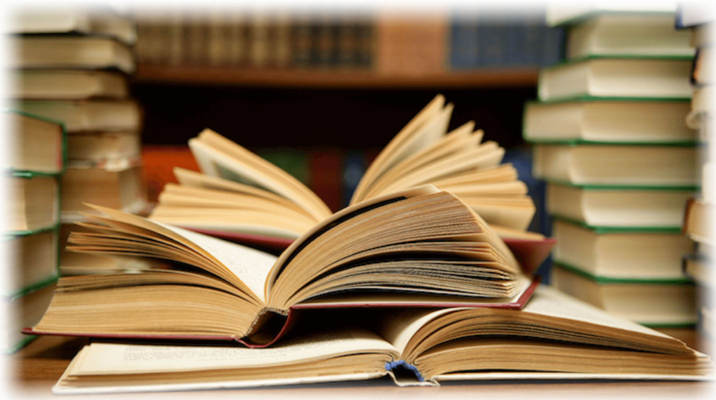


Math Preliminaries

- Norms and unit balls
- Inequalities
- Linear Algebra

Norm Definition

- Norm

$$\mathbb{F} = \mathbb{R} \text{ or } \mathbb{C} \quad , \quad V = \mathbb{F}^n \quad , \quad \|\cdot\| : V \to \mathbb{R}$$

1. $\forall\, a \in \mathbb{F},\ \mathbf{v} \in V: \quad \|a\mathbf{v}\| = |a|\,\|\mathbf{v}\|.$
2. $\forall\, \mathbf{v}_1, \mathbf{v}_2 \in V: \quad \|\mathbf{v}_1 + \mathbf{v}_2\| \leq \|\mathbf{v}_1\| + \|\mathbf{v}_2\|.$
3. $\forall\, \mathbf{v} \in V: \quad \|\mathbf{v}\| \geq 0.$
4. $\|\mathbf{v}\| = 0 \quad \Leftrightarrow \quad \mathbf{v} = 0.$

- Example $\quad V = \mathbb{R}, \quad \|\mathbf{v}\| \triangleq \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$

Well-known Norms

- l_p norms

$$\|\mathbf{v}\|_{\ell_p} \text{ or } \|\mathbf{v}\|_p \triangleq \sqrt[p]{|v_1|^p + |v_2|^p + \cdots + |v_n|^p}$$

$$\|\mathbf{v}\|_{\ell_\infty} \text{ or } \|\mathbf{v}\|_\infty \triangleq \lim_{p\to\infty} \|\mathbf{v}\|_p = \max_{1\le i\le n} |v_i|$$

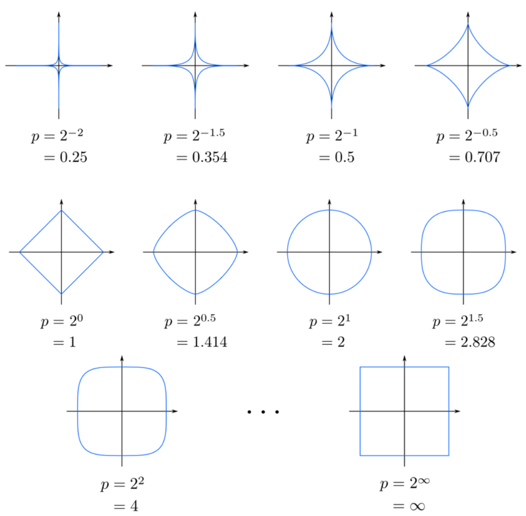
$$\|\mathbf{v}\|_{\ell_0} \text{ or } \|\mathbf{v}\|_0 \triangleq \big|\{i,\ v_i \neq 0\}\big|$$

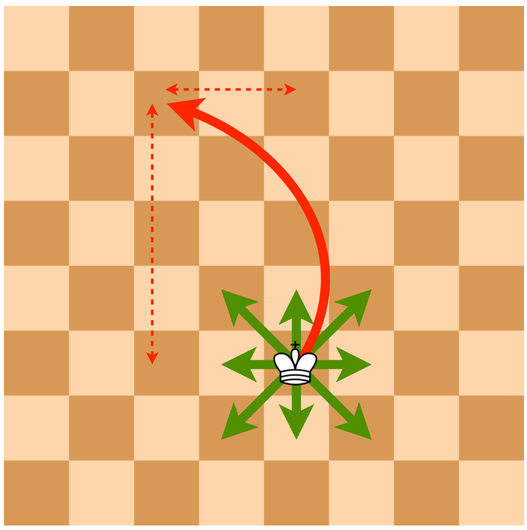
- ℓ_p is a norm for $p \geq 1$.
- Triangle inequality is not generally satisfied for $0 \leq p < 1$.

