سوال ① (۸۱۰۱۰۰۰۸٤)

$\nabla^2 u = 0 \qquad r_0 < r < r_1 \qquad 0 < \theta < \pi/2$

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استاد: طامع

$$\begin{cases} u_\theta(r,0)=0 \\ u(r,\pi/2)=0 \end{cases}$$

$u(r_0,\theta)=2/\pi\,\theta$

$u(r_1,\theta)=\cos 3\theta-\cos\theta$

$u(r,\theta)=R(r)\,Y(\theta)$

$$\Rightarrow \frac{r^2R''(r)+rR'(r)}{R(r)}+\frac{Y''(\theta)}{Y(\theta)}=\lambda=k^2$$

$Y''(\theta)+k^2Y(\theta)=0 \Rightarrow \underline{Y_k(\theta)=A_k\cos k\theta+B_k\sin k\theta}$

$$r^2R''(r)+rR'(r)-k^2R(r)=0 \Rightarrow \begin{cases} R_0(r)=C_0\ln r+D_0 \\ R_k(r)=C_kr^k+D_kr^{-k} \end{cases}$$

$u_\theta(r,0)=R(r)\,Y'(0)=0 \longrightarrow Y'(0)=0 \Rightarrow Y'_k(0)=0$

$u(r,\pi/2)=R(r)\,Y(\pi/2)=0 \Rightarrow R(r)\neq 0 \Rightarrow Y(\pi/2)=0$

$Y_k(\frac{\pi}{2})=0 \Rightarrow A_k\cos k\frac{\pi}{2}=0 \Rightarrow k=2n-1$

$n=1,2,\dots$

$u(r,\theta)=R(r)\,Y(\theta) \longrightarrow u_n(r,\theta)=R_n(r)\,Y_n(\theta)$

$u(r,\theta)=R_0(r)\,Y_0(\theta)+\sum_{n=1}^{\infty}R_n(r)\,Y_n(\theta)=\sum_{n=1}^{\infty}\left(C_nr^{2n-1}+D_nr^{-(2n-1)}\right)$

$\times A_n\cos\left((2n-1)\theta\right)$

$$u(r_1,\theta)=\cos 3\theta-\cos\theta \Rightarrow \begin{cases} C_1r_1+D_1=-1 \to D_1=-C_1r_1^2-r_1 \\ C_nr_1^{2n-1}+D_nr_1^{-(2n-1)}=0 \Rightarrow D_n=-C_nr_1^{2(2n-1)} \end{cases}$$

ادامه ①

$$u(r_0,\theta)=\frac{2\theta}{\pi}=\sum_{n=1}^{\infty}\left(C_n r_0^{2n-1}+D_n r^{-(2n-1)}\right)\cos\left((2n-1)\theta\right)$$

$$C_1 r_0+D_1 r_0^{-1}=\frac{4}{\pi^2}(\pi-2)\rightarrow C_1 r_0+\left(-C_1 r_1^{2}-r_1\right)r_0^{-1}=\frac{4}{\pi}(\pi-2)$$

$$C_n r_0^{2n-1}+D_n r_0^{-(2n-1)}=\frac{4}{\pi^2}\,\frac{(2n-1)\pi*(-1)^n+2}{(2n-1)^2}$$

$$C_n r_0^{2n-1}-\left(C_n r_1^{2(2n-1)}\right)r_0^{-(2n-1)}=\frac{4}{\pi^2}\left(\frac{(2n-1)\pi(-1)^n+2}{(2n-1)^2}\right)$$

سوال ۲

$$\begin{cases} u_t - u_{xx} = e^{-a|x|} & -\infty < x < \infty \\ u(x,0) = f(x) \\ \lim\limits_{x \to \infty} u(x,t) = 0 \end{cases}$$

$$\xrightarrow{F} F(u_t) - F(u_{xx}) = F(e^{-a|x|})$$

$$\Rightarrow \hat{u}_t + w^2 \hat{u} = \frac{10}{25+w^2}$$

$$\hat{u}(w,t) = c e^{-w^2 t} + \frac{10}{25w^2+w^4}$$

$$u(x,0) = f(x) \xrightarrow{F} u(x,0) = F(f(t)) \Rightarrow \hat{u}(w,0) = F(w)$$

$$\hat{u}(w,0) = c + \frac{10}{25w^2+w^4} = F(w) \to c = F(w) - \frac{10}{w^2(25+w^2)}$$

$$\hat{u}(w,t) = c e^{-w^2 t} + \frac{10}{25w^2+w^4} = \left(F(w) - \frac{10}{w^2(25+w^2)}\right) \times e^{-w^2 t} + \frac{10}{(25+w^2)w}$$

