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معادله موج زیر را حل کنید

سوال اول)

$$\begin{cases} u_{tt} = u_{xx} & 0 < x < 4 \\ u_x(0,t) = 0 = u_x(4,t) & (1) \\ u(x,0) = 2\cos(\pi x) + 5 \quad u_t(x,0) = 1 - \cos(3\pi x) & (2) \end{cases}$$

با توجه به شرایط مرزی (1)

شکل کلی جواب به طور زیر نوشته می شود.

$u_{tt} = u_{xx}$

$$u(x,t) = f(x)G(t) \Rightarrow \frac{f\ddot{G}}{fG} = \frac{f''G}{fG} = \frac{\ddot{G}}{G} = \frac{f''}{f} = k \to f'' - kf = 0, \quad \ddot{G} - kG = 0$$

$f_n(x) = \cos(\frac{n\pi}{4}x)$ $n = 0, 1, 2, \ldots$ ⇐ neumann شرایط مرزی

$$u(x,t) = \sum_{n=0}^{\infty} G_n(t) f_n(x)$$

$$\ddot{G} - KG = 0 \qquad K = -\left(\frac{n\pi}{4}\right)^2 = \lambda_n^2 \to \ddot{G} + \lambda_n^2 G = 0$$

$$\Rightarrow u(x,t) = \sum_{n=0}^{\infty} \left(A_n \sin(\lambda_n t) + B_n \cos(\lambda_n t)\right) \cos\left(\frac{n\pi}{4}x\right)$$

$$u(x,0) = 2\cos(\pi x) + 5 = \sum_{n=0}^{\infty} B_n \cos\left(\frac{n\pi}{4}x\right) \quad B_0/5 + \sum_{n=1}^{\infty} B_n \cos\left(\frac{n\pi}{4}x\right)$$

ضریب کسینوسی

سری فوریه

$$B_n = \frac{2}{4}\int_0^4 \cos(\pi x)\cos\left(\frac{n\pi}{4}x\right)dx = \frac{2n\sin(n\pi)[\text{illegible}]\cos(n\pi)}{\pi(n^2-16)}$$

clips

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$$u_t(x,0)=\sum_{n=0}^{\infty} A_n \lambda_n \cos\left(\frac{n\pi}{۴}x\right)=1-\cos(۲\pi x)=A_0\lambda_0$$

$$+\sum_{n=1}^{\infty} A_n \lambda_n \cos\left(\frac{n\pi}{۴}x\right)$$

$$A_n=\frac{1}{۴\lambda_n}\int_0^{۴}\left(1-\cos(۲\pi x)\right)\cos\left(\frac{n\pi}{۴}x\right)dx=\frac{1}{۴\lambda_n}$$

$$\frac{۲۵۶\sin(\pi n)}{۶۴\pi n-\pi n^{۳}}$$

(ادامه سوال بعد)

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معادله حرارت زیر را حل کنید.

$$u_t - 4u_{xx} = 0 \qquad 0 < x < \pi$$

$$u_x(0,t) = 0 \qquad u(\pi,t) = 0$$

$$u(x,0) = 2x^3 - 3\sin(3x)$$

$u_t - 4u_{xx} = 0$

$u(x,t) = f(x)G(t) \Rightarrow \frac{f\dot{G}}{fG} = 4\frac{f''G}{fG} = k \Rightarrow \frac{\dot{G}}{4G} = \frac{f''}{f} = k \to f'' - kf = 0$

$\quad \to \dot{G} - 4kG = 0$

$f_n(x) = \sin(nx) \qquad n = 1, 2, 3, \dots$ شرایط مرزی diricklet مع

$$u(x,t) = \sum_{n=1}^{\infty} G_n(t)\sin(nx)$$

$$\dot{G} - 4kG = 0 \quad k = -n^2 \Rightarrow \dot{G} + \underbrace{(2n)^2}_{\lambda_n} G = 0 \Rightarrow G_n(t) = B_n e^{-\lambda n^2 t}$$

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{\lambda n^2 t}\sin(nx)$$

$$u(x,0) = 2x^3 - 3\sin(3x) = \sum_{n=1}^{\infty} B_n \sin(nx)$$

$$\Rightarrow B_n = \frac{2}{\pi}\int_0^{\pi} \left(2x^3 - 3\sin(3x)\right)\sin(nx)\,dx$$

$$= \frac{2}{\pi}\left(\frac{4(n^2x^2-2)\sin(nx)}{n^4} - \frac{2x(n^2x^2-6)\cos(nx)}{n^3} - \frac{3\sin((n-3)x)}{2(n-3)} + \frac{3\sin((n+3)x)}{2(n+3)}\right)$$

۳) معادلات زیر را به روش دالامبر حل کنید:　　　　الف)