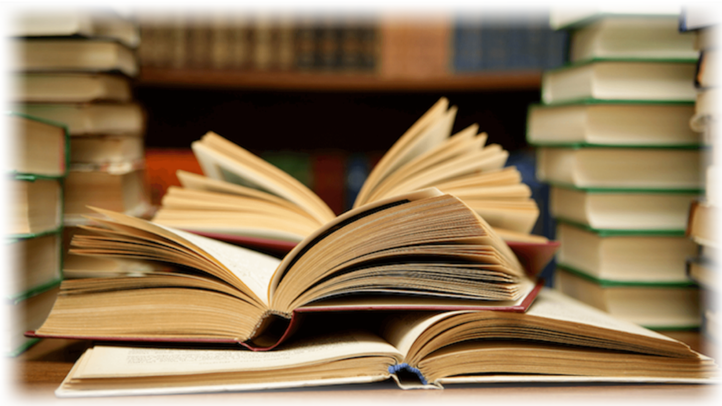
Unit ball of Well-Known Norms

- Unit ball

$$B_1(\mathbb{F}^n; \ell_p) \triangleq \{\mathbf{x} \in \mathbb{F}^n \mid \|\mathbf{x}\|_p = 1\}$$







Math Preliminaries

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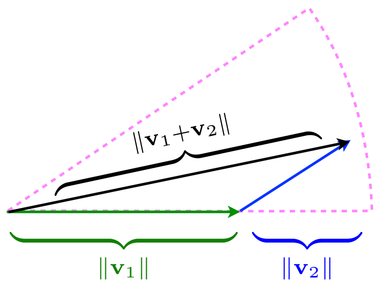
Inequalities related to Norms

- Triangle inequality

$$\|\mathbf{v}_1 + \mathbf{v}_2\| \leq \|\mathbf{v}_1\| + \|\mathbf{v}_2\|$$

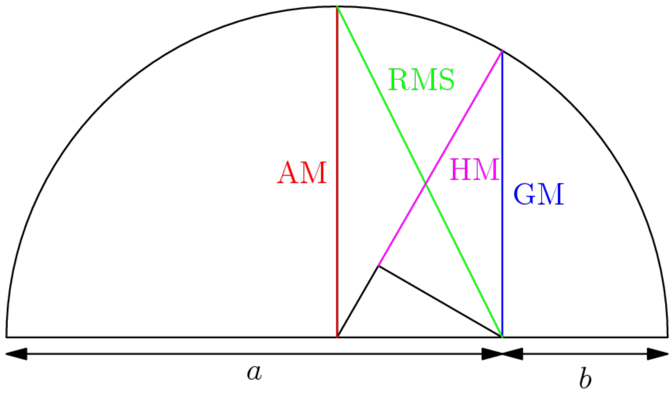
$$\Rightarrow \quad \|\mathbf{v}_1 - \mathbf{v}_2\| \leq \|\mathbf{v}_1\| + \|\mathbf{v}_2\| \quad , \quad \|\mathbf{v}_1\| - \|\mathbf{v}_2\| \leq \|\mathbf{v}_1 \pm \mathbf{v}_2\|$$



Inequalities related to Norms

- HM-GM-AM-RMS inequality

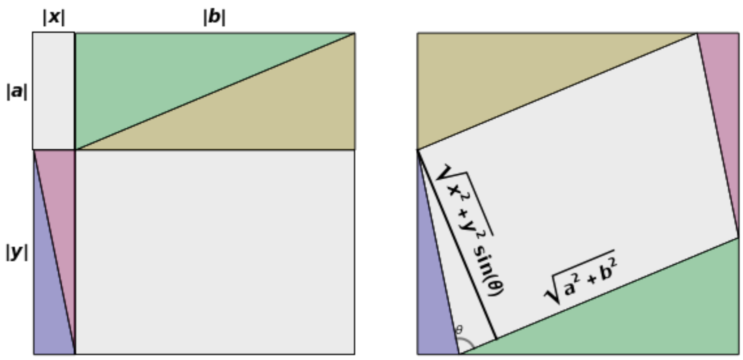
$$\underbrace{\frac{n}{\frac{1}{|x_1|}+\cdots+\frac{1}{|x_n|}}}_{\text{HM}} \le \underbrace{\sqrt[n]{|x_1 \ldots x_n|}}_{\text{GM}} \le \underbrace{\frac{|x_1|+\cdots+|x_n|}{n}}_{\text{AM}} \le \underbrace{\sqrt{\frac{|x_1|^2+\cdots+|x_n|^2}{n}}}_{\text{RMS}}$$



