تمرینات بخش ۱-۲-۱

(۱)

$y' = 2xy^2 \Rightarrow \frac{dy}{dx} = 2xy^2 \Rightarrow \frac{dy}{y^2} = 2x\,dx \Rightarrow \frac{-1}{y} = x^2 + C$

$\Rightarrow y = \frac{-1}{x^2 + C} \Rightarrow y(0) = 1 \Rightarrow C = -1 \Rightarrow \boxed{y = \frac{1}{1 - x^2}}$

(۲)

(1) $y' + 3y = 8 \Rightarrow y' = 8 - 3y \Rightarrow \frac{dy}{dx} = 8 - 3y \Rightarrow \frac{dy}{8 - 3y} = dx$

$\Rightarrow x + C = \frac{1}{3}\ln(8 - 3y)$

$\Rightarrow -3x - 3C = \ln(8 - 3y) \Rightarrow e^{-3x + C'} = 8 - 3y \Rightarrow C_0 e^{-3x} - 8 = -3y$

$\Rightarrow y = \frac{8}{3} - ce^{-3x}$ (I) $(y(0)) = 2 \Rightarrow \frac{8}{3} - c = 2 \Rightarrow C = \frac{2}{3} \Rightarrow y = (8 - 2e^{-3x})/3$

(II) $y(0) = 4 \Rightarrow \frac{8}{3} - C = 4 \Rightarrow C = -4/3 \Rightarrow y = (8 + 4e^{-3x})/3$

(۴) $x^2 y y' = e^y \Rightarrow x^2 y\,dy = e^y \Rightarrow ye^{-y}dy = \frac{dx}{x^2} \Rightarrow \frac{1}{x} = (ye^{-y} + e^{y})$

$\Rightarrow 1 = -Cx + (1+y)x = e^{-y} \Rightarrow 1 + (x + (1+y)) = e^{-y} \Rightarrow \boxed{(1 + Cx)e^{y} = (1+y)x}$

(۳) $y' = 1 + yxe^{x-y} \Rightarrow e^{x-y} = u \Rightarrow u' = yxu + 1$

$\Rightarrow u' = (1 - y')e^{x-y} = (1 - y')u \Rightarrow u' = -6xu^2 \Rightarrow \frac{-du}{u^2} = 6x\,dx \Rightarrow$

$\frac{1}{u} = 3x^2 + C \Rightarrow \frac{1}{e^{x-y}} = 3x^2 + C \Rightarrow e^{y-x} = 3x^2 + C \Rightarrow \boxed{3x^2 - e^{y-x} = C}$

تمرینات بخش ۲-۳-۱

① M, N همگن

① $y\,dx - (x + \sqrt{xy})\,dy = 0$

$y = ux \Leftarrow \frac{y}{x} = u$

$ux\,dx - (x + \sqrt{ux^2})\,dy = 0 \Rightarrow u\,dx = (\sqrt{u} + 1)\,dy$

$\Rightarrow u\,dx = (\sqrt{u} + 1)(x\,du + u\,dx) \Rightarrow u\,dx = x\sqrt{u}\,du + x\,du$

$u\sqrt{u}\,dx + u\,dx \Rightarrow -u\sqrt{u}\,dx = x\sqrt{u}\,du + x\,du = (\sqrt{u} + 1)x\,du$

$\frac{dx}{x} = \frac{\sqrt{u} + 1}{u\sqrt{u}}\,du \Rightarrow \ln x + C = -\ln u + \frac{2}{\sqrt{u}} \Rightarrow \ln x + C_1 = \frac{x}{y} + 2\sqrt{\frac{x}{y}}$

$\Rightarrow \ln y + C = 2\sqrt{\frac{x}{y}} \Rightarrow \ln y + \ln|C| = 2\sqrt{\frac{x}{y}} \Rightarrow \ln Cy = 2\sqrt{\frac{x}{y}}$

② $y' = -4x + 3y + 15$

$x = X - 3 \Leftarrow (x_0, y_0) = (-3, -1)$ (نقطه برخورد معادلات خطوط صورت و مخرج (صفحه))

$y = Y - 1$

$y' = \frac{-4X - 12 + 3Y - 3 + 15}{2X - 6 + Y - 1 + 7} \Rightarrow y' = \frac{-4x - 3y}{2x + y} \Rightarrow \frac{dy}{dx} = \frac{-4 - 3u}{2 + u}$

$(-4 - 3u)\,dx = (2 + u)(x\,du + u\,dx) \Rightarrow (2 + u)\,x\,du = -(4 + u^2 + 5u)\,dx$