$4u_{tt} = 25u_{xx} \qquad 0 < x < \pi \qquad t > 0$

$u(0,t) = u(\pi,t) = 0 \qquad u(x,0) = \sin(2x)$

$u_t(x,0) = 0$　　ب) $u_{tt} = c^2 u_{xx}$

$0 < x < L \qquad u(0,t) = 0 \qquad u(L,t) = 0$

$u(x,0) = -4\sin(\frac{5\pi x}{L}) + 3\sin(\frac{\pi x}{L}) \qquad u_t(x,0) = 0$

$\Rightarrow \frac{f''(x)}{4f} = -p^2 \Rightarrow f''(x) + 4p^2 f(x) = 0 \Rightarrow F(x) = A\cos 2px + B\sin px$

$\Rightarrow u(0,t) = 0 \Rightarrow f(x) = A\cos 2p(0) + B\sin 2p(0) = 0 \Rightarrow \boxed{A=0}$

$u(\pi,t) = 0 \Rightarrow f(x) = B\sin 2p\pi = 0 \Rightarrow \sin 2p\pi = 0 \Rightarrow p = \frac{n}{2}x$

$\Rightarrow f_n = B_n \sin\frac{n}{2}x$

$\frac{G''(t)}{25G(t)} = -p^2 \Rightarrow G''(t) + 25p^2 G(t) = 0 \Rightarrow 25p^2 = \lambda_n^2$

$$\Rightarrow G_n(t) = H_n \cos \alpha^2 n t + T_n \sin \alpha^2 n t \}$$

$$u(x,t) = \sum_{n=1}^{\infty} \left(H_n \cos \alpha^2 n t + T_n \sin \alpha^2 n t\right) B_n \sin \frac{n}{2} x$$

$$u(x,0) = \sum_{n=1}^{\infty} H_n B_n \sin \frac{n}{2} x = \sin 2x \Rightarrow$$

$$u_t(x,0) = \sum_{n=1}^{\infty} \left(-H_n \alpha^2 n \sin \alpha^2 n t + T_n \alpha^2 n \cos \alpha^2 n t\right) B_n \sin \frac{n}{2} x$$

$$\Rightarrow T_n \alpha^2 n \sin \frac{n}{2} x = 0 \Rightarrow T_n = 0 \Rightarrow u(x,t) = \sum_{n=1}^{\infty} \left(H_n \cos \alpha^2 n t\right) \sin \frac{n}{2} x$$

ب) $u_{tt} = c^2 u_{xx} \qquad 0 < x < L$

$$u(0,t) = u(L,t) = 0$$

$$u(x,0) = -4 \sin\left(\frac{5\pi x}{L}\right) + 3 \sin\left(\frac{\pi}{L} x\right)$$

$u_t(x,0) = 0 \rightarrow g(x) = 0$ یعنی براساس حل دالامبر داریم

$$u(x,t) = \frac{1}{2}\left(f(x+ct) + f(x-ct)\right)$$

$$= \frac{1}{2}\left(-4 \sin\left(\frac{5\pi(x+ct)}{L}\right) + 3 \sin\left(\frac{\pi}{L}(x+ct)\right) - 4 \sin\left(\frac{5\pi(x-ct)}{L}\right) + 3 \sin\left(\frac{\pi}{L}(x-ct)\right)\right)$$

$$= -4 \sin \frac{5\pi x}{L} \cos \frac{5\pi ct}{L} + 3 \sin \frac{\pi x}{L} \cos \frac{\pi ct}{L}$$

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۲- معادلات حرارت زیر را حل کنید.

$$u_t = u_{xx} + t e^{-t} + \cos(3\pi x)$$

$$u_x(0,t) = 1 \;;\; u_x(1,t) = 1$$

$$u(x,0) = x + \sin^2(\pi x)$$

$$u_t = u_{xx} + x - \frac{1}{2} + 2t - \sin^2\left(\frac{5\pi x}{4}\right)$$

$$0 < x < 1 \,,\; t > 0$$

$$u_x(0,t) = t \quad , \; u(1,t) = t^2$$

$$u(x,0) = 0$$

الف) $u_t = u_{xx} + e^{-t} + \cos 3\pi x$

$u_x(0,t) = 1 \quad u_x(1,t) = 1 \quad u(x,0) = x + \sin^2 \pi x$

$u(x,t) = w(x,t) + v(x)$

$$\Rightarrow u_t = u_{xx} + v''(x) + e^{-t} + \cos 3\pi x$$

$$\begin{cases} v''(x) = -\cos 3\pi x \rightarrow v'(x) = -\frac{1}{3\pi}\sin(3\pi x) + C \\ v'(0) = 1 \qquad v'(0) = C = 1 \\ v'(1) = 1 \qquad v'(1) = C = 1 \end{cases}$$

$$v(x) = \frac{1}{9\pi}\cos(3\pi x) + x$$

$$\Rightarrow \begin{cases} u_t = u_{xx} + e^{-t} \\ u_x(0,t) = 0 = u_x(1,t) = 0 \\ u(x,0) = -\frac{1}{9\pi}\cos(3\pi x) + \sin^2 \pi x \end{cases}$$

