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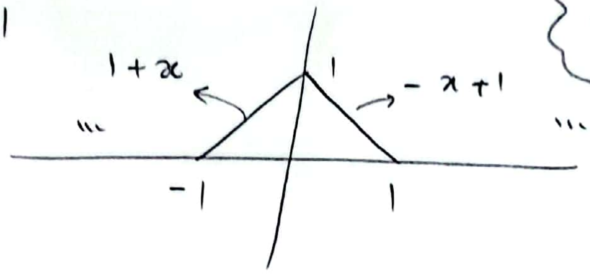
تکلیف سوم ریاضی مهندسی

1 استاد طالع

$g(x) =$



$T = 2$

$L = \frac{T}{2} = 1$

ابتدا سری فوریه $g(x)$ در یک دوره تناوب را بدست می‌آوریم:

$$C_n = \frac{1}{2\ell}\int_{-\ell}^{\ell} \tilde{g}(x)\, e^{-i\frac{n\pi}{\ell}x}\, dx \qquad \ell = 1$$

$$= \frac{1}{2}\left[\int_{-1}^{0} (1+x)\, e^{-in\pi x}\, dx + \int_{0}^{1} (-x+1)\, e^{-in\pi x}\, dx\right]$$

$$= \frac{1}{2}\left[\left(\frac{-(x+1)e^{-in\pi x}}{in\pi} + \frac{1}{n^2\pi^2}e^{-in\pi x}\right)\Bigg|_{-1}^{0}\right.$$

$x+1$	$-x+1$	$e^{-in\pi x}$
1	-1	$\frac{-1}{in\pi}e^{-in\pi x}$
0	0	$\frac{-1}{n^2\pi^2}e^{-in\pi x}$

$$\left. + \left(\frac{(x-1)e^{-in\pi x}}{in\pi} - \frac{1}{n^2\pi^2}e^{-in\pi x}\right)\Bigg|_{0}^{1}\right]$$

$$= \frac{1}{2}\left[\left(\cancel{\frac{-1}{in\pi}} + \frac{1}{n^2\pi^2} - \frac{e^{in\pi}}{n^2\pi^2}\right) + \left(\frac{-e^{-in\pi}}{n^2\pi^2} + \cancel{\frac{1}{in\pi}} + \frac{1}{n^2\pi^2}\right)\right]$$

$$= \frac{1}{2}\left[\frac{2}{n^2\pi^2} - \frac{2e^{\pm in\pi}}{n^2\pi^2}\right] = \frac{1-(-1)^n}{n^2\pi^2} \quad ; n \neq 0$$

$C_0 = \frac{1}{2\ell}\int_{-\ell}^{\ell} \tilde{g}(x)\, dx = \frac{1}{2}[$ مساحت زیر نمودار $] = \frac{1}{2}\left[\frac{2\times 1}{2}\right] = \frac{1}{2}$

$$\rightarrow \mathcal{F}\; G(\omega) = 2\pi \sum_{n=-\infty}^{+\infty} c_n\, \delta(\omega - n\omega_0) \quad ; \omega_0 = \frac{2\pi}{T} = \pi$$

$$\rightarrow G(\omega) = \pi\,\delta(\omega) + \boxed{\sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} \frac{1-(-1)^n}{n^2\pi^2}\,\delta(\omega - n\pi)}$$

(2)

یادآوری : $e^{-a|x|} \xrightarrow{F} \dfrac{2a}{a^2+\omega^2}$

$$\xrightarrow{a=3} \quad e^{-3|x|} \xrightarrow{F} \frac{6}{9+\omega^2}$$

قضیه دوگانی

$$\longrightarrow \quad \frac{6}{9+x^2} \xrightarrow{F} 2\pi\left(e^{-3|-\omega|}\right) = 2\pi e^{-3|\omega|}$$

$$\div 6$$

$$\longrightarrow \quad F\left\{\frac{1}{9+x^2}\right\} = \boxed{\frac{\pi}{3} e^{-3|\omega|}}$$

(3)

آ) $X(\omega) = \displaystyle\int_{-\infty}^{+\infty} x(t)\, e^{-i\omega t}\, dt$

$$\xrightarrow{\omega \to 0} \quad X(0) = \int_{-\infty}^{+\infty} x(t)\, dt = x(t) \text{ مساحت } = \frac{2\times 1}{2} - \frac{2\times 1}{2} = 0$$

