به نام خدا

پاسخ تمرین سری دوم ریاضی مهندسی

دکتر طاهری

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تبدیلات زیر را به دست آورید.

$$\mathcal{F}\left(\frac{x}{b^2+x^2}\right)=i\left(\frac{\pi}{b}e^{-b|\alpha|}\right)'=-i\pi\frac{\alpha}{|\alpha|}e^{-b|\alpha|}$$

$$\mathcal{F}^{-1}\left(\frac{1}{(b+i\alpha)^2}\right)=\frac{1}{-i}\mathcal{F}^{-1}\left(\underbrace{\frac{-i}{(b+i\alpha)^2}}_{\hat{f}'(\alpha)}\right)=\frac{1}{-i}\frac{1}{i}x\underbrace{e^{-bx}H(x)}_{f(x)}=xe^{-bx}H(x)$$

انتگرال فوریه توابع زیر را به دست آورید.

$$f(x)=\begin{cases}0 & |x|>\pi\\ x & |x|<\pi\end{cases}$$

$$A(\omega)=\frac{1}{\pi}\int_{-\pi}^{\pi}x\,\mathrm{Cos}(\omega x)dx=0$$

$$B_{(\omega)}=\frac{1}{\pi}\int_{-\pi}^{\pi}x\sin(\omega x)dx=\frac{2\sin(\pi\omega)-2\pi\omega\cos(\omega\pi)}{\pi\omega^2}$$

$$\Rightarrow f(x)=\int_0^{\pi}\frac{2\sin(\pi\omega)-2\pi\omega\cos(\pi\omega)}{\pi\omega^2}\cdot\sin(\omega x)d\omega \quad \square$$

$$f(x)=\begin{cases}0 & |x|>1\\ Sinh(x) & |x|<1\end{cases}$$

$$A(w)=0;B(w)=\frac{1}{\pi}\int_{-1}^{1}\sinh(x)\sin(\omega x)dx$$

$$\begin{cases}u=\sin(\omega n)\to du=\omega\cos(\omega x)dx\\ dv=\sinh(x)dx\to v=\cosh(x)\end{cases}$$

$$B(\omega)=\sin(\omega x)\mathrm{Cosh}(x)-w\int_{-1}^{1}\cosh(x)\cdot\cos(\omega x)\,dx$$

$$\begin{cases}u=\cos(\omega x)\\ dv=\cosh(x)\,dx\end{cases}$$

$$B(\omega)=\frac{2\cosh(1)\,\mathrm{Sin}(\omega)-2\omega\sinh(1)\cos(\omega)}{(\omega^2+1)\pi}$$

$$f(x)=\int_0^1 B(w)\mathrm{Sin}(\omega x)d\omega$$

انتگرال فوریه تابع را بدست آورده و سپس درستی انتگرال I را نشان دهید .

$$f(x) = \begin{cases} Sin(x) & 0 < x < \pi \\ 0 & other\ wise \end{cases} \quad ; \qquad I = \int_0^\infty \frac{cos^2(\frac{\pi x}{2})}{1-x^2}dx = 0$$

$$A(\omega) = \frac{1}{\pi}\int_0^\pi \mathrm{Sin}\,(x)\cos\,(\omega x)dx = \frac{\mathrm{Cos}\,(\pi\omega)+1}{\pi(1-\omega^2)}$$

$$B(\omega) = \frac{1}{\pi}\int_0^\pi \sin\,(x)\sin\,(\omega x)dx = \frac{\sin\,(\pi\omega)}{\pi(1-\omega^2)}$$

$$\Rightarrow f(x) = \int_0^\infty (A(\omega)\mathrm{Cos}\,(\omega x) + B(\omega)\mathrm{Sin}\,(\omega x))d\omega$$

$$I = \int_0^\infty \frac{\cos^2\,((\frac{\pi}{2})x)}{1-x^2}dx \overset{x=\omega}{\rightarrow} \int_0^\infty \frac{\cos^2\,((\frac{\pi}{2})\omega)}{1-\omega^2}d\omega$$

$$\cos^2\,\left(\frac{\pi}{2}\omega\right) = 1-\sin^2\,\left(\frac{\pi}{2}\omega\right) = \frac{\cos\,(\pi\omega)}{2}+\frac{1}{2}$$

$$\rightarrow I = \frac{1}{2}\int_0^\infty \frac{Cos(\pi w)+1}{1-w^2}dw$$

$$x = \pi \rightsquigarrow f(\pi) = \sin\,(\pi) = 0 = \int_0^\infty \frac{cos^2(\pi\omega)+\cos\,(\pi\omega)+sin^2(\pi\omega)}{1-\omega^2}d\omega$$

$$0 = \int_0^\infty \frac{1+\cos\,(\pi\omega)}{1-\omega^2} : 0 = I$$

$$f(x) = \begin{cases} 1 - x^2 & |\mathrm{x}| < 1 \\ 0 & \lceil x \rceil > 1 \end{cases} \qquad I = \int \frac{(xcosx - sinx)^2}{x^6} dx = \frac{\pi}{15}$$

