به نام خدا

پاسخ تمرین سری سوم ریاضی مهندسی

دکتر طاهری

بهار ۱۴۰۲

تبدیل لاپلاس توابع زیر را به دست آورید.

سوال اول

تبدیل لاپلاس

a) $f(t)=4\sin(3t)+2\cos(9t) \overset{L}{\Rightarrow} F(s)=\left[4\times\frac{3}{s^2+(3)^2}\right]+\left[2\times\frac{s}{s^2+(9)^2}\right]$

$\Rightarrow F(s)=\frac{12}{s^2+9}+\frac{2s}{s^2+81}$

b) $f(t)=3t^2+5t+2 \overset{L}{\Rightarrow} F(s)=3\times\frac{2}{s^3}+5\times\frac{1}{s^2}+2\times\frac{1}{s}$ $\qquad t^n \xrightarrow{L} \frac{n!}{s^{n+1}}$

c) $f(t)=\delta(t-2)\cos(t) \overset{L}{\Rightarrow} F(s)=L\{\cos(2)\ \delta(t-2)\}=\cos(2)\,e^{-2s}$

d) $f(t)=t^2e^{-t}\cos(t) \xrightarrow{\text{نکته}}$ ① $f(t)\,e^{kt} \overset{L}{\Rightarrow} F(s-k)$, ② $t^n f(t)=(-1)^n\frac{d^nF(s)}{ds^n}$

$h(t)=t^2\cos(t) \xrightarrow{L} H(s)=(-1)^2\frac{d^2}{ds^2}\left(\frac{s}{s^2+1}\right) \Rightarrow \frac{d}{ds}\left(\frac{s}{s^2+1}\right)=\frac{s^2+1-2s^2}{(s^2+1)^2}=\frac{1-s^2}{(1+s^2)^2}$

$\Rightarrow \frac{d}{ds}\left(\frac{1-s^2}{(1+s^2)^2}\right)=\frac{-2s(1+s^2)^2-(1-s^2)2(1+s^2)2s}{(1+s^2)^{\cancel{4}3}}$

$=\frac{-2s(1+s^2)-(1-s^2)4s}{(1+s^2)^3}=\frac{-2s-2s^3-4s+4s^3}{(1+s^2)^3}=\frac{2s^3-6s}{(1+s^2)^3}$

$\Rightarrow H(s)=\frac{2s^3-6s}{(1+s^2)^3} \Rightarrow F(s)=\frac{2(s+1)^3-6(s+1)}{(1+(s+1)^2)^3}$

e) $f(t)=\frac{\sin(4t)}{t} \Rightarrow$ نکته: $g(t)=\frac{\sin(t)}{t} \Rightarrow f(t)=4\,g(4t)$

$L\left\{\frac{f(t)}{t}\right\}=\int_s^\infty F(u)du \longleftrightarrow$ دقیقا برعکس $t^n f(t)$ که از $F(s)$ مشتق گرفته می‌شد. $\Rightarrow \int_s^\infty\frac{1}{s^2+1}ds=\tan^{-1}(s)\Big|_s^\infty$

$=\frac{\pi}{2}-\tan^{-1}(s)=\cot^{-1}(s)=\tan^{-1}\left(\frac{1}{s}\right)$

$\Rightarrow f(t)=4g(4t)=4\times\frac{1}{4}\left(\frac{\pi}{2}-\tan^{-1}\left(\frac{s}{4}\right)\right)=\frac{\pi}{2}-\tan^{-1}\left(\frac{s}{4}\right)=\cot^{-1}\left(\frac{s}{4}\right)=\tan^{-1}\left(\frac{4}{s}\right)$

f) $f(t)=\sin(at)-at\cos(at) \Rightarrow F(s)=\frac{a}{s^2+a^2}-a\frac{d}{ds}\left[\frac{s}{s^2+a^2}\right](-1)$

$\Rightarrow F(s)=\frac{a}{s^2+a^2}+a\frac{s^2+a^2-2s^2}{(s^2+a^2)^2}=\frac{a}{s^2+a^2}+a\frac{(a^2-s^2)}{(s^2+a^2)^2}=\frac{2a^3}{(s^2+a^2)^2}$

تبدیل لاپلاس معکوس توابع زیر را حساب کنید.

