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تمرینات بخش ۴-۱

(1) $y'' - xy = 0 \quad , \quad y = \sum_{n=0}^{\infty} a_n x^n \longrightarrow y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \longrightarrow y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

جایگذاری در معادله $\longrightarrow \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0 \longrightarrow \sum_{n=-1}^{\infty} (n+3)(n+1) a_{n+3} x^{n+1} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$

$$\longrightarrow 2a_2 + \sum_{n=0}^{\infty} (n+3)(n+1) a_{n+3} x^{n+1} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$$\longrightarrow 2a_2 + \sum_{n=0}^{\infty} \left[(n+3)(n+2) a_{n+3} - a_n\right] x^{n+1} = 0 \longrightarrow a_2 = 0 \; , \; a_{n+3} = \frac{a_n}{(n+3)(n+2)}$$

$a_0 \; , \; a_3 = \frac{a_0}{6} \; , \; a_6 = \frac{a_3}{6\times 5} \; , \; a_9 = \frac{a_6}{9\times 8} = \frac{a_0}{9\times 8\times 6\times 5\times 3\times 2}$

$a_1 \; , \; a_4 = \frac{a_1}{4\times 3} \; , \; a_7 = \frac{a_4}{7\times 6} = \frac{a_1}{7\times 6\times 4\times 3} \; , \; a_{10} = \frac{a_7}{10\times 9} = \frac{a_1}{10\times 9\times 7\times 6\times 4\times 3}$

$a_2 = 0 = a_5 = a_8 = a_{11} = \dots$

$$y(x) = a_0\left(1 + \frac{x^3}{3\times 2} + \frac{x^6}{6\times 5\times 3\times 2} + \dots\right) + a_1\left(x + \frac{x^4}{4\times 3} + \frac{x^7}{7\times 6\times 4\times 3} + \dots\right) + \dots$$

$$y = a_0 y_1 + a_1 y_2 \longrightarrow y_1 = 1 + \frac{x^3}{3\times 2} + \frac{x^6}{6\times 5\times 3\times 2} + \dots = 1 + \sum_{n=1}^{\infty} \frac{x^{3n}}{(2\times 3)(5\times 6)\dots(3n-1)(3n)}$$

$$y_2 = x + \frac{x^4}{4\times 3} + \frac{x^7}{4\times 3\times 7\times 6} + \dots = x + \sum_{n=1}^{\infty} \frac{x^{3n+1}}{(3\times 4)(6\times 7)\dots(3n)(3n+1)}$$

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(9) $y' = 2xy \qquad x_0 = 0$

تمرینات بخش ۴-۲

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$$

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \dots$$

$$y' - 2xy = \left(a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \dots\right) - \left(2a_0 x + 2a_1 x^2 + 2a_2 x^3 + 2a_3 x^4 + \dots\right) \quad (1)$$

ادامه $y'-2xy = a_1 + (2a_2-2a_0)x + (3a_3-2a_1)x^2 + (4a_4-2a_2)x^3 + (5a_5-2a_3)x^4 + \cdots$

$$\rightarrow \begin{cases} a_1=0 \\ 2a_2-2a_0=0 \\ 3a_3-2a_1=0 \\ 4a_4-2a_2=0 \\ 5a_5-2a_3=0 \\ 6a_6-2a_4=0 \end{cases} \longrightarrow \begin{cases} a_1=0 \\ a_0=a_2 \\ a_3=a_1=0 \\ a_4=\frac{a_2}{2}=\frac{a_0}{2} \\ a_5=0 \\ a_6=\frac{a_4}{3}=\frac{a_0}{3!} \end{cases}$$

$a_1=0=a_3=a_5=\cdots$

$a_{2K}=\frac{a_0}{K!}$

$$y=a_0+a_2x^2+a_4x^4+\cdots=a_0+a_0x^2+\frac{a_0}{2!}x^4+\frac{a_0}{3!}x^6+\frac{a_0}{4!}x^8+\cdots$$

$$=a_0\left(1+x^2+\frac{x^4}{2!}+\frac{x^6}{3!}+\frac{x^8}{4!}+\cdots\right)=\boxed{a_0e^{x^2}}$$

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(2) $y'-y=x \quad , \quad y(0)=1$ $\qquad y=\sum_{n=0}^{\infty} a_nx^n = a_0+a_1x+a_2x^2+a_3x^3+\cdots$

$y'=a_1+2a_2x+3a_3x^2+\cdots$

$y'-y=x \rightarrow (a_1+2a_2x+3a_3x^2+\cdots)-(a_0+a_1x+a_2x^2+a_3x^3+\cdots)=x$