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}(w,t) e^{jwx} \, dw$$

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(\left(F(w) - \frac{10}{w^2(25+w^2)}\right) e^{jw^2 t} + \frac{10}{25w^2+w^4}\right) e^{jwx} \, dw$$

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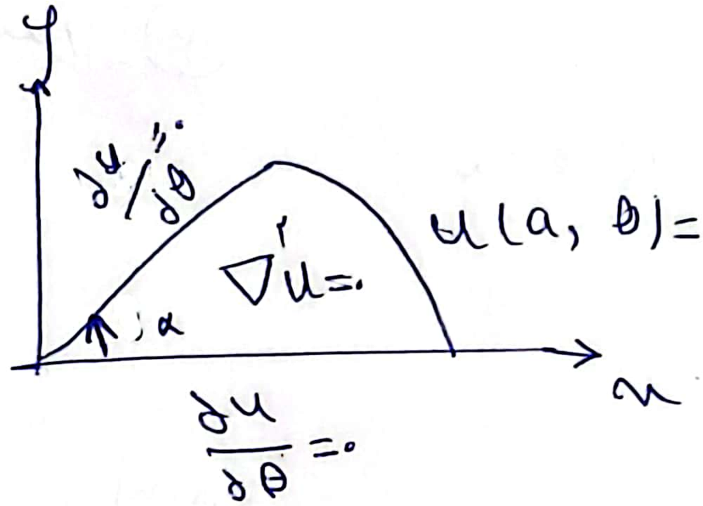


$$u(r,\theta) = R(r)\,Y(\theta) \Rightarrow \frac{r^2 R''(r) + r R'(r)}{R(r)} = -\frac{Y''(\theta)}{Y(\theta)} = \lambda = k^2$$

$\Rightarrow$

معادله θ: $Y''(\theta) + k^2 Y(\theta) = 0 \Rightarrow Y_k(\theta) = A_n \cos(k\theta) + B_k \sin_n k\theta$

معادله r: $r^2 R''(r) + r R'(r) - k^2 R(r) = 0 \Rightarrow$

$$R_0(0) = C_0 \ln r + D_0 \qquad R_k(r) = C_n r^n + D_k r^{-n}$$

$u_\theta(r,0) = R(r)\,Y'(0) = 0 \longrightarrow Y'(0) = 0 \longrightarrow B_k = 0$

$u_\theta(r,\alpha) = R(r)\,Y'(\alpha) = 0 \longrightarrow Y'(\alpha) = 0 \rightarrow A_n k \sin k\alpha = 0$

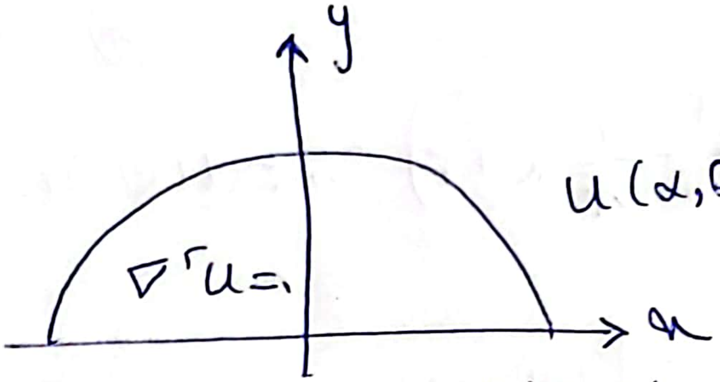
$\alpha k = n\pi \longrightarrow k = \frac{n\pi}{\alpha}$ $\quad u(r,\theta) = R(r)\,Y(\theta) \rightarrow u_n(r,\theta) = R_n(r)\,Y_n(\theta)$

$$u(r,\theta) = R_0(r)\,Y_0(\theta) + \sum_{n=1}^{\infty} R_n(r)\,Y_n(\theta)$$

$$u(r,\theta) = (C_0 \ln r + D_0)A_0 + \sum_{n=1}^{\infty}\left(C_n r^{\frac{n\pi}{\alpha}} + D_n r^{-\frac{n\pi}{\alpha}}\right) A_n \cos\frac{n\pi}{\alpha}\theta$$