عکس تبدیل لاپلاس

a) $F(s) = \dfrac{s-1}{s^2-2s+\underbrace{5}_{1+4}} = \dfrac{s-1}{(s-1)^2+2^2} \xrightarrow{L^{-1}} e^{t}\cos(2t)$

b) $F(s) = \dfrac{1+e^{-s}}{s^2} = \dfrac{1}{s^2} + \dfrac{e^{-s}}{s^2} \xrightarrow{L^{-1}} t - (t-1)\,u_1(t)$

c) $F(s) = \dfrac{(s+2)(s+4)}{s(s+1)(s+3)} = \dfrac{A}{s} + \dfrac{B}{s+1} + \dfrac{C}{s+3} \Longrightarrow A = \dfrac{8}{3}$

$\Rightarrow B = -\dfrac{3}{2}$

$\Rightarrow C = -\dfrac{1}{6}$

$\xrightarrow{L^{-1}} \dfrac{8}{3} - \dfrac{3}{2}e^{-t} - \dfrac{1}{6}e^{-3t}$

d) $F(s) = \dfrac{s+3}{(s+1)^2(s+2)} = \dfrac{A}{(s+1)} + \dfrac{B}{(s+1)^2} + \dfrac{C}{(s+2)} \Rightarrow B = 2$

$\Rightarrow C = 1$

$\Rightarrow A = -1$

$\xrightarrow{L^{-1}} f(t) = -e^{-t} + 2te^{-t} + e^{-2t}$

تجزیه کسر

e) $F(s) = \dfrac{s^2-2}{(s^2+3)^2} = \dfrac{A}{(s^2+3)} + \dfrac{B}{(s^2+3)^2} \Rightarrow B = -5$

$\Rightarrow A \Rightarrow \dfrac{s^2-2}{(s^2+3)^2} + \dfrac{5}{(s^2+3)^2} = \dfrac{s^2+3}{(s^2+3)^2} = \dfrac{1}{s^2+3}$

$\Rightarrow A = 1$

$\Rightarrow F(s) = \dfrac{1}{(s^2+3)} - \dfrac{5}{(s^2+3)^2} \xrightarrow{L^{-1}} f(t) = \dfrac{1}{\sqrt{3}}\sin(\sqrt{3}t)$

$$\sin(at) - at\cos(at) \xrightarrow{L} \frac{2a^3}{(s^2+a^2)^2} \longleftarrow$$ لال لسؤال f البحث لتوصر : e حل الماسن

$$\Rightarrow \frac{-5}{(s^2+3)^2} \xrightarrow[L^{-1}]{a=\sqrt{3}} \frac{-5}{2\cdot 3\sqrt{3}}\left(\sin(\sqrt{3}t) - \sqrt{3}\,t\cos(\sqrt{3}t)\right)$$

$$\Rightarrow f(t) = \frac{1}{\sqrt{3}}\sin(\sqrt{3}t) - \frac{5}{6\sqrt{3}}\left(\sin\sqrt{3}t - \sqrt{3}t\cos\sqrt{3}t\right)$$

f) $$F(s) = \frac{s+1}{(s^2+s+1)(s+2)} = \frac{As+B}{s^2+s+1} + \frac{C}{s+2} \Rightarrow C = \frac{-1}{4-2+1} = -\frac{1}{3}$$

$$\frac{s+1}{(s^2+s+1)(s+2)} + \frac{1/3}{(s+2)} = \frac{(s+1)+\frac{1}{3}(s^2+s+1)}{(s^2+s+1)(s+2)} = \frac{s+1+\frac{1}{3}s^2+\frac{1}{3}s+\frac{1}{3}}{''}$$

$$= \frac{\frac{1}{3}(s^2+4s+4)}{''} = \frac{\frac{1}{3}(s+2)^2}{''} = \frac{1}{3}\,\frac{(s+2)}{(s^2+s+1)} \Rightarrow A = \frac{1}{3}$$

$$\Rightarrow B = \frac{2}{3}$$

$$\Rightarrow F(s) = \frac{1}{3}\left(\frac{s+2}{\underbrace{s^2+s+1}_{(s+\frac{1}{2})^2+\frac{3}{4}}} - \frac{1}{s+2}\right) = \frac{1}{3}\left(\frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2+\frac{3}{4}} + \frac{\frac{3}{2}}{(s+\frac{1}{2})^2+\frac{3}{4}} - \frac{1}{s+2}\right)$$

