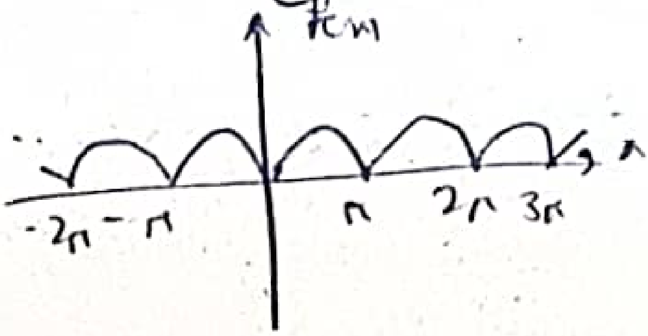
محمد اماتلو ۸۱۰۱۰۰۰۸۴ تکلیف سری اول ریاضی مهندسی

(۱) این تابع زوج است پس $b_n=0$ داریم

$L=\pi$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi}{L}x$$



$$a_0=\frac{2}{L}\int_0^L f(x)\,dx=\frac{2}{\pi}\int_0^{\pi}\sin x\,dx=-\frac{2}{\pi}\cos x\Big]_0^{\pi}$$
$$=\frac{4}{\pi}$$

$$a_n=\frac{2}{L}\int_0^L f(x)\cos\frac{n\pi}{L}x\,dx=\frac{2}{\pi}\int_0^{\pi}\sin x\cos nx\,dx$$

$$=\frac{1}{\pi}\int_0^{\pi}\left[\sin(1+n)x+\sin(1-n)x\right]dx$$

$$=\frac{1}{\pi}\left[-\frac{1}{1+n}\cos(1+n)x-\frac{1}{1-n}\cos(1-n)x\right]_0^{\pi}$$

$$=\frac{1}{\pi}\left[\frac{1}{1+n}\cos n\pi+\frac{1}{1-n}\cos n\pi+\frac{1}{1+n}+\frac{1}{1-n}\right]$$

$$=\frac{1}{\pi}\left[\frac{2}{1-n^2}\right]\left[1+\cos n\pi\right]=\frac{2}{\pi}\left[\frac{1+(-1)^n}{1-n^2}\right]\quad n\neq 1$$

$$a_1=\frac{2}{\pi}\int_0^{\pi}\sin x\cos x\,dx=\frac{1}{\pi}\int_0^{\pi}\sin 2x\,dx$$

$$=\frac{-1}{2\pi}\cos 2x\Big]_0^{\pi}=0$$

$$f(x) = \frac{2}{\pi} + \sum_{n=2}^{\infty} \frac{2}{\pi}\left(\frac{1+(-1)^n}{1-n^2}\right)\cos nx$$

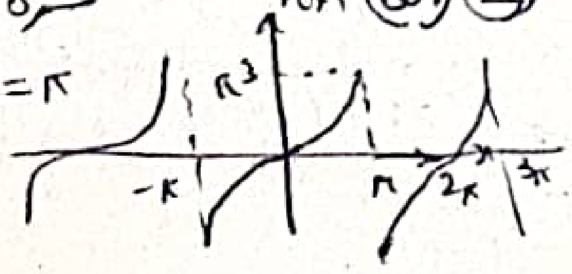
$$x = \pi \rightarrow f(\pi) = 0 = \frac{2}{\pi} + \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{2(-1)^k}{1-4k^2}$$

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{4k^2-1} = \frac{1}{2}$$

(اول) (2)

سری فوریه سینوسی

$$f(x) = x^3 = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x \qquad L = \pi$$



