پروژه شماره ۴- درس الکترومغناطیس، پاییز ۹۹

پروژه شماره ۴ می تواند در گروههای ۳ نفری انجام شود. ارزش نمره ایی این پروژه شامل ۰/۵ نمره از بیست و تا سقف ۰/۵ نمره تشویقی خواهد داشت. در این پروژه شما با بکارگیری دانش الکترومغناطیس به شبیه سازی یک موتور ساده DC خواهید پرداخت. گزارش پروژه شامل توضیحات فرمولبندی، نتایج و کدهای شبیه سازی (در صورت کد نویسی) است. **مهلت تحویل این پروژه تا پایان روز 99/10/26 است**.

برای انجام این پروژه ابتدا با خواندن متنی که در ادامه می آید با فیزیک کارکرد یک موتور ساده آشنا شوید و سپس در پاسخ به مساله زیر به شبیه سازی بپردازید. پارامترها، شکلها و فرمولهای مورد اشاره در مساله زیر در متن بعدی توضیح داده شده است.

Problem Simulation of the DC Motor

Let $V_{max} = 40\ V$, $I_{max} = 5\ A$, $K_b = K_T = 0.07\ V/rad/sec\ (= N\text{-}m/A)$, $J = 6\times10^{-5}\ kg\text{-}m^2$, $R = 2\ ohms$, $L = 2\ mH$, and $f = 0.0004\ N\text{-}m/rad/sec$. Develop a simulation of the DC motor that includes the motor model given by (1.6) and a voltage saturation model of the amplifier as illustrated in Figure 1.20. Put a step input of $V_S(t) = 10\ V$ into the motor and plot out (a) $\theta(t)$, (b) $\omega(t)$, (c) $i(t)$, and (d) $V_S(t)$.

1

The Physics of the DC Motor

The principles of operation of a direct current (DC) motor are presented based on fundamental concepts from electricity and magnetism contained in any basic physics course. The DC motor is used as a concrete example for reviewing the concepts of magnetic fields, magnetic force, Faraday's law, and induced electromotive forces (emf) that will be used throughout the remainder of the book for the modeling of electric machines. All of the Physics concepts referred to in this chapter are contained in the book *Physics* by Halliday and Resnick [34].

1.1 Magnetic Force

Motors work on the basic principle that magnetic fields produce forces on wires carrying a current. In fact, this experimental phenomenon is what is used to define the magnetic field. If one places a current carrying wire between the poles of a magnet as in Figure 1.1, a force is exerted on the wire. Experimentally, the magnitude of this force is found to be proportional to both the amount of current in the wire and to the length of the wire that is between the poles of the magnet. That is, F_{magnetic} is proportional to ℓi. The direction of the magnetic field $\vec{\mathbf{B}}$ at any point is defined to be the direction that a small compass needle would point at that location. This direction is indicated by arrows in between the north and south poles in Figure 1.1.

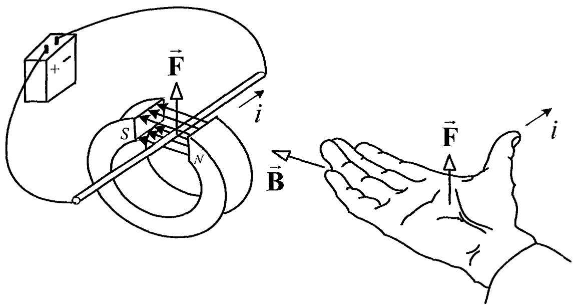

FIGURE 1.1. Magnetic force law. From *PSSC Physics*, 7th edition, by Haber-Schaim, Dodge, Gardner, and Shore, published by Kendall/Hunt, 1991.

With the direction of $\vec{\mathbf{B}}$ perpendicular to the wire, the strength (magnitude) of the *magnetic induction field* $\vec{\mathbf{B}}$ is defined to be

$$B = |\vec{\mathbf{B}}| \triangleq \frac{F_{\text{magnetic}}}{\ell i}$$

where F_{magnetic} is the magnetic force, i is the current, and ℓ is the length of wire perpendicular to the magnetic field carrying the current. That is, B is the proportionality constant so that $F_{\text{magnetic}} = i\ell B$. As illustrated in Figure 1.1, the direction of the force can be determined using the right-hand rule. Specifically, using your right hand, point your fingers in the direction of the magnetic field and point your thumb in the direction of the current. Then the direction of the force is out of your palm.

Further experiments show that if the wire is parallel to the $\vec{\mathbf{B}}$ field rather than perpendicular as in Figure 1.1, then no force is exerted on the wire. If the wire is at some angle θ with respect to $\vec{\mathbf{B}}$ as in Figure 1.2, then the force is proportional to the *component* of $\vec{\mathbf{B}}$ perpendicular to the wire; that is, it is proportional to $B_\perp = B\sin(\theta)$. This is summarized in the *magnetic force law*: Let $\vec{\boldsymbol{\ell}}$ denote a vector whose magnitude is the length ℓ of the wire in the magnetic field and whose direction is defined as the positive direction of current in the bar; then the magnetic force on the bar of length ℓ carrying the current i is given by

$$\vec{\mathbf{F}}_{\text{magnetic}} = i\vec{\boldsymbol{\ell}} \times \vec{\mathbf{B}}$$

or, in scalar terms, $F_{\text{magnetic}} = i\ell B\sin(\theta) = i\ell B_\perp$. Again, $B_\perp \triangleq B\sin(\theta)$ is the component of $\vec{\mathbf{B}}$ perpendicular to the wire.[1]

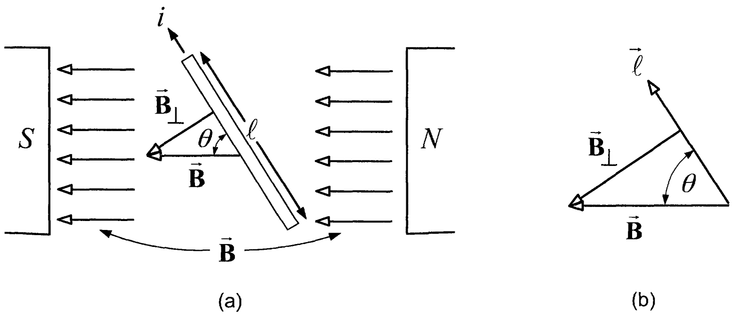

FIGURE 1.2. Only the component $B_\perp$ of the magnetic field which is perpendicular to the wire produces a force on the current.

[1] Motors are designed so that the conductors are perpendicular to the external magnetic field.

Example *A Linear DC Machine* [19]

Consider the simple linear DC machine in Figure 1.3 where a sliding bar rests on a simple circuit consisting of two rails. An external magnetic field is going through the loop of the circuit up out of the page indicated by the $\otimes$ in the plane of the loop. Closing the switch results in a current flowing around the circuit and the external magnetic field produces a force on the bar which is free to move. The force on the bar is now computed.

