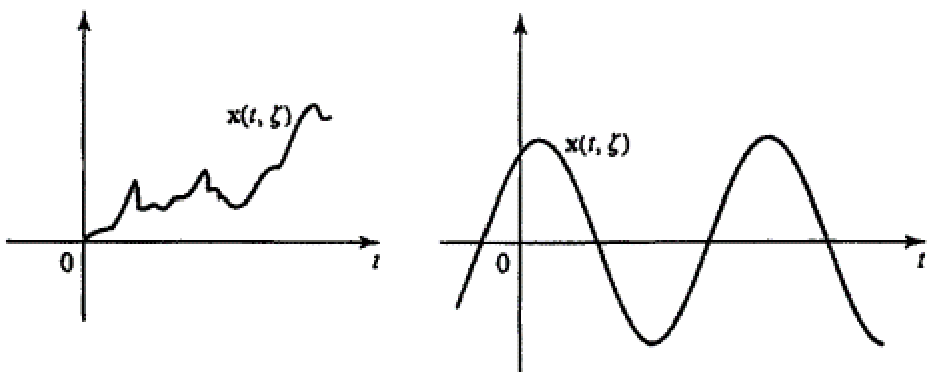
* DEFINITIONS As we recall, a random variable $\mathbf{x}$ is a rule for assigning to every outcome ζ of an experiment S a number $\mathbf{x}(\zeta)$. A stochastic process $\mathbf{x}(t)$ is a rule for assigning to every ζ a function $\mathbf{x}(t, \zeta)$. Thus a stochastic process is a family of time functions depending on the parameter ζ or, equivalently, a function of t and ζ. The domain of ζ is the set of all experimental outcomes and the domain of t is a set R of real numbers.

If R is the real axis, then $\mathbf{x}(t)$ is a continuous-time process. If R is the set of integers, then $\mathbf{x}(t)$ is a discrete-time process. A discrete-time process is, thus, a sequence of random variables. Such a sequence will be denoted by $\mathbf{x}_n$, or, to avoid double indices, by $\mathbf{x}[\boldsymbol{n}]$.

We shall say that $\mathbf{x}(t)$ is a discrete-state process if its values are countable. Otherwise, it is a continuous-state process.

Most results in this investigation will be phrased in terms of continuous-time processes. Topics dealing with discrete-time processes will be introduced either as illustrations of the general theory, or when their discrete-time version is not self-evident.

We shall use the notation $\mathbf{x}(t)$ to represent a stochastic process omitting, as in the case of random variables, its dependence on ζ. Thus $\mathbf{x}(t)$ has the following interpretations: . 1. It is a family (or an ensemble) of functions $x(t, \zeta)$. In this interpretation, t and ζ are variables. 2. It is a single time function (or a sample of the given process). In this case, t is a variable and ζ is fixed. 3. If t is fixed and ζ is variable, then $\mathbf{x}(t)$ is a random variable equal to the state of the given process at time t. 4. If t and ζ are fixed, then $\mathbf{x}(t)$ is a number.



(a) (b)

A physical example of a stochastic process is the motion of microscopic particles in collision with the molecules in a fluid (Brownian motion). The resulting process $\mathbf{x}(t)$ consists of the motions of all particles (ensemble). A single realization $\mathbf{x}(t, \zeta_i)$ of this process is the motion of a specific particle (sample). Another example is the voltage

$$\mathbf{x}(t) = \mathbf{r}\cos(\omega t + \varphi)$$

of an ac generator with random amplitude $\mathbf{r}$ and phase φ. In this case, the process x(t) consists of a family of pure sine waves and a single sample is the function

$$\mathbf{x}\left(t, \zeta_{i}\right)=\mathbf{r}\left(\zeta_{i}\right) \cos \left[\omega t+\varphi\left(\zeta_{i}\right)\right]$$

According to our definition, both examples are stochastic processes. There is, however, a fundamental difference between them. The first example (regular) consists of a family of functions that cannot be described in terms of a finite number of parameters Furthermore, the future of a sample $\mathbf{x}(t, \zeta)$ of $\mathbf{x}(t)$ cannot be determined in terms of its past. Finally, under certain conditions, the statistics [1] of a regular process $\mathbf{x}(t)$ can be determined in terms of a single sample . The second example (predictable) consists of a family of pure sine waves and it is completely specified in terms of the random variables r and φ. Furthermore, if x(t, ζ) is known for $t \leq t_{o}$, then it is determined for $t>t_{0}$. Finally, a single sample $\mathbf{x}(t, \zeta)$ of $\mathbf{x}(t)$ does not specify the properties of the entire process because it depends only on the particular values $r(\zeta)$ and $\varphi(\zeta)$ of $\mathbf{r}$ and φ.