$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx = \frac{2}{\pi}\int_0^{\pi} x^3 \sin nx\,dx$$

$$= \frac{2}{\pi}\Big[-\frac{x^3}{n}\cos nx + \frac{3x^2}{n^2}\sin nx + \frac{6x}{n^3}\cos nx - \frac{6}{n^4}\sin nx\Big]_0^{\pi}$$

x^3	$\sin nx$
$3x^2$	$-\frac{1}{n}\cos nx$
$6x$	$\frac{-1}{n^2}\sin nx$
6	$\frac{1}{n^3}\cos nx$
0	$\frac{1}{n^4}\sin nx$

$$= \frac{2}{\pi}\Big[-\frac{\pi^3}{n}\cos n\pi + \frac{6\pi}{n^3}\cos n\pi\Big]$$

$$= \frac{2}{\pi}(-1)^n\Big[-\frac{\pi^3}{n} + \frac{6\pi}{n^3}\Big] = \frac{2(-1)^n}{\pi}\left(\frac{6\pi - \pi^3 n^2}{n^3}\right)$$

$$f(x) = x^3 \quad 0<x<\pi = \sum_{n=1}^{\infty} \frac{2}{\pi}(-1)^n\Big[-\frac{\pi^3}{n} + \frac{6\pi}{n^3}\Big]\sin nx$$

با مشتق گیری داریم

$$3x^2 = \sum_{n=1}^{\infty} \frac{2(-1)^n}{n^3\pi}\left[6n\pi - \pi^3 n^3\right]\cos nx$$

ب) به ازای $x=\frac{\pi}{2}$ در بند (الف) داریم

$$\frac{3\pi^2}{4} = \frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^n\frac{6n\pi - n^3\pi^3}{n^3}\cos\frac{n\pi}{2}$$

$$\cos\frac{n\pi}{2} = \begin{cases} 0 & \text{فرد } n = 2m+1 \\ (-1)^n & \text{زوج } n = 2m \end{cases}$$

پس داریم

$$\sum_{m=1}^{\infty}\frac{2(-1)^{2m}\left[6\pi(2m) - (2m)^3\pi^3\right]}{8m^3} = \frac{3\pi^2}{4}$$

$$\sum_{n=1}^{\infty}\frac{2(-1)^n\left[6n\pi - 4n^3\pi^3\right]}{n^3} = 3\pi^3$$

3) برای $f(x)$ داریم

$$f(x) = \sum_{n=-\infty}^{+\infty} c_n e^{jn\frac{\pi}{L}x} \qquad L=2$$

$$c_n = \frac{1}{2L}\int_{-L}^{+L} f(x) e^{-jn\frac{\pi}{L}x}dx = \frac{1}{4}\int_{-2}^{+2} f(x)\, e^{-jn\frac{\pi}{2}x}dx$$

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$$c_n = \frac{1}{4}\int_{-2}^{+2} \underbrace{f(x)}_{\text{فرد}} [\underbrace{\cos\frac{n\pi}{2}x}_{\text{زوج}} - j\underbrace{\sin\frac{n\pi}{2}x}_{\text{فرد}}]\,dx$$

$$= -\frac{j}{2}\int_0^2 f(x)\sin\frac{n\pi}{2}x\,dx = -\frac{j}{2}\int_0^2 e^{-\frac{x}{2}}\sin 5\pi x\,\sin\frac{n\pi}{2}x\,dx$$

$$= +\frac{j}{4}\int_0^2 e^{-\frac{x}{2}}[\cos(5\pi+\frac{n\pi}{2})x - \cos(5\pi-\frac{n\pi}{2})x]\,dx$$

$$\boxed{\begin{aligned} \int e^{ax}\cos bx\,dx &= \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)+c \\ \int e^{ax}\sin bx\,dx &= \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)+c \end{aligned}}$$

$$= \frac{j}{4}\frac{e^{-x/2}}{1+\pi^2(5+\frac{n}{2})^2}\left[-\frac{1}{2}\cos(5+\frac{n}{2})\pi x + (5+\frac{n}{2})\pi\sin(5+\frac{n}{2})\pi x\right]_0^2$$

$$-\frac{j}{4}\frac{e^{-\frac{x}{2}}}{1+\pi^2(5-\frac{n}{2})^2}\left[-\frac{1}{2}\cos(5-\frac{n}{2})\pi x + (5-\frac{n}{2})\pi\sin(5-\frac{n}{2})\pi x\right]_0^2$$

با جایگذاری کران انتگرال ها جواب c_n بدست می آید.

