استاد فیلی — تکلیف سری #1 DS — ۸۱۰۱۰۰۰۸۴ — محمد رامیلو

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۱)

الف) $\sum_{i=0}^{n}\sum_{j=0}^{i}\sum_{k=1}^{\log n} c2^{-k} = \sum_{i=0}^{n}\sum_{j=0}^{i} c\log n = \sum_{i=0}^{n} ic\log n$

$= c(n^2+n)\log = \boxed{O(n^2\log n)}$

ب) $\sum_{i=1}^{n}\sum_{j=1}^{[\sqrt{i}]} c = \sum_{i=1}^{n} c[\sqrt{i}] \leq \sum_{i=1}^{n} ci = \boxed{O(n^2)}$

$\sum_{i=0}^{n} c[\sqrt{i}] \to \underbrace{1+1+1}_{3}+\underbrace{2+2+2+2+2}_{5}+\underbrace{3+3+3+3+3+3+3}_{7}+\dots = (2n+1)n$

$= \boxed{O(n^2)}$

ج) $\sum_{i=0}^{n}\sum_{j=0}^{i} c = \sum_{i=0}^{n} ci = \boxed{O(n^2)}$

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۲)

الف) $(\log n)^2 = O(\log^2 n)$

$\log n! = \log(n\times(n-1)(n-2)(\dots)1) = \log n + \log(n-1) \dots + 1 \leq n\log n$

$\Rightarrow \log n! = O(n\log n)$

$n^2 = O(n^2)$

$(5/2)^n = \underline{O(2^n)}$

$\Rightarrow \underline{(5/2)^n > n^2 > \log n! > \log^2 n}$ | همچنین حدود هم نیز خود پاسخ است

ب) $(\sqrt{2})^{\log^2 n} > n^{\log\log n} > (\log n)! > \log(n!)$

$\log(n!) \xrightarrow{\log} \log\log(n!) < \log(n\log n)$ (I)

$(\log n)! \xrightarrow{\log} \log((\log n)!) = \log(\log n) + \log(\log n - 1) + \cdots \leq n\log\log n$

$= O(n\log\log n)$ (II)

(I), (II) $\quad \log(n\log n) < \log n \log\log n$ [redacted] $< n\log\log n$

$\Rightarrow \quad \log(n!) < (\log n)!$

$n^{\log\log n} \xrightarrow{\log} \log n \log\log n$ (I')

$(\sqrt{2})^{\log^2 n} \xrightarrow{\log} \log^2 n \log\sqrt{2} < \log^2 n$ (II')

(I') (II') $\Rightarrow \log n \log\log n < \log n \times \log n \Rightarrow (\sqrt{2})^{\log^2 n} > n^{\log\log n}$

$n > \log n \quad , \quad n^{\log\log n} > (\log n)!$

$\Rightarrow (\sqrt{2})^{\log^2 n} > n^{\log\log n} > (\log n)! > (\log n!)$

ج) $\sum_{k=0}^{n} \frac{n^k}{k!} > 2^n > \sum_{j=1}^{n} \sum_{i=1}^{j} i > n^{1/\log n}$

بسط تیلور ، $\sum_{k=0}^{n} \frac{n^k}{k!} \leq ce^x \Rightarrow \sum_{k=0}^{n} \frac{n^k}{k!} = O(e^n)$

$\sum_{j=1}^{n} \sum_{i=1}^{j} i = \sum_{j=1}^{n} \frac{(j+1)j}{2} = \sum_{j=1}^{n} j^2 = O(n^3)$

$n^{1/\log n} \xrightarrow{\log} \log n^{1/\log n} = \frac{1}{\log n} \log n = 1 \rightarrow n^{1/\log n} = O(1)$

$2^n = O(2^n)$

$$\boxed{\Rightarrow \sum_{k=0}^{n} \frac{n^k}{k!} > 2^n > \sum_{j=1}^{n} \sum_{i=1}^{j} i > n^{1/\log n}}$$

۳) الف) قابل رد کردن است. مثال نقض،

$g(n) = 0$

$f(n) = \tan(n)$

ب) ثابت می‌شود

$f(n) \leq cg(n) \xrightarrow{\log} \log_2 f(n) \leq \overset{c}{\cancel{\log_2}}\, c + \log_2 g(n)$

$\Rightarrow \log_2 f(n) \leq c \log_2 g(n) \Rightarrow \log_2 f(n) \in O(\log_2 g(n))$

ج) $\left.\begin{array}{l} f(n) \leq cg(n) \\ g(n) \leq c_2 h(n) \end{array}\right\} \Rightarrow f(n) \leq ch(n) \Rightarrow f(n) = O(h(n))$