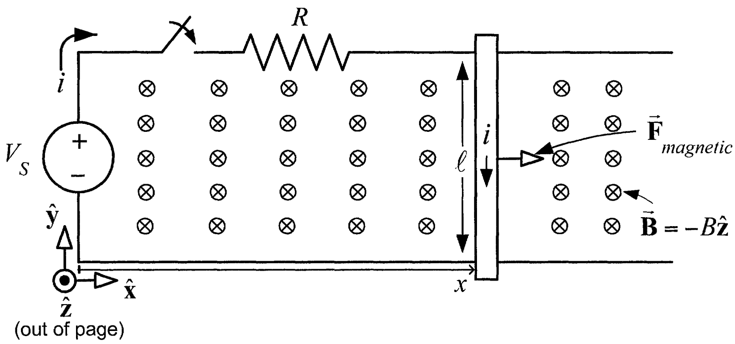

FIGURE 1.3. A linear DC motor.

The magnetic field is constant and points into the page (indicated by $\otimes$) so that written in vector notation, $\vec{\mathbf{B}} = -B\hat{\mathbf{z}}$ with $B > 0$. By the right hand rule, the magnetic force on the sliding bar points to the right. Explicitly, with $\vec{\boldsymbol{\ell}} = -\ell\hat{\mathbf{y}}$, the force is given by

$$\begin{aligned}\vec{\mathbf{F}}_{\text{magnetic}} &= i\vec{\boldsymbol{\ell}} \times \vec{\mathbf{B}} = i(-\ell\hat{\mathbf{y}}) \times (-B\hat{\mathbf{z}}) \\ &= i\ell B\hat{\mathbf{x}}.\end{aligned}$$

To find the equations of motion for the bar, let f be the coefficient of viscous (sliding) friction of the bar so that the friction force is given by $F_f = -f dx/dt$. Then, with m_ℓ denoting the mass of the bar, Newton's law gives

$$i\ell B - f dx/dt = m_\ell d^2x/dt^2.$$

Just after closing the switch at $t = 0$, but before the bar starts to move, the current is $i(0^+) = V_S(0^+)/R$. However, it turns out that as the bar moves the current does *not* stay at this value, but instead decreases due to electromagnetic induction. This will be explained later.

1.2 Single-Loop Motor

As a first step to modeling a DC motor, a simplistic single-loop motor is considered. It is first shown how torque is produced and then how the current in the single loop can be reversed (commutated) every half turn to keep the torque constant.

1.2.1 Torque Production

Consider the magnetic system in Figure 1.4, where a cylindrical core is cut out of a block of a permanent magnet and replaced with a soft iron core. The term "soft" iron refers to the fact that material is easily magnetized (a permanent magnet is referred to as "hard" iron).

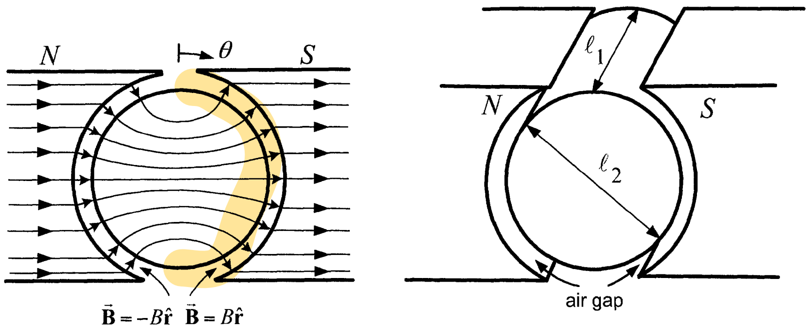

FIGURE 1.4. Soft iron cylindrical core placed inside a hollowed out permanent magnet to produce a radial magnetic field in the air gap.

An important property of soft magnetic materials is that the magnetic field at the surface of such materials tends to be normal (perpendicular) to the surface. Consequently, the cylindrical shape of the surfaces of the soft iron core and the stator permanent magnet has the effect of making the field in the air gap *radially* directed; furthermore, it is reasonably constant (uniform) in magnitude. A mathematical description of the magnetic field in the air gap due to the permanent magnet is simply

$$\vec{\mathbf{B}} = \begin{cases} +B\hat{\mathbf{r}} & \text{for } 0 < \theta < \pi \\ -B\hat{\mathbf{r}} & \text{for } \pi < \theta < 2\pi \end{cases}$$

where $B > 0$ is the magnitude or strength of the magnetic field and θ is an arbitrary location in the air gap.[2]

Figure 1.5 shows a rotor loop wound around the iron core of Figure 1.4. The length of the rotor is ℓ_1 and its diameter is ℓ_2. The torque on this rotor loop is now calculated by considering the magnetic forces on sides a and a' of the loop. On the other two sides of the loop, that is, the front and

[2]Actually it will be shown in a later chapter that the magnetic field must be of the form $\vec{\mathbf{B}} = \pm B(r_0/r)\hat{\mathbf{r}}$ in the air gap, that is, it varies as $1/r$ in the air gap. However, as the air gap is small, the $\vec{\mathbf{B}}$ field is essentially constant across the air gap.

back sides, the magnetic field has negligible strength so that no significant force is produced on these sides. As illustrated in Figure 1.5(b), the rotor angular position is taken to be the angle θ_R from the vertical to side a of the rotor loop.

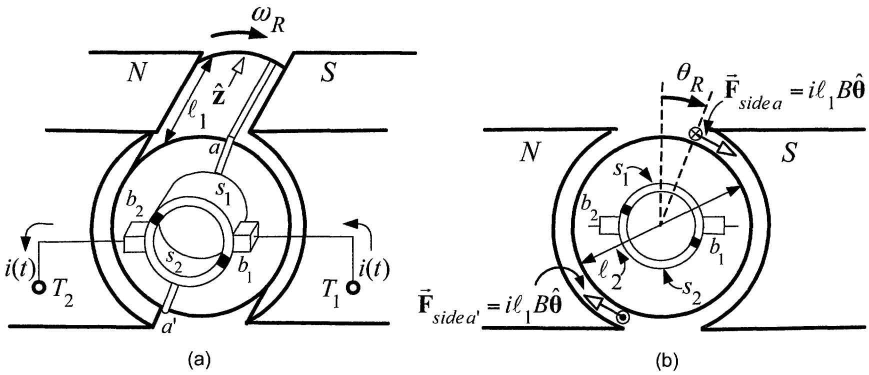

FIGURE 1.5. A single-loop motor. From *Electromagnetic and Electromechanical Machines*, 3rd edition, L. W. Matsch and J. Derald Morgan, 1986. Reprinted by permisson of John Wiley & Sons.

Figure 1.6 shows the cylindrical coordinate system used in Figure 1.5. Here $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\mathbf{z}}$ denote unit cylindrical coordinate vectors. The unit vector $\hat{\mathbf{z}}$ points along the rotor axis into the paper in Figure 1.5(b), $\hat{\boldsymbol{\theta}}$ is in the direction of increasing θ, and $\hat{\mathbf{r}}$ is in the direction of increasing r.

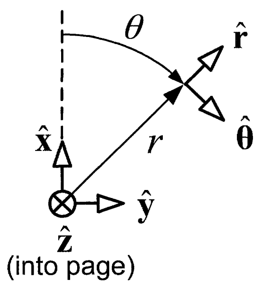

FIGURE 1.6. Cylindrical coordinate system used in Figure 1.5.