Inequalities related to Norms

- Cauchy-Schwarz inequality

$$\left(|x_1 y_1|+\cdots+|x_n y_n|\right)^2 \le \left(|x_1|^2+\cdots+|x_n|^2\right)\left(|y_1|^2+\cdots+|y_n|^2\right)$$



Inequalities related to Norms

- Young's inequality

$$\frac{1}{p} + \frac{1}{q} = 1 \quad \Rightarrow \quad |xy| \leq \frac{|x|^p}{p} + \frac{|y|^q}{q}$$

$$\log(\cdot) = \text{concave} \quad \Rightarrow \quad \forall \begin{cases} 0 < \alpha < 1 \\ 0 < w,\, z \end{cases} : \quad \alpha \log(z) + (1-\alpha)\log(w) \leq \log\left(\alpha z + (1-\alpha) w\right)$$

$$\Rightarrow \quad z^{\alpha} \cdot w^{(1-\alpha)} \leq \alpha z + (1-\alpha) w$$

$$\alpha = \tfrac{1}{p},\ z = |x|^p,\ w = |y|^q \ \Rightarrow \ \checkmark$$

Inequalities related to Norms

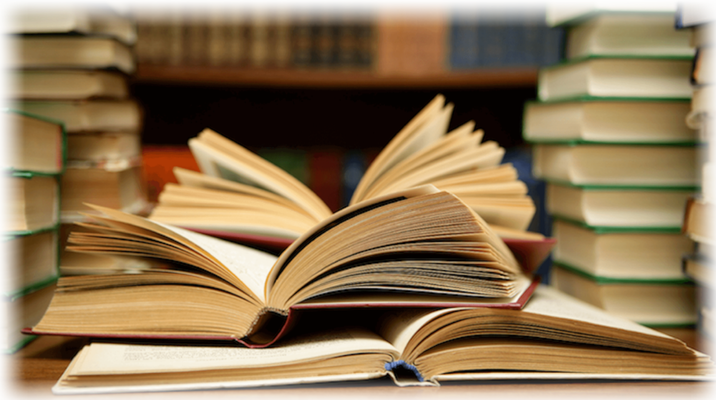
- Holder's inequality

$$\frac{1}{p} + \frac{1}{q} = 1 \quad \Rightarrow \quad |x_1 y_1| + \cdots + |x_n y_n| \leq \sqrt[p]{|x_1|^p + \cdots + |x_n|^p} \cdot \sqrt[q]{|y_1|^q + \cdots + |y_n|^q}$$

$$\langle \mathbf{x}\,,\, \mathbf{y} \rangle \leq \|\mathbf{x}\|_{\ell_p} \cdot \|\mathbf{y}\|_{\ell_q}$$

$$\left| \frac{x_i}{\|\mathbf{x}\|_p} \cdot \frac{y_i}{\|\mathbf{y}\|_q} \right| \overset{\text{Young}}{\leq} \frac{1}{p} \frac{|x_i|^p}{\|\mathbf{x}\|_p^p} + \frac{1}{q} \frac{|y_i|^q}{\|\mathbf{y}\|_q^q}$$

$$\Rightarrow \quad \frac{\sum_{i=1}^n |x_i y_i|}{\|\mathbf{x}\|_p \, \|\mathbf{y}\|_q} \leq \frac{1}{p} \underbrace{\frac{\sum_{i=1}^n |x_i|^p}{\|\mathbf{x}\|_p^p}}_{=1} + \frac{1}{q} \underbrace{\frac{\sum_{i=1}^n |y_i|^q}{\|\mathbf{y}\|_q^q}}_{=1} = 1 \quad \Rightarrow \quad \checkmark$$



Math Preliminaries

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Eigenvalues and Eigenvectors

