تمرینات بخش ۸-۲ و ۸-۳ جزوه

(۱) $\cos x - 1 = -\frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + O(x^7)$

$\ln(1+u) = u - \frac{u^2}{2} + \frac{u^3}{3} + O(u^4)$

$\Rightarrow \ln\left(1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + O(x^7)\right) = A$

$\Rightarrow A = -\frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \left(-\frac{x^4}{8} + \frac{x^6}{48} - \frac{x^6}{24}\right) + O(x^7)$

$\Rightarrow \boxed{A = -\frac{x^2}{2} - \frac{x^4}{12} - \frac{x^6}{45}}$

(۲) الف) $\frac{1}{1-g(x)} = \frac{(1-g^3)+g^3}{1-g} = g^2+g+1+\frac{g^3}{1-g}$

$\lim_{x\to a} \frac{g^3/(1-g)}{g^2} = \lim_{x\to a} \frac{g}{1-g} = 0 \Rightarrow \frac{g^3}{1-g} = O(g^2)$

$\Rightarrow \frac{1}{1-g} = g^2+g+1+O(g^2)$

ب) $\sin x = x - \frac{x^3}{6} + \frac{x^5}{5!} + O(x^5)$

$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} \Rightarrow \frac{1}{\cos x} = \frac{1}{1 - \frac{x^2}{2} + \frac{x^4}{4!}}$

$\frac{1}{1-g} = g^2+g+1+O(g^2)$

$\Rightarrow \frac{1}{1-\cos x} = \left[1 + \left(\frac{x^2}{2} - \frac{x^4}{4!}\right) + \frac{x^4}{4}\right]$

$\tan x = x + \frac{x^3}{3} + \frac{5x^5}{24} - \frac{x^5}{12} - \frac{x^3}{6} + \frac{x^5}{5!} + O(x^5)$

$\Rightarrow \tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5$

③ $f(x) = \cos\left(\frac{x}{1-x^2}\right)$, $\frac{x}{1-x^2} = \frac{1}{2}\left(\frac{1}{1-x} - \frac{1}{1+x}\right)$

$$\frac{1}{1-x} = 1+x+x^2+x^3+x^4+O(x^4)$$

$$\frac{-1}{1+x} = -1+x-x^2+x^3-x^4+O(x^4)$$

$$\Rightarrow \frac{1}{1-x} - \frac{1}{1+x} = 2x+2x^3+2O(x^4)$$

$$\cos\left(\frac{1}{2}\left(\frac{1}{1-x} - \frac{1}{1+x}\right)\right) = 1 - x^2/2 - x^4 + x^4/24 = 1 - x^2/2 - \frac{23}{24}x^4$$

④ $\lim\limits_{x\to 0} \frac{\cos x - \sqrt[3]{\cos x}}{x^n}$

$$\cos x = 1 - x^2/2 + x^4/24 - \frac{x^6}{720}$$

$$\sqrt[3]{\cos x} = 1 - x^2/6 + \dots$$

$$\lim_{x\to 0} \frac{1 - x^2/2 - 1 + x^2/6}{x^n} \Rightarrow \boxed{n = 2}$$

⑤ $\lim\limits_{x\to 0} \frac{1-\sqrt{1+x^2}\cos x}{\tan^4 x} = \frac{1}{3}$ اثبات

$$\sqrt{1+u} = 1 + \frac{u}{2} - u^2/8 \Rightarrow \sqrt{1+x^2} = 1 + x^2/2 - x^4/8$$

$$\cos x = 1 - x^2/2 + x^4/24 \Rightarrow \sqrt{1+x^2}\cos x = (1 + x^2/2 - x^4/8)\left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right)$$

$$\sqrt{1+x^2}\cos x = 1 - x^2/2 + x^4/24 + x^2/2 - x^4/4 - x^4/8 = 1 - x^4/3$$

$$\lim_{x\to 0} \frac{1-\sqrt{1+x^2}\cos x}{\tan^4 x} = \frac{1 - 1 + x^4/3}{x^4} = \boxed{1/3}$$

(۵) ۲) $\lim\limits_{x\to\infty}\left(x^2-\frac{x}{\sin\frac{1}{x}}\right)=-\frac{1}{6}$

$x=\frac{1}{t} \Rightarrow t=\frac{1}{x} \longrightarrow x\to\infty \Rightarrow t\to 0$

$\lim\limits_{t\to 0}\left(\frac{1}{t^2}-\frac{1}{t\sin t}\right)=\lim\limits_{t\to 0}\left(\frac{t\sin t-t^2}{t^2\times t\sin t}\right)$

$t\left(t-\frac{t^3}{6}\right)-t^2=-\frac{t^4}{6}$

$\boxed{\Rightarrow \lim\limits_{t\to 0}\frac{-t^4/6}{t^4}=-\frac{1}{6}}$

۳) $\lim\limits_{x\to 0}\frac{e-(1+x)^{1/x}}{x}=\frac{e}{2}$

$\Rightarrow (1+x)^{1/x}=e^{\frac{1}{x}\ln(1+x)}\ ,\ \ln(1+x)=x+\frac{x^2}{2}+O(x^2)$