Referring back to Figure 1.5, for $i > 0$, the current in side a of the loop is going into the page (denoted by $\otimes$) and then comes out of the page (denoted by $\odot$) on side a'. Thus, on side a, $\vec{\boldsymbol{\ell}} = \ell_1\hat{\mathbf{z}}$ (as $\vec{\boldsymbol{\ell}}$ points in the direction of positive current flow) and the magnetic force $\vec{\mathbf{F}}_{\text{side }a}$ on side a

is then

$$\begin{aligned}\vec{\mathbf{F}}_{\text{side } a} &= i\vec{\boldsymbol{\ell}}\times\vec{\mathbf{B}}\\ &= i(\ell_1\hat{\mathbf{z}})\times(B\hat{\mathbf{r}})\\ &= i\ell_1 B\hat{\boldsymbol{\theta}}\end{aligned}$$

which is tangential to the motion as shown in Figure 1.5(b). The resulting torque is

$$\begin{aligned}\vec{\boldsymbol{\tau}}_{\text{side } a} &= (\ell_2/2)\hat{\mathbf{r}}\times\vec{\mathbf{F}}_{\text{side } a}\\ &= (\ell_2/2)i\ell_1 B\hat{\mathbf{r}}\times\hat{\boldsymbol{\theta}}\\ &= (\ell_2/2)i\ell_1 B\hat{\mathbf{z}}.\end{aligned}$$

Similarly, the magnetic force on side a' of the rotor loop is

$$\begin{aligned}\vec{\mathbf{F}}_{\text{side } a'} &= i\vec{\boldsymbol{\ell}}\times\vec{\mathbf{B}}\\ &= i(-\ell_1\hat{\mathbf{z}})\times(-B\hat{\mathbf{r}})\\ &= i\ell_1 B\hat{\boldsymbol{\theta}}\end{aligned}$$

so that the corresponding torque is then

$$\begin{aligned}\vec{\boldsymbol{\tau}}_{\text{side } a'} &= (\ell_2/2)\hat{\mathbf{r}}\times\vec{\mathbf{F}}_{\text{side } a'}\\ &= (\ell_2/2)i\ell_1 B\hat{\mathbf{r}}\times\hat{\boldsymbol{\theta}}\\ &= (\ell_2/2)i\ell_1 B\hat{\mathbf{z}}.\end{aligned}$$

The total torque on the rotor loop is then

$$\begin{aligned}\vec{\boldsymbol{\tau}}_m &= \vec{\boldsymbol{\tau}}_{\text{side } a}+\vec{\boldsymbol{\tau}}_{\text{side } a'}\\ &= 2(\ell_2/2)i\ell_1 B\hat{\mathbf{z}}\\ &= \ell_1\ell_2 Bi\hat{\mathbf{z}}.\end{aligned}$$

The torque points along the z axis, which is the axis of rotation. In scalar form,

$$\tau_m = K_T i$$

where $K_T \triangleq \ell_1\ell_2 B$. The force is proportional to the strength B of magnetic field $\vec{\mathbf{B}}$ in the air gap due to the permanent magnet.

In order to increase the strength of the magnetic field in the air gap, the permanent magnet can be replaced with a soft iron material with wire wound around the periphery of the magnetic material as shown in Figure 1.7(a). This winding is referred to as the *field winding*, and the current it carries is called the *field current*. In normal operation, the field current is held constant. The strength of the magnetic field in the air gap is then proportional to the field current i_f at lower current levels (i.e., $B = K_f i_f$) and then saturates as the current increases. This may be written as $B =$

$f(i_f)$ where $f(\cdot)$ is a saturation curve satisfying $f(0) = 0, f'(0) = K_f$ as shown in Figure 1.7(b).

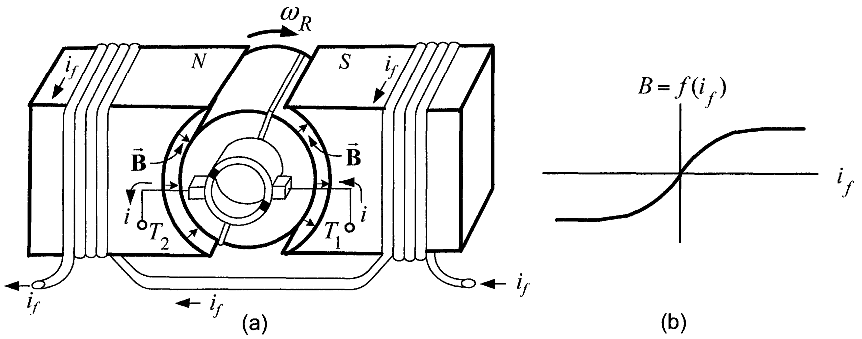

FIGURE 1.7. (a) DC motor with a field winding. (b) Radial magnetic field strength in the air gap produced by the field current.

1.2.2 Commutation of the Single-Loop Motor

The above derivation for the torque $\tau_m = K_T i$ assumes that the current in the side of the rotor loop[3] under the south pole face is into the page and the current in the side of the loop under the north pole face is out of the page as in Figure 1.8(a). In order to make this assumption valid, the direction of the current in the loop must be changed each time the rotor loop passes through the vertical.

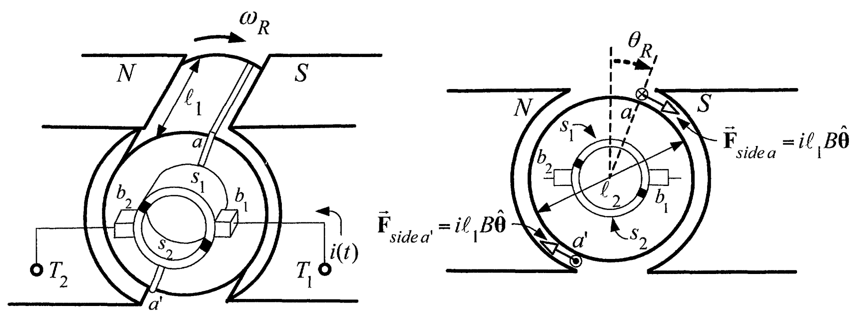

FIGURE 1.8. (a) $0 < \theta_R < \pi$. From *Electromagnetic and Electromechanical Machines,* 3rd edition, L. W. Matsch and J. Derald Morgan, 1986. Reprinted by permisson of John Wiley & Sons.

[3]The rotor loop is also referred to as the *armature* winding and the current in it as the *armature* current.

The process of changing the direction of the current is referred to as *commutation* and is done at $\theta_R = 0$ and $\theta_R = \pi$ through the use of the slip rings s_1, s_2 and brushes b_1, b_2 drawn in Figure 1.8. The slip rings are rigidly attached to the loop and thus rotate with it. The brushes are fixed in space with the slip rings making a sliding electrical contact with the brushes as the loop rotates.

