HW4 Compressed sensing

تمرین سری ۴ حسگری فشرده

Subject:

Year: Month: Date:

سوال ۱:

$\forall t \in \mathbb{R}^+ \cup \{0\}$

(I) $\exists c_1, c_2 \in \mathbb{R}^+ : \mathbb{P}[|X - \mathbb{E}X| \geq t] \leq c_1 \cdot e^{-c_2 t^2}$ $\Longrightarrow \mathrm{Var}(X) \leq \frac{c_1}{c_2}$

let $\mu = \mathbb{E}X$

$$\mathrm{Var}(X) = (\mathbb{E}X^2) - \mu^2 = \mathbb{E}[\underbrace{X^2 - \mu^2}_{\geq 0}] = \int_0^\infty \mathbb{P}[X^2 - \mu^2 \geq t']\, dt'$$

$$= \int_0^\infty \mathbb{P}[X^2 \geq \mu^2 + t']\, dt' = \int_0^\infty \mathbb{P}[|X| \geq \underbrace{\sqrt{\mu^2 + t'}}_{t}]\, dt'$$

$$= \int_\mu^\infty 2t\,\mathbb{P}[|X| \geq t]\, dt = \int_0^\infty 2t\,\mathbb{P}[|X - \mathbb{E}X| \geq t]\, dt$$

$$\leq 2c_1 \int_0^\infty t e^{-c_2 t^2}\, dt = \int_0^\infty 2c_1 \sqrt{\frac{u}{c_2}}\, e^{-u} \cdot \frac{1}{2\sqrt{c_2 u}}\, du$$

let $u = ct^2$

$$= \int_0^\infty \frac{2c_1}{2c_2} e^{-u}\, du = \frac{c_1}{c_2} \underbrace{\int_0^\infty e^{-u}\, du}_{1} = \frac{c_1}{c_2}$$

Thus, we proved that $\boxed{\mathrm{Var}(X) \leq \frac{c_1}{c_2}}$ Q.E.D. ∎

(II) $m_X$: median $\longrightarrow \begin{cases} \mathbb{P}[X \geq m_X] \geq 1/2 \\ \mathbb{P}[X \leq m_X] \leq 1/2 \end{cases}$

if $\mathbb{P}[|X - \mathbb{E}X| \geq t] \leq c_1 \cdot e^{-c_2 t^2}$ $\underset{\exists c_3, c_4 \in \mathbb{R}^+}{\Longrightarrow}$ $\forall t \geq 0$: $\mathbb{P}[|X - m_X| \geq t] \leq c_3 \cdot e^{-c_4 t^2}$

$\exists c_1, c_2 \in \mathbb{R}^+$:

let $\Delta = |\mathbb{E}X - m_X|$, if $0 \leq \Delta < \frac{t}{2}$:

$$\mathbb{P}(|X - \mathbb{E}X| \geq t) \leq \mathbb{P}(|X - \mathbb{E}X| + |\mathbb{E}X - m_X| \geq t) \leq \mathbb{P}[|X - \mathbb{E}X| \geq \tfrac{t}{2} + \Delta]$$

$$\leq \mathbb{P}[|X - \mathbb{E}X| \geq \tfrac{t}{2}] \leq c_1 e^{-c_2 t^2/4}$$

if $\Delta \geq \frac{t}{2}$:

$$\frac{1}{2} \leq \mathbb{P}[|X - \mathbb{E}X| > \Delta] \leq c_1 e^{-c_2 \Delta^2} \leq c_1 e^{-c_2 \frac{t^2}{4}} \rightarrow 2c_1 e^{-c_2 t^2/4} \geq 1$$

thus, at both cases $(\Delta < \frac{t}{2})$ & $(\Delta \geq t/2)$ the bound

$$\mathbb{P}[|X - m_X| \geq t] \leq 2c_1 e^{-c_2 \frac{t^2}{4}} = c_3 \cdot e^{-c_4 t^2} \quad \begin{cases} c_3 = 2c_1 \\ c_4 = \frac{c_2}{4} \end{cases}$$

Q.E.D. ∎

HW4

Subject:

Year: $\exists c_3, c_4$ Month: Date:

ادامه سوال 1:

III if $\mathbb{P}\left[|X-m_X|\geq t\right] \leq c_3 \cdot e^{-c_4 t^2} \Rightarrow \begin{cases} \exists c_1, c_2 \\ \mathbb{P}\left[|X-\mathbb{E}X|\geq t\right] \leq c_1 \cdot e^{-c_2 t^2} \\ 2c_3+1 = c_1, \quad c_2 = c_4/4 \end{cases}$

حل اول

$X, Y \overset{iid}{\sim} P_X$

$\mathbb{P}\left(|X-\mathbb{E}X|\geq t\right) = \mathbb{P}\left(|X-\mathbb{E}X|^2 \geq t^2\right) \leq e^{-\lambda t^2} \mathbb{E}\left[e^{\lambda |\mathbb{E}_Y Y - X|^2}\right]$

$\underset{\text{Jensen}}{\leq} e^{-\lambda t^2} \mathbb{E}_{X,Y}\left[e^{\lambda (X-Y)^2}\right] = e^{-\lambda t^2} \cdot \int_0^\infty \mathbb{P}\left[e^{\lambda (X-Y)^2} \geq u\right] du$

let $u = e^{\lambda v} \rightarrow \leq e^{-\lambda t^2}\left[\alpha + \lambda \int_{\frac{\log \alpha}{\lambda}}^\infty e^{\lambda v}\, \mathbb{P}\left[|X-Y| \geq \sqrt{v}\right] dv\right]$

$\leq e^{-\lambda t^2}\left[\alpha + 2\lambda \int_{\frac{\log \alpha}{\lambda}}^\infty e^{\lambda v} \cdot \mathbb{P}\left[|X-m_X| \geq \frac{v}{2}\right] dv\right] \quad (*)$

$(*)$ was due to: $\mathbb{P}\left[|X-Y|\geq t\right] \leq 1-\left(1-\mathbb{P}\left[X-m_X \geq \frac{t}{2}\right]\right)\left(1-\mathbb{P}\left[Y-m_Y \geq \frac{t}{2}\right]\right)$

$\leq 2\mathbb{P}\left[|X-m_X| \geq t/2\right]$

$\Rightarrow \mathbb{P}\left[|X-\mathbb{E}X| \geq t\right] \leq e^{-\lambda t^2}\left[\alpha + 2\lambda \int_{\frac{\log \alpha}{\lambda}}^\infty e^{\lambda v} \cdot c_3 e^{-c_4 \frac{v}{4}} dv\right]$

$\hookrightarrow \mathbb{P}\left[|X-\mathbb{E}X| \geq t\right] \leq e^{-\lambda t^2}\left[\alpha + 2\lambda c_3 \frac{\alpha^{1-\frac{c_4}{4\lambda}}}{\frac{c_4}{4}-\lambda}\right]$

we set $\alpha, \lambda$ such that $2\lambda c_3 e^{v(\lambda - \frac{c_4}{4})} = 1$ (we optimize $\alpha, \lambda$)

$\left.\begin{matrix} \alpha = 1 \\ \lambda = \frac{c_4}{8} \end{matrix}\right\}$ $\boxed{\mathbb{P}\left[|X-\mathbb{E}X| \geq t\right] \leq e^{-\frac{c_4 t^2}{8}} (1+2c_3)}$

$(c_1 = 1+2c_3, \; c_2 = c_4/8)$ Q.E.D.