با فرض $y_p(t) = \sum_{n=-\infty}^{+\infty} c_n e^{j\frac{n\pi}{2}x}$ داریم $\longleftarrow$ جواب خصوصی

$$y_p'(t) = \sum_{n=-\infty}^{+\infty} j\frac{n\pi}{2} c_n e^{j\frac{n\pi}{2}x} \rightarrow y_p''(t) = \sum_{n=-\infty}^{+\infty} -\frac{n^2\pi^2}{4} c_n e^{j\frac{n\pi}{2}x}$$

با جایگذاری در معادله داریم

$$y'' + \alpha y' - y = f(x)$$

$$\sum_{n=-\infty}^{+\infty}\left[\left(\frac{jn\pi}{2}\right)^2 - \alpha\, j\frac{n\pi}{2} + 1\right] c'_n e^{j\frac{n\pi}{2}x} = \sum_{n=-\infty}^{+\infty} c_n e^{j\frac{n\pi}{2}x}$$

$$c'_n = \frac{c_n}{1 - j\frac{n\pi}{2}\alpha - \frac{n^2\pi^2}{4}}$$

که c_n قبلاً حساب شد.

(4) (الف)

$$f(t) = \sum_{n=-\infty}^{+\infty} c_n e^{+j\frac{n\pi}{L}x} \qquad 2L = T \rightarrow L = T/2$$

$$c_n = \frac{1}{2L}\int_0^{2L} f(x) e^{-j\frac{n\pi}{L}x} dx = \frac{1}{T}\int_0^{T} \frac{A}{T} x e^{-j\frac{2n\pi}{T}x} dx$$

$$= \frac{1}{T}\left[\frac{-A}{j2n\pi} x e^{-j\frac{2n\pi}{T}x} + \frac{AT}{4n^2\pi^2} e^{-j\frac{2n\pi}{T}x}\right]_0^T$$

$$= \frac{1}{T}\left[\frac{-A}{j2n\pi} T + \frac{AT}{4n^2\pi^2} - \frac{AT}{4n^2\pi^2}\right]$$

$$= \frac{jA}{2n\pi}$$

(5)

$\frac{A}{T}x$		$e^{-j\frac{2n\pi}{T}x}$
$\frac{A}{T}$		$\frac{-T}{j2n\pi} e^{-j\frac{2n\pi}{T}x}$
0		$\frac{-T^2}{4n^2\pi^2} e^{-j\frac{2n\pi}{T}x}$

$$f(t) = \sum_{n=-\infty}^{+\infty} \frac{jA}{2n\pi} e^{j\frac{2n\pi}{T}t}$$

ب

$$\sum_{n=-\infty}^{+\infty} |c_n|^2 = \frac{1}{2L}\int_0^{2L} (f(x))^2 dx$$ طبق رابطه ی پارسوال

$$= \frac{1}{T}\int_0^{T} \frac{A^2}{T^2} t^2 dt = \frac{A^2}{T^3}\,\frac{1}{3}T^3 = A^2/3$$

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$$\boxed{\sin 3\theta = 3\sin\theta - 4\sin^3\theta}$$

$$\sin\theta = \frac{1}{j2}(e^{j\theta} - e^{-j\theta})$$

$$\cos\theta = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$$

$$\sin^3\theta = \frac{1}{4}(3\sin\theta - \sin 3\theta)$$

$$f(x) = \frac{1}{4}(3\sin x - \sin 3x)\cos 2x$$

$$= \frac{3}{4}\sin x\cos 2x - \frac{1}{4}\sin 3x\cos 2x$$

$$= \frac{3}{8}\sin 3x - \frac{3}{8}\sin x - \frac{1}{8}\sin 5x - \frac{1}{8}\sin x$$

$$= -\frac{1}{2}\sin x + \frac{3}{8}\sin 3x - \frac{1}{8}\sin 5x$$

$T_1 = 2\pi$

$T_2 = \frac{2\pi}{3}$ $\longrightarrow T = 2\pi \Longrightarrow 2L \longrightarrow L = \pi$

