تمرینات بخش ۷-۲ جزوه

(۱)

$a)\ |\bar{z}_1+\bar{z}_2|^2 = |x_1+x_2+(y_1+y_2)i|^2 = (x_1+x_2)^2+$

$(y_1+y_2)^2 \Rightarrow |\bar{z}_1+\bar{z}_2|^2 = \underbrace{x_1^2+x_2^2+2x_1x_2+y_1^2+y_2^2+2y_1y_2}$

$= |z_1|^2+|z_2|^2+2(x_1x_2+y_1y_2) = z_1\bar{z}_2 = (x_1x_2+y_1y_2)+$

$(-x_1y_2+y_1x_2)i \Rightarrow Re(z_1\bar{z}_2) = x_1x_2+y_1y_2 \Rightarrow$

$|z_1+z_2|^2 = |z_1|^2+|z_2|^2+2\,Re(z_1\bar{z}_2)$

(۲) $z_1 = x_1+iy_1$ و $z_2 = x_2+iy_2$

(۲)

چون z_1 و z_2 دو بردار غیر هم خط و نا موازی هستند

$\frac{x_1}{x_2} \neq \frac{y_1}{y_2}$

$az_1+az_2=0 \Rightarrow ax_1+iay_1+ax_2+iby_2=0 \Rightarrow$

$$\begin{cases} ax_1+bx_2=0 \\ ay_1+by_2=0 \end{cases}$$

حالا فرض کنیم که a و b مخالف صفر باشند

$* \Rightarrow \begin{matrix} ax_1=-bx_2 \\ ay_1=-by_2 \end{matrix} \Rightarrow \frac{ax_1}{ay_1} = \frac{-bx_2}{-by_2} \Rightarrow$

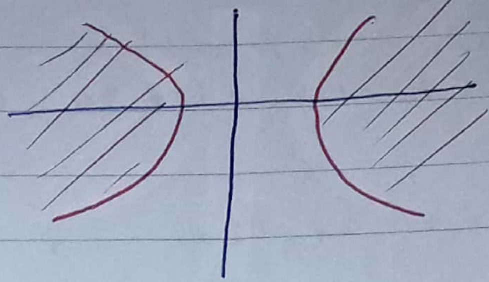
$\frac{x_1}{y_1} = \frac{x_2}{y_2} \Rightarrow$ حکم بدست آمده مخالف فرض است

۴

الزاماً $a=0 \,,\, b=0 \Leftarrow az_1+bz_2>0$

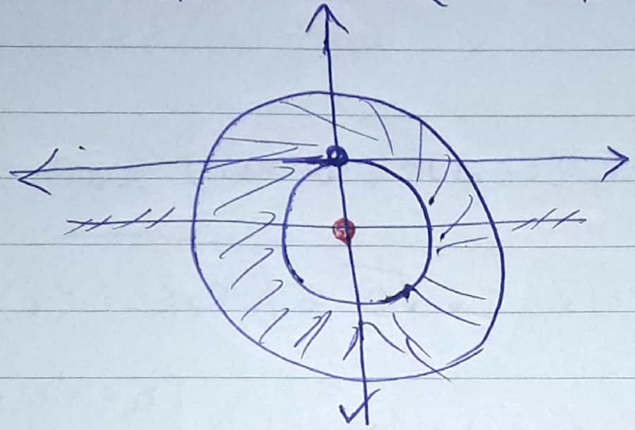
a) $(z_1)^2 = (x_1^2-y_1^2+2xy_1 i) \Rightarrow Re(z^2) = x^2-y^2$

$Re(z^2)>1 \Rightarrow x^2-y^2>1$



b) $1 < |z+i| \leq 2 \Rightarrow$

$1 < |x+(y+1)i| \leq 2 \Rightarrow 1 < x^2+(y+1)^2 \leq 4$



۵

$kz^2+akz+a^2=0 \Rightarrow z = \dfrac{-ak \pm \sqrt{a^2k^2-4ka^2}}{2k} \Rightarrow$

$z = \dfrac{-a\cancel{k}}{2\cancel{k}} \pm \dfrac{(\sqrt{4k-k^2}\,a)i}{2k} \Rightarrow k = \dfrac{\sqrt{4k-k^2}\,a}{2k}$