$\forall t \geq \gamma\Delta$

III حل دوم $\mathbb{P}\left(|X-\mathbb{E}X| \geq t\right) \leq \mathbb{P}\left(|X-\mathbb{E}X| \geq \Delta + (1-\frac{1}{\gamma}t)\right)$

let $\Delta = |\mathbb{E}X - m_X|$

$\leq \mathbb{P}\left[|X-m_X| \geq (1-\frac{1}{\gamma})t\right] \leq c_3 \cdot e^{-c_4 \frac{t^2}{\left(\frac{\gamma}{\gamma-1}\right)^2}}$

$e^{\lambda \Delta^2} = e^{\lambda |m_X - \mathbb{E}X|^2} \underset{\text{Jensen}}{\leq} \mathbb{E}\left[e^{\lambda |m_X - \mathbb{E}X|^2}\right] = \int_0^\infty \mathbb{P}\left[e^{\lambda |m_X - \mathbb{E}X|^2} \leq t\right] dt$

$\leq \int_0^\infty c_3 \cdot e^{-c_4 \left(\sqrt{\frac{\log(t)}{\lambda}}\right)^2} = \int_0^\infty c_3 t^{\frac{-c_4}{\lambda}} dt \leq 1 + \int_1^\infty c_3 t^{\frac{-c_4}{\lambda}} dt = 1 + \frac{c_3}{\frac{c_4}{\lambda}-1} \underset{\lambda = \frac{2}{3}c_4}{=} 1+2c_3$

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ادامه سوال 2:

راه دوم:

(III) $e^{\lambda x} = e^{\frac{sx^2}{2\sigma^2}} \cdot e^{-\frac{(\lambda - \frac{sx}{\sigma^2})^2}{2s/\sigma^2}} e^{\frac{\lambda^2\sigma^2}{2s}}$

$$\int \mathbb{E}[e^{\lambda X}]\, d\lambda \leq \int_{\mathbb{R}^+} e^{\frac{\lambda^2\sigma^2}{2} - \frac{\lambda^2 \sigma^2}{2s}}\, d\lambda = \int_{\mathbb{R}^+} e^{\frac{-\lambda^2(1-s)\sigma^2}{2s}}\, d\lambda = \sqrt{\frac{2\pi s}{(1-s)\sigma^2}}$$

$$\mathbb{E}\int_{\mathbb{R}^+} e^{\frac{-(\lambda - \frac{sx}{\sigma^2})^2 \sigma^2}{2s}} e^{\frac{sx^2}{2\sigma^2}}\, d\lambda = \sqrt{\frac{2\pi s}{\sigma^2}}\, \mathbb{E}\left[e^{\frac{sx^2}{2\sigma^2}}\right]$$

$$\Rightarrow \sqrt{\frac{2\pi s}{\sigma^2}}\, \mathbb{E}\left[e^{\frac{sx^2}{2\sigma^2}}\right] \leq \sqrt{\frac{2\pi s}{(1-s)\sigma^2}} \xrightarrow{\forall s \in [0,1)} \mathbb{E}\left[e^{\frac{sx^2}{2\sigma^2}}\right] \leq \frac{1}{\sqrt{1-s}}$$

Q.E.D

(IV) $X_i \overset{iid}{\sim} subg(\sigma_i^2)$, $Y = X_1 + \dots + X_n$

$$\mathbb{E}[e^{\lambda Y}] = \prod_{i=1}^{n} \mathbb{E}(e^{\lambda X_i}) \leq \prod_{i=1}^{n} e^{\frac{\lambda^2 \sigma_i^2}{2}} = \exp\left(\frac{\lambda^2}{2} \sum_{i=1}^{n} \sigma_i^2\right)$$ Q.E.D

$\Rightarrow Y = X_1 + \dots + X_n$ is subgaussian with param $\sigma^2 = \sigma_1^2 + \dots + \sigma_n^2$

(V) $\|x\|_2 = 1$, $A_{ij} \overset{iid}{\sim} subg(\sigma^2)$. $\vec{y} = A\vec{x}$

we know that $y_{(i)} = \sum_j A_{ij} x_j$ and $A_{ij}$'s are $\sigma^2$-subgaussian. thus according to (IV) we can say that $\forall i: y_{(i)}$ is $\sum_i \sigma^2 x_i^2 = \sigma^2$ subgaussian. thus $y_{(i)}$'s are independent & subgaussian. $\Rightarrow$ thus:

$$\mathbb{P}\left[\|Ax\|_2^2 \geq a\right] = \mathbb{P}\left[e^{s\sum_{i=1}^{n} y_{(i)}^2} \geq e^{sa}\right] \overset{\text{By Markov}}{\leq} e^{-sa} \cdot \mathbb{E}\left[e^{s\sum y_{(i)}^2}\right]$$

$$\leq e^{-sa}\, \mathbb{E}\left[e^{s\sum_i y_{(i)}^2}\right] \overset{\text{independence}}{=} e^{-sa} \prod_i \mathbb{E}\left[e^{s y_{(i)}}\right] \overset{\text{according to (III)}}{\leq} e^{-sa} \cdot (1 - 2\sigma^2 s)^{-\frac{m}{2}} \quad \left(0 < s < \frac{1}{2\sigma^2}\right)$$

$$\Rightarrow \mathbb{P}\left(\|Ax\|_2^2 \geq a\right) \leq e^{-sa} \cdot (1 - 2\sigma^2 s)^{-m/2} \quad \forall\, 0 < s < \frac{1}{2\sigma^2}$$

$$\mathbb{P}\left[\|Ax\|_2^2 \leq a\right] = \mathbb{P}\left[e^{-s\sum_i y_{(i)}^2} \geq e^{-sa}\right] \leq e^{sa} \prod_i \mathbb{E}\left[e^{-s y_{(i)}^2}\right] \leq \frac{e^{sa}}{(1 + 2\sigma^2 s)^{\frac{m}{2}}}$$

$$= \frac{e^{sa}}{(1+2\sigma^2)^{\frac{m}{2}}} \quad s > 0$$

Q.E.D.