- Linear operators

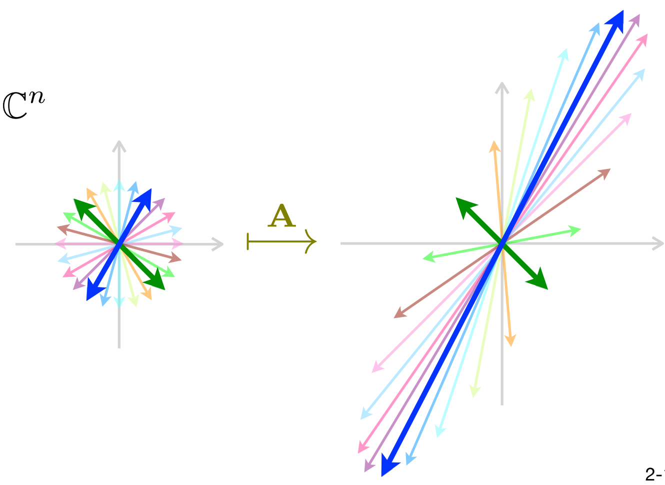
$$\mathbf{A} \in \mathbb{C}^{n\times n}$$

$$\text{If} \quad \lambda \in \mathbb{C},\ \mathbf{v} \in \mathbb{C}^n \ \ \&\ \ \mathbf{A}\cdot\mathbf{v} = \lambda\,\mathbf{v} \quad \Rightarrow \quad \begin{cases} \lambda & : \ \text{eignevalue} \\ \mathbf{v} & : \ \text{eignevector} \end{cases}$$

- Linear systems theory

$$\mathbf{v} \in \mathbb{C}^n \longrightarrow \boxed{\mathrm{L}} \longrightarrow \mathbf{A}\mathbf{v} \in \mathbb{C}^n$$

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} \quad \Rightarrow$$



Eigenvalues and Eigenvectors

- Symmetric matrices

$$\mathbf{A} \in \mathbb{R}^{n\times n}: \ \mathbf{A}^T = \mathbf{A} \qquad \text{or} \qquad \mathbf{A} \in \mathbb{C}^{n\times n}: \ \mathbf{A}^H = \mathbf{A}$$

$$\mathbf{A} = \begin{bmatrix} -1 & -2 & 1 \\ -2 & 2 & 3 \\ 1 & 3 & 1 \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} -1 & \mathrm{j} & 7 \\ -\mathrm{j} & 2 & 2-\mathrm{j} \\ 7 & 2+\mathrm{j} & 1 \end{bmatrix}$$

$$\mathbf{A}\mathbf{v} = \lambda\,\mathbf{v} \quad \Rightarrow \quad \lambda\|\mathbf{v}\|_2^2 = \mathbf{v}^H(\lambda\mathbf{v}) = \mathbf{v}^H\mathbf{A}\mathbf{v} = \mathbf{v}^H\mathbf{A}^H\mathbf{v} = (\mathbf{A}\mathbf{v})^H\mathbf{v} = \overline{\lambda}\|\mathbf{v}\|_2^2$$

$$\Rightarrow \quad \lambda = \overline{\lambda} \quad \Rightarrow \quad \boxed{\lambda \in \mathbb{R}}$$

$$\begin{cases} \mathbf{A}\mathbf{v}_1 = \lambda_1\mathbf{v}_1 \\ \mathbf{A}\mathbf{v}_2 = \lambda_2\mathbf{v}_2 \end{cases} \quad \Rightarrow \quad \lambda_2\mathbf{v}_1^H\mathbf{v}_2 = \mathbf{v}_1^H\mathbf{A}\mathbf{v}_2 = (\mathbf{A}\mathbf{v}_1)^H\mathbf{v}_2 = \lambda_1\mathbf{v}_1^H\mathbf{v}_2$$

$$\Rightarrow \quad \mathbf{v}_1^H\mathbf{v}_2 = 0 \quad \Rightarrow \quad \boxed{\langle \mathbf{v}_1, \mathbf{v}_2\rangle = 0}$$

