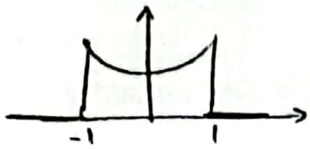
HW2 ریاضیات مهندسی 810100084 محمد امانلو

① الف)

$$f(x) = \begin{cases} \cosh(x) & |x|<1 \\ 0 & |x|>1 \end{cases}$$



$f(x)$ زوج است

$$f(x) = \int_0^{\infty} \left(A(w)\cos(wx) + B(w)\sin(wx)\right) dw$$

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \overbrace{\underbrace{f(x)}_{\text{زوج}}\ \underbrace{\cos(wx)}_{\text{زوج}}}^{\text{زوج}} dx = \frac{2}{\pi}\int_0^{+\infty} f(x)\cos(wx)\,dx$$

$$B(w) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \underbrace{\underbrace{f(x)}_{\text{زوج}}\ \underbrace{\sin(wx)}_{\text{فرد}}}_{\text{فرد}} dx = 0$$

$$\rightarrow f(x) = \int_0^{+\infty} A(w)\cos(wx)\,dw \quad , \quad A(w) = \frac{2}{\pi}\int_0^{+\infty} f(x)\cos(wx)\,dx = \frac{2}{\pi}\int_0^{+\infty}\cosh(x)\cos(wx)\,dx$$

$$I = \int_0^1 \cosh(x)\cos(wx) = \frac{\cosh(x)}{w}\sin(wx)\Big|_0^1 - \frac{1}{w}\int_0^1 \sinh(x)\sin(wx)\,dx$$

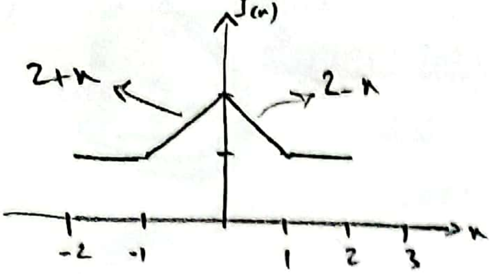
$$= \frac{\cosh(1)\sin(w)}{w} - \cancel{\frac{\cosh(0)\sin(0)}{w}} - \frac{1}{w}\left[\frac{-\sinh(x)\cos(wx)}{w}\Big|_0^1 + \frac{1}{w}\underbrace{\int_0^1 \cosh(x)\cos(wx)\,dx}_{I}\right]$$

$$I = \frac{\cosh(1)\sin(w)}{w} + \frac{1}{w^2}\left[\sinh(1)\cos(w) - \cancel{\sinh(0)\cos(0)}\right] - \frac{1}{w^2} I$$

$$\rightarrow I = \frac{\dfrac{\cosh(1)\sin(w)}{w} + \dfrac{\sinh(1)\cos(w)}{w^2}}{1+w^2} = \frac{w\cosh(1)\sin(w) + \sinh(1)\cos(w)}{1+w^2}$$

$$\rightarrow A(w) = \frac{2}{\pi}\,\frac{w\cosh(1)\sin(w) + \sinh(1)\cos(w)}{1+w^2}$$

$$\rightarrow f(x) = \int_0^{\infty} \frac{2}{\pi}\,\frac{w\cosh(1)\sin(w) + \sinh(1)\cos(w)}{1+w^2}\cos(wx)\,dw$$



$$f(x) = \begin{cases} 2-|x| & |x|<1 \\ 1 & 1<|x|<2 \\ 0 & |x|>2 \end{cases}$$

$f(x)$ زوج است.

$$f(x) = \int_0^{+\infty} A(w)\cos(wx) + B(w)\sin(wx)\,dw$$

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \underbrace{\underbrace{f(x)}_{\text{زوج}}\underbrace{\cos(wx)}_{\text{زوج}}}_{\text{زوج}}dx = \frac{2}{\pi}\int_0^{+\infty} f(x)\cos(wx)\,dx$$

$$B(w) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \underbrace{\underbrace{f(x)}_{\text{زوج}}\underbrace{\sin(wx)}_{\text{فرد}}}_{\text{فرد}}dx = 0$$

$$A(w) = \frac{2}{\pi}\int_0^{+\infty}(2-|x|)\cos(wx)\,dx = \frac{2}{\pi}\left[\int_0^1 (2-x)\cos(wx)\,dx + \int_1^2 \cos(wx)\,dx\right]$$