$$B(\omega) = 0$$

$$A(\omega) = \int_{-1}^{1} \frac{(1 - x^2)\cos(\omega x)}{\pi} dx = \frac{2}{\pi} (\int_0^1 \cos(\omega x)\, dx - \int_0^1 x^2 \cos(\omega x)\, dx) =$$

$$= \frac{2}{\pi} (\frac{\sin(\omega x)}{\omega} - (\frac{x^2 \sin(\omega x)}{\omega} - \int_0^1 2x \frac{\sin(\omega x)}{\omega} dx)) = \frac{4(\omega - \omega\cos(w))}{\pi\omega^3}$$

$$\int_{-1}^{1} (1 - x^2)^2 dx = \int_{-1}^{1} (x^4 - 2x^2 + 1) dx = \left(x + \frac{x^5}{5} - \frac{2x^3}{3}\right) = \frac{16}{15}$$

از پارسوال می دانیم

$$\frac{16}{15} = \frac{1}{2\pi} \int_{-\infty}^{\infty} 16 \left(\frac{\sin(\omega) - \omega\cos(\omega)}{\omega^3}\right)^2 d\omega = \frac{16}{\pi} \int_0^{\infty} \left(\frac{\sin(\omega) - \omega\cos(\omega)}{\omega^3}\right)^2 d\omega$$

$$I = \frac{\pi}{15}$$

تابع $f(x)$ را با استفاده از تبدیل فوریه معکوس به دست آورید.

$$\hat{f}(\alpha) = \frac{1}{(i\alpha + 4)(i\alpha - 4)}$$

$$\hat{f}(\alpha) = \frac{1}{(i\alpha + 4)(i\alpha - 4)} = \frac{A}{i\alpha + 4} + \frac{B}{i\alpha - 4}$$
$$\longrightarrow\longrightarrow (i\alpha - 4)A + B(ia + 4) = 1$$
$$\Longrightarrow Ai\alpha - 4A + Bi\alpha + 4B = 1$$
$$\Longrightarrow A + B = 0 : A = -B : 4B + 4B = 1 = 8B$$
$$-4A + 4B = 1 : B = \frac{1}{8}\ A = -\frac{1}{8}$$
$$-\frac{1}{8} \times f^{-1}\left(\frac{1}{4 + i\alpha}\right) + \frac{1}{8} \times f^{-1}\left(\frac{1}{\alpha - 4}\right) =$$

$$= -\frac{1}{8} e^{-4x} H(x) + \frac{-1}{8} e^{4x} H(-x) =$$
$$-\frac{1}{8}\left(e^{-4x} H(x) + e^{4x} H(-x)\right) = \frac{-1}{8}\left(e^{-4|x|}\right)$$

تبدیل فوریه معکوس توابع زیر را بدست آورید.

$$F^{-1}\left(\frac{1}{w^2+8w+32}\right)$$

$$F^{-1}\left(\frac{1}{w^2+6w+21.25}\right)$$

$$F^{-1}\left(e^{-|w|}Cosw\right)$$

I $F^{-1}\left(\frac{1}{w^2+8w+32}\right)$

$\rightsquigarrow \hat{f} = \frac{1}{(w+4)^2+4^2} \rightsquigarrow f(x) = e^{-4ix} \cdot F^{-1}\left(\frac{1}{w^2+4^2}\right)$

$f(x) = \frac{e^{-4ix}}{8} \cdot F^{-1}\left(\frac{8}{w^2+4^2}\right)$

$\rightsquigarrow f(x) = \frac{e^{-4ix} \cdot e^{-4|x|}}{8} = \frac{e^{-4ix-4|x|}}{8}$ ☑

II $F^{-1}\left(\frac{1}{w^2+6w+21{,}25}\right) \rightsquigarrow \hat{f}(w) = \frac{1}{(w+3)^2+12{,}25}$

$\rightsquigarrow f(x) = e^{-3ix} F^{-1}\left(\frac{1}{w^2+12{,}25}\right) = \frac{e^{-3ix}}{7} \cdot F^{-1}\left(\frac{7}{w^2+3{,}5^2}\right)$

$\rightsquigarrow f(x) = \frac{e^{-3ix-3{,}5|x|}}{7}$ ☑

III $F^{-1}\left(e^{-|w|}\cos(w)\right)$ تقارن $\rightsquigarrow F\left(\frac{2}{1+x^2}\right) = e^{-|w|}$

$F(f(x)\cos x) = \frac{1}{2}\left(\hat{f}(w-1)+\hat{f}(w+1)\right)$

$F^{-1}\left(e^{|w|}\cos(w)\right) = \frac{1}{2\pi}\hat{f}(x)$

$\rightsquigarrow \frac{1}{2\pi}F\left(e^{|w|}\cos(w)\right) = \frac{1}{4\pi}\left(\frac{2}{1+(x-1)^2}+\frac{2}{1+(x+1)^2}\right)$

$\rightsquigarrow f(x) = \frac{1}{2\pi}\left(\frac{1}{x^2+2x+1}+\frac{1}{x^2-2x+1}\right)$ ☑