$, b = \pm\dfrac{a}{2} \Rightarrow \sqrt{3}\,\dfrac{a\sqrt{4k-k^2}}{\cancel{2}k} = \dfrac{a}{\cancel{2}} \Rightarrow k = \sqrt{3(4k-3k^2)} \Rightarrow$

$k^2 = 12k-3k^2 \Rightarrow 4k^2 = 12k \,,\, k=0$ یا $k=3$

$$A = z^3 - (1+4i)z^2 - (3-9i)z + 14 - 2i = 0$$

چون گفته ریشه محض دارد

$$(xi)^3 - (1+4i)(xi)^2 + (9i-3)xi + 14 - 2i = 0$$

$$\Rightarrow xi + x^2 + 4ix^2 - 9x - 3xi + 14 - 2i = 0$$

$$\Rightarrow x^2 - 9x + 14 = 0 \longrightarrow \begin{cases} x = 2 \\ \alpha = 7 \end{cases} \Rightarrow \boxed{\alpha = 2i}$$

$$-x^3 + 4x^2 - 3x - 2 = 0$$

$$\Rightarrow A = (z-2i)(z^2 - (2i+1)z + (1+7i)) = 0$$

$$z^2 - (2i+2)z + (1+7i) = 0 \quad \sqrt{\Delta} = \sqrt{-3+4i-4-28i} =$$

$$i\sqrt{24i+7} \Rightarrow i(4+3i) = (4i-3) \Rightarrow \begin{cases} z_1 = \frac{2i+1+4i-3}{2} = 3i-1 \\ z_2 = \frac{2i+1-4i+3}{2} = 2-i \end{cases}$$

$a+bi$ و $a=4$ $b=3$ $a+bi$ $a^2+b^2 = 25$

تمرینات کتبی صفحه ۸ کتاب:

$$z = \frac{1}{2i\left(\frac{1}{2}-i\right)} = \frac{1}{i+2} = \frac{1}{2+i} \times \frac{2-i}{2-i} =$$

$$\frac{2-i}{4+1} = \frac{2}{5} - \frac{1}{5}i$$

(10) $\frac{3}{4-2i} = \frac{3}{4-2i} \times \frac{4+2i}{4+2i} = \frac{12+6i}{16+4} = \frac{12}{20} + \frac{6}{20} i$

(11) a) $\overline{z+w} = \bar{z} + \bar{w} \Rightarrow (a+c)+(b+d)i = a+c-(b+d)i =$

$\underbrace{(a-bi)}_{\bar{z}} + \underbrace{(c-di)}_{\bar{w}} = \bar{z}+\bar{w}$

b) $\overline{zw} = \bar{z}\bar{w} \Rightarrow \overline{zw} = \overline{((ac-bd)+(ad+cb)i)} =$

$ac-bd-(ad+cb)i = A$

$\bar{z}\bar{w} = (a-bi)(c-di) = ac-bd+(-ad-cb)i = B$

$A = B \Rightarrow \overline{wz} = \bar{z}.\bar{w}$

c) $\overline{zz} = \bar{z} \times \bar{z} = (\bar{z})^2 \Rightarrow \overline{\underbrace{z \times z \times z \cdots \times z}_{n}} = (\bar{z}) \times (\bar{z}) \times$

$\times \bar{z} \times \cdots \bar{z} = (\bar{z})^n \Rightarrow$

$z^2 + \frac{1}{2} z + \frac{1}{4} = 0 \rightarrow 4z^2 + 2z + 1 = 0 \rightarrow z = \frac{-2 \pm \sqrt{4-16}}{8}$

$\Rightarrow z_1 = \frac{-2+\sqrt{12}\, i}{8}$ $\qquad z_2 = \left(\frac{-2-\sqrt{12}\, i}{8}\right)$

$z_1 = \frac{-1+\sqrt{3}\, i}{4}$ $\qquad z_2 = \frac{-1-\sqrt{3}\, i}{4}$

تمرینات بخش ۷-۴ تا ۷-۵ جزوه

۱

$z = -15 + 8i \rightarrow r = 17, \quad \cos\theta = \frac{-15}{17} \quad \sin\theta = \frac{8}{17}$

$\sqrt{z} = \sqrt{r}\,\text{cis}\left(\frac{\theta}{2} + \frac{2k\pi}{2}\right) \quad k = 0, 1 \Rightarrow$

$\sqrt{z} = \sqrt{r}\left(\text{cis}\,\frac{\theta}{2}\right) = \sqrt{17}\cos\frac{\theta}{2} + i\sqrt{17}\sin\frac{\theta}{2} = 1 + 4i$ ✓