۱)

$$\log_k n = \frac{\log_2 n}{\log_2 k} = c_1 \log_2 n \rightarrow c_2 > \frac{1}{\log_2 k} \; , \; c_3 < \frac{1}{\log_2 k}$$

$$\Rightarrow \boxed{c_3 \log_2 n < \log_k n < c_2 \log_2 n} \Rightarrow \boxed{\log_k n = \theta(\log_2 n)}$$

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۴)

الف) $T(n) = 9T(n/3) + n^2 \log^3 n$

$a = 9 , \; b = 3 \qquad n^{\log_3 9} = n^2$

$f(n) = \Omega(n^2)$ و $\cancel{9} \times \frac{n^2}{\cancel{9}} \log^3 n/3 < c n^2 \log^3 n$

$\Rightarrow T(n) = \theta(n^2 \log^3 n)$

ب) $T(n) = 2^n T(n/2) + n^n$

طرفین را در 2^{-2n} ضرب $\rightarrow 2^{-2n} T(n) = 2^{-n} T(n/2) + 2^{-2n} n^n \xrightarrow{T'(n) = 2^{-2n} T(n)}$

$\Rightarrow T'(n) = T'(n/2) + (n/4)^n$

$a = 1 \quad b = 2 \rightarrow g(n) = n^{\log_2 1} = 1 \; , \; f(n) = (\frac{n}{4})^n$

$f(n) = \Omega(1) \; , \; 1 \times (\frac{n}{8})^{n/2} < c(\frac{n}{4})^n$ ✓

$\Rightarrow T'(n) = \theta((n/4)^n) \Rightarrow 2^{-2n} T(n) = \theta((n/4)^n)$

$\Rightarrow T(n) = 4^n \theta((n/4)^n) = \boxed{O(n^n)}$

ج) $T(n) = \sqrt{n}\, T(\sqrt{n}) + n$

$$= \frac{T(n)}{n} = \frac{T(\sqrt{n})}{\sqrt{n}} + 1 \longrightarrow F(n) = F(\sqrt{n}) + 1$$

$\xrightarrow{n = ۲^m}$ $F'(۲^m) = F'(۲^{m/۲}) + 1 \longrightarrow G(m) = G(\frac{m}{۲}) + 1$

$a = 1 \quad b = ۲ \quad f = 1 \quad g = n^{\log_۲^1} = 1$

$\Rightarrow G(m) = \Theta(1 \times \log m) = \Theta(\log m)$

$\Rightarrow F'(۲^m) = \Theta(\log m) \xRightarrow{m = \log_۲^n} F(۲^{\log_۲^n}) = \Theta(\log \log_۲^n)$

$\Rightarrow F(n) = \Theta(\log \log n)$

د) $T(n) = T(n/۲) + n(۵ - \cos(n))$

$a = 1 \quad b = ۲ \quad f(n) = n(۵ - \cos(n)) \quad g(n) = n^{\log_۲^1} = c$

$T(n) = \Theta(۵n - n\cos(n)) = O(n)$

ه) برای انتقال A به B،

```
void tower(int n, char A, char B, char C) {
    if (n==1) {
        move(A, C);
        move(C, B);
    }
    else {
        tower(n-1, A, B, C);
        move(A, C);
        Tower(n-1, B, A, C);
        move(C, B);
        tower(n-1, A, B, C);
    }
```

$T(n) = ۳T(n-1) + ۲$

$= ۳^۲ T(n-۲) + ۳ \times ۲ + ۲$

$= ۳^۳ T(n-۳) + ۳^۲ \times ۲ + ۳ \times ۲ + ۲$

$\Rightarrow T(n) = ۳^k T(n-k) + ۳^{k-1} \times ۲ + \dots + ۲$

$\Rightarrow T(n) = ۳^{n-1} T(1) + ۲(۳^{n-۲} + \dots + ۳^0)$

$\Rightarrow T(n) = \boxed{O(۳^n)}$

(4)

```
input  divider                                   T(1)
input  divisor                                   T(1)
n = sizeof(divider) , quotient = []              T(1)
for (int i=(n-1); i>=0 ; i--) {                  
    a += divider[i]                              T(1)
    b = a/divisor                                T(1)
    Remain = a % divisor
    if ( len(quotient)==0) {
        a x= 10 ;
        continue; }
    else {
        quotient.append(b)
        a = 0
        }
}
output quotient.to_str & Remain
```

$$\sum_{i=n-1}^{0} c = O(n)$$

$$\Rightarrow \boxed{T(n) = O(n)}$$

Finish :)