Equality. We shall say that two stochastic processes $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are equal (everywhere) if their respective samples $\mathbf{x}(t, \zeta)$ and $\mathbf{y}(t, \zeta)$ are identical for every ζ. Similarly, the equality $\mathbf{z}(t)=\mathbf{x}(t)+\mathbf{y}(t)$ means that $z(t, \zeta)=\mathbf{x}(t, \zeta)+\mathbf{y}(t, \zeta)$ for every ζ. Derivatives, integrals, or any other operations involving stochastic processes are defined similarly in terms of the corresponding operations for each sample.

As in the case of limits, the above definitions can be relaxed. We give below the meaning of MS equality and in App. 9A we define MS derivatives and integrals. Two processes $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are equal in the MS sense iff

$$E\left\{|\mathbf{x}(t)-\mathbf{y}(t)|^{2}\right\}=0$$

for every t. Equality in the MS sense leads to the following conclusions: We denote by A_{t} the set of outcomes ζ such that $\mathbf{x}(t, \zeta)=\mathbf{y}(t, \zeta)$ for a specific t, and by A_{∞} the set of outcomes ζ such that $\mathbf{x}(t, \zeta)=\mathbf{y}(t, \zeta)$ for every t. From (9-1) it follows that $\mathbf{x}(t, \zeta)-\mathbf{y}(t, \zeta)=0$ with probability 1 ; hence $P\left(A_{t}\right)=P(S)=1$. It does not follow, however, that $P\left(A_{\infty}\right)=1$. In fact, since A_{∞} is the intersection of all sets A_{t} as t ranges over the entire axis, $P\left(A_{\infty}\right)$ might even equal 0 .

EXAMPLE 1 POISSON PROCESS In "Hands On" file we introduced the concept of Poisson points and we know that these points are specified by the following properties: P_1 : The number $\mathbf{n}(t_1, t_2)$ of the points t_i in an interval (t_1, t_2) of length $t = t_2 - t_1$ is a Poisson random variable with parameter λt :

$$P\{\mathbf{n}(t_1, t_2) = k\} = \frac{e^{-\lambda t}(\lambda t)^k}{k!}$$

P_2 : If the intervals (t_1, t_2) and (t_3, t_4) are nonoverlapping, then the random variables $\mathbf{n}(t_1, t_2)$ and $\mathbf{n}(t_3, t_4)$ are independent. Using the points t_i, we form the stochastic process

$$\mathbf{x}(t) = \mathbf{n}(0, t)$$

This is a discrete-state process consisting of a family of increasing staircase functions with discontinuities at the points t. For a specific $t, \mathbf{x}(t)$ is a Poisson random variable with parameter λt; hence

$$E\{\mathbf{x}(t)\} = \eta(t) = \lambda t$$

We shall show that its autocorrelation equals

$$R(t_1, t_2) = \begin{cases} \lambda t_2 + \lambda^2 t_1 t_2 & t_1 \geq t_2 \\ \lambda t_1 + \lambda^2 t_1 t_2 & t_1 \leq t_2 \end{cases}$$

or equivalently that

$$C(t_1, t_2) = \lambda \min(t_1, t_2) = \lambda t_1 U(t_2 - t_1) + \lambda t_2 U(t_1 - t_2)$$

