دانشکده فنی دانشگاه تهران

دانشکده مهندسی برق و کامپیوتر

تمرین چهارم درس ریاضیات مهندسی

طراح
آرمان مجیدی

نیمسال دوم ۱۴۰۲-۱۴۰۳

سوال ۱

معادله موج داده شده را حل کنید.

$$9u_{xx} = u_{tt}, \quad 0 < x < \pi$$

$$\begin{cases} u_x(0,t) = 0, \quad u_x(\pi,t) = 3, \;\; t > 0 \\ u(x,0) = 0, \quad u_t(x,0) = \cos(3x) + \sin(2x) \end{cases}$$

پاسخ سوال ۱

$$u(x,t) = v(x,t) + w(x,t) \Rightarrow w(x,t) = 0 + \frac{x^2}{2\pi}(3-0)$$

$$\Rightarrow u(x,t) = v(x,t) + \frac{3x^2}{2\pi}$$

$$\Rightarrow v(x,0) = -\frac{3x^2}{2\pi} \quad ; \quad v_t(x,0) = \cos(3x) + \sin(2x)$$

$$\Rightarrow 9v_{xx} + \frac{27}{\pi} = v_{tt} \quad \begin{cases} v(0,t) = 0, \quad v(\pi,t) = 0 \\ v(x,0) = -\frac{3x^2}{2\pi}, \quad v_t(x,0) = \cos(3x) + \sin(2x) \end{cases}$$

$$\text{B.C. : Neumann} \xrightarrow{\text{جواب حدسی}} v(x,t) = \sum_{n=0}^{\infty} T_n(t)\cos(nx)$$

$$9v_{xx} + \frac{27}{\pi} = v_{tt} \Rightarrow \sum_{n=0}^{\infty}\left[\ddot{T}_n(t) + 9n^2 T_n(t)\right]\cos(nx) = \frac{27}{\pi}$$

$$\Rightarrow \ddot{T}_n(t) + 9n^2 T_n(t) = \frac{2}{\pi}\int_0^{\pi}\left(\frac{27}{\pi}\right)\cos(nx)dx \xrightarrow{n=۰} \ddot{T}_0(t) = \frac{27}{\pi} \Rightarrow T_0(t) = \frac{27}{2\pi}t^2 + Ct + D$$

$$\xrightarrow{n \neq 0} \lambda^2 + 9n^2, 0 \to \lambda = \pm i(3n)$$

$$\Rightarrow T_n(t) = A_n\cos(3nt) + B_n\sin(3nt), \quad n \neq 0$$

$$\Rightarrow v(x,t) = T_0(t) + \sum_{n=1}^{\infty} A_n\cos(3nt)\cos(nx) + \sum_{n=1}^{\infty} B_n\sin(3nt)\cos(nx)$$

$$v(x,0) = -\frac{3x^2}{2\pi} \to D + \sum_{n=0}^{\infty} A_n\cos(nx) = -\frac{3x^2}{2\pi} \Rightarrow D = \frac{1}{\pi}\int_0^{\pi} -\frac{3x^2}{2\pi}dx = -\frac{\pi}{2}$$

$$A_n = \frac{2}{\pi}\int_0^{\pi}\left(-\frac{3x^2}{2\pi}\right)\cos(nx)dx = -\frac{6(-1)^n}{\pi n^2}$$

$$\Rightarrow v_t(x,t) = \dot{T}_0(t) + \sum_{n=1}^{\infty} 3n\left[B_n\cos(3nt) - A_n\sin(3nt)\right]\cos(nx)$$

$$v_t(x,0) = \cos(3x) + \sin(2x) \to C + \sum_{n=1}^{\infty} 3nB_n\cos(nx) = \cos(3x) + \sin(2x)$$

$$\Rightarrow C = \frac{1}{\pi}\int_0^{\pi}\cos(3x) + \sin(2x)dx = 0$$

$$\Rightarrow 3nB_n = \frac{2}{\pi}\int_0^{\pi}(\cos(3x) + \sin(2x))\cos(nx)dx = \begin{cases} 0 & \text{if } n : \text{even} \\ \frac{8}{\pi(n^2-4)} & \text{if } n : \text{odd} \end{cases} \quad , \quad n \neq 2,3$$