$\sqrt{z} = \sqrt{r}\,\text{cis}\left(\frac{\theta}{2} + \pi\right) = -\sqrt{17}\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\sqrt{17} = -1 - 4i$ ✓

$\theta \simeq 2.8 \quad \frac{\theta}{2} = 1.4$

$\theta > 90$

$\frac{\cos\theta + 1}{2} = \cos^2\frac{\theta}{2} \Rightarrow \cos^2\frac{\theta}{2} = \frac{1}{17} \Rightarrow \cos\frac{\theta}{2} = \frac{1}{\sqrt{17}}$

$\theta > 90$

$\frac{1 - \cos\theta}{2} = \sin^2\frac{\theta}{2} \Rightarrow \sin^2\frac{\theta}{2} = \frac{16}{17} \Rightarrow \sin\frac{\theta}{2} = \frac{4}{\sqrt{17}}$

۲

$x^4 + x^3 + x^2 + x + 1 = 0$

طرفین تساوی را در $(x-1)$ ضرب می‌کنیم اما باید دقت کنیم $x = 1$ را در آخر حذف کنیم.

$(x-1)(x^4 + x^3 + x^2 + x + 1) = x^5 - 1 = 0 \rightarrow x^5 = 1$

حالت شماره ۰ ام به علت اضافه است

$\sqrt[5]{z_0} = 1\,\text{cis}\left(\frac{\theta}{5} + \frac{2k\pi}{5}\right) \Rightarrow \begin{cases} z_1 = (\text{cis}\,0) = 1 \Rightarrow \\ z_2 = 1\,\text{cis}\left(\frac{2\pi}{5}\right) = \frac{\sqrt{5} - 1}{4} + i\frac{\sqrt{2(\sqrt{5}+5)}}{4} \\ z_3 = 1\,\text{cis}\,\frac{4\pi}{5} = -\frac{\sqrt{5} - 1}{4} + i\frac{\sqrt{10 - 2\sqrt{5}}}{4} \end{cases}$

$$\begin{cases} z_4 = 1\,cis\,\frac{7\pi}{5} = -\frac{\sqrt5-1}{4} - \frac{\sqrt{10-2\sqrt5}}{4}i \\ z_5 = 1\,cis\,\frac{8\pi}{5} = \frac{\sqrt5-1}{4} - \frac{\sqrt{10+2\sqrt5}}{4}i \end{cases}$$

5

$$2r^2 cis(2\theta) + r^2 cis(-2\theta) = 3 \Rightarrow$$

$$r^2(3\cos(2\theta) + \sin(2\theta)i) = 3 + 0i \Rightarrow$$

$$\sin 2\theta = 0 \rightarrow \cos(2\theta) = 1 \text{ یا } -1 \Rightarrow r^2 = 1,\ r^2 = -1 \Rightarrow r = 1 \text{ یا } r = -1$$

$$\Rightarrow \cos\theta = 1,\ \frac12,\ -\frac12 \qquad \sin\theta = 0,\ \frac{\sqrt3}{2},\ -\frac{\sqrt3}{2}$$

$$\theta = 0 \Rightarrow z = 1 \checkmark \qquad \theta = \frac{2\pi}{3} \Rightarrow z = -\frac12 + \frac{\sqrt3}{2}i \checkmark$$

$$\theta = \frac{4\pi}{3} \Rightarrow z = -\frac12 - \frac{\sqrt3}{2}i$$

$$\theta = \frac{2}{3} = \frac12 + \frac{\sqrt3}{2}i \text{ یا } \theta = \pi \Rightarrow z = -1 \text{ یا } \theta = \frac{5}{3} \Rightarrow z = \frac12 - \frac{\sqrt3}{2}i \text{ یا}$$

$$\theta = 0 \text{ یا } \frac{2\pi}{3} \text{ یا } \frac{4\pi}{3} \Rightarrow z = 1 \text{ یا } -\frac12 \pm \frac{\sqrt3}{2}$$

۸

$(x+1)^4 + (x-1)^4 = 0 \quad . \quad (x+1)^4 + (x-1)^4 =$

$\underbrace{\left((x+1)^2 + (x-1)^2 i\right)}_{B} \left((x+1)^2 - (x-1)^2 i\right) =$