$$\xrightarrow{L^{-1}} \frac{1}{3}\left(e^{-\frac{t}{2}}\cos\left(\sqrt{\tfrac{3}{4}}t\right) + \sqrt{3}\,e^{-\frac{t}{2}}\sin\left(\sqrt{\tfrac{3}{4}}t\right) - e^{-2t}\right)$$

a) $\ddot{y}+2\dot{y}+y=\delta(t-2)\ ,\ y(0)=0\ ,\ \dot{y}(0)=0$

$$\xrightarrow{L} s^2Y(s)+2sY(s)+Y(s)=e^{-2s} \Longrightarrow Y(s)=\frac{e^{-2s}}{s^2+2s+1}=\frac{e^{-2s}}{(s+1)^2}\xrightarrow{L^{-1}}$$

$$e^{-\alpha s}F(s)\xrightarrow{L^{-1}} f(t-\alpha)u(t-\alpha) \longrightarrow F(s)=\frac{1}{(s+1)^2}\xrightarrow{L^{-1}} te^{-t}$$

$$\Longrightarrow e^{-\alpha s}F(s)=\frac{e^{-2s}}{(s+1)^2}\xrightarrow{L^{-1}}(t-2)e^{-(t-2)}u(t-2)$$

b) $\ddot{y}+4y=\sin(t)-u_{2\pi}(t)\sin(t-2\pi)\ ,\ y(0)=0\ ,\ \dot{y}(0)=0 \xrightarrow{L}$

$$s^2Y(s)+4Y(s)=\frac{1}{s^2+1}-\frac{e^{-2\pi s}}{s^2+1}=\frac{1-e^{-2\pi s}}{s^2+1}\Longrightarrow Y(s)=\frac{1-e^{-2\pi s}}{(s^2+1)(s^2+4)}$$

$$\Longrightarrow Y(s)=\Bigg[\underbrace{\frac{1/3}{s^2+1}-\frac{1/3}{s^2+4}}_{G(s)}\Bigg]-e^{-2\pi s}\left[\frac{\frac{1}{3}}{s^2+1}-\frac{1/3}{s^2+4}\right]=\frac{1}{3}\left[G(s)-e^{-2\pi s}G(s)\right]$$

ادامه b:

$$\Longrightarrow y(t)=\frac{1}{3}\left[g(t)-g(t-2\pi)u(t-2\pi)\right]$$

$$g(t)=\sin(t)+\frac{1}{2}\sin(2t)$$

c) $\ddot{y}+5\dot{y}+6y=te^{-t}\ ,\ y(0)=1\ ,\ \dot{y}(0)=0 \xrightarrow{L}$ $f'(t)\xrightarrow{L} sF(s)-f(0)$

$f''(t)\xrightarrow{L} s^2F(s)-sf(0)-f'(0)$

$$\xrightarrow{L} s^2Y(s)-s+5Y(s)-5+6Y(s)=\frac{1}{(s+1)^2}$$

$$\Longrightarrow Y(s)(s^2+5+6)=\frac{1}{(s+1)^2}+s+1 \Longrightarrow Y(s)=\frac{1}{(s+1)^2(s^2+5s+6)}+\frac{s+1}{s^2+5s+6}$$

$$\xrightarrow{\text{تجزیه کسر}} Y(s)=\frac{-3/4}{s+1}+\frac{1/2}{(s+1)^2}+\frac{4}{s+2}+\frac{-\frac{9}{4}}{s+3}$$

$$y(t)=-\frac{3}{4}e^{-t}+\frac{1}{2}e^{-t}t+4e^{-2t}-\frac{9}{4}e^{-3t}$$

برای تابع تبدیل زیر معادله ی دیفرانسیل معادل با آن را بنویسید.

$$\frac{X(s)}{F(s)} = \frac{s+3}{s^3+11s^2+12s+18}, \Rightarrow (s^3+11s^2+12s+18)\,X(s) = (s+3)\,F(s)$$

$$\rightarrow x^{(3)}(t) + 11\,x^{(2)}(t) + 12\,\dot{x}(t) + 18\,x(t) = \dot{f}(t) + 3\,f(t)$$