$$\text{if} \quad n = 2 \to 6B_2 = \frac{2}{\pi}\int_0^{\pi}(\cos(3x) + \sin(2x))\cos(2x)dx = 0 \Rightarrow B_2 = 0$$

$$\text{if} \quad n=3 \to 9B_3 = \frac{2}{\pi}\int_0^{\pi} (\cos(3x)+\sin(2x))\cos(3x)dx = 1-\frac{8}{5\pi} \Rightarrow B_3 = \frac{1}{9}-\frac{8}{45\pi}$$

$$\Rightarrow v(x,t) = T_0(t) + \sum_{n=0}^{\infty}\left(-\frac{6(-1)^n}{\pi n^2}\right)\cos(3nt)\cos(nx)$$

$$+\sum_{n:odd,n\neq 3}^{\infty}\left(\frac{8}{3n\pi\left(n^2-4\right)}\right)\sin(3nt)\cos(nx) + \left(\frac{1}{9}-\frac{8}{45\pi}\right)\sin(9t)\cos(3x); \quad T_0(t) = \frac{27}{2\pi}t^2 - \frac{\pi}{2}$$

$$u(x,t) = v(x,t) + w(x,t) = v(x,t) + \frac{3x^2}{2\pi}$$

سوال ۲

معادله گرما داده شده را حل کنید.

$$\frac{1}{4}\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2} \qquad 0<x<2\pi,\ \ 0<t$$

$$\begin{cases} u(0,t)=0 \ \ u(2\pi,t)=0 \\ u(x,0)=\delta(x-\frac{1}{2}) \end{cases}$$

پاسخ سوال ۲

$$\xrightarrow{\text{جواب حدسی}} u(x,t)=\sum_{n=1}^{\infty}T_n(t)\sin\left(\frac{n}{2}x\right)$$

$$\frac{1}{4}\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}\rightarrow\sum_{n=1}^{\infty}\left[\dot{T}_n(t)+n^2T_n(t)\right]\sin\left(\frac{n}{2}x\right)=0$$

$$\Rightarrow \dot{T}_n(t)^2+T_n(q)=0\Rightarrow ODE\rightarrow\lambda_n+n^2=0\rightarrow\lambda_n=-n^2$$

$$\Rightarrow T_n(t)=A_ne^{-n^2t}$$

$$u(x,t)=\sum_{n=1}^{\infty}A_ne^{-n^2t}\sin\left(\frac{n}{2}x\right)\Rightarrow u(x,0)=\sum_{n=1}^{\infty}A_n\sin\left(\frac{n}{2}x\right)$$

$$\Rightarrow\sum_{n=1}^{\infty}A_n\sin\left(\frac{n}{2}x\right)=\delta\left(x-\frac{1}{2}\right)$$

$$\longrightarrow A_n=\frac{1}{\pi}\int_0^{2\pi}\delta\left(x-\frac{1}{2}\right)\sin\left(\frac{n}{2}x\right)dx\rightarrow A_n=\frac{1}{\pi}\sin\left(\frac{n}{4}\right)$$

$$\Rightarrow u(x,t)=\sum_{n=1}^{\infty}\sin\left(\frac{n}{4}\right)e^{-n^2t}\sin\left(\frac{n}{2}x\right)$$

سوال ۳

معادله گرما غیرهمگن داده شده را حل کنید.

$$u_t = 4u_{xx} + \Pi(\frac{x-\pi}{2\pi}), \;\; 0 < x < 2\pi, \;\; 0 < t$$

$$\begin{cases} u(0,t) = 0, \;\; u(2\pi, t) = 1 \\ u(x,0) = \Pi(\frac{x}{2\pi}) + \frac{x}{2\pi} \end{cases}$$

