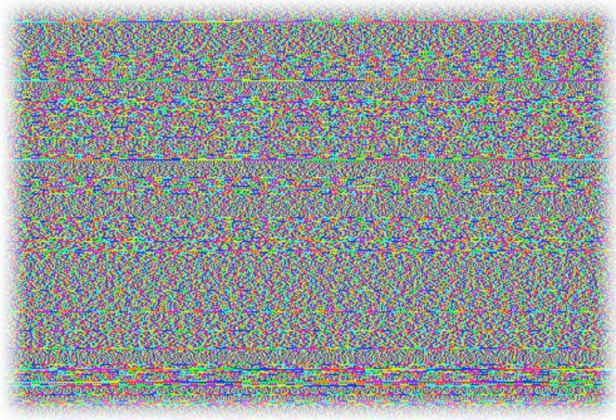


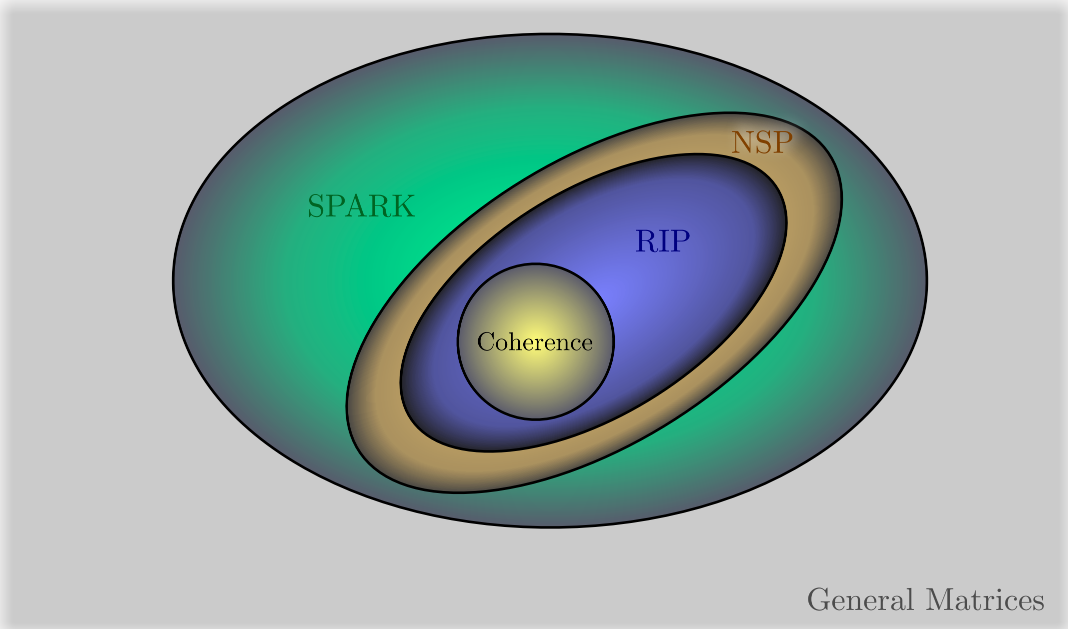
Deterministic Matrix Design

- Design Technique
- Equiangular Tight Frames
- Some of the known designs

Recovery Guarantees

Type of Guarantee	Robustness against model mismatch	Robustness against noise	Computationally affordable to verify
SPARK	✗	✗	✗
NSP	✓	✗	✗
RIP	✓	✓	✗
Coherence	✓	✓	✓

Recovery Guarantees



Recovery Guarantees

- Coherence

$$\mu(\mathbf{\Phi}_{m\times n}) = \max_{\substack{1\leq i,j\leq n\\ i\neq j}} \frac{|\langle \phi_i\,,\,\phi_j\rangle|}{\|\phi_i\|_2\ \|\phi_j\|_2}$$

- Matrix design technique

Design $\mathbf{\Phi} = [\phi_1\,,\,\phi_2\,,\,\dots\,,\,\phi_n]_{m\times n}$ such that:

- $\|\phi_i\|_2 = 1$,
- $\max_{i\neq j} |\langle \phi_i\,,\,\phi_j\rangle|$ is small.

