سوال ۱ ۸۱۰۱۰۰۰۸۴ محمد امانلو

تمرینات بخش ۳-۱

(۲) $y''' + 2y'' - 4y' + 8y = e^{2x}(2\cos x + 4\sin x)$

$y_h : y''' + 2y'' - 4y' + 8y = 0 \to r^3 + 2r^2 - 4r + 8 = 0$

$\to (r+4)(r^2 - 2r + 2) = 0 \to \begin{cases} r_1 = -4 \\ r_2 = i+1 \\ r_3 = i-1 \end{cases} \to \boxed{y_h = c_1 e^{-4x} + e^{x}(c_1\cos x + c_2 \sin x)}$

$$W = \begin{vmatrix} e^{-4x} & \cos x & \sin x \\ -4e^{-4x} & -\sin x & \cos x \\ 16e^{-4x} & -\cos x & -\sin x \end{vmatrix} = e^{-4x}(\sin^2 x + \cos^2 x) + e^{-4x}(\cancel{-\sin x\cos x + \sin x \cos x})$$

$$16e^{-4x}(\cos^2 x + \sin^2 x) = e^{-4x} + 16e^{-4x} = \boxed{20e^{-4x}}$$

$y_p = e^{2x}\{A_1\cos x + A_2 \sin x\} \to y_p' = 2e^{2x}\{A_1\cos x + A_2\sin x\}$
$+ e^{2x}\{-A_1\sin x + A_2\cos x\}$

$y_p'' = 4e^{2x}[A_1\cos x + A_2\sin x] + 2e^{2x}[A_1\cos x + A_2\sin x] + 2e^{2x}\{-A_1\sin x + A_2\cos x\}$
$+ 2e^{2x}[-A_1\cos x - A_2\sin x]$

$y_p''' = 8e^{2x}\{A_1\cos x + A_2\sin x\} + 4e^{2x}\{\cancel{A_1\cos x +} - A_1\sin x + A_2\cos x\}$
$+ 4e^{2x}\{A_1\cos x + A_2\sin x\} + 2e^{2x}\{-A_1\sin x + A_2\cos x\}$
$+ 4e^{2x}[-A_1\sin x + A_2\cos x] + 2e^{2x}[-A_1\cos x - A_2\sin x]$
$+ 4e^{2x}[-A_1\cos x - A_2\sin x] + 2e^{2x}[A_1\sin x - A_2\cos x]$

$\to \begin{cases} A_1 = -\frac{14}{37} \\ A_2 = \frac{10}{37} \end{cases} \quad \boxed{\to y = c_1 e^{-4x} + c_2 e^{x}\cos x - \left[\frac{14}{37}\cos x + \frac{10}{37}\sin x\right] e^{2x}}$

(۱)

3 $\quad y''' - 3y'' + 2y' = x + e^{x}$ $\qquad (r-1)(r-2)=0$

$$y''' - 3y'' + 2y' = 0 \rightarrow r^3 - 3r^2 + 2r = 0 \rightarrow r(r^2 - 3r + 2) = 0 \rightarrow \begin{cases} r_1 = 0 \\ r_2 = 1 \\ r_3 = 2 \end{cases}$$

$$\boxed{y_h = c_1 + c_2 e^{x} + c_3 e^{2x}}$$

$$\boxed{y_p = \frac{1}{4}x^2 + \frac{1}{2}e^{x} + x + 1 - xe^{x} - \frac{x}{4} - \frac{1}{8} - \cancel{\frac{e^{x}}{2}}}$$

$$\boxed{y = 1 + \frac{3}{4}x + \frac{1}{4}x^2 - xe^{x}}$$

$$W = \begin{vmatrix} 1 & e^{x} & e^{2x} \\ 0 & e^{x} & 2e^{2x} \\ 0 & e^{x} & 4e^{2x} \end{vmatrix} = 1\left(4e^{3x} - 2e^{3x}\right) = \boxed{2e^{3x}}$$