Eigenvalues and Eigenvectors

- Symmetric matrices

$$\mathbf{A} \in \mathbb{R}^{n\times n}: \ \mathbf{A}^T = \mathbf{A} \qquad \text{or} \qquad \mathbf{A} \in \mathbb{C}^{n\times n}: \ \mathbf{A}^H = \mathbf{A}$$

$$\begin{cases} \lambda_1 \to \mathbf{v}_1 \\ \lambda_2 \to \mathbf{v}_2 \\ \quad\vdots \\ \lambda_n \to \mathbf{v}_n \end{cases}, \quad \lambda_i \in \mathbb{R} \quad , \quad \langle \mathbf{v}_i, \mathbf{v}_j\rangle = \mathbf{v}_i^H\mathbf{v}_j = \delta[i-j]$$

$$\mathbf{V} = [\mathbf{v}_1, \dots, \mathbf{v}_n] \quad \Rightarrow \quad \mathbf{A}\mathbf{V} = [\underbrace{\mathbf{A}\mathbf{v}_1}_{\lambda_1\mathbf{v}_1}, \dots, \underbrace{\mathbf{A}\mathbf{v}_n}_{\lambda_n\mathbf{v}_n}] = \mathbf{V}\begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

$$\mathbf{V}^H\mathbf{V} = \mathbf{I}_{n\times n} = \mathbf{V}\,\mathbf{V}^H$$

$$\Rightarrow \quad \mathbf{A}\underbrace{\mathbf{V}\mathbf{V}^H}_{\mathbf{I}_{n\times n}} = \boxed{\mathbf{A} = \mathbf{V}\,\mathrm{diag}(\lambda_1, \dots, \lambda_n)\,\mathbf{V}^H}$$

Eigenvalues and Eigenvectors

- Symmetric matrices

$$\mathbf{A} \in \mathbb{R}^{n\times n}: \;\; \mathbf{A}^T = \mathbf{A} \qquad \text{or} \qquad \mathbf{A} \in \mathbb{C}^{n\times n}: \;\; \mathbf{A}^H = \mathbf{A}$$

$$\mathbf{V} = \big[\mathbf{v}_1 \,, \,\dots \,, \,\mathbf{v}_n\big] \quad , \quad \mathbf{A} = \mathbf{V} \, \underbrace{\mathrm{diag}\big(\lambda_1 \,, \,\dots \,, \,\lambda_n\big)}_{\mathbf{\Lambda}} \, \mathbf{V}^H$$

$$\mathbf{x} \in \mathbb{C}^{n\times 1} \quad \Rightarrow \quad \frac{\mathbf{x}^H \mathbf{A}\,\mathbf{x}}{\mathbf{x}^H \mathbf{x}} = \frac{\mathbf{x}^H \mathbf{V}\,\mathbf{\Lambda}\,\mathbf{V}^H \mathbf{x}}{\mathbf{x}^H \mathbf{V}\;\mathbf{V}^H \mathbf{x}} = \frac{\big(\mathbf{V}^H \mathbf{x}\big)^H \mathbf{\Lambda} \big(\mathbf{V}^H \mathbf{x}\big)}{\big(\mathbf{V}^H \mathbf{x}\big)^H \big(\mathbf{V}^H \mathbf{x}\big)}$$

$$= \frac{\sum_{i=1}^{n} \lambda_i \, \big|\langle \mathbf{v}_i \,, \,\mathbf{x}\rangle\big|^2}{\sum_{i=1}^{n} \big|\langle \mathbf{v}_i \,, \,\mathbf{x}\rangle\big|^2}$$

$$\Rightarrow \boxed{\lambda_{\min} \;\leq\; \frac{\mathbf{x}^H \mathbf{A}\,\mathbf{x}}{\mathbf{x}^H \mathbf{x}} \;\leq\; \lambda_{\max}}$$

Eigenvalues and Eigenvectors

- Symmetric matrices

$$\mathbf{A} \in \mathbb{R}^{n\times n}: \;\; \mathbf{A}^T = \mathbf{A} \qquad \text{or} \qquad \mathbf{A} \in \mathbb{C}^{n\times n}: \;\; \mathbf{A}^H = \mathbf{A}$$

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$$

$$\text{Min-Max Theorem:} \begin{cases} \lambda_i = \displaystyle\min_{\substack{\mathcal{U} \\ \dim(\mathcal{U}) = i}} \left(\max_{0 \neq \mathbf{x} \in \mathcal{U}} \frac{\mathbf{x}^H \mathbf{A}\,\mathbf{x}}{\mathbf{x}^H \mathbf{x}} \right) \\[2ex] \lambda_i = \displaystyle\max_{\substack{\mathcal{U} \\ \dim(\mathcal{U}) = n-i+1}} \left(\min_{0 \neq \mathbf{x} \in \mathcal{U}} \frac{\mathbf{x}^H \mathbf{A}\,\mathbf{x}}{\mathbf{x}^H \mathbf{x}} \right) \end{cases}$$