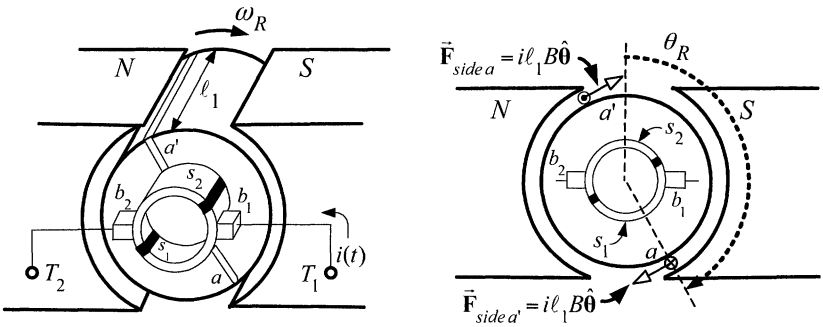

Figure 1.8(b) Rotor loop just prior to commutation where $0 < \theta_R < \pi$.

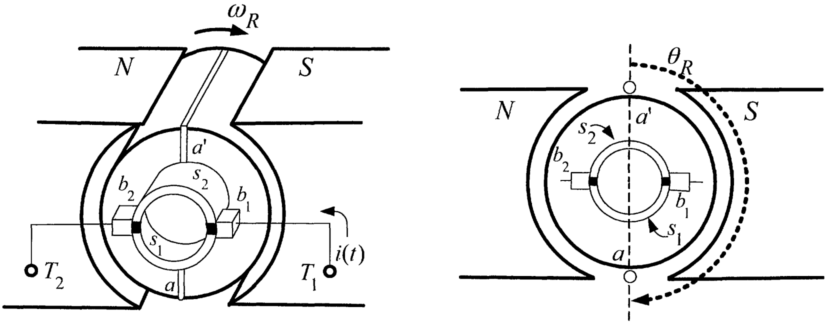

Figure 1.8(c) The ends of the rotor loop are shorted when $\theta_R = \pi$.

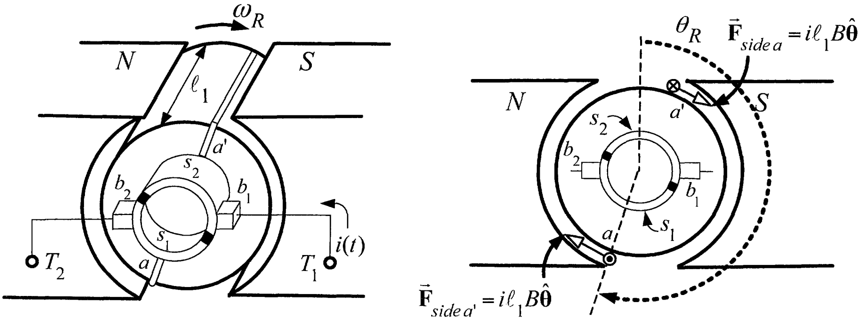

Figure 1.8(d) Rotor loop just after commutation where $\pi < \theta_R < 2\pi$.

To see how the commutation of the current is accomplished using the brushes and slip rings, consider the sequence of Figures 1.8(a)−(d). As shown in Figure 1.8(a), the current goes through brush b_1 into the slip ring s_1. From there, it travels down (into the page $\otimes$) side a of the loop, comes back up side a' (out of the page $\odot$) into the slip ring s_2, and, finally, comes out the brush b_2. Note that side a of the loop is under the south pole face while side a' is under the north pole face. Figure 1.8(b) shows the rotor loop just before commutation where the same comments as in Figure 1.8(a) apply.

Figure 1.8(c) shows that when $\theta_R = \pi$, the slip rings at the ends of the loop are shorted together by the brushes forcing the current in the loop to drop to zero. Subsequently, as shown in Figure 1.8(d), with $\pi < \theta_R < 2\pi$, the current is now going through brush b_1 into slip ring s_2. From there, the current travels down (into the page $\otimes$) side a' of the loop and comes back up (out of the page $\odot$) side a. In other words, the current has *reversed* its direction in the loop from that in Figures 1.8(a) and 1.8(b). This is precisely what is desired, as side a is now under the north pole face and side a' is under the south pole face. As a result of the brushes and slip rings, the current direction in the loop is reversed every half-turn.

1.3 Faraday's Law

Figure 1.9 shows a magnet moving upwards into a wire loop producing a changing magnetic flux in the loop.

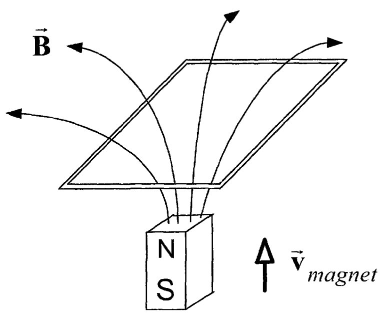

FIGURE 1.9. A magnet moving upwards produces a changing flux in the loop which in turn results in an induced emf and current in the loop.

Recall that a changing flux within a loop produces an induced *electro-*

motive force (emf) ξ in the loop according to Faraday's law.[4] That is,

$$\xi = -d\phi/dt$$

where

$$\phi = \int_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}}$$

is the flux in the loop and S is any surface with the loop as its boundary. Faraday's law is now reviewed in some detail.

1.3.1 The Surface Element Vector $d\vec{\mathbf{S}}$

The surface element $d\vec{\mathbf{S}}$ is a vector whose magnitude is a differential (small) element of area dS and whose direction is normal (perpendicular) to the surface element. As there are two possibilities for the normal to the surface, one must choose the normal in a consistent manner. In particular, depending on the particular normal chosen, a convention is used to characterize the positive and negative directions of travel around the surface boundary. To describe this, consider Figure 1.10(a) which shows a small surface element with the normal direction taken to be up in the positive z direction. In this case, with $\hat{\mathbf{n}} = \hat{\mathbf{z}}$, $dS = dxdy$, the surface element vector is defined by

$$d\vec{\mathbf{S}} \triangleq dxdy\hat{\mathbf{z}}.$$

The corresponding direction of travel around the surface boundary is indicated by the curved arrow in the figure.

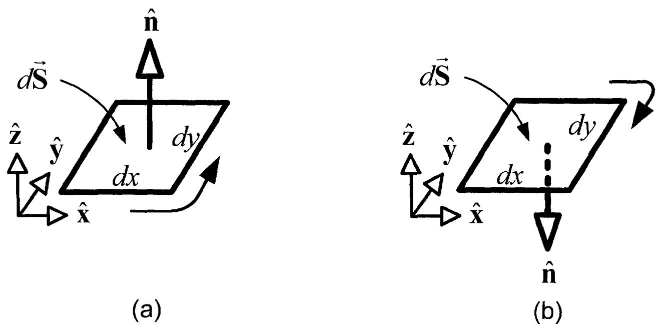

FIGURE 1.10. (a) Positive direction of travel around a surface element with the normal up. (b) Positive direction of travel around a surface element with the normal down.

[4] ξ is the Greek letter "xi" and is pronounced "ksi".

In Figure 1.10(b) a surface element with the normal direction taken to be down in the negative z direction is shown. In this case $\hat{\mathbf{n}} = -\hat{\mathbf{z}}, dS = dxdy$ so that the surface element vector is defined as

$$d\vec{\mathbf{S}} = -dxdy\hat{\mathbf{z}}.$$

The direction of positive travel around the surface element is indicated by the curved arrow in Figure 1.10(b) and is opposite to that of Figure 1.10(a).