$$A(w) = \frac{2}{\pi}\left[\frac{2\sin(wx)}{w}\Big|_0^1 - \frac{x\sin(wx)}{w}\Big|_0^1 + \frac{-1}{w}\frac{\cos(wx)}{w}\Big|_0^1 + \frac{\sin(wx)}{w}\Big|_1^2\right]$$

$$= \frac{2}{\pi}\left[\frac{2\cancel{\sin(w)}}{\cancel{w}} - \frac{\cancel{\sin(w)}}{\cancel{w}} - \frac{\cos w}{w^2} + \frac{1}{w^2} + \frac{\sin(2w)}{w} - \frac{\cancel{\sin(w)}}{\cancel{w}}\right]$$

$$= \frac{2}{\pi w}\left(\frac{1-\cos w}{w} + \sin(2w)\right)$$

$$\longrightarrow f(x) = \int_0^{+\infty}\frac{2}{\pi w}\left(\sin(2w) + \frac{1-\cos(w)}{w}\right)\cos(wx)\,dw$$

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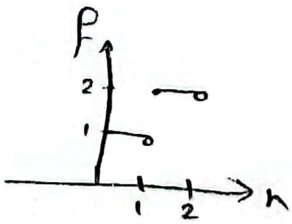
$$\int_0^{+\infty} Y(x)\sin(wt)\,dx = \begin{cases} 1 & 0\le t<1 \\ 2 & 1\le t<2 \\ 0 & t>2 \end{cases}$$

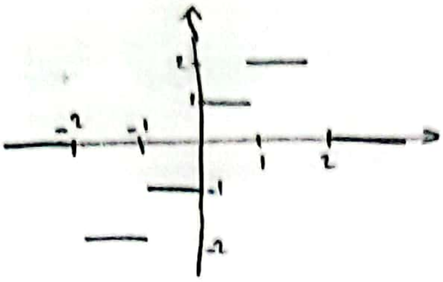
$x \to w$, $t \to x$

$$\int_0^{+\infty} Y(w)\sin(xw)\,dw = \begin{cases} 1 & 0\le x<1 \\ 2 & 1\le x<2 \\ 0 & x>2 \end{cases}$$



گسترش فرد این تابع

$$h(x) = \begin{cases} 0 & x>2 \\ 2 & 1<x\le 2 \\ 1 & 0<x\le 1 \\ -1 & -1\le x<0 \\ -2 & -2\le x<-1 \\ 0 & x<-2 \end{cases}$$



$$h(x) = \int_0^{+\infty} A(w)\cos(wx) + B(w)\sin(wx)\,dw$$

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \underbrace{\underbrace{h(x)}_{\text{فرد}}\underbrace{\cos(wx)}_{\text{زوج}}}_{\text{فرد}}\,dx = 0 \qquad B(w) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \underbrace{\underbrace{h(x)}_{\text{فرد}}\underbrace{\sin(wx)}_{\text{فرد}}}_{\text{زوج}}\,dx = \frac{2}{\pi}\int_0^{+\infty} h(x)\sin(wx)\,dx$$

$$h(x) = \int_0^{+\infty} B(w)\sin(wx)\,dw$$

$$B(w) = \frac{2}{\pi}\int_0^{+\infty} h(x)\sin(wx)\,dx = \frac{2}{\pi}\left[\int_0^1 \sin(wx)\,dx + \int_1^2 2\sin(wx)\,dx + 0\right]$$

$$= \frac{2}{\pi}\left[\frac{-\cos(wx)}{w}\Big|_0^1 + \frac{-2\cos(wx)}{w}\Big|_1^2\right] = \frac{2}{\pi w}\left[-\cos w + \cos 0 - 2\cos(2w) + 2\cos w\right]$$

$$= \frac{2}{\pi w}\left(1+\cos(w) - 2\cos(2w)\right)$$

$$\longrightarrow f(x) = h(x)\ (x>0) \longrightarrow h(x) = \int_0^{+\infty} \frac{2}{\pi w}\left(1+\cos(w) - 2\cos(2w)\right)\sin(wx)\,dw$$

$$\longrightarrow f(x) = \int_0^{+\infty} Y(w)\sin(wx)\,dw = \int_0^{+\infty} \frac{2}{\pi w}\left(1+\cos(w) - 2\cos(2w)\right)\sin(wx)\,dw$$

$$\longrightarrow Y(x) = \frac{2}{\pi x}\left(1 + 2\cos x - 2\cos(2x)\right)$$

$$f(x) = \begin{cases} 0 & x<\alpha \\ c & \alpha<x<\beta \\ 0 & x>\beta \end{cases} \qquad \alpha<0\ ,\ \beta>0\ ,\ c$$