پاسخ سوال ۳

$$u(x,t) = v(x,t) + w(x,t) \Rightarrow w(x,t) = 0 + \frac{x}{2\pi}(1-0)$$

$$\Rightarrow u(x,t) = v(x,t) + \frac{x}{2\pi}$$

$$\Rightarrow v(x,0) = \Pi\left(\frac{x}{2\pi}\right)$$

$$\Rightarrow v_t = 4v_{xx} \quad \begin{cases} v(0,t) = 0, \;\; v(2\pi,t) = 0 \\ v(x,0) = \Pi(\frac{x}{2\pi}) \end{cases}$$

$$\xrightarrow{\text{جواب حدسی}} v(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin\left(\frac{n}{2}x\right)$$

$$\Rightarrow \sum_{n=1}^{\infty} \left[\dot{T}_n(t) + n^2 T_n(t)\right] \sin\left(\frac{n}{2}x\right) = \Pi\left(\frac{x-\pi}{2\pi}\right)$$

$$\Rightarrow \dot{T}_n(t) + n^2 T_n(t) = \frac{1}{\pi}\int_0^{2\pi} \Pi\left(\frac{x-\pi}{2\pi}\right) \sin\left(\frac{n}{2}x\right) dx$$

$$\Rightarrow \dot{T}_n(t) + n^2 T_n(t) = \frac{2}{n\pi}\left[1 - \cos(\pi n)\right] = \begin{cases} 0 & \text{if } n : \text{even} \\ \frac{4}{n\pi} & \text{if } n : \text{odd} \end{cases}$$

$$T_n(t) = A_n e^{-n^2 t} + \frac{4}{\pi n^3}$$

$$\Rightarrow v(x,t) = \sum_{n=1}^{\infty} A_n e^{-n^2 t} \sin\left(\frac{n}{2}x\right) + \sum_{n:odd}^{\infty} \frac{4}{\pi n^3} \sin\left(\frac{n}{2}x\right)$$

$$v(x,0) = \Pi(\frac{x}{2\pi}) \Rightarrow \sum_{n=1}^{\infty} A_n \sin\left(\frac{n}{2}x\right) + \sum_{n:odd}^{\infty} \frac{4}{\pi n^3} \sin\left(\frac{n}{2}x\right) = \Pi(\frac{x}{2\pi})$$

$$\left[A_n + \frac{2}{\pi n^3}(1 - (-1)^n))\right] = \frac{1}{\pi}\int_0^{2\pi} \Pi(\frac{x}{2\pi}) \sin\left(\frac{n}{2}x\right) dx = \frac{1}{\pi}\int_0^{\pi} \sin\left(\frac{n}{2}x\right) dx$$

$$\Rightarrow A_n + \begin{cases} 0 & \text{if } n : \text{even} \\ \frac{4}{\pi n^3} & \text{if } n : \text{odd} \end{cases} = \begin{cases} \frac{2}{n\pi}\left[1 - (-1)^{\frac{n}{2}}\right] & \text{if } n : \text{even} \\ \frac{2}{n\pi} & \text{if } n : \text{odd} \end{cases}$$

$$A_n = \begin{cases} \frac{2}{n\pi}\left[1 - (-1)^{\frac{n}{2}}\right] & \text{if } n : \text{even} \\ \frac{2}{n\pi} - \frac{4}{\pi n^3} & \text{if } n : \text{odd} \end{cases}$$

$$v(x,t) = \sum_{n:even}^{\infty} \frac{2}{n\pi}\left[1-(-1)^{\frac{n}{2}}\right] e^{-n^2 t}\sin\left(\frac{n}{2}x\right) + \sum_{n:odd}^{\infty}\left[\left(\frac{2}{n\pi}-\frac{4}{\pi n^3}\right)e^{-n^2 t}+\frac{4}{\pi n^3}\right]\sin\left(\frac{n}{2}x\right)$$

سوال ۴

معادله موج زیر را حل کنید

$$u_{tt} = u_{xx}, \;\; 0 < x < 1, \;\; 0 < t$$

$$\begin{cases} u_x(0,t) = t - 6, \;\; u(1,t) = 7t \\ u(x,0) = 6 - 6x, \;\; u_t(x,0) = \Lambda\,(x-1) \end{cases}$$