$\Rightarrow \frac{(2 + u)}{u^2 + 5u + 4}\,du = \frac{dx}{x}$

$$\Rightarrow \frac{1}{3} \ln \frac{1}{(u+1)(u+4)^2} = \ln cx$$

$$\Rightarrow cx^3 = \frac{1}{(\frac{y}{x}+1)(\frac{y}{x}+4)^2} \Rightarrow C(x+3)^3 = \frac{1}{(\frac{(y+1)}{x+3}+1)(\frac{y+1}{x+3}+4)^2}$$

$$\Rightarrow C = \frac{1}{(y+x+4)(y+4x+13)^2} \Rightarrow \boxed{C' = \frac{1}{C} = (y+x+4)(y+4x+13)^2}$$

3) $x\sin(\frac{y}{x})y' = y\sin(\frac{y}{x}) + x \xRightarrow{u = y/x}$

$$\sin u\,(u\,dx + x\,du) = (u\sin u + 1)dx \Rightarrow dx = x\sin u\,du$$

$$\frac{dx}{x} = \sin u\,du \Rightarrow \ln x + C = -\cos u \Rightarrow \boxed{\ln x + C = -\cos\frac{y}{x}}$$

$(x^2y^2 - 1)dy + 2xy^3dx = 0 \qquad y = u^\alpha$ 4)

$$(x^2u^{3\alpha-1} - u^{\alpha-1})\alpha\,du + 2xu^{3\alpha} = 0 \Rightarrow$$

$\alpha = -1 \Leftarrow 3\alpha + 1 = \alpha - 1$ برای همگن بودن لازم و کافی است که

$$\Rightarrow (x^2u^{-4} - u^{-2})\alpha\,du + 2xu^{-3}dx = 0 \Rightarrow (u^{-2} - x^2u^{-4})\alpha\,du = 2xu^{-3}du$$

$$\alpha(1 - \frac{x^2}{u^2})du = 2\frac{x}{u}dx \Rightarrow \frac{x}{u} > 0, \quad \frac{x}{u} = t$$

$$\Rightarrow (1+t)\,du = 2(u\,dt + t\,du) \Rightarrow (-2t^2 + t - 1)du = 2tu\,dt$$

$$\Rightarrow \ln(u) = \frac{-\ln((2t+1)(t-1)^2)}{3}$$

$$= C(u^2) = \frac{1}{\frac{|2x+u|}{|u|} \times \frac{(x-u)^2}{u^2}} \Rightarrow |2x+u|(x-u)^2 = cte$$

$$\Rightarrow \left|2x+\frac{1}{y}\right|\left(x-\frac{1}{y}\right)^2 = \boxed{\text{const} = C}$$

تمرینات بخش ۱-۳-۳ جزوه

① $(2xy - \tan y)dx + (x^2 - x\sec^2 y)dy = 0$

$$\frac{du}{dy} = x^2 - x\sec^2 y + A'(y) \Rightarrow (A')(y) = 0 \Rightarrow (A(y)) = C$$

$$\Rightarrow u = yx^2 - x\tan y + C_1$$

$$\boxed{yx^2 - x\tan y} = C_2 - C_1 = C \Leftarrow u = C_2 \text{ بعد }$$

② $(ye^{xy} - 2y^3)dx + (xe^{xy} - 6xy^2 - 2y)dy = 0$

$$\frac{\delta u}{\delta x} = ye^{xy} - 2y^3 \Rightarrow u(x,y) = \int (ye^{xy} - 2y^3)dx = e^{xy} - 2y^3x + A(y)$$

$$\Rightarrow A'(y) = -2y \Rightarrow A(y) = -y^2 + C_1$$

$$\boxed{e^{xy} - 2y^3x - y^2 = C}$$

بعد $Q = C_2$ بعد

③ $\left(\frac{xy}{\sqrt{1+x^2}} + 2xy - \frac{y}{x}\right)dx + \left(\sqrt{1+x^2} + x^2 - \ln x\right)dy$

عبارت دیفرانسیل کامل است

$$u = \int\left(\frac{xy}{\sqrt{1+x^2}} + 2xy - \frac{y}{x}\right)dx = y\sqrt{x^2+1} + yx^2 - y\ln x + A(y)$$

$\Rightarrow A(y) = C_1 \,,\, A'(y) = 0 \;|\; du = 0 \Rightarrow u = C_2 \Rightarrow$

$$y\sqrt{x^2+1} + yx^2 - y\ln x = C$$

۲

$$y' = \frac{x+3y-1}{-3x+2y+4} \Rightarrow \frac{dy}{dx} = \frac{x+3y-1}{-3x+2y+4}$$

$$\Rightarrow (x+3y-1)\,dx = (-3x+2y+4)\,dy \Rightarrow$$

$$(x+3y-1)\,dx + (-2y+3x-4)\,dy = 0$$

عبارت دیفرانسیل کامل است!

$$u = \int (x+3y-1)\,dx = \frac{1}{2}x^2 + 3yx - x + A(y)$$

$$\Rightarrow (A'(y)) = -4$$

$$\Rightarrow A(y) = -4y - y^2$$

$$\Rightarrow u = \boxed{\frac{1}{2}x^2 + 3yx - xy^2 - 4y = C}$$