371
1.5	0.399393720261875	0.000606279738125104
1.75	0.363093223093708	0.000543140542655474
2	0.332856698405724	0.000476634927609076
2.25	0.307275379864898	0.000416927827409819
2.5	0.285348560745011	0.000365724969274783
2.75	0.266344338750366	0.000322327916300547
3	0.249714359989685	0.000285640010314864
3.25	0.235039572778197	0.000254544868861728
3.5	0.221994157574661	0.000228064647561477
3.75	0.210320928542152	0.000205387247321637
4	0.199814148371759	0.000185851628241196
4.25	0.190307266913553	0.00016892356263698
4.5	0.181664010227673	0.000154171590509317
4.75	0.173771797163448	0.000141246314812521
5	0.166536803125264	0.000129863541403125

(b) $y' = (y/4)(1 - y/20)$ $\quad y(0) = 1$, $Y(x) = \dfrac{20}{1 + 19e^{-x/4}}$

x for h=0.5	y	error
0	1	0
0.5	1.12565449532978	0
1	1.26604595518932	0
1.5	1.42262725525114	0
2	1.5969230934279	2.69608247949193e-07
2.5	1.79051489416285	6.68326814112774e-07
3	2.005018449316	1.21725333901423e-06
3.5	2.24205541653797	1.94490046556695e-06
4	2.50321708449075	2.87547595378967e-06
4.5	2.79002006524455	4.02735757099038e-06
5	3.10385384697172	5.40858828568247e-06

x for h=0.25	y	error
0	1	0
0.25	1.06107279313458	0
0.5	1.12565449532978	0
0.75	1.1939191373468	0
1	1.2660459489019	6.28741303465574e-09
1.25	1.34221908219595	1.44764742415759e-08
1.5	1.42262723050754	2.47436049427563e-08
1.75	1.50746318669777	3.73642652373718e-08
2	1.59692331043058	5.26055712324336e-08
2.25	1.69120690121377	7.07435288127556e-08
2.5	1.79051547043183	9.20578353547086e-08
2.75	1.89505190568234	1.16826822660343e-07
3	2.00501952124784	1.45321501054951e-07
3.25	2.12062098924519	1.77798517242422e-07
3.5	2.24205714694644	2.14491996608501e-07
3.75	2.3695256769848	2.55604271082177e-07
4	2.50321965867116	3.01295551174974e-07
4.25	2.64332599046904	3.51672630127808e-07
4.5	2.79002368582533	4.06776796246788e-07
4.75	2.94348204703529	4.66571188795228e-07
5	3.10385872463209	5.30927921627011e-07

c) $y = -y + 2\cos(x)$ $\quad y(0) = 1$ $\quad Y(x) = \cos x + \sin x$

x for h=0.5	y	error
0	1	0
0.5	1.35700810049458	0
1	1.38177329067604	0
1.5	1.06823218827176	0
2	0.492016908047532	0.00113368223100768
2.5	-0.206151331309105	0.00347985986612731
3	-0.854644337188891	0.00577184864831237
3.5	-1.29434645936877	0.00710654438835445
4	-1.41747900197117	0.00703288579962646
4.5	-1.19376136789174	0.00543545079586405
5	-0.677880893592819	0.00261880439290718

x for h=0.25	y	error
0	1	0
0.25	1.21631638096517	0
0.5	1.35700810049458	0
0.75	1.41332762889716	0
1	1.38179034704177	-1.70563657362344e-05
1.25	1.26431702528021	-1.00435293541867e-05
1.5	1.06821696645609	1.5221815663935e-05
1.75	0.805687483078439	5.24081460058223e-05
2	0.493053804701304	9.67855772353032e-05
2.25	0.149756285874213	0.000143288290968657
2.5	-0.202858676896749	0.000187205453772099
2.75	-0.54286579896385	0.000224412383717731
3	-0.849123993890679	0.000251505350101211
3.25	-1.10259076073035	0.000265950119699321
3.5	-1.28750610221433	0.000266187233917359
3.75	-1.39237236296677	0.000251686884870317
4	-1.41066906569469	0.00022294952315427
4.25	-1.34125829768799	0.000181449545615608
4.5	-1.18845544107569	0.000129523979817359
4.75	-0.961760847974099	7.02118866976464e-05
5	-0.67526914269792	7.05349800800281e-06

