(1)

تمرین ۲-۳ (جزوه)

1) $f(x) = \begin{cases} [\sin(\frac{1}{x})] & x \neq 0 \\ 0 & x = 0 \end{cases}$

$x_0 < x_1 < x_2 < \cdots < x_n \quad ; \quad I_i = [x_{i-1}, x_i] \quad ; \quad L(I_i) = x_i - x_{i-1} = \Delta x$

$\forall c_i \in I_i \quad m_i = -1 \quad M_i = 1 \rightarrow L(p,f) = \sum_{i=1}^{n} m_i \Delta x = -\Delta x$

$\neq$

$U(p,f) = \sum_{i=1}^{n} M_i \Delta x = \Delta x$

در یکی از ادامه‌ها تابع ما دارای $S(p,f)$ نبود پس تابع ما انتگرال پذیر نیست

2) $g(x) = \begin{cases} \frac{1}{x} - [\frac{1}{x}] & x \neq 0 \\ 0 & x = 0 \end{cases}$

$x_0 < x_1 < \cdots < x_n \quad ; \quad I_i = [x_{i-1}, x_i] \quad ; \quad L(I_i) = x_i - x_{i-1} = \Delta x$

$\forall c_i \in I_i \quad m_i = 0 \quad M_i = 1 \quad U(p,f) = \lim_{n \to \infty} \frac{b-a}{n} \sum_{i=1}^{n} f(a + i\frac{b-a}{n}) \xrightarrow{a = m_i} = f(1) = 0$

$L(p,f) = f(0) = 0$

$\Rightarrow L(p,f) = U(p,f) \Rightarrow$ تابع انتگرالپذیر است

(2) $\int_a^b x^2 dx = \frac{b^3 - a^3}{3}$

$\int_a^b f(x) dx = \lim_{\|p\| \to 0} \sum_{i=1}^{n} f(c_i) \Delta x_i = \lim_{n \to \infty} \frac{b-a}{n} \sum_{i=1}^{n} f(a + i\frac{b-a}{n})$

$= \frac{b-a}{n} \sum_{i=1}^{n} (a + i\frac{b-a}{n})^2 = \sum a^2 + \frac{i^2}{n^2}(b-a)^2 = \sum_{i=1}^{n} a^2 + \frac{(b-a)^2}{n^2} \sum i^2 \sum_{i=1}^{n} 2ai\frac{b-a}{n}$

$\frac{b-a}{\cancel{n}} \times \frac{\cancel{n}(n+1)(2n+1)}{6n^2} \times a^2 + b^2 + ab = \boxed{\frac{b^3 - a^3}{3}}$

تمرین بخش ۳_۲ جزوه

(۱۳) $\int_1^3 f(x)\,dx = 8$

$$\int_1^3 f(x)\,dx = \lim_{\|P\|\to 0}\sum_{i=1}^{n} f(c_i)\Delta x_i = \lim_{n\to\infty}\frac{3-1}{n}\sum_{i=1}^{n} f\left(1+i\,\frac{3-1}{n}\right)$$

$$\frac{2}{n}\sum f\left(1+\frac{2i}{n}\right) = 8 \to 4n = \sum_{i=1}^{n} f\left(1+\frac{2i}{n}\right)$$

می‌دانیم $\exists c\in(a,b),\ \int_a^b f(x)\,dx = f(c)(b-a)$

$$\Rightarrow 8 = f(c)(2) \to \boxed{f(c)=4}\ ✓$$

تمرین بخش ۳_۲ المعارف بخش ۴_۲

(۲۴) $$\int_0^2 (4x-x^3)\,dx = \lim_{n\to\infty}\frac{2-0}{n}\sum_{i=1}^{n} f\left(0+i\,\frac{2-0}{n}\right)$$

$$=\frac{2}{n}\sum_{i=1}^{n} 4\times\frac{2i}{n}-\left(\frac{2i}{n}\right)^3 = \frac{2}{n}\times\left(4\times\frac{(1+n)}{\cancel{n}}\times\cancel{n}-\frac{(4n(2n+1))^2}{4n^3}\right)$$

$$=\frac{8}{\cancel{n}}+8-\frac{2}{n}\left(\frac{16n^4+4n^2+16n^3}{4n^4}\right) = 8-8=\boxed{0}$$

