Exploring Mellin Transform: Properties, Examples, and Applications

Your Name

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Contents

1 Introduction

The Mellin Transform is a powerful integral transform widely used in mathematics, physics, and engineering. It plays a key role in problems involving asymptotics, signal processing, and number theory. This report explores the properties of the Mellin Transform, demonstrates specific cases, and visualizes results using Python implementations.

2 Mathematical Background

The Mellin Transform of a function $f(x)$ is defined as:

$$\mathcal{M}\{f(x)\}(s) = \int_0^\infty x^{s-1} f(x)\, dx, \tag{1}$$

where $s = \sigma + i\omega$ is a complex variable. For the transform to exist, $f(x)$ must satisfy certain conditions. A key function associated with the Mellin Transform is the Gamma function, defined as:

$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x}\, dx. \tag{2}$$

3 Proving Mellin Transform of $f(x) = e^{-x}$

To demonstrate the relationship between the Mellin Transform and the Gamma function, consider the function $f(x) = e^{-x}$. Mathematically, the transform is:

$$\mathcal{M}\{e^{-x}\}(s) = \int_0^\infty x^{s-1} e^{-x}\, dx = \Gamma(s). \tag{3}$$

Python was used to compute this transform numerically and compare it with the analytical Gamma function. The following plot illustrates the comparison:

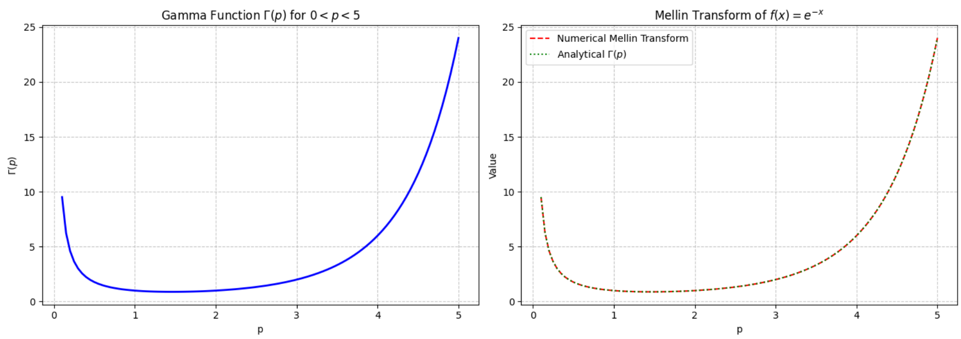

Figure 1: Comparison of numerical Mellin Transform and analytical Gamma function for $f(x) = e^{-x}$.

4 Verification for $f(x) = \frac{1}{x+1}$

The Mellin Transform of $f(x) = \frac{1}{x+1}$ is given by:

$$\int_0^\infty \frac{x^{p-1}}{x+1}\, dx = \frac{\pi}{\sin(\pi p)}. \tag{4}$$

This can also be expressed as:

$$\Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin(\pi p)}. \tag{5}$$

Python code was used to verify this identity numerically. The results of the Mellin Transform, Gamma product, and analytical solution are shown in the plot below:

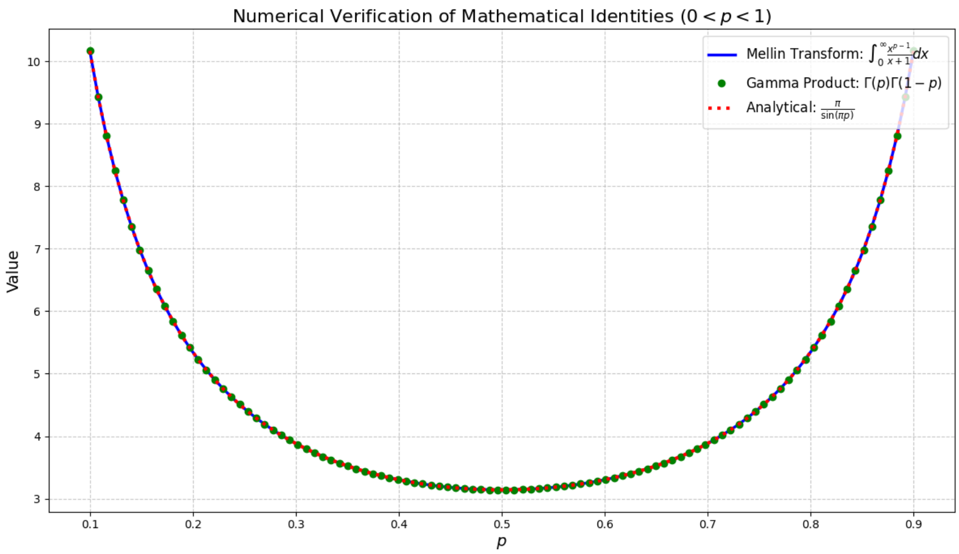

Figure 2: Numerical verification of mathematical identities for $0 < p < 1$.