$T_3 = \frac{2\pi}{5}$

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$$f(x)=\sum_{n=-\infty}^{+\infty} c_n e^{j\frac{n\pi}{L}x}$$

$$= -\frac{1}{j4}e^{jx} + \frac{1}{j4}e^{-jx} + \frac{3}{j16}e^{j3x} - \frac{3}{j16}e^{-j3x}$$

$$-\frac{1}{j16}e^{j5x}+\frac{1}{j16}e^{-j5x} \to \begin{cases} c_1 = -\frac{1}{j4} = c_{-1}^* \\ c_3 = c_{-3}^* = \frac{3}{j16} \\ c_5 = c_{-5}^* = -\frac{1}{j16}\end{cases}$$

$$\frac{1}{2L}\int_{-L}^{+L}(f(x))^2dx = \sum_{n=-\infty}^{+\infty}|c_n|^2$$

$$\frac{1}{2\pi}\int_{-\pi}^{+\pi} \underbrace{\sin^6 x\cos^2 2x}_{\text{زوج}}\, dx = \frac{1}{16}+\frac{1}{16}+\left(\frac{9}{(16)^2}\right)2 + \frac{1}{(16)^2}\times 2$$

$$\int_0^{\pi} \sin^6 x\cos^2 2x\, dx = \pi\left[\frac{1}{8}+\frac{18+2}{162}\right]$$

(6)

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{L}x+b_n\sin\frac{n\pi}{L}x\right)$$

$L=\pi \to a_0=2$

$a_n=\frac{1}{n^3+1}$

$b_n=\frac{n}{n^3+1}$ (7)

$$a_n = \frac{1}{L}\int_{-L}^{+L} f(x)\cos\frac{n\pi}{L}x \quad , \quad b_n = \frac{1}{L}\int_{-L}^{+L} f(x)\sin\frac{n\pi}{L}x\, dx$$

$$\text{(I)} = \int_{-\pi}^{+\pi} f(x)\left[\sin^2 x + \cos 5x\right]^2 \sin 3x\, dx$$

$$= \int_{-\pi}^{+\pi} f(x)\left[\frac{1}{2} - \frac{1}{2}\cos 2x + \cos 5x\right]^2 \sin 3x\, dx$$

$$= \int_{-\pi}^{+\pi} f(x)\left[\frac{1}{4} + \frac{1}{4}\cos^2 2x + \cos^2 5x + \cos 5x - \frac{1}{2}\cos 2x\right.$$

$$\left. - \cos 2x \cos 5x\right] \sin 3x\, dx$$

$$\frac{1}{2}\left[\sin(5x) + \sin x\right]$$

$$= \frac{1}{4}\int_{-\pi}^{+\pi} f(x)\sin 3x\, dx + \frac{1}{8}\int_{-\pi}^{+\pi} f(x)\left[1 + \cos 2x\right]\sin 3x\, dx$$

$$+ \frac{1}{2}\int_{-\pi}^{+\pi} f(x)\left[1 + \cos 10x\right]\sin 3x\, dx + \frac{1}{2}\int_{-\pi}^{+\pi} \cos 2x \sin 3x\, dx$$

$$- \frac{1}{2}\int_{-\pi}^{+\pi} f(x)\left(\cos 7x + \cos 3x\right]\sin 3x\, dx$$

$$= \pi\left[\frac{1}{4} b_3\right] + \frac{1}{8}\pi b_3 + \frac{1}{16}\pi\left[b_5 + b_1\right] + \frac{1}{2}\pi b_3$$