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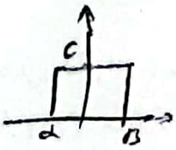
$$f(x) = \int_0^{+\infty} \frac{\sin(\lambda)}{\lambda}\cos(x\lambda)\,d\lambda \xlongequal{\lambda\to w} \int_0^{\infty} \underbrace{\frac{\sin(w)}{w}}_{A(w)}\cos(xw)\,dw \longrightarrow$$ فرم انتگرال کسینوسی

$\alpha = -\beta$



پس تابع f، تابع زوج است. پس f به صورت زیر است

$$\rightarrow \frac{\sin(w)}{w} = A(w) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(x)\cos(wx)\,dx = \frac{2}{\pi}\int_{0}^{+\infty} f(x)\cos(wx)\,dx = \frac{2}{\pi}\int_{0}^{B} c\cos(wx)\,dw$$

$$= \frac{2c}{\pi w}\sin(wx)\Big|_{0}^{B} = \frac{2c}{\pi w}\sin(wB)$$

$$\rightarrow \frac{\sin(w)}{w} = \frac{2c}{\pi w}\sin(wB) \longrightarrow \begin{cases} B=1 \rightarrow \alpha = 1 \\ \frac{2c}{\pi} = 1 \rightarrow c = \frac{\pi}{2} \end{cases} \longrightarrow f(x) = \begin{cases} 0 & x<-1 \\ \frac{\pi}{2} & -1<x<1 \\ 0 & x>1 \end{cases}$$

$$f(x) = \begin{cases} \sin(x) & 0<x<\pi \\ 0 & \text{o.w} \end{cases}$$

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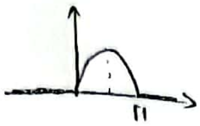

$$I = \int_{0}^{+\infty} \frac{\cos^2\left(\frac{x\pi}{2}\right)}{1-x^2}\,dx = 0 \xrightarrow{x \to w} \int_{0}^{+\infty} \frac{\cos\left(\frac{w\pi}{2}\right)}{1-w^2}\,dw = \frac{1}{2}\int_{0}^{+\infty} \frac{1+\cos(w\pi)}{1-w^2}\,dw$$

$$f(x) = \int_{0}^{+\infty} A(w)\cos(wx) + B(w)\sin(wx)\,dw$$

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(x)\cos(xw)\,dx = \frac{1}{\pi}\int_{0}^{\pi} \sin x\cos(wx)\,dx = \frac{1}{2\pi}\int_{0}^{\pi} \sin((1+w)x) + \sin((1-w)x)\,dx$$

$$B(w) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(x)\sin(wx)\,dx = \frac{1}{\pi}\int_{0}^{+\infty} \sin x\sin(wx)\,dx = \frac{1}{2\pi}\int_{0}^{\pi} -\cos((1+w)x) + \cos((1-w)x)\,dx$$

$$\rightarrow A(w) = \frac{1}{2\pi}\int_{0}^{\pi} \sin(x(1+w)) + \sin(x(1-w))\,dx = \frac{1}{2\pi}\left[\frac{-1}{1+w}\cos(x(1+w))\Big|_{0}^{\pi} + \frac{-1}{1-w}\cos(x(1-w))\Big|_{0}^{\pi}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{1+w}\left(-\cos(\pi(1+w)) + \cos 0\right) + \frac{1}{1-w}\left(-\cos(\pi(1-w)) + \cos 0\right)\right] = \frac{1}{2\pi}\left(\frac{\cos(\pi w)+1}{1+w} + \frac{1+\cos(\pi w)}{1-w}\right)$$

$$= \frac{1}{\pi}\,\frac{\cos(\pi w)+1}{1-w^2}$$

$$\rightarrow B(w) = \frac{1}{2\pi}\int_{0}^{\pi} \left(-\cos((w+1)x) + \cos(x(1-w))\right)dx = \frac{1}{2\pi}\left[-\frac{\sin(x(1+w))}{1+w}\Big|_{0}^{\pi} + \frac{\sin(x(1-w))}{1-w}\Big|_{0}^{\pi}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{1+w}\left(-\sin(\pi(1+w)) + 0\right) + \frac{1}{1-w}\left(\sin(\pi(1-w)) + 0\right)\right] = \frac{1}{2\pi}\left(\frac{\sin(\pi w)}{1+w} + \frac{\sin(\pi w)}{1-w}\right)$$

$$= \frac{1}{\pi}\,\frac{\sin(\pi w)}{1-w^2}$$

$$f(x) = \frac{1}{\pi}\int_0^{+\infty} \frac{(1+\cos(\pi w))\cos(wx) + \sin(\pi w)\sin(wx)}{1-w^2}\,dw = \frac{1}{\pi}\int_0^{\infty} \frac{\cos(wx) + \cos(w\pi - wx)}{1-w^2}\,dw$$