$0 < r < \alpha \Rightarrow u(r,\theta) = \mathcal{E}_0 + \sum_{n=1} \mathcal{E}_n r^{\frac{n\pi}{\alpha}} \cos\frac{n\pi}{\alpha}\theta \Rightarrow \begin{cases} u(\alpha,\theta) = f(\theta) \\ \mathcal{E}_0 = \frac{1}{\alpha}\int_0^\alpha f(\theta)\,d\theta \end{cases}$

سوال ۴



$$u(a,\theta) = f(\theta)$$

$$\left.\frac{\partial u}{\partial \theta}\right|_{\theta=\pi} = 0 \qquad u(r,0) = 0$$

$$u(r,\theta) = R(r)\, Y(\theta) \rightarrow \frac{r^2 R''(r) + r R'(r)}{R(r)} = \lambda = k^2$$

$$-\frac{Y''(\theta)}{Y(\theta)} = \lambda = k^2 \Rightarrow Y''(\theta) + k^2 Y(\theta) = 0 \rightarrow Y_k(\theta) = A_k \cos k\theta + B_k \sin(k\theta)$$

$$r^2 R''(r) + r R'(r) - k^2 R = 0 \Rightarrow R_0(r) = C \ln r + D$$

$$R_k(r) = C_k r^k + D_k r^{-k}$$

$$u(r,0) = R(r)\, Y(0) = 0 \longrightarrow Y(0) = 0 \longrightarrow \boxed{A_k = 0}$$

$$u_\theta(r,\pi) = R(r)\, Y_\theta(\pi) = 0 \longrightarrow Y_\theta(\pi) = 0 \quad \left.\frac{\partial Y}{\partial \theta}\right|_{\theta=\pi} = 0 \rightarrow B_k k \cos(k\pi) = 0 \quad k = \frac{2n-1}{2}$$

$$u(r,\theta) = R(r)\, Y(\theta) \rightarrow u_n(r,\theta) = R_n(r)\, Y_n(\theta)$$

$$u(r,\theta) = R_0(r)\, Y_0(\theta) + \sum_{n=0}^{\infty} R_n(r)\, Y_n(\theta) = \sum_{n=1}^{\infty} \left(C_n r^{\frac{2n-1}{2}} + D_n r^{-\frac{2n-1}{2}}\right) B_k \sin\left(\frac{2n-1}{2}\theta\right)$$

$0 < r < a$

$$u(r,\theta) = \sum E_n r^{\frac{2n-1}{2}} \sin\left(\frac{2n-1}{2}\theta\right)$$

$$u(a,\theta) = f(\theta) = \sum_{n=1}^{\infty} E_n a^{\frac{2n-1}{2}} \sin\left(\frac{2n-1}{2}\theta\right) \Rightarrow E_n = \frac{2}{\pi a^{\frac{2n-1}{2}}} \times \int f(\theta) \times \sin\left(\frac{2n-1}{2}\theta\right)$$

ⓐ حل

$$f_{xx}-\frac{1}{\pi^2}f_{tt}=\left(\frac{1}{\pi^2}x^2-\frac{1}{\pi}x+2\right)\sin t\,u(t)-2t\,u(t)$$