$(2c_3+1)e^{-\frac{c_4 t^2}{(\frac{\delta}{\gamma-1})^2}} \underset{t \geq \gamma\Delta}{\geq} (2c_3+1)e^{-c_4\Delta^2(\gamma-1)^2} \underset{(\gamma-1)^2 = 2/3}{\geq} \frac{1+2c_3}{1+\frac{c_3}{\frac{1}{(\gamma-1)^2}-1}} \geq 1 \Big\} \; (2c_3+1) \times e^{-\frac{c_4 t^2}{4.95}}$ ✓

HW

$\forall s \geq 0$

$X$ is $\text{subg}(\sigma^2) \longleftrightarrow \mathbb{E}[e^{sX}] \leq e^{\frac{s^2\sigma^2}{2}}$ $\quad$ ? $2\sigma^2$

Ⅰ $\mathbb{E}X = 0 \longrightarrow \mathbb{E}[e^{sX}] = \sum_{k=0}^{\infty} \frac{\mathbb{E}(X^k)s^k}{k!} = 1 = 1 + s\,\mathbb{E}X + O(s^2)$

let $s \to 0$ ($s \ll 1$)

$\Rightarrow$ which suggests $\boxed{\mathbb{E}X = 0}$ Q.E.D. ■

Ⅱ $\mathbb{P}[|X| \geq t] \leq 2e^{-t^2/2\sigma^2}$

$\mathbb{P}(X \geq t] \leq \frac{\mathbb{E}[e^{sX}]}{e^{st}} \leq e^{-st + \frac{s^2\sigma^2}{2}} \xrightarrow{(*)} \mathbb{P}[X \geq t] \leq \inf_{s \geq 0} e^{-st + \frac{s^2\sigma^2}{2}}$

$\frac{\partial}{\partial s}\left(-st + \frac{s^2\sigma^2}{2}\right) = 0 \longrightarrow s = \frac{t}{\sigma} \overset{(*)}{\Longrightarrow} \mathbb{P}[X \geq t] \leq e^{-\frac{t^2}{2\sigma^2}}$ (**)

↳ is convex

likewise : $\mathbb{P}[X \leq -t] \leq e^{-\frac{t^2}{2\sigma^2}}$ (***)

Applying union bound : $\mathbb{P}[|X| \geq t] \leq \mathbb{P}[X \geq t] + \mathbb{P}[X \leq -t]$

using (**), (***) $\longrightarrow \boxed{\mathbb{P}[|X| \geq t] \leq 2e^{-\frac{t^2}{2\sigma^2}}}$ Q.E.D. ■

Ⅲ $\mathbb{E}\left[e^{\frac{sX^2}{2\sigma^2}}\right] \leq \frac{1}{\sqrt{1-s}} \quad \forall s \in [0,1)$

$$\mathbb{E}\left[e^{\frac{sX^2}{2\sigma^2}}\right] = \sum_{k=0}^{\infty} \frac{\mathbb{E}[X^{2k}]\,s^k}{k!}$$

$$\mathbb{E}[X^{2k}] \leq \int_0^{\infty} t^{2k-1}\,\mathbb{P}[|X| \geq t]\,dt \leq \int_0^{\infty} (2k-1) \cdot 2e \quad dt$$

$$\Rightarrow \mathbb{E}[X^{2k}] \leq \int_0^{\infty} \quad (2\sigma^2)^k$$

$$\xrightarrow{(*), (**)} \mathbb{E}\left[e^{\frac{sX^2}{2\sigma^2}}\right] \leq \sum_{k=0}^{\infty} \frac{2(k!)\cdot(2\sigma^2)^k \cdot s^k}{(2\sigma^2)^k} = 2\sum_{k=0}^{\infty} s^k$$

$$= \frac{2}{1-s} - 1$$

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ادامه سوال 1:

$$f \text{ is } 1\text{-lipschitz} \Rightarrow \mathbb{P}\left[\,|f(\cdot) - M[f]| \geq t\right] \leq c_1 \cdot e^{-c_2 \frac{t^2}{2}}$$

$$\frac{c_1}{\sqrt{c_2}} \Gamma\left(\tfrac{3}{2}\right)\sqrt{p}$$

$$\mathbb{E}[X^p] \geq \left(\sqrt[p]{\mathbb{E}\|f(x)\|^p} - M[f(\cdot)]\right)^p \Rightarrow \left(\sqrt[p]{\mathbb{E}\|f(x)\|^p} - M[f(\cdot)]\right) \leq \text{Const}$$

$$\mathbb{P}\left[\,\|f(x)\|_p - f(x) \geq t\right] \leq \mathbb{P}\left[\,\|f_p - f\| + \|f - M_f\| > t_{\text{const}}\right]$$

$$\leq \mathbb{P}[X > t] \leq c_1 \cdot e^{-(t - \text{const})^2}$$

$$\leq c_1 \cdot e^{-c_2\left(\frac{t^2}{2} - \text{const}\right)} = \left(c_1 \cdot e^{c_1^2 \Gamma\left(\frac{3}{2}\right)^2 p}\right) e^{-c_2 \frac{t^2}{2}}$$

thus under any moment we've concentration.

Q.E.D. ∎

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السؤال ٤:

VI $m(n,\varepsilon)$? $\rightarrow \lim_{n} \mathbb{P}\left[\exists \frac{1}{\sqrt{m}} A,\ \varepsilon\text{-embedding}\right] = 1$

we'll prove that $\left(\frac{1}{\sqrt{m}}\|Ax\|^2\right)^2$ has concentration around $16\sigma^2$ and hence for $\sigma^2=1$ or transformation $\frac{1}{m\sigma^2}\|Ax\|_2^2$ has $\varepsilon$-embedding property.

$s \leq 0.34$

By V

(i) $\mathbb{P}\left[\frac{1}{m\sigma^2}\|Ax\|_2^2 > 1+\varepsilon\right] \leq \frac{e^{-sm\sigma^2(1+\varepsilon)}}{(\sqrt{1-2s})^{m\sigma^2}} = \left(\frac{e^{-s}}{\sqrt{1-2s}} \cdot e^{-s\varepsilon}\right)^{m\sigma^2} \leq e^{(2s^2-s\varepsilon)m\sigma^2}$

$s>0$

(ii) $\mathbb{P}\left[\frac{1}{m\sigma^2}\|Ax\|_2^2 < 1-\varepsilon\right] \leq \frac{e^{sm\sigma^2(1-\varepsilon)}}{(\sqrt{1+2s})^{m\sigma^2}} = \left(\frac{e^{s}}{\sqrt{1+2s}} \cdot e^{s\varepsilon}\right)^{m\sigma^2} \leq e^{(2s^2-s\varepsilon)m\sigma^2}$

if $\varepsilon<1$ and we let $s=\frac{\varepsilon}{4}$, with union bound we get:

$$\mathbb{P}\left[-\varepsilon < \left|\frac{\|Ax\|_2^2}{m\sigma^2} - 1\right| < \varepsilon\right] \geq 1-2e^{-\frac{m\sigma^2\varepsilon^2}{8}}$$

Now we compute the $\varepsilon$-embedding probability for the points $x_1, x_2, \dots, x_n$:

$$\mathbb{P}[\varepsilon\text{-embedding}] = \mathbb{P}\left(\forall_{\substack{i\neq j\\ i,j\in[n]}} : 1-\varepsilon \leq \frac{\|T(x_i)-T(x_j)\|_2}{\|x_i-x_j\|_2} \leq 1+\varepsilon\right]$$

$$= \mathbb{P}\left[\forall i j : (1-\varepsilon) \leq T(z_{ij}) \leq (1+\varepsilon)\right] \overset{\text{Union Bound}}{\leq} 1-\binom{n}{2} 2e^{-\frac{m\sigma^2\varepsilon^2}{8}} = 1-\delta$$

let $z_{ij} = \frac{x_i-x_j}{\|x_i-x_j\|_2}$

Now let $\delta = 2\binom{n}{2} e^{-\frac{m\sigma^2\varepsilon^2}{8}} \rightarrow \boxed{m(n,\varepsilon) \geq 16 \cdot \frac{\log(n/\delta)}{\sigma^2\varepsilon^2}}$

Q.E.D. ■

Hence, if the dimension of the projected space is $O\left(\frac{\log N}{\varepsilon^2}\right)$, then with high probability close to 1, we have the $\varepsilon$-embedding property.