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با توجه به شرایط مرزی D $\quad u(x,t)=\sum_{n=1}^{\infty} G_n(t)\cos n\pi x + G_0(t)$

$$\Rightarrow \sum_{n=1}^{\infty} G_n'(t)\cos n\pi x + G_0'(t) = \sum_{n=1}^{\infty} -n^2\pi^2 G_n(t)\cos n\pi x + e^{-t}$$

$$\Rightarrow G_0'(t) = e^{-t} \longrightarrow \boxed{G_0(t) = -e^{-t} + c}$$

$$\Rightarrow \frac{G_n'(t)}{G_n(t)} = -n^2\pi^2 \Rightarrow G_n(t) = c_n e^{-n^2\pi^2 t}$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} c_n \cos n\pi x - 1 + c = -\frac{1}{9\pi}\cos(3\pi x) + \sin^2 \pi x = \frac{1}{9\pi}\cos(3\pi x) + \frac{1}{2}(1-\cos 2\pi x)$$

$$-1 + C = \frac{1}{2} \quad C = \frac{3}{2}$$

$$c_3 = -\frac{1}{9\pi} \qquad c_2 = -\frac{1}{2} \qquad c_i = 0 \quad i \neq 2,3$$

$$\Rightarrow u(x,t) = -e^{-t} + \frac{3}{2} - \frac{1}{2} e^{-4\pi^2 t}\cos 2\pi x - \frac{1}{9\pi} e^{-9\pi^2 t}\cos 3\pi x$$

$$\Rightarrow u(x,t) = e^{-t} + \frac{3}{2} - \frac{1}{2} e^{-4\pi^2 t}\cos 2\pi x - \frac{1}{9\pi} e^{9\pi^2 t}\cos 3\pi x + \frac{1}{9\pi}\cos 3\pi x + x$$

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ب-۴) $u_t = u_{xx} + x - \frac{1}{2} + 2t - \sin^2\left(\frac{2\pi x}{2}\right)$

$0 < x < 1,\ T > 0,\ u_x(0,t) = t,\ u(1,t) = t^2$

$u(x,0) = 0$

$\frac{G'(x)}{G(x)} = \frac{f''(x)}{f(x)} = P^2 = f''(x) + P^2 f(x) = 0$

$f(x) = A\cos P\pi + B\sin P\pi = u(0,t) = f(x)$

$= A = t \Rightarrow u(1,t) \to f(x) = t\cos P + B\sin P = t^2$

$B = t^2 \to f(x) = t\cos Px + t^2 \sin P\pi$

$G'(t) + P^2 G(t) = 0 \to G(t) = De^{x^2 n t}$

$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} b e^{-x^2 n t} (t\cos P\pi + t^2 \sin Px)$

$= u(x,0) = \sum_{n=1} + \cos Px + t^2 \sin Px = 0$

سوال ۵)

معادله موج زیر را حل کنید:

$u_{tt} = c^2 u_{xx},\ -\infty < x < \infty,\ t > 0$

$u(x,0) = \sin x,\ u_t(x,0) = \cos x$ I.C

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$$L\{u_{tt}\} = s^2 U(x,s) - s u(x,0) - u_t(x,0)$$

$$L\{u_{xx}\} = U_{xx}$$

$$\Rightarrow s^2 U(x,s) - s\sin x - \cos x = c^2 U_{xx}$$

$$\Rightarrow U_{xx} - \frac{s^2}{c^2} U = \frac{1}{c^2}(-s\sin x - \cos x)$$

$$U(x,s) = Ae^{-s/c\,x} + Be^{s/c\,x} + D\sin x + E\cos x$$

$$\Rightarrow -D\left(1 + \frac{s^2}{c^2}\right) = \frac{-s}{c^2}, \quad -E\left(1 + \frac{s^2}{c^2}\right) = \frac{-1}{c^2}$$

$$\Rightarrow D = \frac{s}{c^2(1 + s^2/c^2)} \qquad E = \frac{1}{c^2\left(1 + \frac{s^2}{c^2}\right)}$$

$$\Rightarrow D = \frac{s}{c^2 + s^2} \qquad E = \frac{1}{c^2 + s^2}$$

$$U(x,s) = Ae^{-s/c\,x} + Be^{s/c\,x} + \frac{s}{c^2+s^2}\sin x + \frac{1}{c^2+s^2}\cos x$$

$$\Rightarrow u(x,t) = L^{-1}[U(x,s)]$$

$$= L^{-1}\left[Ae^{-s/c\,x} + Be^{s/c\,x} + \frac{s}{c^2+s^2}\sin x + \frac{1}{c^2+s^2}\cos x\right]$$

$$\cos(ct)\sin x + \frac{1}{c}\sin(ct)\cos x$$

$$+ AL^{-1}\left[e^{-s/c\,x}\right] + BL^{-1}\left[e^{s/c\,x}\right]$$