$0<x<\pi \quad t>0.$

$$\begin{cases} f(0,t)=e^{-t}u(t) \\ f(\pi,t)=e^{-(t-1)}u(t-1) \end{cases} \qquad \begin{cases} f(x,0)=0 \\ f_t(x,0)=0 \end{cases}$$

$$\xrightarrow{\mathcal{L}} \mathcal{L}\{f_{xx}\}-\frac{1}{\pi^2}\mathcal{L}\{f_{tt}\}=\mathcal{L}\left\{\frac{x^2}{\pi^2}-\frac{x}{\pi}+2\right\}\mathcal{L}\{\sin t\}$$
$$-2\mathcal{L}\{t\,u(t)\}$$

$$F_{xx}-\frac{1}{\pi^2}\left[s^2F(x,s)-sf(x,0)-f_t(x,0)\right]$$
$$=\left(\frac{x^2}{\pi^2}-\frac{x}{\pi}+2\right)\frac{1}{1+s^2}-2\frac{1}{s^2}\Rightarrow F_{xx}-\frac{s^2}{\pi^2}F=\left(\frac{x^2}{\pi^2}-\frac{x}{\pi}+2\right)\frac{1}{s^2+1}-\frac{2}{s^2}$$

$$F=ax^2+bx+c\Rightarrow 2a-\frac{s^2}{\pi^2}(ax^2+bx+c)=\frac{x^2}{\pi^2(1+s^2)}$$
$$-\frac{x}{\pi(1+s^2)}+\frac{2}{s^2+1}-\frac{2}{s^2}\Rightarrow a=\frac{-1}{s^2(1+s^2)}\quad b=\frac{\pi}{s^2(1+s^2)}$$

$$c=0\Rightarrow F=\frac{\pi x-x^2}{s^4+s^2}\Rightarrow F(x,s)=c\,e^{\frac{sx}{\pi}}+e\,e^{-\frac{sx}{\pi}}+F$$

$$f(0,t)=e^{-t}\to F(0,s)=\frac{1}{s+1}\to c+d=\frac{1}{s+1}$$

$$f(\pi,t)=e^{-(t-1)}u(t-1)\to F(\pi,s)=\frac{e^{-s}}{s+1}\Rightarrow c\,e^{s}+e\,e^{-s}=\frac{e^{-s}}{s+1}$$

$$c=0,\ e=\frac{1}{s+1}\Rightarrow f(x,t)=\mathcal{L}^{-1}\left\{\frac{e^{-\frac{sx}{\pi}}}{s+1}+\frac{-x^2+\pi x}{s^4+s^2}\right\}$$

سوال ۷

$$u(x,y) = ax^{۳} + bx^{۲} + ۲۰x + cxy^{۲} + ۲۹y^{۲} - ۱۰$$

$$\frac{\partial^{۲} u}{\partial x^{۲}} + \frac{\partial^{۲} u}{\partial y^{۲}} = ۰$$

$$۶xa + ۲b + ۰ + ۲xc + ۵۸ = ۰$$

$$۶a + ۲c = ۰ \qquad b = -\frac{۵۸}{۲} \Rightarrow \boxed{b = -۲۹}$$

$$a = -\frac{c}{۳}$$

$$u(x,y) = ax^{۳} + ۲۰x - ۲۹x^{۲} - ۳axy^{۲} + ۲۹y^{۲} - ۱۰$$

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = ۳ax^{۲} + ۲۰ - ۵۸x - ۳ay^{۲}$$

$$\int \Rightarrow v(x,y) = ۳ax^{۲}y - ۵۸xy + ۲۰y - ay^{۳} + g(x)$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \Rightarrow ۶axy + g'(x) - ۵۸y = ۶axy - ۵۸y$$

$$\rightarrow g'(x) = ۰ \rightarrow g(x) = c$$

$$v(x,y) = ۳ax^{۲}y - ۵۸xy + ۲۰y - ay^{۳} + c$$

$$f(x,y) = ax^{۳} - ۲۹x^{۲} + ۲۰x - ۳axy^{۲} + ۲۹y^{۲} - ۱۰ + i(۳axy - ay^{۳} + c - ۵۸xy)$$

$$f(۰) = -۱۰ \Rightarrow f(۰,۰) = -۱۰ + i(c) \rightarrow \boxed{c = ۰}$$

$$f''(i) = ۵۸ + ۶ia$$

$$f(x) = ax^{۳} - ۲۹x^{۲} + ۲۰x - ۱۰ \rightarrow f'(x) = ۳ax^{۲} - ۵۸x + ۲۰ \rightarrow f''(x) = ۶ax - ۵۸$$