$\left((x+1)(x-1)(-i)\right) \cdot \left((x+1)^2 + i(x+1)(x-1) - (x-1)^2\right)$

$\left(x+i + (x-1)i\right) = (x+1)^2 - (-i)(x+1)(x-1) - (x-1)^2 = 0$

$A = x^2 + 2x + 1 - (x^2-1)i - (x^2 - 2x + 1) = -x^2 i + 4x + i$

$B = x^2 + 2x + 1 + (x^2-1)i - (x^2 - 2x + 1) = x^2 i + 4x - i$

$\Rightarrow$

① $x - xi + 1 + i = 0 \Rightarrow \boxed{x = i}$

② $x + xi + 1 - i = 0 \Rightarrow x = i$

③

④ $A = 0 \Rightarrow -x^2 i + 4x + i = 0$, $\Delta = 16 + 4i^2 = 12$

$\frac{-4+\sqrt{12}}{-2i} = \sqrt{3}i - 2i$

$\frac{-4-\sqrt{12}}{-2i} = -\sqrt{3}i - 2i$

⑤

⑥ $x^2 i + 4x - i = 0$, $\Delta = 16 + 4i^2 = 12$

$\frac{-4+\sqrt{12}}{2i} = 2i - \sqrt{3}i$

$\frac{-4-\sqrt{12}}{2i} = 2i + \sqrt{3}i$

جوابها مطابق فرمول دبیرستان بوده

if $x=1 \Rightarrow x = \pm i \cot(\frac{\pi}{۲}) = \pm i \Rightarrow$ ۱و۲ ✓

if $x=۰ \Rightarrow x = \pm i \cot(\frac{\pi}{۱۲}) = \pm i(۲+\sqrt{۳}) = \begin{cases} -۲i-\sqrt{۳}i & ۴ \\ \sqrt{۳}i+۲i & ۳ \end{cases}$

if $x=۲ \Rightarrow x = \pm i(\cot\frac{۵\pi}{۱۲}) = \pm i(۲-\sqrt{۳}) \begin{cases} ۲i-\sqrt{۳}i & ۵ \\ \sqrt{۳}i-۲i & ۶ \end{cases}$

بنابراین همه جوابها در فرمول داده شده صدق و فرمول درست است.

تمرینات بخش صفحه ۱۱ کتاب

۳) $z = ۴\sqrt{۳}+۴i$ $\quad$ $w=-۲+۲i$

$\Downarrow$ $\quad$ $\Downarrow$

$\theta = \frac{\pi}{۶}$ $\quad$ $\theta = \frac{۳\pi}{۴}$, $r = ۲\sqrt{۲}$ $\Rightarrow z = ۸\,cis(\frac{\pi}{۶})$

$r=۸$ $\quad$ $w = ۲\sqrt{۲}\,cis\,\frac{۳\pi}{۴}$

$zw = r_1r_2\,cis(\theta_1+\theta_2) = ۱۶\sqrt{۲}(cis(\frac{۱۱\pi}{۱۲}))$

$\frac{z}{w} = \frac{r_1}{r_2}\,cis(\theta_1-\theta_2) = \frac{۸}{۲\sqrt{۲}}\,cis(\frac{-۷\pi}{۱۲})$

$\frac{1}{z} = \frac{1}{r_1}\,cis(۰-\theta_1) = \frac{1}{۸}\,cis(-\frac{\pi}{۶})$

۲ج) $(۱-\sqrt{۳}i)^۵ = ۲\,cis(-\frac{\pi}{۳})^۵ = ۳۲\,cis(-\frac{۵\pi}{۳}) = ۳۲\,cis(-\frac{\pi}{۳})$

$\Downarrow$

$r=۲$ $\quad \theta = -\frac{\pi}{۳}$ $\qquad$ $۳۲-\sqrt{۳}۲۲i$

۳۸) the fift root of ۳۲ , $۳۲ = ۳۲\, cis(۰)$

$\sqrt[۵]{۳۲} = ۲\, cis\left(\frac{۰+۲k\pi}{۵}\right) =$

(۱) $= ۲\, cis\, ۰ = ۲$

(۲) $cis\, \frac{۲\pi}{۵} = \frac{\sqrt{۵}-۱}{۲} + \frac{\sqrt{۱۰+۲\sqrt{۵}}}{۲} i$

(۳) $۲\, cis\left(\frac{۴\pi}{۵}\right) = -\frac{\sqrt{۵}-۱}{۲} + \frac{\sqrt{۱۰-۲\sqrt{۵}}}{۲} i$