(۲۶) $$\int_\pi^{2\pi}(x-2\sin x)\,dx = \lim \frac{2\pi-\pi}{n}\sum_{i=1}^{n} f\left(\pi+i\,\frac{2\pi-\pi}{n}\right)$$

$$\frac{\pi}{n}\sum\left(\pi+\frac{\pi}{n}-2\sin\left(\pi+\frac{\pi}{n}\right)\right)=$$

$$\frac{\pi}{n}\sum\left(\pi+\frac{\pi i}{n}+2\sin\left(\frac{\pi i}{n}\right)\right) = \frac{\pi^2}{\cancel{n}}+\frac{\pi^2}{n^2}\frac{n(n+1)}{2}+\cancel{\frac{\pi}{n}\ldots} = \boxed{\frac{\pi^2}{2}}$$

parsnote

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بخش ۴-۳ جزوه ۱۰- مسئلات ۳-۴ ، ۴-۴

۱۶) $g(x) = \int_1^{\sqrt{x}} \frac{z^2}{z^4+1}\, dz$

$$g'(x) = \frac{(\sqrt{x})^2}{(\sqrt{x})^4+1} = \boxed{\frac{x}{x^2+1}}$$

۲۸) $\int_0^4 (4-t)\sqrt{t}\, dt \rightarrow \frac{4}{n}\sum_{i=1}^{n} f\left(a+\frac{b-a}{n}i\right)$

$$\cancel{\frac{4}{n}\sum_{i=1}^{n}\left(4-\frac{4}{n}i\right)\sqrt{\frac{4}{n}i} = \frac{4}{n}\left(4-4\times\frac{n(n+1)}{2n}\right)\sqrt{\frac{4}{n}\frac{n(n+1)}{2}}}$$

$$\cancel{\frac{16}{n} - \frac{n(n+1)}{1}\sqrt{2n+2} = \frac{4}{n}(4-2n+2)\sqrt{2n+2}}$$

$$\int_0^4 (4-t)\sqrt{t}\, dt = \int_0^4 (4-t)\, dt \int_0^4 \sqrt{t}\, dt$$

$$= \left(4t-\frac{1}{2}t^2\right)\frac{2}{3}t^{3/2} = (16-8)\,\frac{2}{3}\times\sqrt{4^3} = \boxed{\frac{128}{3}}$$

۳۸) $\int_{-2}^{2} f(x)\, dx \qquad f(x) = \begin{cases} 2 & -2\le x\le 0 \\ 4-x^2 & 0< x\le 2\end{cases}$

$$\int_{-2}^{0} 2\, dx + \int_0^2 (4-x^2)\, dx = 4+8-\frac{8}{3} = \frac{36}{3}-\frac{8}{3} = \boxed{\frac{28}{3}}$$

$(4x-\frac{1}{3}x^3)$

۵۶) $f(x) = \int_0^{\sin x}\sqrt{1+t^2}\, dt \qquad g(y) = \int_3^{y} f(x)\, dx$

$$g'(y) = y'\times f(y) - 0 = y' f(y) \rightarrow g''(y) = f'(y)$$

$$f'(y) = \cos y\times\sqrt{1+\sin^2 y} = \cos\frac{\pi}{6}\times\sqrt{1+\frac{1}{4}} = \frac{\sqrt{5}}{2}\times\frac{\sqrt{3}}{2} = \frac{\sqrt{15}}{4}$$

parsnote

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۲۴) $6+\int_a^x \frac{f(t)}{t^2}\,dt = 2\sqrt{x}$

$\rightarrow \frac{f(t)}{t^2} = 2\times\frac{1}{2}\times\frac{1}{\sqrt{t}} \Rightarrow f(x) = \frac{x^2}{\sqrt{x}} = \frac{x^2\sqrt{x}}{x} = \boxed{x\sqrt{x}}$

$6+\int_a^x \frac{x\sqrt{x}}{x^2}\,dx = 6+\int_a^x x^{-1/2}\,dx = 6+2x^{1/2}-2a^{1/2} = 2x^{1/2}$ $\quad \frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}}$