d) $y' = y - 2\sin x$, $y(0) = 1$, $Y(x) = \cos x + \sin x$

x for h=0.5	y	error
0	1	0
0.5	1.35700810049458	0
1	1.38177329067604	0
1.5	1.06823218827176	0
2	0.493055974072602	9.4616205936926e-05
2.5	-0.202073665098435	-0.000597806344541985
3	-0.846323381895174	-0.0025491066454042
3.5	-1.2811080102816	-0.00613190469881464
4	-1.39851733464687	-0.0119287815246742
4.5	-1.16738920815059	-0.0209367089452877
5	-0.640337432673628	-0.0349246565262838

x for h=0.25	y	error
0	1	0
0.25	1.21631638096517	0
0.5	1.35700810049458	0
0.75	1.41332762889716	0
1	1.38176402124684	9.26942919221574e-06
1.25	1.26428719261471	1.97891361497504e-05
1.5	1.06820166920773	3.0519064031731e-05
1.75	0.805698370483418	4.15207410267016e-05
2	0.4930978854009	5.27048776394756e-05
2.25	0.149835430401902	6.41437632797892e-05
2.5	-0.202747602493471	7.61310504939217e-05
2.75	-0.542730597000988	8.92104208558564e-05
3	-0.848976711658143	0.000104223117564639
3.25	-1.10244717082712	0.000122360216465722
3.5	-1.28738513694132	0.000145221960907493
3.75	-1.39229556571018	0.000174889628279606
4	-1.41066013236439	0.000214016192848776
4.25	-1.34134279208333	0.000265943940949809
4.5	-1.18866077657836	0.00033485948248102
4.75	-0.962116635403721	0.000425999316319481
5	-0.675808011636841	0.000545922436928969

A1 :

$$y' = f(x,y) \rightarrow y_{n+1} = y_n + \int_{x_n}^{x_{n+1}} f(x)\,dx$$

$$\boxed{f_{n-k} = f(x_{n-k}, y_{n-k})} \quad \text{for} \quad k = -1, 0, 1, 2, 3 \rightarrow \text{AM5}$$

$$\rightarrow y_{n+1} = y_n + h\left[a * f_{n+1} + b * f_n + c * f_{n-1} + d * f_{n-2} + e * f_{n-3}\right]$$

$\rightarrow$ to get a ---

$$f(x,y) = f_n + (x-x_n) f_n' + \frac{1}{2!}(x-x_n)^2 f_n'' + \frac{1}{3!}(x-x_n)^3 f_n^3 + \cdots$$

$$\rightarrow \int_{x_n}^{x_{n+1}} f(x)\,dx = f_n(x_{n+1}-x_n) + \frac{1}{2} f_n'(x-x_n)^2\Big|_{x_n}^{x_{n+1}} + \frac{1}{6} f_n''(x-x_n)^3\Big|_{x_n}^{x_{n+1}}$$

$$+ \frac{1}{4!} f_n^3 (x-x_n)^4\Big|_{x_n}^{x_{n+1}} + \frac{1}{5!} f_n^4 (x-x_n)^5\Big|_{x_n}^{x_{n+1}}$$

$$\rightarrow \int_{x_n}^{x_{n+1}} f(x)\,dx = f_n(x_{n+1}-x_n) + \frac{1}{2} f_n' \times (x_{n+1}-x_n)^2 + \frac{1}{6} f_n'' \times (x_{n+1}-x_n)^3 + \frac{1}{4!} f_n^3 h^4 + \frac{1}{5!} f_n^4 h^5$$

$$\rightarrow \int_{x_n}^{x_{n+1}} f(x)\,dx = h f_n + \frac{1}{2} h^2 f_n' + \frac{1}{3!} h^3 f_n'' + \frac{1}{4!} h^4 f_n^3 + \frac{1}{5!} h^5 f_n^4$$

$$\rightarrow \boxed{y_{n+1} = y_n + h f_n + \frac{1}{2} h^2 f_n' + \frac{1}{3!} h^3 f_n'' + \frac{1}{4!} h^4 f_n^3 + \frac{1}{5!} h^5 f_n^4} \quad ★$$