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HW4

$\forall \lambda \geq 0 : \mathbb{E}\left[e^{\lambda X_i}\right] \leq e^{\frac{\lambda^2 \sigma^2}{2}} \Rightarrow \mathbb{E}\left[\max_i X_i\right] \leq \sigma\sqrt{2\log n}$ :3 ([illegible])

(I) $X_1, \ldots, X_n \sim subg(\sigma^2)$, not nessacarily independent!

$$\mathbb{E}\left[\max_i X_i\right] = \mathbb{E}\left[\frac{1}{\lambda}\log\left(\max_i e^{\lambda X_i}\right)\right] \leq \mathbb{E}\left[\frac{1}{\lambda}\log\left(\sum_{i=1}^{n} e^{\lambda X_i}\right)\right] \quad \lambda \in \mathbb{R}^+$$

$$\underset{\text{Jensen}}{\leq} \frac{1}{\lambda}\log\left(\mathbb{E}\left(\sum_{i=1}^{n} e^{\lambda X_i}\right)\right) \leq \frac{1}{\lambda}\log\left(n e^{\frac{\lambda^2\sigma^2}{2}}\right)$$

$$\underset{\forall \lambda \geq 0}{\Longrightarrow} \mathbb{E}\left[\max_i X_i\right] \leq \underbrace{\frac{\log(n)}{\lambda} + \frac{\lambda \sigma^2}{2}}_{f(\lambda)} \xrightarrow{\text{(convex wrt } \lambda)} \lambda^* = \sqrt{\frac{2\log n}{\sigma^2}}$$

$$\Rightarrow \mathbb{E}\left[\max_i X_i\right] \leq \inf_{\lambda} f(\lambda) = \max\{f(0), f(\lambda^*)\} = f(\lambda^*) = \boxed{\sigma\sqrt{2\log n}}$$

Q.E.D.

(II) $P$ is a n-gon with vertices $\{\vec{v}_1, \ldots, \vec{v}_n\} \in \mathbb{R}^d$

$\vec{X}$ an RV in $\mathbb{R}^d$ $\rightarrow$ let $Y_i = \vec{v}_i^T \vec{X} \sim subg(\sigma^2)$

$\Rightarrow$ Prove that $\mathbb{E}\left[\max_{\theta \in P}\{\vec{\theta}^T \vec{X}\}\right] \leq \sigma\sqrt{2\log n}$

We assume that the n-gon $P$ is convex, thus $\exists\, \alpha_1, \ldots, \alpha_n \geq 0$ such that $\forall \theta \in P : \theta = \sum_{i=1}^{n} \alpha_i v_i, \; \sum_{i=1}^{n} \alpha_i = 1$

$$\Rightarrow \mathbb{E}\left[\max_{\theta \in P}\{\vec{\theta}^T \vec{X}\}\right] = \mathbb{E}\left[\max_{\alpha_i}\left\{\sum_{i=1}^{n} \alpha_i v_i^T X\right\}\right] \leq \mathbb{E}\left[\left(\sum_{i=1}^{n}\alpha_i\right) \cdot \max_i\{v_i^T X\}\right]$$

$$\Rightarrow \mathbb{E}\left[\max_{\theta \in P}\{\theta^T X\}\right] \leq \mathbb{E}\left[\max_i \underbrace{v_i^T X}_{subg(\sigma^2)}\right] \leq \sigma\sqrt{2\log n}$$

in accordance with (I)

Q.E.D.

(III) $X_1, \ldots, X_n$, $X_i \overset{iid}{\sim} \mathcal{N}(0, \sigma^2) \Rightarrow \mathbb{E}\left[\max_i X_i\right] \geq \frac{\sigma\sqrt{\log n}}{\sqrt{\pi \log 2}}$

Subject: $\Phi(t) = \int_{-\infty}^{t} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx$

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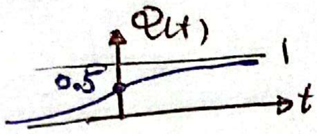


:3 OMG

**III** $\mathbb{E}\left[\max_i X_i\right] \geq t \cdot \mathbb{P}\left[\max_i X_i \geq t\right] = t\left(1 - \left[1 - \Phi(\frac{t}{\sigma})\right]^n\right)$

let $t = \sqrt{\frac{4\sigma^2 \log N/2}{\pi \log 2}} \Rightarrow$ therefore:

$$\mathbb{E}\left(\max_i X_i\right) \leq \sqrt{\frac{4\sigma^2 \log N}{\pi \log 2}}\left(1 - \left[1 - \log 2 \cdot e^{-\log n}\right]^n\right)$$

we also know that $\forall x > 1.4: \log 2 \cdot e^{-\frac{\pi}{4}\log 2 \cdot x^2} \leq \Phi(x)$

$$\Rightarrow \mathbb{E}\left[\max_i X_i\right] \leq \sqrt{\frac{4\sigma^2 \log N}{\pi \log 2}}\left(1 - e^{-\log 2}\right) = \sqrt{\frac{\sigma^2}{\pi}\log_2^N} \quad \text{Q.E.D}$$

just keep in mind that $x > 1.4 \Leftrightarrow N \geq 3 \equiv \sqrt{\frac{4}{\pi}\log_2^N} > 1.4$ ✓

which is true!

**IV** $\{X_t\}_{t=1}^{\infty}$, $X_i \sim \text{subG}\left(\frac{\sigma_i}{\sqrt{\log(i+1)}}\right)^2 \Longrightarrow$ Prove that: $\mathbb{E}\left[\max_{1 \leq i} X_i\right] < c \cdot \max_i \sigma_i$

$$\mathbb{E}\left[\max_i \frac{|X_i|}{\sqrt{1+\log i}}\right] = \int_0^{\infty} \mathbb{P}\left[\max_i \frac{|X_i|}{\sqrt{1+\log i}} \geq t\right] dt \leq t_0 + \int_{t_0}^{\infty} \sum_i \mathbb{P}\left[X_i \leq t\sqrt{1+\log i}\right]$$

$$\rightarrow \mathbb{E}\left[\max_i \frac{|X_i|}{\sqrt{1+\log i}}\right] \leq t_0 + \int_{t_0}^{\infty} \sum_i \mathbb{P}\left[X_i \leq t\sqrt{1+\log i}\right] dt$$

$\sigma = \max_i \sigma_i$

$$\leq t_0 + \sum_i \int_{t_0}^{\infty} 2e^{-\frac{t^2(1+\log i)}{2\sigma_i^2}} dt \leq t_0 + 2\sum_i \int_{t_0}^{\infty} e^{-\frac{t^2}{2\sigma^2}} i^{-\frac{t^2}{2\sigma^2}} dt$$

let $t_0 = 2\sigma$

$$\leq 2\sigma + \left(2\int_{t_0}^{\infty} e^{-\frac{t^2}{2\sigma^2}}\right)\left(\sum_{i=1}^{\infty} i^{-\frac{t_0^2}{2\sigma^2}}\right) \overset{(*)}{=} 2\sigma + \frac{\pi^2}{6} \times 2\int_0^{\infty} e^{-\frac{t^2}{2\sigma^2}} dt$$

$$\Rightarrow \mathbb{E}\left[\max_i \frac{|X_i|}{\sqrt{1+\log i}}\right] \leq 2\sigma + \frac{\pi^2}{6}\sqrt{2\pi\sigma^2} = \boxed{\left(2 + \frac{\pi^2\sqrt{2\pi}}{6}\right)\sigma} \quad \text{Q.E.D.}$$