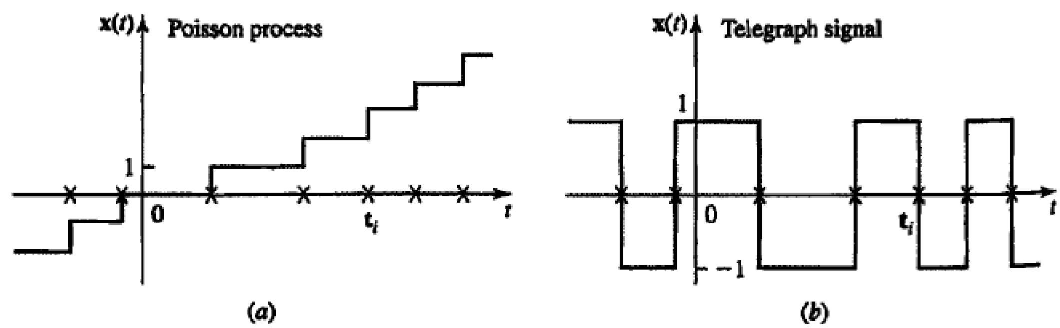


FIGURE 9-3

Proof. The preceding is true for $t_1 = t_2$ because

$$E\{\mathbf{x}^2(t)\} = \lambda t + \lambda^2 t^2$$

Since $R(t_1, t_2) = R(t_2, t_1)$, it suffices to prove for $t_1 < t_2$. The random variables $\mathbf{x}(t_1)$ and $\mathbf{x}(t_2) - \mathbf{x}(t_1)$ are independent because the intervals $(0, t_1)$ and (t_1, t_2) are nonoverlapping. Furthermore, they are Poisson distributed with parameters λt_1 and $\lambda(t_2 - t_1)$ respectively. Hence

$$E\{\mathbf{x}(t_1)[\mathbf{x}(t_2) - \mathbf{x}(t_1)]\} = E\{\mathbf{x}(t_1)\} E\{\mathbf{x}(t_2) - \mathbf{x}(t_1)\} = \lambda t_1 \lambda(t_2 - t_1)$$

Using the identity

$$\mathbf{x}(t_1)\,\mathbf{x}(t_2)=\mathbf{x}(t_1)\left[\mathbf{x}(t_1)+\mathbf{x}(t_2)-\mathbf{x}(t_1)\right]$$

we conclude from the above and that

$$R(t_1,t_2)=\lambda t_1+\lambda^2 t_1^2+\lambda t_1\lambda(t_2-t_1)$$

and results. Nonuniform case If the points t_4 have a nonuniform density $\lambda(t)$ as in (4-124), then the preceding results still bold provided that the product $\lambda(t_2-t_1)$ is replaced by the integral of $\lambda(t)$ from t_1 to t_2. Thus

$$E\{\mathbf{x}(t)\}=\int_0^t \lambda(\alpha)d\alpha$$

and

$$R(t_1,t_2)=\int_0^{t_1}\lambda(t)dt\left[1+\int_0^{r_2}\lambda(t)dt\right]\quad t_1\leq t_2$$

EXAM1PLE 2 TELEGRAPH SIGNAL Using the Poisson points t_i, we form a process $\mathbf{x}(t)$ such that $\mathbf{x}(t)=1$ if the number of points in the interval $(0,t)$ is even, and $\mathbf{x}(t)=-1$ if this number is odd Denoting by $p(k)$ the probability that the number of points in the interval $(0,t)$ equals k, we conclude that

$$\begin{aligned}
P(\mathbf{x}(t)=1)&=p(0)+p(2)+\cdots\\
&=e^{-\lambda t}\left[1+\frac{(\lambda t)^2}{2!}+\cdots\right]=e^{-\lambda t}\cosh\lambda t\\
P(\mathbf{x}(t)=-1)&=p(1)+p(3)+\cdots\\
&=e^{-\lambda t}\left[\lambda t+\frac{(\lambda t)^3}{3!}+\cdots\right]=e^{-\lambda t}\sinh\lambda t
\end{aligned}$$

Hence

$$E\{\mathbf{x}(t)\}=e^{-\lambda t}(\cosh\lambda t-\sinh\lambda t)=e^{-2\lambda t}$$

To determine $R(t_1,t_2)$, we note that, if $t=t_1-t_2>0$ and $\mathbf{x}(t_2)=1$, then $\mathbf{x}(t_1)=1$ if the number of points in the interval (t_1,t_2) is even. Hence

$$P\{\mathbf{x}(t_1)=1\mid\mathbf{x}(t_2)=1\}=e^{-\lambda t}\cosh\lambda t\quad t=t_1-t_2$$

Multiplying by $P\{\mathbf{x}(t_2)=1\}$, we obtain

$$P\{\mathbf{x}(t_1)=1,\mathbf{x}(t_2)=1\}=e^{-\lambda t}\cosh\lambda t e^{-\lambda t_2}\cosh\lambda t_2$$