$$+ \frac{1}{4}\pi\left[b_{13} - b_7\right] + \frac{1}{4}\pi\left(b_{13} - b_7\right] - \frac{1}{4}\pi\left[b_5 + b_1\right]$$

$$- \frac{1}{4}\pi\left[b_{10} - b_4\right] - \frac{1}{4}\pi b_6$$ 8

است $\frac{n}{n^3+1} = b_n$ که

$\frac{n\pi}{L}x = nDx$ $\qquad T = 2\pi \to L = \pi$ (الف) 7

$$f(x) = \sin^7 x = \sin x \sin^6 x = \sin x (\sin^2 x)^3$$

$$= \frac{\sin x}{8}(1 - \cos 2x)^3 = \frac{1}{8}\sin x (1 - 3\cos 2x + 3\cos^2 2x + \cos^3 2x)$$

$$\boxed{\cos 3\theta = 4\cos^3\theta - 3\cos\theta \to \cos^3\theta = \frac{1}{4}(\cos 3\theta + 3\cos\theta)}$$

$$= \frac{1}{8}\sin x\left(1 - 3\cos 2x + \frac{3}{2} + \frac{3}{2}\cos 4x + \frac{1}{4}\cos 6x + \frac{3}{4}\cos 2x\right)$$

$$= \frac{5}{16}\sin x - \frac{9}{32}\cos 2x \sin x + \frac{3}{16}\sin x \cos 4x$$

$$+ \frac{1}{32}\sin x \cos 6x$$

$$= \frac{5}{16}\sin x - \frac{9}{64}(\sin 3x - \sin x) + \frac{3}{32}(\sin 5x - 3\sin 3x)$$

$$+ \frac{1}{64}(\sin 7x - \sin 5x)$$

این تابع فرد است پس $a_0 = a_n = 0$ داریم

$$b_1 = \frac{5}{16} + \frac{9}{64}, \quad b_3 = -\frac{9}{64} - \frac{3}{32}, \quad b_5 = \frac{3}{32} + \frac{1}{64}$$

$$b_7 = \frac{1}{64}$$

باقی ضرایب صفرند

(ب) در این صورت $2L = \pi \to L = \frac{\pi}{2}$

$$\frac{n\pi}{L} = \frac{n\pi}{\frac{\pi}{2}} = 2n$$

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$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx = \frac{2}{\pi}\int_0^{\pi}\sin^7 x\sin 2nx\,dx$$

که $\sin^7 x$ را از بند قبل محاسبه شده قرار می‌دهیم در آخر نیز

با استفاده از روابط

$$\begin{cases} \sin\alpha\cos\beta = \frac{1}{2}\left(\sin(\alpha+\beta)+\sin(\alpha-\beta)\right) \\ \cos\alpha\cos\beta = \frac{1}{2}\left(\cos(\alpha+\beta)+\cos(\alpha-\beta)\right] \\ \sin\alpha\sin\beta = -\frac{1}{2}\left[\cos(\alpha+\beta)+\cos(\alpha-\beta)\right] \end{cases}$$

ضرایب را حساب می‌کنیم.

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$$f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{L}x+b_n\sin\frac{n\pi}{L}x\right) \quad \begin{cases} L=\pi \\ a_0=0 \end{cases}$$

$$=\sum_{n=1}^{\infty} a_n\cos nx+b_n\sin nx$$

$$a_n = \frac{1}{L}\int_{-L}^{+L} f(x)\cos\frac{n\pi}{L}x\,dx = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos nx\,dx$$

$$f(x) = \sum_{n=1}^{\infty}\frac{1}{\pi}\int_{-\pi}^{+\pi} f(x')\left[\cos nx\cos nx' + \sin nx\sin nx'\right]dx'$$

$$=\frac{1}{\pi}\sum_{n=1}^{\infty} f(x')\cos(n(x'-x))\,dx'$$