$\Rightarrow (1+x)^{1/x}=e\left(1-\frac{x}{2}\right)=e-e\frac{x}{2}$

$\Rightarrow \lim\limits_{x\to 0}\frac{e-(1+x)^{1/x}}{x}=\lim\frac{e-e+e\,x/2}{x}=\frac{e}{2}$

۵) $\lim\limits_{x\to 0}\frac{1-x^2/2-\cos\left(\frac{x}{1-x^2}\right)}{x^4}=\frac{23}{24}$

$\frac{x}{1-x^2}=\frac{1}{2}\left(\frac{1}{1-x}+\frac{-1}{1-x}\right)$

$\left.\begin{array}{l}\frac{1}{1-x}\left(1+x+x^2+x^3+x^4+O(x^4)\right)\\ \frac{-1}{1+x}\left(1+x-x^2+x^3-x^4+O(x^4)\right)\end{array}\right\} \Rightarrow \frac{1}{2}\left(\frac{1}{1-x}-\frac{1}{1+x}\right)=x+x^3+O(x^4)$

$\Rightarrow \cos\left(\frac{x}{1-x^2}\right)=1-\frac{(x+x^3)^2}{2!}+\frac{(x+x^3)^4}{4!}+O(x^4)$

$$\Rightarrow \cos\left(\frac{x}{1+x^2}\right) = 1 - \frac{x^2}{2} - \frac{23}{24}x^4$$

$$\Rightarrow \lim_{x\to 0} \frac{1 - x^2/2 - \cos\left(\frac{x}{1+x^2}\right)}{x^4} = \boxed{\frac{23}{24}}$$

تمرین بخش ۸-۱ جزوه:

(1) $A = f(x)$

$$f(x) = (1+x)^n \Rightarrow f'(x) = n(1+x)^{n-1} \Rightarrow f''(x) = n(n-1)(1+x)^{n-2}$$

$$\Rightarrow f^{(k)}(x) = \frac{n!}{(n-k)!}(1+x)^{n-k} \xrightarrow{k<n} f^{(n)}(x) = \frac{n!}{(n-n)!}(1+x)^0 = n!$$

$$\Rightarrow f^{(n+1)}(x) = 0 \Rightarrow f^{(k)}(0) = \frac{n!}{(n-k)!} \times 1^{(n-k)} = \frac{n!}{(n-k)!}$$

$$R_{n+1} = \frac{f^{(n+1)}(c)}{(n+1)!} = \frac{0}{(n+1)!}$$

$$A = f(0) + f'(0)x + \dots + \frac{f^{(n)}(0)}{n!}x^n + R_n = f(0) + f'(0)x + \dots \frac{f^{(k)}(0)x^k}{k!}$$

$$A = \sum_{0}^{n} \frac{n!}{(n-k)!}\frac{x^k}{k!} = \sum_{k=0}^{n}\binom{n}{k}x^k$$

(2) $$\tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = \frac{1}{2}\underbrace{\left(\ln(1+x) - \ln(1-x)\right)}_{f(x)}$$

$$f(x) = f(0) + f'(0) + \frac{f''(0)x^2}{2!} + \dots + \frac{f^{(n-1)}(0)x^{(n-1)}}{(n-1)!} + R(n) =$$ [illegible]

$$f^{(n)}(x) = \left(g^{(n)} - h^{(n)}(x)\right) = (-1)^{n-1}\frac{(n-1)!}{(1+x)} - \left((-1)^{2n-1}\frac{(n-1)!}{(1-x)^n}\right)$$

$$\Rightarrow f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n} + \frac{(-1)^{2n}(n-1)!}{(1-x)^n} =$$

$$f^{(n)}(0) = (-1)^{n-1}\frac{(n-1)!}{1} + \frac{(n-1)!}{1} \Rightarrow f^{(k)}(0) = 0$$

$$2(n-1)!$$

$f(0)=0$

$$\overset{2n+2(0,)}{\Rightarrow} f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \cdots + \frac{f^{(2n+2)}(0)x^{2n+2}}{(2n+2)!} + R_{2n+3}$$

$$f(x) = 2(x + \frac{2!\,x^3}{3!} + \frac{4!\,x^5}{5!} + \cdots + \frac{2n!\,x^{2n+1}}{(2n+1)!})$$

$$A = \frac{1}{2} f(x) = x + \frac{2!\,x^3}{3!} + \frac{4!\,x^5}{5!} + \cdots \boxed{\frac{2n!}{(2n+1)!} x^{2n+1}}$$