عیب $f(x)$ با دیریکله

$$f(0) = \frac{f(0^+) + f(0^-)}{2} = \frac{\sin(0^+) + 0^-}{2} = 0 = \frac{1}{\pi}\int_0^{+\infty} \frac{\cos(0) + \cos(w\pi)}{1-w^2}\,dw = \frac{1}{\pi}\int_0^{+\infty} \frac{1+\cos(w\pi)}{1-w^2}\,dw$$

$$\xrightarrow{\times \frac{\pi}{2}} 0 \times \frac{\pi}{2} = \frac{1}{2}\int_0^{\infty} \frac{1+\cos(w\pi)}{1-w^2}\,dw = \int \frac{\cos^2(w\pi)}{1-w^2}\,dw = I$$

$$\longrightarrow \boxed{I = 0 \times \frac{\pi}{2} = 0}$$

$$\frac{6}{\pi}\int_0^\infty \frac{2+w^2}{4+5w^2+w^4}\cos(wx)\,dw = e^{-x}+e^{-2x} \qquad x>0$$

در سمت چپ تساوی ضریب $B(w)$ برابر صفر است $\longleftarrow$ $e^{-x}+e^{-2x}$ را بسط زوج می‌دهیم

$$f(x) = \begin{cases} e^{-x}+e^{-2x} & x>0 \\ e^{x}+e^{2x} & x<0 \end{cases} \longrightarrow \text{زوج} \Longrightarrow \begin{matrix} A(w)\neq 0 \\ B(w)=0 \end{matrix}$$

$$f(x) = \int_0^\infty A(w)\cos(wx)\,dw \qquad A(w) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \underbrace{\underbrace{f(x)}_{\text{زوج}}\underbrace{\cos(wx)}_{\text{زوج}}}_{\text{زوج}}dx = \frac{2}{\pi}\int_0^{+\infty} f(x)\cos(wx)\,dx$$

$$= \frac{2}{\pi}\int_0^\infty (e^{-x}+e^{-2x})\cos(wx)\,dx$$

$$\underset{x>0}{f(x)} = \int_0^\infty\left(\frac{2}{\pi}\int_0^\infty (e^{-x}+e^{-2x})\cos(wx)\,dx\right)\cos(wx)\,dw = e^{-x}+e^{-2x}$$

$$\frac{2}{\pi}\int_0^\infty (e^{-x}+e^{-2x})\cos(wx)\,dx \overset{?}{=} \frac{6}{\pi}\cdot\frac{2+w^2}{4+5w^2+w^4} \longrightarrow \text{باید اثبات شود}$$

$$A(w) = \frac{2}{\pi}\int_0^\infty e^{-x}\cos(wx)\,dx + \frac{2}{\pi}\int_0^\infty e^{-2x}\cos(wx)\,dx = \frac{2}{\pi}\left[\mathcal{L}(\cos(wx))\Big|_{s=1} + \mathcal{L}(\cos(wx))\Big|_{s=2}\right]$$

$$= \frac{2}{\pi}\left[\frac{s}{s^2+w^2}\Big|_{s=1} + \frac{s}{s^2+w^2}\Big|_{s=2}\right] = \frac{2}{\pi}\left[\frac{1}{1+w^2}+\frac{2}{4+w^2}\right] = \frac{2}{\pi}\left[\frac{3w^2+6}{4+5w^2+w^4}\right] = \frac{6}{\pi}\left(\frac{w^2+2}{4+5w^2+w^4}\right) \checkmark$$

بنابراین

$$\frac{6}{\pi}\int_0^{\infty} \frac{2+w^2}{4+5w^2+w^4}\cos(wx)\,dw = e^{-x}+e^{-2x} \qquad x>0$$

(6)

$$f(x) = \int_{-\infty}^{+\infty} \Big(\underbrace{\underbrace{\frac{1}{w^2+4}}_{\text{زوج}} \underbrace{\cos(wx)}_{\text{زوج}}}_{\text{زوج}} + \underbrace{\underbrace{\frac{w}{w^2+4}}_{\text{فرد}} \underbrace{\sin(wx)}_{\text{فرد}}}_{\text{زوج}} \Big)\,dw \qquad M = \int_{-\infty}^{+\infty} f(x)\left(2\cos^3 x + 3\sin^3 x\right)dx$$

$$f(x) = 2\int_0^{+\infty} \left(\frac{1}{w^2+4}\cos(wx) + \frac{w}{w^2+4}\sin(wx)\right)dw$$

$$\frac{f(x)}{2} = \int_0^{+\infty} \left(\frac{1}{w^2+4}\cos(wx) + \frac{w}{w^2+4}\sin(wx)\right)dw$$

