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۸۱۰۱۰۰۰۸۴

۱- درصورتی‌که تابع $f(x)$ در $-\pi < x < \pi$ به‌صورت $f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n^2}$ ← $\int_0^{\pi} f(x)\sin^2 x\, dx$

$f(x)$ زوج است. $a_0 = \frac{1}{T}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{T}\int_0^{\pi} f(x)\,dx = 0$     $\sin^2 x = \frac{1-\cos 2x}{2}$

$$\int_0^{\pi} f(x)\,dx = 0$$

$$a_n = \frac{2}{T}\int_{-\pi}^{\pi} f(x)\cos nx = \frac{(-1)^n}{n^2} \Longrightarrow \frac{4}{2\pi}\int_0^{\pi} f(x)\cos nx = \frac{(-1)^n}{n^2}$$

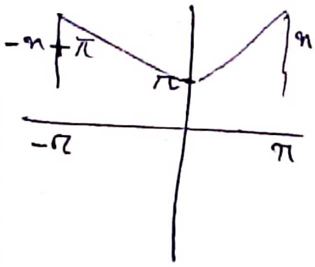
$$\int_0^{\pi} f(x)\sin^2 x\, dx = \int_0^{\pi} f(x)\frac{(1-\cos 2x)}{2}\,dx = \underbrace{\int_0^{\pi}\frac{f(x)}{2}\,dx}_{0} - \frac{1}{2}\int_0^{\pi} f(x)\cos 2x\, dx = \frac{-1}{2}$$

$$\frac{-1}{2}\int_0^{\pi} f(x)\cos 2x\, dx = \frac{-1}{2}\times\frac{(-1)^2}{2^2}\times\frac{\pi}{2} = \boxed{\frac{-\pi}{16}}$$

۲- سری فوریه کسینوسی توابع بند الف و بند ب و سری فوریه سینوسی توابع بند ج و بند د را به دست آورید.

الف) $f(x) = x + \pi \quad 0 \leqslant x < \pi$

سری کسینوسی باید بسط نیم‌دامنه زوج بدهیم




$f(x) = a_0 + \sum_{n=1} a_n = \cos(nt)$

$$a_0 = \frac{1}{T}\int_{-\pi}^{\pi} \underbrace{f(x)}_{\text{زوج}}\,dx = \frac{1\times 2}{2\pi}\int_0^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi}(x+\pi)\,dx$$

$$a_0 = \left(\frac{x}{2}^2 + \pi x\right)\times\frac{1}{\pi}\Big|_0^{\pi} = \left(\frac{\pi}{2}^2 + \pi^2\right) + \frac{1}{\pi} = \boxed{\frac{3}{2}\pi}$$

$$a_n = \frac{2}{\pi}\left(\frac{x+\pi}{n}\sin(nx) + \frac{\cos nx}{n^2}\right)\Big|_0^{\pi}$$

$$a_n = \frac{2}{\pi}\left(\left(\frac{2\pi}{n}\sin(n\pi)\right) + \frac{\cos(n\pi)}{n^2}\right) = \left(\frac{1}{n^2}\right) = \frac{2}{\pi}\left(\frac{\cos(n\pi - 1)}{n^2}\right)$$

$$a_n \begin{cases} 0 & n \text{ زوج} \\ \frac{-4}{n^2\pi} & n \text{ فرد} \end{cases}$$

$$f(x) = \frac{3}{2}\pi + \sum_{n=1}^{\infty}\frac{-4}{(2n-1)^2\pi}\cos(nx)$$

ب) $f(x) = \sin 3x \quad 0 \leqslant x < \pi$

نسبت نیم دامنه زوج به این دلیل = درصورت سوال گفته است کسینوسی

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1\times 2}{2\pi}\int_0^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi}\sin 3x\,dx = \frac{-\cos 3x}{3\pi}\Big|_0^{\pi} = \frac{1+1}{3\pi} = \frac{2}{3\pi}$$

$$a_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \quad \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx = \frac{2}{\pi}\int_0^{\pi}\sin(3x)\cos(nx)\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi}\frac{\sin(3+n)x - \sin(n-3)x}{2}\,dx = \frac{1}{\pi}\left(\frac{-\cos(3+n)x}{3+n}\right) + \left(\frac{\cos(n-3)x}{n-3}\right)\Big|_0^{\pi}$$