۳

$$\ln \sqrt{(1-x^2)^x} = \frac{x}{2}\ln(1-x^2) \Rightarrow f = g(x)$$

$$g(x) = f'(x) = \left(\frac{\ln(1-x^2)}{2} + \frac{2x^2}{2(x^2-1)}\right) = \frac{1}{2}(\ln(1+x) + \ln(1-x)$$

$$\times 2 - \frac{2}{x^2-1}) = \frac{1}{2}(\ln(1+x) + \ln(1-x) + 2(\frac{1}{1-x} + \frac{1}{1+x})$$

$$\Rightarrow f'(0) = \cancel{0} \Rightarrow g^{(n)}(x) =$$

$$\frac{1}{2}(-1)^{x-2}\frac{(x-1)!}{(1+x)^x} + (-1)^{2x-1}\frac{(x+1)!}{(1-x^x)} - ((-1)^{2x}\frac{x!}{(1-x)^{x+1}} + (-1)^x\frac{x!}{(1+x)^{x+1}}$$

$$f^{(n)}(0) = (-1)^{x-2}\frac{(x-2)!}{1} - 1\frac{(x-2)!}{1} - (-1)^{x-1}(n-1)! - (n-1)!$$

برای n های زوج صفر است! اما برای n های فرد :

$$f^{(n)}(0) = \frac{1}{2}(-(x-2)!) - (x-2)! - (x-1)! - (x-1)! =$$

$$((x-2)! + (x-1)!) = \boxed{-\frac{x!}{(x-1)}}$$

درس رو ببینید

$$f(x) = f(0) + f'(0)x + \dots + \frac{f^{(2n+2)}(0)\,x^{2n+2}}{(2n+2)!} + R_{2n+2}$$

$$= \sum_{1}^{n} \frac{f^{(2n+1)}(0)\,x^{2n+1}}{(2n+1)!} \Rightarrow \sum_{n=1}^{n} \frac{\cancel{(2n+1)!}\,x^{2n+1}}{2n\,\cancel{(2n+1)!}} + R_{2n+2}$$

$$\boxed{\Rightarrow f(x) = \frac{x^3}{2} - \frac{x^5}{4} - \frac{x^7}{6} - \dots - \frac{x^{2k+1}}{2k} + R_{2n+1}}$$

(4) $f(x) = \cos(\sin x) \Rightarrow f(0) = 1$

$f'(x) = -\cos x \sin(\sin x) \Rightarrow f'(0) = 0$

$f''(x) = \cos x \cdot \cos(\sin x)(-\cos x) \Rightarrow f''(0) = 1$

$f''' = 0 \qquad f^{(4)} = 5$

$$\Rightarrow T_4 = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \frac{f^{(4)}(0)x^4}{4!}$$

$$f(x) = 1 - \frac{x^2}{2} + \frac{5x^4}{24} + R_5, \quad T_4 = 1 - \frac{x^2}{2} + \frac{5x^4}{24}$$

(7) $f(x) = \cos x \qquad f'''(x) = \sin x$

$f'(x) = -\sin x \qquad f^{(4)}(x) = \cos x$

$f''(x) = -\cos x \qquad f^{(5)}(x) = -\sin x$

$$f(x) = f(\tfrac{\pi}{2}) + f'(\pi/2)(x - \pi/2) + \frac{f''(x)(x-\pi/2)^2}{2} + \dots + \frac{f^{(5)}(0)}{5!}$$

$$\Rightarrow f(x) = 0 - (x - \pi/2) + 0 + \frac{1(x-\pi/2)^3}{} + \dots + \frac{f^{(5)}(x)(x-\pi/2)^5}{5!}$$

$$F(x_0) = \frac{\sin(c)(\pi/18)^5}{120} < \frac{(\pi/18)^5}{120} = 10^{-6} \Rightarrow \boxed{\cos(80) = 0.17365}$$

(۸) 1) $f(x) = \sqrt{1+x}$ [illegible]

$$f(x) = f(0) + f'(0)x + \frac{f''(c)x^2}{2}$$

$$f'(x) = \frac{1}{2\sqrt{x+1}}, \quad f''(x) = \frac{-1}{4\sqrt{(x+1)^3}}$$

$$\Rightarrow f(x) = 1 + \frac{x}{2} + \frac{-x^2}{8\sqrt{(1+c)^3}} \Rightarrow \mathrm{Max}\left(\frac{-x^2}{8\sqrt{1+c}^3}\right) = 0 \quad c \to \infty$$

$$\min\left(\frac{-x^2}{8\sqrt{1+c}^3}\right) = -\frac{x^2}{8} \quad c \to 0$$

$$\Rightarrow 1 + \frac{x}{2} - \frac{x^2}{8} < f(x) < 1 + x/2$$

۳/ $f(x) = \cosh x$ ۱۳

$f'(x) = \sinh x$

$f''(x) = \cosh x$

$f'''(x) = \sinh x$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + f'''(c)\frac{x^3}{6}$$

$$\Rightarrow f(x) = 1 + 0 + 0.5x^2 + \frac{\sinh(c)}{6}x^3 \Rightarrow$$

$$f(x) = 1 + 0.5x^2 \Rightarrow \cosh(x) + 0.5x^2 \xrightarrow{c>0}$$

$$\mathrm{Min}\ \frac{\sinh(c)}{6}x^3 = 0$$

$$\Rightarrow \boxed{f(x) = 1 + 0.5x^2}$$