ب) $x(t) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{+\infty} X(\omega)\, e^{i\omega t}\, dt$

$$\xrightarrow{t \to 0} \quad x(0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)\, d\omega \longrightarrow \boxed{\int_{-\infty}^{+\infty} X(\omega)\, d\omega = 2\pi x(0) = 2\pi}$$

ج) قضیه پارسوال : $\displaystyle\int_{-\infty}^{+\infty} |X(\omega)|^2\, d\omega = 2\pi \int_{-\infty}^{+\infty} |x(t)|^2\, dt$

$$= \int_{-\infty}^{\infty} |X(\omega)|\,|X(\omega)|\, d\omega = \int_{-\infty}^{+\infty} |X^2(\omega)|\, d\omega$$

$$\rightarrow \int_{-\infty}^{+\infty} |X^2(\omega)|^2 d\omega = 2\pi \times \left[\int_{-1}^{0} (1+x)^2 dx + \int_{0}^{2} (1-x)^2 dx + \int_{2}^{3} (x-3)^2 dx \right]$$

$$= 2\pi \left[\left(\frac{(1+x)^3}{3} \right)_{-1}^{0} + \left(-\frac{(1-x)^3}{3} \right)_{0}^{2} + \left(\frac{(x-3)^3}{3} \right)_{2}^{3} \right] = \frac{4\pi}{3}$$

④ $\xrightarrow{F\{\cdot\}}$ $\left[(j\omega)^2 + j\omega - 2 \right] Y(j\omega) = X(j\omega) = \frac{1}{j\omega + 3}$

$$\rightarrow Y(j\omega) = \frac{X(j\omega)}{(j\omega+2)(j\omega-1)} = \frac{1}{(j\omega+3)(j\omega+2)(j\omega-1)}$$

$$= \frac{A}{j\omega+3} + \frac{B}{j\omega+2} + \frac{C}{j\omega-1}$$

$$\rightarrow A = (j\omega+3) Y(j\omega) \Big|_{j\omega=-3} = \frac{1}{(-1)(-4)} = \frac{1}{4}$$

$$B = (j\omega+2) Y(j\omega) \Big|_{j\omega=-2} = \frac{1}{(1)(-3)} = \frac{-1}{3}$$

$$C = (j\omega-1) Y(j\omega) \Big|_{j\omega=1} = \frac{1}{(4)(3)} = \frac{1}{12}$$

$$\rightarrow Y(j\omega) = \frac{\frac{1}{4}}{j\omega+3} + \frac{-1/3}{j\omega+2} + \frac{1/12}{j\omega-1}$$

$$\xrightarrow{F^{-1}} y(t) = \left[\frac{1}{4} e^{-3t} - \frac{1}{3} e^{-2t} \right] u(t) - \frac{1}{2} e^{t} u(-t)$$

$+\infty$

⑤ ㄱ) $e^{-a|x|} \xrightarrow{\mathcal{F}} \dfrac{2a}{a^2+\omega^2}$

$\xrightarrow{a=2}$ $e^{-2|x|} \xrightarrow{\mathcal{F}} \dfrac{4}{4+\omega^2}$

$\rightarrow e^{-2|\sqrt{2\pi}\,x|} \xrightarrow{\mathcal{F}} \dfrac{1}{\sqrt{2\pi}} \dfrac{4}{4+\left(\frac{\omega}{\sqrt{2\pi}}\right)^2} = \dfrac{1}{\sqrt{2\pi}} \dfrac{4}{4+\frac{\omega^2}{2\pi}}$

$\rightarrow \mathcal{F}^{-1}\left\{ \dfrac{1}{4+\frac{\omega^2}{2\pi}} \right\} = \dfrac{\sqrt{2\pi}}{4} e^{-2\sqrt{2\pi}|x|}$

ㄴ) $X(\omega) = \dfrac{A}{j\omega+4} + \dfrac{B}{j\omega-4} = \dfrac{1}{8}\left(\dfrac{-1}{j\omega+4} - \dfrac{1}{4-j\omega} \right)$

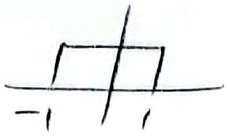
$$\begin{cases} A = (j\omega+4)\,X(j\omega)\Big|_{j\omega=-4} = \dfrac{1}{-4-4} = \dfrac{-1}{8} \\ B = (j\omega-4)\,X(j\omega)\Big|_{j\omega=4} = \dfrac{1}{4+4} = \dfrac{1}{8} \end{cases}$$