(۴) $۲\, cis\left(\frac{۶\pi}{۵}\right) = -\frac{\sqrt{۵}-۱}{۲} - \frac{\sqrt{۱۰-۲\sqrt{۵}}}{۲} i$

(۵) $۲\, cis\left(\frac{۸\pi}{۵}\right) = \frac{\sqrt{۵}-۱}{۲} - \frac{\sqrt{۱۰+۲\sqrt{۵}}}{۲} i$

تمرینات کتاب ۶_۷ جزوه ۵

(۱) $\sin^۴\theta$ $\quad (۲i\sin\theta)^۴ = (e^{in\theta} - e^{-in\theta})^۴$ و $n=۱$

$\Rightarrow ۱۶\sin^۴\theta = e^{۴in\theta} + e^{-۴in\theta} - ۴(e^{۳in\theta} e^{-in\theta} + e^{in\theta} e^{-۳in\theta}) +$

$۶e^{۲in\theta} e^{-۲in\theta} \Rightarrow ۶\times۱ \Rightarrow ۱۶\sin^۴\theta = ۲\cos۴\theta - ۸\cos(۲\theta + ۶) \Rightarrow$

$\sin^۴\theta = \frac{۱}{۸}\cos۴\theta - \frac{۱}{۲}\cos۲\theta + \frac{۳}{۸}$

$۱۶\cos^۴\theta = (e^{i\theta} + e^{-i\theta})^۴ = e^{۴i\theta} + e^{-۴i\theta} + ۴(e^{۲i\theta} + e^{-۲i\theta}) + ۶ \Rightarrow$

$\cos^۴\theta = \frac{۱}{۸}\cos۴\theta + \frac{۱}{۲}\cos۲\theta + \frac{۳}{۸} \Rightarrow (۲\cos\theta)^۵ = (e^{i\theta} + e^{-i\theta})^۵ =$

$e^{۵i\theta} + e^{-۵i\theta} + ۱۰(e^{-i\theta} + e^{i\theta}) + ۵(e^{۳i\theta} + e^{-۳i\theta}) =$

$۲\cos۵\theta + ۲۰\cos\theta + ۱۰\cos۳\theta = ۳۲\cos^۵\theta = ۳۲\cos۵\theta + ۲۰\cos\theta + ۱۰\cos۳\theta$

$$\Rightarrow \cos^5\theta = \frac{1}{16}\cos 5\theta + \frac{5}{8}\cos\theta + \frac{5}{16}\cos^3\theta$$

(2) $$A = \frac{1-\cos(nx) - i\sin(nx)}{1-\cos x - i\sin x} \times \frac{1-\cos x + i\sin x}{1-\cos x + i\sin x}$$

$$\frac{1-\cos x + i\sin x - \cos nx + \cos nx\cos x - i\sin x\cos nx - i\sin nx + \sin nx \sin x}{2 - 2\cos x}$$

$$Re(A) = \frac{1-\cos x - \cos nx + \cos x(\cos nx) + \sin nx \sin x}{2(1-\cos x)} =$$

$$Re(A) = \frac{(1-\cos x) - \cos nx(1-\cos x)}{2(1-\cos x)} + \frac{2\sin nx \cdot \cos\frac{x}{2} \cdot \sin\frac{x}{2}}{2(1-\cos x) = 4\cos^2\frac{x}{2}} =$$

$$= \frac{1-\cos nx}{2} + \frac{\sin nx \sin\frac{x}{2}}{2\cos\frac{x}{2}} = -\cos\frac{nx}{2} - \frac{\cos\frac{x}{2}\sin\frac{x}{2}}{\cos\frac{x}{2}} =$$

$$-\cos\frac{nx}{2}\left(1 + \frac{\sin\frac{x}{2} \cdot \sin\frac{nx}{2}}{\cos\frac{x}{2}}\right) = \frac{-\cos\frac{nx}{2}}{\cos\frac{x}{2}}\left(\cos\frac{x}{2} + \sin\frac{x}{2}\sin\frac{nx}{2}\right) = \frac{\sin\frac{nx}{2}}{\sin\frac{x}{2}}$$

$$\cos\frac{(n-1)x}{2}$$

۲

$$\left(\frac{1-\tan x}{1+\tan x}\right)^n = \frac{1-\tan x}{1+\tan x} = \frac{i\cos x-\sin x}{i\cos x+\sin x} \;\underline{\underline{x\,i}}$$

$$\frac{\cos x+i\sin x}{\cos x-i\sin x} \Rightarrow \left(\frac{1-\tan x}{1+\tan x}\right)^n = \left(\frac{\cos x+i\sin x}{\cos x-i\sin x}\right)^n \underline{\underline{x\,\cos x+i\sin x}}$$

$$\left(\frac{\cos x+i\sin x}{\cos^2 x+\sin^2 x}\right)^n = (\cos x+i\sin x)^{2n} = \left(e^{ix}\right)^{2n} = e^{2inx}$$