$$y = c_1 + c_2 e^{x} + c_3 e^{2x} + \frac{1}{4}x^2 + x + 1 - xe^{x} - \frac{x}{4} - \frac{1}{8}$$

$$y' = c_2 e^{x} + 2c_3 e^{2x} + \frac{1}{2}x + 1 - \frac{1}{4} - e^{x} - xe^{x}$$

$$y'' = c_2 e^{x} + 4c_3 e^{2x} + \frac{1}{2} - e^{x} - e^{x} - xe^{x}$$

$$W_1 = \begin{vmatrix} 0 & e^{x} & e^{2x} \\ 0 & e^{x} & 2e^{2x} \\ x+e^{x} & e^{x} & 4e^{2x} \end{vmatrix} = (x+e^{x})\left(2e^{3x} - e^{3x}\right) = e^{3x}(x+e^{x})$$

$$W_2 = \begin{vmatrix} 1 & 0 & e^{2x} \\ 0 & 0 & 2e^{2x} \\ 0 & x+e^{x} & 4e^{2x} \end{vmatrix} = (-1)(x+e^{x})2e^{2x}$$

$$W_3 = \begin{vmatrix} 1 & e^{x} & 0 \\ 0 & e^{x} & 0 \\ 0 & e^{x} & x+e^{x} \end{vmatrix} = e^{x}(x+e^{x})$$

$$u_1' = \frac{W_1}{W} = \frac{e^{3x}(x+e^{x})}{2e^{3x}}$$

$$u_1' = \frac{1}{2}(x+e^{x})$$

$$u_2' = \frac{-\cancel{2}(x+e^{x})e^{2x}}{\cancel{2}e^{3x}} = \boxed{\frac{-x-e^{x}}{e^{x}}}$$

$$u_3' = \frac{e^{x}(x+e^{x})}{2e^{3x}} = \boxed{\frac{x+e^{x}}{2e^{2x}}}$$

2 $\rightarrow u_1 = \frac{1}{4}x^2 + \frac{1}{2}e^{x}$, $u_2 = \frac{x}{e^{x}} + \frac{1}{e^{x}} - x$, $u_3 = -\frac{x}{4}e^{-2x} - \frac{1}{8}e^{-2x} - \frac{1}{2e^{x}}$

۴ $\quad y''' - ۳y' + ۲y = e^{-۲x} + ۲\sinh x = e^{-۲x} + e^{x} - e^{-x}$

$\rightarrow y_h: \quad r^{۳} - ۳r + ۲ = ۰ \rightarrow (r+۲)(r-۱)^{۲} = ۰ \rightarrow \begin{cases} r_۱ = -۲ \\ r_{۲,۳} = ۱ \end{cases}$

$$\boxed{y_h = c_۱ e^{-۲x} + c_۲ e^{x} + x c_۲ e^{x}}$$

$y_p = x A_۱ e^{-۲x} + x^{۲} A_۲ e^{x} - A_۳ e^{-x}$

$\rightarrow y_p' = A_۱ e^{-۲x} - ۲x A_۱ e^{-۲x} + ۲x A_۲ e^{x} + x^{۲} A_۲ e^{x} + A_۳ e^{-x}$

$y_p'' = -۲A_۱ e^{-۲x} - ۲A_۱ e^{-۲x} + ۴x A_۱ e^{-۲x} + ۲A_۲ e^{x} + ۲x A_۲ e^{x} + ۲x A_۲ e^{x} + x^{۲} A_۲ e^{x}$
$- A_۳ e^{-x}$

$y_p''' = ۴A_۱ e^{-۲x} + ۴A_۱ e^{-۲x} + ۴A_۱ e^{-۲x} - ۸x A_۱ e^{-۲x} + ۲A_۲ e^{x} + ۲x A_۲ e^{x}$
$+ ۲A_۲ e^{x} + ۲x A_۲ e^{x} + ۲x A_۲ e^{x} + x^{۲} A_۲ e^{x} + A_۳ e^{-x}$