پاسخ سوال ۴

$$u(x,t)=v(x,t)+w(x,t)\rightarrow w(x,t)=(x-1)(t-6)+7t$$
$$u(x,t)=v(x,t)+(x-1)(t-6)+7t$$
$$\Rightarrow v(x,0)=(6-6x)-((x-1)(-6))=0,\quad v_t(x,0)=\Lambda(x-1)-(x+6)$$

$$v_{tt}=v_{xx}\quad \begin{cases} v_x(0,t)=0, & v(1,t)=0\\ v(x,0)=0, & v_t(x,0)=\Lambda\left(x-1\right)-x-6\end{cases}$$

$$\xrightarrow{\text{جواب حدسی}} v(x,t)=\sum_{n=1}^{\infty}T_n(t)\cos\left(\frac{2n-1}{2}\pi x\right)$$
$$\Rightarrow \sum_{n=1}^{\infty}\left[\ddot{T}_n(t)+\left(\frac{2n-1}{2}\pi\right)^2 T_n(t)\right]\cos\left(\frac{2n-1}{2}\pi x\right)=0$$
$$\Rightarrow \ddot{T}_n(t)+\left(\frac{2n-1}{2}\pi\right)^2 T_n(t)=0\rightarrow \lambda_n=\pm i\left(\frac{2n-1}{2}\pi\right)$$
$$\Rightarrow T_n(t)=A_n\cos\left(\frac{2n-1}{2}\pi t\right)+B_n\sin\left(\frac{2n-1}{2}\pi t\right)$$
$$v(x,t)=\sum_{n=1}^{\infty}A_n\cos\left(\frac{2n-1}{2}\pi t\right)\cos\left(\frac{2n-1}{2}\pi x\right)+\sum_{n=1}^{\infty}B_n\sin\left(\frac{2n-1}{2}\pi t\right)\cos\left(\frac{2n-1}{2}\pi x\right)$$
$$\Rightarrow v(x,0)=\sum_{n=1}^{\infty}A_n\cos\left(\frac{2n-1}{2}\pi x\right)=0\rightarrow A_n=0$$
$$v_t(x,t)=\sum_{n=1}^{\infty}\left(\frac{2n-1}{2}\pi\right)B_n\cos\left(\frac{2n-1}{2}\pi t\right)\cos\left(\frac{2n-1}{2}\pi x\right)$$
$$\Rightarrow v_t(x,0)=\sum_{n=1}^{\infty}\left(\frac{2n-1}{2}\pi\right)B_n\cos\left(\frac{2n-1}{2}\pi x\right)=\Lambda\left(x-1\right)-x-6$$
$$\Rightarrow \left(\frac{2n-1}{2}\pi\right)B_n=2\int_0^1\left[\Lambda\left(x-1\right)-x-6\right]\cos\left(\frac{2n-1}{2}\pi x\right)dx$$
$$\Rightarrow \left(\frac{2n-1}{4}\pi\right)B_n=\int_0^1 -6\cos\left(\frac{2n-1}{2}\pi x\right)dx$$

$$\Rightarrow B_n = \left(\frac{4}{\pi(2n-1)}\right)\left(-\frac{12\cos(\pi n)}{\pi - 2\pi n}\right) = \begin{cases} \frac{48}{(\pi-2\pi n)^2} & \text{if } n : \text{even} \\ \frac{-48}{(\pi-2\pi n)^2} & \text{if } n : \text{odd} \end{cases}$$

$$v(x,t) = \sum_{n:even}^{\infty} \frac{48}{(\pi-2\pi n)^2} \sin\left(\frac{2n-1}{2}\pi t\right)\cos\left(\frac{2n-1}{2}\pi x\right)$$
$$+ \sum_{n:odd}^{\infty} \frac{-48}{(\pi-2\pi n)^2} \sin\left(\frac{2n-1}{2}\pi t\right)\cos\left(\frac{2n-1}{2}\pi x\right)$$

$$u(x,t) = v(x,t) + w(x,t) = \sum_{n:even}^{\infty} \frac{48}{(\pi-2\pi n)^2} \sin\left(\frac{2n-1}{2}\pi t\right)\cos\left(\frac{2n-1}{2}\pi x\right)$$
$$+ \sum_{n:odd}^{\infty} \frac{-48}{(\pi-2\pi n)^2} \sin\left(\frac{2n-1}{2}\pi t\right)\cos\left(\frac{2n-1}{2}\pi x\right) + (x-1)(t-6) + 7t$$

سوال ۵

معادله گرما داده شده را حل کنید.