۴

با توجه به قضیه دموآور داریم که مجموع همه ی ریشه های مختلط عدد حقیقی ۱ ریشه n ام آن صفر است.

$$\sqrt[n]{z}=1 \Rightarrow z=\operatorname{cis}\left(0+\frac{2\pi k}{n}\right) \to k=0,\dots,n-1$$

$r=1 \quad \theta=0$

$$\Rightarrow z_1=1$$

$$z_2=\operatorname{cis}\left(\frac{2\pi}{n}\right)=\cos\frac{2\pi}{n}+i\sin\left(\frac{2\pi}{n}\right)$$

$$z_3=\operatorname{cis}\left(\frac{4\pi}{n}\right)=\cos\frac{4\pi}{n}+i\sin\left(\frac{4\pi}{n}\right)$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$z_n=\operatorname{cis}\left(\frac{2(n-1)\pi}{n}\right)=\cos\left(\frac{n-1}{n}\right)2\pi+i\sin\left(\frac{2(n-1)\pi}{n}\right)$$

ریشه های n ام ساده

$$\sum_{i=0}^{n} z_i = 0 \Rightarrow$$

چون تمام ریشه ها مجموع صفر دارند پس مجموع قسمت حقیقی هم صفر است

$$\Rightarrow \sum_{i=0}^{n-1} \cos\left(\frac{2i\pi}{n}\right) = 1 + \cos\frac{2\pi}{n} + \cos\frac{4\pi}{n} + \dots + \cos\frac{2(n-1)\pi}{n} = 0$$

$$\Rightarrow \cos\frac{2\pi}{n} + \cos\frac{4\pi}{n} + \dots + \cos\frac{2(n-1)\pi}{n} = -1$$

$$\sum_{i=0}^{n-1} \sin\frac{2i\pi}{n} = 0 \Rightarrow 0 + \sin\frac{2\pi}{n} + \sin\frac{4\pi}{n} + \dots + \sin\frac{2(n-1)\pi}{n} = 0$$

$$\Rightarrow \sin\frac{2\pi}{n} + \sin\frac{4\pi}{n} + \dots + \sin\frac{2(n-1)\pi}{n} = 0$$

۹

با توجه به اینکه $e^{\frac{2\pi i}{5}}$ یکی از ریشه ۵ واحد است و مخالف ۱ است داریم

$$1+z+z^2+z^3+z^4 = \frac{z^5-1}{z-1} \xrightarrow{z=e^{2\pi i/5}} 0$$

$$z+1+z^2+z^3+5z^4+4z^5+4z^6+4z^7+4z^8+5z^9 =$$

$$\underbrace{z+1+z^2+z^3+z^4}_{0}+4(z^4+z^5+z^6+z^7+z^8)+5z^9 =$$

$$0+4z^4(1+z+z^2+z^3+z^4)+5z^9 = 5z^9 =$$

$$5e^{\frac{18\pi i}{5}} = 5e^{\frac{(18-10)\pi i}{5}} = 5e^{\frac{8\pi i}{5}} = -5e^{\frac{3\pi i}{5}}$$

$$z^9 = \frac{z^{10}}{z} \Rightarrow 5z^9 = \frac{5z^{10}}{z} = \frac{5\times e^{4\pi i}}{e^{2\pi i/5}} = \frac{5\times(-1)^4}{e^{2\pi i/5}} = \frac{5}{e^{2\pi i/5}}$$

$$\frac{5}{\cos\frac{2\pi}{5} + i\sin\left(\frac{2\pi}{5}\right)} \times \frac{\cos\frac{2\pi}{5} - i\sin\frac{2\pi}{5}}{\cos\frac{2\pi}{5} - i\sin\frac{2\pi}{5}} =$$

$$\frac{5(\operatorname{cis}(-\frac{2\pi}{5}))}{1} = 5\left(-\cos\frac{3\pi}{5} - i\sin\frac{3\pi}{5}\right) =$$

$$-5\operatorname{cis}\frac{3\pi}{5} = -5e^{\frac{3\pi i}{5}}$$

تمرینات کتاب H.W (ضعیف)