$\rightarrow \begin{cases} A_۳ = \frac{۱}{۴} \\ A_۲ = \frac{۱}{۶} \\ A_۱ = \frac{۱}{۹} \end{cases}$

$$\boxed{y = c_۱ e^{-۲x} + c_۲ e^{x} + x c_۲ e^{x} + \frac{x}{۹} e^{-۲x} + \frac{x^{۲}}{۶} e^{x} - \frac{۱}{۴} e^{-x}}$$

تمرینات بخش ۳_۱

سوال ۲

$$y''' - 3y'' + 2y' = \frac{e^{x}}{1+e^{-x}}$$

$$y_h: \quad y''' - 3y'' + 2y' = 0 \rightarrow r^3 - 3r^2 + 2r = 0 \rightarrow r(r^2 - 3r + 2) = 0$$

$$(r-1)(r-2)$$

$$\rightarrow \begin{cases} r_1 = 0 \\ r_2 = 1 \\ r_3 = 2 \end{cases}$$

$$\boxed{y_h = c_1 + c_2 e^{x} + c_3 e^{2x}}$$

$$W = \begin{vmatrix} 1 & e^{x} & e^{2x} \\ 0 & e^{x} & 2e^{2x} \\ 0 & e^{x} & 4e^{2x} \end{vmatrix} = \left(4e^{3x} - 2e^{3x}\right) = \boxed{2e^{3x}}$$

$$y = c_1 + c_2 e^{x} + c_3 e^{2x}$$

$$W_1 = \begin{vmatrix} 0 & e^{x} & e^{2x} \\ 0 & e^{x} & 2e^{2x} \\ \frac{e^{x}}{1+e^{-x}} & e^{x} & 4e^{2x} \end{vmatrix} = \frac{e^{x}}{1+e^{-x}}\left(2e^{3x} - e^{3x}\right) = \boxed{\frac{e^{4x}}{1+e^{-x}}}$$

$$W_2 = \begin{vmatrix} 1 & 0 & e^{2x} \\ 0 & 0 & 2e^{2x} \\ 0 & \frac{e^{x}}{1+e^{-x}} & 4e^{2x} \end{vmatrix} = \boxed{\frac{-2e^{3x}}{1+e^{-x}}}$$

$$\rightarrow u_1' = \frac{W_1}{W} = \frac{e^{4x}}{2e^{3x}(1+e^{-x})}$$

$$u_2' = \frac{W_2}{W} = \frac{-2\cancel{e^{3x}}}{2\cancel{e^{3x}}(1+e^{-x})}$$

$$W_3 = \begin{vmatrix} 1 & e^{x} & 0 \\ 0 & e^{x} & 0 \\ 0 & e^{x} & \frac{e^{x}}{1+e^{-x}} \end{vmatrix} = \boxed{\frac{e^{2x}}{1+e^{-x}}}$$

$$u_3' = \frac{W_3}{W} = \frac{e^{2x}}{2e^{3x}(1+e^{-x})}$$

۴

$$u_1 = \frac{-\ln(e^{-x}+1) + e^{x} - x}{2} \quad , \quad u_2 = -\ln(e^{-x}+1) - x$$

$$u_3 = \frac{e^{x} + x}{2}$$

$$y_p = \frac{-\ln(e^{-x}+1) + e^{x} - x}{2} - e^{x}\ln(e^{-x}+1) - xe^{x} + \frac{1}{2}e^{2x}(e^{x}+x)$$

