Let the frontal slices of a tensor, $\mathcal{X} \in \mathrm{R}^{3\times4\times2}$ be:

$$\mathbf{X_1} = \begin{bmatrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & 9 & 12 \end{bmatrix}, \quad \mathbf{X_2} = \begin{bmatrix} 13 & 16 & 19 & 22 \\ 14 & 17 & 20 & 23 \\ 15 & 18 & 21 & 24 \end{bmatrix}$$

In this example we have three modes, because the dimension of the tensor is three.
The size of the tensor is $(|I_1| = 3, |I_2| = 4, |I_3| = 2)$.

Matricization: (transforming a tensor into a matrix)

- **mode-2 unfolding:** The number of rows and columns of mode-2 unfolded matrix, $X_{(2)}$, is $|I_2| = 4$, and $|I_1| \times |I_3| = 3 \times 2 = 6$ respectively. Then its size will be $(4, 6)$. Suppose we want to specify the position of (2, 2, 2)-th element of the tensor $\mathcal{X}$, 17, in the matrix $X_{(2)}$.

$$\mathbf{X_1} = \begin{bmatrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & 9 & 12 \end{bmatrix}, \quad \mathbf{X_2} = \begin{bmatrix} 13 & 16 & 19 & 22 \\ 14 & \boxed{17} & 20 & 23 \\ 15 & 18 & 21 & 24 \end{bmatrix}$$

position of the element in tensor: $i_1 = 2, i_2 = 2, i_3 = 2$

position of row: $i_2 = 2$

position of column: must be calculated!!

$$\begin{aligned} j &= 1 + \sum_{\substack{k=1 \\ k\neq n}}^{N} (i_k - 1)J_k \quad \mathit{with} \quad J_k = \prod_{\substack{m=1 \\ m\neq n}}^{k-1} I_m \\ &= 1 + \sum_{\substack{k=1 \\ k\neq 2}}^{3} (i_k - 1)J_k \quad \mathit{with} \quad J_k = \prod_{\substack{m=1 \\ m\neq 2}}^{k-1} I_m \\ &= 1 + (i_1 - 1)J_1 + (i_3 - 1)J_3 \\ &= 1 + (i_1 - 1) + (i_3 - 1)I_1 \\ &= 1 + (i_1 - 1) + (i_3 - 1)3 \\ &= \boxed{i_1 + (i_3 - 1)3} \\ &= 2 + (2 - 1)3 \\ &= 5 \end{aligned}$$

$$X_{(2)} = \begin{bmatrix} 1 & 2 & 3 & 13 & 14 & 15 \\ 4 & 5 & 6 & 16 & \boxed{17} & 18 \\ 7 & 8 & 9 & 19 & 20 & 21 \\ 10 & 11 & 12 & 22 & 23 & 24 \end{bmatrix}$$

The n-Mode Product: The n-mode (matrix) product of a tensor $\mathcal{X} \in \mathrm{R}^{I_1\times I_2\times\cdots\times I_n\times\cdots\times I_N}$ with a matrix $U \in \mathrm{R}^{J\times I_n}$ is denoted by $\mathcal{X} \times_n U$ and is of size $I_1 \times \cdots \times I_{n-1} \times J \times I_{n+1} \times \cdots \times I_N$. Elementwise, we have

$$(\mathcal{X} \times_n U)_{i_1\cdots i_{n-1} j i_{n+1}\cdots i_N} = \sum_{i_n=1}^{I_n} x_{i_1 i_2\cdots i_N} u_{j i_n}.$$

Each mode-n fiber is multiplied by the matrix U. The idea can also be expressed in terms of unfolded tensors:

$$\mathcal{Y} = \mathcal{X} \times_n U \iff Y_{(n)} = UX_{(n)}.$$

Example: Let $U = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \in \mathrm{R}^{2\times 3}$ and $\mathcal{X}$ be the tensor defined above, $\mathcal{X} \in \mathrm{R}^{3\times 4\times 2}$.
The product $\mathcal{Y} = \mathcal{X} \times_1 U \in \mathrm{R}^{2\times 4\times 2}$ is

$$\mathbf{Y_1} = \begin{bmatrix} 22 & 49 & 76 & 103 \\ 28 & 64 & 100 & 136 \end{bmatrix}, \quad \mathbf{Y_2} = \begin{bmatrix} 130 & 157 & 184 & 211 \\ 172 & 208 & 244 & 280 \end{bmatrix}$$

Suppose we want to obtain (1, 3, 2)-th element of $\mathcal{Y}$. So we have $j = 1, i_2 = 3, i_3 = 2$.

- According to definition:

$$(\mathcal{X} \times_1 U)_{ji_2i_3} = \sum_{i_1=1}^{3} x_{i_1i_2i_3} u_{ji_1}$$

  Then by replacing the values,

$$\begin{aligned}(\mathcal{X} \times_1 U)_{132} &= \sum_{i_1=1}^{3} x_{i_132} u_{1i_1} \\ &= x_{132}u_{11} + x_{232}u_{12} + x_{332}u_{13} \\ &= 19 * 1 + 20 * 3 + 21 * 5 = 184\end{aligned}$$

- Second method:

$$Y_{(1)} = UX_{(1)}$$

  $\mathcal{Y}$ will be obtaind by folding $Y_{(1)}$ with respect to mode-1. The (1, 3, 2)-th element is 184.

A Definition:
The n-Rank

For $\mathcal{X} \in \mathrm{R}^{I_1\times\cdots\times I_N}$, the n-rank of $\mathcal{X}$, denoted by $rank_n(\mathcal{X})$ is the column rank of $\mathcal{X}_{(n)}$ which is the size of the vector space spanned by the mode-n fibers. If $R_n = rank_n(\mathcal{X}), n = 1, \cdots, N$, we can say that $\mathcal{X}$ is a $rank-(R_1, R_2, \cdots, R_N)$ tensor.

Tensor $\mathcal{X}$ in our example is $rank-(2, 2, 2)$.

HOSVD

The method *HOSVD* is convincing generalization of the matrix *SVD* and capable of computing the left singular vectors of $\mathcal{X}_{(n)}$. When $R_n < rank_n(\mathcal{X})$ for one or more n, the decompostion is called truncated *HOSVD*.

HOSVD():

Input: $\mathcal{X} \in \mathbb{R}^{I_1 \times I_2 \times \cdots I_N}$ **and** n-rank $(R_1, R_2, \cdots, R_N)$

Output: $\mathcal{G}, \boldsymbol{U}^{(\mathbf{1})}, \boldsymbol{U}^{(\mathbf{2})}, \cdots, \boldsymbol{U}^{(\boldsymbol{N})}$

for $n = 1, \cdots, N$ **do**

  $\boldsymbol{U}^{(\boldsymbol{n})} \leftarrow R_n$ Leading left singular vectors of $\mathcal{X}_{(n)}$

// End For

return $\mathcal{G} \leftarrow \mathcal{X} \times_1 \boldsymbol{U}^{(1)T} \times_2 \boldsymbol{U}^{(2)T} \cdots \times_N \boldsymbol{U}^{(N)T}$

For more information read this article.

Figure 1: This image is related to **Face Recognition** Exercise.