$(a_1-a_0)+(2a_2-a_1)x+(3a_3-a_2)x^2+(4a_4-a_3)x^3+\cdots=x$

$$\begin{cases} a_1-a_0=0 \\ 2a_2-a_1=1 \\ 3a_3-a_2=0 \\ 4a_4-a_3=0 \\ \vdots \end{cases} \longrightarrow \begin{cases} a_1=a_0 \\ a_2=\frac{a_1+1}{2} \\ a_3=\frac{a_2}{3}=\frac{a_1+1}{3!} \\ a_4=\frac{a_3}{4}=\frac{a_1+1}{4!} \end{cases}$$

$a_1=a_0$

$a_n=\frac{1+a_1}{n!}=\frac{1+a_0}{n!} \qquad n>1$

$$y=a_0+a_1x+a_2x^2+a_3x^3+\cdots=a_0+a_0x+\frac{1+a_0}{2}x^2+\frac{1+a_0}{3!}x^3+\cdots$$

$$=a_0+a_0x+(1+a_0)\left(\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)$$

$$y=a_0+a_0x+(1+a_0)(e^x-x-1)\xrightarrow{y(0)=1} a_0=1$$

$$y=1+x+2(e^x-x-1)=2e^x-x-1$$

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$$xy' - y = x\sin x \qquad y = \sum_{n=0}^{\infty} a_n x^n , \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

روش ۱

$$xy' - y = x\sin x \longrightarrow (a_1x + 2a_2x^2 + 3a_3x^3 + \cdots) - (a_0 + a_1 + a_2x^2 + a_3x^3) =$$

$$= x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \cdots \longrightarrow -a_0 + a_1x + a_2x^2 + 2a_3x^3 + 3a_4x^4 + \cdots = x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} \cdots$$

$$\rightarrow \begin{cases} a_0 = 0 \\ a_1 = 0 \longrightarrow a_1 = C \\ a_2 = 1 \\ a_3 = 0 \\ a_4 = -\frac{1}{3} \times \frac{1}{3!} \\ a_5 = 0 \\ a_6 = \frac{1}{5 \times 5!} \end{cases} \qquad a_0 = 0 \qquad a_{2n} = \frac{(-1)^{n+1}}{(2n-1)(2n-1)!} , \quad n \geq 2$$

$$y = Cx + x^2 - \frac{1}{3} \times \frac{1}{3!} x^4 + \frac{1}{5} \times \frac{1}{5!} x^6 - \cdots$$

روش ۲

$$xy' - y = x\sin x \longrightarrow y' - \frac{1}{x} y = \sin x$$

$$\mu(x) = e^{-\int \frac{1}{x} dx} = e^{-\ln x} = \frac{1}{x} \longrightarrow y(x) = x\left[\int \frac{\sin x}{x} dx + C\right]$$

$$\frac{y'}{x} - \frac{y}{x^2} = \frac{\sin x}{x} \qquad \frac{-1}{x^2} = \frac{d}{dx}\left(\frac{1}{x}\right)$$

$$\frac{\frac{dy}{dx}}{x} - \frac{d}{dx}\left(\frac{1}{x}\right) y = \frac{\sin x}{x} \rightarrow \frac{d}{dx}\left(\frac{y(x)}{x}\right) = \frac{\sin x}{x}$$

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$$y(x) = x\left[\int \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right) dx\right] + C$$

$$\boxed{y(x) = Cx + x\left[x - \frac{x^3}{3 \times 3!} + \frac{x^5}{5 \times 5!} - \cdots\right]}$$

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تمرینات بخش ۴-۳

(1) $(1+x^2)y'' - 4xy' + 6y = 0$

$$y = \sum_{n=0}^{\infty} a_n x^n \to y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \to y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$\to \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} n(n-1)a_n x^n - \sum_{n=0}^{\infty} 4n a_n x^n + \sum_{n=0}^{\infty} 6a_n x^n = 0$$

$$\to \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} (n^2 - 5n + 6) a_n x^n = 0$$

$$\to \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=2}^{\infty} (n-5)(n-4) a_{n-2} x^{n-2} = 0$$

$$0 + 0 + \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=2}^{\infty} (n-5)(n-4) a_{n-2} x^{n-2} = 0$$

$$\to \sum_{n=2}^{\infty} \left(n(n-1)a_n + (n-5)(n-4)a_{n-2}\right) x^{n-2} = 0 \to a_n = -\frac{(n-5)(n-4)a_{n-2}}{n(n-1)}$$