$\boxed{\text{سوال ۳}}$ $y''' + y' = \sin x$

الف) $y''' + y' = 0 \to r^3 + r = 0 \to (r^2+1)r = 0 \to \begin{cases} r_1 = 0 \\ r_{2,3} = \pm i \end{cases}$

$$y_h = c_1 + c_2 \cos x + c_3 \sin x$$

$$y_p = x(A_0 \cos x + A_1 \sin x)$$

$$\to y_p' = A_0 \cos x + A_1 \sin x + x(-A_0 \sin x + A_1 \cos x)$$

$$y_p'' = -A_0 \sin x + A_1 \cos x - A_0 \sin x + A_1 \cos x + x(-A_0 \cos x - A_1 \sin x)$$

$$y_p''' = -A_0 \cos x - A_1 \sin x - A_0 \cos x - A_1 \sin x - A_0 \cos x - A_1 \sin x + x(A_0 \sin x - A_1 \cos x)$$

$\to$

ب) $z = y' \to z' = y'' \to z'' = y'''$

$$z'' + z = 0 \to r^2 + 1 = 0 \to r_1, r_2 = \pm i$$

$z_h = c_1 \cos x + c_2 \sin x$

$y' = z$

$\boxed{z = z_h + z_p}$

سوال ۴

$r_{1,2} = \pm 2i \qquad r_{3,4} = 1 \quad , r_5 = -1$

$(r-1)^2(r+1)(r^2+4) = 0 \rightarrow (r-1)^2(r^3+4r+r^2+4) = 0$

$r^5 - r^4 + 3r^3 - 3r^2 - 4r + 4 = 0 \rightarrow$

$$\boxed{y^{(5)} - y^{(4)} + 3y^{(3)} - 3y^{(2)} - 4y' + 4y = 0}$$

سوال ۱

تمرینات بخش ۳_۲

$(2x+3)^3 y''' + 3(2x+3)y' - 6y = 0$

$(2x+3)y' = Dy \rightarrow (2x+3)^3 y''' = D(D-1)(D-2)y$

$\rightarrow$

$r(r-1)(r-2) + 3r - 6 = 0 \rightarrow r^3 - 3r^2 + 5r - 6 = 0$

$(r^2 - 3r + 2)$

$\rightarrow$

سوال ۳   کاهش مرتبه

$$y_1 = x \rightarrow y = Vx \rightarrow \begin{cases} y' = V'x + V \\ y'' = V''x + 2V' \\ y''' = V^{(3)}x + 3V'' \end{cases}$$

$$x^3 \sin x \left(V^{(3)}x + 3V''\right) - \left(3x^2 \sin x + x^3 \cos x\right)\left(V''x + 2V'\right) + \left(6x \sin x + 2x^2 \cos x\right)$$

$$\left(V'x + V\right) - \left(6 \sin x + 2x \cos x\right) x V = 0$$

سوال ۴ $y^{(4)} - y'' = 3x^2 - \sin 2x$

$$y_h: \quad r^4 - r^2 = 0 \rightarrow r^2\left(r^2 - 1\right) = 0 \rightarrow \begin{cases} r_{1,2} = 0 \\ r_{3,4} = \pm 1 \end{cases}$$

$$\boxed{y_h = c_1 + x c_2 + c_3 e^{x} + c_4 e^{-x}}$$

$$y_p = x^2\left(A_0 x^2 + A_1 x + A_3\right) + A_4 \sin 2x$$

$$y_p = A_0 x^4 + A_1 x^3 + A_3 x^2 + A_4 \sin 2x \rightarrow y_p' = 4A_0 x^3 + 3A_1 x^2 + 2A_3 x + 2A_4 \cos 2x$$

$$\rightarrow y_p'' = 12A_0 x^2 + 6A_1 x + 2A_3 - 4A_4 \sin 2x$$

$$y''' = 24A_0 x + 6A_1 - 8A_4 \cos 2x \rightarrow y^{(4)} = 24A_0 + 16A_4 \sin 2x$$

$$\rightarrow \begin{cases} A_4 = \dfrac{1}{320} \\ A_3 = -3 \\ A_2 = 0 \end{cases} \text{ و } A_1 = \boxed{\dfrac{-1}{4}}$$

V