$\boxed{6 = 2a^{1/2} \Rightarrow 3 = \sqrt{a} \Rightarrow a = 9}$

آزمون جزوه ۴-۳

1) $\int_0^{\ln(n)} [e^x]\,dx = n\ln(n) - \ln(n!)$

$\ln(1)+\ln(2)+\dots+\ln(n)$

$\int [e^x]\,dx = 1(\ln 2 - \cancel{\ln 1}) + 2(\cancel{\ln 3} - \cancel{\ln 2}) + \dots + (n-1)(\ln(n))$

$\int_0^{\ln(n)} e^x\,dx \geq \int_0^{\ln(n)} [e^x]\,dx \Rightarrow n-1 \geq n\ln(n) - \ln(n!)$

$$\boxed{e^{n-1} \geq \frac{e^{n\ln(n)}}{e^{\ln(n!)}} \Rightarrow e^{n-1} \geq \frac{n^n}{n!} \Rightarrow n! \geq \frac{n^n}{e^{n-1}} \Rightarrow n! \geq n^n e^{1-n}}$$

تمرین از تست ۴-۴

۲۵) $\int_0^{\pi} (4\sin\theta - 2\cos\theta)\,d\theta$

$-4\cos\theta - 2\sin\theta \Big|_0^{\pi} \rightarrow +4+4 = \boxed{8}$

(۳)

ادامه تمرین ۳-۶

$$\left.\begin{array}{l} f'(x) = g'(x) \times \dfrac{1}{\sqrt{1+g^2(x)}} - 0 \\ g'(x) = -\sin x \times (1+\sin^2\cos x) \end{array}\right\} \Rightarrow$$

$$f'(x) = \frac{-\sin x(1+\sin^2(\cos x))}{\sqrt{1+g^2(x)}} \xrightarrow{x \to \pi/2}$$

$$\frac{-1 \times (1+\sin^2(0))}{\sqrt{1+g^2(x)}} = \frac{-1}{\sqrt{1+g^2(\frac{\pi}{2})}} = \frac{-1}{1} = \boxed{-1}$$

$$g(\frac{\pi}{2}) = \int_0^{\cos\frac{\pi}{2}} (1+\sin^2 t)\,dt = 0$$

تمرین های تمرین ۳-۷

(۱)
$$A = \lim_{n\to\infty} \left(\frac{1}{\sqrt{n^4+3(1\times n)^2}} + \frac{2}{\sqrt{n^4+3(2n)^2}} \cdots\right)$$

$$A = \frac{1}{n} \lim \sum \left(\frac{n i}{\sqrt{n^4+3(n i)^2}}\right) = \frac{1}{n} \times \frac{n^2 i}{n^2\sqrt{n^2+3i^2}} = \frac{1}{n} \times \frac{i}{\sqrt{n^2+3i^2}}$$

$$= \frac{1}{n} \sum \frac{i}{n} \times \frac{1}{\sqrt{1+3(\frac{i}{n})^2}} = \frac{1}{n} f(\frac{i}{n}) \xrightarrow{f = \frac{x}{\sqrt{1+3x^2}}} \int_0^1 \frac{x}{\sqrt{1+3x^2}}\,dx = \boxed{\frac{1}{3}}$$

(۲)
$$\lim_{n\to\infty} \left(\frac{n^2}{n^4+1^4} + \frac{2n^2}{n^4+2^4} + \cdots + \frac{(n-1)n^2}{n^4+(n-1)^4}\right) = \int_0^1 \frac{x}{1+x^4}\,dx$$

$$\frac{1}{n} \sum_{i=1}^{n-1} \frac{n^3 \times i}{n^4+i^4} = \frac{i/n}{1+i^4/n^4} \longrightarrow f(x) = \frac{x}{1+x^4} \rightarrow \boxed{\int_0^1 \frac{x}{1+x^4}\,dx}$$

$$C = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{n^2+k^2x^2} = \int_0^1 \frac{1}{1+y^2x^2}\,dy$$

$$\frac{1}{n} \times \frac{n^2}{n^2+k^2x^2} \rightarrow \frac{1}{n} \times \frac{1}{1+\frac{k^2x^2}{n^2}} \Rightarrow f(x,k) = \frac{1}{1+k^2x^2}$$

$$\int_0^1 \frac{1}{1+y^2x^2}\,dy$$

parsnote

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(۳۸) $\int_1^2 (x+\frac{1}{x})^2 dx = \int_1^2 (x^2+2+\frac{1}{x^2})\,dx =$