Similarly,

$$\begin{aligned}
P\{\mathbf{x}(t_1)=-1,\mathbf{x}(t_2)=-1\}&=e^{-\lambda t}\cosh\lambda t e^{-\lambda t_2}\sinh\lambda t_2\\
P\{\mathbf{x}(t_1)=1,\mathbf{x}(t_2)=-1\}&=e^{-\lambda t}\sinh\lambda t e^{-\lambda t_2}\sinh\lambda t_2\\
P\{\mathbf{x}(t_1)=-1,\mathbf{x}(t_2)=1\}&=e^{-\lambda t}\sinh\lambda t e^{-\lambda t_2}\cosh\lambda t_2
\end{aligned}$$

Since the product $\mathbf{x}(t_1)\,\mathbf{x}(t_2)$ equals 1 or -1 , we conclude omitting details that

$$R(t_1, t_2) = e^{-2\lambda|t_1 - t_2|}$$

This process is called semirandom telegraph signal because its value $\mathbf{x}(0) = 1$ at $t = 0$ is not random. To remove this certainty, we form the product

$$\mathbf{y}(t) = \mathbf{a}\mathbf{x}(t)$$

where $\mathbf{a}$ is a random variable taking the values +1 and -1 with equal probability and is independent of $\mathbf{x}(t)$. The process $\mathrm{y}(t)$ so formed is called random telegraph signal. Since $E\{\mathbf{a}\} = 0$ and $E\left\{\mathbf{a}^2\right\} = 1$, the mean of $\mathbf{y}(t)$ equals $E\{\mathbf{a}\}E\{\mathbf{x}(t)\} = 0$ and its autocorrelation is given by

$$E\left\{\mathbf{y}(t_1)\,\mathbf{y}(t_2)\right\} = E\left\{\mathbf{a}^2\right\} E\left\{\mathbf{x}(t_1)\,\mathbf{x}(t_2)\right\} = e^{-2\lambda|t_1 - t_2|}$$

We note that as $t \to \infty$ the processes $\mathbf{x}(t)$ and $\mathbf{y}(t)$ have asymptotically equal statistics.

More on Poisson processes. If $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ represent two independent Poisson processes with parameters $\lambda_1 t$ and $\lambda_2 t$, respectively, it easily follows [as in (6-86)] that their sum $\mathrm{x}_1(t) + \mathrm{x}_2(t)$ is also a Poisson process with parameter $(\lambda_1 + \lambda_2)\,t$. What about the difference of two independent Poisson processes? What can we say about the distribution of such a process? Let

$$\mathbf{y}(t) = \mathbf{x}_1(t) - \mathbf{x}_2(t)$$

where $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are two independent Poisson processes as just defined. Then

$$\begin{aligned}
P\{\mathbf{y}(t) = n\} &= \sum_{k=0}^{\infty} P\left\{\mathbf{x}_1(t) = n + k\right\} P\left\{\mathbf{x}_2(t) = k\right\} \\
&= \sum_{k=0}^{\infty} e^{-\lambda_1 t} \frac{(\lambda_1 t)^{n+k}}{(n+k)!} e^{-\lambda_2 t} \frac{(\lambda_2 t)^k}{k!} \\
&= e^{-(\lambda_1 + \lambda_2)t} \left(\frac{\lambda_1}{\lambda_2}\right)^{n/2} \sum_{k=0}^{\infty} \frac{\left(t\sqrt{\lambda_1 \lambda_2}\right)^{n+2k}}{k!(n+k)!} \\
&= e^{-(\lambda_1 + \lambda_2)t} \left(\frac{\lambda_1}{\lambda_2}\right)^{n/2} I_{|n|}\left(2\sqrt{\lambda_1 \lambda_2 t}\right) \quad n = 0, \pm 1, \pm 2, \ldots
\end{aligned}$$

where

$$I_n(x) \sum_{k=0}^{\infty} \frac{(x/2)^{n+2k}}{k!(n+k)!}$$

represents the modified Bessel function of order n. it follows that

$$E\{\mathbf{y}(t)\} = (\lambda_1 - \lambda_2)\,t \quad \mathrm{Var}\{\mathbf{y}(t)\} = (\lambda_1 + \lambda_2)\,r$$

Thus the difference of two independent Poisson processes is not Poisson. However, it is easy to show that a random selection from a Poisson process yields a Poisson process!