(b)

$$y_{n+1} = y_n + h\left[a * f_{n+1} + b * f_n + c * f_{n-1} + d * f_{n-2} + e * f_{n-3}\right]$$

$$\boxed{f_{n-k} = f(x_{n-k}, y_{n-k})}$$

$$f_{n-k} = f_n - kh f_n' + \frac{1}{2}(kh)^2 f_n'' - \frac{1}{3!}(kh)^3 f_n^3 + \frac{1}{4!}(kh)^4 f_n^4 - \frac{1}{5!}(kh)^5 f_n^5$$

$$h\Big[a\left(f_n + h f_n' + \frac{1}{2} h^2 f_n'' + \frac{1}{3!} h^3 f_n^3 + \frac{1}{4!} h^4 f_n^4 + \frac{1}{5!} h^5 f_n^5\right) + b * f_n$$

$$c\left(f_n - h f_n' + \frac{1}{2} h^2 f_n'' - \frac{1}{3!} h^3 f_n^3 + \frac{1}{4!} h^4 f_n^4 - \frac{1}{5!} h^5 f_n^5\right)$$

$$d\left(f_n - 2h f_n' + \frac{4}{2} h^2 f_n'' - \frac{8}{3!} h^3 f_n^3 + \frac{16}{4!} h^4 f_n^4 - \frac{32}{5!} h^5 f_n^5\right)$$

$$e\left(f_n - 3h f_n' + \frac{9}{2} h^2 f_n'' - \frac{27}{3!} h^3 f_n^3 + \frac{3^4}{4!} h^4 f_n^4 - \frac{3^5}{5!} h^5 f_n^5\right)\Big]$$

$$= h\Big[(a+b+c+d+e) f_n + h(a-c-2d-3e) f_n'$$

$$+ h^2\left(\tfrac{1}{2}a + \tfrac{1}{2}c + 2d + \tfrac{9}{2}e\right) f_n'' + h^3\left(\frac{a}{3!} - \frac{c}{3!} - \frac{8d}{3!} - \frac{27e}{3!}\right) f_n^3$$

$$+ h^4\left(\frac{a}{4!} + \frac{c}{4!} + \frac{16d}{4!} + \frac{3^4 e}{4!}\right) f_n^4 + h^5\left(\frac{a}{5!} - \frac{c}{5!} - \frac{32d}{5!} - \frac{3^5 e}{5!}\right) f_n^5\Big]$$

→ matching The coefficient for corresponding powers of h with equation $\star\star$

$$\rightarrow \boxed{y_{n+1} = y_n + h f_n + \frac{1}{2} h^2 f_n' + \frac{1}{3!} h^3 f_n'' + \frac{1}{4!} h^4 f_n^3 + \frac{1}{5!} h^5 f_n^4} \quad \star\star$$

$$\boxed{a+b+c+d+e = 1}$$

$$\boxed{a - c - 2d - 3e = \frac{1}{2}}$$

$$\boxed{\tfrac{1}{2}a + \tfrac{1}{2}c + \frac{4}{2}d + \frac{9}{2}e = \frac{1}{3!}}$$

$$\boxed{\frac{a - c - 8d - 27e}{3!} = \frac{1}{4!}}$$

$$\boxed{\frac{a + c + 16d + 81e}{4!} = \frac{1}{5!}}$$

→ system of equations

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & -2 & -3 \\ \tfrac{1}{2} & 0 & \tfrac{1}{2} & 2 & \tfrac{9}{2} \\ \tfrac{1}{3!} & 0 & -\tfrac{1}{3!} & -\tfrac{8}{3!} & -\tfrac{27}{3!} \\ \tfrac{1}{4!} & 0 & \tfrac{1}{4!} & \tfrac{16}{4!} & \tfrac{81}{4!} \end{bmatrix} \begin{Bmatrix} a \\ b \\ c \\ d \\ e \end{Bmatrix} = \begin{Bmatrix} 1 \\ \tfrac{1}{2} \\ \tfrac{1}{3!} \\ \tfrac{1}{4!} \\ \tfrac{1}{5!} \end{Bmatrix}$$

→ Solving it :

$$\begin{Bmatrix} a \\ b \\ c \\ d \\ e \end{Bmatrix} = \begin{Bmatrix} 0.348611111111111 \\ 0.897222222222222 \\ -0.366666666666667 \\ 0.147222222222222 \\ -0.026388888888889 \end{Bmatrix} \begin{matrix} \checkmark \\ \checkmark \\ \checkmark \\ \checkmark \\ \checkmark \end{matrix} = \frac{1}{720} \begin{Bmatrix} 251 \\ 646 \\ -264 \\ 106 \\ -19 \end{Bmatrix}$$

using the last term of expanding eq 2

$$\to h^6 \left(\frac{a}{5!} - \frac{c}{5!} - \frac{32d}{5!} - \frac{3^5 e}{5!} \right) f_n^5$$

→ plugging the values for a ... e into the last term

$$\boxed{\text{error}_{\text{local}} = \frac{h^6}{(720)(120)} \left(251 + 264 - (106)(32) + (243)(19) \right) f_n^5}$$