$\frac{1}{3}x^3+2x-\frac{1}{x}\Big|_1^2 \rightarrow \frac{1}{3}\times 8+4-\frac{1}{2}-(\frac{1}{3}+2-1)$

$\frac{8}{3}-\frac{4}{3}+4-\frac{1}{2} = \frac{4}{3}+\frac{7}{2} = \boxed{\frac{29}{6}}$

(۴۰) $\int_0^2 |2x-1|\,dx \Rightarrow \frac{2-0}{n}\sum_{i=1}^{n}\left|2\times\frac{2}{n}i-1\right|$

$\frac{2}{n}\times\sum\left|\frac{4i}{n}-1\right| = \begin{cases} i\geq\frac{n}{4} & \frac{2}{n}\sum_{\frac{n}{4}}^{n}(\frac{4i}{n}-1) \\ i<\frac{n}{4} & \frac{2}{n}\sum^{\frac{n}{4}}(1-\frac{4i}{n}) \end{cases}$

$= \frac{2}{n}\times\left(\cancel{4}\times\frac{\frac{n}{4}+n}{2n}\times\frac{3n}{\cancel{4}}-\frac{3n}{4}\right)+\frac{2}{n}\times\left(\frac{3n^2}{4n}+\frac{3n^2}{2n}-\frac{3n}{4}\right)$

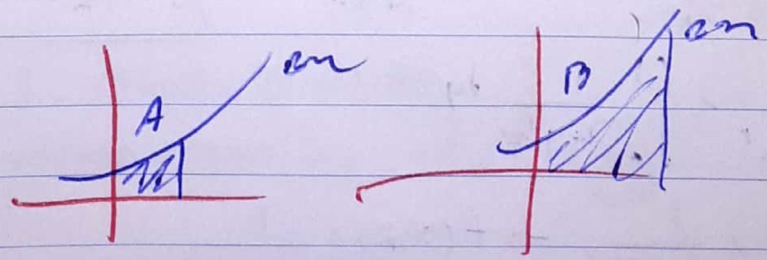
$= \frac{2}{n}\times\left(\frac{n}{4}-\frac{4}{n}\,\frac{\frac{n}{4}(\frac{n}{4}+1)}{2}\right)+\frac{2}{n}\times\left(\frac{n}{4}-\frac{n^2+4n}{8n}\right)$

$= \frac{3}{4}+3-\frac{3}{2}+\frac{1}{2}-\frac{1}{4}-0 = \boxed{2/5}$

(۴۸) $\int_{-10}^{10}\frac{2e^x}{\sinh x+\cosh x}dx = \frac{2e^x}{e^x} = 2$

$\int_{-10}^{10} 2\,dx = 2x\Big|_{-10}^{10} = +20-(-20) = 40$

(۷۲)



$B = 3A$

$3\int_0^a e^x dx = \int_0^b e^x dx \rightarrow 3e^a-3 = e^b-1$

$3e^a = e^b+2 \Rightarrow \boxed{b = \ln(3e^a-2)}$

تمرینات بخش ۴-۵ جزوه

1) الف) $2 \le \int_{-1}^{1} \sqrt{1+x^4}\,dx \le 2\sqrt{2}$

$m(b-a) \le \int_{-1}^{1} \sqrt{1+x^4}\,dx \le M(b-a) \Rightarrow \begin{matrix} m=1 \\ M=\sqrt{2} \end{matrix} \Rightarrow$