$$a_n = \frac{1}{\pi}\left(\left(\frac{\cos(n-3)\pi}{n-3} - \frac{\cos(3+n)\pi}{n+3}\right) - \left(\frac{-1}{3+n} + \frac{1}{n-3}\right)\right)$$

$$a_n = \frac{1}{\pi}\left(\left(\frac{\cos(n+1)\pi}{n-3} - \frac{\cos(n+1)\pi}{n+3}\right) - \left(\frac{6}{n^2-9}\right)\right) = \frac{1}{\pi}\left(\frac{\cos(n+1)\pi\times 6}{n^2-9} - \frac{6}{n^2-9}\right)$$

$$a_n = \begin{cases} \dfrac{-12}{(n^2-9)\pi} & \text{زوج} \\ 0 & \text{فرد} \end{cases} \qquad f(x) = \frac{2}{3\pi} + \sum_{n=1}^{\infty}\frac{-12}{(4n^2-9)\pi}\cos(nx)$$

ج) $f(x) = x^2 \quad 0 \le x < \pi$

بسط نیم‌دامنه فرد. ولی ضریب سینوسی است.

$$b_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} \underset{\text{فرد}}{f(x)}\,\underset{\text{فرد}}{\sin(nx)}\,dx \qquad b_n = \frac{2}{\pi}\int_0^{\pi} x^2\sin(nx)\,dx = \frac{2}{\pi}\left(\frac{2x\sin(nx)}{n^2} - \frac{x^2\cos(nx)}{n} - \frac{2\cos(nx)}{n^3}\right)\Big|_0^{\pi}$$

$$b_n = \frac{2}{\pi}\left(\left(\frac{-\pi^2\cos(n\pi)}{n} - \frac{2\cos(n\pi)}{n^3}\right) - \left(\frac{-2}{n^3}\right)\right)$$

$$b_n = \frac{2}{\pi}\left(-\cos n\pi\left(\frac{n^2\pi^2+2}{n^3}\right) + \frac{2}{n^3}\right) = \frac{2}{\pi}\left(\frac{(-1)^{n+1}(n^2\pi^2+2)+2}{n^3}\right)$$

$$f(x) = \sum \frac{2}{\pi}\left(\frac{(-1)^{n+1}(n^2\pi^2+2)+2}{n^3}\right)\sin(nx)$$

د) $f(x) = e^{2x} \quad 0 \le x < \pi$

بسط سینوسی ← بسط نیم دامنه فرد

$$b_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} \underset{\text{فرد}}{f(x)}\,\underset{\text{فرد}}{\sin(nx)}\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx = \frac{2}{\pi}\int_0^{\pi} e^{2x}\sin(nx)\,dx$$

$$b_n = \frac{2e^{2\pi}}{\pi} \frac{(2\sin(nx) - n\cos(nx))}{n^2+4}\Big|_0^{\pi} = \frac{2}{\pi} \frac{(n + e^{2\pi}(-n\cos(n\pi))}{(n^2+4)})$$

$$= \frac{2n(1 - e^{2\pi}\cos(n\pi))}{\pi(n^2+4)} \qquad b_n = \frac{2n(1-e^{2\pi}(-1)^n)}{\pi(n^2+4)}$$

$$f(x) = \sum_{n=1}^{\infty} \frac{2n}{\pi} \cdot \frac{(1-e^{2\pi}(-1)^n)}{(n^2+4)}$$

۳ - $|x|$ یک تابع زوج است. پس ضریب سینوسی آن در سری فوریه صفر است. پس فقط مقدار ثابت $(a_0)$ و ضریب کسینوسی داریم $(a_n)$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{f(x)}_{\text{زوج}} dx = \frac{1\times 2}{2\pi}\int_0^{\pi} x\,dx = \frac{1}{\pi} \frac{x^2}{2}\Big|_0^{\pi} = \boxed{\frac{\pi}{2}}$$

$$a_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} \underbrace{f(x)}_{\text{زوج}} \underbrace{\cos(nx)}_{\text{زوج}} dx = \frac{2}{\pi}\int_0^{\pi} x\cos(nx)\,dx = \frac{2}{\pi}\left(x\frac{\sin(nx)}{n} + \frac{\cos(nx)}{n^2}\right)\Big|_0^{\pi}$$

$$a_n = \frac{2}{\pi}\left(\frac{\cos(n\pi)}{n^2} - \frac{1}{n^2}\right) \begin{cases} 0 & n \text{ زوج} \\ \dfrac{-4}{\pi n^2} & n \text{ فرد} \end{cases} \qquad a_n = \frac{-4}{\pi(2n+1)^2} \quad n=0 \text{ تا } \infty \qquad f(x) = \frac{\pi}{2} + \sum_{n=0}^{\infty} \frac{-4}{\pi(2n+1)^2}$$