As illustrated in Figure 1.10, the vector differential surface element $d\vec{\mathbf{S}}$ is defined to be a vector whose magnitude is the area of the differential surface element and whose direction is normal to the surface. One may choose either normal, and the corresponding direction of positive travel around the surface is then determined.

Two surface elements may be connected together as in Figure 1.11 and travel around the total surface is defined as shown. Note that along the common boundary of the two joined surface elements, the directions of travel "cancel" out each other, resulting in a net travel path around both surface elements. The normals for the surface elements must both be up or both be down; that is, the normal must be continuous as one goes from one surface element to the next.

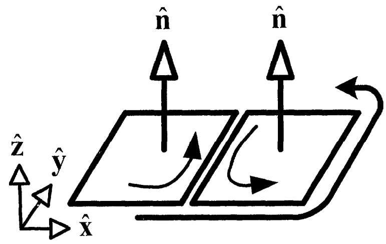

FIGURE 1.11. Positive direction of travel around two joined surface elements.

1.3.2 *Interpreting the Sign of* ξ

The interpretation of positive and negative values of the induced electromotive force ξ is now explained. Faraday's law says that the induced emf (voltage) in a loop is given by

$$\xi = -d\phi/dt$$

where

$$\phi = \int_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}}.$$

If $\xi > 0$, the induced emf will force current in the positive direction of travel around the surface while if $\xi < 0$, the induced emf will force current in the opposite direction. As illustrated in problems 1 and 2, this sign convention for Faraday's law is just a precise mathematical way of describing Lenz's law: *"In all cases of electromagnetic induction, an induced voltage will cause a current to flow in a closed circuit in such a direction that the magnetic field which is caused by that current will oppose the change that produced the current"* (pages 873–877 of Ref. [34]).

Faraday's law is now illustrated by some examples. Specifically, it is used to compute the induced emf in the linear DC machine, the induced emf in the single-loop machine and the self-induced voltage in the single-loop machine.

1.3.3 Back Emf in a Linear DC Machine

Figure 1.12 shows the linear DC machine where the back emf it generates is now computed. The magnetic field is constant and points into the page, that is, $\vec{\mathbf{B}} = -B\hat{\mathbf{z}}$, where $B > 0$. The magnetic force on the bar is $\vec{\mathbf{F}}_{\text{magnetic}} = i\ell B\hat{\mathbf{x}}$. To compute the induced voltage in the loop of the circuit, let $\hat{\mathbf{n}} = \hat{\mathbf{z}}$ be the normal to the surface so that $d\vec{\mathbf{S}} = dxdy\hat{\mathbf{z}}$, where $dS = dxdy$.

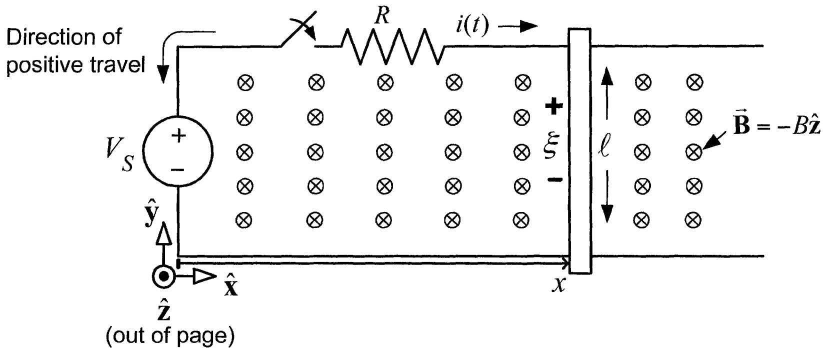

FIGURE 1.12. With $d\vec{\mathbf{S}} = dxdy\hat{\mathbf{z}}$, the direction of positive travel around the flux surface is in the counterclockwise direction.

Then

$$\phi = \int_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} = \int_0^\ell \int_0^x (-B\hat{\mathbf{z}}) \cdot (dxdy\hat{\mathbf{z}}) = \int_0^\ell \int_0^x -Bdxdy = -B\ell x.$$

The induced (back) emf is therefore given by

$$\xi = -d\phi/dt = -d(-B\ell x)/dt = B\ell v.$$

In the flux computation, the normal for the surface was taken to be in $+\hat{\mathbf{z}}$ direction. By putting together the differential flux surfaces $d\vec{\mathbf{S}}$ in a fashion similar to Figure 1.11, the positive direction of travel around the surface is counterclockwise around the loop as indicated in Figure 1.12. Here the sign conventions for source voltage V_S and the back emf ξ are opposite so that, as the back emf $\xi = B\ell v > 0$, it is *opposing* the applied source voltage V_S.

Remark $\phi = -B\ell x$ is the flux in the circuit due to the *external* magnetic field $\vec{\mathbf{B}} = -B\hat{\mathbf{z}}$. There is also a flux $\psi = Li$ due to the current i in the circuit. For this example, the inductance is small and one just sets $L = 0$.

Electromechanical Energy Conversion

As the back emf $\xi = Blv$ opposes the current i, electrical power is being absorbed by this back emf. Specifically, the electrical power absorbed by the back emf is $i\xi = iB\ell v$ while the mechanical power produced is $F_{\text{magnetic}}v = i\ell Bv$. That is, the electrical power absorbed by the back emf reappears as mechanical power, as it must by conservation of energy. Another way to view this is to note that $V_S i$ is the electrical power delivered by the source and, as $V_S - B\ell v = Ri$, one may write

$$V_S i = Ri^2 + i(B\ell v) = Ri^2 + F_{\text{magnetic}}v.$$

In words, the power from the source $V_S i$ is dissipated as heat in the resistance R while the rest is converted into mechanical power.

Equations of Motion for the Linear DC Machine

The equations of motion for the bar in the linear DC machine are now derived. With the inductance L of the circuit loop taken to be zero, m_ℓ the mass of the bar, f the coefficient of viscous friction, it follows that

$$\begin{aligned} V_S - B\ell v &= Ri \\ m_\ell \frac{dv}{dt} &= i\ell B - fv. \end{aligned}$$

Eliminating the current i, one obtains

$$m_\ell \frac{d^2x}{dt^2} = \ell B(V_S - B\ell v)/R - fv = -\left(\frac{B^2\ell^2}{R} + f\right)\frac{dx}{dt} + \frac{\ell B}{R}V_S$$

or

$$m_\ell \frac{d^2x}{dt^2} + \left(\frac{B^2\ell^2}{R} + f\right)\frac{dx}{dt} = \frac{\ell B}{R}V_S.$$

This is the equation of motion for the bar with V_S as the control input and the position x at the measured output.