$n = 2, 3, \ldots$

$$\begin{cases} n=2 \to a_2 = -3a_0 \\ n=3 \to a_3 = -\frac{1}{3}a_1 \\ n=4 \to a_4 = 0 \\ n=5 \to a_5 = 0 \\ n=6 \to a_6 = 0 \end{cases}$$

$a_n = 0$

$n \geq 4$

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 = a_0 + a_1 x - 3a_0 x^2 - \frac{1}{3} a_1 x^3$$

$$= a_0(1 - 3x^2) + a_1\left(x - \frac{1}{3}x^3\right)$$

(3) $y'' + xy' + y = 0$

$$y = \sum_{n=0}^{\infty} a_n x^n \to y' = \sum_{n=0}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\to \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} (n+1)a_n x^n = 0 \to \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=2}^{\infty} (n-1) a_{n-2} x^{n-2} = 0$$

$$0 + 0 + \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=2}^{\infty} (n-1)a_{n-2} x^{n-2} = 0$$

$$\sum_{n=2}^{\infty} \left(n(n-1)a_n + (n-1)a_{n-2}\right) x^{n-2} = 0 \to a_n = \frac{-a_{n-2}}{n} \quad n = 2, 3, \ldots$$

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ادامه

$$a_2 = -\frac{a_0}{2} \qquad a_3 = \frac{-a_1}{3}$$

$$a_4 = \frac{-a_2}{4} = \frac{a_0}{4\times 2} \qquad a_5 = \frac{-a_3}{5} = \frac{a_1}{5\times 3}$$

$$a_6 = \frac{-a_0}{6\times 4\times 2} \qquad a_7 = \frac{-a_1}{7\times 5\times 3}$$

$$a_{2n} = \frac{a_0}{2^n n!}(-1)^n \qquad a_{2n+1} = \frac{a_1}{\frac{(2n+1)!}{2^n n!}} = \frac{2^n n!}{(2n+1)!} a_1 (-1)^n$$

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 = a_0 - \frac{a_1}{3}x - \frac{a_0}{2}x^2 + \frac{a_1 x^3}{5\times 3} + \frac{a_0}{4\times 2}x^4 + \cdots$$

$$y = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^n n!} + a_2 \sum_{n=0}^{\infty} \frac{(-1)^n 2^n n! \, x^{2n+1}}{(2n+1)!}$$

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(3) $(1-x^2)y'' - 6xy' - 4y = 0 \qquad y = \sum_{n=0}^{\infty} a_n x^n$

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1} \rightarrow y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

$$(1-x^2)\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} 6n a_n x^n - 4\sum_{n=0}^{\infty} 4 a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} n(n-1)a_n x^n - \sum_{n=0}^{\infty} 6n a_n x^n - \sum_{n=0}^{\infty} 4 a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} (n^2+5n+4) a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} (n+2)(n-1) a_{n-2} x^{n-2} = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} (n+2)(n-1) a_{n-2} x^{n-2} = 0$$

$$\sum_{n=2}^{\infty} \left(n(n-1)a_n - (n+2)(n-1)a_{n-2}\right) x^{n-2} = 0 \qquad \begin{cases} a_n = \frac{n+2}{n}(a_{n-2}) \\ n = 2, 3, \ldots \end{cases}$$

$$\begin{cases} a_2 = 2a_0 \\ a_4 = 3a_0 \\ a_6 = 4a_0 \end{cases} \qquad a_3 = \frac{5}{3}a_1 \quad a_5 = \frac{7}{3}a_1 \quad a_7 = \frac{9}{3}a_1 \qquad \begin{cases} a_{2n} = (n+1)a_0 \\ a_{2n+1} = \frac{2n+3}{3}a_1 \end{cases}$$

ادامه

$$y = a_0 + a_1 + a_2 x^2 + a_3 x^3 + \cdots$$

$$y = a_0 \sum_{n=1}^{\infty} (n+1) x^{2n} + a_1 \sum_{n=0}^{\infty} \frac{(2n+3)}{3} x^{2n+1}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad |x|<1 \longrightarrow \begin{cases} \dfrac{1}{1-x^2} = \sum\limits_{n=0}^{\infty} x^{2n} \\ \left(\dfrac{1}{1-2x}\right)' = \dfrac{2x}{(1-x^2)^2} = \sum\limits_{n=0}^{\infty} 2n x^{2n-1} \end{cases}$$