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \;\; 0 < x, \;\; t > 0$$

$$\begin{cases} u(0,t) = e^{-6t} \\ u(x,0) = xsinc(x) \end{cases}$$

سوال ۶

معادله موج داده شده را با شرایط زیر حل کنید.

$$u_{xx} - u_{tt} = 7xt, \quad 0 < x < 2$$

$$\begin{cases} u(0,t) = 4, \quad u(2,t) = 7 \\ u(x,0) = x^2 + \frac{3}{2}x, \quad u_t(0,x) = 2 \end{cases}$$

پاسخ سوال ۶

$$u_{xx} - u_{tt} = 7xt$$

$$u(x,t) = v(x,t) + w(x,t)$$

$$w(x,t) = 4 + \frac{x}{2}(7-4) = \frac{3x}{2} + 4$$

$$\Rightarrow u(x,t) = v(x,t) + \frac{3x}{2} + 4$$

$$\Rightarrow v_{xx} - v_{tt} = 7xt$$

$$\begin{cases} v(0,t) = 0, \quad v(2,t) = 0 \\ v(x,0) = x^2 - 4, \quad v_t(x,0) = 2 \end{cases}$$

$$\xrightarrow{\text{جواب حدسی}} v(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin\left(\frac{n\pi}{2}x\right)$$

$$\Rightarrow \sum_{n=1}^{\infty} \left[\ddot{T}_n(t) + \left(\frac{n\pi}{2}\right)^2 T_n(t)\right] \sin\left(\frac{n\pi}{2}x\right) = -7xt$$

$$\ddot{T}_n(t) + \left(\frac{n\pi}{2}\right)^2 T_n(t) = \int_0^2 -7xt \sin\left(\frac{n\pi}{2}\right)\, dx$$

$$\Rightarrow \ddot{T}_n(t) + \left(\frac{n\pi}{2}\right)^2 T_n(t) = \begin{cases} \frac{28t}{\pi t} & \text{if } n : \text{even} \\ \frac{-28t}{\pi t} & \text{if } n : \text{odd} \end{cases}$$

$$\Rightarrow T_n(t) = A_n \cos\left(\frac{n\pi}{2}t\right) + B_n \sin\left(\frac{n\pi}{2}t\right) + \begin{cases} \frac{56t}{(\pi t)^3} & \text{if } n : \text{even} \\ \frac{-56t}{(\pi t)^3} & \text{if } n : \text{odd} \end{cases}$$

$$\rightarrow v(x,t) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{2}t\right) \sin\left(\frac{n\pi}{2}x\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{2}t\right) \sin\left(\frac{n\pi}{2}x\right)$$

$$+ \sum_{n=1}^{\infty} (-1)^n \frac{56t}{(\pi n)^3} \sin\left(\frac{n\pi}{2}\pi\right)$$

$$v(x,0) = x^2 - 4$$

$$\Rightarrow \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{2}x\right) = x^2 - 4$$

$$\Rightarrow A_n = \int_0^2 \left(x^2 - 4\right) \sin\left(\frac{n\pi}{2}x\right) dx \Rightarrow A_n = -\frac{8\left(\pi^2 n^2 - 2\cos(n\pi) + 2\right)}{\pi^3 n^3}$$

$$\Rightarrow A_n = \begin{cases} \frac{-8}{\pi n} & \text{if } n : \text{even} \\ \frac{-8}{\pi n} + \frac{-32}{(\pi n)^3} & \text{if } n : \text{odd} \end{cases}$$

$$v_t(x,t) = \sum_{n=1}^{\infty} \frac{n\pi}{2} B_n \cos\left(\frac{n\pi}{2}t\right) \sin\left(\frac{n\pi}{2}x\right) + \sum_{n=1}^{\infty} (-1)^n \frac{56}{(\pi n)^3} \sin\left(\frac{n\pi}{2}x\right)$$