44) $e^{x+i} = e^x(e^i) = e^x(\cos 1 + i\sin 1) = e^x\cos 1 + e^x\sin 1\, i$

$\cos 1 = 0.540$ $\quad\quad$ $\sin 1 = 0.1841$

50) $\int e^{(1+i)x}\,dx = \frac{1}{i+1} e^{(1+i)x} = \frac{e^x}{2}(1-i)(\cos x + i\sin x)$

$= \frac{e^x}{2}(\cos x + \sin x + i(\sin x - \cos x))$

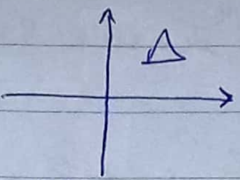
★ $\int e^{(1+i)x} = \int e^x(\cos x) + i\int e^x \sin x \Rightarrow \int e^x \cos x = \frac{e^x}{2}(\cos x + \sin x)$

$\int e^x \sin x = \frac{e^x}{2}(\sin x - \cos x)$

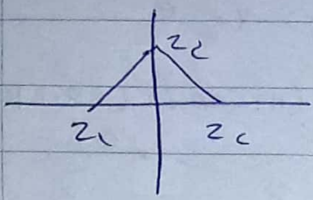
۴

تمرینات بخش ۷-۷ جزوه

۱- مثلث دلخواهی را فرض می‌کنیم



محورهای مختصات را انتقال می‌دهیم تا



محور حقیقی بر $z_1 z_2$ و محور موهومی از z_3 گذر کند

$z_2 = x$

$z_1 = -x \qquad z_2 = x$

$z_3 = \sqrt{3}\,xi \qquad z_0 = \frac{\sqrt{3}}{3}\,xi$

$3z_0^2 = 3\left(-\frac{1}{3}x^2\right) = -x^2$

$\Longrightarrow z_1^2 + z_2^2 + z_3^2 = x^2 + x^2 - 3x^2 = -x^2 \Longrightarrow$

$z_1^2 + z_2^2 + z_3^2 = 3z_0^2$

$\frac{1}{z_1 - z_2} + \frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} = 0 \Longrightarrow 0 = (z_2 - z_3)(z_3 - z_1) + (z_1 - z_2)$

$(z_3 - z_2) + (z_1 - z_2)(z_1 - z_3) \Longrightarrow$

$z_2 z_3 - z_2 z_1 - z_3^2 + z_3 z_1 + z_1 z_3 - z_1^2 + z_2 z_1 - z_2 z_3 + z_1 z_2 - z_1 z_2$

$-z_2^2 + z_3 z_2 \Longrightarrow -z_1^2 - z_2^2 - z_3^2 + z_1 z_2 + z_1 z_3 + z_2 z_3 = 0 \Longleftrightarrow$

$\cancel{-x^2 - x^2 + 3x^2 - x^2 + 2\sqrt{3}\,xi - 2\sqrt{3}\,xi} = 0$

3

$y+\frac{1}{y}=2\cos x \Rightarrow y=e^{ix}=\cos x+i\sin x \Rightarrow y^n=e^{nix}$

$x+\frac{1}{x}=2\cos\theta \Rightarrow x=e^{i\theta}=\cos\theta+i\sin\theta \Rightarrow x^n=e^{ni\theta}$

a) $\frac{y^n}{x^n}+\frac{x^n}{y^n}=\frac{e^{nix}}{e^{ni\theta}}+\frac{e^{ni\theta}}{e^{nix}}=\underbrace{e^{(nx-n\theta)i}}_{z}+\underbrace{e^{(n\theta-nx)i}}_{\bar z}$

$z+\bar z=2\operatorname{Re}z=2\cos(nx-n\theta)=2\cos(n\theta-nx)$

b) $x^n+y^n+\frac{1}{x^n y^n}=\underbrace{e^{(n\theta+nx)i}}_{z}+\underbrace{e^{-(n\theta-nx)i}}_{\bar z}=$