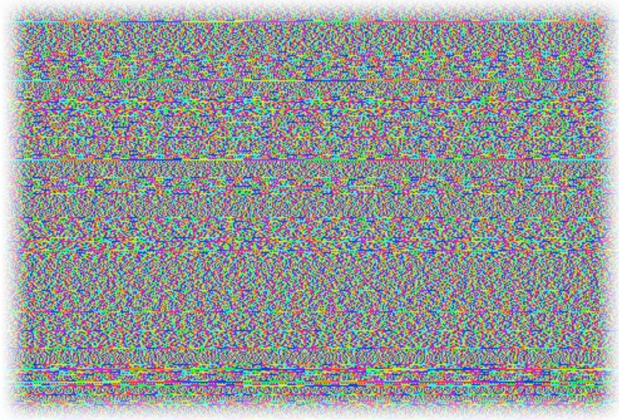
Recovery Guarantees

$$\text{Welch bound:} \quad n \geq m \quad \Rightarrow \quad \mu(\boldsymbol{\Phi}_{m\times n}) \geq \sqrt{\frac{n-m}{m(n-1)}}$$

$$n \gg m \quad \Rightarrow \quad \mu_{\text{Welch}} \approx \frac{1}{\sqrt{m}}$$

$\boldsymbol{\Phi}_{m\times n}$ designed by coherence minimisation : $\begin{cases} k_{\text{garntd}} & \approx & \frac{1}{2\mu(\boldsymbol{\Phi})} \\ \mu(\boldsymbol{\Phi}) & \gtrapprox & \frac{1}{\sqrt{m}} \end{cases} \quad \Rightarrow \quad m \gtrapprox \mathcal{O}\big(k_{\text{garnted}}^2\big)$

$\boldsymbol{\Phi}_{m\times n}$ is a random [Gaussian] matrix $\quad \Rightarrow \quad m \gtrapprox \mathcal{O}\big(k \log(\frac{n}{k})\big)$



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Equiangular Tight Frames (ETF)

$$\boxed{\boldsymbol{\Phi}_{m\times n} = \text{ETF} \quad \Leftrightarrow \quad \begin{cases} \|\phi_i\|_2 = 1, \ \forall\, 1 \leq i \leq n, \\ \mu(\boldsymbol{\Phi}) = \mu_{\text{Welch}} = \sqrt{\frac{(n-m)}{m(n-1)}} \end{cases}}$$

$\Rightarrow$ Optimal coherence value!

$$\boldsymbol{\Phi}_{m\times n} = \text{ETF} \quad \Rightarrow \quad \forall\, i \neq j : \ \big|\langle \phi_i \,,\, \phi_j \rangle\big| = \mu_{\text{Welch}}$$

$$\boldsymbol{\Phi}_{m\times n} = \begin{array}{c}\text{real-valued}\\ \text{ETF}\end{array} \quad \Rightarrow \quad \begin{cases} n \leq \frac{m(m+1)}{2}, \\ \sqrt{\frac{m(n-1)}{n-m}} \ \text{is an odd integers,} \\ \sqrt{\frac{(n-m)(n-1)}{m}} \ \text{is an odd integers.} \end{cases}$$

$$\boldsymbol{\Phi}_{m\times n} = \begin{array}{c}\text{complex-valued}\\ \text{ETF}\end{array} \quad \Rightarrow \quad n \leq m^2$$

Existence of ETFs

- Do we have ETFs for all n≤m²?
    - No, not in general.
- Do we know all the pairs (m,n) for which an ETF exists?
    - No!
- Are there specific constructions?
    - Yes: based on Difference Sets, Strongly regular graphs, ...

Difference Sets

- Let $\mathcal{D} = \{d_1, \dots, d_m\}$ and $n \in \mathbb{N}$ be given. Define:

$$\forall\, 0 \le k < n: \quad c_k = \left|\Big\{(i,j) \mid 1 \le i,j \le m,\ d_i - d_j \overset{n}{\equiv} k\Big\}\right|$$

$(\mathcal{D}\,,\, n)$ forms a difference set iff:

$$c_1 = c_2 = \cdots = c_{n-1}.$$

- Example $\qquad \mathcal{D} = \{0, 1, 3, 9\},\ n = 13$

$0 - 1 \overset{13}{\equiv} 12$	$1 - 0 \overset{13}{\equiv} 1$	$3 - 0 \overset{13}{\equiv} 3$	$9 - 0 \overset{13}{\equiv} 9$
$0 - 3 \overset{13}{\equiv} 10$	$1 - 3 \overset{13}{\equiv} 11$	$3 - 1 \overset{13}{\equiv} 2$	$9 - 1 \overset{13}{\equiv} 8$
$0 - 9 \overset{13}{\equiv} 4$	$1 - 9 \overset{13}{\equiv} 5$	$3 - 9 \overset{13}{\equiv} 7$	$9 - 3 \overset{13}{\equiv} 6$