Random selection of Polsson points. Let $\mathbf{x}(t) \sim P(\lambda t)$ represent a Poisson process with parameter λt as before, and suppose each occurrence of $\mathbf{x}(t)$ gets tagged independently with probability p. Let $\mathbf{y}(t)$ represent the total number of tagged events in the interval $(0, t)$ and let $\mathbf{z}(t)$ be the total number of untagged events in $(0, t)$. Then

$$\mathbf{y}(t) \sim P(\lambda p t) \quad \mathbf{z}(t) \sim P(\lambda q t)$$

where $q = 1 - p$. Proof. Let A_n represent the event " n events occur in $(0, t)$ and k of them are tagged." Then

$$\begin{aligned} P\left(A_n\right) &= P\{k \text{ events are tagged } \mid \mathbf{x}(t) = n\} P\{\mathbf{x}(t) = n\} \\ &= \left(\begin{array}{c} n \\ k \end{array}\right) p^k q^{n-k} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \end{aligned}$$

Also the event $\{y(t) = k\}$ represents the mutually exclusive union of the events A_k, $A_{k+1}, \ldots$ Thus

$$\{\mathbf{y}(t) = k\} = \bigcup_{n=k}^{\infty} A_n$$

so that

$$\begin{aligned} P\{\mathbf{y}(t) = k\} &= \sum_{n=k}^{\infty} P\left(A_n\right) = e^{-\lambda t} \sum_{n=k}^{\infty} \frac{(\lambda t)^n}{k!(n-k)!} p^k q^{n-k} \\ &= e^{-\lambda t} \frac{(\lambda p t)^k}{k!} \sum_{r=0}^{\infty} \frac{(\lambda q t)^r}{r!} \\ &= e^{-\lambda(1-q) t} \frac{(\lambda p t)^k}{k!} = e^{-\lambda p t} \frac{(\lambda p t)^k}{k!} \quad k = 0, 1, 2, \ldots \end{aligned}$$

represents a Poisson process with parameter $\lambda p t$. Similarly the untagged events $\mathbf{z}(t)$ form an independent Poisson process with parameter $\lambda q t$. For example, if customers arrive at a counter according to a Poisson process with parameter λt, and the probability of a customer being male is p, then the male customers form a Poisson process with parameter $\lambda p t$, and the female customers form an independent Poisson process with parameter $\lambda q t$. for a deterministic selection of Poisson points.)

Next we will show that the conditional probability of a subset of a Poisson event is in fact binomial.

Poisson points and binomial distribution. For $t_1 < t_2$ consider the conditional probability

$$
\begin{aligned}
P\{\mathbf{x}(t_1) &= k \mid \mathbf{x}(t_2) = n\} \\
&= \frac{P\{\mathbf{x}(t_1) = k, \mathbf{x}(t_2) = n\}}{P\{\mathbf{x}(t_2) = n\}} \\
&= \frac{P\{\mathbf{x}(t_1) = k, \mathbf{n}(t_1, t_2) = n-k\}}{P(\mathbf{x}(t_2) = n\}} \\
&= \frac{e^{-\lambda t_1}(\lambda t_1)^k}{k!} \frac{e^{-\lambda(t_2-t_1)}[\lambda(t_2-t_1)]^{n-k}}{(n-k)!} \frac{n!}{e^{-\lambda t_2}(\lambda t_2)^n} \\
&= \binom{n}{k} \left(\frac{t_1}{t_2}\right)^k \left(1 - \frac{t_1}{t_2}\right)^{n-k} \sim B\left(n, \frac{t_1}{t_2}\right) \quad k = 0, 1, 2, \ldots, n
\end{aligned}
$$

which proves our claim. In particular, let $k = n = 1$, and let Δ be the subinterval in the beginning of an interval of length T. Then from we obtain

$$P[\mathbf{n}(\Delta) = 1 \mid \mathbf{n}(t, t+T) = 1\} \frac{\Delta}{T}$$

But the event $\{\mathbf{n}(\Delta) = 1\}$ is equivalent to $\{t < \mathbf{t}_i < t + \Delta\}$, where $\mathbf{t}_i$ denotes the random arrival instant. Hence the last expression represents

$$P(t < \mathbf{t}_i < t + \Delta \mid \mathbf{n}(t, t+T\} = 1) = \frac{\Delta}{T}$$

i.e., given that only one Poisson occurrence has taken place in an interval of length T, the conditional p.d.f. of the corresponding arrival instant is uniform in that interval. In other words, a Poisson arrival is equally likely to happen anywhere in an interval T, given that only one occurrence has taken place in that interval.