$$v_t(x,0) = 2 \longrightarrow \left[\frac{n\pi}{2} B_n + (-1)^n \frac{56}{(\pi n)^3}\right] = \int_0^2 2\sin\left(\frac{n\pi}{2}x\right)$$

$$\Rightarrow \frac{n\pi}{2} B_n = -(-1)^n \frac{56}{(\pi n)^3} + 4\left(\frac{1 - \cos(n\pi)}{n\pi}\right)$$

$$\Rightarrow B_n = \begin{cases} \frac{-112}{(\pi n)^4} & \text{if } n : \text{even} \\ \frac{8}{(\pi n)^2} + \frac{112}{(\pi n)^4} & \text{if } n : \text{odd} \end{cases}$$

$$\rightarrow v(x,t) = \sum_{n:even}^{\infty} \left[\frac{-8}{\pi n}\right] \cos\left(\frac{n\pi}{2}t\right) \sin\left(\frac{n\pi}{2}x\right) + \sum_{n:odd}^{\infty} \left[\frac{-8}{\pi n} + \frac{32}{\pi^3 n^3}\right] \cos\left(\frac{n\pi}{2}t\right) \sin\left(\frac{n\pi}{2}x\right)$$

$$+ \sum_{n:even}^{\infty} \left[\frac{-112}{\pi^4 n^4}\right] \sin\left(\frac{n\pi}{2}t\right) \sin\left(\frac{n\pi}{2}x\right) + \sum_{n:odd}^{\infty} \left[\frac{8}{\pi^2 n^2} + \frac{112}{\pi^4 n^4}\right] \sin\left(\frac{n\pi}{2}t\right) \sin\left(\frac{n\pi}{2}x\right)$$

$$+ \sum_{n=1}^{\infty} (-1)^n \frac{56t}{(\pi n)^3} \sin\left(\frac{n\pi}{2}\pi\right)$$

$$\rightarrow u(x,t) = v(x,t) + w(x,t)$$

$$\Rightarrow u(x,t) = \sum_{n:even}^{\infty} \left[\frac{-8}{\pi n}\right] \cos\left(\frac{n\pi}{2}t\right) \sin\left(\frac{n\pi}{2}x\right) + \sum_{n:odd}^{\infty} \left[\frac{-8}{\pi n} + \frac{32}{\pi^3 n^3}\right] \cos\left(\frac{n\pi}{2}t\right) \sin\left(\frac{n\pi}{2}x\right)$$

$$+ \sum_{n:even}^{\infty} \left[\frac{-112}{\pi^4 n^4}\right] \sin\left(\frac{n\pi}{2}t\right) \sin\left(\frac{n\pi}{2}x\right) + \sum_{n:odd}^{\infty} \left[\frac{8}{\pi^2 n^2} + \frac{112}{\pi^4 n^4}\right] \sin\left(\frac{n\pi}{2}t\right) \sin\left(\frac{n\pi}{2}x\right)$$

$$+ \sum_{n=1}^{\infty} (-1)^n \frac{56t}{(\pi n)^3} \sin\left(\frac{n\pi}{2}\pi\right) + \left(\frac{3x}{2} + 4\right)$$

نکات کلی درباره تمرین

- در صورتی که در تمرین هر گونه ابهام و یا پرسشی دارید می‌توانید با آرمان مجیدی در ارتباط باشید.
- در صورتی که سوالی از تمرین دارید که ممکن است برای دیگران نیز مفید باشد،آن را در گروه درس مطرح کنید.
- مشورت و همفکری با دوستان خود هنگام نوشتن تمرین کاری مفید و سازنده است و از انجام آن پرهیز نکنید، اما این کار باید در راستای فهم درس و تمرین باشد و از کپی‌کردن تمارین یکدیگر خودداری کنید.
- پاسخ های خود را به صورت یک فایل به فرمت PDF در سامانه درس با فرمت نامگذاری Engmath-HWNum-SID بارگذاری نمایید.