$(*)$ since $\sum_{i=1}^{\infty} i^{-\frac{t_0^2}{2\sigma^2}} = \sum_{i=1}^{\infty} \frac{1}{i^2} = \boxed{\frac{\pi^2}{6}}$



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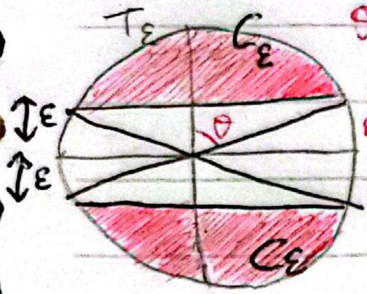
HW4

$(**)\quad 1-x \le e^{-x}$

Subject:
Year: let ~~three~~ points pick on the sphere $S^{n-1}$ uniformly. سوال 5:

$0 \le \varepsilon \le 1 \longrightarrow \{x_1, \dots, x_N\} \subseteq \mathbb{R}^n \quad , \|x_i\|_2 = 1$




$\theta = \cos^{-1}(\varepsilon)$

$$\frac{\sigma(C_\varepsilon)}{\sigma(S^{n-1})} = \frac{\mathrm{vol}(T_\varepsilon)}{\mathrm{vol}(B_2^n)} = \frac{(r^2-\varepsilon^2)^{d/2}}{r^d} \le e^{-\frac{\varepsilon^2 d}{2r^2}}$$

$$r=1 \Rightarrow \sigma(C_\varepsilon) \le e^{-\frac{n\varepsilon^2}{2}} \quad (*)$$

$$\mathbb{P}\left(\forall i,j : |\langle x_i, x_j\rangle| \le \varepsilon\right] = 1 - \mathbb{P}\left[\exists i,j : |\langle x_i, x_j\rangle| \ge \varepsilon\right]$$

$$\ge 1 - \sum_{i,j} \mathbb{P}\left[|\langle x_i, x_j\rangle| \ge \varepsilon\right] \ge 1 - N^2\, \mathbb{P}\left[|\langle x_i, x_j\rangle| \ge \varepsilon\right]$$

Union Bound $\quad \binom{N}{2} \le N^2 \quad x_i, x_j \sim \mathrm{Unif}(S^{n-1})$

$$\Rightarrow \mathbb{P}\left[\forall i,j : |\langle x_i, x_j\rangle| \le \varepsilon\right] \ge 1 - 2N^2 \sigma(C_\varepsilon) \overset{(*)}{\ge} 1 - 2N^2 e^{-\frac{n\varepsilon^2}{2}} = 1-\delta$$

$1-\delta$ here means with high probability

$$\Rightarrow \delta = 2N^2 e^{-\frac{n\varepsilon^2}{2}} \to N = \sqrt{\delta} \cdot e^{\frac{n\varepsilon^2}{4}} = e^{n \cdot \frac{\varepsilon^2 \log \delta}{8}}$$

$$\Rightarrow \text{thus } N = e^{n\, c(\varepsilon)} \quad \text{where } c(\varepsilon) = \frac{\varepsilon^2 \log(\delta)}{8}$$

Q.E.D. ■

سوال 6:

$C \cap D = \emptyset, \quad C, D \in \mathbb{R}^n$

$$\to \left[\mathrm{vol}(\lambda C + \bar{\lambda} D)\right]^{1/n} \ge \lambda \left[\mathrm{vol}(C)\right]^{1/n} + \bar{\lambda} \left[\mathrm{vol}(D)\right]^{1/n} \quad \forall \lambda \in [0,1]$$

$$\to \equiv \mathrm{vol}(\lambda C + \bar{\lambda} D) \ge \left[\mathrm{vol}(C)\right]^{\lambda} \cdot \left[\mathrm{vol}(D)\right]^{1-\lambda} \quad \forall \lambda \in [0,1]$$

I) $B_2^n = \{x \in \mathbb{R}^n : \|x\|_2 \le 1\}$

$\forall A \subseteq \mathbb{R}^n, \varepsilon \ge 0 ;\ \mathrm{vol}(A) = \mathrm{vol}(B_2^n)$ $\Bigg\} \Rightarrow \mathrm{vol}(A^\varepsilon) \ge \mathrm{vol}((B_2^n)^\varepsilon)$

$X^\varepsilon = X + \varepsilon \cdot B_2^n$ (Minkowski addition)

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$x \in A_{\frac{\varepsilon}{L}} \cap A'_{\frac{\varepsilon}{L}}$ which suggests:

$f$ is $L$-lip

$$\left.\begin{array}{l} \Rightarrow \exists x_A \in A : \|x - x_A\|_2 < \frac{\varepsilon}{L} \longrightarrow |f(x) - f(x_A)| \leq \varepsilon \\ \Rightarrow \exists x_{A'} \in A' : \|x - x_{A'}\|_2 < \frac{\varepsilon}{L} \longrightarrow |f(x) - f(x_{A'})| \leq \varepsilon \end{array}\right\} \Longrightarrow$$

$$\Rightarrow |f(x) - M(f)| \leq \varepsilon \quad \rightarrow \boxed{\mathbb{P}\left[A_{\frac{\varepsilon}{L}} \cap A'_{\frac{\varepsilon}{L}}\right] \geq 1 - 2e^{-\frac{2\varepsilon^2}{nL^2}}}$$

Q.E.D. ■



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Harper theorem: if $A, B \subseteq P(\mathcal{X})$, then there exists $A'$ and $B'$ such that
$$\begin{cases} |A| = |A'| \text{ and } |B| = |B'| \\ |A'| \leq r_A \text{ and containing all } (r_A - 1) \text{ sized sets} \\ |B'| \leq r_B \text{ and containing all } (r_B - 1) \text{ sized sets} \\ d(A, B) \leq d(A', B') = \min_{\substack{a \in A' \\ b \in B'}} |a \Delta b| \end{cases}$$

let $A \subseteq \{0,1\}^n$ such that $P(A) = \frac{|A|}{2^n} < \frac{1}{2} \Rightarrow |A| > 2^{n-1}$
and let $B = A_\varepsilon^c$. We wish to prove that $\mathbb{P}[B] \leq e^{-\frac{2\varepsilon^2}{n}}$.