$\cos(2n+1)x$ انتگرال میگیریم

$$\int_{-\infty}^{0} |x|\,dx = \frac{\pi}{2}x + \sum_{n=0}^{\infty} \frac{-4}{\pi(2n+1)^3} \sin(2n+1)x$$

$$x \geq 0 \longrightarrow \frac{x^2}{2} = \frac{\pi}{2}x + \sum_{n=0}^{\infty}\frac{-4}{\pi(2n+1)^3}\sin(2n+1)x$$

$$x = \frac{\pi}{2} \longrightarrow \frac{\pi^2}{8} = \frac{\pi^2}{4} + \sum_{n=0}^{\infty} \frac{-4}{\pi(2n+1)^3}(-1)^n \qquad \frac{3\pi^2}{8} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} \times \frac{4}{\pi}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} = \frac{\pi^3}{32}$$

$0 < x < \pi$ $\qquad \sum_{n=1}^{\infty} \dfrac{1}{(4n^2-1)^2}$ $\qquad a_0 = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} \underbrace{f(x)}_{\text{زوج}} = \dfrac{1}{\pi}\int_0^{\pi} f(x)\,dx = \dfrac{1}{\pi}\int_0^{\pi}$

$\sin(x)\,dx$ $\qquad a_0 = -\dfrac{\cos(x)}{\pi}\Big|_0^{\pi} = \dfrac{1+1}{\pi} = \dfrac{2}{\pi}$

$$a_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} \underbrace{f(x)}_{\text{زوج}} \underbrace{\cos(nx)}_{\text{زوج}}\,dx = \frac{2}{\pi}\int_0^{\pi} \sin(x)\cos(nx)\,dx$$

$$a_n=\frac{2}{\pi}\left(\frac{n\sin(n)\sin(nx)+\cos(x)\cos(nx)}{n^2-1}\right)\Big|_0^{\pi} \qquad a_n=\frac{2}{\pi}\left(\left(\frac{-\cos n\pi}{n^2-1}\right)-\frac{dx}{n^2-1}\right)=$$

$$\frac{-2}{\pi}\left(\frac{\cos(n\pi)+1}{n^2-1}\right) \qquad n \begin{cases} \text{زوج} & \frac{-4}{\pi(n^2-1)} \\ \text{فرد} & 0 \end{cases} \qquad f(x)=\frac{2}{\pi}+\sum_{n=1}^{\infty}\frac{-4}{\pi(4n^2-1)}\cos(nx)$$

قانون بارسوال

$$p=\frac{1}{2\pi}\int_{-\pi}^{\pi}\underbrace{f(x)^2}_{\text{زوج}}dx=\frac{1}{\pi}\int_0^{\pi}\sin^2 x\,dx=\frac{1}{\pi}\times\frac{\pi}{2}=\boxed{\frac{1}{2}}$$

$$\frac{1}{2}=\frac{a_0^2}{\frac{4}{\pi^2}}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-4)^2}{\pi^2(4n^2-1)^2} \qquad \sum_{n=1}^{\infty}\frac{1}{(4n^2-1)^2}=\left(\frac{1}{2}-\frac{4}{\pi^2}\right)\times\frac{2}{16}$$

$$=\frac{\pi^2}{8}\left(\frac{1}{2}-\frac{4}{\pi^2}\right)$$

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$$x=\frac{\pi}{2}-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos(2n-1)x$$

انتگرال :
$$\frac{x^2}{2}=\frac{\pi}{2}x-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{(2n-1)^3}$$

$$\longrightarrow \frac{\pi x}{2}-\frac{x^2}{2}=\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3} \qquad \sin(2n-1)x \longrightarrow g(x)=x(\pi-x)$$

$$=\frac{8}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{(2n-1)^3}$$