More generally if $t_1 < t_2 < \cdots < t_n < T$ represents the n arrival instants of a Poisson process in the interval $(0, T)$, then the joint conditional distribution of $t_1, t_2, \ldots, t_n$ given $\mathbf{x}(T) = n$ simplifies into

$$
\begin{aligned}
P\{\mathbf{t}_1 &\le x_1, \mathbf{t}_2 \le x_2, \ldots, \mathbf{t}_n \le x_n \mid \mathbf{x}(T) = n\} \\
&= \frac{P\{\mathbf{t}_1 \le x_1, \mathbf{t}_2 \le x_2, \ldots, \mathbf{t}_n \le x_n, \mathbf{x}(t) = n\}}{P\{\mathbf{x}(T) = n\}} \\
&= \frac{1}{e^{-\lambda T}\frac{(\lambda T)^n}{n!}} \sum_{\{m_1, m_2 \ldots, \ldots, m_n\}} \prod_{t=1}^{n} e^{-\lambda(x_1 - x_{i-1})} \frac{[\lambda(x_1 - x_{i-1})]^{m_i}}{m_i!} \\
&= \sum_{m_1, m_2, \ldots, m_n} \frac{n!}{m_1! m_2! \cdots m_n!} \left(\frac{x_1}{T}\right)^{m_1} \left(\frac{x_2 - x_1}{T}\right)^{m_2} \cdots \left(\frac{x_n - x_{n-1}}{T}\right)^{m_n}
\end{aligned}
$$

with $x_0 = 0$. The summation is over all nonnegative integers $\{m_1, m_2, \ldots, m_n\}$ for which $m_1 + m_2 + \cdots + m_n = n$ and $m_1 + m_2 + \cdots + m_k \ge k = 1, 2, \ldots, n-1$. the above formula represents the distribution of n independent points arranged in increasing order; each of which is uniformly distributed over the interval $(0, T)$. It follows that a Poisson process $\mathbf{x}(t)$ distributes points at random over the infinite interval $(0, \infty)$ the same way the uniform random variable distributes points in a finite interval.

General Properties The statistical properties of a real stochastic process $\mathbf{x}(t)$ are completely determined [2] in terms of its n th-order distribution

$$F\left(x_1, \ldots, x_n; t_1, \ldots, t_n\right) = P\left(\mathbf{x}\left(t_1\right) \leq x_1, \ldots, \mathbf{x}\left(t_n\right) \leq x_n\right\}$$

The joint statistics of two real processes $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are determined in terms of the joint distribution of the random variables

$$\mathbf{x}\left(t_1\right) \ldots, \mathbf{x}\left(t_n\right), \mathbf{y}\left(t_1^{\prime}\right), \ldots, \mathbf{y}\left(t_m^{\prime}\right)$$

The complex process $\mathbf{z}(t) = \mathbf{x}(t) + j\mathbf{y}(t)$ is specified in terms of the joint statistics of the real processes $\mathbf{x}(t)$ and $\mathbf{y}(t)$. A vector process (n-dimensional process) is a family of n stochastic processes.

CORRELATION AND COVARLANCE. The autocorrelation of a process $\mathbf{x}(t)$, real or complex, is by definition the mean of the product $\mathbf{x}\left(t_1\right) \mathbf{x}^*\left(t_2\right)$. This function will be denoted by $R\left(t_1, t_2\right)$ or $R_x\left(t_1, t_2\right)$ or $R_{xx}\left(t_1, t_2\right)$. Thus

$$R_{xr}\left(t_1, t_2\right) = E\left\{\mathbf{x}\left(t_1\right) \mathbf{x}^*\left(t_2\right)\right\}$$

where the conjugate term is associated with the second variable in $R_{xx}\left(t_1, t_2\right)$. From this it follows that

$$R\left(t_2, t_1\right) = E\left\{\mathbf{x}\left(t_2\right) \mathbf{x}^*\left(t_1\right)\right\} = R^*\left(t_1, t_2\right)$$