ETF construction via CDS

Theorem. If $\mathcal{D} = \{d_1, \dots, d_m\}$ is a difference set mode n, then,

$$\mathbf{\Phi}_{m\times n} = \frac{1}{\sqrt{m}} \begin{bmatrix} e^{-j\frac{2\pi}{n}0\times d_1} & e^{-j\frac{2\pi}{n}1\times d_1} & \dots & e^{-j\frac{2\pi}{n}(n-1)\times d_1} \\ e^{-j\frac{2\pi}{n}0\times d_2} & e^{-j\frac{2\pi}{n}1\times d_2} & \dots & e^{-j\frac{2\pi}{n}(n-1)\times d_2} \\ \vdots & \vdots & \ddots & \vdots \\ e^{-j\frac{2\pi}{n}0\times d_m} & e^{-j\frac{2\pi}{n}1\times d_m} & \dots & e^{-j\frac{2\pi}{n}(n-1)\times d_m} \end{bmatrix}$$

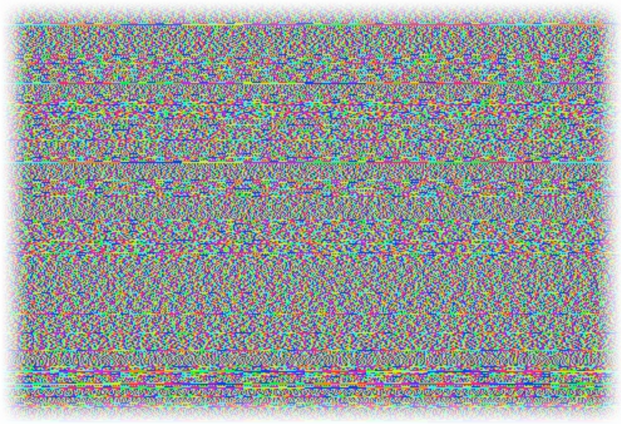
is an ETF.

- Proof

$$\langle \boldsymbol{\phi}_i, \boldsymbol{\phi}_j \rangle = \frac{1}{m} \sum_{l=1}^{m} e^{j\frac{2\pi}{n}(i-j)d_l}$$

$$\Rightarrow \quad \left|\langle \boldsymbol{\phi}_i, \boldsymbol{\phi}_j \rangle\right|^2 = \frac{1}{m^2} \sum_{l_1,l_2=1}^{m} e^{j\frac{2\pi}{n}(i-j)(d_{l_1}-d_{l_2})} = \frac{1}{m^2} \sum_{l=0}^{n-1} c_l \, e^{j\frac{2\pi}{n}(i-j)l} \underset{\uparrow}{=} \frac{n-m}{m(n-1)} \quad ✓$$

$$c_0 = m, \ c_1 = c_2 = \dots = c_{n-1} = \frac{m^2-m}{n-1}$$



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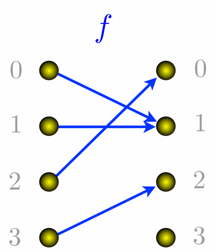
Deterministic Designs

- DeVore's design

$$q = \text{prime power} = p^l \quad , \quad \mathbb{F} = \mathrm{GF}(q)$$

$$f : \mathbb{F} \mapsto \mathbb{F} \quad \Rightarrow \quad \chi(f) \triangleq \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{q^2-1} \end{bmatrix}, \qquad \underset{0\le a,b<q}{v_{aq+b}} = \begin{cases} 1, & f(a) = b, \\ 0, & f(a) \neq b. \end{cases}$$

- Example



$$q = 4 \qquad \Rightarrow \quad \chi(f) = [\, \overbrace{0\ \ 1\ \ 0\ \ 0}^{f(0)}\ \ \overbrace{0\ \ 1\ \ 0\ \ 0}^{f(1)}\ \ \overbrace{1\ \ 0\ \ 0\ \ 0}^{f(2)}\ \ \overbrace{0\ \ 0\ \ 1\ \ 0}^{f(3)} \,]^T$$

$$\begin{array}{cccccccccccccccc} 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 \end{array}$$