5 Exploring Scaling and Shifting Properties

5.1 Scaling Property

The scaling property states that:

$$\mathcal{M}\{f(ax)\}(s) = a^{-s}\mathcal{M}\{f(x)\}(s). \tag{6}$$

Python was used to demonstrate this property for $f(x) = e^{-x}$ with $a = 2$. The results are compared numerically and analytically.

5.2 Shifting Property

The shifting property states that:

$$\mathcal{M}\{x^a f(x)\}(s) = \mathcal{M}\{f(x)\}(s+a). \tag{7}$$

Both properties are verified numerically and visualized in the plots below:

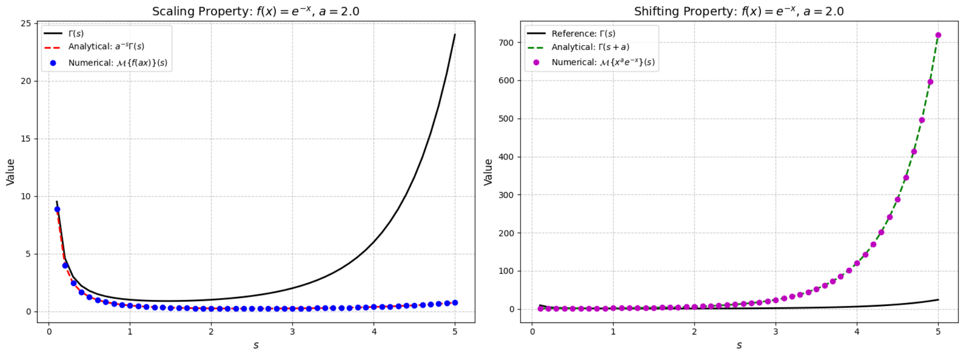

Figure 3: Numerical verification of scaling and shifting properties.

6 Visualizing $\phi(r, \theta)$

The convolution-based approach was used to compute $\phi(r, \theta)$:

$$\phi(r, \theta) = \int_0^\infty f(\xi) \frac{h(r/\xi, \theta)}{\xi} \, d\xi, \tag{8}$$

where $h(r, \theta)$ is defined as:

$$h(r, \theta) = \frac{r^n (1 + r^{2n}) \cos(n\theta)}{1 + 2r^{2n} \cos(2n\theta) + r^{4n}}, \quad n = \frac{\pi}{2\alpha}. \tag{9}$$

The resulting $\phi(r, \theta)$ was plotted for $\theta \in [-\pi/4, \pi/4]$ and $r \in [0.1, 5]$:

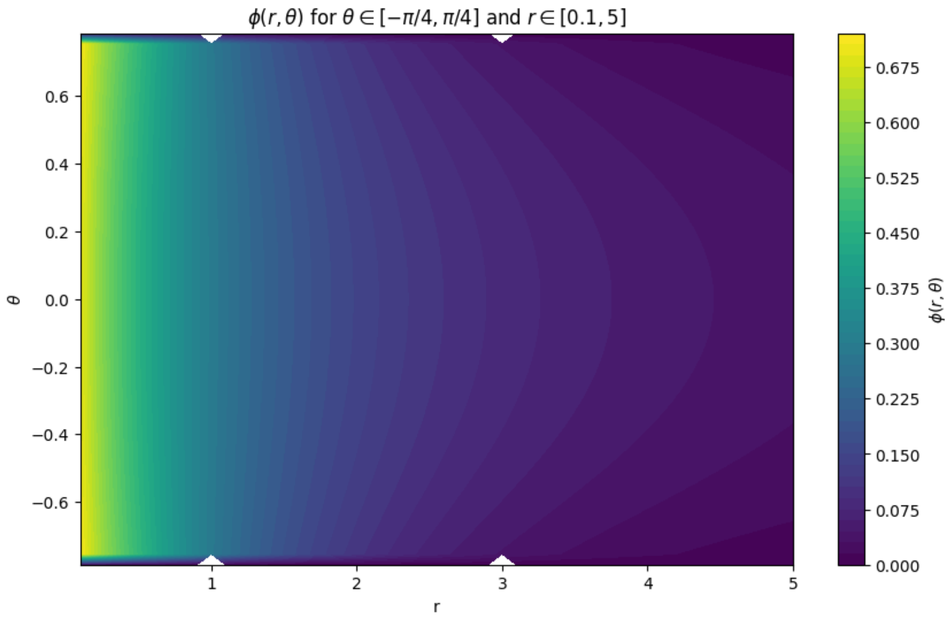

Figure 4: Contour plot of $\phi(r, \theta)$ for $\theta \in [-\pi/4, \pi/4]$ and $r \in [0.1, 5]$.

7 Conclusion

This report demonstrates the application of the Mellin Transform to various functions, verifies mathematical identities, and explores key properties such as scaling and shifting.

Python was instrumental in numerically validating these concepts and visualizing their results. Potential extensions could involve exploring higher-dimensional transforms or real-world applications in data science and engineering.

References

1. Mellin Transform Theory and Applications.
2. SciPy Documentation: `https://scipy.org`
3. Python Libraries: NumPy, SciPy, Matplotlib.