We note, further, that

$$R(t, t) = E\left\{|\mathbf{x}(t)|^2\right\} \geq 0$$

The last two equations are special cases of this property: The autocorrelation $R\left(t_1, t_2\right)$ of a stochastic process $\mathbf{x}(t)$ is a positive definite (p.d.) function, that is, for any a_i and a_j :

$$\sum_{i,j} a_i a_j^* R\left(t_i, t_j\right) \geq 0$$

This is a consequence of the identity

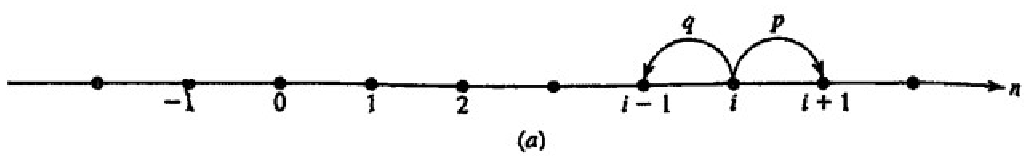
$$0 \leq E\left\{\left|\sum_i a_i \mathbf{x}\left(t_i\right)\right|^2\right\} = \sum_{i,j} a_i a_j^* E\left\{\mathbf{x}\left(t_i\right) \mathbf{x}^*\left(t_j\right)\right\}$$

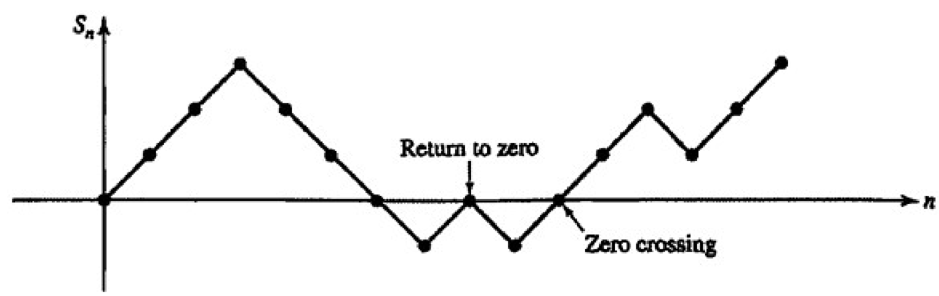
We show later that the converse is also true: Given a p.d. function $R\left(t_1, t_2\right)$, we can find a process $\mathbf{x}(t)$ with autocorrelation $R\left(t_1, t_2\right)$.

EXAMPLE 3 RANDOM WALK

In the concept of random walks, we think of a series of independent random variables that can take either +1 or -1 with probabilities p and q (where q =

1 - p). A common example is a sequence of Bernoulli trials, where each trial has a success probability of p, resulting in Xi = +1 for a successful trial and Xi = -1 otherwise. We define the partial sum Sn as the accumulated positive or negative excess by the nth trial, starting with S0 = 0. In a random walk scenario, a particle moves up or down by one unit at regular intervals, and Sn denotes the particle's position after the nth step. The random walk is symmetric if p = q = 1/2 and asymmetric if p doesn't equal q. For instance, in the gambler's ruin problem, Sn depicts the accumulated wealth of a player at the nth stage. Random walks are utilized to model various real-life phenomena such as gas molecule motion in diffusion processes, thermal noise occurrences, and fluctuations in stock values due to random events like collisions. This model helps us analyze the long-term patterns emerging from a series of individual observations. Specifically, we focus on events like "return to the origin (zero)" within n consecutive steps, which indicate when the random walk returns to its starting point.





random walk to the starting point is a noteworthy event since the process starts all over again from that point onward. In particular, the events "the first return (or visit) to the origin," and more generally "the r th return to the origin," "waiting time for the first gain (first visit to +1)," "first passage through $r > 0$ (waiting time for r th gain)" are also of interest. In addition, the number of sign changes (zero crossings), the level of maxima and minima and their corresponding probabilities are also of great interest.

To compute the probabilities of these events, let [$s_n = r$] represent the event "at stage n, the particle is at the point r," and $p_{n,r}$ its probability. Thus

$$p_{n,t} P \{ \ s_n = r\} = \binom{n}{k} p^k q^{n-k}$$

where k represents the number of successes in n trials and $n - k$ the number of failures. But the net gain

$$r = k - (n-k) = 2k - n$$

or $k = (n+r)/2$, so that

$$p_{n,} = \binom{n}{(n+r)/2} p^{(n+r)/2} q^{(n-r)/2}$$

where the binomial coefficient is understood to be zero unless $(n+r)/2$ is an integer between 0 and n, both inclusive. Note that n and r must be therefore odd or even together.