1.3.4 Back Emf in the Single-Loop Motor

The back emf induced in the single loop motor by the external magnetic field of the permanent magnet is now computed. To do so, consider the flux surface for the rotor loop shown in Figure 1.13. The surface is a half-cylinder of radius $\ell_2/2$ and length ℓ_1 with the rotor loop as its boundary. The cylindrical surface is in the air gap, where the magnetic field is known to be radially directed and constant in magnitude, that is,

$$\vec{\mathbf{B}} = \begin{cases} +B\hat{\mathbf{r}} & \text{for } 0 < \theta < \pi \\ -B\hat{\mathbf{r}} & \text{for } \pi < \theta < 2\pi. \end{cases} \tag{1.1}$$

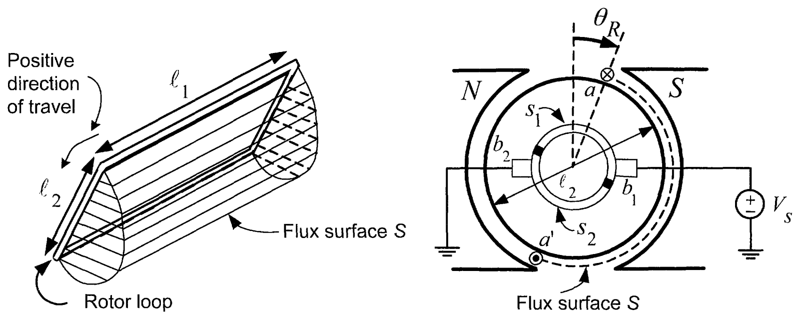

FIGURE 1.13. Flux surface for the single loop motor.

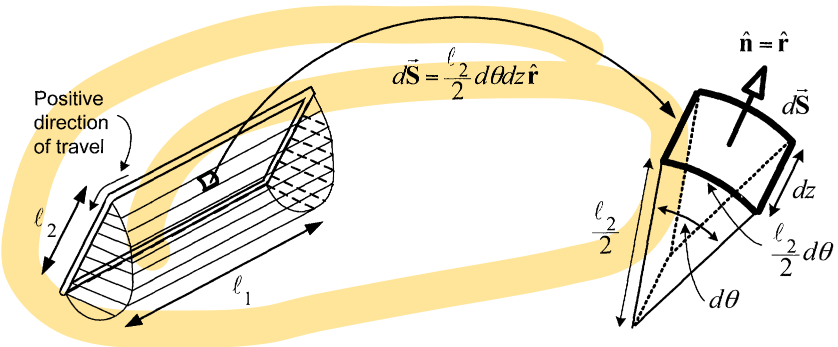

FIGURE 1.14. Surface element vector for the flux surface of Figure 1.13. The positive direction of travel around this surface is indicated by the curved arrow.

On the cylindrical part of the surface, the surface element is chosen as

$$d\vec{\mathbf{S}} = (\ell_2/2)d\theta dz\hat{\mathbf{r}}$$

which is directed outward from the axis of the cylinder as illustrated in Figure 1.14. The corresponding direction of positive travel is also indicated in Figure 1.14. On the two ends (half-disks) of the cylindrical surface, the $\vec{\mathbf{B}}$ field is quite weak making the flux through these two half-disks negligible.

Then, neglecting the flux through the two ends of the surface, the flux $\phi(\theta_R)$ for $0 < \theta_R < \pi$ is given by

$$\begin{aligned}
\phi(\theta_R) &= \int_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} \\
&= \int_0^{\ell_1} \int_{\theta=\theta_R}^{\theta=\pi} (B\hat{\mathbf{r}}) \cdot (\frac{\ell_2}{2} d\theta dz \hat{\mathbf{r}}) + \int_0^{\ell_1} \int_{\theta=\pi}^{\theta=\pi+\theta_R} (-B\hat{\mathbf{r}}) \cdot (\frac{\ell_2}{2} d\theta dz \hat{\mathbf{r}}) \\
&= \int_0^{\ell_1} \int_{\theta=\theta_R}^{\theta=\pi} B\frac{\ell_2}{2} d\theta dz + \int_0^{\ell_1} \int_{\theta=\pi}^{\theta=\pi+\theta_R} -B\frac{\ell_2}{2} d\theta dz \\
&= \frac{\ell_1\ell_2 B}{2}(\pi - \theta_R) - \frac{\ell_1\ell_2 B}{2}\theta_R \\
&= -\ell_1\ell_2 B\left(\theta_R - \frac{\pi}{2}\right). \qquad (1.2)
\end{aligned}$$

This derivation is based on the fact that the $\vec{\mathbf{B}}$ field is directed radially outward over the length $(\ell_2/2)(\pi - \theta_R)$ and radially inward over the length $(\ell_2/2)\theta_R$ (see Figure 1.13). In problem 7, the reader is asked to show that

$$\phi(\theta_R) = -\ell_1\ell_2 B\left(\theta_R - \pi/2 - \pi\right) \text{ for } \pi < \theta_R < 2\pi. \qquad (1.3)$$

A plot of the flux versus the rotor angle θ_R is given in Figure 1.15.

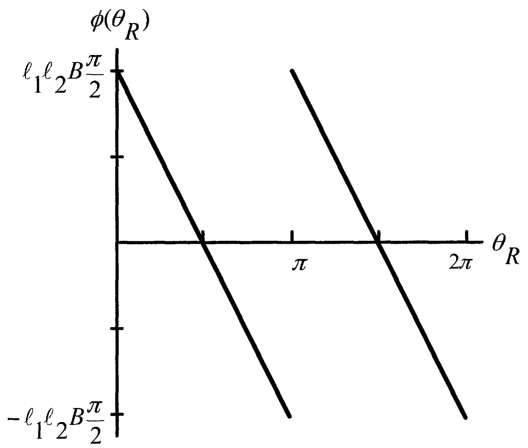

FIGURE 1.15. The rotor flux $\phi(\theta_R)$ due to the external magnetic field vs. θ_R.

Equations (1.2) and (1.3) may be combined into one expression as[5]

$$\phi(\theta_R) = -\ell_1\ell_2 B\left(\theta_R \bmod \pi - \frac{\pi}{2}\right) \qquad (1.4)$$

[5] $\theta_R \bmod \pi$ is the remainder after θ_R is divided by π. For example, $\theta_R = 5\pi/2 = 4 \times \pi + \pi/2$ so that $5\pi/2 \bmod \pi = \pi/2$.

which is a correct expression for any angle θ_R. By (1.2) and (1.3), the induced emf in the rotor loop is calculated as

$$\xi = -\frac{d\phi}{dt} = (\ell_1\ell_2 B)\frac{d\theta_R}{dt} = K_b\omega_R$$

where $K_b \triangleq \ell_1\ell_2 B$ is called the back emf constant.

The total emf in the rotor loop due to the voltage source V_S and external magnetic field is $V_S - K_b\omega_R$. How does one know to subtract ξ from the applied voltage V_S? As shown in Figure 1.14, the positive direction of travel around the loop is in *opposition* to V_S, so that if $\xi > 0$, it is opposing the applied voltage V_S. The standard terminology is to call $\xi \triangleq K_b\omega_R$ the *back emf* of the motor.

1.3.5 Self-Induced Emf in the Single-Loop Motor

The computation of the flux in the rotor loop produced by its own (armature) *current* is now done. To do so, consider the flux surface shown in Figure 1.16.