- for $\varepsilon = 1$, $\mathbb{P}(B) \leq \frac{1}{2}$ and it's obvious.
- for $\varepsilon > 1$, According to the lemma above there exists $A', B'$ such that
$$\begin{cases} |A'| = |A| \geq 2^{n-1} \,;\, |B| = |B'| \\ B_{r_A - 1}(0) \subseteq A' \subseteq B_{r_A}(0) \,;\, B_{r-1}(1) \subseteq B' \subseteq B_r(1) \\ d(A', B') > d(A, B) \end{cases}$$

Since $A'$ has at least half of the space $\{0,1\}^n$, it's inside the sphere $B_{r_A}(0)$. thus, all the points has at most $\frac{n}{2}$ ones!
On the other hand $d(A', B') > d(A, B) > \varepsilon$; therefore, $B'$ has ~~at most~~ at least $\frac{n}{2} + \varepsilon$ ones! which suggests
$$e^{-\frac{2\varepsilon^2}{n}} \geq \mathbb{P}\left[\sum_{i=1}^{n} X_i > \frac{n}{2} + \varepsilon\right] \geq \mathbb{P}[B']$$ and now using the Hoeffding Bound we'll get:
$$\Rightarrow \mathbb{P}(B') \leq e^{-\frac{2\varepsilon^2}{n}}$$

Now let
$$A \triangleq \{x \in \mathcal{X} : f(x) \leq M(f)\} \rightarrow \mathbb{P}(A) \geq \frac{1}{2} \Rightarrow \mathbb{P}[A_{\frac{\varepsilon}{L}}] \geq 1 - e^{-\frac{2\varepsilon^2}{nL^2}}$$
$$A' \triangleq \{x \in \mathcal{X} : f(x) \geq M(f)\} \rightarrow \mathbb{P}(A') \geq \frac{1}{2} \Rightarrow \mathbb{P}\left(A'_{\frac{\varepsilon}{L}}\right) \geq 1 - e^{-\frac{2\varepsilon^2}{nL^2}}$$

$\forall (x, x') \in \mathcal{X}^2 \rightarrow |f(x) - f(x')| \leq d(x, x')$

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if $\mathbb{P}_{X \sim \mu}\left[ f(x) - \mathrm{med}(f) \geq t \right] \leq Ce^{-\frac{t^2}{\sigma^2}}$ $\quad \forall t \geq 0, \ f \in 1\text{-lip}(\mathcal{X})$ $\quad (*)$

$\Rightarrow$ if $\mu(A) \geq \frac{1}{2}, \forall A \subseteq \mathcal{X}, \forall \varepsilon \geq 0 \rightarrow \mu(A^\varepsilon) \geq 1 - Ce^{-\frac{t^2}{2\sigma^2}}$

if $(*)$ holds for any 1-lipschitz function, then we can easily see that $f(x) = d(x, A) = \inf_{a \in A} \{d(x, a)\}$ is 1-lipschitz because:

$\forall x_1, x_2 \in \mathcal{X} \rightarrow f(x_1) - f(x_2) = d(x_1, A) - d(x_2, A) \leq d(x_1, a_1) - d(x_2, a_2)$
$\exists a_1, a_2 \in A$

$\Rightarrow f(x_1) - f(x_2) \leq d(x_1, a_2) - d(x_2, a_2) - d(x_2, a_2)$

since $a_1$ is the minimizer of the distance $d(x_1, A)$ $\quad \leq d(x_1, x_2)$ triangle inequality

$\Rightarrow |f(x_1) - f(x_2)| \leq d(x_1, x_2) \Rightarrow f \in 1\text{-lip}(\mathcal{X})$

if $\mu(A) \geq \frac{1}{2}$ since $\left(\forall x \in A : f(x) = 0 \rightarrow f(A) = 0\right)$, thus $\rightarrow \mathrm{med} f = 0$

$\mathbb{P}_{X \sim \mu}\left[ f(x) - \mathrm{med}(f) \geq t \right] = \mathbb{P}_{X \sim \mu}\left[ f(x) \geq t \right] \leq C \cdot e^{-t^2/\sigma^2} \quad (\forall t \geq 0)$

(VII)

full

let's assume that for each 1-lipschitz function $f$ we have $\mathbb{P}\left[ f(\cdot) - M(f) \geq \varepsilon \right] \leq C \cdot e^{-\frac{t^2}{2\sigma^2}}$. then if we let $F(x) = d(x, A)$ $F$ will be 1-lipschitz as well since:

$F(x_1) - F(x_2) = d(x_1, A) - d(x_2, A) = |x_1 - a_1| - |x_2 - a_2|$

$\leq |x_1 - a_2| - |x_2 - a_2| \leq |x_1 - x_2|$

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(IV) $\sup_{A \subseteq B_2^n} \left\{ 1 - \mathbb{P}[A^\varepsilon] \mid \mathbb{P}(A) \geq \frac{1}{2} \right\} \leq 2e^{-\frac{n\varepsilon^2}{4}}$

from (III) we got $\mathbb{P}(A)\ \mathbb{P}[(A^\varepsilon)^c] \leq (1-\frac{\varepsilon^2}{8})^{2n}$

since $\mathbb{P}[A] \geq \frac{1}{2} \Rightarrow \frac{1}{2}(\underbrace{1-\mathbb{P}[A^\varepsilon]}_{\mathbb{P}((A^\varepsilon)^c)}) \leq (1-\frac{\varepsilon^2}{8})^{2n} \leq e^{-\frac{\varepsilon^2 n}{4}}$

$1-x \leq e^{-x}$

$\Rightarrow \boxed{\mathbb{P}[(A^\varepsilon)^c] \leq 2e^{-\frac{n\varepsilon^2}{4}}}$ Q.E.D. ■

which proves & confirms that if we have a uniformly distributed sphere & $\mathbb{P}[A] \geq \frac{1}{2} \overset{\text{then}}{\Longrightarrow} \boxed{\mathbb{P}[A^\varepsilon] \geq 1-2e^{-\frac{n\varepsilon^2}{4}}}$

(V) for a metric space $\rightarrow (\mathcal{X}, d)$, and the probability measure $\mu \rightarrow$ the isometric inequality $\Rightarrow \forall \varepsilon > 0,\ A \subseteq \mathcal{X},\ \mu(A) \geq \frac{1}{2} : \mu(A^\varepsilon) \geq 1 - Ce^{-\frac{t^2}{2\sigma^2}}$

$\Rightarrow$ Prove that if $\mu$ holds the isometric inequality then $\rightarrow \mathbb{P}_{X\sim\mu}[f - \mathrm{med}(f) \geq t] \leq Ce^{-\frac{t^2}{\sigma^2}}$ ($\forall t > 0,\ f \in \mathrm{Lip}(F)$) $\forall x, x' \in \mathcal{X}$

we assume that $f$ is 1-lipschitz $\Rightarrow |f(x) - f(x')| \leq d(x, x')$

let $A = \{x \in \mathcal{X} : f(x) - m_f \leq 0\}$ since $m_f = \mathrm{med}_\mu(f)$, then $\mu(A) \geq \frac{1}{2}$. Consider: $A^\varepsilon = \{x \in \mathcal{X} : f(x) - m_f \leq \varepsilon\}\ d(x, A) \leq \varepsilon\}$

by isometric inequality $\Rightarrow \mu(A^\varepsilon) \geq 1 - Ce^{-\frac{\varepsilon^2}{2\sigma^2}}$

$\mu((A^\varepsilon)^c) = 1 - \mu(A^\varepsilon) \leq Ce^{-\frac{\varepsilon^2}{2\sigma^2}}$