$\xrightarrow{\mathcal{F}^{-1}} x(t) = \boxed{\dfrac{-1}{8}\left[e^{-4t}u(t) + e^{4t}u(-t) \right]} = \cancel{\frac{1}{4}\sinh(4t)\,u(t)}$

$\mathcal{F}\{e^{-at}u(t)\} = \dfrac{1}{j\omega+a}$

$\xrightarrow{t\to -t}$ $\mathcal{F}\{e^{at}u(-t)\} = \dfrac{1}{-j\omega+a}$

$\rightarrow x(t) = \dfrac{-1}{8} e^{-4|t|}$

⑥ $f(x) =$ 

$$\rightarrow \mathcal{F}\{f(x)\} = F(\omega) = \int_{-\infty}^{+\infty} f(x) e^{-i\omega x}\, dx = \int_{-1}^{1} e^{-i\omega x}\, dx$$

$$= \frac{-1}{i\omega} e^{-i\omega x}\Big)_{-1}^{1} = \frac{e^{i\omega} - e^{-i\omega}}{i\omega} = \frac{2i\sin(\omega)}{i\omega} = \frac{2\sin\omega}{\omega}$$

$$I = \int_0^{\infty} \frac{\sin^3(\omega)}{\omega}\, d\omega = \int_0^{\infty} \frac{1}{4}\, \frac{3\sin\omega - \sin 3\omega}{\omega}\, d\omega$$

از طرفی

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{2\sin\omega}{\omega} e^{i\omega x}\, d\omega$$

$$\xrightarrow{x=0} \pi \underbrace{f(0)}_{1} = \int_{-\infty}^{+\infty} \frac{\sin\omega}{\omega}\, d\omega \longrightarrow \frac{\pi}{2} = \int_0^{\infty} \frac{\sin\omega}{\omega}\, d\omega$$

$$\xrightarrow{\text{تغییر متغیر}} \int_0^{\infty} \frac{\sin(3\omega)}{3\omega}\, 3\, d\omega = \int_0^{\infty} \frac{\sin 3\omega}{\omega}\, d\omega = \frac{\pi}{2}$$

$$\longrightarrow I = \frac{3}{4}\int_0^{\infty} \frac{\sin\omega}{\omega}\, d\omega - \frac{1}{4}\int_0^{\infty} \frac{\sin 3\omega}{\omega}\, d\omega = \frac{3}{4}\left[\frac{\pi}{2}\right] - \frac{1}{4}\left[\frac{\pi}{2}\right] = \frac{\pi}{4}$$

⑦ $$e^{-b|x|} \xrightarrow{\mathcal{F}} \frac{2b}{\omega^2 + b^2}$$

$$\Rightarrow \underbrace{x e^{-b|x|}}_{g(x)} \xrightarrow{\mathcal{F}} j \frac{d}{d\omega}\left(\frac{2b}{\omega^2 + b^2}\right) = j\frac{-4b\omega}{(\omega^2 + b^2)^2} = G(\omega)$$

با استفاده از قضیه پارسوال:

$$\int_{-\infty}^{+\infty} \frac{16 b^2 \omega^2}{(\omega^2+b^2)^4}\, d\omega = 2\pi \int_{-\infty}^{+\infty} x^2 e^{-2b|x|}\, dx$$

$$\Rightarrow 2\int_{0}^{\infty} \frac{16 b^2 \omega^2}{(\omega^2+b^2)^4}\, d\omega = 4\pi \int_{0}^{\infty} x^2 e^{-2bx}\, dx$$

$$\longrightarrow \int_{0}^{\infty} \frac{\omega^2}{(\omega^2+b^2)^4}\, d\omega = 2\pi \int_{0}^{\infty} x^2 e^{-2bx}\, dx$$

$$= 2\pi\, \mathcal{L}\{x^2\}\Big|_{s=+2b} = \frac{2\pi \times 2}{s^3}\Big|_{s=2b} = \frac{\pi}{2b^3}$$

$$b^2 = 0.25 \longrightarrow \int_{0}^{\infty} \frac{\omega^2}{(\omega^2+0.25)^4}\, d\omega = \frac{\pi}{4(\frac{1}{4})} = \pi$$