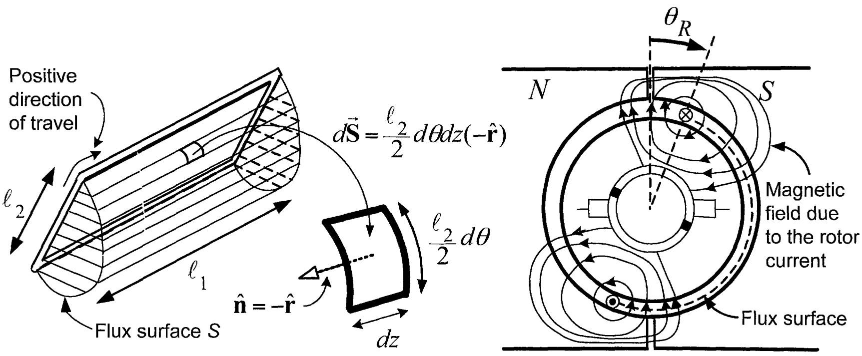

FIGURE 1.16. Computation of the inductance of the rotor loop. The surface element vector is $d\vec{\mathbf{S}} = -r_R d\theta dz\hat{\mathbf{r}}$ with a resulting positive direction of travel as indicated by the curved arrow. This direction coincides with the direction of positive current, that is, $i > 0$.

With reference to Figure 1.16, note that the magnetic field on the flux surface *due to the armature current* has the form

$$\vec{\mathbf{B}}(r_R, \theta - \theta_R, i) = iK(r_R, \theta - \theta_R)\,(-\hat{\mathbf{r}})$$

where

$$\begin{aligned} K(r_R, \theta - \theta_R) &> 0 \qquad \text{for } 0 \le \theta - \theta_R \le \pi \\ K(r_R, \theta - \theta_R) &< 0 \qquad \text{for } \pi \le \theta - \theta_R \le 2\pi. \end{aligned}$$

The exact expression for $K(r_R, \theta - \theta_R)$ is not easy to compute, but it is not needed for the analysis here. Rather, the point is that with $i > 0$, the magnetic field $\vec{\mathbf{B}}(r_R, \theta - \theta_R, i)$ due to the current in the rotor loop is radially in on the flux surface shown in Figure 1.16, that is, for $\theta_R \le \theta \le \theta_R + \pi$. For convenience, the surface element is chosen to be $d\vec{\mathbf{S}} = r_R d\theta dz(-\hat{\mathbf{r}})$ so that positive direction of travel around the surface coincides with the positive direction of the current i in the loop. The flux ψ in the rotor loop is then computed as[6]

$$\begin{aligned}
\psi(i) &= \int_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} = \int_{\theta_R}^{\theta_R+\pi} \int_0^{\ell_1} iK(r_R, \theta - \theta_R)\,(-\hat{\mathbf{r}}) \cdot (-r_R d\theta dz \hat{\mathbf{r}}) \\
&= i \int_{\theta_R}^{\theta_R+\pi} \int_0^{\ell_1} K(r_R, \theta - \theta_R) r_R d\theta dz \\
&= Li
\end{aligned}$$

where

$$L \triangleq \int_{\theta_R}^{\theta_R+\pi} \int_0^{\ell_1} K(r_R, \theta - \theta_R) r_R d\theta dz > 0. \tag{1.5}$$

This last equation just says the flux in the loop (due to the current in the loop) is proportional to the current i in the loop. The proportionality constant L is the called the *inductance* of the loop.[7] If $-d\psi/dt = -Ldi/dt > 0$, then the induced emf will force current into the page $\otimes$ on side a and out of the page $\odot$ of side a' in Figure 1.16. That is, this induced emf has the same sign convention as the armature current i and the source voltage V_S.

With the rotor locked at some angle θ_R so that the external magnetic field cannot induce an emf in the rotor loop, the equation describing the current i in the rotor loop is given by Kirchhoff's voltage law

$$V_S - Ri - L\frac{di}{dt} = 0$$

or

$$V_S = Ri + L\frac{di}{dt}.$$

Here R is the resistance of the loop and V_S is the source voltage applied to the loop. The loop and its equivalent circuit are shown in Figure 1.17.

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[6]The notation ψ is used to distinguish this flux from the flux ϕ in the loop due to the *external* permanent magnet. However, the total flux using an inward normal would be $\psi - \phi$ as the *outward* normal was used to compute ϕ in Section 1.3.4.

[7]It appears from equation (1.5) and Figure 1.16 that L can vary with θ_R. However, in an actual motor, there are loops spread evenly around the complete periphery of the rotor and, due to symmetry, the total self-inductance does not depend on θ_R.

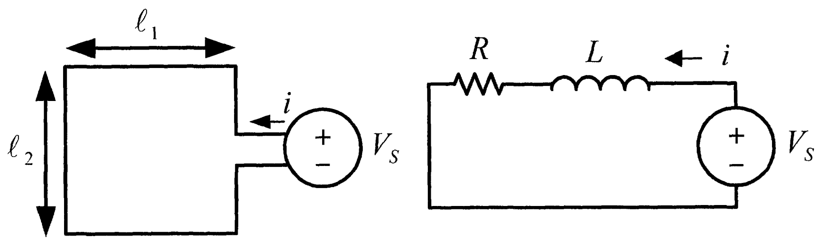

FIGURE 1.17. Left: Rotor loop. Right: Equivalent circuit.

The reader should convince himself/herself that Lenz's law holds as it must. For example, suppose a voltage $V_S > 0$ is applied to the loop resulting in both $i > 0$ and $di/dt > 0$, that is, the flux $\psi = Li$ is positive and increasing. The induced voltage is $-Ldi/dt < 0$ and opposes the current i producing the increasing flux $\psi = Li$. In this circumstance, the voltage source V_S is forcing the current i against this induced voltage $-Ldi/dt$ and the power absorbed by the induced voltage is $-iLdi/dt = -d\left(\frac{1}{2}Li^2\right)/dt$. This power is stored in the energy $\frac{1}{2}Li^2$ of the magnetic field surrounding the loop.

Arcing Between the Commutator and Brushes

Suppose the single-loop motor is rotating at constant speed ω_0 with a constant current i_0 in the rotating loop. Let L be the inductance of the loop. Now, every half-turn, the current in the loop reverses direction as shown in Figures 1.8(b)–(d). During this commutation, the current in the loop goes from i_0 to 0 to $-i_0$ (or vice versa) with a corresponding change in the loop's flux given by $\Delta\psi = L(-i_0) - Li_0 = -2Li_0$. By Faraday's law, the self-induced emf is then $-\Delta\psi/\Delta t = 2Li_0/\Delta t$ where Δt is the time for the current to change direction. Note that this time Δt decreases as the motor speed increases, so that, even if L is small, the induced emf in the loop (due to the reversal of current in the loop) can be quite large at high motor speeds. Large electric fields are produced by the induced voltage Ldi/dt when the loop is shorted which in turn ionizes the surrounding air. As the free electrons collide and recombine with the ionized air, light is given off and seen as arcing or sparking. These large voltages which cause the arcing between the slip rings and the brushes can damage the brushes as well as produce unwanted transient currents in the armature circuit.