$Ce^{-\frac{t^2}{2\sigma^2}} \geq \mu((A^\varepsilon)^c) \geq \mu[f(x) - m_f \geq t]$ Q.E.D. ■

since $A^\varepsilon \subseteq \{x \in \mathcal{X} : f(x) - m_f \leq \varepsilon\}$ $\Longleftarrow$ (*)

$(A^\varepsilon)^c \supseteq \{x \in \mathcal{X} : f(x) - m_f \geq \varepsilon\}$

$\mu((A^\varepsilon)^c) \geq \mu(f(x) - m_f \geq \varepsilon)$

HW4

let $\lambda = \frac{1}{2}$ $\quad C \to A$, $\quad D \to B_\varepsilon^n = \varepsilon B_2$

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$$\mathrm{vol}\left(\frac{A^\varepsilon}{2}\right)^{1/n} = \left[\frac{\mathrm{vol}(A^\varepsilon)}{2^n}\right]^{\frac{1}{n}} = \mathrm{vol}\left(\frac{1}{2}A + \frac{1}{2}B_{\varepsilon_2}^n\right) \geq \left[\left(\frac{1}{2}\right)^n \mathrm{vol}(A)\right]^{\frac{1}{n}} + \left[\left(\frac{\varepsilon}{2}\right)^n \mathrm{vol}(B_2^n)\right]^{1/n}$$

$$\Rightarrow \left[\mathrm{vol}(A^\varepsilon)\right]^{1/n} \geq \mathrm{vol}(A) + \varepsilon \cdot \left[\mathrm{vol}(B_2^n)\right]^{1/n} = (1+\varepsilon)\left[\mathrm{vol}(B_2^n)\right]^{1/n}$$

raising to the power of $n$, results in:

$$\Rightarrow \mathrm{vol}(A^\varepsilon) \geq (1+\varepsilon)^n \mathrm{vol}(B_2^n) = \mathrm{vol}((1+\varepsilon)B_2^n) = \mathrm{vol}((B_2^n)^\varepsilon)$$

$$\Rightarrow \boxed{\mathrm{vol}(A^\varepsilon) \geq \mathrm{vol}((B_2^n)^\varepsilon) = (1+\varepsilon)^n \mathrm{vol}(B_2^n)}$$ Q.E.D ■

II $A \subseteq B_2^n \Rightarrow (\forall a \in A, b \in (A^\varepsilon)^c): \frac{1}{2}\|a+b\|_2 \leq 1 - \frac{\varepsilon^2}{8}$

$(A^\varepsilon)^c = B_2^n \setminus A^\varepsilon$ $\qquad 1 \geq \|a\|_2, \|b\|_2 \leq 1$

since $\forall a \in A, \forall b \in B_2^n \setminus A^\varepsilon \Rightarrow \|a-b\|_2 \geq \varepsilon$ & also the fact that $\|a+b\|_2^2 = 2\|a\|_2^2 + 2\|b\|_2^2 - \|a-b\|_2^2 \leq 4 - \varepsilon^2$

$$\Rightarrow \frac{1}{2}\|a+b\|_2 \leq \sqrt{1 - \frac{\varepsilon^2}{4}} \leq 1 - \frac{\varepsilon^2}{8} \quad (\forall \varepsilon \in (0,1))$$ Q.E.D ■

III $\mathbb{P}$ is uniform on $B_2^n$, $A \subseteq B_2^n$ $\Rightarrow \mathbb{P}[A](1 - \underbrace{\mathbb{P}[A^\varepsilon]}) \leq (1 - \frac{\varepsilon^2}{8})^{2n}$

$\mathbb{P}[(A^\varepsilon)^c]$

from the inequality BM and for $\lambda = \frac{1}{2}$, $C = A$, $D = (A^\varepsilon)^c$, we get:

$$\mathrm{vol}\left(\frac{1}{2}A + \frac{1}{2}(A^\varepsilon)^c\right) \geq \mathrm{vol}(A)^{\frac{1}{2}} \mathrm{vol}((A^\varepsilon)^c)^{1/2} = \sqrt{\mathbb{P}[A]\,\mathbb{P}[(A^\varepsilon)^c]}$$

since from II we saw that $\forall a \in A, \forall b \in (A^\varepsilon)^c \Rightarrow \frac{1}{2}\|a+b\|_2$ always is in a sphere centered at origin and with radius $(1 - \frac{\varepsilon^2}{8})$.

so, we can deduce that $X = \frac{1}{2}A + \frac{1}{2}(A^\varepsilon)^c \subseteq (1 - \frac{\varepsilon^2}{8})B_2^n$

and hence $\mathbb{P}[\frac{1}{2}A + \frac{1}{2}(A^\varepsilon)^c] \leq \mathbb{P}[(1 - \frac{\varepsilon^2}{8})B_2^n] = \mathrm{vol}((1 - \frac{\varepsilon^2}{8})B_2^n)$

$$\Rightarrow \mathbb{P}[A]\,\mathbb{P}[(A^\varepsilon)^c] \leq \left[(1 - \frac{\varepsilon^2}{8})^n \mathrm{vol}(B_2^n)\right]^2 \leq (1 - \frac{\varepsilon^2}{8})^{2n}$$

$$\Rightarrow \boxed{\mathbb{P}[A]\,\mathbb{P}[(A^\varepsilon)^c] \leq (1 - \frac{\varepsilon^2}{8})^{2n}}$$ Q.E.D. ■



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likewise $|f(x_1) - f(x_2)| \le |x_1 - x_2|$ since $f \in 1$-lip.

if $\mu(A) \ge \frac{1}{2}$. we know that $f(A) = 0 \to M[f] = 0$,

thus:

$$\mu((A_\varepsilon)^c) = \mathbb{P}[f(x) \ge \varepsilon] = \mathbb{P}[f - M(f) \ge \varepsilon] \le c \cdot e^{-\frac{t^2}{2\sigma^2}}$$

which prove that $\boxed{\mu((A^\varepsilon)^c \le c \cdot e^{-\frac{t^2}{2\sigma^2}} \Rightarrow \mu(A^\varepsilon) \ge 1 - c\, e^{-\frac{t^2}{2\sigma^2}}}$

Q.E.D ■

VIII $\mu$ holds in isometric inequality $\Rightarrow \forall f \in \mathrm{Lip}(\mathcal{X})$

Prove that $M(f) \le \mathbb{E}_\mu[f] + C\sigma\sqrt{\frac{\pi}{2}}$

if we write the inequalities for the sets $\{f \le M(f)\}$ & $\{f \ge M(f)\}$

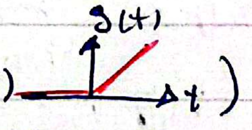
According to isometric inequality & VII:

$$\mathbb{P}[|f(\cdot) - M(f)| \ge t] \le 2C \cdot e^{-\frac{t^2}{2\sigma^2}}$$

By Jensen

$$\Rightarrow |\mathbb{E}(f) - M(f)| \le \mathbb{E}|f(x) - M(f)| = \int_0^\infty \mathbb{P}[|f - M_f| \ge t]\, dt$$

$$\le \int_0^\infty 2c \cdot e^{-\frac{t^2}{2\sigma^2}} = c \cdot \sqrt{2\pi\sigma^2}$$

on the other hand: (let $g(t) = \max(t, 0)$  )



$$\mathbb{E}[f] - M(f) \le g(\mathbb{E}f - M(f)) \le \mathbb{E}[g(\mathbb{E}f - M(f))]$$

$$\le \int_0^\infty \mathbb{P}[\mathbb{E}f(\cdot) - M(f) \ge t]\, dt = \int_0^\infty c \cdot e^{-\frac{t^2}{2\sigma^2}}\, dt = C\sqrt{\frac{\pi\sigma^2}{2}}$$

likewise, the same bound will result for $M(f) - \mathbb{E}f$

thus: $c\sqrt{\frac{\pi}{2}}\sigma < \mathbb{E}f(x) - M(f) < C\sqrt{\frac{\pi}{2}}\sigma$