$$\left(\frac{1}{1-x}\right)' = \frac{1}{(1-x)^2} = \sum n x^{n-1}$$

$$y = a_0 \sum_{n=0}^{\infty} n x^{2n} + a_0 \sum_{n=0}^{\infty} x^{2n} + \frac{a_1}{3} \sum_{n=0}^{\infty} 2n x^{2n+1} + a_1 \sum_{n=0}^{\infty} x^{2n+1}$$

$$y = \frac{a_0}{2} x \sum_{n=0}^{\infty} 2n x^{2n-1} + a_0 \sum_{n=0}^{\infty} x^{2n} + \frac{a_1}{3} x^2 \sum_{n=0}^{\infty} 2n x^{2n-1} + a_1 x \sum_{n=0}^{\infty} x^{2n}$$

$$y = \frac{a_0 x}{2} \times \frac{2x}{(1-x^2)^2} + a_0 \times \frac{1}{1-x^2} + \frac{a_1}{3} x^2 \times \frac{2x}{(1-x^2)^2} + a_1 x \cdot \frac{1}{1-x^2}$$

$$y = a_0 \left(\frac{x^2}{(1-x^2)^2} + \frac{1}{1-x^2} \right) + a_1 \left(\frac{2x^3}{3} \cdot \frac{1}{(1-x^2)^2} + \frac{x}{1-x^2} \right)$$

$$y = a_0 \frac{1}{(1-x^2)^2} + a_1 \times \frac{3x - x^3}{3(1-x^2)^2}$$

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3) $(x^2 - 2x) y'' - 5(x-1) y' - 7y = 0 \quad , \quad y(1) = 1 \quad , \quad y'(1) = 2$

$$t = x - 1 \rightarrow y' = y'_t \rightarrow y'' = y''_t$$

$$(t+1)(t-1) y''_t - 5t\, y'_t - 7y = 0 \quad , \quad y(0) = 1 \quad , \quad y'(0) = 2$$

$$(t^2 - 1) \sum_{n=0}^{\infty} n(n-1) a_n t^{n-2} - \sum_{n=0}^{\infty} 5n\, a_n t^n - \sum_{n=0}^{\infty} 7 a_n t^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1) a_n t^n - \sum_{n=0}^{\infty} n(n-1) a_n t^{n-2} - \sum_{n=0}^{\infty} 5n\, a_n t^n - \sum_{n=0}^{\infty} 7 a_n t^n = 0$$

ادامه

$$\sum_{n=۰}^{\infty} (n^۲ - ۶n - ۷) a_n t^n - \sum_{n=۰}^{\infty} n(n-۱) a_n t^{n-۲} = ۰$$

$$\sum_{n=۲}^{\infty} (n-۹)(n-۱) a_{n-۲} t^{n-۲} - \sum_{n=۰}^{\infty} n(n-۱) a_n t^{n-۲} = ۰$$

$$\sum_{n=۲}^{\infty} \Big((n-۹)(n-۱) a_{n-۲} - n(n-۱) a_n \Big) t^{n-۲} = ۰ \quad , \quad a_n = \frac{n-۹}{n} a_{n-۲}$$

$$a_۲ = -\frac{۷}{۲} a_۰ , \quad a_۴ = -\frac{۵}{۴} a_۲ = \frac{۳۵}{۸} a_۰ , \quad a_۶ = -\frac{۳۵}{۱۶} a_۰$$

$$a_۳ = -۲a_۱ , \quad a_۵ = \frac{۸}{۵} a_۱ \ldots$$

$$y(t) = a_۰ + a_۱ t + a_۲ t^۲ + a_۳ t^۳ + \ldots$$

$$y(t) = a_۰ + a_۱ t - \frac{۷}{۲} a_۰ t^۲ - ۲a_۱ t^۳ + \frac{۳۵}{۸} a_۰ t^۴ + \ldots$$

$$y(t) = a_۰ + a_۱ (x-۱) - \frac{۷}{۲} a_۰ (x-۱)^۲ - ۲a_۱ (x-۱)^۳ + \frac{۳۵}{۸} a_۰ (x-۱)^۴ + \ldots$$

$$y(۱) = ۱ \longrightarrow a_۰ = ۱ \qquad y'(۱) = ۲ \longrightarrow a_۱ = ۲$$

$$y(x) = ۱ + ۲(x-۱) - \frac{۷}{۲}(x-۱)^۲ - ۴(x-۱)^۳ + \frac{۳۵}{۸}(x-۱)^۴ + \ldots$$