$\boxed{2 \le \int_{-1}^{1} \sqrt{1+x^4}\,dx \le 2\sqrt{2}}$

ب) $\frac{\pi}{12} \le \int_{\pi/4}^{\pi/3} \tan x\,dx \le \frac{\pi\sqrt{3}}{12} \rightarrow b-a \le \int_{\pi/4}^{\pi/3} \tan x\,dx \le \sqrt{3}(b-a)$

$\Rightarrow \frac{\pi}{12} \le \int_{\pi/4}^{\pi/3} \tan x\,dx \le \frac{\pi\sqrt{3}}{12}$

ج) $\frac{\sqrt{3}}{8} \le \int_{\pi/4}^{\pi/3} \frac{\sin x}{x}\,dx \le \frac{\sqrt{2}}{2}$ $\quad m = \frac{\sqrt{2}/2}{\pi/4}$ $\quad M = \frac{\sqrt{3}/2}{\pi/3}$

$m: \frac{2\sqrt{2}}{\pi} \times \left(\frac{\pi}{3} - \frac{\pi}{4}\right) \le \int_{\pi/4}^{\pi/3} \frac{\sin x}{x}\,dx \le \frac{3\sqrt{3}}{2\pi} \times \left(\frac{\pi}{3} - \frac{\pi}{4}\right)$ $\quad \frac{\pi}{12}$

$\boxed{\frac{\sqrt{3}}{8} \le \int_{\pi/4}^{\pi/3} \frac{\sin x}{x}\,dx \le \frac{\sqrt{2}}{2}}$

2) $I = \int_0^1 \sqrt{1+x^3}\,dx$ $\quad 1 \le f(c) \le \sqrt{2}$

$I = \int_0^1 \sqrt{x^3+1} = f(c)(1-0) = \sqrt{1+c^3}(1-0) = \sqrt{1+c^3}$

$\Rightarrow \boxed{1 \le I \le \sqrt{2}}$ $\quad x \ge -1 \Rightarrow 1 \le \sqrt{1+x^3} \le 1 + \frac{x^3}{2} \xrightarrow{\int_0^1}$

$1 \le I \le x + \frac{1}{5\times 2} x^5 \Big|_0^1 \Rightarrow \boxed{1 \le I \le 1.1}$ $\quad x + \frac{1}{4}x^4 \Big|_0^1 = \frac{5}{4}$

$\left(\int_0^1 \sqrt{1+x^3} \times \frac{1}{9}\,dx\right)^2 \le \left(\int_0^1 (1+x^3)\,dx\right)\left(\int_0^1 1\,dx\right) = 1.25 \Rightarrow |I| \le \sqrt{1.25}$ $\quad 1.11$

parsnote

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$$(1-0)\,f(0) < \int_0^1 f(x)\,dx < (1-0)\,\frac{f(0)+f(1)}{2}$$

$$\frac{f(0)}{1} < \int_0^1 f(x)\,dx < 1.1207 \Rightarrow$$

$$\boxed{1 < I < 1.1}$$

تمرینات بخش ۳-۶

(1) $f(x) = \frac{1}{x}\int_1^x f(t)\,dt + 1 \quad x>0$

$$f(x) - 1 = \frac{1}{x}\int_1^x f(t)\,dt \Rightarrow x f(x) - x = \int_1^x f(t)\,dt$$

$$f'(x)\,x + \cancel{f(x)} - 1 = \cancel{f(x)} \Rightarrow f'(x) = \frac{1}{x} \Rightarrow$$

$$\left.\begin{array}{l} f(x) = \ln x + C \\ f(1) = 1 \end{array}\right\} \Rightarrow C = 1 \Rightarrow \boxed{f(x) = \ln x + 1}$$

(2) $\int_x^{1/x} \frac{\ln t}{1+t^2}\,dt \Rightarrow \left(\frac{1}{x}\right)'\left(\frac{\ln \frac{1}{x}}{1+1/x^2}\right) - \frac{\ln x}{1+x^2}$

$$-\frac{1}{x^2} \times \frac{-\ln x \cdot x^2}{x^2+1} - \frac{\ln x}{1+x^2} = \frac{\ln x}{x^2+1} - \frac{\ln x}{x^2+1} = 0 = f'(x)$$

$$f(1) = 0 \Rightarrow f(x) = 0$$

parsnote

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