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$$\Rightarrow \mathbb{P}[f-\mathbb{E}f \geq t] = \mathbb{P}[f-M(f) > t + \mathbb{E}f - M(f)] \leq \mathbb{P}\left[f-M_f \geq t - C\sqrt{\tfrac{\pi}{2}}\sigma\right]$$

$$\leq Ce^{-\frac{(t-u)^2}{2\sigma^2}} \leq Ce^{\frac{c^2\pi}{12}}e^{-\frac{t^2}{8\sigma^2}} \leq e^{\frac{c^2\pi}{6}}e^{\frac{c^2\pi}{12}}e^{-\frac{t^2}{8\sigma^2}}$$

$(t-u)^2 \geq \frac{t^2}{4} - \frac{u^2}{3}$

$$\Rightarrow \boxed{\mathbb{P}[f(\cdot) - \mathbb{E}f(\cdot) \geq t] \leq e^{\frac{\pi c^2}{4}} e^{-\frac{t^2}{8\sigma^2}}} \quad \forall t \geq 0$$

■ Q.E.D.

**(IX)** $\forall f \in Lip(\mathcal{X}) \longrightarrow \mathbb{P}_\mu[f - \mathbb{E}_\mu(f) \geq t] \leq Ce^{-\frac{t^2}{2\sigma^2}} \quad \forall t \geq 0$

$\quad \& \ \mathbb{P}_\mu[f - M(f) \geq t] \leq 2Ce^{-t^2/8\sigma^2} \quad \forall t \geq 0.$

We know that $\mathbb{P}[f - M(f) \geq t] \leq Ce^{-t^2/2\sigma^2} \ \forall t$. let $t = M(f) - \mathbb{E}f$

we get $\rightarrow \frac{1}{2} \leq \mu[f \geq M(f)] \leq C \cdot \exp\left(-\frac{(M-\mathbb{E}f)^2}{2\sigma^2}\right)$

$\rightarrow |M(f) - \mathbb{E}f| \leq \sqrt{2\sigma^2 \log 2C} \Rightarrow \mathbb{E}f \leq M(f) + \sigma\sqrt{2\log(2C)}$

$$\mathbb{P}[f - M(f) \geq t] = \mathbb{P}[f - \mathbb{E}f \geq t + M(f) - \mathbb{E}f] \leq \mathbb{P}\left[f - \mathbb{E}f \geq t - \sigma\sqrt{2\log 2C}\right]$$

$$\hookrightarrow \leq C\exp\left(-\frac{(t-z)^2}{2\sigma^2}\right) \leq C \cdot e^{-\frac{t^2}{8\sigma^2}} \cdot e^{\frac{2\log 2C}{4}} \leq 2Ce^{-t^2/8\sigma^2}$$

■ Q.E.D.

**(X)** $W_1(\mu, \nu) = \sup_{\|f\|_{Lip} \leq 1} \left\{ \mathbb{E}_\mu[f(X)] - \mathbb{E}_\nu[f(X)] \right\}, \ D(\nu, \mu) = \mathbb{E}_\nu\left[\log \frac{d\nu}{d\mu}\right]$

let $W_1(\mu, \nu) \leq \sqrt{2\sigma^2 D(\nu\|\mu)} \Longrightarrow$ Prove that $\left(d(A,B) = \min_{x \in A, y \in B}\{d(x,y)\}\right)$

$$d(A,B) \leq W_1(\mu(\cdot|A), \mu(\cdot|B)) \leq \sqrt{2\sigma^2 \log\left(\frac{1}{\mu(A)}\right)} + \sqrt{2\sigma^2 \log\left(\frac{1}{\mu(B)}\right)}$$

For ... $B = (A^\varepsilon)^c$ $(d(A,B) \geq \varepsilon)$

$(*): (a-b)^2 \geq \frac{a^2}{2} - b^2$

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So $\mu(B) \leq 2e^{-\frac{\varepsilon^2}{4\sigma^2}}$ which suggests:

$$\mu(A^\varepsilon) = 1 - \mu(B) \geq 1 - 2e^{-\frac{\varepsilon^2}{4\sigma^2}} \quad \text{Q.E.D.}$$

we Proved a better bound!

سوال 7:

(I) Since we have median concentration for all 1-lipschitz functions $f$, we assume $f$ to be the n-th elements as $f(\vec{x}) = x_n$. this function is 1-lipschitz cause:

$\forall x,y:$ $|f(x) - f(y)| \leq \sqrt{(x_1-y_1)^2 + \dots + (x_n - y_n)^2} = |x_n - y_n| = |x-y|$

since the distribution is uniform $M(f) = 0$. therefore:

$$\mathbb{P}\left[\,|f(x)| > \varepsilon\,\right] \leq 2 e^{-\frac{n\varepsilon^2}{2}} \xrightarrow[\text{is uniform}]{\text{distrib.}} \mathbb{P}\left[\,f(x) > \varepsilon\,\right] \leq e^{-\frac{n\varepsilon^2}{2}}$$

thus: $\mu((A^\varepsilon)^c) \leq \mathbb{P}[f(x) > \varepsilon] \leq e^{-\frac{n\varepsilon^2}{2}} \Rightarrow \mu(A^\varepsilon) \geq 1 - e^{-\frac{n\varepsilon^2}{2}}$

Now we Proved for all $A \subseteq \mathbb{R}^n$ that are in the shape of half sphere.

thus, due to the lévi lemma if $\sigma(A) = \sigma(A') \rightarrow \sigma(A^\varepsilon) \leq \sigma(A'^\varepsilon)$

thus for all $A$ such that $\mu(A) \geq \frac{1}{2} \rightarrow \mu(A^\varepsilon) \geq 1 - e^{-\frac{n\varepsilon^2}{2}}$ Q.E.D. ■

(II) As we discussed in problem 6 part (2), if we take $f(x) = d(x, A)$, with the assumption $\mu(A) \geq \frac{1}{2}$, we have $M(f) = 0$

thus: $\mathbb{P}\left[\,|f(x) - M| \geq \varepsilon\,\right] = \mathbb{P}(d(x, A) \geq \varepsilon) = \mu\left[(A^\varepsilon)^c\right] \leq 2e^{-\frac{n\varepsilon^2}{2}}$

cause the function $f$ is 1-lipschitz. therefore we proved $\mu(A^\varepsilon) \geq 1 - 2e^{-\frac{n\varepsilon^2}{2}}$ which suggest we have concentration in geometric aspect regardless of metric $d$ and distribution $\mu$. Q.E.D ■

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