$2\operatorname{Re}(e^{n\theta+nx})=2\cos(n\theta+nx) \Longrightarrow$

7

$x=\frac{1}{x^4+4}=\frac{1}{(x^2+2x+2)(x^2-2x+2)}=$

$\frac{1}{(x+1)(x+1+i)(x-1-i)(x-1+i)} \to$

$\frac{A}{x-1-i}+\frac{B}{x-1+i}+\frac{C}{x+1+i}+\frac{D}{x+1-i}=\frac{1}{x^4+4}$

$$\Longrightarrow A(x^2+2x+2)(x-1+i)+B(x^2+2x+2)(x-1-i)+C$$

$$(x^2-2x+2)(x+1-i)+D(x^2-2x+2)(x+1+i)$$

$$x=1+i \to 2i(4+4i)A=1 \Rightarrow 8(i-1)A=1 \Rightarrow 8A=\frac{1}{i-1}=8A=\frac{1}{1-i} \Rightarrow$$

$$\boxed{\frac{-1-i}{2} \Rightarrow A=\frac{1+i}{16}}$$

$$x=1-i \Rightarrow -2i(4-4i)B=1 \Rightarrow (-8i-8)B=1 \Rightarrow B=\frac{-1}{8i+8}=B=\boxed{\frac{-1+i}{16}}$$

$$x=-1-i \Rightarrow (4+4i)(-2i)C=1 \to (8-8i)C=1 \to C=\frac{1}{8-8i} \Rightarrow \boxed{C=\frac{1+i}{16}}$$

$$x=-1+i \Rightarrow (4-4i)(2i)D=1 \quad (8i+8)D=1 \to D=\frac{1-i}{16}$$

$$\Longrightarrow \alpha=\frac{1}{16}\left(\frac{-1-i}{x-1-i}+\frac{i-1}{x-1+i}+\frac{1+i}{x+1+i}+\frac{1-i}{x+1-i}\right)=$$

$$\frac{1}{16}\left(\frac{-2x+4}{x^2-2x+2}-\frac{+2x+2}{x^2+2x+2}\right)=\frac{-\frac{1}{8}x+\frac{1}{4}}{x^2-2x+2}+\frac{\frac{1}{8}x+\frac{1}{4}}{x^2+2x+2}$$

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$$C=\cos\alpha+\tfrac{1}{2}\cos 2\alpha+\dots \longrightarrow \sum_{n=0}^{+\infty}\frac{\cos(n+1)\alpha}{2^n}=$$

$$\frac{1}{2}\sum_{0}^{+\infty} e^{(n+1)\alpha i}+e^{-(n+1)\alpha i}$$

$$= \frac{1}{2}\left(\sum_{0}^{+\infty} \frac{e^{(2n+1)\alpha i}}{2^{n}} + \sum_{0}^{+\infty} \frac{e^{-(2n+1)\alpha i}}{2^{n}}\right) =$$

$$\frac{1}{2}\left(e^{\alpha i} + \sum_{1}^{+\infty} e^{\alpha i}\left(\frac{e^{2\alpha i}}{2}\right)^{n} + e^{-\alpha i} + \sum_{1}^{+\infty}\left(\frac{e^{-2\alpha i}}{2}\right)^{n} e^{-\alpha i}\right)$$

$$= \frac{1}{2}\left(e^{\alpha i} + e^{\alpha i}\frac{\frac{e^{2\alpha i}}{2}}{1-\frac{e^{2\alpha i}}{2}} + e^{-\alpha i} + e^{-\alpha}\frac{\frac{e^{-2\alpha i}}{2}}{1-2e^{2\alpha i}}\right) =$$

$$\frac{1}{2}\left(e^{\alpha i} + \frac{e^{2\alpha}}{2-e^{2\alpha i}} + e^{\alpha i} + \frac{e^{-2\alpha i}}{2-e^{-2\alpha i}}\right) =$$

$$\frac{1}{2}\left(\frac{2e^{\alpha i}}{2-e^{2\alpha i}} + \frac{2e^{-\alpha i}}{2-e^{-2\alpha i}}\right) = \frac{2e^{\alpha i} - e^{-\alpha i} + 2e^{-\alpha i} - e^{\alpha i}}{4+1-2e^{2\alpha i}+e^{-2\alpha i}} =$$

$$\frac{2^{\alpha i} + 2^{-\alpha i}}{5-2(2\cos 2\alpha)} = \boxed{\frac{2\cos\alpha}{5-4\cos 2\alpha}}$$