Deterministic Designs

- DeVore's design

$$q = \text{prime power} = p^l \quad , \quad \mathbb{F} = \mathrm{GF}(q)$$

$$P : \mathbb{F} \mapsto \mathbb{F} \ \text{ is a polynomial of degree [at most] } d \ \text{ if } \ P(x) = \sum_{i=0}^{d} p_i\, x^i, \quad p_i \in \mathbb{F}$$

$$\{P_1, P_2, \ldots, P_{q^{d+1}}\} = \text{all polynomials of degree [at most] } d$$

$$\mathbf{\Phi}_{q^2 \times q^{d+1}} \triangleq \tfrac{1}{\sqrt{q}} \Big[\chi(P_1) \quad \chi(P_2) \quad \ldots \quad \chi(P_{q^{d+1}}) \Big]$$

$$\Rightarrow \quad \mu\big(\mathbf{\Phi}_{q^2 \times q^{d+1}}\big) = \tfrac{d}{q}$$

Deterministic Designs

- Channel Coding technique

$$\mathcal{C}\big(n_{\text{cc}}, k_{\text{cc}}\big) = \begin{array}{c}\text{linear binary}\\ \text{channel code}\end{array} , \quad \begin{cases} d_{\min} = \text{min. Hamming dist.,} \\ (1, \ldots, 1)^T = \text{valid code-word.} \end{cases}$$

$$\mathbf{c}_{n_{\text{cc}} \times 1} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{n_{\text{cc}}} \end{bmatrix} \in \mathcal{C}\big(n_{\text{cc}}, k_{\text{cc}}\big) \quad \Rightarrow \quad \overline{\mathbf{c}}_{n_{\text{cc}} \times 1} = \begin{bmatrix} c_1 \oplus 1 \\ c_2 \oplus 1 \\ \vdots \\ c_{n_{\text{cc}}} \oplus 1 \end{bmatrix} \in \mathcal{C}\big(n_{\text{cc}}, k_{\text{cc}}\big)$$

$c_i \in \{0, 1\}$

complement code

replacing 0s with -1

$$\mathbf{c}_{n_{\text{cc}} \times 1} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{n_{\text{cc}}} \end{bmatrix} \quad \Rightarrow \quad \widetilde{\mathbf{c}}_{n_{\text{cc}} \times 1} \triangleq \begin{bmatrix} 2c_1 - 1 \\ 2c_2 - 1 \\ \vdots \\ 2c_{n_{\text{cc}}} - 1 \end{bmatrix}$$

Deterministic Designs

- Channel Coding technique

$$\mathcal{C}(n_{\rm cc}, k_{\rm cc}) = \begin{array}{l}\text{linear binary}\\ \text{channel code}\end{array}, \quad \begin{cases} d_{\min} = \text{min. Hamming dist.,} \\ (1, \dots, 1)^T = \text{valid code-word.} \end{cases}$$

Divide all $2^{k_{\rm cc}}$ codewords into $2^{k_{\rm cc}-1}$ complement pairs:

$$(\mathbf{c}_1, \overline{\mathbf{c}}_1),\ (\mathbf{c}_2, \overline{\mathbf{c}}_2),\ \dots, (\mathbf{c}_{2^{k_{\rm cc}-1}}, \overline{\mathbf{c}}_{2^{k_{\rm cc}-1}})$$

$$\boldsymbol{\Phi}_{n_{\rm cc} \times 2^{k_{\rm cc}-1}} \triangleq \frac{1}{\sqrt{n_{\rm cc}}} \begin{bmatrix} \widetilde{\mathbf{c}}_1 & \widetilde{\mathbf{c}}_2 & \dots & \widetilde{\mathbf{c}}_{2^{k_{\rm cc}-1}} \end{bmatrix}$$

$$\Rightarrow \quad \mu\big(\boldsymbol{\Phi}_{n_{\rm cc} \times 2^{k_{\rm cc}-1}}\big) \le \frac{n_{\rm cc} - 2d_{\min}}{n_{\rm cc}}$$

Deterministic Designs

- Channel Coding technique

$$\mathcal{C}(n_{\rm cc}, k_{\rm cc}) = \begin{array}{l}\text{linear binary}\\ \text{channel code}\end{array}, \quad \begin{cases} d_{\min} = \text{min. Hamming dist.,} \\ (1, \dots, 1)^T = \text{valid code-word.} \end{cases}$$

$$\Rightarrow \quad \mu\big(\boldsymbol{\Phi}_{n_{\rm cc} \times 2^{k_{\rm cc}-1}}\big) \le \frac{n_{\rm cc} - 2d_{\min}}{n_{\rm cc}}$$

useful only when $d_{\min} \approx \frac{1}{2} n_{\rm cc}$

There exist BCH codes with

$$n_{\rm cc} = 2^l - 1$$
$$d_{\min} \ge 2^{l-1} - 2^t, \ t < l - 1 \quad ✓$$