1.4 Dynamic Equations of the DC Motor

Based on the simple single-loop DC motor analyzed above, the complete set of equations for a DC motor can be found. The total emf (voltage) in the loop due to the voltage source V_S, the external permanent magnet and the changing current i in the rotor loop is

$$V_S - K_b\omega_R - L\frac{di}{dt}.$$

This voltage goes into building up the current in the loop against the loop's resistance, that is,

$$V_S - K_b\omega_R - L\frac{di}{dt} = Ri$$

or

$$L\frac{di}{dt} = -Ri - K_b\omega_R + V_S.$$

This relationship is often illustrated by the equivalent circuit given in Figure 1.18. Recall that the torque τ_m on the loop due to the external magnetic field acting on the current in the loop is

$$\tau_m = K_T i$$

where $K_T \triangleq \ell_1\ell_2 B$ is called the torque constant. By connecting a shaft and gears to one end of the loop, this motor torque can be used to do work (lift weight, etc.). Let $-f\omega_R$ model the friction torque (due to the brushes, bearings, etc.) where f is the coefficient of viscous friction and let τ_L be the load torque (e.g., due to a weight being lifted).

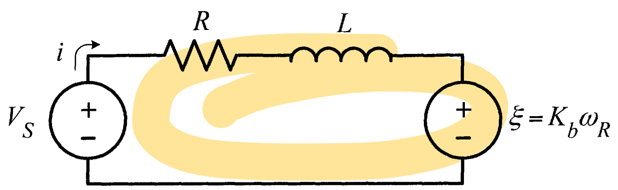

FIGURE 1.18. Equivalent circuit of the armature electrical dynamics.

Then, by Newton's law,

$$\tau_m - \tau_L - f\omega_R = J\frac{d\omega_R}{dt}$$

where J is the moment of inertia of the rotor (See the appendix of this chapter). The system of equations characterizing the DC motor is then

$$\begin{aligned} L\frac{di}{dt} &= -Ri - K_b\omega_R + V_S \\ J\frac{d\omega_R}{dt} &= K_T i - f\omega_R - \tau_L \\ \frac{d\theta_R}{dt} &= \omega_R. \end{aligned} \tag{1.6}$$

A picture of a DC motor servo system and its associated schematic is shown in Figure 1.19. In the schematic, R is the resistance of the rotor loop, L is the inductance of the rotor loop, $\xi = K_b\omega_R$ is the back emf, $\tau_m = K_T i$ is the motor torque, J is the rotor moment of inertia, and f is the coefficient of viscous friction. The positive directions for τ_m, θ_R, and τ_L are indicated by the curved arrows. The fact that the curved arrow for τ_L is opposite to that of τ_m just means that if the load torque is positive then it opposes a positive motor torque τ_m.

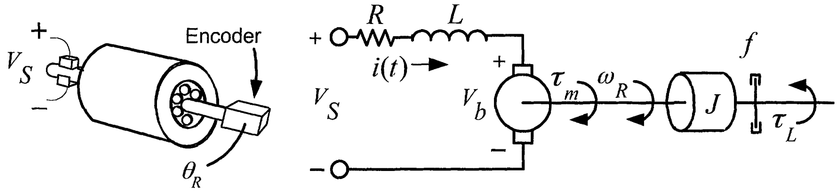

FIGURE 1.19. DC motor drawing and schematic.

Electromechanical Energy Conversion

The mechanical power produced by the DC motor is $\tau_m\omega_R = K_T i\omega_R = i\ell_1\ell_2 B\omega_R$ while the electrical power absorbed by the back emf is $i\xi = iK_b\omega_R = i\ell_1\ell_2 B\omega_R$. The fact that $K_T = K_b = \ell_1\ell_2 B$ must be for conservation of energy to hold. That is, the electrical power absorbed by the back emf equals (is converted to) the mechanical power produced. Another way to view this energy conversion is to write the electrical equation as

$$V_S = Ri + L\frac{di}{dt} + \xi.$$

The power out of the voltage source $V_S(t)$ is given by

$$\begin{aligned} V_S(t)i(t) &= Ri^2(t) + Li\frac{di}{dt} + iK_b\omega_R \\ &= Ri^2 + \frac{d}{dt}\left(\frac{1}{2}Li^2\right) + K_T i\omega_R \\ &= Ri^2 + \frac{d}{dt}\left(\frac{1}{2}Li^2\right) + \tau_m\omega_R. \end{aligned}$$

Thus the power $V_S(t)i(t)$ delivered by the source goes into heat loss in the resistance R, into stored magnetic energy in the inductance L of the loop and the amount $i\xi$ goes into the mechanical energy $\tau_m\omega_R$.

Remark *Voltage and Current Limits*

The amount of voltage V_S that may be applied to the input terminals T_1, T_2 of the motor is limited by capabilities of the amplifier supplying the voltage, that is, $|V_S| \leq V_{\max}$. Let $V_c(t)$ be the voltage commanded to the amplifier, then the actual voltage V_S out of the amplifier to the motor is limited by $V_{\max}$ as illustrated in Figure 1.20.

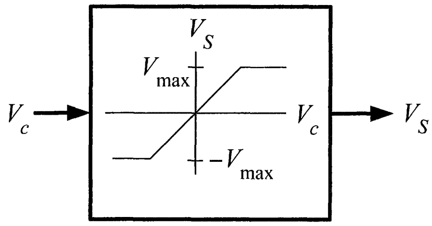

FIGURE 1.20. Saturation model of an amplifier.

In addition, there is a limit to the amount of current the rotating loop can handle before overheating or causing problems with commutation as previously mentioned. Typically there are two current limits (ratings), the *continuous* current limit $I_{\max\_\text{cont}}$ and the *peak* current limit $I_{\max\_\text{peak}}$. The continuous current limit $I_{\max\_\text{cont}}$ is the amount of current the motor can handle if left in use indefinitely. That is, the amount of heat dissipated in the rotor windings due to ohmic losses is equal to the amount of heat taken away by thermal conduction through the brushes and thermal convection with the air so as to be in a thermal equilibrium. The peak current limit $I_{\max\_\text{peak}}$ is the amount of current the motor can handle for short periods of time (typically only a few seconds).

1.5 Microscopic Viewpoint

Additional insight into the back emf ξ is found by calculating it from a microscopic point of view using the ideas given in Ref. [34] (page 887). To illustrate this approach, the back emf in the linear DC machine is recomputed from the microscopic point of view. To proceed, recall that the magnetic force on a charged particle q is $\vec{\mathbf{F}}_{\text{magnetic}} = q\vec{\mathbf{v}} \times \vec{\mathbf{B}}$, where $\vec{\mathbf{v}}$ is the velocity of the charge (see Ref. [34], page 816).

Example *A Linear DC Machine*

In this example, the linear DC machine is reanalyzed from the microscopic point of view. As before, $\vec{\mathbf{B}} = -B\hat{\mathbf{z}}$ where $B > 0$. Suppose the motor (bar) is moving to the right with a constant speed v_m. Each charge q in the sliding bar has total velocity $\vec{\mathbf{v}} = v_m\hat{\mathbf{x}} - v_d\hat{\mathbf{y}}$, where v_d is the drift speed of the charges down the wire. The magnetic force on the charge q is

$$\vec{\mathbf{F}}_{\text{magnetic}} = q\vec{\mathbf{v}}\times\vec{\mathbf{B}} = q(v_m\hat{\mathbf{x}} - v_d\hat{\mathbf{y}}) \times (-B\hat{\mathbf{z}}) = qv_mB\hat{\mathbf{y}} + qv_dB\hat{\mathbf{x}}.$$