سوال ۷

$$f(z)=\frac{x^2+xy+x+i(x^2y+y^3-y)}{x^2+y^2}$$

$$u(x,y)=\frac{x^2+xy+x}{x^2+y^2}=Re\{f(z)\}$$

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\Rightarrow\frac{(2x+y+1)(x^2+y^2)-2x(x^2+xy+x)}{(x^2+y^2)^2}$$

$$V(x,y)=\frac{x^2y+y^3-y}{x^2+y^2}\rightarrow\frac{\partial v}{\partial y}=\frac{(x^2+3y^2-1)(x^2+y^2)-2y(xy+y^3-y)}{(x^2+y^2)^2}$$

$$\Rightarrow\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\ ✓$$

$$\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}\Rightarrow\begin{cases}\dfrac{\partial u}{\partial y}=\dfrac{x(x^2+y^2)-2y(x^2+xy+x)}{x^2+y^2}\\[2ex]\dfrac{\partial v}{\partial x}=\dfrac{2x(x^2+y^2)-2x(x^2y+y^3-y)}{x^2+y^2}\end{cases}$$

$\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$ ✓ پس معادلات کوشی ریمان برقرار است

f در تمام نقاط بجز $x,y=0$ که تعریف نشده تحلیلی است

$$f(z)=\frac{z^2+z}{z^2}\Rightarrow f'(z)=1-\frac{1}{z^2}\rightarrow f''(z)=+\frac{2}{z^3}$$

$$f(z)=z+\frac{1}{z}\qquad f'''(z)=\frac{-6}{z^4}\rightarrow f'''(i)=\frac{-6}{i^4}=-6$$

سوال (۱)

الف)

$$u(x,y) = 3x^2y + 2x^2 - y^3 - 2y^2$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6y + 4 - 6y - 4 = 0 \checkmark$$

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 6xy + 4x \longrightarrow v(x,y) = \int (6xy + 4x)$$

$$v(x,y) = 3xy^2 + 4xy + f(x)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow 3x^2 - 3y^2 - 4y = -3y^2 - 4y - f'(x)$$

$$f'(x) = -3x^2 \rightarrow f(x) = -x^3 + C$$

$$v(x,y) = 3xy^2 + 4xy - x^3 + C$$

ب)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow \frac{2(x^2+y^2) - 4x^2}{(x^2+y^2)^2} + \frac{2(x^2+y^2) - 4y^2}{x^2+y^2} = 0$$

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = \frac{2x}{x^2+y^2} \longrightarrow v(x,y) = 2\tan^{-1}\left(\frac{y}{x}\right) + f(x)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \rightarrow \frac{2y}{x^2+y^2} = \frac{2y}{x^2+y^2} - f'(x) \Rightarrow f'(x) = 0 \Rightarrow f(x) = C$$

$$v(x,y) = 2\tan^{-1}(y/x) + C$$

$f(z) = u(r,\theta), iv(r,\theta)$ ⑨ سوال

$u(r,\theta) = r\cos\theta \ln r - r\theta\sin\theta$

$\frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r} = r\cos\theta \ln r + r\cos\theta - r\theta\sin\theta$

$v(r,\theta) = r\sin\theta \ln r + r\theta\cos\theta + f(r)$

$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} = \sin\theta \ln r + \sin\theta + \theta\cos\theta + f'(r)$

$= \sin\theta \ln r + \sin\theta + \theta\cos\theta \rightarrow f'(r) = 0 \rightarrow f(r) = c$

$v(r,\theta) = r\sin\theta \ln r + r\theta\cos\theta + c$

$f(r,\theta) = u(r,\theta) + i\,v(r,\theta) \Rightarrow \underline{r = z} \Big| \underline{\theta = 0}$

$f(z) = z\ln z + ic \rightarrow f'(z) = \ln z + \frac{1}{z} \times z = \ln z + 1$

$f'(z) = \ln z + 1 \rightarrow f''(z) = \frac{1}{z} \rightarrow f''(i) = -i$

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