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{f(x)}{2}\cos(wx)\,dx = \frac{1}{w^2+4} \longrightarrow \int_{-\infty}^{+\infty} f(x)\cos(wx)\,dx = \frac{2\pi}{w^2+4}$$

$$B(w) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{f(x)}{2}\sin(wx)\,dx = \frac{w}{w^2+4} \longrightarrow \int_{-\infty}^{+\infty} f(x)\sin(wx)\,dx = \frac{2\pi w}{w^2+4}$$

$$\cos^3 x = \frac{1}{4}\left(\cos 3x + 3\cos x\right) \quad , \quad \sin^3 x = \frac{1}{4}\left(-\sin(3x) + 3\sin x\right)$$

$$M = \int_{-\infty}^{+\infty} f(x)\left(\frac{2}{4}\left(\cos 3x + 3\cos x\right) + \frac{3}{4}\left(-\sin 3x + 3\sin x\right)\right)dx$$

$$M = \frac{1}{2}\int_{-\infty}^{\infty} f(x)\underbrace{\cos 3x\,dx}_{w=3} + \frac{3}{2}\int_{-\infty}^{+\infty} f(x)\underbrace{\cos x\,dx}_{w=1} - \frac{3}{4}\int_{-\infty}^{+\infty} f(x)\underbrace{\sin(3x)\,dx}_{w=3} + \frac{9}{4}\int_{-\infty}^{+\infty} f(x)\underbrace{\sin x\,dx}_{w=1}$$

$$M = \left(\frac{1}{2}\times\frac{2\pi}{9+4}\right) + \left(\frac{3}{2}\times\frac{2\pi}{1+4}\right) - \left(\frac{3}{4}\times\frac{2\pi\times 3}{9+4}\right) + \left(\frac{9}{4}\times\frac{2\pi\times 1}{1+4}\right)$$

$$M = \frac{\pi}{13} + \frac{3\pi}{5} - \frac{9\pi}{26} + \frac{9\pi}{10} = \boxed{\frac{16\pi}{13}}$$

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$3\int_0^{+\infty} f(x)\cos(ax)\,dx - \int_0^{\infty} x f(x)\sin(ax)\,dx = 0$

$f(0)=1$ $\quad$ $f(x)$ زوج

$f(x)$ زوج $\longrightarrow f(x)=f(-x)$

$A(a)=\frac{2}{\pi}\int_0^{+\infty} f(x)\cos(ax)\,dx \qquad A'(a)=\frac{2}{\pi}\int_0^{+\infty} -x f(x)\sin(ax)\,dx$

$3\int_0^{+\infty} f(x)\cos(ax)\,dx - \int_0^{+\infty} x f(x)\sin(ax)\,dx = 0 \longrightarrow 3\frac{\pi}{2}A(a)+\frac{\pi}{2}A'(a)=0$

$\longrightarrow \frac{\pi}{2}\left(3A(a)+A'(a)\right)=0 \longrightarrow 3A(a)=-A'(a) \longrightarrow 3A(a)=-\frac{dA(a)}{da}$

$\longrightarrow \int -\frac{dA}{3A} = \int da \longrightarrow -\frac{1}{3}\ln A = a+c \longrightarrow \ln A = -3a+c \longrightarrow A = e^{-3a}e^{c}$

$A(a)=c\,e^{-3a}$

$f(x)=\int_0^{+\infty} A(a)\cos(ax)\,da = \int_0^{+\infty} ce^{-3a}\cos(ax)\,da \xrightarrow{x=0} f(0)=\int_0^{+\infty} ce^{-3a}\times 1\,da$

$\int_0^{+\infty} ce^{-3a}\,da=1 \longrightarrow \frac{c}{-3}e^{-3a}\Big|_0^{+\infty} = -\frac{c}{3}(0-1)=\frac{c}{3}=1 \longrightarrow c=3$

$f(x)=\int_0^{+\infty}3e^{-3a}\cos(ax)\,da = 3\,\mathcal{L}\{\cos(ax)\}\Big|_{s=3} = 3\,\frac{s}{s^2+x^2}\Big|_{s=3} = 3\times\frac{3}{9+x^2} = \boxed{\frac{9}{9+x^2}}$